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A sufficient condition for Hamiltonicity in locally finite graphs

Karl Heuer

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A SUFFICIENT CONDITION FOR HAMILTONICITY IN LOCALLY FINITE GRAPHS

KARL HEUER

ABSTRACT. Using topological circles in the Freudenthal compactification of a locally finite connected graph for infinite cycles, we extend a result of Oberly and Sumner for Hamiltonicity in finite graphs to locally finite graphs. This answers a question of Stein positively and gives a sufficient condition for Hamiltonicity in locally finite graphs.

1. Introduction

It was and is still a very active field in graph theory to determine whether a finite graph is Hamiltonian. The problem to decide whether a finite graph is Hamiltonian is difficult. This indicates that it should not be easy to find a necessary and sufficient condition for a graph to be Hamiltonian which can easily be checked. On the other hand, there are a lot of conditions for graphs which are either necessary or sufficient for Hamiltonicity. One certain kind of sufficient conditions for Hamiltonicity uses forbidden induced subgraphs. In this paper we consider only one result of this type, due to Oberly and Sumner from 1979. In order to state their theorem, we need two definitions. We call a graph *locally connected* if the neighbourhood of each vertex induces a connected subgraph. A graph is called *claw-free* if it does not contain the graph $K_{1,3}$ as an induced subgraph. Now the theorem of Oberly and Sumner is as follows:

Theorem 1.1. [19, Thm. 1] *Every finite, connected, locally connected, claw-free graph on at least three vertices is Hamiltonian.*

Most Hamiltonicity results consider only finite graphs. The reason for this is that it is not clear what a Hamilton cycle in an infinite graph should be. We follow the idea of Diestel and Kühn [10, 11] who used a topological approach to replace infinite cycles by circles, i.e., homeomorphic images of the unit circle $S^1 \subseteq \mathbb{R}^2$ in the Freudenthal compactification $|G|$ of a locally finite connected graph G. Since every vertex of a locally finite connected graph G is also a point in the space $|G|$, it is natural to call a locally finite connected graph G *Hamiltonian* if there is a circle in $|G|$ that contains all vertices of G. Using this notion of Hamiltonicity for locally finite connected graphs, some Hamiltonicity results for finite graphs could be partially or even completely generalized to locally finite graphs, see [3, 5, 14, 16]. The most natural candidates for Hamiltonicity theorems which could generalize to locally finite graphs are probably such theorems which involve local conditions like in Theorem 1.1.

This paper deals with a question about Hamiltonicity in locally finite graphs. In [20, Question 5.1.3] Stein asks whether Theorem 1.1 can be generalized to locally finite graphs. We answer this question positively by the following theorem, which is the main result of this paper.

Theorem 1.2. *Every locally finite, connected, locally connected, claw-free graph on at least three vertices is Hamiltonian.*

The structure of this paper is as follows. In Section 2 we recall some basic definitions and fix some further notation we shall need in this paper. Section 3 contains some facts, lemmas and theorems which we shall need in the proof of the main result. The proof of Theorem 1.2 together with some corollaries is the content of Section 4.

During the writing of this paper I noticed that Hamann, Lehner and Pott [15] are investigating similar questions.

2. Basic definitions and notation

In this section, important definitions and notation are listed. Furthermore, some basic definitions are recalled in order to avoid confusion with regard to the notation. In general, we will follow the graph theoretical notation of [7] in this paper if nothing different is stated. For basic facts about graph theory, the reader is also referred to [7]. Beside finite graph theory, a topologically approach to locally finite graphs is covered in [7, Ch. 8.5]. For a wider survey in this field, see [8].

All graphs which are considered in this paper are undirected and simple. In general, we do not assume a graph to be finite. For this section, we fix an arbitrary graph $G = (V, E)$.

The graph G is called *locally finite* if every vertex of G has only finitely many neighbours.

For a vertex set X of G, we denote by $G[X]$ the induced subgraph graph of G with vertex set X. For vertex sets with up to four vertices, we omit the set brackets and write $G[v, w, x, y]$ instead of $G[\{v, w, x, y\}]$ where $\{v, w, x, y\} \subseteq V$. We write $G - X$ for the graph $G[V \setminus X]$ and for singleton sets, we omit the set brackets and write just $G - v$ instead of $G - \{v\}$ where $v \in V$. For the cut which consists of all edges of G that have one endvertex in X and the other endvertex in $V \setminus X$, we write $\delta(X)$.

Let C be a cycle of G and u be a vertex of C. Then we write u^+ and u^- for the neighbour of u in C in positive and negative, respectively, direction of C using a fixed orientation of C. Later on we will not mention that we fix an orientation for the considered cycle using this notation. We implicitly fix an arbitrary orientation of the cycle.

Let P be a path in G and T be a tree in G. We write $\mathring{P}$ for the subpath of P which we obtain from P by removing the endvertices of P. If s and t are vertices of T, we write sTt for the unique path in T with endvertices s and t. Note that this covers also the case where T is a path. If $P_v = v_0 \dots v_n$ and $P_w = w_0 \dots w_k$ are paths in G with $n, k \in \mathbb{N}$ where v_n and w_0 may be equal but apart from that these paths are disjoint and the vertices v_n, w_0 are the only vertices of P_v and P_w which lie in T, we write $v_0 \dots v_n T w_0 \dots w_k$ for the path with vertex set $V(P_v) \cup V(v_n T w_0) \cup V(P_w)$ and edge set $E(P_v) \cup E(v_n T w_0) \cup E(P_w)$.

Let v be a vertices of G. Then we define the *distance* between v and a vertex set $X \subseteq V$ in G to be the least distance between v and a vertex of X in G. Similarly, we define the distance between two vertex sets $X \subseteq V$ and $Y \subseteq V$ in G to be the least distance between a vertex in X and a vertex in Y. For a subgraph H of G, we set the distance between v (or $X \subseteq V$) and H in G to be the distance between v (or X) and $V(H)$ in G. If it is clear that we consider distances in G, we will just write distance instead of distance in G.

For a vertex set $X \subseteq V$ and an integer $k \geq 1$, we denote with $N_k(X)$ the set of vertices in G from which the distance is at least 1 and at most k to X in G. We write $N(X)$ instead of $N_1(X)$, which denotes the usual neighbourhood of X in G. For a singleton set $\{v\} \subseteq V$, we omit the set brackets and write just $N_k(v)$ and

$N(v)$ instead of $N_k(\{v\})$ and $N(\{v\})$, respectively. If H is a subgraph of G, we just write $N_k(H)$ and $N(H)$ instead of $N_k(V(H))$ and $N(V(H))$, respectively.

For an integer $k \geq 1$, the *k-th power* G^k of G is the graph with vertex set $V(G)$ where two vertices are adjacent if and only if the distance between them in G is at least 1 and at most k. In the proof of Theorem 1.2, we define also certain graphs with superscript indices but do not mean powers of graphs. It should be clear in the context of the proof.

We call G *locally connected* if for every vertex $v \in V$ the subgraph $G[N(v)]$ is connected.

We refer to the graph $K_{1,3}$ also as *claw*. The graph G is called *claw-free* if it does not contain the claw as an induced subgraph.

A one-way infinite path in G is called a *ray* of G. Now an equivalence relation can be defined on the set of all rays of G by saying that two rays in G are equivalent if they cannot be separated by finitely many vertices. It is easy to check that this relation really defines an equivalence relation. The corresponding equivalence classes of rays under this relation are called the *ends* of G. We denote the set of all ends of G by $\Omega(G)$.

For the rest of this section, we assume G to be locally finite and connected. Next we define a topology on G together with its ends to obtain a topological space which we call $|G|$. For this, we follow the definition in [7]. We begin by defining the point set of $|G|$. All elements of $V(G) \cup \Omega(G)$ are in the point set of $|G|$. Furthermore, we add for every edge $e = uv$ of G a set $\mathring{e} = (u, v)$ which is bijective to the open real unit interval $(0, 1)$ to the point set of $|G|$ where $\mathring{e}$ is disjoint from $V(G) \cup \Omega(G)$ and for any two edges $e, f \in E$, the sets $\mathring{e}$ and $\mathring{f}$ are disjoint. Additionally, we fix for every edge e of G a bijection between the set $[u, v] := \{u\} \cup \mathring{e} \cup \{v\}$ and the closed real unit interval $[0, 1]$. Each such bijection induces a metric on the corresponding set $[u, v]$. For every edge $e = uv$ of G, we also refer to the set $[u, v]$ as *edge* and denote it by e. To keep clarity, we point out whether we talk about edges of G or the topological objects. The elements of $\mathring{e}$ are called *inner points* of the edge e. For an edge set $F \subseteq E(G)$, we define $\mathring{F} = \bigcup\{\mathring{e}\ ;\ e \in F\}$. If $H = (V_H, E_H)$ is a subgraph of G and we talk about H in $|G|$, we use this as an abbreviation for considering the point set $V_H \cup \mathring{E}_H$ in $|G|$.

Based on the point set of the space $|G|$, we define now the corresponding topology by stating a basis of open sets. For every edge uv of G, we define all subsets of (u, v) to be open that correspond to an open subset of $(0, 1)$ using the fixed bijection between $[u, v]$ and $[0, 1]$.

For every $u \in V(G)$ and $\epsilon > 0$, we declare the set as open which consists of all points on edges $[u, v]$, where $uv \in E(G)$, whose distance is less than ϵ to u in $[u, v]$ measured by the metric the fixed bijection between $[u, v]$ and $[0, 1]$ induces.

For $\omega \in \Omega(G)$ and a finite set $S \subseteq V(G)$, let $C(S, \omega)$ be the unique component of $G - S$ which contains rays from ω. Furthermore, we set

$$\Omega(S, \omega) = \{\omega' \in \Omega(G)\ ;\ C(S, \omega') = C(S, \omega)\}$$

and define for every $\epsilon > 0$ the set $\mathring{E}_\epsilon(S, \omega)$ as the set of all inner points of edges $[u, v]$ with $u \in S$ and $v \in C(S, \omega)$ which have distance less than ϵ to v in $[u, v]$ always measured by the metric induced by the fixed bijection between $[u, v]$ and $[0, 1]$. Finally, we declare for every $\omega \in \Omega(G)$, every finite set $S \subseteq V(G)$ and $\epsilon > 0$ the point set

$$C(S, \omega) \cup \Omega(S, \omega) \cup \mathring{E}_\epsilon(S, \omega)$$

as open. This completes the definition of the basis of open sets. Every open set of $|G|$ is obtained as a union of the stated open sets. Note at this point that the last type of open sets is often defined in a slightly different way, namely by fixing

$\epsilon = 1$. Since the graph G is locally finite, there are only finitely many edges uv with $u \in S$ and $v \in C(S, \omega)$. Therefore, it is easy to check that both bases define the same topology.

An equivalent way of defining the topological space $|G|$ is to endow G with the topology of a 1-complex (also called CW complex of dimension 1) and consider the Freudenthal compactification G. This connection was examined in [9]. For the original paper of Freudenthal about the Freudenthal compactification, see [13].

For a point set X in $|G|$, we denote its closure in $|G|$ by $\overline{X}$.

A subspace Z of $|G|$ is called *standard subspace* of $|G|$ if $Z = \overline{H}$ where H is a subgraph of G.

A *circle* in $|G|$ is the image of a homeomorphism which maps from the unit circle S^1 in $\mathbb{R}^2$ to $|G|$. The graph G is called *Hamiltonian* if there exists a circle in $|G|$ which contains all vertices of G. We call such a circle a *Hamilton circle* of G. For G being finite, this coincides with the usual meaning, namely that there is a cycle in G which contains all vertices of G. Such cycles are called *Hamilton cycles* of G. For a circle C in $|G|$, we call the edge set $\{uv \in E(G)\ ;\ [u, v] \subseteq C\}$ the *circuit* of C.

The image of a homeomorphism which maps from the closed real unit interval $[0, 1]$ to $|G|$ is called an *arc* in $|G|$. For an arc α in $|G|$, we call a point x of $|G|$ an *endpoint* of α if 0 or 1 is mapped to x by the homeomorphism which defines α. Furthermore, we say that α *ends* in a point x of $|G|$ if x is an endpoint of α.

Let ω be an end of G and Z be a standard subspace of $|G|$. Then we define the *degree* of ω in Z as a value in $\mathbb{N} \cup \{\infty\}$, namely the supremum of the number of edge-disjoint arcs in Z that end in ω. The next definition is due to Bruhn and Stein (see [4]) and allows us to distinguish the parity of degrees of ends also in the case where their degrees are infinite. We call the degree of ω *even* in Z if there exists a finite set $S \subseteq V$ such that for every finite set $S' \subseteq V$ with $S \subseteq S'$ the maximum number of edge-disjoint arcs in Z whose endpoints are ω and some $s \in S'$ is even. Otherwise, we call the degree of ω *odd* in Z.

3. Toolkit

This section covers some facts which we shall need later or for the proofs of the last two lemmas of this section, Lemma 3.10 and Lemma 3.11. These two lemmas are very important tools for the proof of the main result. We begin with a basic proposition about infinite graphs.

Proposition 3.1. [7, Prop. 8.2.1] *Every infinite connected graph has a vertex of infinite degree or contains a ray.*

Especially for infinite connected graphs which are locally finite, Proposition 3.1 shows the existence of rays. The proof of this proposition bases on a compactness argument and is not difficult. Anyhow, we do not state a proof here.

We proceed with a couple of lemmas about connectedness in the topological space $|G|$ of a locally finite connected graph G.

Lemma 3.2. [12, Thm. 2.6] *If G is a locally finite connected graph, then every closed topologically connected subset of $|G|$ is arc-connected.*

It follows from this lemma that being connected is equivalent to being arc-connected for closed topologically connected subsets of $|G|$. We shall use this fact for topologically connected standard subspaces, which are closed by definition.

The next lemma is a basic tool and gives a necessary condition for the existence of certain arcs in $|G|$.

Lemma 3.3. [7, Lemma 8.5.3] *Let G be a locally finite connected graph and $F \subseteq E(G)$ be a cut with the sides V_1 and V_2.*

(i) *If F is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \setminus \mathring{F}$ with one endpoint in V_1 and the other in V_2.*
(ii) *If F is infinite, then $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, and there may be such an arc.*

The following lemma states a graph-theoretical characterization of topologically connected standard subspaces. By Lemma 3.2, we get also a characterization of arc-connected standard subspaces.

Lemma 3.4. [7, Lemma 8.5.5] *If G is a locally finite connected graph, then a standard subspace of $|G|$ is topologically connected (equivalently: arc-connected) if and only if it contains an edge from every finite cut of G of which it meets both sides.*

Now we state a theorem which helps us to verify when every vertex and every end of a graph has even degree in a standard subspace.

Theorem 3.5. [8, Thm. 2.5] *Let G be a locally finite connected graph. Then the following are equivalent for $D \subseteq E(G)$:*

(i) *D meets every finite cut in an even number of edges.*
(ii) *Every vertex and every end of G has even degree in $\overline{D}$.*

In [8, Thm. 2.5] are actually four equivalent statements involved, but we need only two of them here. Theorem 3.5 follows from a result of Diestel and Kühn [10, Thm. 7.1] together with a result of Berger and Bruhn [2, Thm. 5].

Bruhn and Stein showed the following characterization of circles in terms of vertex and end degrees.

Lemma 3.6. [4, Prop. 3] *Let C be a subgraph of a locally finite connected graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected and every vertex or end x of G with $x \in \overline{C}$ has degree two in C.*

It should be noted at this point that Theorem 3.5 and Lemma 3.6 are crucial for the proof of Lemma 3.11.

Now we turn towards claw-free graphs and prove two basic facts about minimal vertex separators in such graphs.

Proposition 3.7. *Let G be a connected claw-free graph and S be a minimal vertex separator in G. Then $G - S$ has exactly two components.*

Proof. Suppose $G - S$ has at least three components. Since S is a minimal vertex separator, every vertex of S has at least one neighbour in each component of $G - S$. Now pick an $s \in S$ and neighbours v_1, v_2, v_3 of s which lie in different components of $G - S$. Then $G[s, v_1, v_2, v_3]$ is an induced claw. This contradicts our assumption on G. □

The next lemma is a cornerstone of the constructive proof of Lemma 4.2.

Lemma 3.8. *Let G be a connected claw-free graph and S be a minimal vertex separator in G. For every vertex $s \in S$ and every component K of $G - S$, the graph $G[N(s) \cap V(K)]$ is complete.*

Proof. By Proposition 3.7, we know that $G - S$ has precisely two components, say K_1 and K_2. Now suppose for a contradiction that the statement of the lemma is false. Then there exists a vertex $s \in S$ such that s has two distinct neighbours v_1, v_2 which are not adjacent and lie both in K_1 or K_2, say in K_1. Since S is a minimal separator, it has at least one neighbour in each component of $G - S$. Let v_3 be a neighbour of s in K_2. Now the graph $G[s, v_1, v_2, v_3]$ is an induced claw in G, which is a contradiction. □

Before we turn towards the two main lemmas of this section, let us prove a basic fact about locally connected graphs.

Proposition 3.9. *Every connected, locally connected graph on at least three vertices is* 2*-connected.*

Proof. Let G be a connected, locally connected graph on at least three vertices and suppose it is not 2-connected. Then G can not be complete. So there exists a minimal vertex separator which consists just of one vertex, say s, because G is connected. By minimality, s has neighbours in all components of $G - s$. Let v_1 and v_2 be neighbours of s which lie in different components of $G - s$. Since G is locally connected, there exists a path in the neighbourhood of s from v_1 to v_2. This path does not meet s but connects two components of $G - s$. This contradicts that $\{s\}$ is a separator in G. □

The following lemma deals with the structure of an infinite, locally finite, connected, claw-free graph G. Roughly speaking, the lemma says that if we separate any finite connected subgraph from all ends by a finite set $\mathscr{S}$, then this separator decomposes into minimal vertex separators each of which has neighbours in precisely two components of $G - \mathscr{S}$, namely in the unique finite component of $G - \mathscr{S}$ and in an infinite component.

Lemma 3.10. *Let G be an infinite, locally finite, connected, claw-free graph and X be a finite vertex set of G such that $G[X]$ is connected. Furthermore, let $\mathscr{S} \subseteq V(G)$ be a finite minimal vertex set such that $\mathscr{S} \cap X = \emptyset$ and every ray starting in X has to meet $\mathscr{S}$. Then the following holds:*

(i) *The set $\mathscr{S}$ is the disjoint union of $k \geq 1$ minimal vertex separators S_i in G where $1 \leq i \leq k$ such that each vertex $s \in S_i$ has a neighbour in precisely one infinite component K_i of $G - \mathscr{S}$ for every i with $1 \leq i \leq k$.*

(ii) *$G - \mathscr{S}$ has precisely one finite component K_0. This component contains all vertices of X and every vertex of $\mathscr{S}$ has a neighbour in K_0.*

Proof. Since $G[X]$ is connected, there must be a component of $G - \mathscr{S}$ that contains all vertices of X. Let K_0 be this component. We show first that K_0 is finite. Suppose not for a contradiction. Then there is a ray in K_0 by Proposition 3.1 since K_0 is locally finite and connected. Using the connectedness of K_0 again, there exists also a ray in K_0 that starts in X. No ray in K_0 meets the set $\mathscr{S}$ because K_0 is a component of $G - \mathscr{S}$. This contradicts the definition of $\mathscr{S}$. By the minimality of $\mathscr{S}$, we get that every vertex $s \in \mathscr{S}$ has at least one neighbour in K_0. If this would not be the case, then $\mathscr{S} - s$ would be a proper subset of $\mathscr{S}$ which no ray can avoid that starts in X. By the same argument, we also know that every vertex $s \in \mathscr{S}$ has at least one neighbour in an infinite component of $G - \mathscr{S}$. Furthermore, we know that every vertex $s \in \mathscr{S}$ can have only two neighbours in different components because G is claw-free. Since G is locally finite and $\mathscr{S}$ is finite, $G - \mathscr{S}$ has only finitely many components. So let $K_1, \ldots, K_k$ be the infinite components of $G - \mathscr{S}$. Using that G is infinite, locally finite and connected, the graph G contains a ray by Proposition 3.1, which implies that $k \geq 1$ holds. The previous observations ensure that we can partition the set $\mathscr{S}$ into vertex sets $S_1, \ldots, S_k$ where a vertex $s \in \mathscr{S}$ lies in S_i if and only if s has a neighbour in K_i for every i with $1 \leq i \leq k$. This definition implies that S_i is a separator in G for each i with $1 \leq i \leq k$. Using the minimality of $\mathscr{S}$, we obtain furthermore that each set S_i is a minimal vertex separator in G. This completes the proof of statement (i). It remains to check that K_0 is the only finite component of $G - \mathscr{S}$. Let us consider an arbitrary vertex $s \in \mathscr{S}$. We know that s lies in S_i for some i with $1 \leq i \leq k$. So s has a neighbour in two components of $G - \mathscr{S}$, namely K_0 and K_i. Since G is claw-free, the vertex s cannot have a neighbour in any other component of $G - \mathscr{S}$. As s was chosen arbitrarily, we obtain that $K_0, K_1, \ldots, K_k$ are all components of $G - \mathscr{S}$. Especially, K_0 is the only finite component of $G - \mathscr{S}$ and contains all vertices of X by definition. $\square$

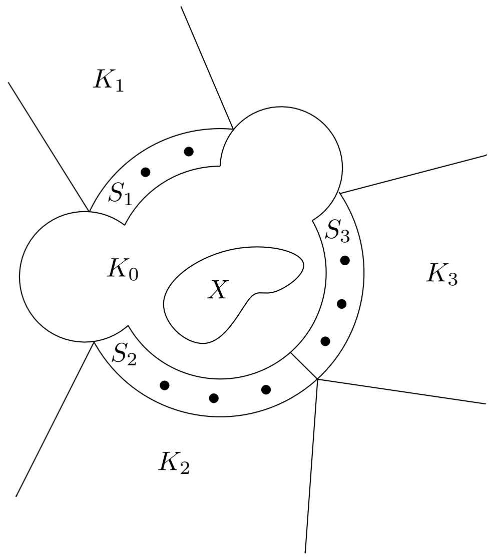

FIGURE 1. Example for Lemma 3.10 with $k = 3$.

The next lemma is our main tool to prove that an infinite, locally finite, connected graph is Hamiltonian. In order to apply this lemma, we need a sequence of cycles and a set of vertex sets which fulfill a couple of conditions. While we obtain a Hamilton circle as a limit object from the sequence of cycles, the vertex sets act as witnesses to verify that the limit object is really a circle. It is not easy to get such cycles and vertex sets. The main work to prove Theorem 1.2 is to construct cycles and vertex sets which fulfill the required conditions. The way of constructing these objects relies on the structure of the graph as described in Lemma 3.10.

Lemma 3.11. *Let G be an infinite, locally finite, connected graph and $(C_i)_{i \in \mathbb{N}}$ be a sequence of cycles of G. Now G is Hamiltonian if there exists an integer $k_i \geq 1$ for every $i \geq 1$ and vertex sets $M_j^i \subseteq V(G)$ for every $i \geq 1$ and j with $1 \leq j \leq k_i$ such that the following is true:*

(i) *For every vertex v of G, there exists an integer $j \geq 0$ such that $v \in V(C_i)$ holds for every $i \geq j$.*
(ii) *For every $i \geq 1$ and j with $1 \leq j \leq k_i$, the cut $\delta(M_j^i)$ is finite.*
(iii) *For every end ω of G, there is a function $f : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{N}$ such that the inclusion $M_{f(j)}^j \subseteq M_{f(i)}^i$ holds for all integers i, j with $1 \leq i \leq j$ and the equation $M_\omega := \bigcap_{i=1}^\infty \overline{M_{f(i)}^i} = \{\omega\}$ is true.*
(iv) *$E(C_i) \cap E(C_j) \subseteq E(C_{j+1})$ holds for all integers i and j with $0 \leq i < j$.*
(v) *The equations $E(C_i) \cap \delta(M_j^p) = E(C_p) \cap \delta(M_j^p)$ and $|E(C_i) \cap \delta(M_j^p)| = 2$ hold for each triple (i, p, j) which satisfies $1 \leq p \leq i$ and $1 \leq j \leq k_p$.*

Proof. We define a subgraph C of G and show that its closure is a Hamilton circle of G. Let

$$V(C) = \bigcup_{i=0}^{\infty} V(C_i),$$
$$E(C) = \{e \in E(G)\ ;\ e \in E(C_i) \text{ for infinitely many } i \geq 0\}.$$

Note that condition (i) implies $V(C) = V(G)$. So the closure $\overline{C}$ contains all ends of G. We get also immediately that $E(C) \neq \emptyset$ by condition (v). Furthermore, condition (iv) implies that for every edge $e \in E(C)$ there exists an integer $j \geq 0$ such that $e \in E(C_i)$ for every $i \geq j$. In order to prove that $\overline{C}$ is a Hamilton circle of G, we want to apply Lemma 3.6. So we need $\overline{C}$ to be topologically connected and that every vertex and every end of $\overline{C}$ has degree two in $\overline{C}$. We prove both of these statements with two claims. Before we can do this, we need the following claim.

Claim 1. *Let $X \subseteq V(G)$ be a finite set of vertices and $D \subseteq E(G)$ be a finite set of edges. Then there exists an integer $j \geq 0$ such that $X \subseteq V(C_i)$ holds for every $i \geq j$ and that each edge of D is either contained in $E(C_i)$ for every $i \geq j$ or not contained in any $E(C_i)$ for $i \geq j$.*

Since X is finite, we can use condition (i) to find an integer q such that $X \subseteq V(C_i)$ holds for every $i \geq q$. Each edge e of D lies either in at most one cycle $C_{\ell-1}$ of the sequence with $\ell \geq 1$ or in at least two, say in C_m and C_n with $m, n \geq 0$. In the first case, we know that e does not lie in any $E(C_i)$ for $i \geq \ell$. Using condition (iv), we obtain in the latter case that e lies in $E(C_i)$ for every $i \geq \ell'$ where $\ell' = \max\{m, n\}$. Using these observations, we can define a function $g : D \longrightarrow \mathbb{N}$ by

$$g(e) = \begin{cases} \ell & \text{if } \ell \text{ is the least integer such that } e \notin E(C_i) \text{ for every } i \geq \ell \\ \ell' & \text{if } \ell' \text{ is the least integer such that } e \in E(C_i) \text{ for every } i \geq \ell'. \end{cases}$$

Since D is finite, we can set $j = \max\{\{q\} \cup \{g(e)\ ;\ e \in D\}\}$. The conditions (i) and (iv) imply that j has the desired properties. This completes the proof of Claim 1.

Now we state and prove the two claims which allow us to apply Lemma 3.6.

Claim 2. *$\overline{C}$ is topologically connected and every vertex as well as every end of G has even degree in $\overline{C}$.*

Since $\overline{C}$ is a standard subspace of $|G|$ and contains all vertices of G, it suffices to show, by Lemma 3.4 and Theorem 3.5, that $E(C)$ meets every nonempty finite cut in an even number of edges and at least twice. So fix any nonempty finite cut D. Now take an integer j such that C_j contains vertices from each side of the partition which induces D and that each edge of D is either contained in $E(C_i)$ for every $i \geq j$ or not contained in any $E(C_i)$ for $i \geq j$. Since D is finite, we obtain by Claim 1 that it is possible to find such an integer j. Now we use that the cycle C_j has vertices in both sides of the partition which induces D. So it must hit the cut D in an even number of edges and at least twice. The same holds for C by its definition. This completes the proof of Claim 2.

Claim 3. *Every vertex and every end in $\overline{C}$ has degree two in $\overline{C}$.*

As noted before, $\overline{C}$ contains all vertices and ends of G. First, we check that every vertex has degree two in C. For this purpose, we fix an arbitrary vertex v of G. Let j be an integer such that v is a vertex of C_j and that each edge which is incident with v is either contained in $E(C_i)$ for every $i \geq j$ or not contained in any $E(C_i)$ for $i \geq j$. We can find such an integer j because of Claim 1 and because G

is locally finite, which implies that the cut $\delta(\{v\})$ is finite. Since C_j is a cycle and v is one of its vertices, v has degree two in C_j and therefore also in C by definition.

For the statement about the ends, we have to show that for every end the maximum number of edge-disjoint arcs in $\overline{C}$ ending in this end is two. We prove first that the degree of every end must be at least two in $\overline{C}$. Afterwards, we show that the degree of every end is less or equal to two in $\overline{C}$.

We already know by Claim 2 that the standard subspace $\overline{C}$ is topologically connected. So $\overline{C}$ is also arc-connected by Lemma 3.2. Therefore, no end has degree zero in $\overline{C}$. Additionally, we know by Claim 2 that every end has even degree in $\overline{C}$. Combining these facts, the degree of every end must be at least two in $\overline{C}$.

Now let us fix an arbitrary end ω of G. In order to bound the degree of ω in $\overline{C}$ from above by two, we use the cuts $\delta(M_j^i)$. We know by condition (iii) that there exists a function f such that $M_\omega = \{\omega\}$ holds. We prove first that for every arc α in $|G|$ which ends in ω there exists an integer i' such that α uses an edge of $\delta(M_{f(i)}^i)$ for every $i \geq i'$. It is an easy consequence of Lemma 3.3 that every arc that does not only consist of inner points of edges must contain a vertex. So we fix a vertex v of α. Now choose i' such that v does not lie in $M_{f(i)}^i$ for any $i \geq i'$. This is possible because of condition (iii). We know by condition (ii) that the cut $\delta(M_{f(i)}^i)$ is finite for every $i \geq i'$. Now α is an arc such that v is on one side of the finite cut $\delta(M_{f(i)}^i)$ and ω is in the closure of the other side for every $i \geq i'$. So by Lemma 3.3, the arc α must use one of the edges of $\delta(M_{f(i)}^i)$ for every $i \geq i'$. Since C contains only two edges of the cut $\delta(M_{f(i)}^i)$ for every $i \geq 1$ by condition (v), there can be at most two edge-disjoint arcs in $\overline{C}$ that end in ω. This completes the proof of Claim 3.

As mentioned before, we can use Claim 2 and Claim 3 to deduce from Lemma 3.6 that $\overline{C}$ is a circle in $|G|$. Furthermore, $\overline{C}$ is a Hamilton circle since $\overline{C}$ contains all vertices of G as we have seen before. □

4. Locally connected claw-free graphs

We begin this section with a lemma that contains the essence of the proof of Theorem 1.1 which Oberly and Sumner presented in [19, Thm. 1].

Lemma 4.1. *Let G be a locally connected claw-free graph and C be a cycle in G. Furthermore, let $v \in N(C)$ and $u \in N(v) \cap V(C)$. Then one of the following holds:*

(i) *For some $x \in \{u^+, u^-\}$, there exists a v–x path P_x in G whose vertices lie completely in $N(u)$ such that $V(P_x) \cap \{u^+, u^-\} = \{x\}$ and for every vertex $z \in V(\mathring{P}_x) \cap V(C)$ the relations $V(P_x) \cap \{z^+, z, z^-\} = \{z\}$ and $z^+z^- \in E(G)$ hold.*

(ii) *The vertices u^+ and u^- are adjacent in G and there exists a certain vertex $w \in (N(u) \cap V(C)) \setminus \{u^+, u^-\}$ together with a v–w path P_w in G whose vertices lie completely in $N(u)$ such that u is adjacent to w^+ or w^-, the vertices u^+, u^-, w^+, w^- do not lie on P_w and for every vertex $q \in V(\mathring{P}_x) \cap V(C)$ the relations $V(P_x) \cap \{q^+, q, q^-\} = \{q\}$ and $q^+q^- \in E(G)$ hold.*

Proof. Since G is locally connected, there exists a v–u^+ path Q whose vertices lie entirely in $N(u)$. Let x be the first vertex on Q, in the direction from v to u^+, which lies in $\{u^+, u^-\}$. Now we set $P_x = vQx$.

Before we proceed, we define the following notation. A vertex $z \in V(\mathring{P}_x) \cap V(C)$ is called *singular* if neither z^+ nor z^- is a neighbour of u. Note that the vertices z^+ and z^- are adjacent in G for every singular vertex $z \in V(\mathring{P}_x) \cap V(C)$ because otherwise $G[z, z^+, z^-, u]$ is a claw, which contradicts the assumption on G.

Now we distinguish two cases:

Case 1. *Every vertex in $V(\mathring{P}_x) \cap V(C)$ is singular.*

In this case, we know that the edge z^-z^+ has to be present for every vertex $z \in V(\mathring{P}_x) \cap V(C)$ as noted above. Now the objects x and P_x verify that statement (i) of the lemma is true.

Case 2. *There exists a vertex in $V(\mathring{P}_x) \cap V(C)$ which is not singular.*

We may assume that u^- and u^+ are adjacent because otherwise the edge vu^- or the edge vu^+ must exist to avoid that $G[u, v, u^-, u^+]$ is a claw. Every such edge corresponds to another path P_y with $V(\mathring{P}_y) \cap V(C) = \emptyset$ which we could use instead of P_x. Hence, we would be done by Case 1.

Let w be the first vertex in $V(\mathring{P}_x) \cap V(C)$ which is not singular by traversing P_x and starting at v. Since w lies in $V(\mathring{P}_x) \cap V(C)$, it cannot be equal to u^+ or u^-. Additionally, w^+ or w^- is adjacent to u because w is not singular. Now set $P_w = vP_xw$. Since $V(P_x) \cap \{u^+, u^-\} = \{x\}$ and x is an endvertex of P_x, we get that neither u^+ nor u^- lie on P_w. Furthermore, we get that $V(P_w)$ does neither contain w^+ nor w^- and each vertex $q \in V(\mathring{P}_w) \cap V(C)$ is singular because w has been chosen as the first vertex in $V(\mathring{P}_x) \cap V(C)$ which is not singular by traversing P_x and starting at v. To prove that statement (ii) of the lemma holds in this case, it remains to show that the edge q^+q^- is present for every vertex $q \in V(\mathring{P}_w) \cap V(C)$. By the choice of w, we know that every vertex $q \in V(\mathring{P}_w) \cap V(C)$ is a singular vertex in $V(\mathring{P}_x) \cap V(C)$. So all required edges are present by the remark from above. $\square$

To each of the statements (i) and (ii) of Lemma 4.1 corresponds a cycle that contains the vertex v and all vertices of C. In statement (i), we get such a cycle using C where we replace the path z^-zz^+ in C by the edge z^-z^+ for every vertex $z \in V(\mathring{P}_x) \cap V(C)$ and the edge ux of C by the path uvP_xx. In statement (ii), we take C and replace the path u^-uu^+ of C by the edge u^-u^+, the edge yw of C by the path $yuvP_ww$ for some $y \in \{w^+, w^-\} \cap N(u)$ and each path q^-qq^+ in C by the edge q^-q^+ for every vertex $q \in V(\mathring{P}_w) \cap V(C)$. We call each of these resulting cycles a *path extension of C*. A finite sequence of cycles (C_i), where $0 \leq i \leq n$ for some $n \in \mathbb{N}$, is called a *path extension sequence of C* if $C_0 = C$ and C_i is a path extension of C_{i-1} for every $i \in \{1, \ldots, n\}$. A path P_x or P_w as in statement (i) or (ii), respectively, of Lemma 4.1 is called *extension path.* The vertices v and u from Lemma 4.1 are called *target* and *base*, respectively, of the path extension.

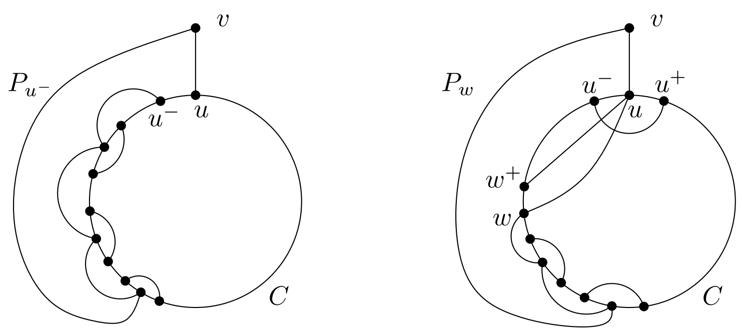

FIGURE 2. Path extensions of a cycle C with target v, base u and extension paths P_{u^-} and P_w, respectively.

Note that Theorem 1.1 can be easily deduced from Lemma 4.1 together with Proposition 3.9.

Now we turn towards the proof of Theorem 1.2, which is the main result of this paper. The plan to prove this theorem is to construct a sequence of cycles together with certain vertex sets carefully such that we obtain a Hamilton circle as a limit object of the sequence of cycles using Lemma 3.11. The next lemma will be the main tool for constructing such objects and relies on path extensions of cycles, which we get as in Lemma 4.1.

Lemma 4.2. *Let $G = (V, E)$ be an infinite locally finite, connected, locally connected, claw-free graph, C be a cycle of G which has a vertex in distance at least 3 to $N(C)$ and $\mathscr{S} \subseteq N(C)$ be a minimal vertex set such that every ray starting in C meets $\mathscr{S}$. Furthermore, let k, S_j and K_j be analogously defined as in Lemma 3.10. Then there exists a cycle C' with the properties:*

(i) *$V(K_0) \cup \mathscr{S} \cup N_3(\mathscr{S}) \subseteq V(C')$.*
(ii) *For every integer j with $1 \leq j \leq k$, there are vertex sets $M_j \subseteq V$ such that $V(K_j) \subseteq M_j \subseteq (V \setminus V(C)) \cup N_2(\mathscr{S})$ and $|E(C') \cap \delta(M_j)| = 2$ hold where the graph $G[M_j]$ is connected and contains either no or all vertices of K_p for each $p \in \{1, \ldots, k\}$.*
(iii) *$E(C - N_2(N(C))) \subseteq E(C')$ and the inclusion $\{u, v\} \subseteq (V \setminus V(C)) \cup N_3(N(C))$ holds for every edge $uv \in E(C') \setminus E(C)$.*

Proof. First we define inductively a sequence of $k+1$ cycles $(C_0, \ldots, C_k)$ and a bijection $f : \{1, \ldots, k\} \longrightarrow \{1, \ldots, k\}$ such that for every $i \in \{1, \ldots, k\}$ the inclusion

$$V(C) \cup \bigcup_{p=1}^{i} (S_{f(p)} \cup (N_3(S_{f(p)}) \cap V(K_{f(p)})) \subseteq V(C_i)$$

holds and C_i contains no vertices from any S_q for $q \in \{1, \ldots, k\} \setminus \{f(1), \ldots, f(i)\}$. Furthermore, we define for all integers $i \in \{1, \ldots, k\}$ and $j \in \{f(1), \ldots, f(i)\}$ vertex sets M_j^i such that the following holds:

- $V(K_j) \subseteq M_j^i \subseteq (V \setminus V(C)) \cup N_2(N(C))$,
- $|E(C_i) \cap \delta(M_j^i)| = 2$,
- all vertices of M_j^i lie either on C_i or in $V \setminus N_4(K_0)$,
- $G[M_j^i]$ is connected,
- M_j^i contains either no or all vertices of K_p for each $p \in \{1, \ldots, k\}$.

We start by setting $C_0 = C$. Note that for C_0 there are no requirements. Now suppose we have already defined the sequence of cycles up to C_m with $0 \leq m < k$, the values of $f(i)$ for every $i \in \{1, \ldots, m\}$ and the sets M_j^i for all $i \in \{1, \ldots, m\}$ and $j \in \{f(1), \ldots, f(m)\}$. The definitions of the cycle C_{m+1}, of the value of $f(m+1)$ and of the sets M_j^{m+1} for each $j \in \{f(1), \ldots, f(m+1)\}$ need some work. We state and prove two claims before we define these objects.

Claim 1. *There are an integer $\ell \in \{1, \ldots, k\} \setminus \{f(1), \ldots, f(m)\}$, a path extension D_1 of C_m which contains precisely one vertex from S_ℓ but no vertices from any S_p with $p \in \{1, \ldots, k\} \setminus \{f(1), \ldots, f(m), \ell\}$ and vertex sets A_j for every $j \in \{f(1), \ldots, f(m)\}$ such that the following is true for every $j \in \{f(1), \ldots, f(m)\}$:*

(a) *$V(K_j) \subseteq A_j \subseteq (V \setminus V(C)) \cup N_2(N(C)$.*
(b) *$|E(D_1) \cap \delta(A_j)| = 2$.*
(c) *All vertices of A_j lie either on D_1 or in $V \setminus N_4(K_0)$.*
(d) *$G[A_j]$ is connected.*
(e) *A_j contains either no or all vertices of K_p for each $p \in \{1, \ldots, k\}$.*

First we pick a vertex $v \in S_{\ell'}$ as target of a path extension of C_m where $\ell' \in \{1, \ldots, k\} \setminus \{f(1), \ldots, f(m)\}$, which is possible since $\mathscr{S} \subseteq N(C)$. Let the vertex $u \in V(C)$ be the base of the extension and P_v be the corresponding extension path with endvertices v and x where s is the last vertex on P_v which lies in $\mathscr{S} \setminus \bigcup_{i=1}^m S_{f(i)}$, say $s \in S_\ell$ with $\ell \in \{1, \dots, k\} \setminus \{f(1), \dots, f(m)\}$. Now we consider the path extension D_1 of C_m where we choose u again as base but with s as target together with the path $P_s = sP_vx$ which we use as extension path. On this way it is ensured that D_1 contains precisely one vertex from $\mathscr{S} \setminus \bigcup_{i=1}^m S_{f(i)}$, namely s. Now we define vertex sets A_j for every $j \in \{f(1), \dots, f(m)\}$ as follows:

$$A_j = \begin{cases} M_j^m \cup V(P_s) \cup \{u\} & \text{if } x \in M_j^m \\ M_j^m \setminus (V(P_s) \cup \{u\}) & \text{otherwise.} \end{cases}$$

We fix an arbitrary $j \in \{f(1), \dots, f(m)\}$ to check that all statements of the claim are true. We begin with statement (a). Note that u was chosen from $V(C) \subseteq V(K_0)$ and has a neighbour in $\mathscr{S}$. Additionally, the inclusion $V(P_s) \subseteq N(u)$ holds. So we get that the inclusion $V(P_s) \cup \{u\} \subseteq \mathscr{S} \cup (N_2(\mathscr{S}) \cap V(K_0))$ is valid. Since the inclusions $\mathscr{S} \subseteq N(C)$ and $V(K_j) \subseteq M_j^m \subseteq (V \setminus V(C)) \cup N_2(N(C))$ hold, the definition of A_j implies that the inclusions $V(K_j) \subseteq A_j \subseteq (V \setminus V(C)) \cup N_2(N(C))$ are true.

Note for statement (b) that D_1 has vertices in A_j and $V \setminus A_j$. To see this, we use that D_1 is a path extension of C_m and the cycle C_m contains vertices with distance at least 3 to $\mathscr{S}$ in $N_2(\mathscr{S}) \cap V(K_j)$ as well as in $V(C)$ by construction and assumption, respectively. These vertices lie in A_j and $V \setminus A_j$, respectively, because of statement (a). So the cycle D_1 hits the cut $\delta(A_j)$ at least twice, but using the equation $|E(C_m) \cap \delta(M_j^m)| = 2$ and the definition of A_j, we get that D_1 hits $\delta(A_j)$ precisely twice.

Note for statement (c) that D_1 is a path extension of C_m. So it contains all vertices from C_m and $V(P_s) \cup \{u\}$. Now the statement follows because all vertices in M_j^m which have distance at most 4 to K_0 are vertices of C_m.

Statement (d) is obvious if $A_j = M_j^m \cup V(P_s) \cup \{u\}$ using that $G[M_j^m]$ is connected. So let us consider the case where $A_j = M_j^m \setminus (V(P_s) \cup \{u\})$. Combining statement (b) and (c), we get that all vertices in A_j with distance at most 4 to K_0 lie on a path P in $G[A_j]$ which is induced by the cycle D_1. Since u is a vertex of C and $V(P_s) \subseteq N(u)$, we get that all vertices in $N(V(P_s) \cup \{u\}) \cap A_j$ lie on the path P. Using that $G[M_j^m]$ is connected, we know that each component of $G[A_j]$ contains a vertex of $N(V(P_s) \cup \{u\}) \cap A_j$. Now the path P ensures that $G[A_j]$ consists of precisely one component, which means that $G[A_j]$ is connected.

To show that statement (e) holds, we use that the corresponding statement with M_j^m instead of A_j is true. Since $V(P_s) \cup \{u\} \subseteq \mathscr{S} \cup (N_2(\mathscr{S}) \cap V(K_0))$ holds, the definition of A_j implies that statement (e) is true. This completes the proof of the Claim 1.

For the rest of the proof of the lemma, we fix an integer ℓ, a cycle D_1 and vertex sets A_j for every $j \in \{f(1), \dots, f(m)\}$ with the properties as in Claim 1. Now we proceed with the next claim, which uses these objects.

Claim 2. *There are a path extension D_2 of D_1 which contains precisely two vertices from S_ℓ and these vertices are adjacent in D_2 but contains no vertices from any S_p where p lies in $\{1, \dots, k\} \setminus \{f(1), \dots, f(m), \ell\}$ and vertex sets B_j for every $j \in \{f(1), \dots, f(m)\}$ such that the following holds for every $j \in \{f(1), \dots, f(m)\}$:*

(a) $V(K_j) \subseteq B_j \subseteq (V \setminus V(C)) \cup N_2(N(C))$.
(b) $|E(D_2) \cap \delta(B_j)| = 2$.
(c) *All vertices of B_j lie either on D_2 or in $V \setminus N_4(V(K_0))$.*
(d) *$G[B_j]$ is connected.*

(e) B_j *contains either no or all vertices of* K_p *for each* $p \in \{1, \dots, k\}$.
(f) B_j *contains either two vertices of* $S_\ell \cap V(D_2)$ *or no vertex of* $S_\ell \cup V(K_\ell)$.

Let s be the only vertex of D_1 which lies in S_ℓ. Now we pick a neighbour v of s in $V(K_\ell)$. There is one because S_ℓ is a minimal separator in G and K_ℓ is one of the components of $G - S_\ell$ by Lemma 3.10. Let P_v be the extension path of a path extension of D_1 with target v and base s and let v and x be the endvertices of P_v. The path P_v must contain vertices in S_ℓ because it starts in $v \in V(K_\ell)$ and ends in x, which lies in another component of $G - S_\ell$ since D_1 contains only the vertex s from S_ℓ and $V(P_v) \subseteq N(s)$. So let t be the last vertex on P_v starting at v which is an element of S_ℓ. Furthermore, let w be the vertex after t on tP_vx. By the choice of t, we know that w and x lie in the same component of $G - S_\ell$. Since s lies in the minimal separator S_ℓ and $V(P_v) \subseteq N(s)$, we get by Lemma 3.8 that w and x are adjacent. Now we define D_2 where we distinguish two cases.

Case 1. *The vertex* w *lies in* $S_{j'}$ *for some* $j' \in \{1, \dots, k\} \setminus \{f(1), \dots, f(m)\}$.

Since D_1 contains no vertices from $S_{j'}$, we know that t and x lie in $N(w)$ and in the same component of $G - S_{j'}$. So we get by Lemma 3.8 that t and x are adjacent. We take the path $P_t = tx$ as extension path of a path extension of D_1 with target t and base s. Furthermore, we set this path extension of D_1 to be D_2. Then s and t are the only vertices from S_ℓ in D_2 and they are adjacent in D_2 too. Additionally, we get that D_2 contains no vertices from any S_p where p lies in $\{1, \dots, k\} \setminus \{f(1), \dots, f(m), \ell\}$ because of the construction of D_2 and because D_1 has this property too. This completes the definition of the cycle D_2 in Case 1.

Case 2. *The vertex* w *does not lies in* S_j *for any* $j \in \{1, \dots, k\} \setminus \{f(1), \dots, f(m)\}$.

Here we take the path $P_t = twx$ as extension path of a path extension of D_1 with target t and base s. We set this path extension of D_1 to be D_2. Note that w cannot be a vertex of S_ℓ due to its choice. So we get that s and t are the only vertices from S_ℓ in D_2 and they are adjacent in D_2 too. Here we get that D_2 contains no vertices from any S_p where p lies in $\{1, \dots, k\} \setminus \{f(1), \dots, f(m), \ell\}$ using the assumption on w together with the construction of D_2 and that D_1 has this property as well. With this we complete the definition of D_2 for Case 2.

It remains to define the sets B_j for every $j \in \{f(1), \dots, f(m)\}$. We do this for both cases in one step where we use the different definitions of the extension path P_t. We set

$$B_j = \begin{cases} A_j \cup V(P_t) \cup \{s\} & \text{if } x \in A_j \\ A_j \setminus (V(P_t) \cup \{s\}) & \text{otherwise.} \end{cases}$$

The statements (a) - (e) are verified in the same way as in Claim 1. Note for statement (d) and (e) that the base of the path extension does not lie in $V(C)$ but in $\mathscr{S} \subseteq N(C)$ and $V(P_t) \cup \{s\} \subseteq \mathscr{S} \cup (N(\mathscr{S}) \cap V(K_0))$. So all vertices in $N(V(P_t) \cup \{s\}) \cap B_j$ have distance at most 4 to K_0 and lie therefore on a path in $G[B_j]$ for every $j \in \{f(1), \dots, f(m)\}$.

To verify statement (f), let us fix an arbitrary $j \in \{f(1), \dots, f(m)\}$. Note that B_j cannot contain vertices from $S_\ell \cup (N_3(S_\ell) \cap V(K_\ell))$ except s and t because each such vertex has distance at most 4 to K_0 and must therefore lie on the cycle D_2 by statement (c). This is not possible since D_2 contains only s and t from S_ℓ. The fact that $G[B_j]$ is connected by statement (d) and contains no vertices from $S_\ell \cup (N_2(S_\ell) \cap V(K_\ell))$ except s and t implies that B_j does not contain any of the vertices in $S_\ell \cup V(K_\ell) \setminus \{s, t\}$. The definition of B_j ensures furthermore that s lies in B_j if and only if t lies in B_j. This completes the proof of Claim 2.

Let us fix a cycle D_2 and vertex sets B_j for every $j \in \{f(1), \dots, f(m)\}$ as in Claim 2 for the rest of the proof. Now we are able to define the desired cycle C_{m+1} together with the value $f(m+1)$ and the vertex sets M_j^{m+1} for every $j \in \{f(1), \dots, f(m+1)\}$. We begin by setting

$$f(m+1) = \ell.$$

Before we build the cycle C_{m+1}, we take a finite tree T_ℓ in K_ℓ such that the inclusion $N_3(S_\ell) \cap V(K_\ell) \subseteq V(T_\ell)$ holds. This is possible because K_ℓ is connected and G is locally finite, which implies together with the finiteness of S_ℓ that $N_3(S_\ell)$ is finite. To build the cycle C_{m+1}, we take first D_2 and replace the edge st of D_2 with $s, t \in S_\ell$ by the path $sn_sT_\ell n_tt$ where n_s and n_t are vertices in $N(s) \cap V(T_\ell)$ and $N(t) \cap V(T_\ell)$, respectively. Let us call the resulting cycle $\tilde{C}$. Now we build a path extension sequence $(\tilde{C}_i)$ of $\tilde{C}$ where we choose the targets always from $S_\ell \cup V(T_\ell)$ and the bases always from $V(T_\ell)$ until a cycle in this sequence contains all vertices of $S_\ell \cup V(T_\ell)$. This is possible by Lemma 4.1 and because $S_\ell \cup V(T_\ell)$ is finite. Let $\tilde{C}_n$ be the last cycle of the sequence. Then we set

$$C_{m+1} = \tilde{C}_n.$$

Since $V(C_m) \cup S_\ell \cup V(T_\ell) \subseteq V(C_{m+1})$ holds by construction, we obtain that

$$V(C) \cup \bigcup_{p=1}^{m+1} (S_{f(p)} \cup (N_3(S_{f(p)}) \cap V(K_{f(p)})) \subseteq V(C_{m+1})$$

is true, which is one of the desired properties. Moreover, we get that C_{m+1} does not contain vertices from S_q for any $q \in \{1, \dots, k\} \setminus \{f(1), \dots, f(m+1)\}$. To see this, note that all vertices of D_2 which lie in $\mathscr{S}$ are contained in some S_j where $j \in \{f(1), \dots, f(m+1)\}$ by definition of D_2 and Claim 2. Since $\tilde{C}$ does not contain any other vertices from $\mathscr{S}$ than D_2 and we choose the bases of the path extensions for the sequence $(\tilde{C}_i)$ always from $V(T_\ell) \subseteq V(K_\ell)$, all vertices of C_{m+1} which lie in $\mathscr{S}$ must also be contained in some S_j where $j \in \{f(1), \dots, f(m+1)\}$.

Next we define the sets M_j^{m+1} for every $j \in \{f(1), \dots, f(m+1)\}$ and verify that they have the desired properties. We begin by setting

$$M_{f(m+1)}^{m+1} = S_\ell \cup V(K_\ell).$$

It is obvious that the inclusions $V(K_\ell) \subseteq M_{f(m+1)}^{m+1} \subseteq (V \setminus V(C)) \cup N_2(N(C))$ hold and that $G[M_{f(m+1)}^{m+1}]$ is connected. Furthermore, the definition of $M_{f(m+1)}^{m+1}$ implies that $M_{f(m+1)}^{m+1}$ contains either no or all vertices of K_p for every $p \in \{1, \dots, k\}$. Since $V(C_{m+1})$ contains all vertices of the set $S_\ell \cup V(T_\ell)$ and $N_3(S_\ell) \cap V(K_\ell)$ is a subset of $V(T_\ell)$, we get that all vertices in $M_{f(m+1)}^{m+1}$ with distance at most 4 to K_0 lie on C_{m+1}. It remains to check that the equation $|E(C_{m+1}) \cap \delta(M_{f(m+1)}^{m+1})| = 2$ holds. Note that $|E(D_2) \cap \delta(M_{f(m+1)}^{m+1})| = 2$ is true because D_2 contains only two vertices of S_ℓ and these vertices are adjacent in D_2 by Claim 2. The construction of C_{m+1} ensures that all edges in $E(C_{m+1}) \setminus E(D_2)$ lie in $G[M_{f(m+1)}^{m+1}]$. So only two edges of C_{m+1} meet the cut $\delta(M_{f(m+1)}^{m+1})$.

By Claim 2, we know that D_2 contains precisely two vertices s, t which lie in S_ℓ. We know furthermore by Claim 2 (f) that either s and t or none of them is an element of B_j for every $j \in \{f(1), \dots, f(m)\}$. Now we make the following definition for every $j \in \{f(1), \dots, f(m)\}$:

$$M_j^{m+1} = \begin{cases} B_j \cup S_\ell \cup V(K_\ell) & \text{if } s, t \in B_j \\ B_j & \text{otherwise.} \end{cases}$$

Let us fix an arbitrary $j \in \{f(1), \dots, f(m)\}$ and verify the desired properties. It is obvious that $G[M_j^{m+1}]$ is connected because $G[B_j]$ is connected by Claim 2 (d). It is also easy to see that the inclusions $V(K_j) \subseteq M_j^{m+1} \subseteq (V \setminus V(C)) \cup N_2(N(C))$ are true since the corresponding result with B_j instead of M_j^{m+1} holds by Claim 2 (a).

To show that all vertices in M_j^{m+1} which have distance at most 4 to K_0 lie on C_{m+1}, it suffices to check the case where $M_j^{m+1} = B_j \cup S_\ell \cup V(K_\ell)$ by Claim 2 (c). Since all vertices in B_j which have distance at most 4 to K_0 lie on C_{m+1} as well as all vertices in $S_\ell \cup V(T_\ell)$ where $N_3(S_\ell) \cap V(K_\ell) \subseteq V(T_\ell)$, we get that all vertices in M_j^{m+1} with distance at most 4 to K_0 lie on C_{m+1}.

Now we check that the cycle C_{m+1} meets precisely two edges of the cut $\delta(M_j^{m+1})$. In the case where $M_j^{m+1} = B_j$ holds, we know that B_j does not contain any vertices of $S_\ell \cup V(K_\ell)$ because of Claim 2 (f). The equation $|E(D_2) \cap \delta(B_j)| = 2$ is also true by Claim 2 (b). Since all edges of $E(C_{m+1}) \setminus E(D_2)$ lie in $G[S_\ell \cup V(K_\ell)]$, the cycle C_{m+1} hits still precisely two edges of the cut $\delta(B_j)$. So let us consider the case where $M_j^{m+1} = B_j \cup S_\ell \cup V(K_\ell)$. Here we know by Claim 2 (f) that B_j contains the vertices s and t from $S_\ell \cup V(K_\ell)$ but no other vertex of this set. By Claim 2 (b), we know additionally that the equation $|E(D_2) \cap \delta(B_j)| = 2$ holds. In this case, the equation implies that also $|E(D_2) \cap \delta(M_j^{m+1})| = 2$ is true. Since all edges of $E(C_{m+1}) \setminus E(D_2)$ lie completely in $G[M_j^{m+1}]$, the cycle C_{m+1} meets the cut $\delta(M_j^{m+1})$ precisely twice.

It remains to prove that M_j^{m+1} has the property that it contains either no or all vertices of K_p for each $p \in \{1, \dots, k\}$. If $j = f(m+1)$, this is obvious by the definition of the set M_j^{m+1}. Otherwise, we know that B_j has this property by Claim 2 (e) and so the definition of M_j^{m+1} implies again that M_j^{m+1} has this property. So the set M_j^{m+1} has all desired properties. This completes the construction of the sequence $(C_0, \dots, C_k)$ of cycles.

To complete the proof of this lemma, we take now a path extension sequence $(\hat{C}_i)$ of C_k where we choose the targets and bases always from $V(K_0)$ until a cycle in this sequence contains all vertices of K_0. Note that $V(K_0)$ is finite by its choice and Lemma 3.10. Let $\hat{C}_M$ be the last cycle of the sequence. Then we set $C' = \hat{C}_M$. The construction of the cycle C_k ensures that $V(C) \cup \mathscr{S} \cup (N_3(\mathscr{S}) \setminus V(K_0)) \subseteq V(C_k)$ holds. Hence, the inclusion $V(K_0) \cup \mathscr{S} \cup N_3(\mathscr{S}) \subseteq V(C')$ is true as desired for statement (i) of this lemma.

For each cycle $\hat{C}_i$, we define now vertex sets $\hat{M}_j^i$ for every $j \in \{1, \dots, k\}$ such that the following holds:

- $V(K_j) \subseteq \hat{M}_j^i \subseteq (V \setminus V(C)) \cup N_2(N(C))$,
- $|E(\hat{C}_i) \cap \delta(\hat{M}_j^i)| = 2$,
- all vertices of $\hat{M}_j^i$ lie either on $\hat{C}_i$ or in $V \setminus N_4(K_0)$,
- $G[\hat{M}_j^i]$ is connected,
- $\hat{M}_j^i$ contains either no or all vertices of K_p for each $p \in \{1, \dots, k\}$.

If we have constructed these vertex sets, statement (ii) of this lemma holds by setting $M_j = \hat{M}_j^M$ for every $j \in \{1, \dots, k\}$. For $i = 0$, the construction of C_k ensures that $\hat{M}_j^0 = M_j^k$ is a valid choice for every $j \in \{1, \dots, k\}$.

Now assume we have already defined for every cycle $\hat{C}_i$ with $0 \leq i \leq N < M$ the corresponding vertex sets $\hat{M}_j^i$. Let P_N be the extension path of the path extension $\hat{C}_{N+1}$ of $\hat{C}_N$ with target v and base u. Moreover, let v and x be the endvertices of P_N. Then we set for every $j \in \{1, \dots, k\}$

$$\hat{M}_j^{N+1} = \begin{cases} \hat{M}_j^N \cup V(P_N) \cup \{u\} & \text{if } x \in \hat{M}_j^N \\ \hat{M}_j^N \setminus (V(P_N) \cup \{u\}) & \text{otherwise.} \end{cases}$$

We fix an arbitrary $j \in \{1, \dots, k\}$ and check all required properties. At first we show that $V(K_j) \subseteq \hat{M}_j^{N+1} \subseteq (V \setminus V(C)) \cup N_2(N(C))$ holds. Note that the target v lies in the set $V(K_0) \setminus V(C)$ and is a neighbour of the base $u \in V(K_0)$. So $V(P_N) \cup \{u\}$ does not contain a vertex of C with distance at least 3 to $N(C)$. Furthermore, we know that $V(P_N) \cup \{u\}$ is a subset of $V(K_0) \cup \mathscr{S}$. So the inclusions $V(K_j) \subseteq \hat{M}_j^N \subseteq (V \setminus V(C)) \cup N_2(N(C))$ imply that the same inclusions are also true for $\hat{M}_j^{N+1}$ instead of $\hat{M}_j^N$.

Next we check that the equation $|E(\hat{C}_{N+1}) \cap \delta(\hat{M}_j^{N+1})| = 2$ holds. By assumption, the cycle C and, therefore, also the cycle $\hat{C}_{N+1}$ have a vertex with distance at least 3 to $N(C)$. Furthermore, the cycle C_k contains vertices of K_j and so must $\hat{C}_{N+1}$. Using the inclusions $V(K_j) \subseteq \hat{M}_j^{N+1} \subseteq (V \setminus V(C)) \cup N_2(N(C))$, we get that $\hat{C}_{N+1}$ hits the cut $\delta(\hat{M}_j^{N+1})$ at least twice. The definition of the set $\hat{M}_j^{N+1}$ and the equation $|E(\hat{C}_N) \cap \delta(\hat{M}_j^N)| = 2$ force that $\hat{C}_{N+1}$ meets the cut $\delta(\hat{M}_j^{N+1})$ precisely twice.

To show that all vertices of $\hat{M}_j^{N+1}$ lie either on $\hat{C}_{N+1}$ or in $V \setminus N_4(K_0)$, note that the same statement with $N+1$ replaced by N holds. Now the statement follows by the definition of $\hat{M}_j^{N+1}$ and because $\hat{C}_{N+1}$ is a path extension of $\hat{C}_N$ with extension path P_N.

Now we prove that $G[\hat{M}_j^{N+1}]$ is connected. This is immediate in the case where $\hat{M}_j^{N+1} = \hat{M}_j^N \cup V(P_N) \cup \{u\}$ using that $G[\hat{M}_j^N]$ is connected. So we consider the case where $\hat{M}_j^{N+1} = \hat{M}_j^N \setminus (V(P_N) \cup \{u\})$. Since all vertices in $\hat{M}_j^{N+1}$ with distance at most 4 to K_0 lie on $\hat{C}_{N+1}$ and $|E(\hat{C}_{N+1}) \cap \delta(\hat{M}_j^{N+1})| = 2$, we get that that a path Q which is induced by the cycle $\hat{C}_{N+1}$ and lies entirely in $G[\hat{M}_j^{N+1}]$ contains all vertices of $\hat{M}_j^{N+1}$ with distance at most 4 to K_0. Now we use that the base u was chosen from $V(K_0)$. Hence, $V(P_N) \cup \{u\}$ is a subset of $V(K_0) \cup \mathscr{S}$. Using the inclusion $\mathscr{S} \subseteq N(K_0)$, we get that all vertices of the set $N(V(P_N) \cup \{u\})$ have distance at most 2 from K_0 and lie therefore on Q. Since $G[\hat{M}_j^N]$ is connected, every component of $G[\hat{M}_j^{N+1}]$ contains a vertex of $N(V(P_N) \cup \{u\})$. Now the path Q ensures that $G[\hat{M}_j^{N+1}]$ can have at most one component. So $G[\hat{M}_j^{N+1}]$ is connected too.

To see that the property of containing either no or all vertices of K_p for every $p \in \{1, \dots, k\}$ is true for $\hat{M}_j^{N+1}$, we use that $\hat{M}_j^N$ has this property. Since $V(P_N) \cup \{u\}$ is a subset of $V(K_0) \cup \mathscr{S}$ and contains therefore no vertices of any K_p where $p \geq 1$, we obtain that $\hat{M}_j^{N+1}$ has the desired property. This completes the construction of the path extension sequence $(\hat{C}_i)$ and so, as mentioned before, also the proof of statement (ii) of the lemma.

Finally, we have to show that statement (iii) of the lemma is true. Note that we have only lost edges of the cycle C by building path extensions. Edges of some cycle we lose by building a path extension of this cycle have always a neighbour on the extension path or are incident with the base of the path extension. Since the bases we have chosen have always a neighbour which does not lie on C and each extension path lies in the neighbourhood of the corresponding base, we obtain that all edges of $C - N_2(N(C))$ must also be edges of C'.

Note for the other part of statement (iii) that each edge $e = uv \in E(C') \setminus E(C)$ lies either on a path whose endvertices are in $\mathscr{S}$ and whose inner vertices lie in some of the trees T_j with $1 \leq j \leq k$ or lies on a path extension we have built during the construction of C'. In the first case, the inclusion $\{u, v\} \subseteq V \setminus V(C)$ is valid. So let us consider the second case. It is easy to see that for any cycle Z and any path extension Z' of Z with base b each edge $f = v_1v_2 \in E(Z') \setminus E(Z)$ satisfies $\{v_1, v_2\} \subseteq N_2(b)$. Either the edge f lies on the corresponding extension path which lies in the neighbourhood of b or the vertices v_1 and v_2 are the two neighbours of b or of some neighbour of b in Z. Now note that during the construction of C' we have always chosen the bases of the path extensions from the set $N(N(C)) \cup V \setminus V(C)$. So the inclusion $\{u, v\} \subseteq N_3(N(C)) \cup V \setminus V(C)$ holds, which proves the other part of statement (iii) and completes the proof of the lemma. $\square$

Now we are able to prove Theorem 1.2. As remarked earlier, we want to apply Lemma 3.11 to prove this theorem. For this purpose, we will use Lemma 4.2 to obtain a sequence of cycles and vertex sets such that all conditions for the use of Lemma 3.11 are fulfilled.

Proof of Theorem 1.2. Let $G = (V, E)$ be a locally finite, connected, locally connected, claw-free graph on at least three vertices. For the proof, we may assume further that V is infinite because Theorem 1.1 deals with the finite case.

We want to define a sequence $(C_i)_{i \in \mathbb{N}}$ of cycles of G where each cycle C_i has a vertex with distance at least 3 to $N(C_i)$. Furthermore, we define an integer sequence $(k_i)_{i \in \mathbb{N} \setminus \{0\}}$ and vertex sets $M^i_j \subseteq V(G)$ for every $i \in \mathbb{N} \setminus \{0\}$ and j with $1 \leq j \leq k_i$.

We start by taking a finite cycle $\tilde{C}$ in G. There exists one since G is 2-connected by Proposition 3.9. Now we build a path extension sequence of $\tilde{C}$ where we choose the targets always from $N_2(C)$ until a cycle in this sequence contains all vertices of $V(C) \cup N_2(C)$. This is possible by Lemma 4.1 and since G is locally finite, which implies that $V(C) \cup N_2(C)$ contains only finitely many vertices. We define C_0 to be the last cycle in the sequence.

Now assume that we have already defined the sequence of cycles up to the cycle C_m for some $m \geq 0$ together with the integer sequence up to k_m and the vertex sets M^i_j for every $i \leq m$ where j satisfies always $1 \leq j \leq k_i$. Then let $\mathscr{S}^{m+1} \subseteq N(C_m)$ be a finite minimal vertex set such that every ray starting in $V(C_m)$ must meet $\mathscr{S}^{m+1}$. Such a set exists because G is locally finite, which implies that $N(C_m)$ is finite. Hence, we could get $\mathscr{S}^{m+1}$ by sorting out vertices from $N(C_m)$. Next we set k_{m+1} as the integer we get from Lemma 3.10. Furthermore, let $S^{m+1}_1, \ldots, S^{m+1}_{k_{m+1}}$ be the minimal separators and $K^{m+1}_0, \ldots, K^{m+1}_{k_{m+1}}$ be the components of $G - \mathscr{S}^{m+1}$ which we get from Lemma 3.10. With these objects and the cycle C_m, we can apply Lemma 4.2 and obtain a new cycle which we set as C_{m+1} and vertex sets for every j, which we set as M^{m+1}_j where $1 \leq j \leq k_{m+1}$ holds.

In order to prove that G is Hamiltonian, we want to use Lemma 3.11. For this purpose, we show the following claim.

Claim 1.

(a) *For every vertex v of G, there exists an integer $j \geq 0$ such that $v \in V(C_i)$ holds for every $i \geq j$.*
(b) *For every $i \geq 1$ and j with $1 \leq j \leq k_i$, the cut $\delta(M^i_j)$ is finite.*
(c) *For every end ω of G, there is a function $f : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{N}$ such that the inclusion $M^j_{f(j)} \subseteq M^i_{f(i)}$ holds for all integers i, j with $1 \leq i \leq j$ and the equation $M_\omega := \bigcap_{i=1}^{\infty} \overline{M^i_{f(i)}} = \{\omega\}$ is true.*
(d) *$E(C_i) \cap E(C_j) \subseteq E(C_{j+1})$ holds for all integers i and j with $0 \leq i < j$.*

(e) *The equations* $E(C_i) \cap \delta(M_j^p) = E(C_p) \cap \delta(M_j^p)$ *and* $|E(C_i) \cap \delta(M_j^p)| = 2$ *hold for each triple* (i, p, j) *which satisfies* $1 \leq p \leq i$ *and* $1 \leq j \leq k_p$.

We begin the proof of this claim with statement (a). Here we use that the inclusions $V(K_0^i) \cup \mathscr{S}^i \cup N_3(\mathscr{S}^i) \subseteq V(C_i) \subseteq V(K_0^{i+1})$ hold for every $i \geq 1$ by construction and Lemma 4.2 (i). Since $N(K_0^i) = \mathscr{S}^i$ is true by definition and Lemma 3.10, statement (a) follows.

We fix an arbitrary integer $i \geq 1$ and some j which satisfies $1 \leq j \leq k_i$ to prove statement (b). By definition and Lemma 4.2 (ii), we know that M_j^i contains either all or no vertices of K_p^i for every p with $1 \leq p \leq k_i$. Using Lemma 3.10, we obtain that $N(M_j^i) \subseteq V(K_0^i) \cup \mathscr{S}^i \cup N(\mathscr{S}^i)$. Since K_0^i and $\mathscr{S}^i$ are finite by definition and Lemma 3.10 and G is locally finite by assumption, we obtain that $\delta(M_j^i)$ is finite.

Let us fix an arbitrary end ω of G for statement (c). We use that for every $i \geq 1$ the end ω lies in precisely one of the closures $\overline{K_1^i}, \ldots, \overline{K_{k_i}^i}$, say $\omega \in \overline{K_j^i}$ where $1 \leq j \leq k_i$, since K_0^i and $\mathscr{S}^i$ are finite by definition and Lemma 3.10. Then we set $f(i) = j$. Now we show that $M_{f(j)}^j \subseteq M_{f(i)}^i$ holds for all integers i, j with $1 \leq i \leq j$. Note that it suffices to prove the inclusion $M_{f(i+1)}^{i+1} \subseteq M_{f(i)}^i$ for every $i \geq 1$. The definition of $M_{f(i)}^i$ and Lemma 4.2 (ii) ensure that $G[M_{f(i)}^i]$ is connected and that the inclusions $V(K_{f(i)}^i) \subseteq M_{f(i)}^i \subseteq (V \setminus V(C_{i-1})) \cup N_2(\mathscr{S}^i)$ are valid for every $i \geq 1$. Note that the definition of f implies the inclusion $V(K_{f(i+1)}^{i+1}) \subseteq V(K_{f(i)}^i)$ for every $i \geq 1$. We need furthermore the observation that $M_{f(i+1)}^{i+1}$ does not contain a vertex of $\mathscr{S}^i$ for any $i \geq 1$. To see this, note that $V(K_0^i) \cup \mathscr{S}^i \cup N_3(\mathscr{S}^i) \subseteq V(C_i)$ holds for every $i \geq 1$ by definition of C_i together with Lemma 4.2 (i). So the distance between $\mathscr{S}^i$ and $\mathscr{S}^{i+1}$ is at least 4 for every $i \geq 1$. Now the inclusion $M_{f(i+1)}^{i+1} \subseteq (V \setminus V(C_i)) \cup N_2(\mathscr{S}^{i+1})$ implies that $M_{f(i+1)}^{i+1}$ cannot contain a vertex of $\mathscr{S}^i$ for any $i \geq 1$. Since we know for every $i \geq 1$ that $G[M_{f(i+1)}^{i+1}]$ is connected, $V(K_{f(i+1)}^{i+1})$ is a subset of $M_{f(i+1)}^{i+1}$ and $\mathscr{S}^i$ separates $V \setminus (V(K_{f(i)}^i) \cup \mathscr{S}^i)$ from $V(K_{f(i)}^i)$ by definition and Lemma 3.10, we obtain that $M_{f(i+1)}^{i+1} \subseteq V(K_{f(i)}^i) \subseteq M_{f(i)}^i$ holds for every $i \geq 1$.

It remains to prove the equation $M_\omega = \{\omega\}$. As noted above, the inclusions $V(K_{f(i)}^i) \subseteq M_{f(i)}^i \subseteq (V \setminus V(C_{i-1})) \cup N_2(\mathscr{S}^i)$ are true for every $i \geq 1$. So the definition of f ensures that ω is an element of M_ω. Next we show that M_ω does not contain a vertex of G or any other end of G than ω. So let $v \in V(G)$ and $\omega' \neq \omega$ be an end of G. This means we can find a finite set of vertices $F \subseteq V(G)$ such that ω and ω' lie in closures of different components of $G - F$. Let $q \geq 1$ be an integer such that all vertices of $F \cup \{v\}$ are contained in K_0^q. We can find such an integer because each vertex $w \in F \cup \{v\}$ lies in some cycle C_{ℓ_w} with $\ell_w \geq 0$ by statement (a). Using that $V(C_i) \subseteq V(C_{i+1})$ is true for every $i \geq 0$ by construction and Lemma 4.2 (i) together with the fact that the set $F \cup \{v\}$ is finite, we can set $q - 1$ as the maximum of all integers ℓ_w. By definition of K_0^q and Lemma 3.10, we obtain that the inclusion $F \cup \{v\} \subseteq V(K_0^q)$ holds. This implies that ω and ω' lie also in closures of different components of $G - V(K_0^q)$. By construction and Lemma 4.2 (ii), we know that the graph $G[M_{f(q+1)}^{q+1}]$ is connected. Furthermore, we get that the inclusions $M_{f(q+1)}^{q+1} \subseteq (V \setminus V(C_q)) \cup N_2(\mathscr{S}^{q+1})$ and $V(K_0^q) \cup \mathscr{S}^q \cup N_3(\mathscr{S}^q) \subseteq V(C_q)$ are true. Since the distance from K_0^q to $\mathscr{S}^{q+1}$ is at least 5 by construction, the set $M_{f(q+1)}^{q+1}$ cannot contain vertices of K_0^q. So v is no element of $M_{f(q+1)}^{q+1}$. Now the connectedness of $G[M_{f(q+1)}^{q+1}]$ implies that $G[M_{f(q+1)}^{q+1}]$ is a subgraph of the component of $G - V(K_0^q)$ whose closure contains ω. As ω and ω' lie in closures of different components of $G - V(K_0^q)$, the end ω' does not lie in the closure of $M_{f(q+1)}^{q+1}$ and we obtain that v and ω' are no elements of M_ω. Since each $M^i_{f(i)}$ is a vertex set, the intersection M_ω cannot contain inner points of edges. Therefore, the equation $M_\omega = \{\omega\}$ is valid, which shows statement (c).

To prove statement (d), take an edge $e \in E(C_i) \cap E(C_j)$ for arbitrary integers i and j that satisfy $0 \leq i < j$. So both endvertices of e lie in $V(C_i) \subseteq V(K_0^{i+1})$. Additionally, the inclusions $V(K_0^{i+1}) \cup \mathscr{S}^{i+1} \cup N_3(\mathscr{S}^{i+1}) \subseteq V(C_{i+1}) \subseteq V(C_j)$ are true by definition of the cycles and Lemma 4.2 (ii). Using the equality $N(K_0^{i+1}) = \mathscr{S}^{i+1}$, we conclude that $e \in E(C_j - N_2(N(C_j)))$ holds. So we get by definition of the cycles and Lemma 4.2 (iii) that e lies in $E(C_{j+1})$. This completes the proof of statement (d).

Let us fix an arbitrary $p \geq 1$ and j with $1 \leq j \leq k_p$ for statement (e). We know that $|E(C_p) \cap \delta(M_j^p)| = 2$ holds by definition of the cycles and Lemma 4.2 (ii). So it suffices to prove that $E(C_p) \cap \delta(M_j^p) = E(C_i) \cap \delta(M_j^p)$ holds for every $i \geq p$. Next let us consider an arbitrary edge $e = uv \in \delta(M_j^p)$. We prove now that u and v are contained in $V(K_0^p) \cup \mathscr{S}^p \cup N(\mathscr{S}^p)$ where at most one of these two vertices lies in $N(\mathscr{S}^p) \setminus V(K_0^p)$. If one of the endvertices of e lies in $N(\mathscr{S}^p) \setminus V(K_0^p)$, it must be contained in $V(K_{j'}^p) \cap M_j^p$ for some j' which satisfies $1 \leq j' \leq k_p$. Then the other endvertex of e lies in $N(K_{j'}^p) \subseteq \mathscr{S}^p$ because the set M_j^p must contain all vertices of $K_{j'}^p$ by definition and Lemma 4.2 (ii). So the inclusion $\{u, v\} \subseteq \mathscr{S}^p \cup N(\mathscr{S}^p)$ is true. Otherwise, neither u nor v is a vertex of $N(\mathscr{S}^p) \setminus V(K_0^p)$. As precisely one endvertex of e lies in $V \setminus M_j^p$ and M_j^p contains either no or all vertices of K_q^p for each q with $1 \leq q \leq k_p$ by definition and Lemma 4.2 (ii), we get that neither u nor v lies in any K_q^p with $1 \leq q \leq k_p$. So both vertices must lie in $V(K_0^p) \cup \mathscr{S}^p$. Now we prove by induction on i that every edge $e' = u'v' \in E(C_p) \cap \delta(M_j^p)$ is an edge of C_i for every $i \geq p$. For $i = p$, this is obvious. So assume that e' is an edge of C_i for some $i \geq p$. Note that $V(K_0^p) \cup \mathscr{S}^p \cup N_3(\mathscr{S}^p) \subseteq V(C_i)$ is true for every $i \geq p$ by definition and Lemma 4.2 (i). Therefore, the edge e' lies in $E(C_i - N_2(N(C_i)))$, which means that e' is also an edge of C_{i+1} by definition of the cycles and Lemma 4.2 (iii). This completes the induction and shows that e' is an edge of C_i for every $i \geq p$. So the inclusion $E(C_p) \cap \delta(M_j^p) \subseteq E(C_i) \cap \delta(M_j^p)$ is true. We complete the proof of statement (e) by showing by induction on i that for every $i \geq p$ the cycle C_i contains no edges of $\delta(M_j^p)$ but the two which are also edges of C_p. For $i = p$, this is obvious by definition of C_p and Lemma 4.2 (ii). So let $i > p$ and $e = uv \in \delta(M_j^p) \setminus E(C_p)$ be fixed for this purpose. Using the induction hypothesis, we know that $e \notin E(C_{i-1})$. Now suppose for a contradiction that e is an edge of C_i. Then the definition of C_i together with Lemma 4.2 (iii) implies that $\{u, v\} \subseteq (V \setminus V(C_{i-1})) \cup N_3(N(C_{i-1}))$ holds. This leads towards a contradiction because we already know that the inclusion $\{u, v\} \subseteq V(K_0^p) \cup \mathscr{S}^p \cup N(\mathscr{S}^p)$ is valid where $\{u, v\}$ is no subset of $N(\mathscr{S}^p) \setminus V(K_0^p)$. That cannot be true since C_{i-1} contains all vertices of $V(K_0^p) \cup \mathscr{S}^p \cup N_3(\mathscr{S}^p)$ by definition of the cycle and Lemma 4.2 (i). This completes the induction and we get that $E(C_p) \cap \delta(M_j^p) = E(C_i) \cap \delta(M_j^p)$ holds for every $i \geq p$. So the proof of statement (e) is done and the claim is proved.

Using the sequence of cycles $(C_i)_{i \in \mathbb{N}}$, the integer sequence $(k_i)_{i \in \mathbb{N} \setminus \{0\}}$ and the vertex sets M_j^i for every $i \in \mathbb{N} \setminus \{0\}$ and j with $1 \leq j \leq k_i$, we can apply Lemma 3.11 because of Claim 1. So we get that G is Hamiltonian. $\square$

Next we state a couple of corollaries of Theorem 1.2. Their analogous for finite graphs are all known and the proofs for the finite versions are the same as for the locally finite versions. The finite version of the following corollary is due to Matthews and Sumner [17, Cor. 1].

Corollary 4.3. *Let G be a locally finite connected graph with at least three vertices. If G^2 is claw-free, then G^2 is Hamiltonian.*

The following three corollaries deal with line graphs. It is well known that this class of graphs forms a subclass of all claw-free graphs. The finite versions of Corollary 4.4 and Corollary 4.5 are due to Oberly and Sumner (see [19, Cor. 1 and Cor. 3]).

Corollary 4.4. *Let G be a locally finite connected graph with at least three edges. If its line graph $L(G)$ is locally connected, then $L(G)$ is Hamiltonian.*

The proof of the finite version of the next corollary in [19, Cor. 3] shows that for a graph, the property of being locally connected is preserved under taking the line graph.

Corollary 4.5. *For every locally finite, connected, locally connected graph with at least three vertices, its line graph is Hamiltonian.*

The finite analogue of the next corollary has appeared in a paper of Nebesky (see [18, Thm. 1]). In [19, Cor. 5] a proof of the finite version of the next corollary can be found using Theorem 1.1, the finite version of Theorem 1.2.

Corollary 4.6. *Let G be a locally finite connected graph with at least three vertices. Then $L(G^2)$ is Hamiltonian.*

The finite version of the next corollary is due to Chartrand and Wall (see [6]). A proof of the finite result using Theorem 1.1 can be found in [19, Cor. 4].

Corollary 4.7. *Let G be a locally finite connected graph with $\delta(G) \geq 3$. Then $L(L(G))$ is Hamiltonian.*

The following corollary involves another class of graphs. We call a graph *chordal* if it has no induced cycle with more than three vertices. Balakrishnan and Paulraja [1, Thm. 5] proved the finite analogue of the following corollary. They showed first that a graph which is 2-connected and chordal has also the property of being locally connected. Then they applied Theorem 1.1.

Corollary 4.8. *Every locally finite, 2-connected, chordal, claw-free graph is Hamiltonian.*

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Karl Heuer, Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany