

Representations of derived A -infinity algebras

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ABSTRACT. The notion of a derived A -infinity algebra arose in the work of Sagave as a natural generalisation of the classical A -infinity algebra, relevant to the case where one works over a commutative ring rather than a field. We develop some of the basic operadic theory of derived A -infinity algebras, building on work of Livernet-Roitzeim-Whitehouse. In particular, we study the coalgebras over the Koszul dual cooperad of the operad dAs , and provide a simple description of these. We study representations of derived A -infinity algebras and explain how these are a two-sided version of Sagave's modules over derived A -infinity algebras. We also give a new explicit example of a derived A -infinity algebra.

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1. Introduction

Strongly homotopy associative algebras, also known as A_∞ -algebras, were invented at the beginning of the sixties by Stasheff as a tool in the study of group-like topological spaces. Since then it has become clear that A_∞ -structures are relevant in algebra, geometry and mathematical physics. In particular, Kadeishvili used the existence of A_∞ -structures in order to classify differential graded algebras over a

field up to quasi-isomorphism [Kad80]. When the base field is replaced by a commutative ring, however, Kadeishvili's result no longer holds. If the homology of the differential graded algebra is not projective over the ground ring there need no longer be a minimal A_∞ -algebra quasi-isomorphic to the given differential graded algebra.

In order to bypass the projectivity assumptions necessary for Kadeishvili's result, Sagave developed the notion of derived A_∞ -algebras [Sag10]. While classical A_∞ -algebras are graded algebras, derived A_∞ -algebras are bigraded algebras. Sagave establishes a notion of minimal model for differential graded algebras (dgas) whose homology is not necessarily projective by showing that the structure of a derived A_∞ -algebra arises on some projective resolution of the homology of a differential graded algebra.

In this paper, we continue the work of [LRW13], developing the description of these structures using operads. The operads we use are non-symmetric operads in the category BiCompl_0 of bicomplexes with zero horizontal differential. We have an operad dAs in this category encoding bidgas, which are simply monoids in bicomplexes. It is shown in [LRW13] that derived A_∞ -algebras are precisely algebras over the operad

$$dA_\infty = (dAs)_\infty = \Omega((dAs)^i).$$

Here $(dAs)^i$ is the Koszul dual cooperad of the operad dAs , and Ω denotes the cobar construction. In this manner, we view a derived A_∞ -algebra as the infinity version of a bidga, just as an A_∞ -algebra is the infinity version of a dga.

We further investigate the operad dAs , in particular studying $(dAs)^i$ -coalgebras. The structure of an As^1 -coalgebra is well-known to be equivalent, via a suspension, to that of a usual coassociative coalgebra. Analogously, $(dAs)^i$ -coalgebras are equivalent, via a suspension in the vertical direction, to coassociative coalgebras which are equipped with an extra piece of structure.

A substantial part of this paper is concerned with representations of derived A_∞ -algebras. Besides being an important part of the basic operadic theory of these algebras, we will use this theory in subsequent work to develop the Hochschild cohomology of derived A_∞ -algebras with coefficients. In section 4, we give a general result expressing a representation of a \mathcal{P}_∞ -algebra for any Koszul operad \mathcal{P} in terms of a square-zero coderivation. Then we work this out explicitly for the derived A_∞ case. We explain how this relates to Sagave's derived A_∞ -modules: the operadic notion of representation yields a two-sided version of Sagave's modules.

Finally, we present a new, explicit example of a derived A_∞ -algebra. The construction is based on some examples of A_∞ -algebras due to Allocca and Lada [AL10].

The paper is organized as follows. In section 2 we begin with a brief review of previous work on derived A_∞ -algebras and establish our notation and conventions. Sections 3 and 4 cover the material on $(dAs)^i$ -coalgebras, coderivations and representations. Section 5 presents our new example. A brief appendix establishes the relationship between two standard sign conventions and gives details of cooperadic suspension in our bigraded setting.

2. Review of derived A_∞ -algebras

In this section we establish our notation and conventions. We review Sagave's definition of derived A_∞ -algebras from [Sag10] and we explain the operadic approach of [LRW13].

2.1. Derived A_∞ -algebras. Let \mathbf{k} denote a commutative ring unless otherwise stated. We start by considering (\mathbb{Z}, \mathbb{Z}) -bigraded \mathbf{k} -modules

$$A = \bigoplus_{i \in \mathbb{Z}, j \in \mathbb{Z}} A_i^j.$$

We will use the following grading conventions. An element in A_i^j is said to be of bidegree (i, j) . We call i the horizontal degree and j the vertical degree. We have two suspensions:

$$(sA)_i^j = A_i^{j+1} \quad \text{and} \quad (SA)_i^j = A_{i+1}^j.$$

A morphism of bidegree (u, v) maps A_i^j to A_{i+u}^{j+v} , hence is a map of bidegree $(0, 0)$

$$s^{-v} S^{-u} A \rightarrow A.$$

We remark that this is a different convention to that adopted in [LRW13]. The difference is a matter of changing the first grading from homological to cohomological conventions.

Note also that our objects are graded over (\mathbb{Z}, \mathbb{Z}) . The reason for the change will be explained below.

The following definition of (non-unital) derived A_∞ -algebra is that of [Sag10], except that we generalize to allow a (\mathbb{Z}, \mathbb{Z}) -bigrading, rather than an (\mathbb{N}, \mathbb{Z}) -bigrading. (Sagave avoids (\mathbb{Z}, \mathbb{Z}) -bigrading because of potential problems taking total complexes, but this is not an issue for the purposes of the present paper.)

DEFINITION 2.1. A *derived A_∞ -algebra* is a (\mathbb{Z}, \mathbb{Z}) -bigraded \mathbf{k} -module A equipped with \mathbf{k} -linear maps

$$m_{ij} : A^{\otimes j} \rightarrow A$$

of bidegree $(-i, 2 - i - j)$ for each $i \geq 0, j \geq 1$, satisfying the equations

$$(2.1) \quad \sum_{\substack{u=i+p, v=j+q-1 \\ j=1+r+t}} (-1)^{rq+pt+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0$$

for all $u \geq 0$ and $v \geq 1$.

Note that the map m_{ij} maps from $(A^{\otimes j})_\alpha^\beta$ to $(A^{\otimes j})_{\alpha-i}^{\beta+2-i-j}$, just as in [LRW13]. Thus the different convention for bidegrees has no effect on signs.

Examples of derived A_∞ -algebras include classical A_∞ -algebras, which are derived A_∞ -algebras concentrated in horizontal degree 0. Other examples are bicomplexes, bidgas and twisted chain complexes (see below).

We remark that we follow the sign conventions of Sagave [Sag10]. For a derived A_∞ -algebra concentrated in horizontal degree 0, one obtains one of the standard sign conventions for A_∞ -algebras. The appendix contains a discussion of alternative sign conventions, with a precise description of the relationship between them.

2.2. Twisted chain complexes. The notion of *twisted chain complex* is important in the theory of derived A_∞ -algebras. The term *multicomplex* is also used for a twisted chain complex.

DEFINITION 2.2. A *twisted chain complex* C is a (\mathbb{Z}, \mathbb{Z}) -bigraded \mathbf{k} -module with \mathbf{k} -linear maps $d_i^C : C \rightarrow C$ of bidegree $(-i, 1-i)$ for $i \geq 0$, satisfying $\sum_{i+p=u} (-1)^i d_i d_p = 0$ for $u \geq 0$. A *map of twisted chain complexes* $C \rightarrow D$ is a family of maps $f_i : C \rightarrow D$, for $i \geq 0$, of bidegree $(-i, -i)$, satisfying

$$\sum_{i+p=u} (-1)^i f_i d_p^C = \sum_{i+p=u} d_i^D f_p.$$

The composition of maps $f : E \rightarrow F$ and $g : F \rightarrow G$ is defined by $(gf)_u = \sum_{i+p=u} g_i f_p$ and the resulting category is denoted tCh_k .

A derived A_∞ -algebra has an underlying twisted chain complex, specified by the maps m_{i1} for $i \geq 0$.

2.3. Vertical bicomplexes and operads in vertical bicomplexes. The underlying category for the operadic view of derived A_∞ -algebras is the category of vertical bicomplexes.

DEFINITION 2.3. An object of the category of *vertical bicomplexes* BiCompl_v is a bigraded \mathbf{k} -module as above equipped with a vertical differential

$$d_A : A_i^j \rightarrow A_i^{j+1}$$

of bidegree $(0, 1)$. The morphisms are those morphisms of bigraded modules commuting with the vertical differential. We denote by $\text{Hom}(A, B)$ the set of morphisms (preserving the bigrading) from A to B .

The category BiCompl_v is isomorphic to the category of \mathbb{Z} -graded chain complexes of \mathbf{k} -modules.

For the suspension s as above, we have $d_{sA}(sx) = -s(d_A x)$.

The tensor product of two vertical bicomplexes A and B is given by

$$(A \otimes B)_u^v = \bigoplus_{i+p=u, j+q=v} A_i^j \otimes B_p^q,$$

with $d_{A \otimes B} = d_A \otimes 1 + 1 \otimes d_B : (A \otimes B)_u^v \rightarrow (A \otimes B)_u^{v+1}$. This makes BiCompl_v into a symmetric monoidal category.

Let A and B be two vertical bicomplexes. We write $\text{Hom}_{\mathbf{k}}$ for morphisms of \mathbf{k} -modules. We will denote by $\text{Mor}(A, B)$ the vertical bicomplex given by

$$\text{Mor}(A, B)_u^v = \prod_{\alpha, \beta} \text{Hom}_{\mathbf{k}}(A_\alpha^\beta, B_{\alpha+u}^{\beta+v}),$$

with vertical differential given by $\partial_{\text{Mor}}(f) = d_B f - (-1)^j f d_A$ for f of bidegree (l, j) . The reason for the change of grading conventions is that, with the convention adopted here, Mor is now an internal Hom on Bicompl_v .

The following notation will be useful in applying the Koszul sign rule. We denote by $|(r, s)|||(r', s')|$ the integer $rr' + ss'$.

2.4. The operad dAs . We now describe an operad in BiCompl_v . All operads considered in this paper are non-symmetric.

DEFINITION 2.4. A *collection* in BiCompl_v is a sequence $A(n)_{n \geq 1}$ where $A(n)$ is a vertical bicomplex for each $n \geq 1$. We denote by $\mathcal{C}\text{BiCompl}_v$ the category of collections of vertical bicomplexes. This category is endowed with a monoidal structure, the plethysm given by,

$$(M \circ N)(n) = \bigoplus_{k, l_1 + \dots + l_k = n} M(k) \otimes N(l_1) \otimes \dots \otimes N(l_k),$$

for any two collections M and N . The unit for the plethysm is given by the collection

$$I(n) = \begin{cases} 0, & \text{if } n \neq 1 \\ \mathbf{k} \text{ concentrated in bidegree } (0, 0), & \text{if } n = 1. \end{cases}$$

DEFINITION 2.5. A (non-symmetric) *operad* in BiCompl_v is a monoid in $\mathcal{C}\text{BiCompl}_v$.

We adopt standard operad notation, so that $\mathcal{P}(M, R)$ denotes the operad defined by generators and relations $\mathcal{F}(M)/(R)$, where $\mathcal{F}(M)$ is the free (non-symmetric) operad on the collection M .

DEFINITION 2.6. The operad dAs in BiCompl_v is defined as $\mathcal{P}(M_{dAs}, R_{dAs})$ where

$$M_{dAs}(n) = \begin{cases} 0, & \text{if } n > 2, \\ \mathbf{k}m_{02} \text{ concentrated in bidegree } (0, 0), & \text{if } n = 2, \\ \mathbf{k}m_{11} \text{ concentrated in bidegree } (-1, 0), & \text{if } n = 1, \end{cases}$$

and

$$R_{dAs} = \mathbf{k}(m_{02} \circ_1 m_{02} - m_{02} \circ_2 m_{02}) \oplus \mathbf{k}m_{11}^2 \oplus \mathbf{k}(m_{11} \circ_1 m_{02} - m_{02} \circ_1 m_{11} - m_{02} \circ_2 m_{11}),$$

with trivial vertical differential.

The algebras for this operad are easily seen to be the bidgas, that is associative monoids in bicomplexes; see [LRW13, Proposition 2.5]. Note that one differential comes from the vertical differential on objects in the underlying category, while the operad encodes the other differential and the multiplication.

The operad dAs is Koszul and one of the main results of [LRW13] identifies the associated infinity algebras.

THEOREM 2.7. [LRW13, Theorem 3.2] *A derived A_∞ -algebra is precisely a $(dAs)_\infty = \Omega((dAs)^i)$ -algebra.*

3. Coalgebras over the Koszul dual cooperad

In this section we initiate a study of the operad dAs and related objects. In particular we consider the category of coalgebras over the Koszul dual cooperad of dAs and coderivations of such coalgebras. This will allow us to give an operadic explanation of Sagave's reformulation of a derived A_∞ -algebra structure in terms of certain structure on the tensor coalgebra. We begin by setting up cooperads and their coalgebras. Then we recall the classical case for the associative operad As , before considering the derived case.

3.1. Cooperads and coalgebras. We briefly set up our conventions for non-symmetric cooperads and (conilpotent) coalgebras over cooperads.

A non-symmetric cooperad in a monoidal category is a comonoid in the associated category of collections endowed with the monoidal structure given by plethysm \circ of collections; see Definition 2.4. Thus a non-symmetric cooperad \mathcal{C} has a structure map $\Delta : \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}$, satisfying standard coassociative and counital conditions.

A conilpotent coalgebra C over a cooperad \mathcal{C} has a structure map

$$\Delta_C : C \rightarrow \mathcal{C}(C) = \bigoplus_k \mathcal{C}(k) \otimes C^{\otimes k},$$

satisfying the standard compatibility with the cooperad structure of \mathcal{C} .

3.2. Cooperadic suspension. The notion of suspension of an operad as in [GJ94, Section 1.3] can be adapted to collections.

We define the operation ΛR for any collection R in BiCompl_v as follows:

$$\Lambda R(n) = s^{1-n} R(n).$$

If \mathcal{R} is a non-symmetric (co)operad so is $\Lambda \mathcal{R}$ and if $\mathcal{R}(V)$ denotes the free (co)algebra (co)generated by V then

$$(\Lambda \mathcal{R})(sV) \cong s\mathcal{R}(V).$$

Consequently, V is an \mathcal{R} -(co)algebra if and only if sV is a $\Lambda \mathcal{R}$ -(co)algebra. Equivalently V is a $\Lambda \mathcal{R}$ -(co)algebra if and only if $s^{-1}V$ is an \mathcal{R} -(co)algebra. Indeed this construction gives rise to an isomorphism of (co)algebra categories.

Further details about cooperadic suspension can be found in the appendix, explaining in detail the signs involved in our bigraded setting.

3.3. The classical case, As^i -coalgebras. We denote by As the usual operad for associative algebras. This can be viewed either as an operad in differential graded modules, which is the usual classical context, or equivalently in vertical bicomplexes (in which case it is concentrated in horizontal degree zero). In the case of this operad, there is a well-known nice relationship, via suspension, between the operadic notion of coalgebra over the cooperad As^i and ordinary coassociative coalgebras.

Indeed, since $\Lambda As^i = As^*$ we have the following result.

PROPOSITION 3.1. *Cooperadic suspension gives rise to an isomorphism of categories between the category of conilpotent coalgebras over the cooperad As^i and the category of conilpotent coassociative coalgebras.*

Under this isomorphism the notion of coderivation $d : C \rightarrow C$ on a coassociative coalgebra C corresponds to the operadic notion of coderivation on the corresponding As^i -coalgebra, $s^{-1}C$. \square

We note that one can remove the conilpotent hypothesis at the expense of using a completed version of the tensor coalgebra.

Recall that $As^i(A) = s^{-1}\overline{T}^c(sA)$, the shifted reduced tensor coalgebra on sA .

We can see the basic idea of how the isomorphism works on objects very explicitly: given a coassociative coalgebra C with comultiplication $\Delta : C \rightarrow C \otimes C$, this completely determines an As^i -coalgebra structure on $s^{-1}C$

$$\overline{\Delta} : s^{-1}C \rightarrow As^i(s^{-1}C) = s^{-1}\overline{T}^c(C).$$

The components of this map are forced to be (shifted) iterations of the coassociative comultiplication Δ , that is, we have $\overline{\Delta} = \sum_{i=0}^{\infty} s^{-1} \Delta^{(i)}$. (Here we make the conventions $\Delta^{(0)} = s^{-1} 1_C$, $\Delta^{(1)} = s^{-1} \Delta$.)

And, on the other hand, an $\mathcal{A}s^i$ -coalgebra structure has to be of this form.

Now we have the general theorem that for a suitable operad \mathcal{P} , a \mathcal{P}_∞ -algebra structure on A is equivalent to a square-zero coderivation of degree one on the \mathcal{P}^i -coalgebra $\mathcal{P}^i(A)$; see [LV12, 10.1.13].

So in the case $\mathcal{P} = \mathcal{A}s$, we get that an $\mathcal{A}s_\infty = A_\infty$ -structure on A is equivalent to a square-zero coderivation of degree one on the $\mathcal{A}s^i$ -coalgebra $\mathcal{A}s^i(A) = s^{-1} \overline{T}^c(sA)$. And, by the above, this is equivalent to a square-zero coderivation of degree one on the coassociative coalgebra $\overline{T}^c(sA)$.

3.4. The operad of dual numbers. We recall the situation for the operad of dual numbers, since the operad $d\mathcal{A}s$ can be built from the operad $\mathcal{A}s$ and the operad of dual numbers, via a distributive law.

The operad of dual numbers only contains arity one operations, so it can be thought of as just a \mathbf{k} -algebra, and algebras over this operad correspond to (left) modules over this \mathbf{k} -algebra. So let $\mathcal{D} = \mathbf{k}[\epsilon]/(\epsilon^2)$ be the algebra of dual numbers. We consider this as a bigraded algebra, where the bidegree of ϵ is $(-1, 0)$.

Then consider the Koszul dual cooperad \mathcal{D}^i . Again this is concentrated in arity one and can be thought of as just a \mathbf{k} -coalgebra. We have $\mathcal{D}^i = \mathbf{k}[x]$, where $x = s\epsilon$, x has bidegree $(-1, -1)$ and the comultiplication is determined by $\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j$.

A \mathcal{D}^i -coalgebra is a comodule C over this coalgebra and this turns out to just be a pair (C, f) , where C is a \mathbf{k} -module and f is a linear map $f : C \rightarrow C$ of bidegree $(1, 1)$. (Given a coaction $\rho : C \rightarrow \mathcal{D}^i \otimes C = \mathbf{k}[x] \otimes C$, write f_i for the projection onto $\mathbf{k}x^i \otimes C$; then coassociativity gives $f_{m+n} = f_m f_n$, so the coaction is determined by f_1 .) A coderivation is a linear map $d : C \rightarrow C$ of bidegree (r, s) such that $df = (-1)^{|(r,s)|| (1,1)|} fd$, that is $df = (-1)^{r+s} fd$. In particular, if d has bidegree $(0, 1)$ then it anti-commutes with f .

3.5. The derived case, $(d\mathcal{A}s)^i$ -coalgebras. We recall from [LRW13, Lemma 2.6] that the operad $d\mathcal{A}s$ can be built from the operad $\mathcal{A}s$ and the operad of dual numbers, via a distributive law, so that we have an isomorphism of operads $d\mathcal{A}s \cong \mathcal{A}s \circ \mathcal{D}$.

We have, on underlying collections, $(d\mathcal{A}s)^i \cong \mathcal{D}^i \circ (\mathcal{A}s)^i$. Since \mathcal{D}^i is concentrated in arity one, applying Λ gives $\Lambda(d\mathcal{A}s)^i \cong \mathcal{D}^i \circ \Lambda(\mathcal{A}s)^i$. It thus seems natural that a $\Lambda(d\mathcal{A}s)^i$ -coalgebra should correspond to a coassociative coalgebra (coming from the $\Lambda(\mathcal{A}s)^i$ -coalgebra structure), plus a compatible linear map (coming from the \mathcal{D}^i -coalgebra structure). This works out as follows.

Consider triples (C, Δ, f) where (C, Δ) is a conilpotent coassociative coalgebra and $f : C \rightarrow C$ is a linear map of bidegree $(1, 1)$ satisfying $(f \otimes 1)\Delta = (1 \otimes f)\Delta = \Delta f$. A morphism between two such triples is a morphism of coalgebras commuting with the given linear maps.

PROPOSITION 3.2. *Cooperadic suspension gives rise to an isomorphism of categories between the category of conilpotent coalgebras over the cooperad $(d\mathcal{A}s)^i$ and the category of triples (C, Δ, f) as above.*

An operadic coderivation of bidegree $(0, 1)$ of a $(dAs)^i$ -coalgebra $s^{-1}C$ corresponds on (C, Δ, f) to a coderivation of bidegree $(0, 1)$ of the coalgebra C , anti-commuting with the linear map f .

PROOF. We will see that a triple (C, Δ, f) as above corresponds to a $(dAs)^i$ -coalgebra structure on $s^{-1}C$, or equivalently to a $\Lambda(dAs)^i$ -coalgebra structure on C .

The cooperad structure of $(dAs)^i$ is given explicitly in [LRW13, Proposition 2.7] and the corresponding structure of $\Lambda(dAs)^i$ is given in the appendix; see Corollary 6.2. In particular, as a \mathbf{k} -module, it is free on generators α_{uv} , with bidegree $(-u, -u)$.

It follows that we can identify $\Lambda(dAs)^i(C)$ with $\mathbf{k}[x] \otimes \overline{T}^c(C)$, where, for $a \in C^{\otimes v}$, $\alpha_{uv} \otimes a \in \Lambda(dAs)^i(C)$ is identified with $x^u \otimes a \in \mathbf{k}[x] \otimes \overline{T}^c(C)$. That is, we have

$$\Lambda(dAs)^i(C) = \bigoplus_v \Lambda(dAs)^i(v) \otimes C^{\otimes v} = \bigoplus_{u,v} \mathbf{k} \alpha_{uv} \otimes C^{\otimes v} \cong \bigoplus_v \mathbf{k}[x] \otimes C^{\otimes v}.$$

Let C be a coalgebra for the cooperad $\Lambda(dAs)^i$, with coaction

$$\rho : C \rightarrow \Lambda(dAs)^i(C) = \mathbf{k}[x] \otimes \overline{T}^c(C).$$

Write $\rho_{i,j} : C \rightarrow C^{\otimes j}$ for the following composite

$$\rho_{i,j} : C \xrightarrow{\rho} \mathbf{k}[x] \otimes \overline{T}^c(C) \twoheadrightarrow \mathbf{k}x^i \otimes C^{\otimes j} \xrightarrow{\cong} C^{\otimes j}.$$

Define $\Delta = \rho_{0,2} : C \rightarrow C \otimes C$ and $f = \rho_{1,1} : C \rightarrow C$. Then, using coassociativity of the coaction and the computation in Example 6.3, one can check that Δ is coassociative (essentially as in the classical case) and that

$$-\rho_{1,2} = (f \otimes 1)\Delta = (1 \otimes f)\Delta = \Delta f.$$

More generally, one has $\rho_{i,j} = (-1)^{i(j+1)} \Delta^{(j-1)} f^i$. Thus the $\Lambda(dAs)^i$ -coalgebra structure is completely determined by Δ and f .

On the other hand, given a triple (C, Δ, f) as above, we can define

$$\rho_{i,j} = (-1)^{i(j+1)} \Delta^{(j-1)} f^i$$

and let $\rho : C \rightarrow (dAs)^i(C)$ be the corresponding map. Using the fact that $(f \otimes 1)\Delta = (1 \otimes f)\Delta = \Delta f$, we see that $\rho_{i,j} = (-1)^{i(j+1)} (f^i \otimes 1^{j-i}) \Delta^{(j-1)}$ and with this relation we can check that ρ does make C into a $\Lambda(dAs)^i$ -coalgebra.

It is straightforward to check the statement about coderivations; we get a coderivation of the coalgebra as in the classical case, together with compatibility with f . \square

EXAMPLE 3.3. As an example we will compute operadic coderivations of the cofree $\Lambda(dAs)^i$ -coalgebra cogenerated by C . From the proof of Proposition 3.2 we have $\Lambda(dAs)^i(C) \cong \mathbf{k}[x] \otimes \overline{T}^c(C)$.

From the cooperad structure given in Corollary 6.2, the coalgebra structure is given by

$$\Delta(x^i \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n-1} \sum_{r+s=i} (-1)^\epsilon (x^r \otimes a_1 \otimes \cdots \otimes a_k) \otimes (x^s \otimes a_{k+1} \otimes \cdots \otimes a_n),$$

where $\epsilon = rn + ik + (s, s)(|a_1| + \cdots + |a_k|)$.

Note that if one denotes by π_0 the projection of $\mathbf{k}[x] \otimes \overline{T}^c(C)$ onto $\mathbf{k}x^0 \otimes \overline{T}^c(C) \cong \overline{T}^c(C)$ then

$$\Delta\pi_0 = (\pi_0 \otimes \pi_0)\Delta$$

where the first Δ is the usual deconcatenation product defined on $\overline{T}^c(C)$.

The linear map

$$f : \mathbf{k}[x] \otimes \overline{T}^c(C) \rightarrow \mathbf{k}[x] \otimes \overline{T}^c(C)$$

is determined by $f(x^n \otimes a) = (-1)^{j+1}x^{n-1} \otimes a$, for $a \in C^{\otimes j}$.

Now an operadic coderivation of bidegree $(0, 1)$ is a coderivation of the coalgebra $\mathbf{k}[x] \otimes \overline{T}^c(C)$, anti-commuting with the map f . Let $d : \mathbf{k}[x] \otimes \overline{T}^c(C) \rightarrow \mathbf{k}[x] \otimes \overline{T}^c(C)$ and write

$$d(x^n \otimes a) = \sum_i x^i \otimes d^{n,i}(a),$$

where $d^{n,i} : \overline{T}^c(C) \rightarrow \overline{T}^c(C)$ and $a \in C^{\otimes j}$.

Write $d^{n,i}(a) = \sum_k d^{n,i,k}(a)$ with $d^{n,i,k}(a) \in C^{\otimes k}$. Then anti-commuting with f means that

$$d^{n,i,k}(a) = (-1)^{j+k+1}d^{n-1,i-1,k}(a),$$

where $a \in C^{\otimes j}$ and hence that

$$d^{n,i,k}(a) = (-1)^{i(j+k+1)}d^{n-i,0,k}(a)$$

for $i \leq n$ and $d^{n,i,k} = 0$ for $i > n$. So d is completely determined by the family of maps $d^{n,0,k}$.

Define $\delta^n : \overline{T}^c(C) \rightarrow \overline{T}^c(C)$ by $\delta^n(a) = (-1)^{nj}d^{n,0}(a) = (-1)^{nj}\pi_0d(x^n \otimes a)$, where $a \in C^{\otimes j}$.

The coderivation condition for d makes each δ^n a coderivation of $\overline{T}^c(C)$. Indeed one can check that for $a \in C^{\otimes j}$,

$$\begin{aligned} \Delta\delta^n(a) &= (-1)^{nj}\Delta\pi_0d(x^n \otimes a) = \\ &= (-1)^{nj}\pi_0 \otimes \pi_0(d \otimes 1 + 1 \otimes d)\Delta(x^n \otimes a) = (\delta^n \otimes 1 + 1 \otimes \delta^n)\Delta(a). \end{aligned}$$

So we obtain a family of coderivations δ^n on $\overline{T}^c(C)$ of bidegree $(-n, 1-n)$.

Using this we have an operadic explanation of the following formulation of a derived A_∞ -algebra structure; this is part of [Sag10, Lemma 4.1].

PROPOSITION 3.4. *A derived A_∞ -algebra structure on a bigraded \mathbf{k} -module A is equivalent to specifying a family of coderivations $\overline{T}^c(sA) \rightarrow \overline{T}^c(sA)$ making $\overline{T}^c(sA)$ into a twisted chain complex.*

PROOF. As recalled above, a \mathcal{P}_∞ -algebra structure on A is equivalent to a square-zero coderivation on the \mathcal{P}^i -coalgebra $\mathcal{P}^i(A)$. Applying this to the example $\mathcal{P} = \mathbf{dAs}$, and with $A = s^{-1}C$, we see that a coderivation d of $(\mathbf{dAs})^i(A)$ corresponds to a family of coderivations δ^n on $\overline{T}^c(sA)$ of bidegree $(-n, 1-n)$.

Now one can check that if we further impose the condition $d^2 = 0$ on the map $\mathcal{P}^i(A) \rightarrow \mathcal{P}^i(A)$, this corresponds to saying that the maps δ^n make $\overline{T}^c(sA)$ into a twisted chain complex.

In more detail, with $a \in C^{\otimes j}$ and using the same notation as in Example 3.3,

$$d^2(x^n \otimes a) = \sum_{r,s} x^s \otimes d^{r,s}d^{n,r}(a).$$

In particular, considering $s = 0$, we see that $d^2 = 0$ implies:

$$\begin{aligned}
\sum_r d^{r,0} d^{n,r}(a) = 0 &\Leftrightarrow \sum_r \sum_k (-1)^{r(j+k+1)} d^{r,0} d^{n-r,0,k}(a) = 0 \\
&\Leftrightarrow \sum_r \sum_k (-1)^{r(j+k+1)+rk+(n-r)j} \delta^r \delta^{n-r,k}(a) = 0 \\
&\Leftrightarrow \sum_r (-1)^{r+nj} \delta^r \delta^{n-r}(a) = 0 \\
&\Leftrightarrow (-1)^{nj} \sum_r (-1)^r \delta^r \delta^{n-r}(a) = 0.
\end{aligned}$$

Thus $d^2 = 0$ implies the twisted chain complex conditions $\sum_r (-1)^r \delta^r \delta^{n-r}(a) = 0$ on the maps δ^r .

Furthermore, by [LV12, 6.3.8], d^2 is completely determined by its projection to A included in the x^0 part and it follows that the condition $d^2 = 0$ holds if and only if the maps δ^r satisfy the twisted chain complex conditions. \square

4. Representations of derived A_∞ -algebras

The aim of this section is to study representations of $d\mathcal{A}s_\infty$ -algebras. We establish some general results on coderivations of representations of coalgebras and then show that representations of homotopy algebras correspond to square-zero coderivations on a certain cofree object. We then use these results to describe $d\mathcal{A}s_\infty$ -representations in terms of a twisted chain complex of coderivations on the tensor coalgebra. Thus we obtain a description of $d\mathcal{A}s_\infty$ -representations similar in spirit to Proposition 3.4.

4.1. Coderivations on representations of coalgebras. One way to describe \mathcal{P}_∞ -structures is via coderivations on cofree coalgebras. We will see that analogously \mathcal{P}_∞ -representations can be described via coderivations on cofree representations of coalgebras, which we will introduce now. We work in the category BiCompl_v of vertical bicomplexes.

DEFINITION 4.1. Let X and Y be vertical bicomplexes and let \mathcal{M} be a collection in BiCompl_v . The vertical bicomplex $\mathcal{M}(X; Y)$ is given by

$$\mathcal{M}(X; Y) = \bigoplus_{n \geq 1} \mathcal{M}(n) \otimes \left(\bigoplus_{a+b+1=n} X^{\otimes a} \otimes Y \otimes X^{\otimes b} \right).$$

If $f: \mathcal{M} \rightarrow \mathcal{M}'$ is a map of collections and $g: X \rightarrow X'$ and $h: Y \rightarrow Y'$ are maps of vertical bicomplexes, the map

$$f(g; h): \mathcal{M}(X; Y) \rightarrow \mathcal{M}'(X'; Y')$$

is defined as the direct sum of the maps $f(a+b+1) \otimes g^{\otimes a} \otimes h \otimes g^{\otimes b}$.

REMARK 4.2. In this section for convenience we drop the symbol \circ for plethysm of collections and just write $\mathcal{C}\mathcal{C}$ for $\mathcal{C} \circ \mathcal{C}$.

One has to be careful when working with $\mathcal{M}(X; Y)$. For example if \mathcal{N} is another collection, in general

$$(\mathcal{M}\mathcal{N})(X; Y) \not\cong \mathcal{M}(\mathcal{N}(X; Y)).$$

However it is true that $(\mathcal{M}\mathcal{N})(X; Y) \cong \mathcal{M}(\mathcal{N}(X); \mathcal{N}(X; Y))$ and we will make frequent use of this.

Dual to the notion of representation (see e.g. [Fre09]) of an algebra over an operad is the notion of representation of a coalgebra over a cooperad. In the following let $(\mathcal{C}, \Delta, \epsilon)$ be a cooperad and let C be a \mathcal{C} -coalgebra with coalgebra structure map $\rho: C \rightarrow \mathcal{C}(C)$.

DEFINITION 4.3. A bigraded module E is called a *representation of C over \mathcal{C}* if there is a map

$$\omega: E \rightarrow \mathcal{C}(C; E)$$

such that the diagrams

$$\begin{array}{ccc} E & \xrightarrow{\omega} & \mathcal{C}(C; E) \\ \omega \downarrow & & \Delta \downarrow \\ \mathcal{C}(C; E) & \xrightarrow{\mathcal{C}(\rho; \omega)} & \mathcal{C}(\mathcal{C}(C); \mathcal{C}(C; E)) \cong (\mathcal{C}\mathcal{C})(C; E) \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{\omega} & \mathcal{C}(C; E) \\ & \searrow & \downarrow \epsilon \\ & & E \end{array}$$

commute.

EXAMPLE 4.4. The example we will be primarily interested in is the following cofree representation. Let $C = \mathcal{C}(X)$ be the cofree \mathcal{C} -coalgebra cogenerated by X . Then to a bigraded module Y we can associate the representation $\mathcal{C}(X; Y)$. The structure map is given by the comultiplication on \mathcal{C} , i.e.

$$\mathcal{C}(X; Y) \rightarrow (\mathcal{C}\mathcal{C})(X; Y) \cong \mathcal{C}(\mathcal{C}(X); \mathcal{C}(X; Y)).$$

Over an arbitrary \mathcal{C} -coalgebra, cofree representations are not that simple, see for instance the result on free representations in [Fre09, 4.3.2].

REMARK 4.5. In [Fre09, 4.3], Fresse defines the enveloping algebra $U_{\mathcal{P}}(A)$ of an algebra A over an operad \mathcal{P} , so that left modules over $U_{\mathcal{P}}(A)$ are precisely representations of the \mathcal{P} -algebra A . This enveloping algebra is obtained as the space of unary operations of the enveloping operad. In [Yal14], Yalin defines the notion of an enveloping cooperad associated to a coalgebra C over a cooperad \mathcal{C} . Similarly to Fresse, if one takes the space of unary operations of this enveloping cooperad one gets the enveloping coalgebra $U_{\mathcal{C}}(C)$ so that left comodules over $U_{\mathcal{C}}(C)$ are precisely representations of the \mathcal{C} -coalgebra C . However, the constructions of Fresse and Yalin are performed in the category of symmetric (co)operads. In this paper we are dealing with non-symmetric (co)operads. But, the constructions of Yalin can be adapted to our case if one considers first the collection

$$\mathcal{C}[C](n) = \bigoplus_{r \geq 0} \mathcal{C}(n+r) \otimes (C \oplus \mathbf{k}\rho)_n^{n+r}$$

where ρ is a “silent” variable of weight 1 and $(C \oplus \mathbf{k}\rho)_n^{n+r}$ is the component of $(C \oplus \mathbf{k}\rho)^{n+r}$ of weight n . This collection forms a cooperad and the enveloping cooperad is a subcooperad of $\mathcal{C}[C]$ obtained as an equalizer like in [Yal14, 2.2].

Next we will define what a coderivation of a representation is. To do this we need to extend the infinitesimal composite \circ' of maps as defined in [LV12, 6.1.3].

DEFINITION 4.6. Let \mathcal{M} , X and Y be as in Definition 4.1. For $g: X \rightarrow X$ and $h: Y \rightarrow Y$ the map

$$\mathcal{M} \circ''(g; h): \mathcal{M}(X; Y) \rightarrow \mathcal{M}(X; Y)$$

is defined on $\mathcal{M}(a+b+1) \otimes X^{\otimes a} \otimes Y \otimes X^{\otimes b}$ as the sum

$$\sum_{\substack{i=1, \\ i \neq a+1}}^{a+b+1} \mathcal{M} \otimes 1^{\otimes i-1} \otimes g \otimes 1^{\otimes a+b+1-i} + \mathcal{M} \otimes 1^{\otimes a} \otimes h \otimes 1^{\otimes b}$$

with 1 denoting either the identity on X or Y .

Let $d_{\mathcal{C}}$ denote the (vertical) differential of the cooperad \mathcal{C} , (C, ρ) a \mathcal{C} -coalgebra in bigraded modules equipped with a coderivation ∂_C and (E, ω) a bigraded module equipped with a map ω making it a representation of C .

DEFINITION 4.7. A map $g: E \rightarrow E$ is called a coderivation if

$$\begin{array}{ccc} E & \xrightarrow{g} & E \\ \omega \downarrow & & \omega \downarrow \\ \mathcal{C}(C; E) & \xrightarrow{\mathcal{C}\partial_C''(\partial_C; g) + d_{\mathcal{C}}(C; E)} & \mathcal{C}(C; E) \end{array}$$

commutes.

We will need analogues of well known results for coderivations on coalgebras. To simplify formulas we encode coderivations via a distributive law; see [Bec69].

DEFINITION 4.8. Let $(\mathcal{P}, \gamma, \eta)$ be an operad and $(\mathcal{C}, \Delta, \epsilon)$ a cooperad. A mixed distributive law is a morphism of collections

$$\beta: \mathcal{P}\mathcal{C} \rightarrow \mathcal{C}\mathcal{P}$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{P}\mathcal{P}\mathcal{C} & \xrightarrow{\gamma^{\mathcal{C}}} & \mathcal{P}\mathcal{C} \\ \mathcal{P}\beta \downarrow & & \beta \downarrow \\ \mathcal{P}\mathcal{C}\mathcal{P} & \xrightarrow{\beta^{\mathcal{P}}} & \mathcal{C}\mathcal{P}\mathcal{P} \xrightarrow{\mathcal{C}\gamma} \mathcal{C}\mathcal{P} \end{array} \quad \begin{array}{ccc} \mathcal{P}\mathcal{C} & \xrightarrow{\mathcal{P}\Delta} & \mathcal{P}\mathcal{C}\mathcal{C} \xrightarrow{\beta^{\mathcal{C}}} \mathcal{C}\mathcal{P}\mathcal{C} \\ \beta \downarrow & & \mathcal{C}\beta \downarrow \\ \mathcal{C}\mathcal{P} & \xrightarrow{\Delta^{\mathcal{P}}} & \mathcal{C}\mathcal{C}\mathcal{P} \end{array}$$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\eta^{\mathcal{C}}} & \mathcal{P}\mathcal{C} \\ \mathcal{C}\eta \searrow & & \beta \downarrow \\ & & \mathcal{C}\mathcal{P} \end{array} \quad \begin{array}{ccc} \mathcal{P}\mathcal{C} & & \\ \beta \downarrow & \searrow \mathcal{P}\epsilon & \\ \mathcal{C}\mathcal{P} & \xrightarrow{\mathcal{C}\mathcal{P}} & \mathcal{P} \end{array}$$

commute.

The operad $(\mathcal{B}, \gamma_{\mathcal{B}}, \eta_{\mathcal{B}})$ that will help us to describe coderivations is the operad freely generated by a single unary operation y . In all of our examples y will be of bidegree $(0, 1)$.

DEFINITION 4.9. We define a distributive law

$$\beta: \mathcal{B}\mathcal{C} \rightarrow \mathcal{C}\mathcal{B}$$

by requiring that

$$\beta(y; c) = \sum_{i=1}^n (-1)^{|c||y|} c; 1^{\otimes i-1} \otimes y \otimes 1^{\otimes n-i} + d_{\mathcal{C}}(c); 1^{\otimes n}$$

for $c \in \mathcal{C}(n)$.

Since \mathcal{B} is freely generated we need to check only that

$$\begin{array}{ccccc} \mathbf{k}y \otimes \mathcal{C} \subset \mathcal{B}\mathcal{C} & \xrightarrow{\mathcal{B}\Delta} & \mathcal{B}\mathcal{C}\mathcal{C} & \xrightarrow{\beta\mathcal{C}} & \mathcal{C}\mathcal{B}\mathcal{C} \\ \beta \downarrow & & & & \mathcal{C}\beta \downarrow \\ \mathcal{C}\mathcal{B} & \xrightarrow{\Delta\mathcal{B}} & \mathcal{C}\mathcal{C}\mathcal{B} & & \end{array}$$

commutes and that $\epsilon\mathcal{B}(\beta(y; c)) = y(\epsilon c)$, which can be easily calculated. The other two defining conditions of a mixed distributive law determine β on the whole of $\mathcal{B}\mathcal{C}$.

It is possible to characterise coderivations via β . Since a coderivation on a representation depends on the coderivation on the coalgebra we state the corresponding result for coalgebras.

PROPOSITION 4.10. *Giving a coderivation on a \mathcal{C} -coalgebra (C, ρ) is equivalent to defining a \mathcal{B} -algebra structure γ_C on C such that*

$$\begin{array}{ccc} \mathcal{B}(C) & \xrightarrow{\mathcal{B}\rho} & \mathcal{B}\mathcal{C}(C) \\ \gamma_C \downarrow & & \beta_C \downarrow \\ & & \mathcal{C}\mathcal{B}(C) \\ & & \mathcal{C}\gamma_C \downarrow \\ C & \xrightarrow{\rho} & \mathcal{C}(C) \end{array}$$

commutes. Explicitly, the coderivation defined by γ_C is $\gamma_C(y)$.

We omit the proof of this proposition since it is analogous to the proof of the result for representations which we will state and prove now. Again let $\gamma_C: \mathcal{B}C \rightarrow C$ correspond to the coderivation ∂_C .

Observe that since \mathcal{B} is concentrated in arity one we have

$$(\mathcal{B}\mathcal{C})(C; E) \cong \mathcal{B}(\mathcal{C}(C; E)) \quad \text{as well as} \quad (\mathcal{C}\mathcal{B})(C; E) \cong \mathcal{C}(\mathcal{B}(C); \mathcal{B}(E)).$$

PROPOSITION 4.11. *Giving a coderivation on E is equivalent to giving a \mathcal{B} -algebra structure map $\gamma_E: \mathcal{B}(E) \rightarrow E$ such that*

$$(4.1) \quad \begin{array}{ccc} \mathcal{B}(E) & \xrightarrow{\mathcal{B}\omega} & \mathcal{B}(\mathcal{C}(C; E)) = (\mathcal{B}\mathcal{C})(C; E) \\ \gamma_E \downarrow & & \beta(C; E) \downarrow \\ & & (\mathcal{C}\mathcal{B})(C; E) \cong \mathcal{C}(\mathcal{B}(C); \mathcal{B}(E)) \\ & & \mathcal{C}(\gamma_C; \gamma_E) \downarrow \\ E & \xrightarrow{\omega} & \mathcal{C}(C; E) \end{array}$$

commutes. The coderivation defined by γ_E is $\gamma_E(y)$.

PROOF. Since \mathcal{B} is free as an operad generated by y , making E a \mathcal{B} -algebra is equivalent to specifying $\gamma_E(y)$. Observe that the condition that the diagram commutes is trivial when we restrict to $IE \subset \mathcal{B}E$. On the other hand, one easily checks that restricted to $\mathbf{k}y \otimes E$ the diagram expresses exactly that $g = \gamma_E(y)$ is a coderivation: the left hand side composition of the maps in the diagram then

equals ωg , while the right hand side equals $(\mathcal{C} \circ'' (\partial_C; g))\omega + d_{\mathcal{C}}\omega$. To show that this implies the general case we proceed by induction. Suppose that (4.1) holds restricted to \mathcal{B}_n as well as restricted to \mathcal{B}_m , where \mathcal{B}_n is the sub- \mathbf{k} -module of \mathcal{B} spanned by $\{y^i, i \leq n\}$. We need to show that

$$\begin{array}{ccc}
\mathcal{B}_n \mathcal{B}_m(E) & \xrightarrow{\gamma_{\mathcal{B}} E} & \mathcal{B}_{n+m}(E) \\
\gamma_{\mathcal{B}} E \downarrow & & \downarrow \gamma_E \\
\mathcal{B}_{n+m}(E) & & E \\
\mathcal{B}\omega \downarrow & & \downarrow \omega \\
\mathcal{B}_{n+m}(\mathcal{C}(C; E)) = (\mathcal{B}_{n+m}\mathcal{C})(C; E) & & \\
\beta(C; E) \downarrow & & \\
(\mathcal{C}\mathcal{B}_{n+m})(C; E) \cong \mathcal{C}(\mathcal{B}_{n+m}(C); \mathcal{B}_{n+m}(E)) & \xrightarrow{\mathcal{C}(\gamma_C; \gamma_E)} & \mathcal{C}(C; E)
\end{array}$$

commutes. Keep in mind that γ_E defines an algebra structure and note that we have the identities

$$(\mathcal{B}\omega)(\gamma_{\mathcal{B}} E) = (\gamma_{\mathcal{B}} \mathcal{C}(C; E))(\mathcal{B}\mathcal{B}\omega)$$

and

$$\beta(\gamma_{\mathcal{B}} \mathcal{C}) = (\mathcal{C}\gamma_{\mathcal{B}})(\beta\mathcal{B})(\mathcal{B}\beta).$$

Then using that (4.1) holds restricted to \mathcal{B}_m and \mathcal{B}_n we find that the right and the upper square in the diagram

$$\begin{array}{ccccc}
\mathcal{B}_n \mathcal{B}_m(E) & \xrightarrow{\mathcal{B}\gamma_E} & \mathcal{B}_n(E) & \xrightarrow{\gamma_E} & E \\
\mathcal{B}\mathcal{B}\omega \downarrow & & \downarrow \mathcal{B}\omega & & \downarrow \omega \\
\mathcal{B}_n \mathcal{B}_m \mathcal{C}(C; E) & & \mathcal{B}_n \mathcal{C}(C; E) & & \\
\mathcal{B}\beta(C; E) \downarrow & & \downarrow \beta(C; E) & & \\
\mathcal{B}_n(\mathcal{C}\mathcal{B}_m)(C; E) \cong \mathcal{B}_n \mathcal{C}(\mathcal{B}_m(C); \mathcal{B}_m(E)) & \xrightarrow{\mathcal{B}_n \mathcal{C}(\gamma_C; \gamma_E)} & \mathcal{B}_n \mathcal{C}(C; E) & & \\
\beta\mathcal{B}(C; E) \downarrow & & \downarrow \beta(C; E) & & \\
\mathcal{C}\mathcal{B}_n \mathcal{B}_m(C; E) & & \mathcal{C}\mathcal{B}_n(C; E) \cong \mathcal{C}(\mathcal{B}_n(C); \mathcal{B}_n(E)) & & \\
\mathcal{C}\gamma_{\mathcal{B}}(C; E) \downarrow & & \downarrow \mathcal{C}(\gamma_C; \gamma_E) & & \\
\mathcal{C}\mathcal{B}_{n+m}(C; E) \cong \mathcal{C}(\mathcal{B}_{n+m}(C); \mathcal{B}_{n+m}(E)) & \xrightarrow{\mathcal{C}(\gamma_C; \gamma_E)} & \mathcal{C}(C; E) & \xrightarrow{\mathcal{C}(C; E)} & \mathcal{C}(C; E)
\end{array}$$

commute. Commutativity of the lower left square follows from the fact that γ_C and γ_E are \mathcal{B} -algebra structure maps. \square

Let $\text{Coder}(E)$ denote the set of coderivations on the representation (E, ω) . For cofree representations over cofree coalgebras we have the following result.

PROPOSITION 4.12. *Let X and Y be bigraded modules, and let \mathcal{C} be as above. Let $\mathcal{C}(X)$ be equipped with a coderivation $\partial_{\mathcal{C}(X)}$. There is a bijection*

$$\text{Coder}(\mathcal{C}(X; Y)) \cong \text{Hom}(\mathcal{C}(X; Y), Y).$$

Explicitly, the bijection is given by composing a coderivation with $\mathcal{C}(X; Y) \xrightarrow{\epsilon} Y$. To construct a coderivation ∂_f from a map $f: \mathcal{C}(X; Y) \rightarrow Y$, set

$$\partial_f = d_{\mathcal{C}}(X; Y) + (\mathcal{C} \circ_{(1)} (f \vee \epsilon \partial_{\mathcal{C}(X)}))(\Delta_{(1)}(X; Y)),$$

where $\circ_{(1)}$ denotes the infinitesimal composite product of morphisms and $\Delta_{(1)}: \mathcal{C}(X; Y) \rightarrow (\mathcal{C} \circ_{(1)} \mathcal{C})(X; Y)$ denotes infinitesimal decomposition, see [LV12, 6.1.4]. The map $f \vee (\epsilon \partial_{\mathcal{C}(X)})$ is either f or $\epsilon \partial_{\mathcal{C}(X)}$ depending on whether the second copy of \mathcal{C} in $(\mathcal{C} \circ_{(1)} \mathcal{C})(X; Y)$ is decorated by an element in Y or not.

PROOF. Let $f: \mathcal{C}(X; Y) \rightarrow Y$ be given and let $\gamma_{\mathcal{C}(X)}: \mathcal{BC}(X) \rightarrow \mathcal{C}(X)$ correspond to $\partial_{\mathcal{C}(X)}$. Since \mathcal{B} is freely generated by y we can define $\gamma_f: \mathcal{BC}(X; Y) \rightarrow \mathcal{C}(X; Y)$ by requiring that restricted to $\mathbf{ky} \otimes \mathcal{C}(X; Y) \subset \mathcal{BC}(X; Y)$ it is given by

$$\mathbf{ky} \otimes \mathcal{C}(X; Y) \xrightarrow{\mathcal{B}\Delta(X; Y)} (\mathcal{BCC})(X; Y) \xrightarrow{\beta\mathcal{C}(X; Y)} \mathcal{C}(\mathcal{BC}(X); \mathcal{BC}(X; Y)) \xrightarrow{\mathcal{C}(\epsilon\gamma_{\mathcal{C}(X)}; \bar{f})} \mathcal{C}(X; Y),$$

where $\bar{f}: \mathcal{BC}(X; Y) \rightarrow Y$ resembles the sum of f and the counit: It is defined by

$$\bar{f}((y^j; c)(x_1, \dots, z, \dots, x_n)) = \begin{cases} \epsilon(c)(x_1, \dots, z, \dots, x_n), & j = 0, \\ f(c(x_1, \dots, z, \dots, x_n)), & j = 1, \\ 0, & j > 1. \end{cases}$$

We need to show that γ_f corresponds to a coderivation. We saw in the proof of Proposition 4.11 that (4.1) holds if it holds restricted to $\mathbf{ky} \otimes \mathcal{C}(X; Y)$, and hence we only consider that case. First observe that

$$\gamma_{\mathcal{C}(X)}(y) = d_{\mathcal{C}} + (\mathcal{C} \circ_{(1)} \epsilon \gamma_{\mathcal{C}(X)}(y))\Delta_{(1)}$$

according to [LV12, 6.3.8]. Hence restricted to $\mathbf{ky} \otimes \mathcal{C}(X)$ the map $\gamma_{\mathcal{C}(X)}$ equals

$$\mathbf{ky} \otimes \mathcal{C}(X) \xrightarrow{\mathcal{B}\Delta(X)} \mathcal{BCC}(X) \xrightarrow{\beta\mathcal{C}(X)} \mathcal{C}\mathcal{BC}(X) \xrightarrow{\mathcal{C}(\epsilon\gamma_{\mathcal{C}(X)})} \mathcal{C}(X).$$

We conclude that

$$\mathcal{C}(\gamma_{\mathcal{C}(X)}; \gamma_f): \mathcal{C}(\mathcal{BC}(X); \mathcal{BC}(X; Y)) \rightarrow \mathcal{C}(\mathcal{C}(X); \mathcal{C}(X; Y))$$

can be written as

$$(\mathcal{C}\mathcal{BC})(X; Y) \xrightarrow{\mathcal{C}\mathcal{B}\Delta(X; Y)} (\mathcal{C}\mathcal{BCC})(X; Y) \xrightarrow{\mathcal{C}\beta\mathcal{C}(X; Y)} (\mathcal{C}\mathcal{C}\mathcal{BC})(X; Y) \xrightarrow{\mathcal{C}\mathcal{C}(\epsilon\gamma_{\mathcal{C}(X)}; \bar{f})} \mathcal{C}\mathcal{C}(X; Y).$$

Hence we need to examine the diagram

$$\begin{array}{ccc}
y\mathcal{C}(X; Y) \subset \mathcal{BC}(X; Y) & \xrightarrow{\mathcal{B}\Delta(X; Y)} & (\mathcal{BCC})(X; Y) \\
\downarrow \mathcal{B}\Delta(X; Y) & & \downarrow \beta\mathcal{C}(X; Y) \\
(\mathcal{BCC})(X; Y) & & (\mathcal{CBC})(X; Y) \\
\downarrow \beta\mathcal{C}(X; Y) & & \downarrow \mathcal{C}\mathcal{B}\Delta(X; Y) \\
(\mathcal{CBC})(X; Y) & & (\mathcal{CBCCC})(X; Y) \\
\downarrow \cong & & \downarrow \mathcal{C}\beta\mathcal{C}(X; Y) \\
\mathcal{C}(\mathcal{BC}(X); \mathcal{BC}(X; Y)) & & \mathcal{CCBC}(X; Y) \\
\downarrow \mathcal{C}(\epsilon\gamma_{\mathcal{C}(X)}; \bar{f}) & & \downarrow \cong \\
\mathcal{C}(X; Y) & \xrightarrow{\Delta(X; Y)} & \mathcal{C}(X; Y)
\end{array}$$

That Δ commutes with the two lower vertical maps is clear. Using that β is a distributive law and the coassociativity of Δ yields that γ_f indeed defines a coderivation. One easily checks that $\gamma_f(y)$ coincides with $d_e + (\mathcal{C} \circ_{(1)} ((\epsilon\partial_{\mathcal{C}(X)}) \vee f))\Delta_{(1)}$ as claimed.

It remains to show that $\text{Hom}(\mathcal{C}(X; Y), Y)$ and $\text{Coder}(\mathcal{C}(X; Y))$ are in bijection. It is clear that $\epsilon\partial_f = f$. Given a coderivation v , to see that $\partial_{\epsilon v} = v$, calculate that

$$\begin{aligned}
& (\mathcal{C} \circ_{(1)} ((\epsilon\partial_{\mathcal{C}(X)}) \vee (\epsilon v)))\Delta_{(1)}(X; Y) + d_e(X; Y) \\
&= \mathcal{C}(\epsilon(X); \epsilon(X; Y))(\mathcal{C} \circ'' (\partial_{\mathcal{C}(X)}; v))\Delta(X; Y) \\
&\quad + \mathcal{C}(\epsilon(X); \epsilon(X; Y))d_e(\mathcal{C}(X); \mathcal{C}(X; Y))\Delta(X; Y) \\
&= \mathcal{C}(\epsilon(X); \epsilon(X; Y))\Delta(X; Y)v \\
&= (\mathcal{C}\epsilon)(X; Y)\Delta(X; Y)v = v,
\end{aligned}$$

where the expression $\mathcal{C}(\epsilon(X); \epsilon(X; Y))\Delta(X; Y)v$ is interpreted as the composite

$$\mathcal{C}(X; Y) \xrightarrow{v} \mathcal{C}(X; Y) \xrightarrow{\Delta(X; Y)} (\mathcal{CC})(X; Y) \cong \mathcal{C}(\mathcal{C}(X); \mathcal{C}(X; Y)) \xrightarrow{\mathcal{C}(\epsilon(X); \epsilon(X; Y))} \mathcal{C}(X; Y)$$

and the expression $(\mathcal{C}\epsilon)(X; Y)\Delta(X; Y)v$ is interpreted as the composite

$$\mathcal{C}(X; Y) \xrightarrow{v} \mathcal{C}(X; Y) \xrightarrow{\Delta(X; Y)} (\mathcal{CC})(X; Y) \xrightarrow{(\mathcal{C}\epsilon)(X; Y)} \mathcal{C}(X; Y).$$

□

Since we are interested in codifferentials we need to examine squares of coderivations. Recall that in the coalgebra case it is well known that the square of a coderivation of odd vertical degree is again a coderivation.

LEMMA 4.13. *Let $g: E \rightarrow E$ and ∂_C be coderivations of odd vertical degree. Then g^2 is a coderivation for $d_C = 0$ with respect to the coderivation ∂_C^2 on C , i.e. the following diagram commutes.*

$$\begin{array}{ccc} E & \xrightarrow{g^2} & E \\ \omega \downarrow & & \downarrow \omega \\ \mathcal{C}(C; E) & \xrightarrow{\mathcal{C} \circ''(\partial_C^2; g^2)} & \mathcal{C}(C; E) \end{array}$$

PROOF. One calculates that due to our assumptions on the degrees of the maps involved

$$\omega g^2 = (\mathcal{C} \circ''(\partial_C; g))^2 \omega.$$

A closer look at the definitions together with the degree hypothesis shows that $(\mathcal{C} \circ''(\partial_C; g))^2$ maps an element $z \in \mathcal{C}(n) \otimes \bigoplus_{i=1}^n (C^{\otimes i-1} \otimes E \otimes C^{\otimes n-i})$ to

$$\sum_{j=1}^n (\mathcal{C}(n) \otimes 1^{\otimes j-1} \otimes (\partial_C \vee g)^2 \otimes 1^{\otimes k-j})(z),$$

with 1 denoting the identity on C or E . Since $(\partial_C \vee g)^2 = \partial_C^2 \vee g^2$ we find that

$$(\mathcal{C} \circ''(\partial_C; g))^2 = \mathcal{C} \circ''(\partial_C^2; g^2).$$

□

4.2. Representations via coderivations. Let \mathcal{P} be a Koszul operad. We already saw that \mathcal{P}_∞ -algebra structures on a vertical bicomplex A with vertical differential d_A are in bijection with the class of square-zero coderivations $\partial_{h+d_A} \epsilon$ induced by $h: \overline{\mathcal{P}^i}(A) \rightarrow A$ and the internal differential d_A on A . We will now prove a similar result for representations. For background on Koszul duality and the cobar construction we refer the reader to [GK94] and [LV12].

For $M \in \text{BiCompl}_v$ to be a representation of A means that there is a morphism

$$f_\infty: \mathcal{P}_\infty(A; M) \rightarrow M$$

of vertical bicomplexes satisfying certain properties. Since $\mathcal{P}_\infty = \Omega(\mathcal{P}^i)$ is free this is equivalent to giving a map

$$f: \overline{\mathcal{P}^i}(A; M) \rightarrow M$$

of bidegree $(0, 1)$ on the augmentation ideal of $\mathcal{P}^i(A; M)$ such that

$$d_M f + f d_{\overline{\mathcal{P}^i}(A; M)} + f_\infty d_2 s^{-1} = 0,$$

with $d_{\overline{\mathcal{P}^i}(A; M)}$ the differential on $\overline{\mathcal{P}^i}(A; M)$ induced by $d_{\mathcal{P}^i}$, d_A and d_M . Here d_2 denotes the twisting differential of the cobar construction and $s^{-1}: \overline{\mathcal{P}^i}(A; M) \rightarrow s^{-1}\overline{\mathcal{P}^i}(A; M)$ the desuspension map.

By Proposition 4.12 the map $d_M \epsilon + f: \mathcal{P}^i(A; M) \rightarrow M$ gives rise to a coderivation $\partial_{d_M \epsilon + f}$ on $\mathcal{P}^i(A; M)$.

PROPOSITION 4.14. *Given an arbitrary map $f: \overline{\mathcal{P}^i}(A; M) \rightarrow M$ the coderivation $\partial_{d_M \epsilon + f}$ squares to zero if and only if f is constructed from a \mathcal{P}_∞ -representation as above.*

PROOF. The results above yield that we only need to check under which conditions $\epsilon \partial_{d_M \epsilon + f}^2$ vanishes. We have

$$\begin{aligned} \epsilon \partial_{d_M \epsilon + f}^2 &= d_M \epsilon \partial_f + f(d_{\mathcal{P}^i} + (\mathcal{P}^i \circ_{(1)} ((d_M \epsilon + f) \vee (d_A \epsilon + h)))(\Delta_{(1)}(A; M)) \\ &= d_M f + f d_{\mathcal{P}^i} + f(\mathcal{P}^i \circ_{(1)} ((d_M \epsilon) \vee (d_A \epsilon)))(\Delta_{(1)}(A; M)) \\ &\quad + f(\mathcal{P}^i \circ_{(1)} (f \vee h))(\Delta_{(1)}(A; M)). \end{aligned}$$

Note that

$$f(\mathcal{P}^i \circ_{(1)} (d_M \epsilon \vee d_A \epsilon))(\Delta_{(1)}(A; M))$$

equals the differential induced on $\overline{\mathcal{P}^i}(A; M)$ by d_A and d_M . Since f is only non-zero on the augmentation ideal we hence find that

$$\epsilon \partial_{d_M + f}^2 = f d_{\overline{\mathcal{P}^i}(A; M)} + d_M f + f(\mathcal{P}^i \circ_{(1)} (f \vee h))(\Delta_{(1)}(A; M)).$$

But

$$f(\mathcal{P}^i \circ_{(1)} (f \vee h))(\Delta_{(1)}(A; M)) = f_\infty d_2 s^{-1}$$

and the result follows. \square

REMARK 4.15. One could also state the result by saying that for a bigraded module M a map $g: \mathcal{P}^i(A; M) \rightarrow M$ of bidegree $(0, 1)$ induces a square-zero coderivation on $\mathcal{P}^i(A; M)$ if and only if $(M, g|_M)$ viewed as a vertical bicomplex with differential $g|_M$ is a \mathcal{P}_∞ -representation of A with structure map induced by $g|_{\overline{\mathcal{P}^i}(A; M)}$. The formulation above is purely a choice of making the role of the vertical differential on M explicit to emphasize the category we work in rather than keeping it implicit.

4.3. Coderivations of $(dAs)^i$ -representations and representations of derived A_∞ -algebras. In section 3 we saw how to describe As^i -coalgebras and $(dAs)^i$ -coalgebras in terms of traditional conilpotent coalgebras. We will now determine what a $(dAs)^i$ -representation of a $(dAs)^i$ -coalgebra C looks like. The results in this section as well as their proofs are analogous to the results for $(dAs)^i$ -coalgebras in 3.5. In particular it yields more insights to describe the structure on the suspension of a representation rather than the representation itself.

REMARK 4.16. Before we concentrate on the derived case, let us consider representations of ordinary A_∞ -algebras. We know that conilpotent As^i -coalgebras and conilpotent coassociative coalgebras correspond to each other, and so do the notions of As^i -coderivation and traditional coderivation. Recall that under this correspondence an As^i -coalgebra C corresponds to the traditional coalgebra sC . For representations the same reasoning shows that (E, ω) is an As^i -representation of C if and only if sE is a coassociative sC -bicomodule. One easily checks that As^i -coderivations on E coincide with coderivations of sE as a bicomodule.

In particular, for $sC = \overline{T}^c(sA) \cong sAs^i(A)$ equipped with a square-zero coderivation making A an A_∞ -algebra we find that representations of A correspond to codifferentials on the $\overline{T}^c(sA)$ -bicomodule $T^c(sA) \otimes sM \otimes T^c(sA) \cong sAs^i(A; M)$. Hence we retrieve the notion of *two-sided module over an A_∞ -algebra* given by Getzler and Jones [GJ90].

PROPOSITION 4.17. *There is an equivalence between the category of $(dAs)^i$ -representations E of C and the category whose objects are sC -bicomodules*

$$(sE, \Delta^L: sE \rightarrow sC \otimes sE, \Delta^R: sE \rightarrow sE \otimes sC),$$

together with a map $f_{sE}: sE \rightarrow sE$ of bidegree $(1, 1)$ such that

$$(f_{sC} \otimes sE)\Delta^L = \Delta^L f_{sE} = (sC \otimes f_{sE})\Delta^L$$

and

$$(sE \otimes f_{sC})\Delta^R = \Delta^R f_{sE} = (f_{sE} \otimes sC)\Delta^R$$

and whose morphisms are bicomodule morphisms commuting with f_{sE} . Under this equivalence a $(\mathrm{dAs})^i$ -coderivation of E of bidegree $(0, 1)$ corresponds to a coderivation of sE as an sC -bicomodule of the same bidegree anti-commuting with f_{sE} .

PROOF. We recalled that E is a $(\mathrm{dAs})^i$ -representation of C if and only if sE is a $\Lambda(\mathrm{dAs})^i$ -representation of sC , hence we might as well determine what $\Lambda(\mathrm{dAs})^i$ -representations are. Similar considerations hold for coderivations on these structures. So suppose E' is a $\Lambda(\mathrm{dAs})^i$ -representations of C' . Let

$$\rho: C' \rightarrow \Lambda(\mathrm{dAs})^i(C') \quad \text{and} \quad \omega: E' \rightarrow \Lambda(\mathrm{dAs})^i(C'; E')$$

denote the structure maps and let

$$\rho^{i,n}: C' \xrightarrow{\rho} \Lambda(\mathrm{dAs})^i(C') \longrightarrow \mathbf{k}\alpha_{in} \otimes C'^{\otimes n} \xrightarrow{\cong} C'^{\otimes n}$$

and

$$\omega^{i,n}: E' \xrightarrow{\omega} (\mathrm{dAs})^i(C'; E') \longrightarrow \mathbf{k}\alpha_{in} \otimes \left(\bigoplus_{a+b+1=n} C'^{\otimes a} \otimes E' \otimes C'^{\otimes b} \right)$$

$$\xrightarrow{\cong} \bigoplus_{a+b+1=n} C'^{\otimes a} \otimes E' \otimes C'^{\otimes b}$$

be the projections of the structure maps to the indicated components. Here $i \geq 0$ and $n \geq 1$ with $\rho^{0,1}$ and $\omega^{0,1}$ equal to the identity.

Spelling out the coassociativity condition for ω in terms of these projections yields the condition that

$$(4.2) \quad ((\rho/\omega)^{i_1, k_1} \otimes \cdots \otimes (\rho/\omega)^{i_n, k_n})\omega^{i, n} = (-1)^\sigma \omega^{i_1 + \cdots + i_n + i, k_1 + \cdots + k_n}$$

where $\sigma = i(k_1 + \cdots + k_n + n) + \sum_{1 \leq x < y \leq n} (i_x k_y + i_y k_x)$, for all $i, i_1, \dots, i_n \geq 0$ and $n, k_1, \dots, k_n \geq 1$, with $(\rho/\omega)^{r, s}$ denoting $\rho^{r, s}$ or $\omega^{r, s}$ depending on the input; see Corollary 6.2. In particular

$$(4.3) \quad ((\rho/\omega)^{0,2} \otimes 1)\omega^{0,2} = (1 \otimes (\rho/\omega)^{0,2})\omega^{0,2}$$

with 1 denoting either the identity on C' or E' , because both terms coincide with $\omega^{0,3}$ and

$$(4.4) \quad ((\rho/\omega)^{1,1} \otimes 1)\omega^{0,2} = \omega^{0,2}\omega^{1,1} = (1 \otimes (\rho/\omega)^{1,1})\omega^{0,2},$$

because all of these compositions are equal to $-\omega^{1,2}$. Hence sE is an sC -bicomodule with a map $f_{sE} = s\omega^{1,1}$ having the properties claimed above. One also sees that

$$(4.5) \quad \omega^{r, s} = (-1)^{r(s+1)}\omega^{0, s}(\omega^{1,1})^r$$

with

$$\omega^{0, s} = ((\rho/\omega)^{0,2} \otimes 1^{\otimes s-2})((\rho/\omega)^{0,2} \otimes 1^{\otimes s-3}) \cdots ((\rho/\omega)^{0,2} \otimes 1)\omega^{0,2}$$

denoting iterated applications of $(\rho/\omega)^{(0,2)}$. Hence $\omega^{0,2}$ and $\omega^{1,1}$ determine ω and one can calculate that (4.5) together with (4.3) and (4.4) yields (4.2).

Since $\alpha_{0,2}$ has vertical degree 0 and $\alpha_{1,1}$ has vertical degree 1 a $\Lambda(\mathrm{dAs})^i$ -coderivation is a map that is a coderivation with respect to the coproduct $\omega^{0,2}$ and anticommutes with $\omega^{1,1}$. \square

Applying this to $C = (\mathrm{dAs})^i(A)$ and $E = (\mathrm{dAs})^i(A; M)$ we find the following.

PROPOSITION 4.18. *The $(\mathrm{dAs})^i$ -representation $(\mathrm{dAs})^i(A; M)$ corresponds to the $\mathbf{k}[x] \otimes \overline{T}^c(sA)$ -bicomodule structure on $\mathbf{k}[x] \otimes T^c(sA) \otimes sM \otimes T^c(sA)$ given by*

$$\begin{aligned} \Delta^L(x^i \otimes (sa_1, \dots, sa_{j-1}, sm, sa_{j+1}, \dots, sa_n)) \\ &= \sum_{k=1}^{j-1} \sum_{r+s=i} (-1)^\epsilon (x^r \otimes (sa_1, \dots, sa_k)) \otimes (x^s \otimes (sa_{k+1}, \dots, sm, \dots, sa_n)), \\ \Delta^R(x^i \otimes (sa_1, \dots, sa_{j-1}, sm, sa_{j+1}, \dots, sa_n)) \\ &= \sum_{k=j}^n \sum_{r+s=i} (-1)^\epsilon (x^r \otimes (sa_1, \dots, sm, \dots, sa_k)) \otimes (x^s \otimes (sa_{k+1}, \dots, sa_n)), \end{aligned}$$

with $\epsilon = r(n+k) + (s, s)(|a_1| + \dots + |a_k|)$, together with the map

$$\begin{aligned} f : \mathbf{k}[x] \otimes T^c(sA) \otimes sM \otimes T^c(sA) &\rightarrow \mathbf{k}[x] \otimes T^c(sA) \otimes sM \otimes T^c(sA), \\ x^i \otimes (sa_1, \dots, sm, \dots, sa_n) &\mapsto (-1)^{n+1} x^{i-1} \otimes (sa_1, \dots, sm, \dots, sa_n) \end{aligned}$$

with $x^{-1} = 0$.

PROPOSITION 4.19. *Let d be a coderivation of $\Lambda(\mathrm{dAs})^i(sA)$ giving rise to a family d_i of coderivations on $\overline{T}^c(sA)$ as discussed in Example 3.3. Giving a $\Lambda(\mathrm{dAs})^i$ -coderivation g on $\Lambda(\mathrm{dAs})^i(sA; sM)$ is equivalent to specifying a family of maps*

$$g_j : T^c(sA) \otimes sM \otimes T^c(sA) \rightarrow T^c(sA) \otimes sM \otimes T^c(sA), \quad j \geq 0,$$

of bidegree $(-j, 1-j)$ such that g_j is a $\overline{T}^c(sA)$ -bicomodule coderivation with respect to d_j .

PROOF. Denote by $g_{i,j}$ the component

$$\mathbf{k}x^i \otimes T^c(sA) \otimes sM \otimes T^c(sA) \rightarrow \mathbf{k}x^j \otimes T^c(sA) \otimes sM \otimes T^c(sA)$$

of g . Since g has to anti-commute with f we see that

$$f^j g_{i,j} = \begin{cases} (-1)^j g_{i-j,0} f^j, & i \geq j, \\ 0, & j > i \end{cases}$$

and hence that g is completely determined by the maps $g_{r,0}$. Define g_r by

$$g_r(sa_1, \dots, sa_{i-1}, sm, sa_{i+1}, \dots, sa_n) = (-1)^{rn} g_{r,0}(x^r \otimes (sa_1, \dots, sm, \dots, sa_n)).$$

Then the g_r are bicomodule coderivations if and only if g is a $\Lambda(\mathrm{dAs})^i$ -coderivation. \square

Applying Proposition 4.14 to the case where $\mathcal{P} = \mathrm{dAs}$ we get that a representation M of a derived A_∞ -algebra A is entirely determined by a square-zero coderivation g of the representation $(\mathrm{dAs})^i(A; M)$ of the $(\mathrm{dAs})^i$ -coalgebra $(\mathrm{dAs})^i(A)$ (endowed itself with the square-zero derivation d defining the A_∞ -algebra structure on A). In Proposition 4.19 we have described explicitly the coderivation g . In the next theorem, we characterize the square-zero coderivations.

THEOREM 4.20. *Let A be a dA_∞ -algebra, and let $h_i : \overline{T}^c(sA) \rightarrow \overline{T}^c(sA)$ be the corresponding coderivations making $\overline{T}^c(sA)$ a twisted chain complex as discussed*

in Proposition 3.4. Then endowing a bigraded \mathbf{k} -module M with the structure of a dA_∞ -representation of A is equivalent to giving maps

$$g_i: T^c(sA) \otimes sM \otimes T^c(sA) \rightarrow T^c(sA) \otimes sM \otimes T^c(sA), \quad i \geq 0,$$

of bidegree $(-i, 1 - i)$ such that

- the g_i make $T^c(sA) \otimes M \otimes T^c(sA)$ a twisted chain complex,
- for all $i \geq 0$ the map g_i is a bicomodule coderivation with respect to h_i .

PROOF. We saw how to construct the maps g_i from a coderivation $g: \Lambda(dAs)^i(A; M) \rightarrow \Lambda(dAs)^i(A; M)$ in the proof of Proposition 4.19. The g_i define a twisted chain complex if and only if for all $u \geq 0$ and all $(sa_1, \dots, sm, \dots, sa_n) \in T^c(sA) \otimes sM \otimes T^c(sA)$

$$\begin{aligned} 0 &= \sum_{i+p=u} (-1)^i g_i g_p(sa_1, \dots, sm, \dots, sa_n) \\ &= \sum_{i+p=u} (-1)^{i+pn} g_i g_{p,0}(x^p \otimes (sa_1, \dots, sm, \dots, sa_n)) \\ &= \sum_{i+p=u} (-1)^{i+pn+i(n+1)} g_i g_{p,0} f^i(x^{p+i} \otimes (sa_1, \dots, sm, \dots, sa_n)) \\ &= \sum_{i+p=u} (-1)^{pn+i(n+1)} g_i f^i g_{p+i,i}(x^{p+i} \otimes (sa_1, \dots, sm, \dots, sa_n)). \end{aligned}$$

But $g_i f^i = (-1)^i g_{i,0}$ on $\mathbf{k}x^i \otimes T^c(sA) \otimes sM \otimes T^c(sA)$, hence the g_i yield a twisted chain complex if and only if

$$0 = \sum_{i+p=u} (-1)^{un} g_{i,0} g_{p+i,i}.$$

Hence the projection of g^2 to $\mathbf{k}x^0 \otimes T^c(sA) \otimes sM \otimes T^c(sA)$ is zero, and Proposition 4.12 yields that $g^2 = 0$ in general. \square

REMARK 4.21. In [Sag10, 6.2] Sagave defines a module over a dA_∞ -algebra A as a bigraded \mathbf{k} -module M such that $sM \otimes T^c(sA)$ is a twisted chain complex whose i -th structure map g_i is a right $\overline{T}^c(sA)$ -coderivation with respect to h_i . The operadic notion of representation hence yields a two-sided variant of Sagave's definition.

5. New example of a derived A_∞ -algebra

In this section, we will use a family of examples of finite dimensional A_∞ -algebras given by Alloca and Lada in [AL10] in order to build a new example of a 3-dimensional derived A_∞ -algebra.

5.1. Examples of finite dimensional A_∞ -algebras. Alloca and Lada give in [AL10] a family of examples of A_∞ -algebras. Taking a subalgebra, one gets the following result as a corollary of [AL10, Theorem 2.1]. Here, the sign conventions for A_∞ -algebras are those of Loday-Vallette.

PROPOSITION 5.1. *The free graded \mathbf{k} -module V spanned by x of degree 0 and y of degree 1 is an A_∞ -algebra with \mathbf{k} -linear structure maps satisfying:*

$$\begin{aligned} m_1(x) &= y, \\ m_n(x \otimes y^{\otimes k} \otimes x \otimes y^{(n-2)-k}) &= (-1)^k s_n x, \quad \text{for } 0 \leq k \leq n-2, \\ m_n(x \otimes y^{n-1}) &= s_{n+1} y, \end{aligned}$$

where $s_n = (-1)^{(n+1)(n+2)/2}$, and $m_n(z) = 0$ for any n and any basis element $z \in V^{\otimes n}$ not listed above. \square

REMARK 5.2. If we modify the above example so that $m_1 = 0$, but everything else is unchanged, then V is still an A_∞ -algebra. That is, we can construct a minimal example from the one above, where we recall that a minimal A_∞ -algebra A is an A_∞ -algebra such that $m_1 = 0$.

5.2. Example of a derived A_∞ -algebra. We describe an example of a derived A_∞ -structure on a rank 3 free bigraded \mathbf{k} -module V spanned by u, v, w where $|u| = (0, 0)$, $|v| = (-1, 0)$, and $|w| = (0, 1)$.

Note that if V is as above, the bidegree $(-k, l)$ of an element $z \in V^{\otimes j}$ satisfies $0 \leq k, 0 \leq l$ and $k + l \leq j$. Since the structure map $m_{in} : V^{\otimes n} \rightarrow V$ is of bidegree $(-i, 2 - i - n)$, the element $m_{in}(z)$ has bidegree $(-k - i, 2 - i - n + l)$. This has the following consequence.

PROPOSITION 5.3. *If the bigraded \mathbf{k} -module V as above is endowed with a derived A_∞ -structure then, for reasons of bidegree, $m_{in}(z)$ with $z \in V^{\otimes n}$ can be potentially non-zero only if $0 \leq i \leq 1$. Furthermore, letting $z = x_1 \otimes \cdots \otimes x_n$ where each x_i is one of the basis elements of V , we have the following.*

- (1) *If $m_{0n}(z) \neq 0$, then there exist $i \neq j$ such that $x_k = w$ for $k \notin \{i, j\}$ and $(x_i, x_j) \in \{(u, u), (u, w), (w, u), (u, v), (v, u)\}$.*
- (2) *If $m_{1n}(z) \neq 0$, then there exists i such that $x_i = u$ and $x_k = w$ for $k \neq i$.*

\square

PROPOSITION 5.4. *Let V be the rank 3 free bigraded \mathbf{k} -module as above. Then V is endowed with the following derived A_∞ -structure. For $n \geq 2$, we let*

$$\begin{aligned} m_{0n}(u \otimes w^{\otimes k} \otimes u \otimes w^{\otimes (n-2)-k}) &= (-1)^k s_n u, \quad \text{for } 0 \leq k \leq n-2, \\ m_{0n}(u \otimes w^{\otimes n-1}) &= s_{n+1} w, \\ m_{0n}(u \otimes w^{\otimes n-2} \otimes v) &= (-1)^{n-2} s_n v, \end{aligned}$$

and for $n \geq 1$, we let

$$\begin{aligned} m_{11}(u) &= v, \\ m_{1n}(u \otimes w^{\otimes n-1}) &= s_{n+1} v, \end{aligned}$$

where $s_n = (-1)^{(n+1)(n+2)/2}$ and we let $m_{ij}(z) = 0$ for any i, j and for any basis element $z \in V^{\otimes j}$ not covered by the cases above.

PROOF. The proof is just a computation. We will not give full details, but we supply enough ingredients so that the computation can be carried out quickly.

Note that to check that V is a derived A_∞ -algebra we only need to check that, for $l \geq 1$ and $z \in V^{\otimes l+1}$, the following three conditions hold.

$$\begin{aligned} \sum_{j+q=l+1} m_{0j} \star m_{0q}(z) &= 0, \\ \sum_{j+q=l+1} (m_{0j} \star m_{1q} + m_{1j} \star m_{0q})(z) &= 0, \\ \sum_{j+q=l+1} m_{1j} \star m_{1q}(z) &= 0, \end{aligned}$$

with the \star -product defined in the formula (6.1) of the appendix.

We consider the three relations in turn, outlining the checking required for each.

Relation I $\sum_{j+q=l+1} m_{0j} \star m_{0q}(z) = 0.$

Let $V_0 = \langle u, w \rangle$ be the subspace of V spanned by the elements of bidegree $(0, r)$, for $r \in \mathbb{Z}$. If V is a derived A_∞ -algebra, then V_0 is an A_∞ -algebra. As a consequence checking the equation on tensors z not containing v is equivalent to checking that V_0 is an A_∞ -algebra. This is true by Proposition 5.1.

It remains to check the equation on tensors containing v . For terms containing at least one v , $m_{0j}(1^{\otimes*} \otimes m_{0q} \otimes 1^{\otimes*})$ is possibly non-zero only on tensors of the form

$$u \otimes w^{\otimes k} \otimes u \otimes w^{\otimes l-k-3} \otimes v, \text{ for } 0 \leq k \leq l-3, \text{ where } j+q=l+1,$$

and a sign computation shows that the expression vanishes on those terms.

Relation II $\sum_{j+q=l+1} (m_{0j} \star m_{1q} + m_{1j} \star m_{0q})(z) = 0.$

This case is similar to the previous one; $m_{0j}(1^{\otimes*} \otimes m_{1q} \otimes 1^{\otimes*}) + m_{1j}(1^{\otimes*} \otimes m_{0q} \otimes 1^{\otimes*})$ is possibly non-zero only on tensors of the form

$$u \otimes w^{\otimes k} \otimes u \otimes w^{\otimes l-k-2}, \text{ for } 0 \leq k \leq l-2, \text{ where } j+q=l+1.$$

Relation III $\sum_{j+q=l+1} m_{1j} \star m_{1q}(z) = 0.$

Since m_{1n} takes values zero or $\pm v$ on basis elements and since m_{1n} applied to a tensor containing a v vanishes, it follows that $\sum_{j+q=l+1} m_{1j} \star m_{1q}(z) = 0.$ \square

REMARK 5.5. In this example, we have $m_{01} = 0$; that is, we have a minimal derived A_∞ -algebra.

For bidegree reasons, the only alternative would be letting $m_{01}(u)$ be (some multiple of) w . However, modifying the above example so that $m_{01}(u) = w$, with everything else unchanged, does not give a derived A_∞ -algebra. A direct computation shows that we would have

$$\sum_{j+q=4} (m_{0j} \star m_{1q} + m_{1j} \star m_{0q})(u \otimes w \otimes u) = v \neq 0$$

and

$$\sum_{j+q=4} (m_{0j} \star m_{1q} + m_{1j} \star m_{0q})(u \otimes u \otimes w) = -v \neq 0.$$

On the other hand, if we ‘truncate’ the above example, setting $m_{ij} = 0$ for $i + j \geq 3$, then it can be checked, using SAGE, that we get a bidga, both in the case with $m_{01} = 0$ and also in the case where we modify the example so that $m_{01}(u) = w$.

6. Appendix: sign conventions

In this appendix, we compare different sign conventions relating to derived A_∞ -algebras. In the special case of A_∞ -algebras such comparisons have been mentioned in the literature.

6.1. Different conventions for derived A_∞ -algebras. We recall that a derived A_∞ -structure on A consists of \mathbf{k} -linear maps $m_{ij} : A^{\otimes j} \rightarrow A$ of bidegree $(-i, 2 - i - j)$ for each $i \geq 0, j \geq 1$, satisfying the equation (2.1) of Definition 2.1:

$$\sum_{\substack{u=i+p, v=j+q-1 \\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0.$$

Consequently the family of maps m_{0n} satisfies the equation

$$\sum_{\substack{v=j+q-1 \\ j=1+r+t}} (-1)^{rq+t} m_{0j}(1^{\otimes r} \otimes m_{0q} \otimes 1^{\otimes t}) = 0,$$

which is the sign convention of Getzler and Jones in [GJ90]. In the definition of derived A_∞ -algebra if we pick the generators

$$\tilde{m}_{ij} = (-1)^{\frac{i(j-1)}{2}} m_{ij}$$

one gets

$$\sum_{u=i+p, v=j+q-1} \tilde{m}_{ij} \star \tilde{m}_{pq} = 0,$$

with

$$(6.1) \quad \tilde{m}_{ij} \star \tilde{m}_{pq} = \sum_{k=1}^j (-1)^{i+j+(q-1)(k+j)+p(j-1)} \tilde{m}_{ij} \circ_k \tilde{m}_{pq}$$

The family \tilde{m}_{0n} satisfies

$$\sum_{u=i+p, v=j+q-1} \sum_{k=1}^j (-1)^{vq+k(q-1)} \tilde{m}_{0j} \circ_k \tilde{m}_{0q} = 0,$$

which is the original definition of A_∞ -algebras by Stasheff [Sta63].

6.2. Different sign conventions for the cooperad $(dAs)^i$. For any graded cooperad \mathcal{C} , if one has elements $a_{uv} \in \mathcal{C}(v)$ satisfying

$$\Delta(a_{uv}) = \sum_{q_1 + \dots + q_j = v} (-1)^{X(I)} a_{ij}; a_I$$

with $a_I = a_{p_1 q_1} \otimes \dots \otimes a_{p_j q_j}$, then setting $\tilde{a}_{uv} = (-1)^{\frac{v(v-1)}{2}} a_{uv}$, one gets

$$\Delta(\tilde{a}_{uv}) = \sum (-1)^{X(I)} (-1)^{\phi(I)} \tilde{a}_{ij}; \tilde{a}_I,$$

where $\phi(I)$ is obtained modulo 2 as

$$\phi(I) = \frac{1}{2} \left(j(j-1) + \left(\sum_k q_k \right) \left(\left(\sum_l q_l \right) - 1 \right) + \sum_k q_k^2 - \sum_l q_l \right) = \sum_{k=1}^{j-1} k + \sum_{k<l} q_k q_l.$$

Recall that the cooperad $(\mathrm{d}\mathcal{A}s)^i$ has generators μ_{uv} of bidegree $(-u, 1-u-v)$ with structure map given by

$$\Delta(\mu_{uv}) = \sum_{\substack{i+p_1+\dots+p_j=u \\ q_1+\dots+q_j=v}} (-1)^{X((p_1, q_1), \dots, (p_j, q_j))} \mu_{ij}; \mu_{p_1 q_1} \otimes \dots \otimes \mu_{p_j q_j},$$

with $X((p_1, q_1), \dots, (p_j, q_j)) = \sum_{1 \leq k < l \leq j} (p_k + q_k(p_l + q_l + 1))$ (see formula (4) in [LRW13]). Consequently the bigraded \mathbf{k} -module generated by the family $(\tilde{\mu}_{0v})_{v \geq 0}$ is a subcooperad of $(\mathrm{d}\mathcal{A}s)^i$ and satisfies

$$\Delta(\tilde{\mu}_{0v}) = \sum_{q_1+\dots+q_j=v} (-1)^{X'(q_1, \dots, q_j)} \tilde{\mu}_{0j}; \tilde{\mu}_{0q_1} \otimes \dots \otimes \tilde{\mu}_{0q_j},$$

with

$$X'(q_1, \dots, q_j) \equiv \sum_{1 \leq k < l \leq j} (q_k(q_l+1) + q_k q_l) + \sum_{k=1}^{j-1} k \equiv \sum_{k=1}^{j-1} (q_k(k+j) + k) \equiv \sum_{k=1}^{j-1} (q_k+1)(k+j),$$

where the computation is performed modulo 2. We recover the signs obtained by Loday and Vallette in [LV12] in their definition of the cooperad $\mathcal{A}s^i$.

Note that if we choose $\tilde{\mu}_{uv}$ as generators for the cooperad $(\mathrm{d}\mathcal{A}s)^i$, the structure map is given by

$$\Delta(\tilde{\mu}_{uv}) = \sum_{\substack{i+p_1+\dots+p_j=u \\ q_1+\dots+q_j=v}} (-1)^{X'((p_1, q_1), \dots, (p_j, q_j))} \tilde{\mu}_{ij}; \tilde{\mu}_{p_1 q_1} \otimes \dots \otimes \tilde{\mu}_{p_j q_j},$$

where $X'((p_1, q_1), \dots, (p_j, q_j)) = \sum_{k=1}^{j-1} (p_k + q_k + 1)(k + j) + \sum_{k < l} (q_k p_l)$.

6.3. Description of the cooperad $\Lambda(\mathrm{d}\mathcal{A}s)^i$. The notion of suspension of a cooperad was explained in Section 3.2. Here we establish the sign conventions for the cooperad structure of $\Lambda\mathcal{C}$ for a cooperad \mathcal{C} in $\mathrm{BiCompl}_v$.

PROPOSITION 6.1. *Let \mathcal{C} be a cooperad in $\mathrm{BiCompl}_v$. Then $\Lambda\mathcal{C}$ is the cooperad with*

$$(\Lambda\mathcal{C})(n) = s^{1-n}\mathcal{C}(n).$$

The cooperad structure of $\Lambda\mathcal{C}$ is given by

$$s^{1-n}\mathcal{C} \mapsto \sum (-1)^{\sum_{k=1}^j |s^{l_k+1}| |s^{1-j}c'| + \sum_{k=2}^j \sum_{i=1}^{k-1} |s^{l_k+1}| |sc''_i|} s^{1-j}c'; s^{1-l_1}c''_1, \dots, s^{1-l_j}c''_j,$$

where the decomposition map of \mathcal{C} maps $c \in \mathcal{C}(n)$ to the sum $\sum c'; c''_1, \dots, c''_j$ with $c' \in \mathcal{C}(j)$ and $c''_i \in \mathcal{C}(l_i)$ for $1 \leq i \leq j$.

PROOF. We explain the algorithm for distributing s^{1-n} over the different tensor products.

Firstly put s^{1-j} in front of c' . This operation is sign free.

Secondly, distribute one s , that will be in front of c''_k , for k going from 1 to j : first s jumps over $s^{1-j}c'$ and is placed in front of c''_1 ; second s jumps over $s^{1-j}c' \otimes sc''_1$

and is placed in front of c'_2 . The sign involved is obtained as $(-1)^x$ where $x \bmod 2$ is $\sum_{k=1}^j |s| |s^{1-j} c'| + \sum_{k=2}^j |s| (\sum_{l=1}^{k-1} |s c'_l|)$.

Finally, for k going down from j to 1 distribute s^{-l_k} over $s^{1-j} c' \otimes s c'_1 \otimes \cdots \otimes s c'_{k-1}$. The sign involved is obtained as $(-1)^x$ where $x \bmod 2$ is $\sum_{k=1}^j |s^{l_k}| |s^{1-j} c'| + \sum_{k=2}^j |s^{l_k}| \sum_{l=1}^{k-1} |s c'_l|$. \square

COROLLARY 6.2. *The cooperad $\Lambda(\mathrm{dAs})^i$ has generators α_{uv} of bidegree $(-u, -u)$ and the cooperad structure is given by*

$$\Delta(\alpha_{uv}) = \sum_{\substack{i+p_1+\cdots+p_j=u \\ q_1+\cdots+q_j=v}} (-1)^{i(v+j)+\sum_{1 \leq k < l \leq j} p_k q_l + q_k p_l} \alpha_{ij}; \alpha_{p_1 q_1} \otimes \cdots \otimes \alpha_{p_j q_j},$$

Note that if $u = 0, i = 0, p_k = 0$ one gets exactly the cooperad $\mathcal{A}s^*$.

PROOF. This is a short sign computation. Let $I = ((p_1, q_1), \dots, (p_j, q_j))$ and let $S(I)$ be the sum such that $(-1)^{S(I)}$ is the sign defined in Proposition 6.1. We recall that $\alpha_{uv} = s^{1-v} \mu_{uv}$ and that $s \mu_{uv}$ has bidegree $(-u, -u - v)$. Computing mod 2, one gets

$$\begin{aligned} X(I) + S(I) &\equiv \sum_{1 \leq k < l \leq j} \binom{p_k + q_k(p_l + q_l + 1)}{1} + \sum_{k=1}^j (q_k + 1)i + \sum_{1 \leq k < l \leq j} (p_k + q_k)(q_l + 1) \\ &\equiv i(v + j) + \sum_{1 \leq k < l \leq j} q_k p_l + p_k q_l. \end{aligned}$$

\square

EXAMPLE 6.3. As an example one has

$$\begin{aligned} \Delta(\alpha_{12}) &= \alpha_{12}; \alpha_{01} \otimes \alpha_{01} - \alpha_{11}; \alpha_{02} - \alpha_{02}; (\alpha_{11} \otimes \alpha_{01} + \alpha_{01} \otimes \alpha_{11}) + \alpha_{01}; \alpha_{12}, \\ \Delta(\alpha_{ij}) &= (-1)^{i(j+1)} \alpha_{i1}; \alpha_{0j} + \text{other terms.} \end{aligned}$$

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