Transformation Double Categories Associated to 2-Group Actions

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Abstract

Transformation groupoids associated to group actions capture the interplay between global and local symmetries of structures described in set-theoretic terms. This paper examines the analogous situation for structures described in category-theoretic terms, where symmetry is expressed as the action of a 2-group $\mathcal{G}$ (equivalently, a categorical group) on a category $\mathbf{C}$. It describes the construction of a transformation groupoid in diagrammatic terms, and considers this construction internal to **Cat**, the category of categories. The result is a double category $\mathbf{C}/\!\!/\mathcal{G}$ which describes the local symmetries of $\mathbf{C}$. We define this and describe some of its structure, with the adjoint action of $\mathcal{G}$ on itself as a guiding example.

1 Introduction

Symmetry is a fundamental notion both in mathematics and in physics. The usual mathematical representation of symmetry is as a *group action* on a set, or perhaps on a manifold, vector space, or whatever sort of structure is of interest. Both the groups and the objects on which they act may have various sorts of structure - one may be interested in algebraic groups, or Lie groups, for example. In particular, symmetry in this sense is described by the action of a *group object* in an appropriate category: an object equipped with the structure maps, such as the multiplication map $m : G \times G \to G$, satisfying the axioms which define a group.

Even superficially very different constructions, such as quantum groups, fit this pattern to some extent. This is a rather broad concept with many variations, but the starting point is the idea of a Hopf algebra, a bialgebra with a certain structure. Quantum groups play a role in describing symmetry in noncommutative geometry and other algebraic settings. Hopf algebras may be seen as group objects in $\mathbf{Alg}^{op}$, the opposite category of some category of algebras. (Which category of algebras depends on the context within noncommutative geometry).

The notion of a group action, then, describes global symmetries: namely, global operations on the object which supports the symmetry, which leave it in some sense unchanged. The exact sense in which it is to be unchanged, and what a transformation of the object can consist of, is the guiding criterion in finding its group of symmetries. On the other hand, there is also a local concept of symmetry. These situations are more naturally described in terms of *groupoids*. Groupoids are categories in which all morphisms are invertible. Here, the entire set which supports the symmetry forms the objects of the groupoid. The individual elements, such as the points of a space, are the objects, and morphisms represent symmetry relations which relate one part to another, which are "indistinguishable" under the relation in question. As Weinstein notes [23], this makes sense even in cases where the symmetry relation between the two parts does not extend to the entire set. The groupoid approach to symmetry is therefore more fine-grained.

However, the situation of a global symmetry group also corresponds to a groupoid picture, by means of "transformation groupoids", which give a groupoid associated to a group action on a set. This construction makes sense not only the category of sets, but also in other settings. Typical examples would include a Lie group acting on a manifold, a topological group acting on a topological space, or an algebraic group acting on a variety. In reasonable situations - which always apply in a topos, such as the category of sets, or of topological spaces, one can construct a groupoid which captures the local picture implied by such global symmetry groups. These will be, respectively, Lie groupoids, topological groupoids, or algebraic groupoids in the cases mentioned above.

We can say that each of these situations is an instance of a general process, which can be described diagrammatically, independent of what sort of structure plays the role of "sets". That is, there is a general construction of a transformation groupoid, which makes sense in a certain kind of category. Each of the cases we mentioned are examples of this construction "internal to" a chosen ambient category, such as topological spaces, manifolds, varieties, and so forth.

The situation we describe in this paper takes these ideas in a slightly different direction. We are interested in describing symmetry of categories. Now, in particular, there are not only morphisms between categories (namely, functors), but also morphisms between these morphisms (namely, natural transformations). This means that the notion of "symmetry" of a category is somewhat more subtle than for a set, or for any set with an additional structure, such as a topological space. Global symmetries of a category are most naturally expressed in terms of a 2-group.

Now, just as a group may be regarded as a special sort of category, so too a 2-group can be seen as a special kind of 2-category. In particular, given a category with one object, and with the property that all morphisms are invertible, the morphisms form a group, whose product is the composition of morphisms. This is a useful way to look at groups in the context of global symmetry. Given an object in a category, $x \in \mathbf{X}$, there is a sub-category with one object $End(x)$, consisting of all the endomorphisms of x. Restricting to the invertible such morphisms, one gets $Aut(x)$. This is a subcategory of $\mathbf{X}$, but it is also a group in the sense just mentioned. It is, indeed, the full symmetry group of the object x.

Similarly, given a category $\mathbf{X} \in \mathbf{Cat}$, one has endofunctors from $\mathbf{X}$ to itself, but also natural transformations between such endofunctors. So $End(\mathbf{X})$ is a category whose objects are endofunctors and whose morphisms are natural transformations. What is more, $End(\mathbf{X})$ is a monoidal category, since endofunctors of $\mathbf{X}$ can always be composed with each other. If we restrict to only those endofunctors which are invertible, we have $Aut(\mathbf{X})$, which is a (strict) 2-group. It is, indeed, the full symmetry 2-group of $\mathbf{X}$.

Then, just as an action of a group G on $x \in \mathbf{X}$ is a homomorphism (that is, a functor) from G to $Aut(x)$, so is an action of a 2-group $\mathcal{G}$ on $\mathbf{X} \in \mathbf{Cat}$ a homomorphism (that is, a 2-functor) from $\mathcal{G}$ to $Aut(\mathbf{X})$. So 2-groups, seen as 2-categories with one object, give a useful framework for describing global 2-group actions.

The essential point in this paper is that 2-group actions, too, can be understood in terms of an internal construction. The ambient category is $\mathbf{Cat}$. Then we can use the fact that an alternative way to see 2-groups is as categorical groups, namely group objects in $\mathbf{Cat}$. The fact that there are different ways of thinking of 2-groups means that there are several ways of thinking of 2-group symmetries. We shall see that the internal view of 2-groups as categorical groups turns out to be a useful framework for describing the *local* symmetry of categories. In this paper, we will describe what comes out of the construction of the transformation groupoid in $\mathbf{Cat}$, which gives the local picture associated to a 2-group action on a category.

It is perhaps worth remarking that the existence of both internal and external views of symmetry for a category relies on the fact that $\mathbf{Cat}$ is a *closed* category. That is, a category $\mathbf{X}$ with an "internal Hom, so that the morphisms from A to B form an object $\mathrm{Hom}(A, B) \in \mathbf{X}$. In our examples above, sets and topological spaces are closed categories, and are the most natural situations where there is an interplay between local and global symmetry. In other ambient categories, such as manifolds, one must be more careful, and the situation is not quite as nice. With **Cat**, however, we have a straightforward example of a closed category, so that $Aut(\mathbf{X})$ can indeed just be seen as a category, equipped with certain structure maps which make it a 2-group.

Now, the internal, local symmetry picture mentioned above gives rise to a transformation groupoid in the same ambient category as our group object, and the object it acts on. So for categories, we find that this local symmetry structure is most naturally encoded by a category (indeed, it will be a groupoid) internal to **Cat** itself. Such a structure is called **Cat**-category, or often a *double category* (though to avoid ambiguity, we will reserve this name for a distinct but equivalent structure).

In particular, this structure is not generally just a 2-category. An external view of it shows that it has two different kinds of morphism, denoted horizontal and vertical, and higher morphisms which naturally have the shape of squares. The shape of higher morphisms is a typical issue in higher-categorical constructions: see e.g. [6, 12] for discussion of variants of higher categories. What our discussion here demonstrates is that these variations in shape become relevant when we try to extend the concept of symmetry to higher structures.

Our motivation here comes from geometry, although this will play no direct role in this paper. We will, however, describe a little of this motivation here, and in a subsequent paper we will give more details about the example which prompted us to study this question.

The example we had in mind comes from *higher gauge theory* [3]. This is a general program of developing analogs to concepts in gauge theory, for higher-categorical groups. It provides generalizations of, among other things, the theory of connections on bundles over manifolds. These analogs include the study of connections on (non-abelian) gerbes [9, 17, 14, 22]. These entities have been the study of considerable interest in recent years, particularly since they can be used to describe theories in which one can naturally describe the parallel transport of extended objects such as (one-dimensional) strings or (higher-dimensional) "membranes".

Behind the current paper is a desire to better understand the symmetries of the "moduli spaces" for such theories. That is, in ordinary gauge theory, given a manifold M and a Lie group G, there is a space of all connections on principal G-bundles over M. Furthermore, there is a group which acts on this space, the group of all gauge transformations. There is a local picture of this symmetry, in which both connections and gauge transformations are described in terms of certain forms and functions on M. The relation between the global symmetries and this local picture is well-understood. The local picture describes a groupoid whose objects are the connections, which comes from the global group action.

In the case of connections on gerbes, there is also a well-understood local picture in terms of forms and functions. However, these naturally form something more complex than a groupoid. The relationship between this local picture and a notion of global symmetries is less well-established. In our subsequent paper, we will show a simple example of how it arises from a global symmetry 2-group acting on a category which plays the role of the moduli space of connections on gerbes over M. For even higher-categorical structures, denoted n-gerbes, one can describe the transport of higher-dimensional surfaces, or n-branes. This naturally has connections to string theory [21, 2], and to other areas in theoretical physics.

Our most direct motivation relates to work of Yetter [24], and Martins and Porter [15], extending a certain field theory (the Dijkgraaf-Witten model) to 2-groups. One of the ultimate goals in our project is to reproduce for these theories a groupoid-based approach to such a model [16], using the double groupoids we find here, relating these field theories to the representations of the double groupoids.

While this motivation is of interest to us, it is not necessary to understand this paper. For the moment, we are simply interested in understanding better how the relationship between local and global symmetries applies when the entity whose symmetries are involved happens to be a category. Since the notion of symmetry is one of the most fundamental, and categories have proven to be a vital and important part of modern mathematics, this should be of interest well beyond the context of our original example.

We begin in Section 2 with a discussion of 2-groups, and various equivalent definitions which are useful in talking about them. This includes a correspondence with the 2-category of crossed modules and the definition of the double groupoid associated to the 2-group $\mathcal{G}$, which is both an introduction to the notion of double category and a useful 2-dimensional diagrammatic tool for 2-group calculations. This is followed by Section 3 which describes what is meant by the action of a 2-group on a category, and gives as a basic example the adjoint action of a 2-group on itself, analogous to the adjoint action of a group on itself by conjugation. (In the forthcoming paper on higher gauge theory, we show that this is a special case of our geometric example, in the simple situation of the manifold S^1, the circle.)

In Section 4, we describe the construction of the transformation groupoid associated to a group action on a set, in a diagrammatic form, which captures the description of this groupoid in terms of elements. Repeating this construction in **Cat** gives a groupoid internal to **Cat**. We develop a few different points of view on this internal groupoid in Section 4.2, first taking a transposed internal view (as a category internal to **Gpd**, the category of groupoids), and then giving a symmetrical definition as a double category. The transposed viewpoint turns out to have close connections to ordinary transformation groupoids for certain group actions. Further aspects of these viewpoints are developed section 4.3. Our standard example, that of the adjoint action of a 2-group on itself, gives a concrete case throughout, which we develop in detail in 4.4.

Finally, in the appendix, we discuss the case of weak 2-groups and weak actions, and how the preceding analysis would be different in these cases. This is important to develop a general theory of actions of n-groups, but we argue, based on "strictification" results which are special to the case $n = 2$, that the preceding constructions are enough in this case.

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2 2-Groups and 2-Group Actions on a Category

In this paper, we will be describing certain actions of 2-groups. The structures known as 2-groups are also sometimes called categorical groups, **gr**-categories or groupal groupoids. Moreover, all of these can be shown to be equivalent to crossed modules (of groups). As the abundance of terminology suggests, there are several conceptually different, but logically equivalent, definitions which can be given for these structures. We will primarily use the term "2-group", but in fact it will be useful for our discussion to be able to move back and forth between the different definitions. In Section 2.1 we will recall those definitions which will be used, and note how they are equivalent.

This prepares the ground for section 3, where we will consider from two of these points of view how 2-groups can act on categories, and the "transformation" structure which results. This will be the analog of the transformation groupoid associated to a group action on a set.

Note that in the following, to avoid cumbersome notation, we use the notation $\mathbf{X}^{(0)}$ and $\mathbf{X}^{(1)}$ to denote, respectively, the objects and morphisms of a category $\mathbf{X}$, and similar notation for the object and morphism maps of a functor. For a 2-category $\mathbf{X}$, when appropriate, we denote the 2-morphisms by $\mathbf{X}^{(2)}$.

2.1 2-Groups, Categorical Groups, and Crossed Modules

Here we lay out three definitions of equivalent structures: 2-groups, categorical groups, and crossed modules. In this section, we will be careful to distinguish the three, but in the rest of the paper we will generally use the term "2-group". The main point of this section is to highlight the well-known result that there is an equivalence between the three definitions. That is, these definitions specify the objects of three equivalent 2-categories.

2.1.1 2-Groups as 2-Categories

One very useful definition of a 2-group is motivated by analogy to the definition of a group as a (small) one-object category whose morphisms are all invertible. In this case, the elements of the group are the set of morphisms of the category, and the group multiplication is the composition.

This highlights the way the definition of a category generalized the notion of "composition" which begins with algebraic gadgets such as groups. One might equally well define a (small) category in algebraic language as a unital semigroupoid. Taking the definition of category as more basic gives a group as a special case. While of course not the standard definition of a group, it is at least straightforward to generalize to "higher" groups in the sense of higher category theory.

(We will assume some familiarity with 2-categories in what follows, though for readers who are not so familiar we suggest the succinct note by Leinster [11] as a starting point, and the more comprehensive survey by Lack [10] and Chapter 7 of Borceux [4] for more detail.)

Definition 2.1.1 *A* ***2-group*** *$\mathcal{G}$ is a 2-category with one object ($\mathcal{G}^{(0)} = \{\star\}$), for which all 1-morphisms $\gamma \in \mathcal{G}^{(1)}$ and 2-morphisms $\chi \in \mathcal{G}^{(2)}$ are invertible.*

Note that in the case of $\chi \in \mathcal{G}^{(2)}$, we requrie invertibility with respect to both horizontal and vertical composition.

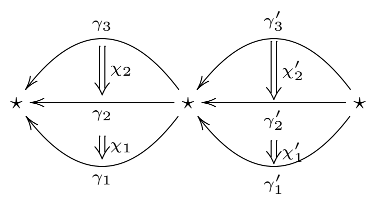
Let us unpack this definition more explicitly.

Suppose $\mathcal{G}$ is a 2-group. There are various 1-morphisms from $\star$ to itself, which have a composition operation, since all such morphisms have matching source and target:

$$\star \xleftarrow[\gamma_1]{} \star \xleftarrow[\gamma_2]{} \star \quad = \quad \star \xleftarrow[\gamma_1 \circ \gamma_2]{} \star \tag{1}$$

Composition of morphisms is the multiplication, which is therefore associative, and every morphism must have an inverse by the definition of a 2-group, so $(\mathcal{G}^{(1)}, \circ)$ forms a group.

Moreover, there are 2-morphisms between the 1-morphisms. They have both a horizontal and a vertical composition. The horizontal composition we denote $\circ$, since this is the same direction as the composition of 1-morphisms. The vertical composition we denote $\cdot$. The two compositions must be compatible, in the sense that the following composite is well-defined:



(2)

This is expressed as the "interchange law"

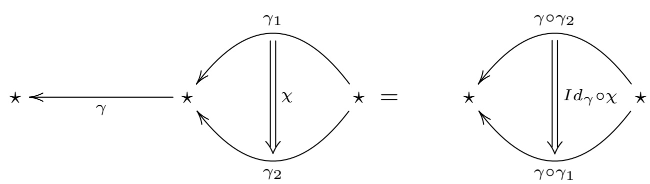
$$(\chi_1 \cdot \chi_2) \circ (\chi_1' \cdot \chi_2') = (\chi_1 \circ \chi_1') \cdot (\chi_2 \circ \chi_2') \tag{3}$$

Again, these 2-morphisms are assumed to be invertible, so it follows that $(\mathcal{G}^{(2)}, \circ)$ forms a group. Note that we do not say that $(\mathcal{G}^{(2)}, \cdot)$ forms a group, since $\cdot$ may not be defined for all pairs of 2-morphisms, unless they have compatible source and target 1-morphisms.

However, one can find a group structure using vertical composition of 2-morphisms, upon choosing a given 1-morphism γ, and considering $Hom(\gamma,\gamma)$, the collection of 2-morphisms from γ to itself. This is a group with operation $\cdot$, since 2-morphisms are invertible under $\cdot$.

(Note that $Hom(\gamma,\gamma)$ does not necessarily close under $\circ$, unless $\gamma = 1_G$. In that case, $Hom(1_G, 1_G)$ is an abelian group by the so-called Eckmann-Hilton argument, which uses the interchange property (3) to show that horizontal and vertical composition agree, and define an abelian group.)

We have noted the composition operations for both types of morphisms. It is also significant that there is also a straightforward interaction between the two types, namely whiskering. That is, a 1-morphism can act on a 2-morphism, either on the left or the right, where it acts by composition with an identity 2-cell. That is, by convention we say:



(4)

Again, all these remarks are only unpacking what is implied by the definition of a 2-category with one object and invertible morphisms. We will want to use this definition occasionally, particularly when describing 2-group actions. However, for the most part is will be useful to think of 2-groups in a way which is more amenable to explicit calculations. The first step toward this is the definition of a categorical group, and then even more explicit is the presentation in terms of a crossed module. We recall these next.

2.1.2 Categorical Groups

Another natural approach to defining a "higher" analog of a group is based on the view that a group is a "group object in **Set**". A group object in any monoidal category $(\mathbf{C}, \otimes)$ is an object $G \in \mathbf{C}$, equipped with structure maps such as $\cdot : G \otimes G \to G$ satisfying the same axioms as those for a group. These axioms can easily be expressed as diagrams which make sense in any monoidal category, and specialize to the usual axioms in $(\mathbf{Set}, \times)$.

Definition 2.1.2 *A* ***categorical group*** *is a (strict) group object in the monoidal category* $(\mathbf{Cat}, \times)$.

That is, categorical groups are (small) categories $\mathbf{G}$, equipped with functors $\otimes : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, and $inv : \mathbf{G} \to \mathbf{G}$ satisfying the group axioms.

Since **Cat** is, in fact, a monoidal 2-category (in which the monoidal product is the cartesian product of categories), our definition specifies that these satisfy the axioms for a group *strictly*. That is, the usual equations such as associativity of multiplication are still equations, rather than 2-isomorphisms. The corresponding weak notion, in which equations are replaced by such 2-isomorphisms, necessarily has additional coherence conditions they must satisfy.

Notice that the "group multiplication" functor, which we have written as $\otimes$, satisfies the properties for a monoidal product, as the notation suggests. Indeed, a (strict) monoidal category is simply a monoid object in **Cat**. A (strict) categorical group is therefore a strict monoidal category with inverses. That is, every object and morphism has an inverse with respect to $\otimes$.

It is standard, and easy to see, that any 2-group as defined in Section 2.1.1 gives rise to a categorical group, and vice versa. We describe the correspondence here to settle notation.

Definition 2.1.3 *Given a 2-group $\mathcal{G}$, the categorical group $(\mathbf{C}(\mathcal{G}), \otimes)$ associated to $\mathcal{G}$ is defined as follows:*

- ***Objects****:* $\mathbf{C}(\mathcal{G})^{(0)} = \mathcal{G}^{(1)}$ *(objects are morphisms of $\mathcal{G}$)*

- ***Morphisms:*** $\mathbf{C}(\mathcal{G})^{(1)} = \mathcal{G}^{(2)}$ *(morphisms are 2-morphisms of* $\mathcal{G}$*)*
- *The composition of morphisms of* $\mathbf{C}(\mathcal{G})$ *is vertical composition of 2-morphisms in* $\mathcal{G}$
- *The monoidal product* $\otimes$ *of* $\mathbf{C}(\mathcal{G})$ *is:*
  - *On* $\mathbf{C}(\mathcal{G})^{(0)}$*, same as composition in* $\mathcal{G}^{(1)}$
  - *On* $\mathbf{C}(\mathcal{G})^{(1)}$*, same as horizontal composition in* $\mathcal{G}^{(2)}$
- *The inverse functor inv :* $\mathbf{C}(\mathcal{G}) \to \mathbf{C}(\mathcal{G})$ *is given by, for each object or morphisms, the inverse of the corresponding 1- or 2- morphism in* $\mathcal{G}$

Given a categorical group $(\mathbf{G}, \otimes)$*, the 2-group* $\star/\!\!/\mathbf{G}$ *has:*

- ***Object:*** *just one,* $\star$
- ***Morphisms:*** $(\star/\!\!/\mathbf{G})^{(1)} = \mathbf{G}^{(0)}$*, the objects of* $\mathbf{G}$
- ***2-Morphisms:*** $(\star/\!\!/\mathbf{G})^{(2)} = \mathcal{G}^{(1)}$*, the morphisms of* $\mathbf{G}$
- *Composition for* $(\star/\!\!/\mathbf{G})^{(1)}$ *and horizontal composition for* $(\star/\!\!/\mathbf{G})^{(2)}$ *are* $\otimes^{(0)}$ *and* $\otimes^{(1)}$ *from* $\mathcal{G}$ *respectively*
- *Vertical composition for* $(\star/\!\!/\mathbf{G})^{(2)}$ *is composition for* $\mathbf{G}^{(1)}$

It is well known, and the unfamiliar reader may easily check, that these two correspondences give an equivalence of the two definitions. For example, the property of invertibility for $\star/\!\!/\mathbf{G}$ follows from the existence of *inv* in $(\mathbf{G}, \otimes)$, and the fact that it satisfies group axioms. It is similarly easy to check that monoidal functors between categorical groups correspond to 2-functors between 2-groups. For example, the interchange law for 2-groups is expressed, in categorical groups, as the fact that the monoidal product is functorial.

(The convention for whiskering in a 2-group is easily translated to the usual convention for the monoidal product of an object with a morphism.)

Another equivalent presentation is a result of the fact that there is a correspondence between groups internal to categories, and categories internal to groups. That is, categories whose sets of objects and sets of morphisms each have the structure of a group, and where source, target, and composition operations are all group homomorphisms. This equivalence follows naturally, since given a categorical group, the structure maps, such as $\otimes$, are functors which satisfy the group axioms. The object and morphism maps for these functors therefore each separately satisfy the same axioms. Thus, the objects of a categorical group form a group, as do the morphisms of the categorical group. Thinking of these separate group structures naturally leads to the third of the equivalent definitions we will use in this paper, namely crossed modules.

2.1.3 Crossed Modules

A well-known theorem [5] says that the 2-category of 2-groups is equivalent to the 2-category of **crossed modules**.

Definition 2.1.4 *A crossed module consists of* $(G, H, \rhd, \partial)$*, where* G *and* H *are groups,* $G \rhd H$ *is an action of* G *on* H *by automorphisms and* $\partial : H \to G$ *is a homomorphism, satisfying the equations:*

$$\partial(g \rhd \eta) = g\partial(\eta)g^{-1} \tag{5}$$

and

$$\partial(\eta) \rhd \zeta = \eta\zeta\eta^{-1} \tag{6}$$

We describe the correspondence with categorical groups, which will be the form we make the most use of, though of course the correspondence with 2-groups follows immediately as well:

Definition 2.1.5 *The categorical group* $\mathbf{G}$ *given by* $(G, H, \rhd, \partial)$ *has:*

- ***Objects:*** $\mathbf{G}^{(0)} = G$

- ***Morphisms:*** $\mathbf{G}^{(1)} = G \times H$*, with source and target maps*

$$s(g, \eta) = g \tag{7}$$

and

$$t(g, \eta) = \partial(\eta) g \tag{8}$$

- ***Identities:***

$$Id_g = (g, 1_H) \tag{9}$$

- ***Composition:***

$$(\partial(\eta)g, \zeta) \circ (g, \eta) = (g, \zeta\eta). \tag{10}$$

That is, as a group, $\mathbf{G}^{(1)} \cong G \ltimes H$*, with multiplication given by:*

$$(g_1, \eta) \otimes (g_2, \zeta) = (g_1 g_2, \eta(g_1 \rhd \zeta)) \tag{11}$$

which corresponds to the horizontal composition of 2-morphisms in $\mathcal{G}$*.*

The theorem alluded to above asserts that any strict 2-group is equivalent to one of this form. We will usually assume that any 2-group we use from now on is presented by a crossed module, and use the corresponding notation for explicit calculations.

The structure of a category which lies behind the definition of a crossed module is somewhat disguised. Much of it is included in the fact that G and H are groups. The explicit action of G on H captures the notion of whiskering, remarked upon above, and together with the "boundary" map ∂ determines the source and target structure maps for the categorical group. However, for example, the vertical composition of 2-morphisms is not explicitly mentioned. As we remarked above, this composition is closely related to the horizontal composition of 2-morphisms via the Eckmann-Hilton argument. We will not, however, give the full details here of how it and other categorical group structures can be reconstructed from a crossed module, as these are well recorded elsewhere.

2.2 Double Groupoid of a 2-Group

A useful construction for calculations later on is the double groupoid of $\mathcal{G}$.

Double categories were introduced by Ehresmann [8, 7], and can be defined in several equivalent ways, three of which are noted by Brown and Spencer [5]. The most pertinent later in this paper is that they are categories *internal* to **Cat**, the category of all categories. This terse definition is the most common in current use, but is somewhat opaque, and obscures the basically symmetric nature of these structures.

One can also describe a (small) double category in a more manifestly symmetric way. Although the two definitions are equivalent, they are superficially rather different. In Section 4.2 we will use both points of view to describe the structure which it is the goal of this paper to construct. Therefore, to avoid confusion, we will reserve "double category" for the symmetric view, and use "**Cat**-category" to mean the equivalent structure seen as a category in **Cat**.

We will not give a full definition of "double category" here, though, since it is equivalent to and can be deduced from the terse definition, but it is intuitively useful to see a double category $\mathcal{D}$ as consisting of:

- a set of objects O
- two sets H and V of morphisms, denoted *horizontal* and *vertical*, together with structure maps making them into horizontal and vertical categories with objects O
- a set of squares S which form the morphisms of two category structures whose objects are H and V respectively

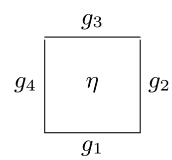
The category structures on the squares S satisfy some extra properties making them compatible with the category structures on H and V. Intuitively, they say that "horizontal" structure maps commute with "vertical" structure maps. For example, one can take a horizontal composite of two squares σ_1 and σ_2, and then take the morphism $s_v(\sigma_1 \circ_h \sigma_2) \in H$ which is its vertical source. This is the same as $s_v(\sigma_1) \circ_h s_v(\sigma_2)$, the horizontal composite of the vertical sources of each square. There are many other conditions of this form, such as the "interchange law" for composition of squares, which has the same form as (3).

A double groupoid is a double category in which all morphisms and squares are invertible. Further discussion of double groupoids is found in Brown and Spencer [5], while an interesting summary of general folklore about this symmetric definition and variations on it can be found online [18].

The above intuition essentially justifies a graphical notation which makes 2-group calculations more straightforward, which we now introduce.

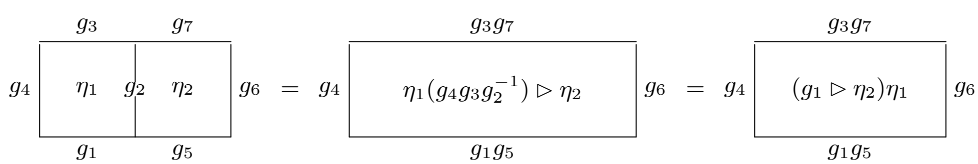
Definition 2.2.1 *Given a 2-group $\mathcal{G}$, the double groupoid of $\mathcal{G}$, denoted $\mathcal{D}(\mathcal{G})$, is the double groupoid with:*

- *A unique object of $\mathcal{D}(\mathcal{G})$: $O = \mathcal{G}^{(0)} = \{\star\}$*
- *Horizontal and vertical morphisms of $\mathcal{D}(\mathcal{G})$ are $H = V = \mathcal{G}^{(1)}$*
- *Squares of $\mathcal{D}(\mathcal{G})$ is given by all squares of the form*

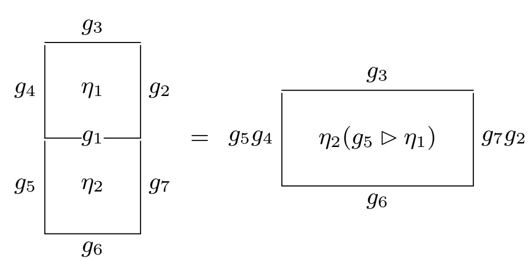


where $g_i \in G$, $i = 1, \ldots, 4$, $\eta \in H$ and $\partial(\eta) = g_1 g_2 g_3^{-1} g_4^{-1}$

- *horizontal and vertical composition of squares is given by*



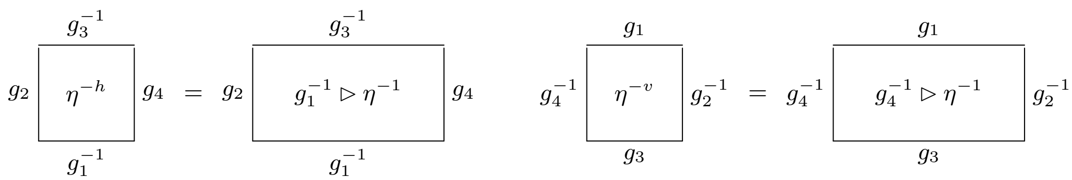
where the two expressions for the H element in the horizontal compositions are the same, using $\partial(\eta_1^{-1}) \rhd \eta_2 = \eta_1^{-1}\eta_2\eta_1$, and



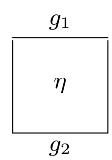
and these operations satisfy the interchange law, i.e. the equality of evaluating a 2 by 2 array of squares in two different ways (first horizontal and then vertical composition, or vice-versa).

In Ehresmann's terminology, this is the double category of *quintets* of the bicategory $\mathcal{G}$: the term refers to the fact that squares are determined by five pieces of data: the four 1-morphisms which are its edges, and the 2-morphism which fills the square.

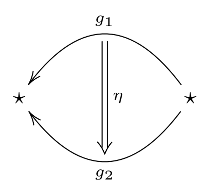
Remark 2.2.2 Due to the interchange law there is a consistent evaluation of any rectangular array of squares, which is independent of the order in which the horizontal and vertical multiplications are performed. Also squares in $\mathcal{D}(\mathcal{G})$ have horizontal and vertical inverses, which are respectively given by (for the square in the definition):



In particular, the double groupoid $\mathcal{D}(\mathcal{G})$ contains a copy of $\mathcal{G}$ by considering only the horizontal morphisms, and the squares for which the vertical source and target are identities. Thus, a typical square of this sort is of the following form:



where here, and henceforth, any unlabelled edge or square is taken to be labelled with the identity of the corresponding group. This can be identified with a 2-morphism in the 2-group $\mathcal{G}$, namely:



(12)

or equivalently, with the morphism (g_1, η) in the categorical group.

3 Actions of 2-Groups on Categories

Our aim is to describe actions of a 2-group $\mathcal{G}$ on a category $\mathbf{C}$. There are two equivalent ways of describing these, depending on whether one views $\mathcal{G}$ as a 2-group or a categorical group. In this section, we will outline these two views and see the relation between them. Then we will consider a natural example, namely the adjoint action of a categorical group on its own underlying category.

There are two different, but equivalent, definitions of group actions on a set, and both will be relevant for us. To begin with, consider a group G, seen as a one-object category whose morphisms are all invertible. Then a G action ϕ on a set X may be described as a functor

$$\phi : G \to \mathbf{Sets} \tag{13}$$

where $X = \phi(\star)$ is the image of the unique object of G, i.e. the image of ϕ is $End(X)$, the full subcategory of **Sets** with the single object X (in fact, since G is a group, the image is $Aut(X)$, consisting of only the invertible endomorphisms of X).

Next we show how to construct the transformation groupoid in a way that naturally generalizes to the 2-group case, by regarding the group G as a set equipped with a multiplication map $m : G \times G \to G$ and an inverse $inv : G \to G$, satisfying the group axioms. Thus we have familiar definition of an action on a set X as a function

$$\hat{\phi} : G \times X \to X \tag{14}$$

This function is related to ϕ by $\hat{\phi}(\gamma, x) = \phi_\gamma(x)$. Functoriality of ϕ means that $\hat{\phi}$ satisfies a compatibility condition with the multiplication map $m : G \times G \to G$, namely that the following commutes:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times Id_X} & G \times X \\ \downarrow{\scriptstyle Id_G \times \hat{\phi}} & & \downarrow{\scriptstyle \hat{\phi}} \\ G \times X & \xrightarrow[\hat{\phi}]{} & X \end{array} \tag{15}$$

This definition is, of course, equivalent to the point of view of an action as a functor. First it determines $\phi(\star) = X$. The two definitions are then related by turning a function $G \to Hom(X, X)$ into an X-valued function of $G \times X$, taking (γ, x) to $\phi_\gamma(x)$. (The term for this in logic is "uncurrying", while "currying" denotes the process which turns a function of n variables into a chain of n one-variable functions, each one returning the next function in the chain).

3.1 Actions of 2-Groups on Categories

Next we extend the two viewpoints of the previous introduction for the action of a group on a set to the action of a 2-group on a category. First, by analogy with (13), regarding a 2-group as a 2-category, an action will be a 2-functor into the 2-category **Cat** (see the appendix, where we consider having a general 2-category $\mathcal{C}$, instead of **Cat**) .

Definition 3.1.1 *A 2-group $\mathcal{G}$ acts (strictly) on a category* $\mathbf{C}$ *if there is a (strict) 2-functor:*

$$\Phi : \mathcal{G} \to \mathbf{Cat} \tag{16}$$

whose image lies in $End(\mathbf{C})$, the full sub-2-category of **Cat** *with the single object* $\mathbf{C}$*.*

Thus $\Phi(*) = \mathbf{C}$, and on 1- and 2-morphisms Φ is given by assignments:

- for each $\gamma \in \mathcal{G}^{(1)}$ we have the endofunctor $\Phi_\gamma : \mathbf{C} \to \mathbf{C}$, acting as

$$(x \xrightarrow{f} y) \mapsto (\Phi_\gamma(x) \xrightarrow{\Phi_\gamma(f)} \Phi_\gamma(y))$$

- for each 2-morphism $(\gamma_1, \chi) \in \mathcal{G}^{(2)}$, we have the natural transformation $\Phi_{(\gamma_1,\chi)} : \Phi_{\gamma_1} \to \Phi_{\gamma_2}$, where $\gamma_2 = \partial(\chi)\gamma_1$, given by assignments $(\mathbf{C}^{(0)} \ni x) \mapsto (\Phi_{(\gamma_1,\chi)}(x) \in \mathbf{C}^{(1)})$ satisfying the naturality condition

$$\begin{array}{ccc} \Phi_{\gamma_1}(x) & \xrightarrow{\Phi_{\gamma_1}(f)} & \Phi_{\gamma_1}(y) \\ \downarrow{\scriptstyle \Phi_{(\gamma_1,\chi)}(x)} & & \downarrow{\scriptstyle \Phi_{(\gamma_1,\chi)}(y)} \\ \Phi_{\gamma_2}(x) & \xrightarrow[\Phi_{\gamma_2}(f)]{} & \Phi_{\gamma_2}(y) \end{array} \tag{17}$$

These assignments must also satisfy the two conditions F1 and F2 of a strict 2-functor (see Appendix A.1 of [22]), which here means:

F1 1) $\Phi_{(\gamma_2,\chi_2)}(x) \circ \Phi_{(\gamma_1,\chi_1)}(x) = \Phi_{(\gamma_1,\chi_2\chi_1)}(x)$ where $\gamma_2 = \partial(\chi_1)\gamma_1$

2) $\Phi_{(\gamma,1)}(x) = \mathrm{id}_{\Phi_\gamma(x)}$

F2 1) $\Phi_{\gamma_1} \circ \Phi_{\gamma_3} = \Phi_{\gamma_1\gamma_3}$

2) $\Phi_{(\gamma_1,\chi_1)} \circ_h \Phi_{(\gamma_3,\chi_2)} = \Phi_{(\gamma_1\gamma_3,\chi_1(\gamma_1 \rhd \chi_2))}$.
Writing out explicitly the horizontal composition of natural transformations on the l.h.s., this condition becomes:

$$\Phi_{(\gamma_1,\chi_1)}(\Phi_{\gamma_4}(x)) \circ \Phi_{\gamma_1}(\Phi_{(\gamma_3,\chi_2)}(x)) = \Phi_{(\gamma_1\gamma_3,\chi_1(\gamma_1 \rhd \chi_2))}(x)$$

where $\gamma_4 = \partial(\chi_2)\gamma_3$. Here the underlying array of squares in $\mathcal{D}(\mathcal{G})$ is

$$\begin{array}{cc} \gamma_1 & \gamma_3 \\ \hline \multicolumn{1}{|c|}{\chi_1} & \multicolumn{1}{c|}{\chi_2} \\ \hline \gamma_2 & \gamma_4 \end{array} \tag{18}$$

The second viewpoint for a 2-group to act on a category is to regard the 2-group $\mathcal{G}$ as a categorical group, i.e. a monoidal category, and proceed by analogy with (14), (15).

Definition 3.1.2 *A strict action of a categorical group $\mathcal{G}$ on a category $\mathbf{C}$ is a functor $\hat{\Phi}$: $\mathcal{G} \times \mathbf{C} \to \mathbf{C}$ satisfying the action square diagram in* **Cat** *(strictly):*

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\otimes \times Id_{\mathbf{C}}} & \mathcal{G} \times \mathbf{C} \\ \Big\downarrow{\scriptstyle Id_{\mathcal{G}} \times \hat{\Phi}} & & \Big\downarrow{\scriptstyle \hat{\Phi}} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[\hat{\Phi}]{} & \mathbf{C} \end{array} \tag{19}$$

Thus, from functoriality, we have the conditions:

$$\hat{\Phi}((\gamma_2,\chi_2),g) \circ \hat{\Phi}((\gamma_1,\chi_1),f) = \hat{\Phi}((\gamma_1,\chi_2\chi_1),g \circ f) \tag{20}$$

$$\hat{\Phi}((\gamma,1_H),\mathrm{id}_x) = \mathrm{id}_{\hat{\Phi}(\gamma,x)} \tag{21}$$

where, in (20), $\gamma_2 = \partial(\chi_1)\gamma_1$. The action square diagram corresponds to the equations (on objects and morphisms respectively):

$$\hat{\Phi}(\gamma_1\gamma_3,x) = \hat{\Phi}(\gamma_1,\hat{\Phi}(\gamma_3,x)) \tag{22}$$

$$\hat{\Phi}((\gamma_1\gamma_3,\chi_1(\gamma_1 \rhd \chi_2)),f) = \hat{\Phi}((\gamma_1,\chi_1),\hat{\Phi}((\gamma_3,\chi_2),f)). \tag{23}$$

Here the underlying array of squares in $\mathcal{D}(\mathcal{G})$ is (18).

The following lemma shows that these two viewpoints are equivalent.

Lemma 3.1.3 *A strict 2-functor $\Phi : \mathcal{G} \to End(\mathbf{C})$ is equivalent to a strict action functor $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$.*

Proof: Given Φ, set: $\hat{\Phi}(\gamma,x) := \Phi_\gamma(x)$ and

$$(\ \hat{\Phi}(\gamma_1,x) \xrightarrow{\hat{\Phi}((\gamma_1,\chi),f)} \hat{\Phi}(\gamma_2,y)\ ) := (\ \Phi_{\gamma_1}(x) \xrightarrow{\Phi_{(\gamma_1,\chi)}(y) \circ \Phi_{\gamma_1}(f)} \Phi_{\gamma_2}(y)\ ) \tag{24}$$

By the naturality condition (17), this is also equal to

$$(\ \Phi_{\gamma_1}(x) \xrightarrow{\Phi_{\gamma_2}(f) \circ \Phi_{(\gamma_1,\chi)}(x)} \Phi_{\gamma_2}(y)\ ) \tag{25}$$

Now functoriality for $\hat{\Phi}$ follows from putting together four naturality squares (17) in the obvious way, and then using functoriality of Φ_γ horizontally and the first F1 condition vertically.

The first action square equation (22) is immediate. The second action square equation (23) follows from:

$$\begin{aligned}\hat{\Phi}((\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)), f) &= \Phi_{(\gamma_1\gamma_3,\chi_1(\gamma_1\rhd\chi_2))}(y) \circ \Phi_{\gamma_1\gamma_3}(f) \\ &= \Phi_{(\gamma_1,\chi_1)}(\Phi_{\gamma_4}(y)) \circ (\Phi_{\gamma_1}(\Phi_{(\gamma_3,\chi_2)}(y)) \circ \Phi_{\gamma_1}(\Phi_{\gamma_3}(f))) \\ &= \Phi_{(\gamma_1,\chi_1)}(\Phi_{\gamma_4}(y)) \circ (\Phi_{\gamma_1}(\Phi_{(\gamma_3,\chi_2)}(y)) \circ \Phi_{\gamma_3}(f))) \\ &= \hat{\Phi}((\gamma_1,\chi_1), \Phi_{(\gamma_3,\chi_2)}(y) \circ \Phi_{\gamma_3}(f)) \\ &= \hat{\Phi}((\gamma_1,\chi_1), \hat{\Phi}((\gamma_3,\chi_2), f))\end{aligned}$$

using the F2 condition for Φ and associativity in the second equality.

Conversely, given $\hat{\Phi}$, set $\Phi_\gamma(x) := \hat{\Phi}(\gamma, x)$ (on objects),

$$(\ \Phi_\gamma(x) \xrightarrow{\Phi_\gamma(f)} \Phi_\gamma(y)\ ) := (\ \hat{\Phi}(\gamma, x) \xrightarrow{\hat{\Phi}((\gamma, 1_H), f)} \Phi(\gamma, y)\ )$$

(on morphisms), and on 2-morphisms the natural transformation $\Phi_{(\gamma_1,\chi)} : \Phi_{\gamma_1} \to \Phi_{\gamma_2}$, where $\gamma_2 = \partial(\chi)\gamma_1$, is given by:

$$(\ \Phi_{\gamma_1}(x) \xrightarrow{\Phi_{(\gamma_1,\chi)}(x)} \Phi_{\gamma_2}(x)\ ) := (\ \hat{\Phi}(\gamma_1, x) \xrightarrow{\hat{\Phi}((\gamma_1,\chi), \mathrm{id}_x)} \hat{\Phi}(\gamma_2, x)\ ) \tag{26}$$

Now, the F1 property for Φ follows in a straightforward manner using the functoriality conditions (20), (21) for $\hat{\Phi}$. The F2 property for Φ follows from:

$$\begin{aligned}\Phi_{(\gamma_1,\chi_1)}(\Phi_{\gamma_4}(x)) \circ \Phi_{\gamma_1}(\Phi_{(\gamma_3,\chi_2)}(x)) &= \hat{\Phi}((\gamma_1,\chi_1), \mathrm{id}_{\hat{\Phi}(\gamma_4,x)}) \circ \hat{\Phi}((\gamma_1, 1_H), \hat{\Phi}((\gamma_3,\chi_2), \mathrm{id}_x)) \\ &= \hat{\Phi}((\gamma_1,\chi_1), \hat{\Phi}((\gamma_3,\chi_2), \mathrm{id}_x)) \\ &= \hat{\Phi}((\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)), \mathrm{id}_x)) \\ &= \Phi_{(\gamma_1\gamma_3,\chi_1(\gamma_1\rhd\chi_2))}(x)\end{aligned}$$

where we use functoriality of $\hat{\Phi}$ (20) in the second equality and the second action square condition (23) applied to $f = \mathrm{id}_x$ in the penultimate equality. □

It is convenient to introduce a succinct notation for a 2-group action analogous to the usual notation $g \rhd x = \phi_g(x)$ for a group action. There are actually three possible maps.

Since Φ_γ is a functor with both object and morphism maps, the objects of $\mathcal{G}$ act on both the set of objects and the set of morphisms of $\mathbf{C}$. Moreover, the action functor $\hat{\Phi}$ also has both object and morphism maps. The object map $\hat{\Phi}(\gamma, -)$ is the same as the object map for Φ_γ, by the above argument. However, the morphism map determines an action of the morphisms of $\mathcal{G}$ on the morphisms of $\mathbf{C}$, which is a different action again.

We will return to the relation between these three actions again in Corollary 4.3.2. For the moment, is convenient to use the symbol $\blacktriangleright$ to denote all three, as follows.

Definition 3.1.4 *If $\mathcal{G}$ is a 2-group classified by the crossed module $(G, H, \rhd, \partial)$, let the notation $\blacktriangleright$ denote the following.*

- *Given $\gamma \in \mathcal{G}^{(0)} = G$ and $x \in \mathbf{C}^{(0)}$, let*

$$\gamma \blacktriangleright x = \Phi_\gamma(x) = \hat{\Phi}(\gamma, x) \tag{27}$$

- *Given $\gamma \in \mathcal{G}^{(0)} = G$ and $f \in \mathbf{C}^{(1)}$, let*

$$\gamma \blacktriangleright f = \Phi_\gamma(f) = \hat{\Phi}((\gamma, 1_H), f) \tag{28}$$

- *Given* $(\gamma,\chi)\in\mathcal{G}^{(1)}=G\ltimes H$ *and* $(f:x\to y)\in\mathbf{C}^{(1)}$, *let*

$$\begin{aligned}(\gamma,\chi)\blacktriangleright f &= \hat{\Phi}((\gamma,\chi),f)\\ &= \Phi_{(\gamma,\chi)}(y)\circ(\gamma\blacktriangleright f)\\ &= (\partial(\chi)\gamma\blacktriangleright f)\circ\Phi_{(\gamma,\chi)}(x)\end{aligned}\tag{29}$$

The last two expressions are the same as those given in the proof of Lemma 3.1.3, written in our new notation. It will be revisited in (65).

Notice that, just as there is a "weakened" notion of a categorical group, satisfying group axioms only up to coherent isomorphism, there is also a weakened notion of a categorical group action. This would similarly satisfy the condition in (15) up to coherent isomorphism. The weak case is of interest in the long run as one continues further from 2-groups to n-groups for $n\geq 3$. However, for the present case, there are various strictification results which we consider in Appendix A which allow us to ignore this complication.

3.2 Example: Adjoint Action of 2-Groups

Here we want to describe the adjoint action of a 2-group $\mathcal{G}$ on itself. This is the analog of the usual adjoint action of a group G on itself by conjugation, which is a functor

$$\Phi: G\to End(G)\tag{30}$$

given by the property that

$$\Phi_{\gamma}(g)=\gamma g\gamma^{-1}\tag{31}$$

We want a 2-functor given "by conjugation", insofar as this makes sense. In fact, as we shall see, this is easy to do in the language of crossed modules, since the axioms (5) and (6) for a crossed module $(G,H,\triangleright,\partial)$ imply that the action $\triangleright$ of G on H resembles conjugation as much as possible. This will be made even clearer by using the square calculus in the double groupoid of $\mathcal{G}$, $\mathcal{D}(\mathcal{G})$.

In accordance with Definition 3.1.1, we take the action of $\mathcal{G}$ on itself to be given by a 2-functor from $\mathcal{G}$ to **Cat** with image in $End(\mathcal{G})$, i.e. the "acting" $\mathcal{G}$ is regarded as a 2-category, whilst the "acted on" $\mathcal{G}$ is regarded as a (monoidal) category. (This is analogous to the situation for the adjoint action of a group G, regarded as a category, acting on G, regarded as a set).

Definition 3.2.1 *Suppose $\mathcal{G}$ is the 2-group given by a crossed module $(G,H,\triangleright,\partial)$. Then we define a strict 2-functor:*

$$\Phi:\mathcal{G}\to\mathbf{Cat}\tag{32}$$

with image in $End(\mathcal{G})$ in the following way. At the object level, $\Phi(*)=\mathcal{G}$, where $\mathcal{G}$ on the right is taken to be a category. For each morphism $\gamma\in\mathcal{G}^{(1)}$

$$\Phi_{\gamma}:\mathcal{G}\to\mathcal{G}\tag{33}$$

is a morphism of $End(\mathcal{G})$, namely an endofunctor of $\mathcal{G}$. Its object map is given by:

$$\Phi_{\gamma}(g)=\gamma g\gamma^{-1}\tag{34}$$

and its morphism map is given by

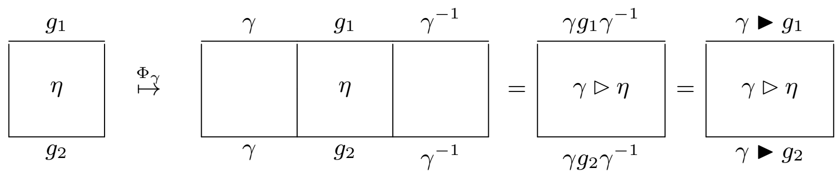
$$\Phi_{\gamma}(g,\eta)=(\gamma g\gamma^{-1},\gamma\triangleright\eta)\tag{35}$$

For each 2-morphism $(\gamma,\chi)\in\mathcal{G}^{(2)}$ there is a natural transformation from Φ_{γ} to $\Phi_{\partial(\chi)\gamma}$, given by:

$$\Phi_{(\gamma,\chi)}(g)=(\gamma g\gamma^{-1},\chi(\gamma g\gamma^{-1})\triangleright\chi^{-1}))\tag{36}$$

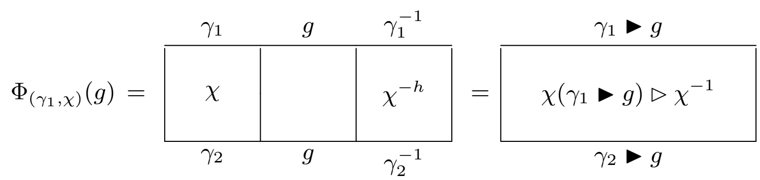
In the action notation of Definition 3.1.4, the first two simply say that $\gamma \blacktriangleright g = \gamma g \gamma^{-1}$ and $\gamma \blacktriangleright (g, \eta) = (\gamma \blacktriangleright g, \gamma \triangleright \eta)$. The third part of that definition will define $(\gamma, \chi) \blacktriangleright (g, \eta)$, which is not directly given by Φ. However, we can understand it using the calculus of squares introduced in Section 2.2 for the double groupoid of quintets $\mathcal{D}(\mathcal{G})$ associated to $\mathcal{G}$.

Represent morphisms (g_1, η) in the category $\mathcal{G}$ as squares, so that the action of Φ_γ on such a square is given by:



(37)

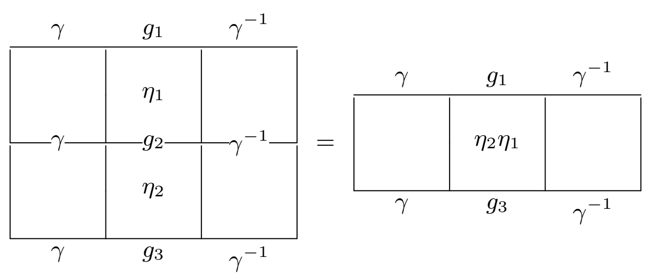
where $g_2 = \partial(\eta) g_1$. Likewise we can represent the morphism $\Phi_{(\gamma_1, \chi)}(g)$ (which has no direct analog in the action notation) as



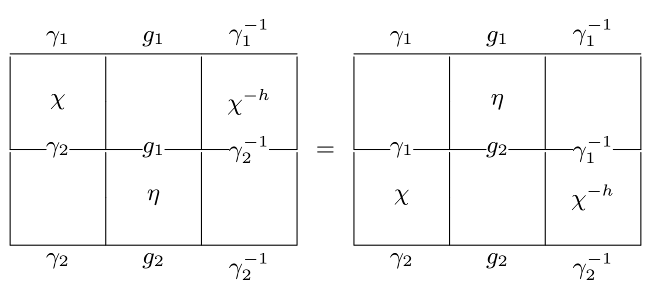
(38)

where $\gamma_2 = \partial(\chi)\gamma_1$.

It follows from (37) that $\Phi_\gamma(g_1, \eta)$ has the correct source and target. Φ_γ is a functor, since it preserves identities, $\Phi_\gamma(g, 1_H) = (\gamma g \gamma^{-1}, 1_H)$ (immediate), and composition, i.e. $\Phi_\gamma(g_2, \eta_2) \circ \Phi_\gamma(g_1, \eta_1) = \Phi_\gamma(g_1, \eta_2 \eta_1)$, where $g_2 = \partial(\eta_1) g_1$, because of:

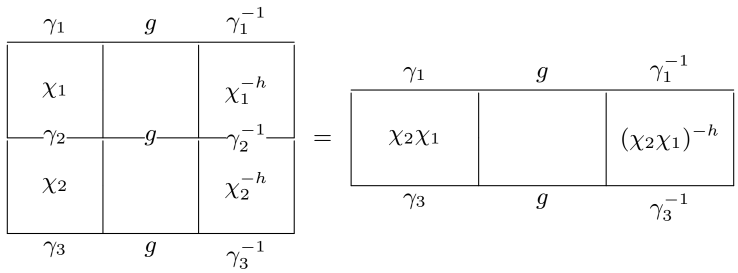


It then follows immediately from (38) that $\Phi_{(\gamma_1, \chi)}(g)$ has the correct source and target. The naturality condition (17), i.e. $\Phi_{\gamma_2}(g_1, \eta) \circ \Phi_{(\gamma_1, \chi)}(g_1) = \Phi_{(\gamma_1, \chi)}(g_2) \circ \Phi_{\gamma_1}(g_1, \eta)$, is the equality:

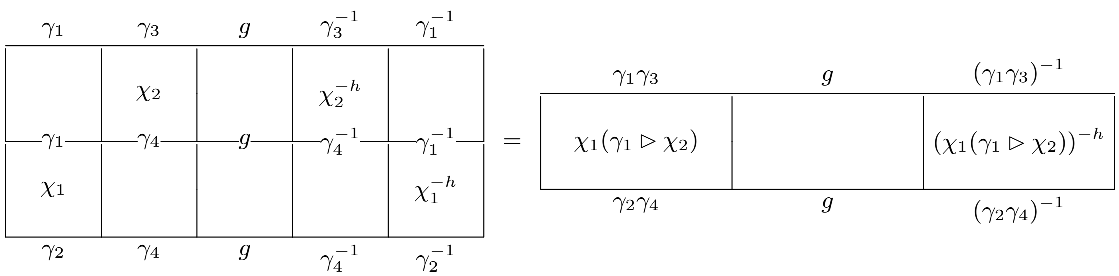


which follows from the interchange law by evaluating the vertical compositions on both sides first.

The first F1 condition, i.e. $\Phi_{(\gamma_2,\chi_2)}(g) \circ \Phi_{(\gamma_1,\chi_1)}(g) = \Phi_{(\gamma_1,\chi_2\chi_1)}(g)$ where $\gamma_2 = \partial(\chi_1)\gamma_1$, follows from the equality:



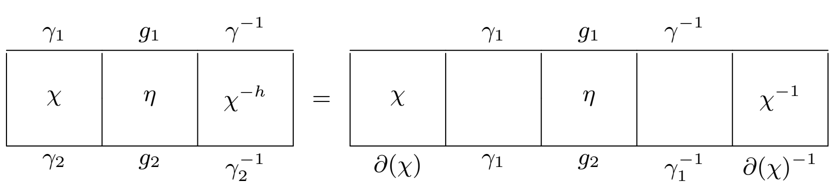
The second F1 condition, i.e. $\Phi_{(\gamma,1_H)}(g) = \mathrm{Id}_{\Phi_\gamma(g)}$, and the first F2 condition, i.e. $\Phi_{\gamma_1}(\Phi_{\gamma_3}(g)) = \Phi_{\gamma_1\gamma_3}(g)$, are both immediate. Finally the second F2 condition, i.e. $\Phi_{(\gamma,\chi_1)}(\Phi_{\gamma_4}(g))\circ\Phi_{\gamma_1}(\Phi_{(\gamma_3,\chi_2)}(g)) = \Phi_{(\gamma_1\gamma_3,\chi_1(\gamma_1\triangleright\chi_2))}(g)$, corresponds to the equality:



This follows from evaluating the two 2 by 2 arrays of squares without g on the left hand side. Thus we have proved:

Lemma 3.2.2 *The 2-group adjoint action of Definition 3.2.1 is a well-defined action in the sense of Definition 3.1.1.*

Remark 3.2.3 We can now display the functor $\hat{\Phi}$ of Definition 3.1.2, in terms of squares, and therefore finish describing this action in the notation of Definition 3.1.4. Namely $(\gamma_1,\chi) \blacktriangleright (g_1,\eta) = \hat{\Phi}((\gamma_1,\chi),(g_1,\eta))$ is given by:



The 5-square array on the right makes it clear that the 2-group adjoint action of (γ_1,χ) can be regarded as the ordinary adjoint action of γ_1 on edges, extended to squares labelled by γ_1 acting on squares, followed by the adjoint action of χ, in the sense of conjugation with the square labelled χ. Evaluating the array and dropping the indices gives an algebraic formula for this action:

$$(\gamma,\chi) \blacktriangleright (g,\eta) = (\gamma g\gamma^{-1}, \chi(\gamma \triangleright \eta)(\gamma g\gamma^{-1}) \triangleright \chi^{-1}) \tag{39}$$

4 Transformation Double Category for a 2-Group Action

In this section, we recall the construction of the transformation groupoid for a group action on a set, and consider the analogous construction for a 2-group action on a category.

If we are given an action of a group on a set, there is a groupoid which corresponds to it.

Definition 4.0.4 *Given a group action $\phi : G \to End(X)$, the **transformation** groupoid $X/\!\!/G$ is the groupoid with:*

- ***Objects**: $x \in (X/\!\!/G)^{(0)} = X$*
- ***Morphisms**: $(\gamma, x) \in (X/\!\!/G)^{(1)} = G \times X$, with source and target maps $s(\gamma, x) = x$, and $t(\gamma, x) = \phi_\gamma(x)$*
- ***Composition**: $(\gamma', \phi_\gamma(x)) \circ (\gamma, x) = (\gamma'\gamma, x)$*

It is clear that this is a groupoid, whose morphisms are invertible since G is a group. The composition then encodes both the group multiplication (in the first component) and the action (in the second component). Formally, the set $P = (X/\!\!/G)^{(1)} \times_{(X/\!\!/G)^{(0)}} (X/\!\!/G)^{(1)}$ of composable pairs of morphisms in $X/\!\!/G$ is then given by the pullback square:

$$\begin{array}{ccc} P & \longrightarrow & G \times X \\ \downarrow & & \downarrow{\scriptstyle \hat{\phi}} \\ G \times X & \xrightarrow[\pi_X]{} & X \end{array} \tag{40}$$

There is an obvious commuting diagram with $G \times G \times X$ replacing P in the above diagram, and hence, by the universal property of this pullback, there is a unique map from $G \times G \times X$ to P, given by:

$$(\gamma', \gamma, x) \mapsto (\gamma', \phi_\gamma(x)), (\gamma, x)) \tag{41}$$

which is clearly an isomorphism. In this way, the composition map from P to $G \times X$ agrees with the map $m \times Id_X$ in (15).

In the next subsections, we will develop a similar construction in **Cat**, which will give a transformation *double category* for a categorical group action. As discussed in Section 2.2, we reserve the term "double category" for a certain symmetric point of view of this structure. To construct it, however, we use an equivalent definition as a **Cat**-category.

4.1 Construction of the Transformation Cat-Category for a 2-Group Action

The most obvious way to define an analog of the transformation groupoid in the situation of a 2-group action comes by simply following the same constructions from an ordinary group action, replacing G with $\mathcal{G}$ and X with $\mathbf{C}$. Thus, one has action diagrams in **Cat**.

Next we will construct a transformation groupoid as we did for group actions on sets, but it will be internal to **Cat**, so we call it a *transformation* **Cat**-*groupoid*.

Thus, now one has a category $(\mathbf{C}/\!\!/\mathcal{G})^{(0)}$ of objects, and a category $(\mathbf{C}/\!\!/\mathcal{G})^{(1)}$ of morphisms. These, of course, have objects and morphisms of their own, but this fact is invisible to the construction and become important only when we want to describe the resulting structure concretely.

Definition 4.1.1 *Given a 2-group $\mathcal{G}$, a category $\mathbf{C}$, and an action of $\mathcal{G}$ on $\mathbf{C}$ as in Definitions 3.1.1 and 3.1.2, the transformation **Cat**-groupoid $\mathbf{C}/\!\!/\mathcal{G}$ is the groupoid internal to **Cat** with:*

- ***Category of objects**: $(\mathbf{C}/\!\!/\mathcal{G})^{(0)} = \mathbf{C}$.*

- ***Category of morphisms***: $(\mathbf{C}/\!\!/\mathcal{G})^{(1)} = \mathcal{G} \times \mathbf{C}$, *with*
  - ***Source*** *functor* $s = \pi_{\mathbf{C}} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$
  - ***Target*** *functor* $t = \hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$
  - ***Identity inclusion*** *functor* $e = 1_{\mathcal{G}} \times Id_{\mathbf{C}} : \mathbf{C} \to \mathcal{G} \times \mathbf{C}$
  - ***Inverse*** *functor* $inv = (inv_{\mathcal{G}}, t) : \mathcal{G} \times \mathbf{C} \to \mathcal{G} \times \mathbf{C}$
- ***Category of composable pairs***: $\mathbf{P}$, *given by the pullback diagram*

$$\begin{array}{ccc} \mathbf{P} & \longrightarrow & \mathcal{G} \times \mathbf{C} \\ \downarrow & & \downarrow{\scriptstyle \hat{\Phi}} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[\pi_{\mathbf{C}}]{} & \mathbf{C} \end{array} \tag{42}$$

- ***Composition functor***: *given by the action square (19)*

The situation is entirely analogous to the transformation groupoid for an ordinary group G, as is seen easily by considering the effect on objects. (Thus the inverse functor, on objects, takes $(\gamma, x) \mapsto (\gamma^{-1}, \gamma \blacktriangleright x)$, just as with an ordinary transformation groupoid.) Analogously with the situation for groups, we can think of $(\mathbf{C}/\!\!/\mathcal{G})^{(1)}$ as a semidirect product 2-group; and there is a canonical isomorphism between $\mathcal{G} \times \mathcal{G} \times \mathbf{C}$ and $\mathbf{P}$. We should check that this definition makes sense.

Lemma 4.1.2 *The transformation* **Cat**-*groupoid* $\mathbf{C}/\!\!/\mathcal{G}$ *is a well-defined groupoid internal to* **Cat**.

Proof: To confirm this, we must check that the source, target, composition, identity inclusion and inverse functors are well defined, and satisfy the usual properties for a groupoid, primarily associativity and the left and right unit laws.

The fact that the source and target maps are functors is obvious, since they are just the projection from a product, and $\hat{\Phi}$ respectively. The first is necessarily a functor, while the second is a functor by definition.

Recall our notation:

$$\gamma \blacktriangleright x = \Phi_{\gamma}(x) = \hat{\Phi}(\gamma, x) \qquad (\gamma, \chi) \blacktriangleright f = \hat{\Phi}((\gamma, \chi), f) \tag{43}$$

and make explicit the source, target and composition functors at the morphism level as follows:

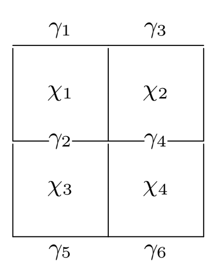
$$s((\gamma, \chi), f) = f \qquad t((\gamma, \chi), f) = (\gamma, \chi) \blacktriangleright f \tag{44}$$

$$((\gamma_1, \chi_1), (\gamma_3, \chi_2) \blacktriangleright f) \circ ((\gamma_3, \chi_2), f) = ((\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)), f) \tag{45}$$

(see (23)). The target of the composition is well-defined due to the commutativity of (19):

$$(\gamma_1, \chi_1) \blacktriangleright ((\gamma_3, \chi_2) \blacktriangleright f) = (\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)) \blacktriangleright f \tag{46}$$

To show the functoriality of composition, we refer to the following array of squares in $\mathcal{D}(\mathcal{G})$, underlying the calculation:



(47)

We denote composition in $\mathcal{G} \times \mathbf{C}$ by $\bar{\circ}$ and composition in $\mathbf{C}$ by $\circ_{\mathbf{C}}$, to distinguish them from the composition functor $\circ$. Then we have:

$$\begin{aligned}
((\gamma_4, \chi_4), g) \,\bar{\circ}\, ((\gamma_3, \chi_2), f) &= ((\gamma_3, \chi_4\chi_2), g \circ_{\mathbf{C}} f) \\
((\gamma_2, \chi_3), (\gamma_4, \chi_4) \blacktriangleright g) \,\bar{\circ}\, ((\gamma_1, \chi_1), (\gamma_3, \chi_2) \blacktriangleright f) &= ((\gamma_1, \chi_3\chi_1), ((\gamma_4, \chi_4) \blacktriangleright g) \circ_{\mathbf{C}} ((\gamma_3, \chi_2) \blacktriangleright f)) \\
&= ((\gamma_1, \chi_3\chi_1), (\gamma_3, \chi_4\chi_2) \blacktriangleright (g \circ_{\mathbf{C}} f))
\end{aligned}$$

where we use (20) in the final equation. We also have:

$$\begin{aligned}
((\gamma_2, \chi_3), (\gamma_4, \chi_4) \blacktriangleright g) \circ ((\gamma_4, \chi_4), g) &= ((\gamma_2\gamma_4, \chi_3(\gamma_2 \rhd \chi_4)), g) \\
((\gamma_1, \chi_1), (\gamma_3, \chi_2) \blacktriangleright f) \circ ((\gamma_3, \chi_2), f) &= ((\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)), f)
\end{aligned}$$

Thus it remains to show the equality of two expressions:

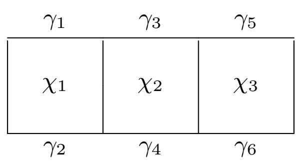
$$((\gamma_2\gamma_4, \chi_3(\gamma_2 \rhd \chi_4)), g) \,\bar{\circ}\, ((\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)), f) = ((\gamma_1\gamma_3, \chi_3(\gamma_2 \rhd \chi_4)\chi_1(\gamma_1 \rhd \chi_2)), g \circ_{\mathbf{C}} f)$$

and

$$((\gamma_1, \chi_3\chi_1), (\gamma_3, \chi_4\chi_2) \blacktriangleright (g \circ_{\mathbf{C}} f)) \circ ((\gamma_3, \chi_4\chi_2), g \circ_{\mathbf{C}} f) = ((\gamma_1\gamma_3, \chi_3\chi_1\gamma_1 \rhd (\chi_4\chi_2)), g \circ_{\mathbf{C}} f).$$

This equality is immediate from the interchange law in $\mathcal{D}(\mathcal{G})$ applied to the array (47).

Associativity of the composition functor follows from the fact that the horizontal composition of three squares in $\mathcal{D}(\mathcal{G})$ has a unique evaluation:



(48)

The identity inclusion e is clearly a functor, and satisfies:

$$\begin{aligned}
((\gamma, \chi), f) \circ ((1_G, 1_H), f) &= ((\gamma, \chi), f) \\
((1_G, 1_H), (\gamma, \chi) \blacktriangleright f) \circ ((\gamma, \chi), f) &= ((\gamma, \chi), f)
\end{aligned}$$

where in the first equation we have a composable pair, since (if the target of f is y):

$$t((1_G, 1_H), f) = (1_G, 1_H) \blacktriangleright f = \Phi_{(1_G, 1_H)}(y) \circ \Phi_{1_G}(f) = \Phi_{(1_G, 1_H)}(y) \circ f = f$$

using (24) in the second equality, functoriality of Φ in the third equality and property F2 (see Def. 3.1.1) in the final equality.

Finally, the inverse functor assigns, to every $((\gamma, \chi), f)$ in the category of morphisms, its inverse $((\gamma^{-1}, \chi^{-h}), (\gamma, \chi) \blacktriangleright f)$, where we note that $(\gamma^{-1}, \chi^{-h}) = (\gamma^{-1}, (\partial(\chi)\gamma)^{-1} \rhd \chi^{-1}) = (\gamma^{-1}, \gamma^{-1} \rhd \chi^{-1})$. This follows immediately from (45). Functoriality of the inverse functor follows easily from (20) and functoriality of the horizontal inverse in $\mathcal{D}(\mathcal{G})$. □

Since **Cat** is actually a 2-category, it could be objected that our construction is implicitly assuming that one is requiring that diagrams commute strictly, and that all pullbacks are strict rather than weak. That is, that we are taking a *strict* internal category in **Cat**. A more subtle approach might consider weakening the axioms to various degrees. However, we will not do this here, for reasons outlined in Appendix A.

On the other hand, there is a different drawback to this construction of a **Cat**-category which we will address, which has to do with the apparent asymmetry of its definition.

4.2 The Transformation Double Category

The construction we have given in the previous section naturally produces a groupoid internal to **Cat**. However, the view of a double category as an internal category in **Cat** obscures the underlying symmetry of this structure, and the structure in the "transverse" direction to the categories of objects and morphisms.

As we remarked in Section 2.2, **Cat**-categories are equivalent to a structure defined in a more symmetric way, having two types of morphisms (horizontal and vertical), and squares. The properties can be deduced from the equivalence: for example, the "interchange law", which says that mixed horizontal and vertical composites can be taken in any order, amounts to the functoriality of the composition $\circ$.

Now, the definition of double categories is symmetric under exchanging the roles of "horizontal" and "vertical" morphisms in our diagrams of squares. One can reflect all squares in a diagonal, and obtain another double category, which we call the *transpose* of the first double category. This result, naturally, still has an interpretation as a category internal to **Cat**.

More technically, we have the following:

Definition 4.2.1 *If $\mathcal{D}$ is an internal category in* **Cat***, then let the* ***transpose*** *of $\mathcal{D}$, which we denote $\widetilde{\mathcal{D}}$, be the internal category in* **Cat** *defined by the following:*

- *The category of objects has*
  - *Objects: the objects of the category $\mathcal{D}^{(0)}$*
  - *Morphisms: the objects of the category $\mathcal{D}^{(1)}$*
- *The category of morphisms has*
  - *Objects: the morphisms of the category $\mathcal{D}^{(0)}$*
  - *Morphisms: the morphisms of the category $\mathcal{D}^{(1)}$*
- *The identity inclusion, source and target, and composition functors $(\widetilde{e}, \widetilde{s}, \widetilde{t}, \widetilde{\circ})$ have as object maps the corresponding structure maps from the category $\mathcal{D}^{(0)}$, and as morphism maps the corresponding structure maps from the category $\mathcal{D}^{(1)}$*

This is a double-category analog of the operation of taking the "opposite" of a category: another category in which morphisms are taken to be oriented in the opposite direction. Indeed, together with such opposite operations in both horizontal and vertical directions, the transpose generates a whole group of operations which take one double category to another: it is plainly isomorphic to the dihedral group D_4, the symmetries of a generic square, since each is determined by the source vertex of a generic square, together with the sense of horizontal and vertical. The opposites do not illustrate much new structure, however, so we will restrict our attention to the transpose.

The fact that the transpose is again an internal category in **Cat** is a standard consequence of basic facts about internalization. We do not want to assume all readers are accustomed to such internal constructions, so will sketch a proof to convey the essential idea.

Lemma 4.2.2 *If $\mathcal{D}$ is a category internal in* **Cat***, so is $\widetilde{\mathcal{D}}$.*

Proof: First, one must check that the structure maps $(\widetilde{e}, \widetilde{s}, \widetilde{t}, \widetilde{\circ})$ are functors. This follows from the compatibility conditions for $\mathcal{D}$. We give the example of the source functor $\widetilde{s}$ here: the others follow similar lines.

We claim that the source functor in $\widetilde{\mathcal{D}}$

$$\widetilde{s} : \widetilde{\mathcal{D}}^{(1)} \to \widetilde{\mathcal{D}}^{(0)} \tag{49}$$

preserves composition, that is, for two composable morphisms χ_1 and χ_2 in $\widetilde{\mathcal{D}}^{(1)}$

$$\widetilde{s}(\chi_1 \circ \chi_2) = \widetilde{s}(\chi_1) \circ \widetilde{s}(\chi_2) \tag{50}$$

where the compositions are in $\widetilde{\mathcal{D}}^{(1)}$ and $\widetilde{\mathcal{D}}^{(0)}$ respectively. Interpreted in $\mathcal{D}$, this is the statement that the composition functor $\circ : \mathcal{D}^{(1)} \times_{\mathcal{D}^{(0)}} \mathcal{D}^{(1)} \to \mathcal{D}^{(1)}$ is compatible with the source maps in $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(0)}$, which is just part of the fact that $\circ$ is a functor. Likewise $\widetilde{s}$ is compatible with the source, target, and identity inclusion maps in $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(0)}$. Similar arguments prove the the functoriality and compatibility of $\widetilde{t}$, $\widetilde{e}$, and $\widetilde{\circ}$. So all the structure maps $(\widetilde{e}, \widetilde{s}, \widetilde{t}, \widetilde{\circ})$ are well-defined morphisms in **Cat**.

Furthermore, since the object and morphism maps of $(\widetilde{e}, \widetilde{s}, \widetilde{t}, \widetilde{\circ})$ satisfy the axioms for a category separately by definition, the functors satisfy them as well. Since its structure maps are well defined and satisfy all the axioms for a category, $\widetilde{\mathcal{D}}$ is indeed an internal category in **Cat**. □

In the above sketch, we deliberately chose to compare the interaction of different maps in the two directions, namely source and composition, to emphasize the differences. It is worth remarking that some conditions are symmetrical: for example, the fact that $\circ$ is functorial, and therefore preserves composition, corresponds in the transpose to exactly the same fact about $\widetilde{\circ}$. This is a form of the *interchange law*.

The above is a special case of a more general duality which can occur with internalization. Suppose we have any two types of objects, X and Y, which can be defined as collections of objects and morphisms satisfying certain diagrammatic axioms. Then it is equivalent to define X-objects internal to the category of Y-objects, and Y-objects internal to the category of X-objects.

Next, we want use the transpose to better understand the structure of the transformation **Cat**-groupoid $\mathbf{C}/\!\!/\mathcal{G}$ associated to a 2-group action. The essential new observation is that its transpose $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$ is built from transformation groupoids associated to group actions in the usual sense.

Theorem 4.2.3 *Given a 2-group $\mathcal{G}$, a category $\mathbf{C}$, and an action of $\mathcal{G}$ on $\mathbf{C}$ given by $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ in* **Cat***, the category $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$ internal to* **Gpd** *has:*

- *Groupoid of Objects: $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)}$, isomorphic to the transformation groupoid $\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}$ associated to the action given by $\hat{\Phi}^{(0)}$.*
- *Groupoid of Morphisms: $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(1)}$, isomorphic to the transformation groupoid $\mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)}$ associated to the action given by $\hat{\Phi}^{(1)}$.*
- *Identity inclusion functor*
$$\widetilde{e} : \mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)} \to \mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)} \tag{51}$$
source and target functors
$$\widetilde{s}, \widetilde{t} : \mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)} \to \mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)} \tag{52}$$
and composition functor
$$\widetilde{\circ} : (\mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)}) \times_{\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}} (\mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)}) \to \mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)} \tag{53}$$
are the functors whose object maps are the corresponding $(e, s, t, \circ)$ for $\mathbf{C}$ and whose morphism maps are those for $\mathcal{G} \times \mathbf{C}$.

Proof: The proof is a special case of the proof of Lemma 4.2.2. First, consider the action as internal to **Cat**, by the inclusion of **Gpd** in **Cat**. Then the case of the groupoid of objects is clear: its objects are the objects of $x \in \mathbf{C}$, and its morphisms are objects of $(\mathbf{C}/\!\!/\mathcal{G})^{(1)} = \mathcal{G} \times \mathbf{C}$, i.e. they are labeled by pairs $(\gamma, x) \in \mathcal{G}^{(0)} \times \mathbf{C}^{(0)}$, and (γ, x) has source x and target $\gamma \blacktriangleright x$. Composition is determined by the object map of the composition functor in $\mathbf{C}/\!\!/\mathcal{G}$ (45) , i.e. by the monoidal product $\otimes$ of $\mathcal{G}^{(0)}$. But this is exactly the transformation groupoid $\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}$, where $\mathcal{G}^{(0)}$ is a group with product $\otimes$, acting on $\mathbf{C}^{(0)}$ by $\hat{\Phi}^{(0)}$.

The argument for the groupoid of morphisms is substantially the same. Its objects are the morphisms of $\mathbf{C}$, i.e. labelled by $f \in \mathbf{C}^{(1)}$, and its morphisms are pairs $((\gamma, \chi), f)$ with source f and target $(\gamma, \chi) \blacktriangleright f$. Composition in the groupoid of morphisms is determined by the morphism part of the composition functor in $\mathbf{C}/\!\!/\mathcal{G}$ (45).

The object and morphism maps of the structure functors must be as given, by definition of the transpose. For instance the identity inclusion functor at the object level is $\widetilde{e}^{(0)}(x) = Id_x$, and at the morphism level is $\widetilde{e}^{(1)}(\gamma, x) = ((\gamma, 1_H), Id_x)$. The target of the morphism on the right-hand side is $(\gamma, 1_H) \blacktriangleright Id_x = \hat{\Phi}((\gamma, 1_H), Id_x) = \Phi_\gamma(Id_x) = Id_{\gamma \blacktriangleright x}$.

By Lemma 4.2.2, this is indeed a category in **Cat**. In particular, these structure maps are functors as required. Moreover, we have already argued that both the category of objects and the category of morphisms are groupoids (indeed, transformation groupoids), so in fact this is a category internal to **Gpd**. □

We would like to emphasize again, that in the transposed perspective of $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$, both the category of objects and the category of morphisms are themselves transformation groupoids in the usual sense, associated to group actions.

We are now ready to combine the perspectives given by $\mathbf{C}/\!\!/\mathcal{G}$ and its transpose to express the whole local symmetry structure implied in the action of $\mathcal{G}$ on $\mathbf{C}$ in a balanced way. This balanced viewpoint is expressed in our definition below of the transformation double category $\mathbf{C}/\!\!/\mathcal{G}$ (abusing notation somewhat by keeping the same name as for the internal category). We recall from subsection 2.2 the notation O for the set of objects, H and V for the sets of horizontal and vertical morphisms, and S for the set of squares. By the preceding discussion, the horizontal category is $(\mathbf{C}/\!\!/\mathcal{G})^{(0)} = \mathbf{C}$ (Def. 4.1.1), the vertical category is $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)} = \mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}$ (Thm. 4.2.3), and for squares, the horizontal composition of squares in S is the morphism map of $\circ$, while vertical composition of squares in S is the morphism map of $\widetilde{\circ}$.

Definition 4.2.4 *Given an action of a 2-group $\mathcal{G}$ on a category $\mathbf{C}$, in terms of Φ or $\hat{\Phi}$ of definitions 3.1.1 and 3.1.2, the transformation double category $\mathbf{C}/\!\!/\mathcal{G}$ is given by:*

- ***Objects**: objects $x \in O = \mathbf{C}^{(0)} = (\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)})^{(0)}$*
- ***Horizontal Category**: same as $\mathbf{C}$, with morphisms $x \xrightarrow{f} y \in H = \mathbf{C}^{(1)}$ composed as usual*
- ***Vertical Category**: same as $\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}$, with morphisms displayed as $x \xrightarrow{(\gamma,x)} \gamma \blacktriangleright x \in V = (\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)})^{(1)}$ with source and target as shown and composed by $(\gamma', \gamma \blacktriangleright x) \circ (\gamma, x) = (\gamma'\gamma, x)$*
- ***Squares**: $S = (\mathbf{C} \times \mathcal{G})^{(1)} = \mathbf{C}^{(1)} \times \mathcal{G}^{(1)}$ consists of pairs of morphisms from $\mathbf{C}$ and $\mathcal{G}$, with elements of S denoted by $\boxed{(\gamma, \chi), f}$, displayed as*

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & y \\ {\scriptstyle (\gamma,x)}\downarrow & \boxed{(\gamma,\chi),f} & \downarrow{\scriptstyle (\partial(\chi)\gamma,y)} \\ \gamma \blacktriangleright x & \xrightarrow[(\gamma,\chi)\blacktriangleright f]{} & (\partial(\chi)\gamma) \blacktriangleright y \end{array} \tag{54}$$

with horizontal and vertical source and target as shown in the display. Horizontal and vertical composition (equivalent to pasting of diagrams of the form (54)) is given by:

$$\boxed{(\gamma_2, \chi_2), g} \circ_h \boxed{(\gamma_1, \chi_1), f} = \boxed{(\gamma_1, \chi_2\chi_1), g \circ f} \tag{55}$$

where $\gamma_2 = \partial(\chi_1)\gamma_1$, and:

$$\boxed{(\gamma_1, \chi_1), (\gamma_3, \chi_2) \blacktriangleright f} \circ_v \boxed{(\gamma_3, \chi_2), f} = \boxed{(\gamma_1\gamma_3, \chi_1(\gamma_1 \rhd \chi_2)), f} \tag{56}$$

where the underlying squares in $\mathcal{D}(\mathcal{G})$ are given in (18).

Theorem 4.2.5 $\mathbf{C}/\!\!/\mathcal{G}$ *is a well-defined double category.*

Proof: Most of the points to be checked are directly implied by the properties of the internal categories $\mathbf{C}/\!\!/\mathcal{G}$ and $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$, e.g. the vertical target of the vertical composition of squares is well-defined due to (46). We note that the vertical target of the horizontal composition of squares, equals the composition of the vertical targets of the two squares, i.e.

$$(\gamma_1, \chi_2\chi_1) \blacktriangleright (g \circ f) = ((\gamma_2, \chi_2) \blacktriangleright g) \circ ((\gamma_1, \chi_1) \blacktriangleright f) \tag{57}$$

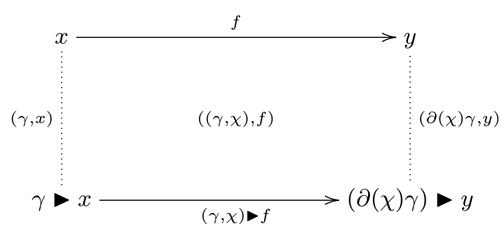
by the functoriality of $\hat{\Phi}$ in (20). Likewise the horizontal target of the vertical composition of squares equals the composition of the horizontal targets of the two squares, i.e.

$$(\partial(\chi_1)\gamma_1\partial(\chi_2)\gamma_3, y) = (\partial(\chi_1(\gamma_1 \triangleright \chi_2))\gamma_1\gamma_3, y). \tag{58}$$

This follows in a straightforward fashion from crossed module properties.

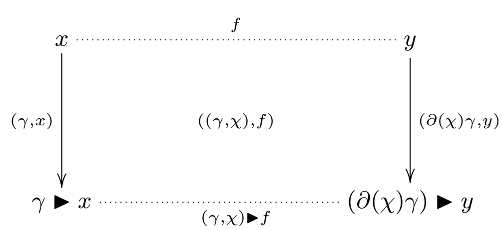
Associativity of the horizontal and vertical composition of squares is an immediate consequence of the associativity of the composition functors in $\mathbf{C}/\!\!/\mathcal{G}$ and $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$. Finally the interchange law for horizontal and vertical composition of squares corresponds to the proof of functoriality of the composition functor in $\mathbf{C}/\!\!/\mathcal{G}$ (Lemma 4.1.2) and reduces to stating the interchange law for four squares (47) in $\mathcal{D}(\mathcal{G})$. □

Remark 4.2.6 From the perspective of the internal category $\mathbf{C}/\!\!/\mathcal{G}$ of Definition 4.1.1, the squares of the double category may be thought of as



(59)

displaying diagrammatically the action of the source and target functors s, t on objects (γ, x), $(\partial(\chi)\gamma, y)$, and morphisms $((\gamma, \chi), f)$. The solid arrows denote the image of the square under source and target functors, while the dotted lines denote its source and target internal to the category of morphisms. Similarly, from the perspective of the internal category $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$ of Theorem 4.2.3, the squares of the double category may be thought of as



(60)

displaying diagrammatically the action of the source and target functors $\widetilde{s}$, $\widetilde{t}$ on objects f, $(\gamma, \chi) \blacktriangleright f$, and morphisms $((\gamma, \chi), f)$. The double category superimposes the two perspectives giving a balanced view, which makes both internal structures transparent at the same time.

4.3 Structure of the Transformation Double Category

Next, we will consider some consequences of the construction of $\mathbf{C}/\!\!/\mathcal{G}$. As constructed, the main information which can be read directly from $\mathbf{C}/\!\!/\mathcal{G}$ itself is precisely the target map, which is just the action $\hat{\Phi}$ itself. Some less obvious consequences are more easily read from the transpose $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$, or from $\mathbf{C}/\!\!/\mathcal{G}$ seen as a double category.

4.3.1 Relation to Ordinary Transformation Groupoids

We have seen, in Theorem 4.2.3, that the groupoid of objects and the groupoid of morphisms of $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$ are both transformation groupoids. The structure maps of $\widetilde{\mathbf{C}/\!\!/\mathcal{G}}$ relate these groupoids to each other. Being functors, these maps consist of two parts, which we can consider separately. Considering the identity-inclusion functor tells us how several transformation groupoids are nested inside one another, and will justify the notation $\blacktriangleright$ we previously introduced for our 2-group action. First, note what these groupoids are in crossed module notation.

Corollary 4.3.1 *If $\mathcal{G}$ is given by a crossed module $(G, H, \partial, \rhd)$, then $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)} = \mathbf{C}^{(0)}/\!\!/G$, and $\mathbf{C}^{(1)}/\!\!/G \subset (\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(1)} = \mathbf{C}^{(1)}/\!\!/(G \ltimes H)$.*

Proof: In the construction of a categorical group from a crossed module, the group $G \ltimes H$ is the group of morphisms, and the group G of objects occurs as the subgroup of elements of the form $(\gamma, 1_H)$. The groupoid inclusions then follow immediately from the identity-inclusion functor in Theorem 4.2.3. □

The fact that the structure maps are functors means that, from an "external" point of view (that is, if we look at underlying sets), $\mathbf{C}/\!\!/\mathcal{G}$ combines three closely related group actions in the standard sense. These three group actions have all been denoted by the symbol $\blacktriangleright$ introduced in Definition 3.1.4. This notation can now be justified because they are all restrictions of the action of $\mathcal{G}^{(1)}$ on $\mathbf{C}^{(1)}$, along the identity inclusion maps for $\mathcal{G}$ and $\mathbf{C}$. In particular, the associated action groupoids are also related by these restrictions. More precisely, we have:

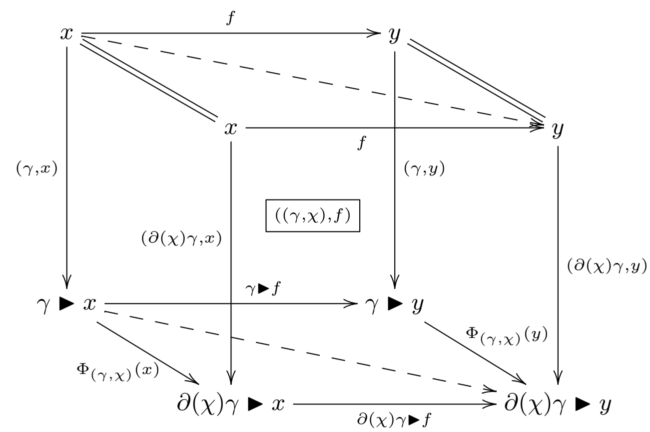
Corollary 4.3.2 *The identity-inclusion functor $\widetilde{e}$ factors into two inclusions:*

$$(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)} \subset \mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(0)} \subset (\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(1)} \tag{61}$$

where the first inclusion is given by the identity inclusion of $\mathbf{C}$, and the second by the identity inclusion of $\mathcal{G}$.

Proof: The fact that objects of $\mathcal{G}$ give endofunctors of $\mathbf{C}$ means that the morphism maps $\Phi^{(1)}_{\gamma}$ determine an action of $\mathcal{G}^{(0)}$ on $\mathbf{C}^{(1)}$. Thus, objects of $\mathcal{G}$ act on both $\mathbf{C}^{(0)}$ and $\mathbf{C}^{(1)}$. By functoriality, the inclusion of objects of $\mathbf{C}$ as identity morphisms in $\mathbf{C}^{(1)}$ is compatible with the action. So the subset $\mathbf{C}^{(0)} \subset \mathbf{C}^{(1)}$ is closed under the action of G. The associated transformation groupoid has as its objects the morphisms $f \in \mathbf{C}$, and as morphisms pairs (γ, f) which compose in the usual way. These correspond to the special squares $\boxed{\quad(\gamma, 1_H), f\quad}$. There is a full inclusion of transformation groupoids induced by the inclusion of their sets of objects $\mathbf{C}^{(0)} \subset \mathbf{C}^{(1)}$.

The group $\mathcal{G}^{(1)} \cong G \ltimes H$ also acts on $\mathbf{C}^{(1)}$. This action is given by $\hat{\Phi}$, and is related to the action of G on $\mathbf{C}^{(1)}$ by the identity inclusion map for $\mathcal{G}$, namely $G \subset G \ltimes H$. In particular, the action of $\mathcal{G}^{(1)}$ on $\mathbf{C}^{(1)}$ combines both natural transformations acting on objects, and functors acting on morphisms. To see this, note that the squares in the double category $\mathbf{C}/\!\!/\mathcal{G}$ occur by construction within a commuting cube which shows the naturality square associated to f, or equivalently the two ways of expressing $(\gamma,\chi) \blacktriangleright f$ in terms of Φ - see (24):



(62)

So the action of morphisms of $\mathcal{G}$ on morphisms of $\mathbf{C}$ involves natural transformations at the source and target objects, as well as functors acting on the morphisms themselves. The outside faces of the cube (62) are themselves special cases of squares in which one of these two parts is an identity. The case $\chi = 1_H$, so that $(\gamma,\chi) = Id_\gamma$ (for example, the front or rear faces of that cube) is precisely the second inclusion of the theorem.

Restricting further to the image of the unit inclusion for $\mathbf{C}$, we have the case $f = Id_x$. In particular, $\gamma \blacktriangleright Id_x = Id_{\gamma\blacktriangleright x}$ is just a consequence of the fact that the action is functorial. (For this reason, it is immediate if we think of the action in terms of $\hat{\Phi}$.)

The action by the subgroup G gives a transformation groupoid with fewer morphisms than the action by the big group $G \ltimes H$ - but the former is strictly a sub-groupoid (by a non-full inclusion) of the latter. $\Box$

Notice that this factorization of $\widetilde{e}$ is given by restricting to the image of the unit inclusions of $\mathbf{C}$ and then $\mathcal{G}$, and not the other way around. There is no well-defined action of $\mathcal{G}^{(1)}$ on $\mathbf{C}^{(0)}$. This can easily be seen, again by taking a special case of a square of the form (62) with $f = Id_x$. Since $\gamma \blacktriangleright x$ and $\partial(\chi)\gamma \blacktriangleright x$ need not be the same object, and even in the case they are, $\Phi_{(\gamma,\chi)}(x)$ need not be an identity, the image of $e : \mathbf{C}^{(0)} \to \mathbf{C}^{(1)}$ is not fixed by the action of $\mathcal{G}^{(1)}$ on $\mathbf{C}^{(1)}$.

4.3.2 Image of the Composition Functor

Next, consider the composition $\widetilde{\circ}$. It is somewhat more complicated than $\circ$, and derives from the 2-group structure of $\mathcal{G}$, just as composition does in transformation groupoids. A little example calculation with this composite will show how 2-group structure and an action must cohere.

Indeed, composition in $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(0)}$ is exactly the composition in the transformation groupoid $\mathbf{C}^{(0)}/\!\!/G$, by Corollary 4.3.1. Similarly, the composition in $(\widetilde{\mathbf{C}/\!\!/\mathcal{G}})^{(1)}$ is the composition in $\mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)}$, which is determined by the group multiplication in $\mathcal{G}^{(1)} \cong G \ltimes H$ (in the crossed-module representation), together with the action on $\mathbf{C}^{(1)}$. So we can write this as:

$$\boxed{(\gamma',\chi'),(\gamma,\chi)\blacktriangleright f}\,\widetilde{\circ}\,\boxed{(\gamma,\chi),f} = \boxed{(\gamma'\gamma,\chi'(\gamma'\rhd\chi)),f} \tag{63}$$

We know by construction and the above theorems that this makes a well-defined double category. It is, however, not given directly by the action Φ seen as a 2-functor. In that point of view, the action of morphisms of $\mathcal{G}$ on morphisms of $\mathbf{C}$ is a consequence of the naturality of $\Phi_{(\gamma,\chi)}$ for all $(\gamma,\chi) \in \mathcal{G}^{(1)}$.

Writing the target of this composite concretely, it is:

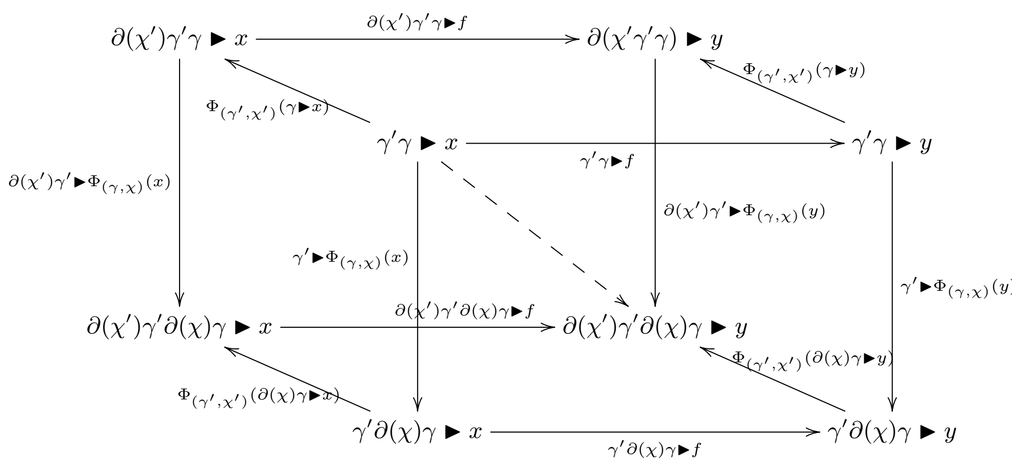
$$(\gamma'\gamma, \chi'(\gamma' \rhd \chi)) \blacktriangleright f \tag{64}$$

However, we have concrete expressions for $(\gamma, \chi) \blacktriangleright f$, shown in the cube (62), namely:

$$\begin{array}{rcccc} (\gamma,\chi) \blacktriangleright f & = & (\partial(\chi)\gamma \blacktriangleright f) & \circ & \Phi_{(\gamma,\chi)}(x) \\ & = & \Phi_{(\gamma,\chi)}(y) & \circ & (\gamma \blacktriangleright f) \end{array} \tag{65}$$

where composition is taken in $\mathbf{C}$. Recall that this is a consequence of the naturality of the bottom square of (62) when we apply $\Phi_{(\gamma,\chi)}$ at the morphism f.

Now, if we then take $(\gamma, \chi) \blacktriangleright f$ as the source of the new square $\boxed{(\gamma', \chi'), (\gamma, \chi) \blacktriangleright f}$, to find its target, we are applying $\Phi_{(\gamma',\chi')}$ at the morphism $(\gamma, \chi) \blacktriangleright f$. To compute this explicitly, note that each morphism in the commuting square at the bottom of (62) gives rise to another commuting square, which form four faces of a cube whose other two faces are the image of the original commuting square under the functors $\Phi_{\gamma'}$ and $\Phi_{\partial\chi'\gamma'}$ respectively:



(66)

We have omitted the images under $\Phi_{\gamma'}$ and $\Phi_{(\partial\chi')\gamma'}$ of the original diagonal $(\gamma, \chi) \blacktriangleright f$ for clarity. However, they are the diagonals on the front and back face of this cube. Together with the corresponding edges between these faces, they form a square whose diagonal is the target we want, just as before.

The target morphism of the square $\boxed{(\gamma', \chi'), (\gamma, \chi) \blacktriangleright f}$, hence also of our composite square, is therefore the dotted arrow across the diagonal of this cube.

The source object of this morphism is $\gamma'\gamma \blacktriangleright x$, and its target object is $\partial(\chi')\gamma'\partial(\chi)\gamma \blacktriangleright y$.

However, we already found that the composite square was $\boxed{(\gamma'\gamma, \chi'(\gamma' \rhd \chi)), f}$. Computing the target of this square directly from (62) shows that its target morphism begins at $\gamma'\gamma \blacktriangleright x$ and ends at $(\partial(\chi'(\gamma' \rhd \chi))\gamma'\gamma \blacktriangleright x$. A straightforward application of the crossed module axioms confirms that indeed:

$$\partial(\chi')\gamma'\partial(\chi)\gamma = \partial(\chi'(\gamma' \rhd \chi))\gamma'\gamma \tag{67}$$

as it must be.

Knowing that we are in a double category where composition is well-defined tells us more. Just as the naturality square at the bottom of (62) gave us two expressions for $(\gamma, \chi) \blacktriangleright f$, the cube of naturality squares (66) gives us a total of six expressions which equal the morphism

$$(\gamma'\gamma, \partial(\chi'(\gamma' \rhd \chi))) \blacktriangleright f \tag{68}$$

As before, each is a composite of morphisms in $\mathbf{C}$, most directly written as:

$$\begin{array}{rcccccc} & \Phi_{(\gamma',\chi')}(\partial(\chi)\gamma \blacktriangleright y) & \circ & (\gamma' \blacktriangleright \Phi_{(\gamma,\chi)}(y)) & \circ & (\gamma'\gamma \blacktriangleright f) \\ = & (\partial(\chi')\gamma') \blacktriangleright \Phi_{(\gamma,\chi)}(y)) & \circ & \Phi_{(\gamma',\chi')}(\gamma \blacktriangleright y) & \circ & (\gamma'\gamma \blacktriangleright f) \\ = & \Phi_{(\gamma',\chi')}(\partial(\chi)\gamma \blacktriangleright y) & \circ & (\gamma'\partial(\chi)\gamma \blacktriangleright f) & \circ & (\gamma' \blacktriangleright \Phi_{(\gamma,\chi)}(x)) \\ = & (\partial(\chi')\gamma'\partial(\chi)\gamma \blacktriangleright f) & \circ & \Phi_{(\gamma',\chi')}(\partial(\chi)\gamma \blacktriangleright x) & \circ & (\gamma' \blacktriangleright \Phi_{(\gamma,\chi)}(x)) \\ = & (\partial(\chi')\gamma' \blacktriangleright \Phi_{(\gamma,\chi)}(y)) & \circ & (\partial(\chi')\gamma'\gamma \blacktriangleright f) & \circ & \Phi_{(\gamma',\chi')}(\gamma \blacktriangleright x) \\ = & (\partial(\chi')\gamma'\partial(\chi)\gamma \blacktriangleright f) & \circ & (\partial(\chi')\gamma' \blacktriangleright \Phi_{(\gamma,\chi)}(x)) & \circ & \Phi_{(\gamma',\chi')}(\gamma \blacktriangleright x) \end{array} \tag{69}$$

The fact that these are all equal is a consequence of the fact that Φ is an action, hence every square in the cube is either a naturality square or the image of a naturality square under a functor, and therefore every square commutes. Some of the terms may also be written differently in light of (67) and similar applications of crossed-module axioms.

In general, a composite of n squares will give rise to many different equivalent expressions for the target morphism, in the same way. In particular, one can easily see by induction that one obtains an $(n+1)$ cube by repeatedly applying Φ at each step. Counting paths around such a cube, one finds $(n+1)!$ possible composites in $\mathbf{C}$ which amount to the same morphism.

Clearly, using the 2-group composition as in (63) to find the target (64) is more more direct, and since we know the transpose of $\mathbf{C}/\!\!/\mathcal{G}$ is a double category, we can use the 2-group action to find the target directly.

4.3.3 Horizontal and Vertical 2-Categories

Finally, we turn to the double category point of view on $\mathbf{C}/\!\!/\mathcal{G}$. When speaking of double categories, one often refers to various smaller structures which arise from specialization. We already know that the horizontal and vertical categories are, respectively, $\mathbf{C}$ itself, and $\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}$. Moreover, there are also two categories which share the same collection of morphisms (the squares of the double category), but have different sets of objects (namely, the horizontal and vertical morphisms). Again, we have already seen that these are, respectively, $\mathcal{G} \times \mathbf{C}$ and $\mathbf{C}^{(1)}/\!\!/\mathcal{G}^{(1)}$.

However, one also speaks of the horizontal and vertical 2-categories of a double category. These are the last aspect of the structure of $\mathbf{C}/\!\!/\mathcal{G}$ we will consider.

Definition 4.3.3 *The horizontal 2-category $H(\mathcal{D})$ of a double category $\mathcal{D}$ has the same objects and 1-morphisms as its horizontal category. The 2-morphisms consist only of the squares of $\mathcal{D}$ for which the vertical morphisms on the boundary are identities. The vertical 2-category $V(\mathcal{D})$ is defined similarly (i.e. it is the horizontal 2-category of $\widetilde{\mathcal{D}}$).*

It is straightforward to check that these form 2-categories, but since it is a standard fact, we will not verify all the details here. However, we will consider the horizontal and vertical 2-categories for $\mathbf{C}/\!\!/\mathcal{G}$. It should be clear from what we have just said that these will amount to $\mathbf{C}$ and $\mathbf{C}^{(0)}/\!\!/\mathcal{G}^{(0)}$, extended by adding certain 2-morphisms.

Theorem 4.3.4 *The horizontal 2-category $H(\mathbf{C}/\!\!/\mathcal{G})$ of the transformation double category associated to a 2-group action has the same objects and 1-morphisms as $\mathbf{C}$. Given objects $x, y \in \mathbf{C}$, and a pair of morphisms $f, f' : x \to y$, the 2-morphisms in $Hom(f, f')$ are labeled by $(1_G, \chi) : 1_G \Rightarrow 1_G$ such that*

$$f' = f \circ \Phi_{(1_G,\chi)}(x) \tag{70}$$

These 2-morphisms compose horizontally and vertically just as the endomorphisms of 1_G in $\mathcal{G}$.

Proof: The objects and 1-morphisms are those of the horizontal category, which is just $\mathbf{C}$ by definition.

The 2-morphisms correspond to squares of the form (54) for which the vertical morphisms are identities. These must be of the form $\boxed{(1_G, \chi), f}$, since the horizontal source of $\boxed{(\gamma, \chi), f}$ is $(\gamma, x) = Id_x$, and thus $\gamma = 1_G$. Moreover, since the target $(\partial(\chi)\gamma, y) =$

$(\partial(\chi), y) = Id_y$, we must also have that $\partial(\chi) = 1_G$. So only 2-endomorphisms of the identity of $\mathcal{G}$ enter as 2-morphisms.

The vertical target is then the horizontal morphism

$$\begin{aligned}(1_G, \chi) \blacktriangleright f &= (\partial(\chi) \blacktriangleright f) \circ \Phi_{(1_G,\chi)}(x) \\ &= f \circ \Phi_{(1_G,\chi)}(x)\end{aligned} \tag{71}$$

When $\gamma_i = 1_G$, the horizontal and vertical composition rules (55) and (56) just amount to multiplication in H: that is, by the composition of 2-endomorphisms of the identity. □

Intuitively, this says that the 2-morphisms with source f correspond exactly to the endomorphisms $(1_G, \chi)$ of 1_G in $\mathcal{G}$. In the language of crossed modules, they correspond to $\chi \in ker(\partial)$. The target is found by precomposing f with the endomorphisms of x these give from the natural transformations $\Phi_{(1_G,\chi)}$. Clearly, these labels χ for 2-morphisms sourced at f will be the same for all f. Depending on the precise details of the action Φ, all such 2-morphisms might be endomorphisms of f itself, or the targets might all be different from f and from each other. This is the type of information about the action Φ which is captured by $H(\mathbf{C} /\!\!/ \mathcal{G})$.

Theorem 4.3.5 *The vertical 2-category $V(\mathbf{C} /\!\!/ \mathcal{G})$ of the transformation double category associated to a 2-group action has the same objects and 1-morphisms as $\mathbf{C}^{(0)} /\!\!/ \mathcal{G}^{(0)}$. Given an object $x \in \mathbf{C}$ and $\gamma, \gamma' \in G$ such that $\gamma \blacktriangleright x = \gamma' \blacktriangleright x$, there will be a 2-morphism from (γ, x) to (γ', x) for each χ such that $\Phi_{(\gamma,\chi)}(x) = Id_{\gamma \blacktriangleright x}$ and $\gamma' = \partial(\chi)\gamma$.*

Proof: The objects and 1-morphisms are those of the vertical category, which is $\mathbf{C}^{(0)} /\!\!/ \mathcal{G}^{(0)}$ by Theorem 4.2.3.

The 2-morphisms correspond to squares of the form (54) in which $x = y$ and the horizontal morphisms on the boundary are both identities. Since the vertical source is $f = Id_x$, they are squares of the form $\boxed{(\gamma, \chi), Id_x}$.

Moreover, the vertical target must also be the identity. Now, to begin with, this must mean that $\gamma \blacktriangleright x = \gamma' \blacktriangleright x$ (taking $\gamma' = \partial(\chi)\gamma$), as stated in the theorem. This vertical target is the horizontal morphism

$$\begin{aligned}(\gamma, \chi) \blacktriangleright f &= (\gamma' \blacktriangleright Id_x) \circ \Phi_{(\gamma,\chi)}(x) \\ &= Id_{\gamma' \blacktriangleright x} \circ \Phi_{(\gamma,\chi)}(x)\end{aligned} \tag{72}$$

The first part of this composite is an identity because γ' acts on $f = Id_x$ by the functor $\Phi_{\gamma'}$, and functors respect identities. Thus, this condition implies that not only are $\gamma \blacktriangleright x$ and $\gamma' \blacktriangleright x$ equal, but indeed the natural transformation $\Phi_{(\gamma,\chi)}$ intertwines them by the identity on $\gamma \blacktriangleright x$.

The compositions again follow (55) and (56). □

Intuitively, the transformation groupoid $\mathbf{C}^{(0)} /\!\!/ \mathcal{G}^{(0)}$ displays the object part of the action $\hat{\Phi}$. The morphism part can potentially appear here also, relating local symmetry transformations which carry x to the same place. However, as we have seen, such a relation need not exist if $\gamma \neq \gamma'$, since even though $\gamma \blacktriangleright x = \gamma' \blacktriangleright x$, there might be no natural transformation intertwining them, or none intertwining them with identities. Conversely, there may be more than one such natural transformation. This is the information about the full action Φ captured by $V(\mathbf{C} /\!\!/ \mathcal{G})$ which the mere groupoid $\mathbf{C}^{(0)} /\!\!/ \mathcal{G}^{(0)}$ cannot see.

We would next like to explicitly describe the above, to examine the structure of the transformation double groupoid $\mathcal{G} /\!\!/ \mathcal{G}$, associated to our example of the adjoint action of a 2-group $\mathcal{G}$ on itself.

4.4 The Transformation Double Category for the Adjoint Action

As in all transformation double groupoids, the horizontal category of $\mathcal{G}/\!\!/\mathcal{G}$ is just the same as $\mathcal{G}$, seen as a monoidal category. The vertical morphisms and squares will always be labelled by pairs of, respectively, objects and morphisms from $\mathcal{G}$.

Summing up the above, we have:

Corollary 4.4.1 *The transformation double groupoid $\mathcal{G}/\!\!/\mathcal{G}$ for the adjoint action of a 2-group classified by the crossed module $(G, H, \triangleright, \partial)$ has:*

- *Objects: labelled by $g \in G$*
- *Horizontal Category: the underlying category of $\mathcal{G}$*
- *Vertical Category: same as $G/\!\!/G$, the transformation groupoid for the adjoint action of G*
- *Squares: labelled by pairs $((\gamma_1, \chi_1), (g_1, \eta_1)) \in (G\times H)\times(G\times H)$, denoted by $\boxed{(\gamma_1, \chi_1), (g_1, \eta_1)}$. As in (54), the horizontal and vertical source and target of this square are displayed as follows:*

$$\begin{array}{ccc} g_1 & \xrightarrow{\eta_1} & g_2 \\ \Phi_{\gamma_1}\big\downarrow & \boxed{(\gamma_1,\chi_1),(g_1,\eta_1)} & \big\downarrow\Phi_{\gamma_2} \\ g_3 & \xrightarrow[\eta_2]{} & g_4 \end{array} \tag{73}$$

where $\gamma_2 = \partial(\chi_1)\gamma_1$. Here η_2 is determined by:

$$\begin{array}{c} g_3 \\ \hline \eta_2 \\ \hline g_4 \end{array} = \begin{array}{c|c|c} \gamma_1 & g_1 & \gamma_1^{-1} \\ \hline \chi_1 & \eta_1 & \chi_1^{-h} \\ \hline \gamma_2 & g_2 & \gamma_2^{-1} \end{array}. \tag{74}$$

- *Horizontal composition of squares is given by composition in H:*

$$\boxed{(\gamma_2, \chi_2), (g_2, \eta_2)} \circ_h \boxed{(\gamma_1, \chi_1), (g_1, \eta_1)} = \boxed{(\gamma_1, \chi_2\chi_1), (g_1, \eta_2\eta_1)} \tag{75}$$

where $g_2 = \partial(\eta_1)g_1$, and vertical composition of squares is given by:

$$\boxed{(\gamma_1, \chi_1), (\gamma_3, \chi_2) \blacktriangleright (g_1, \eta_2)} \circ_v \boxed{(\gamma_3, \chi_2), (g_1, \eta_2)} = \boxed{(\gamma_1\gamma_3, \chi_1(\gamma_1 \triangleright \chi_2)), (g_1, \eta_2)} \tag{76}$$

which makes them into the transformation groupoid $(G \ltimes H)/\!\!/(G \ltimes H)$,

Proof: This is a direct consequence of the structure of $\mathbf{C}/\!\!/\mathcal{G}$ laid out in Definition 4.2.4, with the structure of $\mathcal{G}$ as a categorical group given explicitly for $\mathbf{C}$. $\square$

Now we will find the horizontal and vertical 2-categories, $H(\mathcal{G}/\!\!/\mathcal{G})$ and $V(\mathcal{G}/\!\!/\mathcal{G})$, of the double category associated to the adjoint action of $\mathcal{G}$

Corollary 4.4.2 *If $\mathcal{G}$ is a 2-group classified by the crossed module $(G, H, \triangleright, \partial)$, then the horizontal 2-category $H(\mathcal{G}/\!\!/\mathcal{G})$ associated to the adjoint action of $\mathcal{G}$ has:*

- ***Objects**: $g \in G$*
- ***Morphisms**: $(g, \eta) \in G \ltimes H$ with source, target, and composition as in $\mathcal{G}$*

- ***2-Morphisms**: $((g,\eta),\chi) \in (G \ltimes H) \times \ker(\partial)$, whose source is (g,η), and whose target is*

$$(g,\eta') = (g, \eta \cdot (\chi g \rhd \chi^{-1})) \tag{77}$$

Composition is given by multiplication in the subgroup $\ker(\partial) \subset H$.

Proof: Taking the expression (70) for the target of a 2-morphism, in the case of the adjoint action we use the fact that $\Phi_{(1_G,\chi)}(g)$ is defined in (36). Substituting this gives the target specified in (77). Composition as multiplication in H also follows from Theorem 4.3.4. □

This tells us how to extend $\mathcal{G}$ to a 2-category to reflect part of the symmetry that arises from its action on itself. The vertical 2-category, by contrast, can be better seen as an extension of $G/\!\!/G$, the transformation groupoid for the actions of just the group of objects on itself. In particular, we have the following.

Corollary 4.4.3 *If $\mathcal{G}$ is a 2-group classified by the crossed module $(G, H, \rhd, \partial)$, then the vertical 2-category $V(\mathcal{G}/\!\!/\mathcal{G})$ associated to the adjoint action of $\mathcal{G}$ has:*

- ***Objects**: $g \in G$*
- ***Morphisms**: $(\gamma, g) \in G \times G$ with source, target, and composition maps as in the transformation groupoid for the adjoint action of G on itself*
- ***2-Morphisms**: There will be a 2-morphism from (γ, g) to (γ', g), for each $\chi \in H$ such that*

$$(\gamma g \gamma^{-1}) \rhd \chi^{-1} = \chi^{-1} \tag{78}$$

(That is, χ^{-1} is a fixed point for $\gamma g \gamma^{-1}$ under the action $\rhd$).

Proof: In Theorem 4.3.5, we established when there is a 2-morphism in $V(\mathbf{C}/\!\!/\mathcal{G})$. In our case, taking $x = g$, we find there is such a 2-morphism for any χ such that $\Phi_{(\gamma,\chi)}(g) = Id_{\gamma g \gamma^{-1}}$.

But we have the expression (36) for the maps defining the natural transformation $\Phi_{(\gamma,\chi)}$. So this condition says that

$$\chi(\gamma g \gamma^{-1}) \rhd \chi^{-1} = 1_H \tag{79}$$

or in other words, that

$$(\gamma g \gamma^{-1}) \rhd \chi^{-1} = \chi^{-1} \tag{80}$$

This is exactly the statement above.

As usual, composition follows (55) and (56). □

Intuitively, this tells us that the information about the action Φ captured by the 2-morphisms relating two different symmetry relations γ and γ' taking g to g is fundamentally information about the crossed module structure, and the action $\rhd$ of G on H.

We have included these calculations of the horizontal and vertical 2-categories, partly because these particular slices of the double category $\mathcal{G}/\!\!/\mathcal{G}$ (or $\mathbf{C}/\!\!/\mathcal{G}$ generally) are interesting in their own right. But we also include them to illustrate that many squares in the double category do not appear in either the horizontal or vertical 2-categories. Each is missing a great deal of information about the action Φ which only $\mathcal{G}/\!\!/\mathcal{G}$ (or $\mathbf{C}/\!\!/\mathcal{G}$) contains.

A Weak Actions

In the preceding sections, we have spoken only of strict 2-groups, and strict actions of such 2-groups. This is justifiable because of two "strictification" results, which imply that weakening either of these assumptions does not add anything essentially new. Even so, we will discuss the point here, since there may be applications in which the most natural construction gives weak actions, or actions of weak 2-groups, or both. It will then be useful to understand how these examples relate to the constructions given previously.

A.1 Preliminaries

A "weak" 2-group is a bicategory whose 1-morphisms are weakly invertible (that is, invertible up to isomorphism), and 2-morphisms are all invertible. (Equivalently, one may see this as a monoidal category where objects have weak monoidal inverses and morphisms have strict inverses.) Weak 2-groups, and a mild refinement called "coherent" 2-groups, are discussed at some length by Baez and Lauda [1].

A standard strictification result given by Power [20] (essentially the same as MacLane's coherence result for monoidal categories [13]) states that any bicategory is equivalent to a strict 2-category, that is a bicategory in which the structural 2-morphisms all happen to be identities. Thus, any weak 2-group $\mathcal{W}$ is equivalent as a bicategory to a strict one $\mathcal{G}$. We have defined an action of a 2-group $\mathcal{G}$ on a category $\mathbf{C}$ as a strict functor of 2-categories $\Phi : \mathcal{G} \to \mathbf{Cat}$, with the property that $\Phi(\star) = \mathbf{C}$. Composing with the map $\mathcal{G} \to \mathcal{W}$ from this equivalence, any action of $\mathcal{W}$ induces an action of $\mathcal{G}$. Using the map $\mathcal{W} \to \mathcal{G}$, an action of $\mathcal{W}$ can be recovered from an action of $\mathcal{G}$. If we start with an action of $\mathcal{W}$ and do both of these processes, we get an action of $\mathcal{W}$ which is related to the original one by means of the structure 2-morphisms for the equivalence. For this reason, it is enough to consider actions of strict 2-groups.

The situation for strict and weak actions is slightly different. We can simply weaken the definition of action:

Definition A.1.1 *A weak action of a 2-group $\mathcal{G}$ on a category $\mathbf{C}$ is a pseudofunctor $\Phi : \mathcal{G} \to \mathbf{Cat}$ with $\Phi(\star) = \mathbf{C}$.*

There is another well-known strictification result, again a consequence of results in [20] (though see also discussion in [19]), which states that any pseudofunctor into **Cat** is pseudonaturally equivalent to a strict functor. This is why we have been satisfied to consider strict actions in the present paper. That is, the strictification result is a property specific to the fact that the target of Φ happens to be **Cat**.

This is not a real limitation on our construction of the transformation double groupoid, which already uses the fact that a 2-group may be seen as a categorical group. That is, $\mathcal{G}$ can be seen as $\mathbf{G}$, a group object in **Cat**, so that the category of morphisms, $\mathbf{G} \times \mathbf{C}$ is a well-defined object in **Cat**. Other steps in the construction are well-defined beacuse **Cat** has all necessary limits, for instance, to define an object of composable pairs. This construction of a transformation structure makes sense exactly as a construction internal to **Cat**. Moreover, actions on a category are precisely the context of the application to geometry which we had in mind, and which we will develop in a subsequent article.

Since we have been interested in actions on categories, we have been content to use the result above, and limit our attention to strict actions. On the other hand, it is perfectly natural to define an action of $\mathcal{G}$ on objects in other 2-categories, where no such strictification result need hold. Moreover, it is useful to consider the weak case in order to more fully understand the special case of strict actions. This is true of bicategories, and hence non-strict 2-groups, as well, but we do not wish to stray too far afield from our central point. In this appendix, we simply consider how the use of a weak action of a strict 2-group would affect the previous discussion. That is, we will give up the strictification of *1-morphisms* in **Bicat**, but not of objects.

A.2 Weak Action of a 2-Group on a Category

In Definition A.1.1, we defined a weak action as a kind of pseudofunctor. We recall here the rudiments of a definition of pseudofunctor. We omit the coherence diagrams, though see section 3 of [10], or discussion on the n-categories lab [19]. We will state the definition for bicategories, then remark on the strict case.

Definition A.2.1 *If $\mathcal{X}$ and $\mathcal{Y}$ are bicategories, a pseudofunctor $P : \mathcal{X} \to \mathcal{Y}$ consists of the following data:*

- *A function* $P : \mathcal{X}^{(0)} \to \mathcal{Y}^{(0)}$ *taking* $x \mapsto P(x)$
- *A collection of functors between Hom-categories, for all* $x, x' \in \mathcal{X}^{(0)}$*:*

$$P_{x,x'} : \mathcal{X}(x, x') \to \mathcal{Y}(P(x), P(x')) \tag{81}$$

- *For each* $x \in \mathcal{X}$*, an invertible 2-cell*

$$I_x : Id_{P(x)} \Rightarrow P_{x,x}(Id_x) \tag{82}$$

- *For all* $x, x', x'' \in \mathcal{X}$*, there is a natural transformation*

$$p_{x,x',x''} : \mathcal{Y}(P(x'), P(x'')) \times \mathcal{Y}(P(x), P(x')) \Rightarrow \mathcal{Y}(P(x), P(x'')) \tag{83}$$

There are coherence diagrams for the relation between I_x *and the left and right unitors for* x*, and also for the relation between* $p_{x,x',x''}$ *and the associator.*

We have not given the coherence diagrams for unitors and associator in detail here, but in each case they mean roughly that one can either use the 2-cells from $\mathcal{X}$ and then apply P, or else first apply P and then use the coherence cells from $\mathcal{Y}$, and the results will be equal. In the strict case, these conditions become simpler, since the unitor and associator for both $\mathcal{X}$ and $\mathcal{Y}$ are identities.

If $\mathcal{Y} = \mathbf{Cat}$, and $\mathcal{X}$ is a 2-group $\mathcal{G}$, there is only one object $x = \star$ in $\mathcal{G}$, so these conditions simplify further. Then the pseudofunctor P becomes an action of the sort we previously called Φ, but with weakening. This may be easiest to describe as a weak action of a categorical group action on a category $\mathbf{C}$. That is, it will be a 2-functor Φ from $\mathcal{G}$ into $Aut(\mathbf{C})$, and we have:

- $\mathbf{C} = P(\star)$
- Φ is the unique functor $P_{\star,\star}$
- composition $\circ$ in $\mathcal{G}$ is weakly preserved, with two structure maps:
- the unique unit $I_\star$, written I, is a new structure
- we have a new natural transformation $\phi = p_{\star,\star,\star}$

The unit conditions in this simpler case say that, for our new map I, which relates $\Phi(Id_\star)$ to $Id_{\mathbf{C}} \in Aut(\mathbf{C})$,

$$\Phi(Id_\star) \circ \Phi(\gamma) \underset{\Phi(Id_\star,\gamma)}{\overset{I \circ \Phi(\gamma)}{\rightleftarrows}} \Phi(\gamma) \tag{84}$$

commutes, and similarly for the right-unit property. That is, the conditions are that

$$(I \circ Id_\gamma) = \Phi^{-1}_{Id_\star,\gamma} \tag{85}$$

and

$$(Id_\gamma \circ I) = \Phi^{-1}_{\gamma,Id_\star} \tag{86}$$

Thus, I carries no information not contained in the "compositor"

$$\phi : \hat{\Phi} \otimes \hat{\Phi} \Rightarrow \hat{\Phi} \tag{87}$$

This compositor is a natural transformation between two monoidal functors, so in particular, for each object $(\gamma_1, \gamma_2) \in G \times G$, we get maps:

$$\phi_{(\gamma_1,\gamma_2)} : \Phi(\gamma_1) \circ \Phi(\gamma_2) \to \Phi(\gamma_1 \circ \gamma_2) \tag{88}$$

The coherence condition for the "compositor" ϕ with identity associators amounts to the statement that the diagram

$$\begin{array}{ccc} \Phi \otimes \Phi \otimes \Phi & \xrightarrow{Id_\Phi \circ \phi} & \Phi \otimes \Phi \\ {\scriptstyle \phi \circ Id_\Phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ \Phi \otimes \Phi & \xrightarrow[\phi]{} & \Phi \end{array} \tag{89}$$

commutes.

Now we would like to see a weak equivalent of the commuting square defining an action, giving a weak analog of the functor $\hat{\Phi}$.

A.3 Action As a Weakly Commuting Diagram

Rewriting the above, we have that a weak action Φ corresponds to a functor

$$\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C} \tag{90}$$

equipped with a natural transformation

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\mathcal{G} \times \hat{\Phi}} & \mathcal{G} \times \mathbf{C} \\ {\scriptstyle \otimes \times \mathbf{C}}\downarrow & \Swarrow_{\hat{\phi}} & \downarrow{\scriptstyle \hat{\Phi}} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[\hat{\Phi}]{} & \mathbf{C} \end{array} \tag{91}$$

satisfying a coherence condition. Note that we are using the notation where the name of an object denotes the identity morphism on that object, and the name of a morphism denotes its identity 2-morphism. The horizontal composition $\circ$ also therefore denotes whiskering where appropriate. As before, the monoidal product $\otimes$ represents the composition in the 2-group, since we think of it here as a group object in **Cat**.

The isomorphism ϕ relates multiplication in the 2-group to the action functor Φ. Similarly, the coherence conditions to look for should relate ϕ with the associator. This will involve ways of using this 2-isomorphism to pass between functors from $\mathcal{G}^3 \times \mathbf{C}$ to $\mathbf{C}$. They correspond to the orders in which we can apply the monoidal operation $\otimes$ (the multiplication for the categorical group), and the action Φ. There are six of these, and using the "action" notation $f \blacktriangleright x$ to mean $\Phi(f)(x)$ (or equivalently, $\hat{\Phi}(f, x)$), we can understand these six functors by their effect on objects:

$$\hat{\Phi} \circ (\mathcal{G} \times \hat{\Phi}) \circ (\mathcal{G} \times \mathcal{G} \times \hat{\Phi})(f, g, h, x) = f \blacktriangleright (g \blacktriangleright (h \blacktriangleright x)) \tag{92}$$

$$\hat{\Phi} \circ (\otimes \times \mathbf{C}) \circ (\mathcal{G} \times \mathcal{G} \times \hat{\Phi})(f, g, h, x) = fg \blacktriangleright (h \blacktriangleright x) \tag{93}$$

$$\hat{\Phi} \circ (\mathcal{G} \times \hat{\Phi}) \circ (\mathcal{G} \times \otimes \times \mathbf{C})(f, g, h, x) = f \blacktriangleright (gh \blacktriangleright x) \tag{94}$$

$$\hat{\Phi} \circ (\otimes \times \mathbf{C}) \circ (\mathcal{G} \times \otimes \times \mathbf{C})(f, g, h, x) = f(gh) \blacktriangleright x \tag{95}$$

$$\hat{\Phi} \circ (\mathcal{G} \times \hat{\Phi}) \circ (\otimes \times \mathcal{G} \times \mathbf{C})(f, g, h, x) = fg \blacktriangleright (h \blacktriangleright x) \tag{96}$$

$$\hat{\Phi} \circ (\otimes \times \mathbf{C}) \circ (\otimes \times \mathcal{G} \times \mathbf{C})(f, g, h, x) = (fg)h \blacktriangleright x \tag{97}$$

Since Φ and $\otimes$ are functors, similar expressions hold for morphisms as well.

There are two expressions giving $fg \blacktriangleright (h \blacktriangleright x)$, which amount to finding the composite fg or the image $h \blacktriangleright x$ in two different orders, so these functors are necessarily equal. Thus, we really have five distinct functors here.

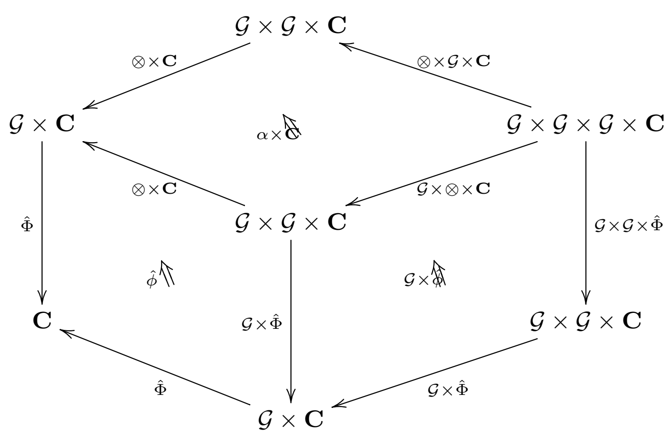
For strict actions of strict 2-groups, just as for ordinary group actions, these functors are all equal. For weak actions, we have the natural transformation $\hat{\phi}$ (and the associator) as "rewrite rules". In the weak case, we have the natural transformation $\hat{\phi}$, with components

$$\hat{\phi}_{(\gamma_1,\gamma_2,x)} = \phi_{(\gamma_1,\gamma_2)}(x) : \Phi(\gamma_1) \circ \Phi(\gamma_2)(x) \to \Phi(\gamma_1 \circ \gamma_2)(x) \tag{98}$$

All of the relations between these five functors involve $\hat{\phi}$ except for one. The associator of $\mathcal{G}$, which we denote α. Then we have, $\alpha \times Id_{\hat{\Phi}} : f(gh) \blacktriangleright x = (fg)h \blacktriangleright x$.

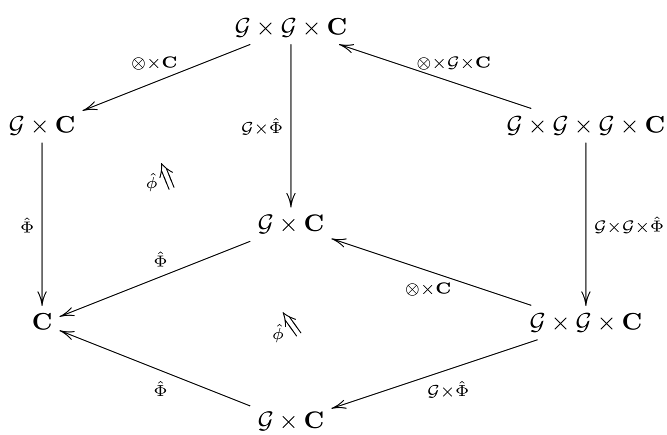
The components of these natural transformations give specific isomorphisms between each of these expressions, given any choice (f, g, h, x). It is perhaps easiest to understand the coherence condition if we think of the case where $\mathbf{C} = \mathcal{G}$, where $\mathcal{G}$ acts on itself, not by the adjoint action we considered earlier, but simply by left multiplication. Then $\hat{\Phi} = \otimes$, and $\hat{\phi} = \alpha$. In this case, the coherence condition for the action will just give the pentagon identity for the monoidal product $\otimes$.

However, if we write $\hat{\phi}$ and α as natural transformations that fill a "commutative" square, we can see that this same coherence condition as a diagram of these 2-cells without making any specific choices of object $(f, g, h, x) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathbf{C}$. The coherence condition for the cell (91) derives from (89). It says that the following diagram of 2-cells



(99)

is equal to the diagram



(100)

These are the front and back faces of a cube, whose edges give functors whose composites are the six paths from $\mathcal{G}^3 \times \mathbf{C}$ to $\mathbf{C}$ noted previously. Each such composite traverses each available direction in this cube, in some order. The faces of the cubes are natural transformations. Four of the faces are compositor cells. One is an associator α for $\mathcal{G}$ together with the identity on $\mathbf{C}$. (In the strict case, this cell is an identity). The identity cell in the second diagram comes from the fact that the result of applying $\hat{\Phi}$ and $\otimes$ to disjoint sets of arguments is independent of the order in which we apply them. (In general, we might assume these are only pseudonaturally equivalent, but we will ignore this cell.)

This relation says that two 2-morphisms in **Cat** are equal, and amount to the same condition as (89). There are two ways to use the compositor to get from the "action first" functor

$$\hat{\Phi} \circ (\mathcal{G} \times \hat{\Phi}) \circ (\mathcal{G} \times \mathcal{G} \times \hat{\Phi}) \tag{101}$$

to the "product first" functor

$$\hat{\Phi} \circ (\otimes \times \mathbf{C}) \circ (\otimes \times \mathcal{G} \times \mathbf{C}) \tag{102}$$

The relation says that these two ways are equal:

$$(\hat{\Phi} \circ (\alpha \times \mathbf{C})) \cdot (\hat{\phi} \circ \mathcal{G} \times \otimes \times \mathbf{C}) \cdot (\hat{\Phi} \circ (\mathcal{G} \times \hat{\phi})) = (\hat{\phi} \circ (\otimes \times \mathcal{G} \times \mathbf{C})) \cdot (\hat{\phi} \circ (\mathcal{G} \times \mathcal{G} \times \hat{\Phi})) \tag{103}$$

This commuting cube says that two natural tranformations are equal: the front face is a composite of three squares involving two $\hat{\phi}$ and one α. The back face has only two nontrivial squares, both made from $\hat{\phi}$. This is exactly the form of the pentagon identity, and is literally the same identity in the special case where $\hat{\phi} = \alpha$. The vertices of the pentagon are the five potentially distinct paths from the source object in this cube to the target object in the cube (since two are necessarily equal). Its edges are the various composites and whiskerings of these squares which relate the paths.

A.4 Weakening the Transformation Double Groupoid Construction

It is natural to ask about weakening the construction we gave for the transformation double groupoid. We described the internal groupoid in **Cat** associated to a strict action, by taking $\mathcal{G} \times \mathbf{C}$ as the category of morphisms, and the functors $\pi_{\mathbf{C}}$ and $\hat{\Phi}$ as source and target. The same construction would apply internally to any category where a group object acts on another object, and in which pullbacks make sense. However, since **Cat** is a bicategory, there are weak versions of most of the constructions given.

Consider the category of composable pairs of morphisms. In the strict case, this is found by a pullback:

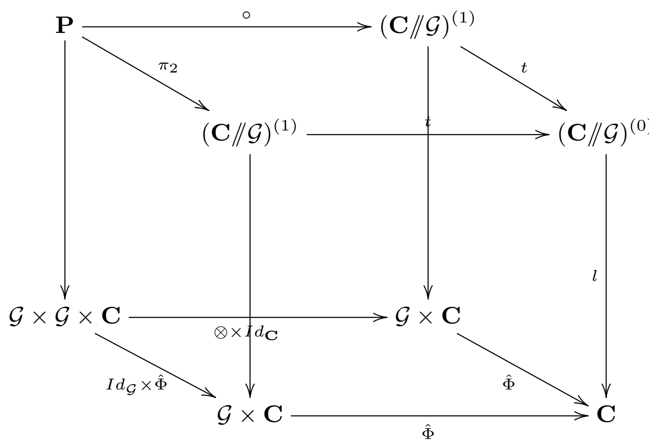
$$\begin{array}{ccc} P & \longrightarrow & \mathcal{G} \times \mathbf{C} \\ \downarrow & & \downarrow{\scriptstyle t=\hat{\Phi}} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[s=\pi_{\mathbf{C}}]{} & \mathbf{C} \end{array} \tag{104}$$

The arrows from P select the first and second elements of a pair.

Now, it happens that $P \cong \mathcal{G} \times \mathcal{G} \times \mathbf{C}$, and we may see the composition of pairs, together with this $\circ : P \to \mathcal{G} \times \mathbf{C}$, as forming a pullback diagram:

$$\begin{array}{ccc} P & \xrightarrow{\circ} & \mathcal{G} \times \mathbf{C} \\ \downarrow & & \| \\ \mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow[m \times Id_{\mathbf{C}}]{} & \mathcal{G} \times \mathbf{C} \end{array} \tag{105}$$

It may be useful to conceptually distinguish the transformation double groupoid itself, and its structure maps, from the objects in (91). To begin with, we can suppose that the category of objects is some category merely equivalent to $\mathbf{C}$ by a "labelling" functor $l : (\mathbf{C}/\!\!/\mathcal{G})^{(0)} \to \mathbf{C}$. The definition of an action, and construction by pullbacks of its associated transformation structure, can be summed up in this cubical diagram in **Cat**:



(106)

(We have omitted the third evident map out of $\mathbf{P}$, namely the first projection map onto $\mathbf{C}/\!\!/\mathcal{G}$.)

The top describes some of the structure maps of the transformation groupoid in **Cat**. The category of morphisms $(\mathbf{C}/\!\!/\mathcal{G})^{(1)}$ is given, on the front and right faces of the cube, by copies of the pullback square (104). Then the category of composable pairs $\mathbf{P}$ and the composition operation is again formed by a pullback of the source and target maps (the top face of the cube). If the bottom face, which is (91), only commutes up to $\hat{\phi}$, then the top face is only a weak pullback.

Since the top face of the cube is necessarily a weak pullback, we may as well suppose *a priori* that the construction of the category of morphisms is also given by a weak pullback. However, since l is assumed to be an equivalence, the weak pullback along l is itself equivalent to the strict pullback. In particular, the "labelling" maps from $(\mathbf{C}/\!\!/\mathcal{G})^{(1)}$ to $\mathcal{G} \times \mathbf{C}$, and from $\mathbf{P}$ to $\mathcal{G} \times \mathcal{G} \times \mathbf{C}$ are also equivalences.

Notice that even if we assume the strict case, and suppose the labelling maps from the objects and morphisms are the identity, the labelling map from $\mathbf{P}$ would not be an identity if we take $\mathbf{P}$ to literally be the category whose objects and morphisms are composable pairs of objects and morphisms of $(\mathbf{C}/\!\!/\mathcal{G})^{(1)}$. So one may as well suppose that none of the vertical arrows are literal identities. However, no matter what l and other labelling maps are chosen, (104) is always equivalent to the strict pullback.

Remark A.4.1 The discussion above shows that there is no real gain in generality by considering a weak construction of the transformation double groupoid. The result will always be equivalent to the construction using strict pullbacks (104), though the pullback giving $\mathbf{P}$ may necessarily be weak if the action is weak.

This, combined with the strictification results mentioned previously, means that, even for weak actions and weak 2-groups, the double groupoid $\mathbf{C}/\!\!/\mathcal{G}$ can always be understood up to equivalence by taking our simple construction, applied to a strict action of a strict 2-group, as in Section 3.

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