

# A note on permutation twist defects in topological bilayer phases

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## ABSTRACT

We present a mathematical derivation of some of the most important physical quantities arising in topological bilayer systems with permutation twist defects as introduced by Barkeshli et al. [BaJQ1]. A crucial tool is the theory of permutation equivariant modular functors developed by Barmeier et al. [BFRS, BaS].

# 1 Introduction

Topological phases of matter have become one of the most important areas of interplay between mathematical physics and condensed matter physics. It has been known for a long time that three-dimensional topological field theories of Reshetikhin-Turaev type describe certain universality classes of robustly gapped systems in 2+1 dimensions, e.g. of quantum Hall fluids. This class of theories includes in particular abelian Chern-Simons theories, which can be defined using integer lattices. In more generality, a basic datum describing a topological phase is a modular tensor category (this notion will be recalled in Section 2).

We wish to study a topological phase described by some modular tensor category  $\mathcal{D}$ , exclusively working in the context of the category  $\mathcal{D}$ . From  $\mathcal{D}$  one can construct an extended topological field theory, as a symmetric monoidal 2-functor

$$\text{tft}_{\mathcal{D}} : \text{cobord}_{3,2,1} \longrightarrow 2\text{-vect} \quad (1)$$

from extended cobordisms to 2-vector spaces, i.e. to finitely semisimple abelian categories; we refer to [Mo] for a review of the relevant notions. The 2-functor  $\text{tft}_{\mathcal{D}}$  provides us with the following structure.

- For a one-dimensional oriented manifold  $S$  one gets a category: for  $S$  the disjoint union of  $n$  circles, one has  $\text{tft}_{\mathcal{D}}(S) = \mathcal{D}^{\boxtimes n}$ , with  $\boxtimes$  the Deligne product of categories enriched over complex vector spaces. The category  $\mathcal{D}$  associated to a circle has the physical interpretation of labels for Wilson lines and for point-like insertions on Wilson lines. Thus in the condensed matter system the isomorphism classes of objects of  $\mathcal{D}$  describe types of quasi-particle excitations in the topological phase.
- For a surface, possibly with marked points – or rather, closed disks excised around the marked points – that are end points of Wilson lines and thus carry the labels of quasi-particles, we get a vector space of conformal blocks. In the context of topological quantum computing these spaces, as subspaces of vector spaces arising in a suitable microscopical model, have been proposed for quantum codes; see e.g. [BMCA, KK] for details in the context of Hopf algebras and of lattice models corresponding to TFTs of Turaev-Viro type, respectively. (In the present paper we are only interested in the behavior of universality classes. We do not touch the very important question of their possible realization in microscopic models.)
- Three-dimensional manifolds with corners give linear maps between the spaces of conformal blocks assigned to their boundaries. On the category  $\mathcal{D}$  this amounts in particular to the structure of a braiding, and on the vector spaces of conformal blocks to representations of mapping class groups. The latter are of interest in the implementation of quantum gates. For example, one needs information about the ‘size’ of the representations of the mapping class group to know whether a given system allows for universal quantum gates.

This structure raises in particular three types of questions about topological phases:

- The problem of classifying the possible types of quasi-particles.
- The problem of computing the dimension of the associated spaces of conformal blocks, which has the interpretation of the number of qubits that can be stored in the corresponding quantum code.
- The problem of understanding the braid group representations on spaces of conformal blocks.

In recent years physical boundaries and surface defects in three-dimensional topological field theories have attracted increasing attention. In this note we do not consider boundaries; we furthermore restrict our attention to surface defects in a single topological phase described by the modular tensor category  $\mathcal{D}$ .

Surface defects lead to a richness of phenomena. Specifically, the three types of problems just stated have natural generalizations to situations in which defects are present. These are the issues we discuss in the present note, for the specific case of twist defects in bilayer systems.

- For any pair  $a, a'$  of topological surface defects there is a category  $\mathcal{W}_{a,a'}$  of Wilson lines confined to the defect surface, which separate a surface region labeled by  $a$  from a surface region labeled by  $a'$ . The objects of this category label defect Wilson lines, while the morphisms label point-like insertions on those Wilson lines. In the application to topological phases, such Wilson lines do not describe intrinsic quasi-particles, but rather extrinsic objects with long-range interaction [BaJQ1]. More generally, there are categories of Wilson lines at which an arbitrary finite number of surface defects end (see the figure (33) below). A first problem is to obtain a concrete description of these categories.
- Surface defects and generalized Wilson lines can intersect transversally surfaces to which one would like to associate appropriate generalizations of conformal blocks. This raises the question of how to define these vector spaces and how to obtain expressions for their dimensions, generalizing the Verlinde formula.
- On these vector spaces, one has the action of appropriate versions of mapping class groups. It is an important task to understand these groups and their actions in detail.

It has been demonstrated in a model-independent analysis [FSV1] that surface defects which separate two regions that are both in the phase labeled by  $\mathcal{D}$  are described by module categories over the modular tensor category  $\mathcal{D}$  (again, the notion of  $\mathcal{D}$ -module category, or  $\mathcal{D}$ -module, will be recalled in Section 2). For a general modular tensor category  $\mathcal{D}$ , the classification of  $\mathcal{D}$ -modules, and thus of surface defects, is out of reach. Notable exceptions are abelian Chern-Simons theories, as studied in [KaS, FSV1, BaJQ2], and Dijkgraaf-Witten theories, for which  $\mathcal{D}$  is the category of finite-dimensional representations of the Drinfeld double of a finite group  $G$ , as well as the category of integrable highest weight representations of the affine Lie algebra based on  $\mathfrak{sl}(2, \mathbb{C})$  at positive integral level. For Dijkgraaf-Witten theories subgroups of  $G$  and certain group cochains enter the classification [Os2] (for the corresponding boundary conditions and surface defects and their geometric interpretation see [FSV2]), while in the  $\mathfrak{sl}(2, \mathbb{C})$  case an A-D-E classification emerges [KO].

Any modular tensor category  $\mathcal{D}$  has at least one indecomposable module, namely the regular  $\mathcal{D}$ -module given by the abelian category  $\mathcal{D}$  itself, with the action being just the ordinary tensor product of  $\mathcal{D}$ . The corresponding defect  $\mathcal{T}_{\mathcal{D}}$  is the *transparent* (or invisible) defect, which is a monoidal unit under fusion of surface defects. (For a discussion of  $\mathcal{T}_{\mathcal{D}}$  in the context of Dijkgraaf-Witten models see [FSV2, Sect. 3.6].) Generically it can be hard to find other  $\mathcal{D}$ -modules besides  $\mathcal{T}_{\mathcal{D}}$ . There is, however, one general situation in which a non-trivial module category can be identified, namely when the modular tensor category  $\mathcal{D}$  is the Deligne product  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$  of two copies of another modular tensor category  $\mathcal{C}$  (with the same structure of modular tensor category, in particular the same braiding, on each copy). In the condensed matter literature this situation is known as a *bilayer system*, see e.g. [BaW, BaQ, BaJQ1] for recent work.

For any bilayer system described by  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$ , in addition to the regular  $\mathcal{D}$ -module there is a second module category, which is defined on a single copy of the abelian category  $\mathcal{C}$ ; its module category structure has been given in [BFRS, Thm. 2.2]. This type of surface defect is the main subject of the present letter. We will present it in detail in Section 2, where we also demonstrate that it corresponds to the permutation twist defect  $\mathcal{P}_{\mathcal{D}}$  [BaJQ1] of the bilayer system, which permutes the different layers. We then also obtain the relevant categories of generalized Wilson lines, thereby providing a model-independent solution of the problem of describing the generalized quasi-particle excitations in the presence of the permutation twist defect. In Section 3 we present the relevant spaces of conformal blocks. In applications to quantum computing their dimensions give the number of qubits that can be stored in a topological code realized by bilayer fractional quantum Hall states in the universality class of the bilayer topological phase.

The results reported in this letter have obvious generalizations to  $n$ -layer systems, corresponding to the situation that  $\mathcal{D} = \mathcal{C}^{\boxtimes n}$  is the Deligne product of any number  $n$  of copies of the same modular tensor category  $\mathcal{C}$ . In this case we find a  $\mathcal{D}$ -module category for every element of the symmetric group  $S_n$  that describes a permutation of the  $n$  layers. The mathematical tools to describe these systems in detail are available [BHS, BFS, Ban, BFRS, BaS]: for the sake of clarity of the exposition we refrain from presenting them in this note.

## 2 The module category $\mathcal{P}_{\mathcal{D}}$ and categories of defect Wilson lines

**Module categories.** A *fusion category* (over  $\mathbb{C}$ ) is a rigid semisimple linear monoidal category, enriched over the category of (complex) vector spaces, with only finitely many isomorphism classes of simple objects, such that the endomorphisms of the monoidal unit form just the ground field,  $\text{End}(\mathbf{1}) = \mathbb{C} \text{id}_{\mathbf{1}}$ . All categories in this letter will be finitely semisimple abelian  $\mathbb{C}$ -linear categories. Usually we assume fusion categories to be strictly monoidal, so that we can drop the associativity constraints. All categories of Wilson lines in topological field theories, including defect and boundary Wilson lines, are in our context described by fusion categories. A *modular tensor category*  $\mathcal{C}$  is a braided fusion category in which the braiding  $c_{U,V}: U \otimes V \xrightarrow{\cong} V \otimes U$  obeys a non-degeneracy condition: the  $|I_{\mathcal{C}}| \times |I_{\mathcal{C}}|$ -matrix with entries

$$s_{i,j} := \text{tr}(c_{S_j, S_i} \circ c_{S_j, S_i}), \quad (2)$$

where  $(S_i)_{i \in I_{\mathcal{C}}}$  is a set of representatives for the isomorphism classes of simple objects of  $\mathcal{C}$ , is invertible. Examples of modular tensor categories arise from Chern-Simons theories, in which case they can be described in terms of finite abelian groups with a quadratic form (abelian Chern-Simons theories) or of integrable highest weight representations of affine Lie algebras (non-abelian Chern-Simons theories).

A *module category*  $\mathcal{M}$  over a monoidal category  $\mathcal{D}$  (or a  $\mathcal{D}$ -module, for short) consists of a category  $\mathcal{M}$  and a  $\mathbb{C}$ -linear bifunctor  $\odot: \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$  together with functorial associativity and unit isomorphisms

$$(X \otimes Y) \odot M \xrightarrow{\cong} X \odot (Y \odot M) \quad \text{and} \quad \mathbf{1} \odot M \xrightarrow{\cong} M \quad (3)$$

for  $X, Y \in \mathcal{D}$  and  $M \in \mathcal{M}$ , obeying coherence conditions. For details see e.g. [Os1, Sect. 2.3]; we refer to the functor  $\odot$  as the mixed tensor product. Thinking about a fusion category as a categorification of a unital associative ring, module categories are categorifications of modules over that ring. Taking the tensor product  $\otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$  and the associated associativity and unit constraints of  $\mathcal{D}$ , any fusion category  $\mathcal{D}$  is, as already mentioned, a module category  $\mathcal{T}_{\mathcal{D}}$  over itself, analogously as any ring is a module over itself. In topological field theories with defects the module category  $\mathcal{T}_{\mathcal{D}}$  describes the transparent defect, separating two regions that both support the same topological phase labeled by  $\mathcal{D}$ .

**A module category for bilayer systems.** In the case of bilayer systems, i.e.  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$  with a modular tensor category  $\mathcal{C}$ , there is another generic module category  $\mathcal{P}_{\mathcal{D}}$ . The abelian category underlying  $\mathcal{P}_{\mathcal{D}}$  is just  $\mathcal{C}$  itself. The mixed tensor product for  $\mathcal{P}_{\mathcal{D}}$  is defined in terms of the tensor product  $\otimes$  in  $\mathcal{C}$  by

$$(U \boxtimes V) \odot M = U \otimes V \otimes M \quad (4)$$

for  $M \in \mathcal{C}$  and  $U \boxtimes V \in \mathcal{D}$ . (In the definition, it suffices to consider objects of  $\mathcal{D}$  of the form  $U \boxtimes V$  with  $U, V \in \mathcal{C}$  only; such objects are called  $\boxtimes$ -factorizable.)

The existence of such a  $\mathcal{D}$ -module generalizes the fact that for a *commutative* ring  $R$ , the tensor product ring  $R \otimes_{\mathbb{Z}} R$  has  $R$  as a module, with action  $(a \otimes b) \cdot m = a b m$  for  $a, b, m \in R$ . Commutativity of  $R$  ensures that this prescription constitutes an action, i.e. the equality of  $[(a_1 \otimes a_2) (b_1 \otimes b_2)] \cdot m = a_1 b_1 a_2 b_2 m$  and  $(a_1 \otimes a_2) \cdot [(b_1 \otimes b_2) \cdot m] = a_1 a_2 b_1 b_2 m$ . The categorification of commutativity of  $R$  is the structure of a braiding; accordingly we take the natural isomorphisms

$$\begin{aligned} \psi_{X, X', M}: \quad (X \otimes X') \odot M &= U \otimes U' \otimes V \otimes V' \otimes M \\ &\xrightarrow{\cong} X \odot (X' \odot M) = U \otimes V \otimes U' \otimes V' \otimes M \end{aligned} \quad (5)$$

(with  $X = U \boxtimes V \in \mathcal{D}$ ,  $X' = U' \boxtimes V' \in \mathcal{D}$  and  $M \in \mathcal{C}$ ) as the associativity constraints for the mixed tensor product of  $\mathcal{P}_{\mathcal{D}}$  that are part of the defining data of the module category to be given by the braiding in  $\mathcal{C}$ , i.e.

$$\psi_{X, X', M} = id_U \otimes c_{U', V} \otimes id_{V' \otimes M}. \quad (6)$$

This mixed associativity constraint has been derived geometrically from the theory of covering surfaces [BaS, Eq. (17)]. There is in fact a whole family of possible constraints for the mixed tensor product, involving higher powers of the braiding  $c$  [BFRS, Thm. 2.2], but they are all equivalent [BFRS, Thm. 2.4].

**$\mathcal{P}_{\mathcal{D}}$  as part of an equivariant topological field theory.** That  $\mathcal{P}_{\mathcal{D}}$  is a twist defect [BaJQ1] has the following mathematical formalization: the module category  $\mathcal{P}_{\mathcal{D}}$  with underlying abelian category  $\mathcal{C}$  is part of a more comprehensive structure [Bar, BaS] – together with the modular category  $\mathcal{D}$  it forms a  $\mathbb{Z}_2$ -equivariant modular category [Tu, Ki]. This amounts to the existence of further mixed fusion functors, including functors

$$\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \quad \text{and} \quad \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D} \quad (7)$$

and constraints which we will need later in our discussion. The structure of a  $\mathbb{Z}_2$ -equivariant modular category can be derived from a  $\mathbb{Z}_2$ -equivariant topological field theory

$$\mathrm{tft}_{\mathcal{D}}^{\mathbb{Z}_2} : \mathrm{cobord}_{3,2,1}^{\mathbb{Z}_2} \longrightarrow 2\text{-vect} \quad (8)$$

which has as domain a cobordism category of manifolds with  $\mathbb{Z}_2$ -covers. The functor  $\mathrm{tft}_{\mathcal{D}}^{\mathbb{Z}_2}$  can be given explicitly by applying the functor  $\mathrm{tft}_{\mathcal{C}}$  (see (1)) for the (non-equivariant) topological field theory based on  $\mathcal{C}$  to the total spaces of the covers,

$$\mathrm{tft}_{\mathcal{D}}^{\mathbb{Z}_2}(-) = \mathrm{tft}_{\mathcal{C}}(\mathrm{cov}(-)). \quad (9)$$

For a detailed discussion in the framework of modular functors we refer to [BaS].

**The Azumaya algebra  $A_{\mathcal{P}}$ .** We need to collect a few facts about the module category  $\mathcal{P}_{\mathcal{D}}$ . As any semisimple indecomposable (left) module category over a fusion category  $\mathcal{D}$ , the category  $\mathcal{P}_{\mathcal{D}}$  describing the permutation twist defect can be realized [Os1] as the category of (right) modules over an algebra  $A_{\mathcal{P}}$  internal in  $\mathcal{D}$ ,

$$\mathcal{P}_{\mathcal{D}} \simeq \mathrm{mod}\text{-}A_{\mathcal{P}}. \quad (10)$$

One possible choice for this algebra is the internal end of the tensor unit  $\mathbf{1} \in \mathcal{C}$ , i.e.  $A_{\mathcal{P}} = \underline{\mathrm{End}}_{\mathcal{C}}(\mathbf{1})$ . As an object of the category  $\mathcal{D}$  this is [BFRS]

$$A_{\mathcal{P}} = \bigoplus_{i \in \mathcal{I}_{\mathcal{C}}} U_i^{\vee} \boxtimes U_i, \quad (11)$$

where the sum is over the isomorphism classes of simple objects of  $\mathcal{C}$ ; its algebra structure is given explicitly in [BFRS, Thm. 5.1]. The algebra  $A_{\mathcal{P}}$  has a natural Frobenius algebra structure, which is presented in [BFRS, Prop. 6.1]. Furthermore, it is an *Azumaya algebra* in  $\mathcal{D}$  [BFRS, Thm. 7.3]. Let us explain the latter notion. For an algebra  $A$  in a braided fusion category  $\mathcal{C}$  there are two *braided induction functors*

$$\alpha_A^{\pm} : \mathcal{C} \rightarrow A\text{-bimod}_{\mathcal{C}}. \quad (12)$$

They associate to an object  $U \in \mathcal{C}$  the bimodule with underlying object  $A \otimes U$  and with the left action  $\rho^{\pm} := m_A \otimes id_U : A \otimes A \otimes U \rightarrow A \otimes U$  given by multiplication  $m_A$  in  $A$ , while the right action is  $\varrho^+ := (m_A \otimes id_U) \circ (id_A \otimes c_{U,A})$  and  $\varrho^- := (m_A \otimes id_U) \circ (id_A \otimes c_{A,U}^{-1})$ , respectively. The functors (12) have a natural structure of a monoidal functor;  $A$  is called an Azumaya algebra iff  $\alpha_A^+$ , or equivalently  $\alpha_A^-$ , is a monoidal equivalence. For  $\mathcal{C}$  the monoidal category of modules over a commutative ring, this coincides with the textbook definition of Azumaya algebras [VZ].

**Remark 1.** (i) Recall that for describing a bilayer system we have to take the same braiding on the two copies of  $\mathcal{C}$  in  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$ . There is another important structure of a braided fusion category on the tensor category  $\mathcal{C} \boxtimes \mathcal{C}$ , namely the ‘enveloping category’  $\mathcal{C}^{\mathrm{env}} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$  in which the second copy of  $\mathcal{C}$  is instead endowed with the inverse braiding. If  $\mathcal{C}$  is modular, then the enveloping category is a modular category as well; in fact, modularity implies that it is equivalent to the Drinfeld center of  $\mathcal{C}$ , i.e.  $\mathcal{C}^{\mathrm{env}} \simeq \mathcal{Z}(\mathcal{C})$ . This structure of a modular tensor category is *not* the one relevant for bilayer systems.

(ii) When regarded as an object of the enveloping category  $\mathcal{C}^{\mathrm{env}} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$  the object (11) of the abelian category  $\mathcal{C} \boxtimes \mathcal{C}$  has again a natural Frobenius algebra structure, which is of interest in various other contexts, see e.g. [Mü, FFRS1, KR, FSS]. This algebra structure on the object (11) is commutative with respect to the braiding of  $\mathcal{C}^{\mathrm{env}}$ , rather than Azumaya.

**Braided induction for tensor products of algebras.** Given two unital associative algebras  $A_1$  and  $A_2$  in a braided monoidal category, their tensor product  $A_1 \otimes A_2$  can be endowed with the structure of a unital associative algebra with multiplication

$$(A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{id_{A_1} \otimes c_{A_2, A_1} \otimes id_{A_2}} A_1 \otimes A_1 \otimes A_2 \otimes A_2 \xrightarrow{m_{A_1} \otimes m_{A_2}} A_1 \otimes A_2. \quad (13)$$

(Replacing the over-braiding  $c_{A_2, A_1}$  by the under-braiding  $c_{A_1, A_2}^{-1}$  yields a different algebra structure on the same object  $A_1 \otimes A_2$ . This algebra is isomorphic, as an associative algebra, to the algebra structure on  $A_2 \otimes A_1$  obtained with the convention chosen here.)

We will now establish a relation between the braided induction functors for the algebras  $A_1$  and  $A_2$  and those for  $A_1 \otimes A_2$ . We first introduce a functor

$$\beta^+ : A_2\text{-bimod}_{\mathcal{C}} \longrightarrow (A_1 \otimes A_2)\text{-bimod}_{\mathcal{C}} \quad (14)$$

that sends  $B \equiv (B, \rho, \varrho) \in A_2\text{-bimod}_{\mathcal{C}}$  to  $A_1 \otimes B$  with the  $(A_1 \otimes A_2)$ -bimodule structure given by the left action

$$(A_1 \otimes A_2) \otimes (A_1 \otimes B) \xrightarrow{id_{A_1} \otimes c_{A_2, A_1} \otimes id_B} A_1 \otimes A_1 \otimes A_2 \otimes B \xrightarrow{m_{A_1} \otimes \rho} A_1 \otimes B \quad (15)$$

and the right action

$$(A_1 \otimes B) \otimes (A_1 \otimes A_2) \xrightarrow{id_{A_1} \otimes c_{B, A_1} \otimes id_{A_2}} A_1 \otimes A_1 \otimes B \otimes A_2 \xrightarrow{m_{A_1} \otimes \varrho} A_1 \otimes B. \quad (16)$$

Again,  $\beta^+$  has a natural monoidal structure. Moreover, one verifies that

$$\alpha_{A_1 \otimes A_2}^+ = \beta^+ \circ \alpha_{A_2}^+ \quad (17)$$

as monoidal functors. Similarly, we introduce another monoidal functor

$$\beta^- : A_1\text{-bimod}_{\mathcal{C}} \rightarrow (A_1 \otimes A_2)\text{-bimod}_{\mathcal{C}}, \quad (18)$$

sending  $B \equiv (B, \rho, \varrho) \in A_1\text{-bimod}_{\mathcal{C}}$  to  $A_2 \otimes B$  with the  $(A_1 \otimes A_2)$ -bimodule structure given by the left action

$$(A_1 \otimes A_2) \otimes (A_2 \otimes B) \xrightarrow{id_{A_1} \otimes m_{A_2} \otimes id_B} A_1 \otimes A_2 \otimes B \xrightarrow{c_{A_2, A_1}^{-1} \otimes id_B} A_2 \otimes A_1 \otimes B \xrightarrow{id_{A_2} \otimes \rho} A_2 \otimes B \quad (19)$$

and the right action

$$(A_2 \otimes B) \otimes (A_1 \otimes A_2) \xrightarrow{id_{A_2} \otimes \varrho \otimes id_{A_2}} A_2 \otimes B \otimes A_2 \xrightarrow{id_{A_2} \otimes c_{A_2, B}^{-1}} A_2 \otimes A_2 \otimes B \xrightarrow{m_{A_2} \otimes id_B} A_2 \otimes B. \quad (20)$$

By direct calculation one sees that the family

$$\nu_U := c_{A_2, A_1} \otimes id_U : A_2 \otimes A_1 \otimes U \longrightarrow A_1 \otimes A_2 \otimes U \quad (21)$$

of isomorphisms, for  $U \in \mathcal{C}$ , furnishes a monoidal natural isomorphism  $\nu : \beta^- \circ \alpha_{A_1}^- \Longrightarrow \alpha_{A_1 \otimes A_2}^-$ . Similarly one verifies that the the same family of isomorphisms gives a monoidal natural isomorphism

$$\tilde{\nu} : \beta^- \circ \alpha_{A_1}^+ \Longrightarrow \beta^+ \circ \alpha_{A_2}^-. \quad (22)$$

This is summarized in the







**Proposition 5.** *The Azumaya algebra  $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$  in  $\mathcal{D}$  is Morita equivalent to the tensor unit  $\mathbf{1}_{\mathcal{D}}$  of  $\mathcal{D}$ .*

*Proof.* Combining (25) and (26) we have  $\Psi_{\text{mod-}A_{\mathcal{P}} \otimes A_{\mathcal{P}}} \cong \Psi_{\text{mod-}A_{\mathcal{P}}} \circ \Psi_{\text{mod-}A_{\mathcal{P}}} = \text{Id}_{\mathcal{D}}$ . Thus by [DNO, Cor. 3.8] the module categories  $\text{mod-}A_{\mathcal{P}} \otimes A_{\mathcal{P}}$  and  $\text{mod-}\mathbf{1}_{\mathcal{D}}$  are equivalent, i.e.  $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$  and  $\mathbf{1}_{\mathcal{D}}$  are Morita equivalent.  $\square$

**Categories of defect Wilson lines.** We are now in a position to find the relevant categories of defect Wilson lines. Consider two types of surface defects separating a topological phase of type  $\mathcal{D}$  from itself, corresponding to two  $\mathcal{D}$ -modules  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . According to [FSV1] the category of surface Wilson lines separating  $\mathcal{M}_1$  from  $\mathcal{M}_2$  is the functor category  $\text{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$  of  $\mathcal{D}$ -module functors. If the left module categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are realized as the categories of right modules over algebras  $A_1$  and  $A_2$  in  $\mathcal{D}$ , respectively, then this functor category is equivalent to the category of  $A_1 \otimes A_2^{\text{op}}$ -modules in  $\mathcal{D}$ . We are interested in the situation that the two  $\mathcal{D}$ -modules in question are either  $\mathcal{T}_{\mathcal{D}}$  or  $\mathcal{P}_{\mathcal{D}}$ , corresponding to the transparent and to the permutation twist defect. Proposition 5 tells us that under forming tensor products the Azumaya algebra  $A_{\mathcal{P}}$  for the twist defect  $\mathcal{P}_{\mathcal{D}}$  has order two up to Morita equivalence; as a consequence, when calculating the functor categories we can work with  $A_{\mathcal{P}}$  in place of  $A_{\mathcal{P}}^{\text{op}}$ . We then find the following categories of defect Wilson lines:

- The category  $\text{Fun}_{\mathcal{D}}(\mathcal{T}_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}})$  of defect Wilson lines separating the transparent surface defect from itself is just  $\mathcal{D}$ , as expected:

$$\text{Fun}_{\mathcal{D}}(\mathcal{T}_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}}) \simeq (\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}})\text{-mod}_{\mathcal{D}} \cong \mathbf{1}_{\mathcal{D}}\text{-mod}_{\mathcal{D}} \cong \mathcal{D}. \quad (27)$$

- There are two categories of defect Wilson lines separating the transparent defect from the twist defect,  $\text{Fun}_{\mathcal{D}}(\mathcal{T}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}})$  and  $\text{Fun}_{\mathcal{D}}(\mathcal{P}_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}})$ ; we find

$$\text{Fun}_{\mathcal{D}}(\mathcal{T}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}}) \simeq (\mathbf{1}_{\mathcal{D}} \otimes A_{\mathcal{P}})\text{-mod}_{\mathcal{D}} \cong A_{\mathcal{P}}\text{-mod}_{\mathcal{D}} \cong \mathcal{C} \quad (28)$$

and, in a similar manner,  $\text{Fun}_{\mathcal{D}}(\mathcal{P}_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}}) \cong \mathcal{C}$ . As shown in [FSV1, Sect. 6.2], each such Wilson line labeled by  $W \in \mathcal{C}$  gives rise to a (special symmetric Frobenius) algebra in the Morita class of  $A_{\mathcal{P}}$ ; this is actually just the internal end  $\underline{\text{End}}_{\mathcal{C}}(W)$  [Os1, Sect. 3].

- Finally, the category of defect Wilson lines separating the twist defect from itself is

$$\text{Fun}_{\mathcal{D}}(\mathcal{P}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}}) \simeq (A_{\mathcal{P}} \otimes A_{\mathcal{P}})\text{-mod}_{\mathcal{D}} \simeq \mathbf{1}_{\mathcal{D}}\text{-mod}_{\mathcal{D}} \cong \mathcal{D}. \quad (29)$$

The category describing Wilson lines which separate the twist defect from the transparent defect provides the labels for a permutation-type “genon” of [BaJQ1]. Genons are thus labeled by objects of the category  $\mathcal{C}$ .

**Fusion of surface defects.** Topological surface defects can be fused. In the particular situation of two surface defects separating a topological phase of type  $\mathcal{C}$  from itself, described by  $\mathcal{C}$ -modules  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , the fusion product is the  $\mathcal{C}$ -module  $\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2$ . The Deligne product  $\boxtimes_{\mathcal{C}}$  of two module categories over a braided fusion category is the categorification of the tensor product  $M_1 \otimes_R M_2$  of two left modules  $M_1$  and  $M_2$  over a commutative ring  $R$  and

has a similar universal property. For a precise definition see [ENOM, Def. 3.3 & Sect. 4.4]. By [ENOM, Prop. 3.5] there is an equivalence

$$\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2 \simeq \text{Func}_{\mathcal{C}}(\mathcal{M}_1^{\text{op}}, \mathcal{M}_2) \quad (30)$$

of abelian categories.

The right hand side of (30) has a natural interpretation [FSV1] as the category of surface Wilson lines that separate the surface defect labeled by  $\mathcal{M}_1^{\text{op}}$  from the one labeled by  $\mathcal{M}_2$ . This is no coincidence: if we realize two module categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$  over a modular tensor category  $\mathcal{C}$  as the categories of right modules over algebras  $A_1$  and  $A_2$  internal in  $\mathcal{C}$ , the fused module category  $\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2$  is realized by the category of right  $A_1 \otimes A_2$ -modules. Now consider a defect surface with the topology of a plane, separated by a Wilson line into two half planes labeled by  $A_1$  and by  $A_2$ , respectively. Wilson lines of this type are labeled by the category of  $A_1$ - $A_2$ -bimodules which equals, as an abelian category,  $(A_1 \otimes A_2^{\text{op}})\text{-mod}_{\mathcal{C}}$ . By folding the plane along the Wilson line we arrive at a configuration in which a Wilson line separates the transparent surface defect  $\mathcal{T}_{\mathcal{C}}$  from the surface defect that is obtained by fusing the surface defect with label  $A_1$  with the orientation-reversed surface defect for  $A_2$ . This is the  $\mathcal{C}$ -module category

$$A_1\text{-mod}_{\mathcal{C}} \boxtimes_{\mathcal{C}} A_2^{\text{op}}\text{-mod}_{\mathcal{C}} \simeq (A_1 \otimes A_2^{\text{op}})\text{-mod}_{\mathcal{C}}. \quad (31)$$

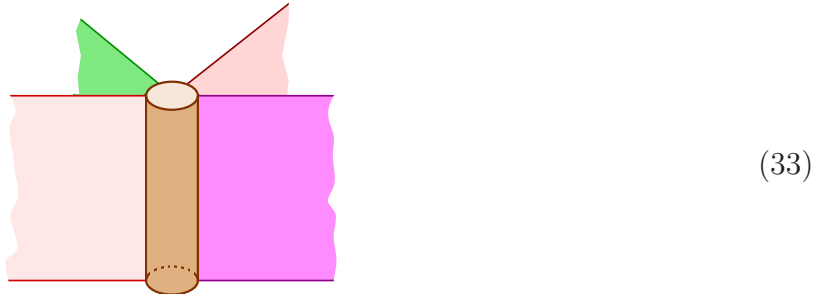
Now for any  $\mathcal{C}$ -module category  $\mathcal{M}$ , the category of surface Wilson lines separating the transparent defect  $\mathcal{T}_{\mathcal{C}}$  from  $\mathcal{M}$  is  $\text{Func}_{\mathcal{C}}(\mathcal{C}, \mathcal{M}) \cong \mathcal{M}$ , and hence in the case at hand the abelian category  $(A_1 \otimes A_2^{\text{op}})\text{-mod}$ . Thus the equality of the two abelian categories can be understood through the folding procedure and provides a consistency check on the description of the fusion of surface defects by the Deligne product.

It follows in particular that the transparent defect, described by the algebra  $\mathbf{1}_{\mathcal{C}}$  in  $\mathcal{C}$ , acts as a (bi)monoidal unit. In the case of permutation twist defects in the bilayer system based on  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$ , we get

$$\mathcal{P}_{\mathcal{D}} \boxtimes_{\mathcal{D}} \mathcal{P}_{\mathcal{D}} \simeq (A_{\mathcal{P}} \otimes A_{\mathcal{P}})\text{-mod}_{\mathcal{D}} \simeq \mathcal{D}, \quad (32)$$

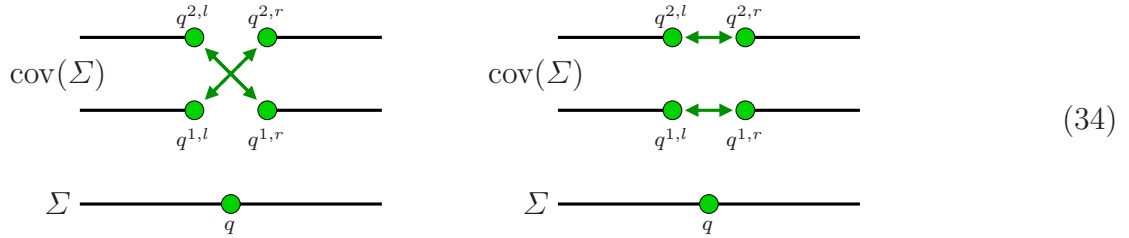
where in the last step we used again that the Azumaya algebra  $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$  is Morita equivalent to the tensor unit. Thus the fusion product of the twist defect  $\mathcal{P}_{\mathcal{D}}$  with itself is the transparent defect; this is certainly not unexpected.

**More general Wilson lines.** A topological field theory of Reshetikhin-Turaev type actually admits more general types of Wilson lines, in which any finite number of surface defects meet. Locally in a three-manifold, the situation looks like in the following picture, in which, as in the formalism used in [FSV2], the locus of the Wilson line in a three-manifold is actually a tube:



In situation at hand, each of the surface defects that meet at the Wilson line can be either the transparent defect or the twist defect. We denote the decorated one-dimensional manifold consisting of a circle with an  $n_{\mathcal{T}}$ -tuple  $\vec{p} \equiv (p_1, p_2, \dots, p_{n_{\mathcal{T}}})$  of points marked with the transparent defect  $\mathcal{T}_{\mathcal{D}}$  and an  $n_{\mathcal{P}}$ -tuple  $\vec{q} \equiv (q_1, q_2, \dots, q_{n_{\mathcal{P}}})$  of points marked by the twist defect  $\mathcal{P}_{\mathcal{D}}$  by  $S_{\vec{p}, \vec{q}}$ , and the associated category of generalized Wilson lines by  $\text{tft}(S_{\vec{p}, \vec{q}})$ . Because of the  $\mathbb{Z}_2$ -fusion rules obeyed by the transparent and twist defects, the category  $\text{tft}(S_{\vec{p}, \vec{q}})$  is equivalent to  $\mathcal{D}$  if  $n_{\mathcal{P}}$  is even, and equivalent to  $\mathcal{C}$  if  $n_{\mathcal{P}}$  is odd.

A geometric understanding of the categories of generalized Wilson lines is provided by the *cover functor*  $\text{cov}$  of [BaS, Prop. 2], which maps the decorated one-manifold  $S_{\vec{p}, \vec{q}}$  as follows to a two-sheeted cover of the circle  $S^1$ . First, take the disjoint union of two copies  $\tilde{S}^{(1)}$  and  $\tilde{S}^{(2)}$  of the non-connected manifold obtained by replacing the open intervals in  $S^1 \setminus (\vec{p} \cup \vec{q})$  by closed intervals. For each marked point  $q_i$  this gives two points  $q_i^{1,l}$  and  $q_i^{1,r}$  on  $\tilde{S}^{(1)}$  and two points  $q_i^{2,l}$  and  $q_i^{2,r}$  on  $\tilde{S}^{(2)}$ . They are associated to the interval on the left hand side and to the one on the right hand side of  $q_i \in S^1$ , respectively. Next we identify  $q_i^{2,l}$  with  $q_i^{1,r}$  and  $q_i^{1,l}$  with  $q_i^{2,r}$ . For the points  $p_i$  a similar construction is performed, but this time we identify  $p_i^{1,l}$  with  $p_i^{1,r}$  and  $p_i^{2,l}$  with  $p_i^{2,r}$ . The two different identifications are illustrated in the left and right hand parts of the following figure:



We have thus associated a two-sheeted cover  $\text{cov}(S) \rightarrow S$  to the decorated one-manifold  $S_{\vec{p}, \vec{q}}$ . As already indicated in (9), the equivariant topological field theory  $\text{tft}_{\mathcal{D}}^{\mathbb{Z}_2}$  is obtained [BaS] by applying the TFT functor associated to  $\mathcal{C}$  to the cover.

Let us check that the categories of generalized Wilson lines for decorated one-manifolds that we have computed in (27)–(29) coincide with the evaluation of the 2-functor  $\text{tft}_{\mathcal{C}}$  on the total space  $\text{cov}(S)$  of the two-sheeted cover of  $S$ . If the number  $n_{\mathcal{P}}$  of twist defects is even, then  $\text{cov}(S) \rightarrow S$  is the trivial cover whose total space has two connected components; we thus get

$$\text{tft}_{\mathcal{D}}^{\mathbb{Z}_2}(S) = \text{tft}_{\mathcal{C}}(S^1 \sqcup S^1) \simeq \text{tft}_{\mathcal{C}}(S^1) \boxtimes \text{tft}_{\mathcal{C}}(S^1) = \mathcal{C} \boxtimes \mathcal{C} = \mathcal{D}. \quad (35)$$

If  $n_{\mathcal{P}}$  is odd, then the total space  $\text{cov}(S)$  is connected and we obtain instead the category  $\text{tft}_{\mathcal{D}}^{\mathbb{Z}_2}(S) = \text{tft}_{\mathcal{C}}(S^1) = \mathcal{C}$ .

### 3 Generalized conformal blocks and their dimensions

To an oriented surface  $\Sigma$  with boundaries, an extended topological field theory assigns a functor. More explicitly, given a decomposition  $\partial\Sigma = -\partial\Sigma_- \sqcup \partial\Sigma_+$  of the boundary into incoming and outgoing parts, we get a functor

$$\text{tft}_{\mathcal{D}}(\Sigma) : \text{tft}_{\mathcal{D}}(\partial\Sigma_-) \longrightarrow \text{tft}_{\mathcal{D}}(\partial\Sigma_+). \quad (36)$$

To achieve a detailed understanding of these functors, two particular perspectives prove to be helpful: First, the functors for particularly simple surfaces can be assembled to obtain those for

more complicated surfaces. Second, by assigning specific objects of  $\mathcal{D}$  to the boundary surfaces, one arrives at vector spaces of (generalized) conformal blocks. We address both of these points of view.

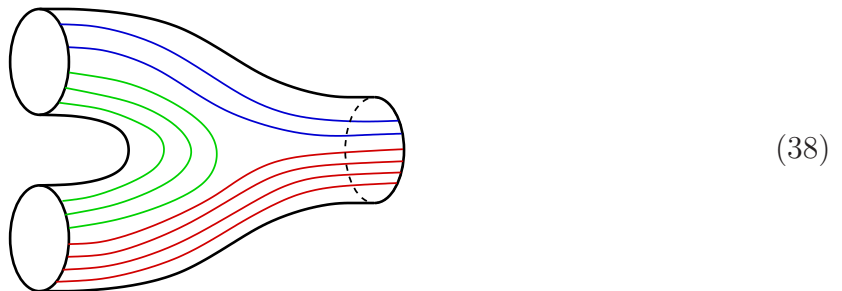
**Mixed tensor products.** The basic observation that allows for the first perspective is that any oriented surface with boundary admits a decomposition into pairs of pants, cylinders and disks. Evaluating the TFT-functor on the pair of pants  $Y$  with two incoming and one outgoing boundary circles gives a functor

$$\mathrm{tft}_{\mathcal{D}}(S^1) \boxtimes \mathrm{tft}_{\mathcal{D}}(S^1) \cong \mathrm{tft}_{\mathcal{D}}(S^1 \sqcup S^1) \longrightarrow \mathrm{tft}_{\mathcal{D}}(S^1). \quad (37)$$

This endows the category  $\mathcal{D}$  associated to the circle with a tensor product functor. (An associativity constraint for this tensor product is then provided by the natural transformation that the TFT associates to a suitable three-manifold with corners; this way  $\mathcal{D}$  becomes a monoidal category.)

Once we allow for non-trivial defects, we get additional categories associated to decorated circles, and thereby additional tensor products; they relate different categories and are thus *mixed* tensor products. In the case of twist defects, these tensor products can be extracted from the underlying equivariant topological field theory. If we deal with a permutation equivariant theory, mixed tensor products have been computed [BaS, Sect. 4.3] with the help of the cover functor, which we already encountered in (9). Proceeding in this way we find:

- Denote by  $n_1$  and  $n_2$  the numbers of twist defects ending on the two ingoing circles and by  $n_3$  the number of twist defects of the outgoing circle. On the pair of pants we then have  $(n_1+n_2+n_3)/2$  lines that connect boundary circles, all labeled by the twist defect. The following picture shows a case with  $n_1=5$ ,  $n_2=7$  and  $n_3=6$  and with each line connecting two different circles:



- If the numbers  $n_1$  and  $n_2$  are both even, then  $n_3$  is necessarily even as well. Thus the categories assigned to the boundary circles are all equivalent to  $\mathcal{D}$ . According to [BaS], the functor associated to any such pair of pants is the tensor product in the monoidal category  $\mathcal{D}$ :

$$\begin{aligned} \mathrm{tft}_{\mathcal{D}}(Y_{n_1, n_2, n_3}) : \quad & \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D} \\ & (U_1 \boxtimes U_2) \times (V_1 \boxtimes V_2) \longmapsto (U_1 \otimes V_1) \boxtimes (U_2 \otimes V_2). \end{aligned} \quad (39)$$

- If  $n_1$  is even and  $n_2$  is odd, then  $n_3$  is necessarily odd, and the functor is the mixed tensor product:

$$\begin{aligned} \mathrm{tft}_{\mathcal{D}}(Y_{n_1, n_2, n_3}) : \quad & \mathcal{D} \times \mathcal{C} \longrightarrow \mathcal{C} \\ & (U_1 \boxtimes U_2) \times M \longmapsto U_1 \otimes U_2 \otimes M, \end{aligned} \quad (40)$$

where on the right hand side the tensor product in  $\mathcal{C}$  appears.

The situation is analogous when  $n_1$  is odd and  $n_2$  is even.

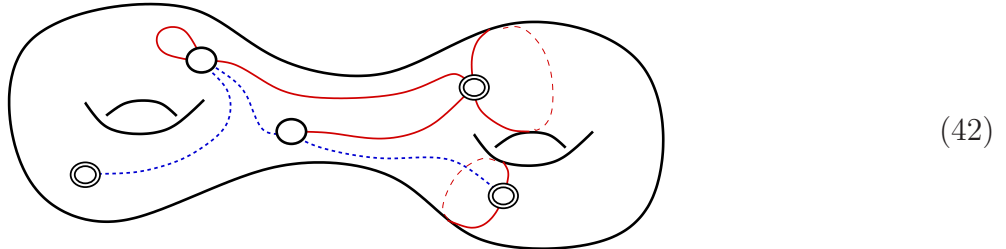
- If both  $n_1$  and  $n_2$  are odd, then  $n_3$  is even. In this case the functor is the one computed in [BaS, Sect. 4.3, p. 314]:

$$\begin{aligned} \text{tft}_{\mathcal{D}}(Y_{n_1, n_2, n_3}) : \quad \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{D} \\ M \times N &\longmapsto \bigoplus_{i \in I_{\mathcal{C}}} (M \otimes N \otimes S_i^{\vee}) \boxtimes S_i. \end{aligned} \quad (41)$$

Again we have a geometric understanding of these functors. Consider a pair of pants  $Y$  with a pattern of non-intersecting surface defect lines, as in the figure (38). And again we take two copies of  $Y$  with the defect lines removed and glue them together according to the prescription in the figure (34), with the identification depending on whether the defect line is labeled by the transparent defect or the twist defect. This way we get a two-sheeted cover  $\tilde{Y}_{n_1, n_2, n_3} \rightarrow Y_{n_1, n_2, n_3}$  which restricts on the boundary circles to the cover constructed in Section 2. The functors  $\text{tft}_{\mathcal{D}}(Y_{n_1, n_2, n_3})$  just described are then the functors  $\text{tft}_{\mathcal{C}}(\tilde{Y}_{n_1, n_2, n_3})$ .

**Spaces of conformal blocks.** We now turn to the second perspective and focus on spaces of conformal blocks. Consider an oriented surface  $\Sigma$ , for the moment without defects, with only ingoing boundary circles, i.e.  $\partial\Sigma = -\partial\Sigma_- \cong (S^1)^{\sqcup n}$  and  $\partial_+\Sigma = \emptyset$ . The extended topological field theory provides a functor  $\text{tft}_{\mathcal{D}}(\Sigma) : \mathcal{D}^{\boxtimes n} \rightarrow \text{vect}$ . Specifying an object in  $\mathcal{D}$  for each boundary circle, we obtain a vector space, known as a space of *conformal blocks*. In applications to topological quantum computing, these spaces are the spaces of ground states and are thus the recipients of qubits. The dimension of the space of conformal blocks, and thus the ground state degeneracy, is computed by the Verlinde formula.

In the presence of surface defects in the three-dimensional theory, the topological surface  $\Sigma$  is endowed with a collection of non-intersecting lines. Such lines are closed or have end points on the boundary circles of  $\Sigma$ . Each segment of a line is labeled either by the transparent defect  $\mathcal{T}_{\mathcal{D}}$  or by the twist defect  $\mathcal{P}_{\mathcal{D}}$ . A typical situation is displayed in the following figure:



Here lines labeled by the transparent defect  $\mathcal{T}_{\mathcal{D}}$  are drawn as dotted lines, while those labeled by the twist defect  $\mathcal{P}_{\mathcal{D}}$  are drawn as solid lines.

A boundary circle is drawn as a double line if an even number of  $\mathcal{P}_{\mathcal{D}}$ -lines ends on it; the associated category is  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$ . Boundary circles with an odd number of  $\mathcal{P}_{\mathcal{D}}$ -lines are drawn as single lines; the corresponding category is  $\mathcal{C}$ . Among the single-line circles are those on which only one  $\mathcal{P}_{\mathcal{D}}$ -line ends; these are “genons”. (Thus in the picture (42) there is one genon, the lower of the two single-line circles.)

Our task is to construct for a surface  $\Sigma$  with  $m_0$  boundary circles having an even number of  $\mathcal{P}_{\mathcal{D}}$ -lines and  $m_1$  boundary circles having an odd number a functor

$$\text{tft}_{\mathcal{D}}(\Sigma) : \quad \mathcal{D}^{\boxtimes m_0} \boxtimes \mathcal{C}^{\boxtimes m_1} \longrightarrow \text{vect} \quad (43)$$

that describes generalized conformal blocks, including their dependence on the labels of the boundary circles. By the axioms of topological field theory, the gluing of surfaces must give rise to the composition of the associated functors. As a consequence, the functor  $\text{tft}_{\mathcal{D}}(\Sigma)$  can be expressed as a composite of the functors arising in a pair-of-pants decomposition of the surface  $\Sigma$ . The latter are already known from the previous discussion: they are provided by the equivariant topological field theory  $\text{tft}_{\mathcal{D}}^{\mathbb{Z}_2}$ . Note that surfaces with  $\mathbb{Z}_2$ -covers can be glued, and in an equivariant topological field theory this translates into the composition of functors.

Hereby we are led to the following generalization of the construction for pairs of pants given above. To a surface  $\Sigma$  with embedded  $\mathcal{T}_{\mathcal{D}}$ -lines and  $\mathcal{P}_{\mathcal{D}}$ -lines we associate a two-sheeted cover  $\text{cov}(\Sigma) \rightarrow \Sigma$  as follows: Again we glue together two copies of  $\Sigma$  with all defect lines removed, in a way that is determined by the surface defect labeling the defect line, as in the figure (34). (A variant is to glue standard disks to the boundary circles of  $\Sigma$  so as to get closed oriented surfaces. In this formulation one gets two-fold *branched* covers, with branch points in disks whose boundaries contain an odd number of twist defects. As mentioned, circles with one twist defect describe genons; they are thus end points of branch cuts, compare [BaJQ3].)

It is crucial that the construction of  $\mathbb{Z}_2$ -covers is compatible with the gluing of surfaces with defects. Indeed, consider the surface  $\Sigma_1 \# \Sigma_2$  that is obtained by gluing together, along appropriate boundary circles, surfaces  $\Sigma_1$  and  $\Sigma_2$  with defects. Then the  $\mathbb{Z}_2$ -cover of  $\Sigma_1 \# \Sigma_2$  furnished by our construction is the same as the surface obtained by gluing the cover  $\text{cov}(\Sigma_1) \rightarrow \Sigma_1$  to the cover  $\text{cov}(\Sigma_2) \rightarrow \Sigma_2$ ,

$$\text{cov}(\Sigma_1 \# \Sigma_2) = \text{cov}(\Sigma_1) \# \text{cov}(\Sigma_2). \quad (44)$$

We thus conclude that the generalized conformal block functor for a general surface as in (43) is obtained by applying  $\text{tft}_{\mathcal{C}}$  to the two-fold cover  $\text{cov}(\Sigma)$ , i.e. we have  $\text{tft}_{\mathcal{D}}(\Sigma) = \text{tft}_{\mathcal{C}}(\text{cov}(\Sigma))$  as in (9). This provides a model independent confirmation of an insight gained in the study [BaJQ1] of several classes of models.

**A generalization of the Verlinde formula.** What we have achieved is to identify the generalized conformal blocks in the bilayer topological field theory based on  $\mathcal{D}$  associated to a surface with defects with ordinary conformal blocks for  $\text{tft}_{\mathcal{C}}$  on the cover of that surface. As a consequence, we can compute the dimension of these spaces with the ordinary Verlinde formula for the theory based on  $\mathcal{C}$ .

Consider a closed surface of genus  $g$ , and thus of Euler characteristic  $\chi = 2 - 2g$ , obtained by gluing disks to the boundary circles of a surface  $\Sigma$ . Let  $\Sigma$  have  $N_0$  boundary circles with an even number of twist defects, labeled with objects  $U_i \boxtimes \tilde{U}_i \in \mathcal{D}$  for  $i = 1, 2, \dots, N_0$ , where  $U_i, \tilde{U}_i \in \mathcal{C}$ , and  $N_1$  boundary circles with an odd number twist defects, labeled with objects  $V_j \in \mathcal{C}$  for  $j = 1, 2, \dots, N_1$ . Then by the Riemann-Hurwitz theorem the cover  $\text{cov}(\Sigma)$  has Euler characteristic  $2\chi - N_1$ , i.e. the genus of the cover increases linearly with  $N_1$ . A boundary circle of  $\Sigma$  with an even number of twist defects has a pre-image on  $\text{cov}(\Sigma)$  consisting of two circles; we label them by the objects  $U_i$  and  $\tilde{U}_i$  of  $\mathcal{C}$ , respectively, with the relevant value of  $i$ . A boundary circle of  $\Sigma$  with an odd number of twist defects has a single circle as its pre-image, which we label by the appropriate object  $V_j \in \mathcal{C}$ . Taking for simplicity the objects  $U_i, \tilde{U}_i$  and  $V_j$  to be simple, we arrive this way at the following formula for the dimensions of spaces of



generalized blocks:

$$\dim_{\mathbb{C}}(\text{tft}_{\mathcal{D}}(\Sigma; \{U_i \boxtimes \tilde{U}_i\}, \{V_j\})) = \sum_{n \in I_{\mathcal{C}}} (S_{0,n})^{2\chi - N_1} \prod_{i=1}^{N_0} \frac{S_{U_i,n}}{S_{0,n}} \frac{S_{\tilde{U}_i,n}}{S_{0,n}} \prod_{j=1}^{N_1} \frac{S_{V_j,n}}{S_{0,n}}. \quad (45)$$

Here  $S$  is the modular S-matrix of the category  $\mathcal{C}$ , i.e. the matrix obtained from  $s = (s_{i,j})$  as given in (2) by rescaling such that  $S$  is unitary and symmetric, and where  $0 \in I_{\mathcal{C}}$  is the isomorphism class of  $\mathbf{1}_{\mathcal{C}}$ .

For instance, for  $\Sigma = S^2$  of genus 0,  $N_0 = 0$  and all  $V_j$  equal, the dimension is

$$\dim_{\mathbb{C}}(\text{tft}_{\mathcal{D}}(S^2; \emptyset, \{V, V, \dots, V\})) = \sum_{n \in I_{\mathcal{C}}} (S_{0,n})^{4-2N_1} (S_{V,n})^{N_1}. \quad (46)$$

### Dependence of the dimension of spaces of conformal blocks on the genon type.

In the context of quantum computing twist defects are of interest because the relevant spaces of conformal blocks are associated with surfaces of higher genus, so that they generically have larger dimension than conformal blocks for surfaces without twist defects. From this perspective it should also be appreciated that each genon comes with the choice of a label, which is an object in the category  $\mathcal{C}$ . Since this datum enters in the dimension formula (45), it constitutes an additional handle on increasing the dimension of the space of conformal blocks.

As a simple instructive example, take a sphere with four genons, i.e.  $\Sigma = S^2$ ,  $N_0 = 0$  and  $N_1 = 4$ , and take  $\mathcal{C}$  to be the modular tensor category of the critical Ising model, describing a free Majorana fermion. (For a related discussion of genons in this model see [BaJQ1, Sect. V].) Then  $\text{cov}(\Sigma) = \mathbb{T}$  is a torus with four boundary circles, and the set  $I_{\mathcal{C}}$  of isomorphism classes of simple objects of  $\mathcal{C}$  has three elements,  $I_{\mathcal{C}} = \{\mathbf{1}_{\mathcal{C}}, \sigma, \epsilon\}$ . We consider two extreme choices for the labels of the genons. First, let all genons be labeled by  $\mathbf{1}_{\mathcal{C}}$ , which is the monoidal unit of  $\mathcal{C}$ . Then the dimension of the space of generalized conformal blocks is

$$\begin{aligned} d_4(\mathbf{1}_{\mathcal{C}}) &:= \dim_{\mathbb{C}}(\text{tft}_{\mathcal{D}}(S^2; \emptyset, \{\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}\})) \\ &= \dim_{\mathbb{C}}(\text{tft}_{\mathcal{C}}(\mathbb{T}; \mathbf{1}_{\mathcal{C}}^{\otimes 4})) = \dim_{\mathbb{C}}(\text{tft}_{\mathcal{C}}(\mathbb{T}; \mathbf{1}_{\mathcal{C}})) = |I_{\mathcal{C}}| = 3. \end{aligned} \quad (47)$$

Second, if all genons are labeled by  $\sigma$ , then by using the fusion rules  $\sigma \otimes \sigma \cong \mathbf{1}_{\mathcal{C}} \oplus \epsilon$  and  $\epsilon \otimes \epsilon \cong \mathbf{1}_{\mathcal{C}}$  we get

$$\begin{aligned} d_4(\sigma) &:= \dim_{\mathbb{C}}(\text{tft}_{\mathcal{D}}(S^2; \emptyset, \{\sigma, \sigma, \sigma, \sigma\})) \\ &= \dim_{\mathbb{C}}(\text{tft}_{\mathcal{C}}(\mathbb{T}; \sigma^{\otimes 4})) = \dim_{\mathbb{C}}(\text{tft}_{\mathcal{C}}(\mathbb{T}; 2\mathbf{1}_{\mathcal{C}} \oplus 2\epsilon)) \\ &= 2 [\dim_{\mathbb{C}}(\text{tft}_{\mathcal{C}}(\mathbb{T}; \mathbf{1}_{\mathcal{C}})) + \dim_{\mathbb{C}}(\text{tft}_{\mathcal{C}}(\mathbb{T}; \epsilon))] = 2(3 + 1) = 8. \end{aligned} \quad (48)$$

To see that these numbers agree with formula (46), just note that for the Ising model we have  $(S_{0,n}) = \frac{1}{2}(1, \sqrt{2}, 1)$  and  $(S_{\sigma,n}) = \frac{1}{2}(\sqrt{2}, 0, -\sqrt{2})$ . The corresponding numbers for arbitrary  $N_1$  are

$$d_{N_1}(\mathbf{1}_{\mathcal{C}}) = 2^{N_1-3} + 2^{N_1/2-2} \quad \text{and} \quad d_{N_1}(\sigma) = 2^{3N_1/2-3}, \quad (49)$$

respectively, and thus grow exponentially with the number of genons. The growth depends, however, explicitly on the choice of label for the genon, and a judicious choice leads to more powerful codes.



**Braiding.** For a  $\mathbb{Z}_2$ -equivariant category, the notion of a braiding has to be replaced by the notion of an *equivariant braiding*. Concretely, part of the data are two autoequivalences  $\tau_{\mathcal{D}}: \mathcal{D} \rightarrow \mathcal{D}$  and  $\tau_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$  of the categories involved. As shown in [BaS],  $\tau_{\mathcal{D}}$  act as  $\tau_{\mathcal{D}}(U \boxtimes V) = V \boxtimes U$ , while  $\tau_{\mathcal{C}}$  is the identity endofunctor of  $\mathcal{C}$ . If  $V \in \mathcal{C}$  labels a circle with an odd number of  $\mathcal{P}_{\mathcal{D}}$ -defects, then the equivariant braiding is given by functorial isomorphisms  $c_{V,W}: V \otimes W \rightarrow \tau(W) \otimes V$ . These isomorphisms have been computed [BaS, Sect. 4.6] from the cover functor and look as follows.

- The braiding of two objects  $V = V_1 \boxtimes V_2 \in \mathcal{D}$  and  $W = W_1 \boxtimes W_2 \in \mathcal{D}$  is just  $c_{V_1, W_1}^{\mathcal{C}} \boxtimes c_{V_2, W_2}^{\mathcal{C}}$  [BaS, Eq. (23)], as one would expect for bilayer systems.
- The equivariant braiding of  $V = V_1 \boxtimes V_2 \in \mathcal{D}$  and  $W \in \mathcal{C}$  is more complicated; it involves the twist  $\theta$  as well as over- and underbraidings:

$$c_{V,W} = (c_{V_1, W}^{\mathcal{C}} \otimes \theta_{V_2}^{-1}) \circ (id_{V_1} \otimes (c_{W, V_2}^{\mathcal{C}})^{-1}) : V_1 \otimes V_2 \otimes W \xrightarrow{\cong} W \otimes V_1 \otimes V_2. \quad (50)$$

- Similarly, for  $V \in \mathcal{C}$  and  $W = W_1 \boxtimes W_2 \in \mathcal{D}$  we have

$$c_{V,W} = (c_{W_1, W_2}^{\mathcal{C}} \otimes id_V) \circ ((c_{W_1, V}^{\mathcal{C}})^{-1} \otimes id_{W_2}) : V \otimes W_1 \otimes W_2 \xrightarrow{\cong} W_2 \otimes W_1 \otimes V. \quad (51)$$

- The equivariant braiding of two objects in  $\mathcal{D}$  is still more complicated; we refer to the last equation-picture in [BaS, Sect. 4.6].

It is an interesting and important problem to obtain the appropriate generalizations of mapping class groups of surfaces with boundary disks and defect lines and the representations of these groups on the spaces of generalized conformal blocks, as well as to relate them to representations of mapping class groups of higher genus surfaces. For results in the condensed matter literature see [BaJQ1, FHN]. This issue is of direct relevance for the problem of implementing universal quantum gates on topological codes described by these spaces of generalized conformal blocks. It has already been demonstrated [FNW, BaJQ1] that the presence of twist defects, via the induced mapping class group actions of higher genus, renders the double layer Ising system with permutation twist defects universal for quantum computing, while it is non-universal without defects in genus zero.

**A remark on orbifolding.** We conclude this note with a speculative remark. As pointed out in [BaJQ1], gauging the symmetry that underlies a twist defect can deconfine the extrinsic defects such that they become intrinsic quasi-particles in a topological phase described by the corresponding orbifold theory. As a physical mechanism for such a gauging, based on the analogy with the emergence of a  $\mathbb{Z}_2$ -gauge theory by a proliferation of double vortices in a superfluid the authors of [BaJQ1] propose a proliferation of double-twist defects. We conjecture that on the level of topological field theory this mechanism is implemented by a three-dimensional analogue of the generalized orbifolds of [FFRS2], in which an orbifold construction is realized with the help of a network of defect lines.

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