Root Systems In Finite Symplectic Vector Spaces

Simon Lentner
Algebra And Number Theory (AZ), University Hamburg
Bundesstraße 55, D-20146 Hamburg

Abstract. We study realizations of root systems in possibly degenerate symplectic vector spaces over finite fields, up to symplectic isomorphisms. The main result of this paper is the classification of such realizations for the field $\mathbb{F}_2$. Thereby, each root system requires a specific degree of degeneracy of the symplectic vector space. Our main motivation for this paper is, that for each such realization of a root system we can construct a Nichols algebra over a nonabelian group.

Contents

1. Introduction

1.1. **Motivation and applications.** For a given $n \times n$ Cartan matrix with entries in $\mathbb{Z}$, a root system of rank n is generated by a basis of a n-dimensional euclidean vector space $V = \mathbb{C}^n$ (the simple roots) with the scalar products between all basis elements prescribed by the Cartan matrix. The Cartan matrix is usually visualized by a generalized Dynkin diagram. One demands stability of the set of all roots under the action of the Weyl/Coxeter group associated to the Cartan matrix. Root systems play among others a prominent role in the theory of Lie algebras as well as Nichols algebras, which appear naturally as Borel parts of quantum groups [AS10], such as $u_q(\mathfrak{g})$.

In our recent study of Nichols algebras over certain nonabelian groups G of nilpotency class 2 in [Len13] we started with a root system over $\mathbb{C}$ with given Cartan matrix for a Nichols algebra over an abelian group $G/[G,G]$. Then, this Nichols algebra was extended to G using an additional root system structure on a symplectic vector space $V = G/G^2$ over the finite field $\mathbb{F}_2$ with the same Cartan matrix. In this application, the symplectic form is induced by the commutator map of G and the prescribed Dynkin diagram is thus a G-decorated commutativity graph.
In the following article, we shall present a definition and classification of *symplectic root system* over the field $\mathbb{F}_2$. As every Cartan matrix can only have entries $0, 1 \in \mathbb{F}_2$, it is sufficient to consider simply-laced Dynkin diagrams and hence ordinary graphs. A symplectic root system over $\mathbb{F}_2$ is then defined as a decoration of the Dynkin diagram graph by simple roots, which are vectors in a (possibly degenerate) symplectic vector space $V = \mathbb{F}_2^n$, such that the decorations of two nodes are (symplectic) orthogonal iff the nodes are non-adjacent. The Coxeter group asociated to the Dynkin diagram over $\mathbb{C}$ acts on the set of all roots by symplectic isomorphisms. If the decorations form a basis of V, the symplectic root system is called minimal.

We achieve a complete classification of symplectic root systems over $\mathbb{F}_2$ up to symplectic isomorphisms on arbitrary graphs. Especially we clarify, which Dynkin diagram admits a symplectic root system for a given nullity, i.e. the degree of degeneracy of the symplectic form and hence the dimension of the nullspace $\dim(V^\perp)$. This nullity turns out to be bounded by the coclique number of the graph. We call symplectic root systems over nondegenerate symplectic vector spaces "extraspecial" as they correspond to extraspecial groups in Section 7.2.

We conclude by giving credit to previous work:

Our notion of a symplectic root system has appeared already in Lusztig's representation theory of finite Lie groups as a technical tool [Lusz84] Chp. 9. It was also extensively studied in singularity theory under the name *vanishing lattices*[1] by Wajnryb [Wa80], Chmutov [Ch82][Ch83] and Jansen [Jan83][Jan85]. The possible

[1] We thank Sergei Chmutov for helpful comments.

groups have been classified by [Jan83] Thm. 4.8 and the number of isomorphism classes of symplectic root systems for a given graph is reduced in [Jan85] Thm. 7.5 to the case $\mathbb{F}_2$. To the best of our knowledge, our combinatorically derived results are complementary and determine the explicit isomorphism classes over $\mathbb{F}_2$, as well as the unique nullity, and apply also for non-minimal symplectic root systems.

1.2. **Structure of the article.** We start with basic definitions in Section 2 and give first examples and properties in Section 3. Most importantly we can prove already at this point a universal property of minimal symplectic root systems, especially they are unique up to isomorphism, as well as their existence. This shows that a minimal symplectic root system of a given graph exists for precisely one isomorphy type of symplectic vector spaces (nullity). However, this result does not yet determine the nullity nor the isomorphy classes of the non-minimal symplectic root systems, which is content of the remaining article.

Corollary (Universal Property)**.** *Suppose (f,V) and (g,W) to be symplectic root systems on the same graph $\mathcal{G}$ and assume moreover (f,V) minimal. Then there exists a homomorphism of symplectic root systems $\phi:(f,V)\to(g,W)$. Especially two minimal symplectic root systems are always isomorphic.*

Lemma (Existence)**.** *For every graph $\mathcal{G}$ there exists a minimal symplectic root system. By the universal property it is unique up to isomorphism.*

We introduce a straightforward notion of quotients and find immediately:

Corollary. *For every graph $\mathcal{G}$ there is up to isomorphism a unique minimal symplectic root system and all symplectic root system of $\mathcal{G}$ are quotients thereof.*

In Section 4 we then consider the restriction of a symplectic root systems to an induced subgraph $\mathcal{G}\subset\mathcal{D}$ and derive bounds for the change in nullity of the symplectic root systems. As an application we prove a bound on the nullity of a symplectic root system in terms of the coclique number of the graph and briefly discuss the extremal cases of the inequality (ADE- vs. complete graphs).

In the main Section 5 of this paper we introduce a construction that extends a given minimal symplectic root system on a subgraph by one node.

Theorem (Minimal Extensions)**.** *Let $\mathcal{D}$ be a graph, $p\in\mathcal{D}$ a node and $G=(g,W)$ a minimal symplectic root system of a spanning subgraph $\mathcal{G}:=\mathcal{D}-p$. Then there exists a unique minimal symplectic root system $F=(f,V)$ on $\mathcal{D}$ extending G.*

Proof. The construction proceeds in the following steps for extending extraspecial, nullspace and finally arbitrary symplectic root systems. From the second step on, the extensions fall into two distinct cases yielding for V either higher or lower nullity than W according to the different cases in the Restriction Theorem 4.3.

- In Section 5.1 we construct almost extraspecial extensions of extraspecial symplectic root systems, i.e. W nondegenerate. The proof uses the minimality of G to express the indicator function λ of a neighbourhood of p in $\mathcal{D}$

as a linear form $\tilde{\lambda}$. Then is uses the assumed nondegeneracy to construct a distinguished element $w_0 \in W$ and a thereof the new decoration $f(p) \in V$.

- In Section 5.2 we classify in contrast extensions of symplectic root systems consisting only of nullspace $W = W^{\perp}$. Especially $\mathcal{G}$ is totally disconnected. Thereby we need surprisingly the choice of an additional nondegenerate symmetric bilinear form $(,)$ on the nullspace regarded as vector space over $\mathbb{F}_2$. As before we construct a linear form $\tilde{\lambda}$, but we yield two cases: Either $\tilde{\lambda} = 0$, then $\mathcal{D}$ is again totally disconnected, $V = V^{\perp}$ is extended by yet another nullvector and has higher nullity then W. For $\tilde{\lambda} \neq 0$ we decompose $W = \ker(\tilde{\lambda}) \oplus x\mathbb{k}$ and extend x by the to-be-definied decoration $f(p) = y$ to a new hyperbolic plane H_1. Especially V then has lower nullity than W
- In Section 5.3 we combine the preceeding results and achieve the final classification result for extending arbitrary sympletic root systems by one point. The crucial ingredient is an artificial nondegenerate, symmetric bilinear form $\langle\langle, \rangle\rangle$ extending the symplectic form. Thereby we effectively write the neighbourhood of the new point as a symmetric difference of two graphs obtained by the two previous methods.
- In Section 5.4 we give an additional very explicit formula for 2-point-extensions of a given extraspecial symplectic root system. The criterium avoids the use of the artificial mixed bilinear form and provides a nice characterization in terms of the two 1-point extensions and their possible interaction. This is used in the example calculations for ADE-type in Section 6.

□

Note that determining the nullity for a given graph is tedious and usually requires to successively apply the extension theorem. As an example, in Section 6 we determine explicitly decoration and nullity for all symplectic root systems corresponding to Dynkin diagrams of finite Cartan type, i.e. in the ADE family. We start with an induction on $A_{2n-2} \to A_{2n}$, which turns out to yield minimal symplectic root systems of extraspecial type. Then we extend by explicit nodes to reach the other diagrams with higher nullity. We indeed find that Cartan type root systems typically require the smallest possible nullity (0 or 1), except D_{2n} needs 2. We then determine all non-minimal quotients up to Dynkin diagram automorphisms.

As further topics, we first connect in Section 7.1 the notion of symplectic root systems to the well-known notion over $\mathbb{C}$. We describe, how a symplectic decoration of the graph as considered here can be additively extended to a full root systems and prove that the Coxeter group accociated to the prescribed Cartan matrix (e.g. the Weyl group) acts on this natural set of symplectic vectors by symplectic isomorphisms.

Section 7.2 explains the application of symplectic root systems to Nichols algebras. In [Len13] we constructed the first Nichols algebras of rank > 2 over nonabelian groups G. The construction starts with a known finite-dimensional Nichols algebra over an abelian group Γ of simply-laced Cartan type with a diagram automorphism $\mathbb{Z}_2$. Then, a symplectic root system over the field $\mathbb{F}_2$ is used to construct a new link-indecomposable finite-dimensional covering Nichols algebra over a central extension $\mathbb{Z}_2 \to G \to \Gamma$ and again over $\mathbb{C}$.

Here, the symplectic vector space over $\mathbb{F}_2$ is $V := \Gamma/\Gamma^2$ with the symplectic form induced by the commutator map on G. The symplectic root systems then provides a basis of V adapted to the existing Dynkin diagram. Especially the nullity of V corresponds to a specific size of the center $Z(G)$. We hope the present classification will enable the classification of diagonal Nichols algebras over nilpotent groups.

2. Main Definition

Throughout this article, we assume all graphs to be finite and all vector spaces to be finite-dimensional.

Definition 2.1. *The following unusually general notion is custom in the theory of p-groups, see e.g. [Hup83] paragraph II.9 (p. 215): A (possibly degenerate) symplectic vector space is a vector space V over a field $\Bbbk$ with a (possibly degenerate) bilinear form*

$$\langle , \rangle : \ V \times V \to \Bbbk$$

that is alternating:

$$\langle v, v \rangle = 0$$

Note that alternating forms are always skew-symmetric, but for char($\Bbbk$) = 2 skew-symmetric is equivalent to symmetric and does not imply alternating. The nullspace (or radical) of V is defined as

$$V^\perp := \{v \in V \mid \forall_{w \in V}\ \langle v, w \rangle = 0\}$$

Note that $W^\perp$ for a subspace denotes the radical $\subset W$, not the complement $\subset V$.

Theorem 2.2. *([Hup83] Satz II.9.6 (p.217)) Every (finite-dimensional) symplectic vector space $V, \langle , \rangle$ possesses a symplectic basis $\{x_1, \ldots x_n, y_1, \ldots y_n, z_1, \ldots z_k\}$ consisting of mutually orthogonal nullvectors $z_i \in V^\perp$ and hyperbolic planes $H_i = x_i\Bbbk \oplus y_i\Bbbk, \quad \langle x_i, y_i \rangle = 1$*

$$V \cong (x_1\Bbbk \oplus y_1\Bbbk) \oplus^\perp (x_2\Bbbk \oplus y_2\Bbbk) \oplus^\perp \cdots \oplus^\perp z_1\Bbbk \oplus^\perp z_2\Bbbk \oplus^\perp \cdots$$

Definition 2.3. *Denote by $k = dimV^\perp$ the nullity, the number of nullvectors z_i in a basis. Further denote by n the conullity, the number of hyperbolic planes $H_i = x_i\Bbbk \oplus y_i\Bbbk$ generating V. Altogether $2n + k = \dim V$. By Theorem 2.2 the type (n, k) precisely characterizes the isomorphism type of a symplectic vector space V.*

- *We call V extraspecial, iff the form is nondegenerate $V^\perp = 0$, i.e. the V is of type $(n, 0)$ and necessarily of even dimension $2n$. The name is chosen as the associated p-groups are usually called extraspecial, see Section 7.2.*

- *We call V almost extrapecial, iff* $\dim V^\perp = 1$*, i.e. V is of type $(n,1)$ and necessarily of odd dimension $2n+1$. The name is chosen as the associated p-groups are usually called almost extraspecial, see Section 7.2.*

The key notion of a symplectic root system over $\mathbb{F}_2$ will now be given purely in terms of graph theory, as the Dynkin diagram is always simply-laced in characteristic 2. Note that we yet have no satisfying general definition for arbitrary characteristic.

Definition 2.4. *A symplectic root system $F = (f, V)$ over the field $\Bbbk = \mathbb{F}_2$ on a (finite) graph $\mathcal{D}$ consists of a (finite-dimensional) symplectic vector space V over the field $\mathbb{F}_2$ together with a node decoration $f : \mathcal{D} \to V$ such that*

- *The image $f(\mathcal{D})$ generates V as a vector space.*
- *Two nodes $p, q \in \mathcal{D}$ are adjacent in the graph $\mathcal{D}$ iff their f-decorations in V are not orthogonal $\langle f(\alpha), f(\beta)\rangle \neq 0$ (then already $= 1_{\mathbb{F}_2}$),*

We call $F = (f, V)$ of type (n, k), iff the symplectic vector space V is of type (n, k). We call F minimal iff the decorations $\{f(\alpha)\}_{\alpha\in\mathcal{D}}$ form a basis of V.

3. First Properties And Examples

Example 3.1. *Let $\mathcal{G} = A_1 = \{p\}$ be an isolated point. Because the node decorations $f(p)$ have to generate V, the only symplectic root systems $F = (f, V)$ are*

- *$V = z\mathbb{F}_2$ with $f(p) = z$ of type $(0, 1)$, which is minimal.*
- *$V = \{0\}$ with $f(p) = 0$ of type $(0, 0)$, which is not minimal.*

Lemma 3.2. *Let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ be a disconnected union of subgraphs. Then any minimal symplectic root system $F = (f, V)$ decomposes as $V = V_1 \oplus^\perp V_2$ with $f(\mathcal{G}_i) \subset V_i$ inducing minimal symplectic root system on each subgraphs. The orthogonal sum especially implies, that if these smaller symplectic root system are of type (n_1, k_1) resp. (n_2, k_2), then F is of type $(n_1 + n_2, k_1 + k_2)$.*

Proof. Because the decorations $\{f(p)\}_{p\in\mathcal{G}}$ by the assumed minimality of F form a basis of V, we may decompose V as direct sum of subspaces V_i with basis $\{f(p)\}_{p\in\mathcal{G}_i}$:

$$V = V_1 \oplus V_2$$

$\mathcal{G}_1$, $\mathcal{G}_2$ were assumed to be mutually disconnected, so the defining property of symplectic root systems implies

$$\forall_{p\in\mathcal{G}_1\ q\in\mathcal{G}_2}\ \langle f(p), f(q)\rangle = 0$$

As the sets $\{f(p)\}_{p\in\mathcal{G}_i}$ generate respectively V_i the sum is orthogonal as asserted. □

Example 3.3. *Let $\mathcal{G}$ be a totally disconnected graph, then the preceeding lemma shows each minimal symplectic root system to be of type $(0, |\mathcal{G}|)$ as V decomposes orthogonally into 1-dimensional nullspaces for each isolated point as in example 3.1.*

If the graph contains proper edges, a pure nullspace $V = V^\perp$ will not suffice:

Example 3.4. *Let $\mathcal{G} = A_2 = \{p, q\}$ be two connected points. The decorations $f(p), f(q)$ have to generate V, thus it can have dimension at most 2. Pure nullspace cases can be discarded, because $\langle f(p), f(q)\rangle = 0$ contrary to the assumed edge pq.*

Hence the only remaining possibility is a single hyperbolic plane $V = x\mathbb{F}_2 \oplus^{\not\perp} y\mathbb{F}_2$, i.e. V of type $(1,0)$. Here, indeed we yield a minimal symplectic root system $f(p) = x$ and $f(q) = y$ with $\langle f(p), f(q)\rangle = 1$. Note that there are many other possibilities, e.g. $f(p) = x + y$ and $f(q) = y$, but these choices obviously just differ by a linear isomorphism that preserves the symplectic form.

The last example shows, that one has to classify symplectic root system according to some isomorphism criterion.

Definition 3.5 (Homomorphisms of symplectic vector spaces)**.** *Let V, W be (possibly degenerate) symplectic vector spaces, then we call a linear map $\phi : V \to W$ a symplectic homomorphism, iff*

$$\forall_{v,w\in V}\ \langle v, w\rangle_V = \langle \phi(v), \phi(w)\rangle_W$$

If moreover ϕ is bijective, we call ϕ symplectic isomorphism. Note that if V is degenerate, a symplectic homomorphism needs not to be bijective. Rather, ϕ might possess a kernel $\ker(\phi) \subset V^\perp$. The conullies of V and $Im(\phi) \subset W$ coincide, but the nullity of V might be higher.

Definition 3.6 (Morphisms)**.** *Let $\mathcal{G}$ be a fixed graph and $F = (f, V),\ G = (g, W)$ symplectic root systems of $\mathcal{G}$. A symplectic root system homomorphism $\phi : F \to G$ is a linear map $\phi : V \to W$ that intertwines the decorations: $\phi \circ f = g$. If moreover ϕ is bijective, we call ϕ a symplectic root system isomorphism.*

Remark 3.7. *Note that we require the graph $\mathcal{D}, \mathcal{G}$ to be fixed sets. Hence a graph automorphism on $\mathcal{D}$ may interchange non-isomorphic symplectic root system in this definition. For nontrivial examples of ADE-type see Remark 6.2.*

The assertion of $\phi \circ f = g$ for symplectic root system morphisms already proves ϕ to be a a symplectic homomorphism, as the defining property of a symplectic root system (f, V) already fixes the entire symplectic form on V:

Lemma 3.8. *Suppose $\phi : (f, V) \to (g, W)$ to be a homomorphism between symplectic root systems. Then ϕ is already a surjective symplectic homomorphism $\langle v, w\rangle_V = \langle \phi(v), \phi(w)\rangle_W$.*

Proof. We calculate that ϕ preserves the symplectic form. First, the defining property of the symplectic root systems (f, V) and (g, W) on the same graph already fixes the value of the symplectic form on the decorations:

$$\forall_{p,q\in\mathcal{G}}\ \ \langle f(p), f(q)\rangle = \langle g(p), g(q)\rangle = 0_{\mathbb{F}_2}, 1_{\mathbb{F}_2}$$

Because we assumed ϕ to be a homomorphism of symplectic root systems $\phi \circ f = g$

$$\forall_{p,q\in\mathcal{G}}\ \ \langle f(p), f(q)\rangle = \langle \phi(f(p)), \phi(f(q))\rangle$$

so ϕ is a symplectic homomorphism $V \to W$. Moreover, by definition of a symplectic root system the images $f(p)$ generate V, hence ϕ is necessarily surjective. $\square$

Because for a minimal symplectic root system (f, V) the decorations $\{f(p)\}_{p\in\mathcal{G}}$ even form by definition a basis of V, we always find in this case a unique linear map $\phi : V \to W$ with $g = \phi \circ f$.

Corollary 3.9 (Universal Property)**.** *Suppose (f, V) and (g, W) to be symplectic root systems on the same graph $\mathcal{G}$ and assume moreover (f, V) minimal. Then there exists a unique homomorphism of symplectic root systems $\phi : (f, V) \to (g, W)$. Especially two minimal symplectic root systems are always isomorphic.*

On the other hand, since $\mathcal{G}$ is supposed finite: If (g, W) is minimal, then any symplectic root system homomorphism $\phi : (f, V) \to (g, W)$ is an isomorphism.

By inducing a symplectic form on the formal vector space generated by the nodes of $\mathcal{G}$ we find morover[2]:

Lemma 3.10 (Existence)**.** *For every graph $\mathcal{G}$ there exists a minimal symplectic root system. By the universal property (Corollary 3.9) it is unique up to isomorphism.*

Proof. Consider the vector space V over $\mathbb{F}_2$ generated by a basis $\{v_p \mid p \in \mathcal{G}\}$ and the symplectic form defined on this basis by

$$\langle v_p, v_q\rangle = \begin{cases} 1_{\mathbb{F}_2}, & \text{if } p \neq q \text{ adjacent} \\ 0_{\mathbb{F}_2}, & \text{else} \end{cases}$$

The bilinear form $\langle , \rangle$ is symmetric and we check it is indeed an alternating form over $\mathbb{F}_2$:

$$\langle \sum_{p\in\mathcal{G}} a_p v_p, \sum_{p\in\mathcal{G}} a_p v_p\rangle = \sum_{p\in\mathcal{G}} a_p^2 \langle v_p, v_p\rangle + 2 \sum_{p\neq q\in\mathcal{G}} a_p a_q \langle v_p, v_q\rangle = 0$$

Moreover, it is clear that by definition $F := (f, V)$ with $f(p) = v_p$ is a symplectic root system. $\square$

Let conversely be $\phi : V \to W$ be a surjective symplectic homomorphism, i.e ϕ surjective and

$$\langle v, w\rangle_V = \langle \phi(v), \phi(w)\rangle_W$$

For a given symplectic root system $F = (f, V)$ on a graph $\mathcal{G}$ we may consider the pair $G := (\phi \circ f, W)$ and verify for G the defining property of a symplectic root system on the same underlying graph $\mathcal{G}$. First we check

$$\forall_{p,q\in\mathcal{G}} \;\; \langle(\phi \circ f)(p), (\phi \circ f)(q)\rangle_W = \langle f(p), f(q)\rangle_V = 0_{\mathbb{F}_2}, 1_{\mathbb{F}_2}$$

Secondly by definition the symplectic root system F requires $\mathrm{Im}(f)$ to generate V and thus by the assumed surjectivity of ϕ the images of $\phi \circ f$ generate again W. Hence we have proven that G is indeed a new symplectic root system. Especially by dimensionality reasons, the new root system cannot be minimal unless ϕ is a

[2]We thank the referee for pointing out this significantly easier approach

symplectic isomorphism; in which case the symplectic root systems G and F are isomorphic.

Definition 3.11 (Quotient). *Let (f, V) be a symplectic root system on a graph $\mathcal{G}$ and $\phi : V \to W$ be a surjective symplectic homomorphism, then we define the quotient symplectic root system on the same graph by $(\phi \circ f, W)$. Thereby ϕ becomes a homomorphism between the symplectic root systems.*

In view of the universal property and the existence of minimal symplectic root systems by Corollary 3.9 and Lemma 3.10) we have altogether:

Corollary 3.12. *For every graph $\mathcal{G}$ there is up to isomorphism a unique minimal symplectic root system and all symplectic root system of $\mathcal{G}$ are quotients thereof.*

4. A Restriction Theorem

Definition 4.1. *Let $F = (f, V)$ be a symplectic root system of a graph $\mathcal{D}$ and $\mathcal{G} \subset \mathcal{D}$ the induced subgraph of a subset of vertices of $\mathcal{D}$. Then we define the restriction $F|_{\mathcal{G}}$ to be $(f|_{\mathcal{G}}, W)$, where f is restricted to $\mathcal{G}$ and $W \subset V$ is the subspace generated by the image $f(\mathcal{G})$. The restriction is clearly a symplectic root system of $\mathcal{G}$.*

Lemma 4.2 (Orthogonal Projection). *This is a generalization of orthogonal projection to the degenerate case (and probably not new): Suppose $W \subset V$ to be symplectic vector spaces and suppose a given $v \in V$ such that $v \perp W^{\perp}$. Then there exists a decomposition $v = v_0 + v_W$ with $v_W \in W$ and $v_0 \perp W$. Moreover, all such decompositions are of the form $v = (v_0 + t) + (v_W - t)$ with some $t \in W^{\perp}$.*

Proof. Denote by x_i, y_i, z_j any symplectic basis of W (Theorem 2.2) and define

$$v_W := \sum_k (\langle v, y_k\rangle x_k - \langle v, x_k\rangle y_k)$$

Then as claimed $v_W \in W$ by construction. Moreover we define $v_0 := v - v_W$ and check on the symplectic basis that as asserted $v_0 \perp W$:

$$\begin{aligned}\langle v_0, x_i\rangle &= \langle v - v_W, x_i\rangle \\ &= \langle v, x_i\rangle - \sum_k (\langle v, y_k\rangle\langle x_k, x_i\rangle - \langle v, x_k\rangle\langle y_k, x_i\rangle) \\ &= \langle v, x_i\rangle + \langle v, x_i\rangle\langle y_i, x_i\rangle = 0\end{aligned}$$

The same calculation proves $v_0 \perp y_i$. Finally we have by construction $v_0 \in v + W$ and hence $v_0 \perp W^{\perp}$ by the additional assumption $v \perp W^{\perp}$.

For the last claim, assume that $v = v_0 + v_W = v_0' + v_W'$ with $v_0, v_0' \in W$ and $v_W, v_W' \perp W$. Then $t := v_0' - v_0 = -(v_W' - v_W)$ is contained in W and $t \perp W$, hence $t \in W^{\perp}$. □

Theorem 4.3 (Restriction). *Let $F = (f, V)$ be a symplectic root system of type (n, k) on a graph $\mathcal{D}$, let $p \in \mathcal{D}$ be any node and denote by $\mathcal{G}$ the induced subgraph on $\mathcal{D} - p$. Then the restricted symplectic root system $F|_{\mathcal{G}} = (f|_{\mathcal{G}}, W)$ is either*

(1) *of same type (n, k) and F was not minimal*

(2) *of type* $(n, k-1)$*, especially* F *was not extraspecial*
(3) *of type* $(n-1, k+1)$*, especially* $F|_{\mathcal{G}}$ *is not extraspecial*

Proof. Consider the subspace $W \subset V$ generated by the image $f(\mathcal{G})$. Either of the following cases applies:

(1) $V = W$, then $F|_{\mathcal{G}}$ has the same type as F. Further, F could not have been minimal, because the decoration $f(p) \in W$ was linear dependent.
(2) $V = f(p)\mathbb{F}_2 \oplus W$. We denote by (n', k') the type of W and aim to determine the two remaining cases as $k' = k \pm 1$. Then V contains at least n' mutually orthogonal hyperbolic planes, hence $n \geq n'$, and the nullspace is $\dim(V^{\perp}) \geq k'-1$ (it depends on whether $f(p) \perp W^{\perp}$). This implies by $\dim(V) = 2 \cdot n + k$ and $\dim(W) = 2 \cdot n' + k'$ that the type (n', k') is as claimed.

□

Corollary 4.4. *For* γ *the size of a maximal coclique* Γ *in a graph* $\mathcal{G}$ *and* (f, V) *a symplectic root system of* $\mathcal{G}$ *of type* (n, k) *we get the following bound:*

$$n \leq |\mathcal{G} - \Gamma| = |\mathcal{G}| - \gamma$$

Proof. We perform induction along $|\mathcal{G} - \Gamma|$. For $\mathcal{G} = \Gamma$ the graph is totally disconnected, hence obviously (see Lemma 3.2) any symplectic root system is of type $n = 0$ and the bound holds with equality.

Now we proceed with the induction step: Suppose (f, V) to be a symplectic root system on some graph $\mathcal{G}$ with a maximal coclique $\Gamma \subsetneq \mathcal{G}$. Choose any $p \in \mathcal{G} - \Gamma$ and consider the restriction $(f|_{\mathcal{G}-p}, W)$ of type (n', k'). By induction,

$$n' \leq |\mathcal{G} - p| - \gamma' = |\mathcal{G}| - 1 - \gamma$$

because Γ is also a coclique in $\mathcal{G} - p$, which is maximal, hence $\gamma' = \gamma$. Now by Theorem 4.3 we have $n = n'$ or $n = n' + 1$, hence $n \leq n' + 1$ and as asserted

$$n \leq n' + 1 \leq |\mathcal{G}| - 1 - \gamma + 1 = |\mathcal{G}| - \gamma$$

□

Note that the bound in the corollary is met for all Cartan type graphs in Section 6 (even D_{2n}). On the other hand, the complete graphs $\mathcal{G} = \mathcal{K}_N$ provide examples with $\gamma = 1$ (the maximal possible value on the right hand side) but still (almost-) extraspecial $n \approx |\mathcal{G}|/2$ – the minimum possible values on the left hand side.

5. An Extension Theorem

Definition 5.1. *Let* $\mathcal{D}$ *be a graph and* $\mathcal{G} \subset \mathcal{D}$ *a subgraph induced by a subset of vertices in* $\mathcal{D}$*. A symplectic root system* $F = (f, V)$ *on* $\mathcal{D}$ *is called extension of a given symplectic root system* $G = (g, W)$ *on* $\mathcal{G}$*, if the restriction* $F|_{\mathcal{G}} = G$*.*

5.1. **Extending Extraspecial Symplectic Root Systems.**

Theorem 5.2 (Extension of Extraspecials)**.** *Let $\mathcal{D}$ be a graph, $p \in \mathcal{D}$ a node and $G = (g, W)$ a minimal extraspecial symplectic root system of $\mathcal{G} := \mathcal{D} - p$. Then there exists an almost extraspecial minimal extension $F = (f, V)$ of G to $\mathcal{D}$.*

Proof. To construct F, take $V = z\mathbb{F}_2 \oplus^\perp W$, which is an almost extraspecial symplectic vector space, and $f|_W = g$ extending G as demanded. To choose the new decoration $f(p)$, we consider the indicator function of the neighbourhood of p in $\mathcal{G}$:

$$\lambda : \mathcal{G} \to \mathbb{F}_2 \qquad \forall_{q \in \mathcal{G}}\ \lambda(q) = 1\ :\Leftrightarrow\ pq \in Edges(\mathcal{D})$$

Because G was assumed minimal i.e. $g(\mathcal{G})$ is a basis of W, there is a unique linear form $\tilde{\lambda}$ on W such that

$$\tilde{\lambda} : W \to \mathbb{F}_2 \qquad \forall_{q \in \mathcal{G}}\ \lambda(q) = \tilde{\lambda}(g(q))$$

Because we assumed G extraspecial, i.e. W nondegenerate, there is a unique element

$$w_0 \in W \qquad \forall_{w \in W}\ \langle w_0, w \rangle = \tilde{\lambda}(w)$$

The choice $f(p) := w_0 + z$ then turns $F = (f, V)$ into a symplectic root system, as

$$\forall_{q \in \mathcal{G}}\ \langle f(p), f(q) \rangle = \langle w_0, g(q) \rangle = \tilde{\lambda}(g(q)) = \lambda(q)$$

while for elements $q, q' \in \mathcal{G}$ the condition was already satisfied in G. Moreover, as the $f(q) = g(q)$ for $q \in \mathcal{G}$ already formed a basis of W by assumption of minimality, together with $f(p) \in z + W$ we get a basis of V. Hence we constructed an almost extraspecial minimal symplectic root system on $\mathcal{D}$. $\square$

5.2. **Extending Nullspace Symplectic Root Systems.** The proof of the last extension theorem crucially relies on the nondegeneracy of the symplectic form. However, for a degenerate vector space W of type (n, k), generally only few edge-configurations to the new point p can be constructed by simply using an existing vector in $w_0 \in W$ together with a new nullvector z as above.

This can be seen already by counting, as there are $|W/W^\perp| = 2^{2n}$ possible w_0 in contrast to 2^{2n+k} possible edge configurations. Note that these cases correspond to the restriction case 2 in Theorem 4.3. The remaining $2^{2n}\left(2^k - 1\right)$ configurations require, in addition to the choice of an element w_0, a new symplectic base-pair in V from a former nullvector in W causing additional new edges – corresponding to the restriction case 3 in Theorem 4.3.

Before turning to the question of extending an arbitrary minimal symplectic root system, let us look at the opposite extreme case: The extension of a symplectic root system of type $(0, k)$, i.e. on a pure nullspace $W = W^\perp$. Because of the Restriction Theorem 4.3, the extension V can only have type $(0, k+1)$ (=new nullvector) or $(1, k-1)$ (=new hyperbolic plane). We show that both cases can be constructed depending on the aspired graph extension. The following proof models similarly to the proof of Theorem 5.2 the indicator function λ of a neighbourhood of p by a linear form $\tilde{\lambda}$ via an artificially chosen nondegenerate symmetric bilinear form on

$W^\perp$. Then in the nontrivial case $\tilde{\lambda} \neq 0$ one constructs V by extending the kernel of $\tilde{\lambda}$ by a new hyperbolic plane $H_1 = x_1\mathbb{k} \oplus y_1\mathbb{k}$.

Theorem 5.3 (Extension of Nullspaces). *Let $G = (g, W)$ be a minimal symplectic root system of type $(0, k)$ on a graph $\mathcal{G}$, i.e. $W = W^\perp$ a pure nullspace. Let $\mathcal{D}$ be a graph with $p \in \mathcal{D}$ a node, such that $\mathcal{G} := \mathcal{D} - p$. Then there exists a minimal extension $F = (f, V)$ of G to $\mathcal{D}$ of either one of the following types:*

- $(0, k+1)$, *if $\mathcal{D}$ is completely disconnected.*
- $(1, k-1)$, *if $\mathcal{D}$ contains at least one edge.*

The extension F is unique up to isomorphism by Corollary 3.12.

Note that in this Theorem the graph $\mathcal{G}$ has to be totally disconnected by definition and $\mathcal{D}$ is a star graph with center p connected to a subset of $\mathcal{G}$, the neighbourhood of $p \in \mathcal{D}$. The first case of the Theorem is then if the neighbourhood of p in $\mathcal{D}$ is empty.

Proof. The first case is a trivial construction, so let's turn to the second case: In order to produce edges at all (see Example 3.4), V of dimension $1 + k$ cannot be a pure nullspace and thus of type $(n, 1 + k - 2n)$ for $n > 0$. The vice-versa restriction to $\mathcal{G} = \mathcal{D} - p$ is the assumed G of type $(0, k)$, so by Theorem 4.3 only $n = 1$ an hence type $(1, k-1)$ remains: This is case 2 in the cited Restriction Theorem: V possesses one hyperbolic plane $H_1 = x\mathbb{k} \oplus y\mathbb{k}$, that is separated by the restriction to $\mathcal{G}$ and W.

We may explicitly construct such a V as follows: Consider again the indicator function λ of the neighbourhood of p in $\mathcal{D}$ (such a neighbourhood is subset of $\mathcal{G}$):

$$\lambda : \mathcal{G} \to \mathbb{F}_2 \qquad \forall_{q \in \mathcal{G}}\ \lambda(q) = 1 \ :\Leftrightarrow\ pq \in Edges(\mathcal{D})$$

By assumption of the second case of the assertion, $\mathcal{D}$ not totally disconnected, so $\lambda \neq 0$. Because G was assumed minimal i.e. $g(\mathcal{G})$ is a basis of W, there is a unique linear form $\tilde{\lambda} \neq 0$ on W such that

$$\tilde{\lambda} : W \to \mathbb{F}_2 \qquad \forall_{q \in \mathcal{G}}\ \lambda(q) = \tilde{\lambda}(g(q))$$

Choose an element $x \in W$ with $\tilde{\lambda}(x) = 1_{\mathbb{F}_2}$, which is possible by $\tilde{\lambda} \neq 0$. Such an x can be used to project $W \to \ker(\tilde{\lambda})$ by $w \mapsto w - \tilde{\lambda}(w) \cdot x$.

With these preperations we define:

$$\begin{aligned}
V &:= \underbrace{\ker\left(\tilde{\lambda}\right)}_{V^\perp} \oplus^\perp (x\mathbb{k} \oplus y\mathbb{k}) = W \oplus y\mathbb{k} \qquad \langle x, y\rangle := 1 \\
f(p) &:= y \\
f(q) &:= g(q) \\
&:= \underbrace{g(q) - \tilde{\lambda}\left(g(q)\right) \cdot x}_{\in \ker(\tilde{\lambda})} + \tilde{\lambda}\left(g(q)\right) \cdot x \qquad \forall q \in \mathcal{G}
\end{aligned}$$

Note that the vice-versa restriction to $\mathcal{G}$ is equal to G. The pair (f, V) defined this way satisfies the axioms of a symplectic root system. This can be seen for node pairs p, q involving the new point p by

$$\begin{aligned}
\forall_{q\in\mathcal{G}}\ \langle f(p), f(q)\rangle &= \langle y, \underbrace{g(q) - \tilde{\lambda}\left(g(q)\right)\cdot x}_{\in\ker(\tilde{\lambda})\subset V^{\perp}} + \tilde{\lambda}\left(g(q)\right)\cdot x\rangle \\
&= \langle y, \tilde{\lambda}\left(g(q)\right)\cdot x\rangle \\
&= \tilde{\lambda}\left(g(q)\right) = \lambda(q)
\end{aligned}$$

while for node pairs $q, q' \in \mathcal{G}$ the condition was already satisfied in G. To show that the new decorations form a basis, note first the $g(q)$ for $q \in \mathcal{G}$ already formed a basis of W by assumption of minimality and the new decoration $f(p) := y$ is linearly independent. $\square$

5.3. **Extending Arbitrary Symplectic Root Systems.** Quite surprisingly, a uniform treatment of the two extremal cases (nondegenerate and nullspace) is possible and yields an equally strong statement as in the cases above. For this, one has to introduce an artificial non-symplectic nondegenerate bilinear form.

Definition 5.4 (Mixed Completion)**.** *Let W, $\langle , \rangle$ be a symplectic vector space over $\Bbbk$ and choose $\pi : W \to W^{\perp}$ a fixed projection to the nullspace and $(,)$ a nondegenerate symmetric bilinear form on the vector space $W^{\perp}$. We define the mixed completion $\langle\langle , \rangle\rangle : W \times W \to \Bbbk$ by*

$$\langle\langle v, w\rangle\rangle := \langle v, w\rangle + (\pi(v), \pi(w))$$

Lemma 5.5. *The mixed completion is a nondegenerate bilinear form on W.*

Proof. Suppose some $v \in W$ to be in the radical, i.e.

$$\forall_{w\in W}\ \langle\langle v, w\rangle\rangle = \langle v, w\rangle + (\pi(v), \pi(w)) = 0$$

Suppose first $w \in W^{\perp}$, then the left (symplectic) term vanishes, hence for the entire term to be 0, the second $(, \pi(w))$-term has to vansh as well. Since $\pi|_{W^{\perp}}$ is the identity and $(,)$ is nondegenerate, we deduce $\pi(v) = 0$ i.e. $v \in \ker(\pi)$. But then already for all $w \in W$ $(\pi(v), \pi(w)) = 0$ and hence the assumption reduces to:

$$\forall_{w\in W}\ \langle v, w\rangle = 0 \qquad \Rightarrow v \in W^{\perp}$$

Because π was a projection the two deductions $v \in \ker(\pi)$ and $v \in W^{\perp}$ amount to $v = 0$. Hence the radical is $\{0\}$ and $\langle\langle , \rangle\rangle$ is indeed nondegenerate as asserted. $\square$

With this tool we combine the techniques for extending extraspecials as well as nullspaces (Sections 5.1 and 5.2) by effectively writing the new graph as a symmetric difference of two graphs obtained by the two methods. Thus we achive a general extension result for arbitrary minimal symplectic root systems:

Theorem 5.6 (Minimal Extensions)**.** *Let $\mathcal{D}$ be a graph, $p \in \mathcal{D}$ a node and $G = (g, W)$ a minimal symplectic root system of $\mathcal{G} := \mathcal{D} - p$. Then there exists a unique minimal extension $F = (f, V)$ of G to $\mathcal{D}$.*

Remark 5.7. *Restriction (Theorem 4.3) shows, that if G is of type (n', k'), then F is either of type $(n, k) = (n', k' + 1)$, which case is similar to extraspecial extension Theorem 5.2, or $(n, k) = (n' + 1, k' - 1)$, which combines the approach with a new symplectic basepair as in Theorem 5.3.*

Which case applies and which precise decoration the new point receives can generally only be decided along the steps of the proof below. However, uniqueness opens the path to directly write down an extended symplectic root system. Also, the case of a double extension of an extraspecial symplectic root system allows for a direct treatment; see Lemma 5.8.

Proof. Choose a fixed projection to the nullspace $\pi : W \to W^{\perp}$, a nondegenerate symmetric bilinear form $(,)$ on the vector space $W^{\perp}$ and a mixed completion of the symplectic form on W (Definition 5.4), which is nondegenerate by Lemma 5.5.

$$\langle\langle v, w\rangle\rangle = \langle v, w\rangle + (\pi(v), \pi(w))$$

The proof proceeds at first largly along the Extraspecial Extension Theorem 5.2:

- Consider the indicator function of the neighbourhood of p $(\subset \mathcal{G})$:

$$\lambda :\ \mathcal{G} \to \mathbb{F}_2 \qquad \forall_{q\in\mathcal{G}}\ \lambda(q) = 1\ :\Leftrightarrow\ pq \in Edges(\mathcal{D})$$

- Because G is minimal, again there is a unique linear form on W such that

$$\tilde{\lambda} :\ W \to \mathbb{F}_2 \qquad \forall_{q\in\mathcal{G}}\ \lambda(q) = \tilde{\lambda}(g(q))$$

- By nondegeneracy of the mixed completion, we find a unique

$$\tilde{w}_0 \in W \qquad \forall_{w\in W}\ \langle \tilde{w}_0, w\rangle = \tilde{\lambda}(\tilde{w})$$

  which we decompose according to the projection $\pi : W \to W^{\perp}$:

$$\tilde{w}_0 = (\tilde{w}_0 - \pi(\tilde{w}_0)) + \pi(\tilde{w}_0) =: w_0 + z_0 \in \ker(\pi) \oplus W^{\perp}$$

At this point, let us go one step in the proof back by rewriting the decomposition for the linear form $\tilde{\lambda} : W \to \mathbb{F}_2$ and even the indicator function $\lambda : \mathcal{G} \to \mathbb{F}_2$:

$$\begin{aligned}
\forall_{w\in W} \qquad \tilde{\lambda}(w) &= \langle\langle \tilde{w}_0, w\rangle\rangle \\
&= \langle\langle w_0, w\rangle\rangle + \langle\langle z_0, w\rangle\rangle \\
&=: \tilde{\lambda}_{W/W^{\perp}}(w) + \tilde{\lambda}_{W^{\perp}}(w) \\
\forall_{q\in\mathcal{G}} \qquad \lambda(q) &= \tilde{\lambda}\,(g(q)) \\
&= \tilde{\lambda}_{W/W^{\perp}}\,(g(q)) + \tilde{\lambda}_{W^{\perp}}\,(g(q)) \\
&=: \lambda_{W/W^{\perp}}(w) + \lambda_{W^{\perp}}(q)
\end{aligned}$$

The subindices $W/W^\perp, W^\perp$ were chosen for the following reason:

$$\begin{aligned}
\tilde{\lambda}_{W/W^\perp}(w) &= \langle\langle w_0, w\rangle\rangle \\
&= \langle w_0, w\rangle + (\pi(w_0), \pi(w)) \\
w_0 \in \ker(\pi) \qquad &= \langle w_0, w\rangle \\
&\quad \Rightarrow \tilde{\lambda}_{W/W^\perp} : W \to W/W^\perp \to \mathbb{F}_2 \\
\tilde{\lambda}_{W^\perp}(w) &= \langle\langle z_0, w\rangle\rangle \\
&= \langle z_0, w\rangle + (\pi(z_0), \pi(w)) \\
z_0 \in W^\perp \qquad &= (\pi(z_0), \pi(w)) \\
\pi \textit{ projection} \qquad &= (\pi(z_0), \pi(w)) \\
&\quad \Rightarrow \tilde{\lambda}_{W^\perp} : W \xrightarrow{\pi} W^\perp \to \mathbb{F}_2
\end{aligned}$$

The intuition behind the upcoming construction is the following: The indicator functions $\lambda_{W/W^\perp}, \lambda_{W^\perp}$ on $\mathcal{G}$ define different graphs $\mathcal{D}_{W/W^\perp}, \mathcal{D}_{W^\perp}$ extending $\mathcal{G}$.

- The first extension can be realized by an element $\langle w_0, -\rangle$, even though this is not generally true if the symplectic form is degenerate. It can thus be realized by adding $f(p) := w_0 \in W$ as in the Extraspecial Extension Theorem 5.2. Note that there we added a nullvector z only to again yield a minimal symplectic root system $V = W \oplus z\mathbb{k}$.
- The second extension can be realized by an element $(z_0, \pi(-))$, even though it is not generally true that an indicator function λ factorizes over π. It can thus be realized by adding as in the Nullspace Extension Theorem 5.3 **either** $f(p) = 0$ for $z_0 = 0$ (and adding a nullvector to achieve a minimal symplectic root system) **or** for $z_0 \neq 0$ by projecting to $\ker(\tilde{\lambda})$ and introducing a new basepair x, y.

The indicator function λ for $\mathcal{D}$ we wish to finally achieve has been written above as the $\mathbb{F}_2$-sum of the indicator functions $\lambda_{W/W^\perp}, \lambda_{W^\perp}$ and hence the neighbourhood of p in $\mathcal{D}$ is constructed as a symmetric difference for $\mathcal{D}_{W/W^\perp}, \mathcal{D}_{W^\perp}$.

We now carry out this proof idea: The precise definition of V and hence the type of symplectic root system we construct depends crucially on z_0:

(1) For $z_0 = 0$ and hence $\lambda_{W^\perp} = 0$ we proceed as in the proof of Theorem 5.2:

$$\begin{aligned}
V &:= W \oplus^\perp z\mathbb{k} \\
f(p) &:= w_0 + z \\
f(q) &:= g(q) \qquad \forall q \in \mathcal{G}
\end{aligned}$$

Note that the vice-versa restriction to $\mathcal{G}$ is equal to G. The pair (f, V) defined this way satisfies the axioms of a symplectic root system of type $(n, k) = (n', k'+1)$ extending G of type (n', k'), which we prove as follows: Obviously, the new decorations generate V and the symplectic root system property for edges $qq' \in \mathcal{G}$ hold already in G.

For the new edges pq adjacent to $p \in \mathcal{D}$ the symplectic root system property is fulfilled as follows:

$$\begin{aligned}
\langle f(p), f(q) \rangle &= \langle w_0 + z, g(q) \rangle \\
z \in V^\perp \qquad &= \langle w_0, g(q) \rangle \\
w_0 \in \ker(\pi) \qquad &= \langle w_0, g(q) \rangle + (w_0, g(q)) \\
&= \langle\langle w_0, g(q) \rangle\rangle \\
&= \lambda_{W/W^\perp}(q) \\
\lambda_{W^\perp} = 0 \qquad &= \lambda(q)
\end{aligned}$$

Hence $\langle f(p), f(q) \rangle = 0$ iff pq non-adjacent in $\mathcal{D}$.

(2) For $z_0 \neq 0$ and hence $\tilde{\lambda}_{W^\perp} \neq 0$ we combine the previous approach using w_0 with the introduction of a new hyperbolic plane H_{n+1} as in the proof of Theorem 5.3: Choose an element $x \in W$ with $\tilde{\lambda}_{W^\perp}(x) = 1_{\mathbb{F}_2}$; such an x can be used to project $W \to \ker(\tilde{\lambda}_{W^\perp})$ by $w \mapsto w - \tilde{\lambda}_{W^\perp}(w) \cdot x$. Then define:

$$\begin{aligned}
V &:= \underbrace{\ker\left(\tilde{\lambda}_{W^\perp}\right)}_{V^\perp} \oplus^\perp (x\mathbb{k} \oplus y\mathbb{k}) = W \oplus y\mathbb{k} \qquad \langle x, y \rangle := 1 \\
f(p) &:= w_0 + y \\
f(q) &:= g(q) \\
&:= \underbrace{g(q) - \tilde{\lambda}_{W^\perp}(g(q)) \cdot x}_{\in \ker(\tilde{\lambda}_{W^\perp})} + \tilde{\lambda}_{W^\perp}(g(q)) \cdot x \qquad \forall q \in \mathcal{G}
\end{aligned}$$

Note that the vice-versa restriction to $\mathcal{G}$ is equal to G. The pair (f, V) defined this way satisfies the axioms of a symplectic root system of type $(n, k) = (n' + 1, k' - 1)$ extending G of type (n', k'), which we prove as follows:

The new decorations generate V and the symplectic root system property for edges $qq' \in \mathcal{G}$ hold already in G. For the new edges pq adjacent to $p \in \mathcal{D}$ the symplectic root system property is fulfilled as follows:

$$\begin{aligned}
\langle f(p), f(q) \rangle &= \langle w_0 + y, g(q) \rangle \\
&= \langle w_0, g(q) \rangle + \langle y, \underbrace{g(q) - \tilde{\lambda}_{W^\perp}(g(q)) \cdot x}_{\in \ker(\tilde{\lambda}_{W^\perp})} + \tilde{\lambda}_{W^\perp}(g(q)) \cdot x \rangle \\
&= \tilde{\lambda}_{W/W^\perp}(g(q)) + \langle y, \tilde{\lambda}_{W^\perp}(g(q)) \cdot x \rangle \\
&= \tilde{\lambda}_{W/W^\perp}(g(q)) + \tilde{\lambda}_{W^\perp}(g(q)) \\
&= \lambda_{W/W^\perp}(q) + \lambda_{W^\perp}(q) \\
&= \lambda(q)
\end{aligned}$$

Hence $\langle f(p), f(q)\rangle = 0$ iff pq non-adjacent in $\mathcal{D}$.

$\square$

5.4. **Tool: Double-Extensions of Extraspecial Symplectic Root Systems.** Finally, we give a practical criterion to obtain the unique minimal symplectic root system which are double extension of extraspecial ones without having to use mixed completions: Let $\mathcal{D}$ be a graph, $p, q \in \mathcal{D}$ vertices and $G = (g, W)$ a minimal extraspecial symplectic root system of $\mathcal{G} := \mathcal{D} - \{p, q\}$ (i.e. of type $(n, 0)$ for $2n = |\mathcal{G}|$). We apply Theorem 5.6 twice: The first extension yields an almost extraspecial extension $(n, 1)$, the second extension allows two cases:

- a type $(n, 2)$ minimal extension $F = (f, V)$ of G to $\mathcal{D}$.
- an extraspecial minimal extension $F = (f, V)$ of G to $\mathcal{D}$ (=type $(n + 1, 0)$)

To determine the case and the precise new decorations $f(p)$, $f(q)$ we now may first apply the much easier Extraspecial Extension Theorem 5.2 to the extensions of G to $\mathcal{D} - p$ resp. $\mathcal{D} - q$, yielding elements $w_p \in W$ resp. $w_q \in W$ for the w_0 in the proof of Theorem 5.2. These elements determine the full extension as follows:

Lemma 5.8 (Double-Extension of Extraspecials)**.** *Depending on w_p, w_q as defined above the extension $F = (f, V)$ has the following form*

	$pq \in \mathit{Edges}(\mathcal{D})$	$pq \notin \mathit{Edges}(\mathcal{D})$
$w_p \perp w_q$	$(n + 1, 0)$	$(n, 2)$
$w_p \not\perp w_q$	$(n, 2)$	$(n + 1, 0)$

with the previous decorations $g(q)$ on $\mathcal{G}$ and new decorations

Case	V	$f(p)$	$f(q)$	
$(n, 2)$	$W \oplus^{\perp} z_p\mathbb{F}_2 \oplus^{\perp} z_q\mathbb{F}_2$	$w_p + z_p$	$w_q + z_q$	
$(n + 1, 0)$	$W \oplus^{\perp} (x\mathbb{F}_2 \oplus y\mathbb{F}_2)$	$w_p + x$	$w_q + y$	*with* $\langle x, y\rangle := 1$

Proof. By the Extraspecial Extension Theorem 5.2 there exist unique extensions $F_p := (F_p, z_p\mathbb{F}_2 \oplus^{\perp} W)$ resp. $F_q := (F_q, z_q\mathbb{F}_2 \oplus^{\perp} W)$ of G to $\mathcal{D}_p := \mathcal{D} - p$ resp. $\mathcal{D}_q := \mathcal{D} - q$ with $f_p(q) := z_p + w_p$ resp. $f_q(p) := z_q + w_q$ (and $\forall_{r \in \mathcal{G}}\ f_p(r) = f_q(r) := g(r)$) for specificly constructed $w_p, w_q \in W$.

Hence we may aim to construct a symplectic root system $F = (f, V)$ for $\mathcal{D}$ such that the restrictions to $\mathcal{D}_p$ resp. $\mathcal{D}_q$ are F_p resp. F_q (conversely we again see that every symplectic root system on $\mathcal{D}$ has to be of this form).

Note first, that any pair of vertices except p, q is contained in $\mathcal{D}_p$ or $\mathcal{D}_q$ and hence by construction the symplectic root system condition has to be checked only for the particular pair pq. Here we calculate

$$\begin{aligned}\langle f(p), f(q)\rangle &= \langle z_q + w_q, z_p + w_p\rangle \\ &= \langle z_q, z_p\rangle + \langle w_q, w_p\rangle\end{aligned}$$

because by construction $z_p, z_q \perp W$. The expression should be $0, 1$ depending on wheather $pq \in \mathit{Edges}(\mathcal{D})$. Hence depending on this and $w_p \perp w_q$, precisely one of the possibilities $\langle z_p, z_q\rangle = 0, 1$ turns F into an symplectic root system of $\mathcal{D}$.

$\square$

6. Example: Symplectic Root Systems for ADE

The unique extension in Theorem 5.2 requires the smaller symplectic root system to be extraspecial. To avoid the general Theorem 5.6 we use in the following proof the double extension of extraspecials in Lemma 5.8. We start with an induction to construct unique minimal symplectic root systems for all A_{2n}, that turn out to be all extraspecial (nullity $k = 0$). Then we add either one or two more vertices to achieve the other diagrams with higher nullity.

Extending the extrapecial minimal A_{2n} by one node by invoking Theorem 5.2 does not require any further consideration and immediately yields almost extraspecial symplectic root systems (nullity $k = 1$) on A_{2n+1}, D_{2n+1}, E_7. On the other hand, when doubly extending an extraspecial A_{2n-2} to A_{2n} (induction) and to D_{2n}, E_6, E_8 using Lemma 5.8 one has to check by explicit calculation, which case of this theorem applies: usually extrapecial and only for D_{2n} of type $(n, 2)$.

Theorem 6.1. *Each simply-laced root system in characteristic 0 of Cartan type $A_n, D_{n\geq 4}, E_{6,7,8}$ admits symplectic root systems precisely of the following types. Each symplectic root system in the lists exists and is unique up to isomorphism. Note that D_{2n+2} has 3 non-isomorphic $(n, 1)$-quotients.*

A_{2n}	$(n,0)$ *minimal*	D_{2n+1}	$(n,1)$ *minimal*	E_6	$(3,0)$ *minimal*
A_{2n+1}	$(n,1)$ *minimal*	D_{2n+1}	$(n,0)$ *quotient*	E_7	$(3,1)$ *minimal*
A_{2n+1}	$(n,0)$ *quotient*	D_{2n+2}	$(n,2)$ *minimal*	E_7	$(3,0)$ *quotient*
		D_{2n+2}	$(n,1)$ 3 *quotients*	E_8	$(4,0)$ *minimal*
		D_{2n+2}	$(n,0)$ *quotient*		

In the proof below, explicit decorations are constructed for each case.

Remark 6.2. *Graph automorphisms preserve the minimal symplectic root systems and hence induce a symplectic isomorphism on V. E.g. the flip of A_{2n} correspond to the symplectic involution $\forall_i\ x_i \leftrightarrow y_i$. The same holds for unique quotients.*

On the other hand, the 3 non-isomorphic quotients of type $(n, 1)$ for D_{2n+2} are permuted by the graph automorphisms as follows:

- *For $n \neq 0$ there is a single order-2 graph automorphism on D_{2n+2} interchanging two of the three quotients while preserving the third.*
- *The graph D_4 has the exceptional automorphism group $\mathbb{S}_3$, which permutes all three quotients.*

The proof will proceed in several steps. We first clarify A_{2n} which is the building block for all other diagrams:

Lemma 6.3. *A_{2n} has a unique symplectic root system, which is minimal and extraspecial. It's precise form is in the natural linear node ordering:*

$x_n + x_{n-1},\ y_{n-1} + y_{n-2},\ x_{n-2} + x_{n-3},\ \ldots,\ y_{n-2} + y_{n-3},\ x_{n-1} + x_{n-2},\ y_n + y_{n-1}$

where the midmost decorations are $\ldots y_2 + y_1,\ x_1,\ y_1,\ x_2 + x_1 \ldots$ or respectively $\ldots x_2 + x_1,\ y_1,\ x_1,\ y_2 + y_1 \ldots$ depending on the genus of n (note that x, y may be switched by the obvious symplectic isomorphism)

Proof. Certainly for $n = 1$ the unique symplectic root system x_1, y_1 of type $(1, 0)$ is minimal and extraspecial, see example 3.4.

Now assume inductively that A_{2n} has the unique extraspecial minimal symplectic root system $G = (g, W)$ as asserted. We wish to invoke Lemma 5.8 to extend this extraspecial minimal symplectic root system on A_{2n} by new vertices p resp. q attached to each end of A_{2n+2}. To prove that we indeed land in the extraspecial case 1 of the Lemma and to determine the precise new decoration, we need to first add one node on each end seperately with decorations $z_p + w_p$ resp. $z_q + w_q$.

This means we have to calculate an actually unique w_p (resp. w_q), such that $\langle w_p, g(r)\rangle = 1$ for the leftmost node r of A_{2n} and $\langle w_p, g(r')\rangle = 0$ for all other vertices. We indeed verify easily, that $w_p = y_n$ has this property:

$$\begin{aligned}
\langle w_p, x_n + x_{n-1}\rangle &= \langle y_n, x_n + x_{n-1}\rangle = 1 \\
\langle w_p, x_i + x_{i-1}\rangle &= 0\ \ \forall_{i<n} \\
\langle w_p, y_i + y_{i-1}\rangle &= 0\ \ \forall_{i<n} \\
\langle w_p, y_n + y_{n-1}\rangle &= \langle y_n, y_n + y_{n-1}\rangle = 0
\end{aligned}$$

for $n > 1$ respectively for $n = 1$:

$$\begin{aligned}
\langle w_p, x_1\rangle &= \langle y_1, x_1\rangle = 1 \\
\langle w_p, y_1\rangle &= \langle y_1, y_1\rangle = 0
\end{aligned}$$

By symmetry we analogously have $w_q = x_n$. Because p, q are not connected and $w_p \not\perp w_q$, Lemma 5.8 applies yielding case 1. Hence it indeed provides V to be extraspecial with new symplectic base-pair z_p, z_q, which we from now on call y_{n+1}, x_{n+1}, and the new decorations

$$f_{A_{2n+2}}(p) = z_p + w_p = y_{n+1} + y_n \qquad f_{A_{2n+2}}(q) = z_q + w_q = x_{n+1} + x_n$$

which is as asserted (after a symplectic isomorphism switching all x_i, y_i). □

For later use, we also wish to calculate the necessary w_0 to attach a node to certain vertice in A_{2n} in:

Lemma 6.4. *When applying Theorem 5.2 to the symplectic root system $G = (g, W)$ of $\mathcal{G} = A_{2n}$ constructed in Lemma 6.3, the following explicit elements $w_0 \in W$ arrise, depending on the neighbourhood in $\mathcal{G}$ of the new to-be-added node p:*

- *$w_0 = y_n$, for attaching p to only the first node of A_{2n}.*

- $w_0 = x_n + x_{n-1}$, *for attaching* p *to only the second node of* A_{2n} $(n \geq 2)$
- $w_0 = y_n + y_{n-1} + y_{n-2}$, *for attaching* p *to only the third node of* A_{2n} $(n \geq 3)$.

(the excluded cases are as follows: for $n = 1$ *the second node is the first node from the right and for* $n = 2$ *the third node is the second node from the right)*

Proof. The first claim (=attaching to the first node) has already been shown during the inductive proof of Lemma 6.3.

For the second claim (=attaching to the second node), note that the newly added p has thus the same neighbourhood as the first node in A_{2n}, hence also the w_0 coincides with the decoration of this first node ($x_n + x_{n+1}$ resp. x_1 for $n = 1$).

For the third claim (=attaching to the third node) we directly calculate, that $w_0 = y_n + y_{n-1} + y_{n-2}$ is orthogonal on all node decorations except the third. Again, all vertices decorated only by y_i or by $x_{i<n-2}$ have obviously orthogonal decoration):

$$\langle y_n + y_{n-1} + y_{n-2}, x_n + x_{n-1} \rangle = \langle y_n, x_n \rangle + \langle y_{n-1}, x_{n-1} \rangle = 0$$
$$\langle y_n + y_{n-1} + y_{n-2}, y_{n-1} + y_{n-2} \rangle = 0$$
$$\langle y_n + y_{n-1} + y_{n-2}, x_{n-2} + x_{n-3} \rangle = \langle y_{n-1}, x_{n-1} \rangle = 1$$
$$\ldots$$
$$\langle y_n + y_{n-1} + y_{n-2}, y_{n-2} + y_{n-3} \rangle = 0$$
$$\langle y_n + y_{n-1} + y_{n-2}, x_{n-1} + x_{n-2} \rangle = \langle y_{n-1}, x_{n-1} \rangle + \langle y_{n-2}, x_{n-2} \rangle = 0$$
$$\langle y_n + y_{n-1} + y_{n-2}, y_n + y_{n-1} \rangle = 0$$

□

We may now continue with the cases D_{2n+1} for $n \geq 2$ resp. E_7 which only require to invoke Theorem 5.2 on the obvious subgraph A_{2n} resp. A_6 with unique extraspecial symplectic root system by the preceeding lemma. The w_0's calculated above yield the precise decorations of the new node p attached to the second resp. third node to be:

$$f_{D_{2n+1}}(p) = z_1 + x_n + x_{n-1}$$
$$f_{E_7}(p) = z_1 + y_3 + y_2 + y_1$$

Finally, as in Lemma 6.3, further calculations are needed when doubly extending the unique minimal extraspecial symplectic root systems on A_{2n} to D_{2n+2}, E_6, E_8 using Lemma 5.8:

We first check that the extensions to E_6 resp. E_8 by attaching vertices p, q to the first and second node of the obvious subgraphs A_4 resp. A_6 again yield extraspecial symplectic root system. The respective w_p, w_q were determined in Lemma 6.4 to be $y_n,\ x_n + x_{n-1}$. As $w_p \not\perp w_q$ and p, q not connected we indeed again land in the extraspecial case 1 of the Lemma and hence the newly added vertices receive the following decorations:

$$
\begin{aligned}
f(p) &:= z_q + w_q = y_{n+1} + y_n & f(q) &:= z_p + w_p = x_{n+1} + x_n + x_{n-1} \\
f_{E_6}(p) &:= y_3 + y_2 & f_{E_6}(q) &:= x_3 + x_2 + x_1 \\
f_{E_8}(p) &:= y_4 + y_3 & f_{E_8}(q) &:= x_4 + x_3 + x_2
\end{aligned}
$$

Then we turn to extending D_{2n+2} from the obvious subgraph A_{2n} by attaching two nodes p, q both to the first node. Again by Lemma 6.4 we thus get $w_p = w_q = y_n$. Still p, q are disconnected, but this time $w_p \perp w_q$, hence we land in case 2 of Lemma 5.8 yielding a type $(n, 2)$ symplectic root system, where we denote the two newly added nullvectors $z_1 := z_p$ and $z_2 := z_q$ and hence yield decorations:

$$
f_{D_{2n+2}}(p) = z_1 + y_n \qquad f_{D_{2n+2}}(p) = z_2 + y_n
$$

We still need to check, that all three possible quotient symplectic root systems of type $(n, 1)$ are non-isomorphic. The decorations are respectively:

$$
\begin{aligned}
f^{(1)}_{D_{2n+2}}(p) &= z + y_n & f^{(1)}_{D_{2n+2}}(q) &= x_n \\
f^{(2)}_{D_{2n+2}}(p) &= y_n & f^{(2)}_{D_{2n+2}}(q) &= z + x_n \\
f^{(3)}_{D_{2n+2}}(p) &= z + y_n & f^{(3)}_{D_{2n+2}}(q) &= z + x_n
\end{aligned}
$$

An isomorphism ϕ intertwining $\phi \circ f^{(1)} = f^{(2)}$ needs to send $\phi : z + y_n \leftrightarrow z + y_n$ and fix z. To keep all other decorations $y_i + y_{i-1}$ we further conclude $\phi : z + y_i \leftrightarrow z + y_i$, which contradicts the stability of the midmost decoration y_1.

An isomorphism ϕ intertwining $\phi \circ f^{(1)} = f^{(3)}$ (or symmetrically $\phi \circ f^{(2)} = f^{(3)}$) needs to send different decorations $y_n, z + y_n$ to the equal decorations $z + y_n, z + y_n$ which is impossible. Hence the three quotients of type $(n, 1)$ are mutually non-isomorphic. Moreover, there is a unique quotient of type $(n, 0)$ with decoration:

$$
f^{(0)}_{D_{2n+2}}(p) = y_n \qquad f^{(0)}_{D_{2n+2}}(q) = x_n
$$

This concludes the proof of Theorem 6.1.

7. Application

7.1. **The Action of the Coxeter/Weyl-Group.** The following notion can be found e.g. in [Hum72] Section 9.2:

Definition 7.1. *Let $E, (,)$ be a euclidean vector space over $\Bbbk = \mathbb{R}$. A subset of $\mathcal{R} \subset E$ is called a (finite) root system, if the following axioms are satisfied:*

- *$\mathcal{R}$ is finite, spans E and $0 \notin \mathcal{R}$*
- *If $\alpha \in \mathcal{R}$, then the only multiples of α in $\mathcal{R}$ are $\pm\alpha$.*

- *If $\alpha \in \mathcal{R}$, then the reflections leaves $\mathcal{R}$ invariant:*

$$\sigma_\alpha(\beta) := \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$$

- *If $\alpha, \beta \in \mathcal{R}$ then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$*

Introduce $d_\alpha = \frac{1}{2}(\alpha,\alpha)$. As usual, we will in the following assume $(,)$ to be normalized such that $d_\alpha \in \mathbb{Z}$ and at least one $d_\alpha = 1$. This implies in particular

$$(\alpha,\beta) = d_\alpha \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$$

In fact, every root system has a basis $\mathcal{R}_0$ of simple roots: A basis means here a basis of the vector space E, such that all roots α are integer linear combinations of $\mathcal{R}_0$, with either all coefficients positive or negative. The Cartan matrix of $\mathcal{R}$ is defined as

$$\forall_{\alpha\neq\beta\in\mathcal{R}_0}\ C_{\alpha,\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = d_\alpha^{-1}(\alpha,\beta) \qquad C_{\alpha,\alpha} := 2$$

Note that there is different convention regarding the order of the Cartan matrix indices α, β; the one above is custom in quantum groups.

Frequently, the Cartan matrix is visualized as a Dynkin diagram with nodes $\mathcal{R}_0$ and edges drawn for $C_{\alpha,\beta} \neq 0$. The Weyl group $\mathcal{W}$ is the Coxeter group determined by the symmetrized Cartan matrix $(,)$. The Weyl group $\mathcal{W}$ is generated by reflections $\sigma_\alpha, \alpha \in \mathcal{R}$. Hence it acts on E as isometries and fixes $\mathcal{R}$. The classification of finite Lie algebras implies $d_\alpha \in \{1,2,3\}$ with $d_\alpha = 1$ for short roots.

Theorem 7.2. *Let C be the Cartan matrix of a root system $\mathcal{R}$, $\mathcal{W}$ the corresponding Weyl group and $(,), d_\alpha$ normalized as in Definition 7.1. Let $\mathcal{G}$ be the graph with nodes $\mathcal{R}_0$ and edges drawn whenever $(\alpha,\beta) \equiv 1 \mod 2$.*
Suppose $F = (f, V)$ is a minimal symplectic root system for $\mathcal{G}$. Then, the decoration $f : \mathcal{R}_0 \to V$ can be additively extended to a map $\tilde{f} : \mathcal{R} \to V$. Furthermore we can define a natural action of $\mathcal{W}$ on V by symplectic isomorphisms, such that $\tilde{f}$ intertwines the $\mathcal{W}$-actions on $\mathcal{R}, V$ and hence especially fixes $\tilde{f}(\mathcal{R}) \subset V$.

Note: If $\mathcal{R}$ is simply-laced, then $\mathcal{G}$ is just the Dynkin diagram of $\mathcal{R}$. For B_n it is the Dynkin diagram $A_1 \times \cdots \times A_1$, for C_n it is $A_1 \times A_{n-1}$, for F_4 it is $A_1 \times A_1 \times A_2$ and for G_2 it is A_2.

Proof. The proof is essentially the invariance of $(,)$ under the action of $\mathcal{W}$: By the assumed minimality the image of the sets $\mathcal{G} = \mathcal{R}_0$ under f is a basis of V. Hence there exists for each simple root $\alpha \in \mathcal{R}_0$ a unique map $\tilde{\sigma}_\alpha : V \to V$ such that this basis $\{f(\beta)\}_{\beta\in\mathcal{R}_0}$ is mapped accordingly:

$$\begin{aligned}\tilde{\sigma}_\alpha\left(f(\alpha)\right) &:= \tilde{f}\left(\sigma_\alpha(\beta)\right) \\ &= \tilde{f}\left(\beta - C_{\alpha,\beta}\alpha\right) \\ &= f(\beta) - C_{\alpha,\beta} f(\alpha)\end{aligned}$$

Note that by additive extension, these formulae hold for non-simple roots α, β as well, but simple reflection of simple roots will suffice here. Note also, that especially for C not simply-laced, $\tilde{f}$ does not need to be injective, as e.g. β and $\beta - 2\alpha$ are mapped to the same vectors in the vector space V.

We yet have to check that the expressions for $\tilde{\sigma}_\alpha$ indeed define a symplectic isomorphism, i.e. preserves the symplectic form $\langle , \rangle$ on V, which we again check on the basis $f(\mathcal{R}_0)$:

$$\begin{aligned}
&\langle f\left(\tilde{\sigma}_\alpha(\beta)\right), f\left(\tilde{\sigma}_\alpha(\gamma)\right)\rangle \\
&= \langle f(\beta) - C_{\alpha,\beta} f(\alpha), f(\gamma) - C_{\alpha,\gamma} f(\alpha)\rangle \\
&= \langle f(\beta), f(\gamma)\rangle - C_{\alpha,\beta}\langle f(\alpha), f(\gamma)\rangle - C_{\alpha,\gamma}\langle f(\beta), f(\alpha)\rangle + \langle f(\alpha), f(\alpha)\rangle \\
&= \langle f(\beta), f(\gamma)\rangle - C_{\alpha,\beta}\cdot d_\alpha C_{\alpha,\gamma} + C_{\alpha,\gamma}\cdot d_\alpha C_{\alpha,\beta} + 0 \\
&= \langle f(\beta), f(\gamma)\rangle
\end{aligned}$$

Here we calculated in $\mathbb{F}_2$ and used that by definition

$$\langle f(\alpha), f(\beta)\rangle = (\alpha, \beta) = d_\alpha C_{\alpha,\beta} = -d_\beta C_{\beta,\alpha}$$

□

7.2. **Commutativity Graphs and Nichols Algebras.** In the application [Len13] of symplectic root systems to Nichols algebras over finite nilpotent groups G, a symplectic root system is used to determine a generating set of elements with commutators prescribed by a fixed graph $\mathcal{G}$. Thereby the commutator map plays the role of the symplectic form. Note that again we so far only consider commutators of order 2.

Definition 7.3. *Let $\mathcal{G}$ be a graph and G a finite group. A decoration $f : \mathcal{G} \to G$ is said to have $\mathcal{G}$ as commutativity graph iff*

- *For $\alpha, \beta \in \mathcal{G}$ the images $f(\alpha), f(\beta)$ commute iff α, β are non-adjacent.*
- *The images $f(\mathcal{G})$ are a generating system of G*

We call such a decoration minimal iff the generating system $f(\mathcal{G})$ is minimal, i.e. no proper subset generates all of G.

Suppose we are given a finite group with $[G, G] = \mathbb{Z}_2$. As usual for p-groups we consider the skew-symmetric, isotropic commutator map $[,]$ (see [Hup83]):

$$G \times G \xrightarrow{[,]} [G, G] = \mathbb{Z}_p$$

$$g, h \mapsto [g, h] = ghg^{-1}h^{-1}$$

$$[h, g] = [g, h]^{-1} \quad [g, g] = 1$$

Because $[G, G]$ is clearly central, the map is bimultiplicative (the right-hand-side argument's works analogously):

$$\begin{aligned}
[g, h][g', h] &= (ghg^{-1}h^{-1})(g'hg'^{-1}h^{-1}) \\
&= g(g'hg'^{-1}h^{-1})hg^{-1}h^{-1} \\
&= gg'hg'^{-1}g^{-1}h^{-1} \\
&= [gg', h]
\end{aligned}$$

Thus the commutator map factorizes to $V := G/G^2 \cong \mathbb{F}_2^n$:

$$V \times V \xrightarrow{\langle,\rangle} \mathbb{F}_2$$

Example 7.4. *Consider the extraspecial groups $G = 2_\pm^{2\cdot n+1}$, which are central products of n groups G_i each isomorphic to*

$$2_+^{2\cdot 1+1} = \mathbb{D}_4 \qquad 2_-^{2\cdot 1+1} = \mathbb{Q}_8$$

Central product means hereby, that $G_i^2 = Z(G_i) = [G_i, G_i] \cong \mathbb{Z}_2$ of all factors are identified, especially $G^2 = Z(G) = [G,G] \cong \mathbb{Z}_2$.

Then $V = G/G^2 = \times_i \mathbb{Z}_2 \times \mathbb{Z}_2$ has a basis of elements x_i, y_i and the commutator map corresponds to the standard symplectic form $\langle x_i, y_i\rangle = \mathbb{F}_2$. Especially extraspecial groups correspond to nondegenerate symplectic vector spaces $V, \langle,\rangle$.

Similary,allowing an additional factor $G_0 = \mathbb{Z}_4$, the resulting groups are known as almost-extraspecial $2_\pm^{2\cdot n+2}$ corresponding to a symplectic vector space of type $(n,1)$.

Remark 7.5. *In [Len13] we start with a group-2-cocycle $u \in Z^2(G/[G,G], \mathbb{Z}_2)$ for to the central extension G, corresponding to the symplectic form as follows*

$$u(\bar{g}, \bar{h})u^{-1}(\bar{h}, \bar{g}) = [g,h] = \langle \bar{g}, \bar{h}\rangle$$

We now show how symplectic root systems can be used to construct generating sets of G with prescribed commutativity graph $\mathcal{G}$. In the application [Len13] the graph $\mathcal{G}$ is a Dynkin diagram of a Nichols algebra $\mathcal{B}(M)$ over the abelian group $\Gamma := G/[G,G]$ and we construct the covering Nichols algebra $\mathcal{B}(\tilde{M})$ over G. The use of a symplectic root system thereby guarantees that the G-decorations in the covering Nichols algebra commute iff nodes are disconnected, which is a necessary condition for finite Nichols algebras, see [HS10], Prop. 8.1.

Note that the next Theorem is restricted to G a group of order 2^n, but every group with $[G,G] = \mathbb{Z}_2$ can be written $G = G_{odd} \times G_2$, where G_{odd} is abelian and of odd order, while G_2 is a 2-group.

Theorem 7.6. *Let G be a 2-group with $[G,G] = \mathbb{Z}_2$ and define as above the following symplectic vector space:*

$$(V, \langle,\rangle) := \left(G/G^2, [,]\right)$$

Let $\mathcal{G}$ be a graph and $F = (f, V)$ a symplectic root system for $\mathcal{G}$, then any lifts $g_\alpha \in G$ of the decoration elements $\{f(\alpha)\}_{\alpha\in\mathcal{G}} \subset V = G/G^2$ has $\mathcal{G}$ as commutativity graph. Moreover, the lifts form a minimal generating system of G iff F was a minimal symplectic root system.

Proof. The main assertion follows almost by definition: For any $\alpha, \beta \in \mathcal{G}$ we show that the lifted G-decoration $\alpha \mapsto g_\alpha$ commute iff the V-decorations are symplectically orthogonal, which happens by the symplectic root system property iff α, β

are non-adjacent in $\mathcal{G}$:

$$\begin{aligned}[g_\alpha, g_\beta] &= \langle \overline{g_\alpha}, \overline{g_\beta} \rangle \\ &= \langle f(\alpha), f(\beta) \rangle\end{aligned}$$

We yet have to prove that the lifted decorations $\{g_\alpha\}_{\alpha \in \mathcal{G}}$ indeed generate G, as the symplectic root system decorations $\{f(\alpha)\}_{\alpha \in \mathcal{G}}$ generate V and especially a minimally G-generating set corresponds to a basis of V. This is the content of the following much more general theorem for p-groups and $V = G/\Phi(G)$:

Theorem 7.7 (Burnside Basis Theorem [Hup83] Satz III.3.15 (p. 273))**.** *Every minimally generating set of a 2-group G (no element may be omitted) $g_1, \dots g_n$ consists precisely of $n = \dim_{\mathbb{F}_2}(V)$ elements for $V := G/G^2$, whose images in V form a basis.*

□

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