# Partially dualized Hopf algebras have equivalent Yetter-Drinfel'd modules

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ABSTRACT. Given a Hopf algebra $H$ and a projection $H \to A$ to a Hopf subalgebra, we construct a Hopf algebra $r(H)$, called the partial dualization of $H$, with a projection to the Hopf algebra dual to $A$. This construction provides powerful techniques in the general setting of braided monoidal categories. The construction comprises in particular the reflections of generalized quantum groups [HS13]. We prove a braided equivalence between the Yetter-Drinfel'd modules over a Hopf algebra and its partial dualization.

## CONTENTS

## 1. Introduction and Summary

1.1. **Introduction.** One of the fundamental observations about a finite-dimensional Hopf algebra $H$ over a field $\Bbbk$ is the fact that the dual vector space $H^*$ has the structure of a Hopf algebra as well. The Hopf algebras $H$ and $H^*$ are, typically, rather different.

In this article we consider a partial dualization of a Hopf algebra $H$ in which the following data $\mathcal{A}$ enter: a projection $\pi : H \to A$ to a Hopf subalgebra $A$, and a Hopf algebra $B$ dual to the subalgebra $A$. The duality is expressed in terms of a non-degenerate Hopf pairing $\omega : A \otimes B \to \mathbf{1}$. these data, we construct another Hopf algebra $r_{\mathcal{A}}(H)$ which has $B$ as a Hopf subalgebra and comes with a projection to $B$. We call $r_{\mathcal{A}}(H)$ the *partially dualized* Hopf algebra. As we will see in Subsection 1.2, such partially dualized Hopf algebras appear in rather different contexts, including the classification of certain pointed Hopf algebras in terms of their Borel parts, the Nichols algebras [AS02, H09, AHS10].

The guiding example of Nichols algebras suggests the following setting for our investigations: for any braided category $\mathcal{C}$, there is a natural notion of a Hopf algebra in $\mathcal{C}$. In the present paper, $H, A$ and $B$ will be Hopf algebras in a braided category $\mathcal{C}$. Working with Hopf algebras in braided categories significantly simplifies the motivating construction for generalized quantum groups [HS13], since it avoids explicit calculations with smash products.

Our second aim is to exhibit a representation-theoretic relation between the Hopf algebras $H$ and $r_{\mathcal{A}}(H)$. Typically $H$ and $r_{\mathcal{A}}(H)$ are not isomorphic, nor even Morita-equivalent. In the present paper, we show the following, more subtle relation: The categories of Yetter-Drinfel'd modules over $H$ and $r_{\mathcal{A}}(H)$ are equivalent as braided categories. This insight is new, even in the case of generalized quantum groups.

The equivalence of categories of Yetter-Drinfel'd modules implies a relation between the Hopf algebra $H$ and and its partial dualization $r_{\mathcal{A}}(H)$ which we discuss in the case of a Hopf algebra over a field: the category of Yetter-Drinfel'd modules is the Drinfel'd center of the category of modules. It is well-known that semisimple algebras with isomorphic centers are Morita equivalent. Replace the notion of an algebra by the one of a monoidal category and, similarly, the notion of a module over a algebra by the notion of a module category over a monoidal category. Then, it is known [ENO11, Thm 3.1] that semisimple tensor categories with braided-equivalent Drinfel'd centers have equivalent bicategories of module categories. This relation has been termed weak monoidal Morita equivalence [Mu03]. It is therefore tempting to speculate that the bicategories of module categories over the monoidal categories $H$-mod and $r_{\mathcal{A}}(H)$-mod are closely related, if not equivalent.

1.2. **Examples.** We discuss several examples of partial dualizations and relate them to known results in the literature; all examples will be discussed in more detail in section 5. The two extremal cases of dualizations are trivial:

- Taking $\pi : A \overset{\sim}{\to} H$ yields a complete dualization: $r_{\mathcal{A}}(H) = B$.
- For the projection $\pi : H \to \mathbf{1}_{\mathcal{C}} =: A$ to the monoidal unit $\mathbf{1}$ of $\mathcal{C}$, we get $r_{\mathcal{A}}(H) = H$.

*Group algebras* of a finite groups already provide examples of non-trivial partial dualizations. Consider the complex group algebra $\mathbb{C}[G]$ of a finite group $G$ which we assume to be a semi-direct product $G = N \rtimes Q$. As a consequence, $\mathbb{C}[G]$ is a Radford biproduct: $\mathbb{C}[G] = \mathbb{C}[N] \rtimes \mathbb{C}[Q]$ with a trivial coaction and a non-trivial action of $\mathbb{C}[Q]$ on $\mathbb{C}[N]$. The (cocommutative) Hopf algebra $\mathbb{C}[Q]$ is dual to the (commutative) Hopf algebra $\mathbb{C}^Q$ of complex functions on $Q$. The partial dualization with respect to the Hopf subalgebra $\mathbb{C}[Q]$ yields a Hopf algebra $\mathbb{C}[N] \rtimes \mathbb{C}^Q$ with a trivial action and a non-trivial coaction of $\mathbb{C}^Q$ on $\mathbb{C}[N]$. The partially dualized Hopf algebra is neither a group algebra nor a dual group algebra.

The monoidal category of modules over the partially dualized Hopf algebra $r_{\mathcal{A}}(\mathbb{C}[G]) = \mathbb{C}[N] \rtimes \mathbb{C}^Q$ turns out to be monoidally equivalent to the category of bimodules over an algebra in the category $\mathrm{vect}_G$ of $G$-graded vector spaces. Our general result thus implies that the Drinfel'd center of the category $\mathbb{C}[G]$-mod and the Drinfel'd center of the category of bimodules are braided equivalent. This equivalence is a special case of [S01, Theorem 3.3].

The *Taft algebra* $T_\zeta$, with $\zeta$ a primitive $d$-th root of unity, is the Hopf algebra generated by a group-like element $g$ of order $d$, and a skew-primitive element $x$ with coproduct $\Delta(x) = g \otimes x + x \otimes 1$. It has a projection $\pi$ to the Hopf subalgebra $A \cong \mathbb{C}[\mathbb{Z}_d]$ generated by $g$. The partial dualization $r_{\mathcal{A}}(H)$ is isomorphic to the Taft algebra itself; the isomorphism depends on a choice of a Hopf pairing $\omega : A \otimes A \to \mathbb{C}$ and thus on a primitive $d$-th root of unity. An example with non-trivial partial dualization is provided by a central extension $\hat{T}_{\zeta,q}$ of the Taft algebra $T_\zeta$ by group-like elements. The partial dualization $r_{\mathcal{A}}(\hat{T}_{\zeta,q}) =: \check{T}_{\zeta,q}$ then does not possess such central group-like elements; instead, the coproduct of the skew-primitive element of $\check{T}_{\zeta,q}$ is modified, leading to additional central characters for the partially dualized Hopf algebra $\check{T}_{\zeta,q}$.

The *reflection of generalized quantum groups* as introduced in [AHS10, HS13] was the original motivation of our construction. In this case, the braided category $\mathcal{C}$ is a category of Yetter-Drinfel'd modules over a complex Hopf algebra $h$, i.e. $\mathcal{C} = {}^h_h\mathcal{YD}(\mathrm{vect}_{\mathbb{C}})$. Usually, the complex Hopf algebra $h$ is the complex group algebra of a finite group, $h = \mathbb{C}[G]$. Next, fix a semisimple object

$$M = M_1 \oplus M_2 \oplus \cdots M_n$$

in $\mathcal{C}$ and consider the Nichols algebra $\mathcal{B}(M)$. It is a Hopf algebra in the braided category $\mathcal{C}$ and plays the role of a quantum Borel part of a pointed Hopf algebra.

For each simple subobject $M_i$ in the direct decomposition of $M$, the Nichols algebra $A_i := \mathcal{B}(M_i)$ is a subalgebra of $\mathcal{B}(M)$; moreover, there is a natural projection of Hopf algebras $\pi_i : \mathcal{B}(M) \to \mathcal{B}(M_i)$. The Nichols algebra $\mathcal{B}(M_i^*)$ for the object $M_i^*$ in $\mathcal{C}$ dual to $M_i$ comes with a non-degenerate Hopf pairing $\omega_i : \mathcal{B}(M_i) \otimes \mathcal{B}(M_i^*) \to \mathbf{1}$. We can thus perform a partial dualization. If the Nichols algebra $\mathcal{B}(M)$ plays the role of a quantum Borel part, the partially dualized Hopf algebra of $\mathcal{B}(M)$ is isomorphic to a quantum Borel part of in $u_q(\mathfrak{g})$ after a reflection on a simple root.

Examples are known that do not correspond to semisimple Lie algebras [H09] and that can have a non-abelian Cartan subalgebra [AHS10, HS10]. In these cases, a Borel subalgebra $H$ and its reflection $r_{\mathcal{A}}(H)$ are not necessarily isomorphic. We exhibit an explicit example in Section 5.3

1.3. **Structure of the article and summary of results.** Section 2 contains an overview of the theory of Hopf algebras in braided categories. Some readers may prefer to skip Sections 2 and 3, assuming that the braided category is the one of complex vector spaces $\mathcal{C} = \mathrm{vect}_{\mathbb{C}}$, and thereby restricting themselves to the case when $H$ is a finite-dimensional complex Hopf algebra. (This setting is not general enough to cover the example of pointed Hopf algebras, though.)

In Section 3, we review the notion of Yetter-Drinfel'd modules over a Hopf algebra $A$ in a braided category $\mathcal{C}$ as defined in [B95]. The category of Yetter-Drinfel'd modules is a braided category $_A^A\mathcal{YD}(\mathcal{C})$. As in case of complex Yetter-Drinfel'd modules, there exists a notion of a Radford biproduct or Majid bosonization for Hopf algebras in $\mathcal{C}$ (see Definition 3.7); it turns a Hopf algebra $K$ in the braided category $_A^A\mathcal{YD}(\mathcal{C})$ into a Hopf algebra $K \rtimes A$ in $\mathcal{C}$.

The Radford projection theorem 3.10 provides a converse: given a projection $\pi : H \to A$ in $\mathcal{C}$ to a Hopf subalgebra $A \subset H$, the coinvariants $K \subset H$ with respect to $\pi$ have a natural structure of a Hopf algebra in the braided category $_A^A\mathcal{YD}(\mathcal{C})$, such that $H$ can be expressed as a Radford biproduct, $H \cong K \rtimes A$. It is then known [B95, Proposition 4.2.3], see also Theorem 3.12, that the following braided categories are isomorphic:

$$_{K \rtimes A}^{K \rtimes A}\mathcal{YD}(\mathcal{C}) \cong \,_K^K\mathcal{YD}(_A^A\mathcal{YD}(\mathcal{C})) \ .$$

We are now ready to describe the construction of partial dualization: suppose that $A$ and $B$ are Hopf algebras and that $\omega : A \otimes B \to \mathbf{1}_{\mathcal{C}}$ is a non-degenerate Hopf pairing. We relate the categories of Yetter-Drinfel'd modules by an isomorphism of braided categories

$$\Omega^{\omega} : \,_A^A\mathcal{YD}(\mathcal{C}) \overset{\sim}{\to} \,_B^B\mathcal{YD}(\mathcal{C}) \ .$$

It is constructed in two steps: we use the Hopf pairing $\omega$ to turn the left $A$-action into a right $B$-coaction and the left $A$-coaction into a right $B$-action. Then the braiding of $\mathcal{C}$ is used in a second step to turn right (co-)actions into left (co-)actions. Schematically, indicating the relevant propositions of the paper, we have

$$\Omega^{\omega} : \ _A^A\mathcal{YD}(\mathcal{C}) \xrightarrow{3.19} \mathcal{YD}_B^B(\mathcal{C}) \xrightarrow{3.16} \,_B^B\mathcal{YD}(\mathcal{C}).$$

Diagrammatically, the $B$-action and $B$-coaction for the Yetter-Drinfel'd module $\Omega^\omega(X)$ are given as follows:
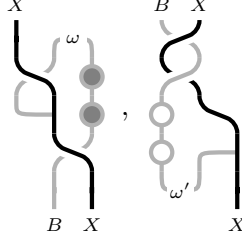


Here, filled circles denote the inverse of the antipode and empty circles the antipode of $B$.

As already explained, the input of our construction is a *partial dualization datum* $\mathcal{A}$: it consists of a Hopf algebra projection $\pi : H \to A$ to a Hopf subalgebra, and a Hopf algebra $B$ with a non-degenerate Hopf pairing $\omega : A \otimes B \to \mathbf{1}_\mathcal{C}$. In Section 4, we construct for a given partial dualization datum $\mathcal{A}$ a new Hopf algebra $r_\mathcal{A}(H)$ in $\mathcal{C}$ as follows:

(1) The Radford projection theorem, applied to the projection $\pi : H \to A$, allows us to write the Hopf algebra $H$ in the form $H \cong K \rtimes A$, with $K$ a Hopf algebra in the braided category ${}^A_A \mathcal{YD}(\mathcal{C})$.

(2) The braided monoidal equivalence $\Omega^\omega$ implies that the image of the Hopf algebra $K$ in the braided category ${}^A_A \mathcal{YD}(\mathcal{C})$ is a Hopf algebra $L := \Omega^\omega(K)$ in the braided category ${}^B_B \mathcal{YD}(\mathcal{C})$.

(3) The partially dualized Hopf algebra $r_\mathcal{A}(H)$ is defined as the bosonization $r_\mathcal{A}(H) := L \rtimes B$ of $L$. This is a Hopf algebra in the braided category $\mathcal{C}$.

To summarize, we dualize a Hopf subalgebra $A$ of $H$ and at the same time covariantly transform the remaining coinvariants $K \subset H$ to $L \subset r_\mathcal{A}(H)$. As a combination of contra- and covariant operations, partial dualization is *not* functorial in $H$.

We list some more results of Section 4:

- the partial dualization datum $\mathcal{A}$ for $H$, one can obtain a partial dualization datum $\mathcal{A}^-$ of the Hopf algebra $r_\mathcal{A}(H)$. There is a canonical isomorphism of Hopf algebras in $\mathcal{C}$ such that

$$r_{\mathcal{A}^-}\left(r_\mathcal{A}(H)\right) \cong H \ ,$$

showing that partial dualization is essentially involutive.

- Theorem 4.4 then asserts that the categories of Yetter-Drinfel'd modules for a Hopf algebra $H$ in $\mathcal{C}$ and its partial dualization $r_\mathcal{A}(H)$ are braided equivalent:

$${}^H_H \mathcal{YD}\left(\mathcal{C}\right) \cong {}^K_K \mathcal{YD}\left({}^A_A \mathcal{YD}\left(\mathcal{C}\right)\right) \xrightarrow{\Omega^\omega} {}^L_L \mathcal{YD}\left({}^B_B \mathcal{YD}\left(\mathcal{C}\right)\right) \cong {}^{r_\mathcal{A}(H)}_{r_\mathcal{A}(H)} \mathcal{YD}\left(\mathcal{C}\right)$$

In Section 5, we finally discuss three classes of examples of partial dualizations.

## 2. Preliminaries

We assume that the reader is familiar with the definition of a braided monoidal category, see e.g. [K95] as a general reference. Denote by $\mathcal{C}$ a

monoidal category with tensor product $\otimes$ and unit object $\mathbf{1}$; without loss of generality, we assume that $\mathcal{C}$ is strict. If $\mathcal{C}$ is endowed with a braiding, we denote it by $c : \otimes \to \otimes^{\mathrm{op}}$. For any braided category $\mathcal{C}$, the monoidal category $\mathcal{C}$ with the inverse braiding $\overline{c}_{X,Y} := c_{Y,X}^{-1}$ is denoted by $\overline{\mathcal{C}}$.

We use the graphical calculus for braided categories for which we fix our conventions as follows: diagrams are read from bottom to top. In Fig. 1 we depict the identity of an object $X$ in $\mathcal{C}$, a morphism $h : X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$, the composition $g \circ f$ of morphisms $f : X \to Y$ and $g : Y \to Z$ and the tensor product of $f$ with $f' : X' \to Y'$ by juxtaposition. The braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ and the inverse braiding $c_{X,Y}^{-1} : Y \otimes X \to X \otimes Y$ are shown in Fig. 2.
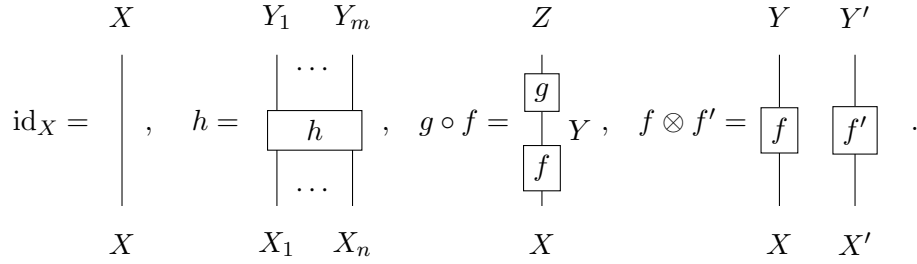


FIGURE 1. Graphical notation for morphisms in monoidal categories

2.1. **Hopf algebras in braided categories.** We recall the definitions of an algebra and of a coalgebra in a (not necessarily braided) monoidal category and the notion of a bialgebra resp. Hopf algebra in a braided category $\mathcal{C}$. For the braided category of $\Bbbk$-vector spaces, these definitions specialize to the textbook definitions.

**Definition 2.1.** *Let $\mathcal{C}$ be a braided category. An object $A$ together with morphisms $\mu_A : A \otimes A \to A$, $\eta_A : \mathbf{1} \to A$, $\Delta_A : A \to A \otimes A$, $\varepsilon_A : A \to \mathbf{1}$ and $S : A \to A$ is called* Hopf algebra in $\mathcal{C}$, *if*

(1) *the triple $(A, \mu_A, \eta_A)$ is a unital, associative algebra in $\mathcal{C}$, i.e.*

$$\mu_A \circ (\mu_A \otimes \mathrm{id}_A) = \mu_A \circ (\mathrm{id}_A \otimes \mu_A)$$
$$\mu_A \circ (\eta_A \otimes \mathrm{id}_A) = \mathrm{id}_A = \mu_A \circ (\mathrm{id}_A \otimes \eta_A),$$
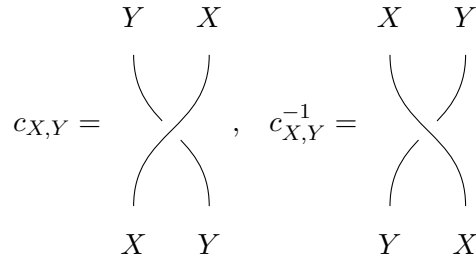


FIGURE 2. Braiding and inverse braiding

(2) *the triple $(A, \Delta_A, \varepsilon_A)$ is a counital, coassociative coalgebra in $\mathcal{C}$, i.e.*

$$(\Delta_A \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta_A) \circ \Delta_A$$
$$(\varepsilon_A \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon_A) \circ \Delta_A,$$

(3) *the morphisms $\mu_A, \eta_A, \Delta_A$ and $\varepsilon_A$ obey the equations*

$$\Delta_A \circ \mu_A = (\mu_A \otimes \mu_A) \circ (\mathrm{id}_A \otimes c_{A,A} \otimes \mathrm{id}_A) \circ (\Delta_A \otimes \Delta_A)$$
$$\varepsilon_A \circ \mu_A = \varepsilon_A \otimes \varepsilon_A$$
$$\Delta_A \circ \eta_A = \eta_A \otimes \eta_A$$
$$\varepsilon_A \circ \eta_A = \mathrm{id}_{\mathbf{1}},$$

(4) *the morphism $S_A : A \to A$ is invertible and obeys*

$$\mu_A \circ (S_A \otimes \mathrm{id}_A) \circ \Delta_A = \eta_A \circ \varepsilon_A = \mu_A \circ (\mathrm{id}_A \otimes S_A) \circ \Delta_A.$$

*We call $\mu_A$ (resp. $\Delta_A$) the* (co)multiplication *of $A$ and $\eta_A$ (resp. $\varepsilon_A$) the* (co)unit *of $A$. The morphism $S_A$ is called the* antipode *of $A$.*

**Remark 2.2.** *The unit, counit and antipode of a Hopf algebra are unique. Thus, to define a Hopf algebra, it is only necessary to specify the multiplication and comultiplication and we can unambiguously talk about the Hopf algebra $(A, \mu, \Delta)$; sometimes, we suppress the structure morphisms in the notation.*

**Example 2.3.** (1) *The monoidal unit $\mathbf{1}$ of $\mathcal{C}$ is a Hopf algebra with all structural morphisms given by $\mathrm{id}_{\mathbf{1}}$.*
(2) *If $A = (A, \mu, \Delta)$ is a Hopf algebra in $\mathcal{C}$, then $A^{\mathrm{op}} := (A, \mu^-, \Delta)$ and $A^{\mathrm{cop}} := (A, \mu, \Delta^-)$ with $\mu^- := \mu \circ c_{A,A}^{-1}$ and $\Delta := c_{A,A}^{-1} \circ \Delta$ are Hopf algebras not in $\mathcal{C}$, but rather in the category $\overline{\mathcal{C}}$ with inverse braiding.*

**Definition 2.4.** *Let $A$ and $B$ be Hopf algebras in $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ is a* Hopf algebra morphism, *if $f$ is an algebra homomorphism,*

$$f \circ \mu_A = \mu_B \circ (f \otimes f) \quad and \quad f \circ \eta_A = \eta_B,$$

*and a coalgebra homomorphism,*

$$\Delta_B \circ f = (f \otimes f) \circ \Delta_A \quad and \quad \varepsilon_B \circ f = \varepsilon_A.$$

**Remark 2.5.** *Standard textbook results continue to hold for Hopf algebras in a braided category: Hopf algebra homomorphisms commute with the antipode, i.e. $f \circ S_A = S_B \circ f$. The antipode $S$ of a Hopf algebra $A$ is an isomorphism of Hopf algebras in $\overline{\mathcal{C}}$*

$$S : A^{\mathrm{op}} \to A^{\mathrm{cop}}.$$

*Since $(A^{\mathrm{op}})^{\mathrm{op}} = A$, we see that $S$ is also an isomorphism between the following Hopf algebras in $\mathcal{C}$*

$$S : A \to (A^{\mathrm{cop}})^{\mathrm{op}}.$$

*Note that $(A^{\mathrm{op}})^{\mathrm{cop}}$ and $(A^{\mathrm{cop}})^{\mathrm{op}}$ are in general different Hopf algebras. Nevertheless, $S^2$ is an isomorphism of Hopf algebras from $(A^{\mathrm{op}})^{\mathrm{cop}}$ to $(A^{\mathrm{cop}})^{\mathrm{op}}$.*

2.2. **Modules and comodules over Hopf algebras.** In a monoidal category, modules over an associative algebra and comodules over an associative coalgebra are defined as usual. Modules, as well as comodules, over a Hopf algebra in a braided category form a monoidal category. A new technical feature are 'side switch' functors $\mathsf{T}$ which establish the equivalence of the categories of left $A$-(co)modules in the braided category $\mathcal{C}$ and right $A^{\mathrm{op}}$-modules (resp. $A^{\mathrm{cop}}$-comodules) in the braided category $\overline{\mathcal{C}}$.

**Definition 2.6.**

(1) *Let $A$ be an algebra in $\mathcal{C}$. A left $A$-module is an object $X \in \mathcal{C}$, together with a morphism $\rho = \rho_X : A \otimes X \to X$, such that*
$$\rho \circ (\mathrm{id}_A \otimes \rho) = \rho \circ (\mu \otimes \mathrm{id}_X) \quad and \quad \rho \circ (\eta \otimes \mathrm{id}).$$

(2) *Let $X$ and $Y$ be two $A$-modules. A morphism $f : X \to Y$ is called $A$-linear, if*
$$f \circ \rho_X = \rho_Y \circ (\mathrm{id}_A \otimes f).$$

(3) *Let $A$ be a coalgebra in $\mathcal{C}$. A left $A$-comodule is an object $Y$ in $\mathcal{C}$, together with a morphism $\delta = \delta_X : X \to A \otimes X$, such that*
$$(\mathrm{id}_A \otimes \delta) \circ \delta = (\Delta \otimes \mathrm{id}_X) \circ \delta \quad and \quad (\varepsilon \otimes \mathrm{id}_X) \circ \delta_X = \mathrm{id}_X.$$

(4) *Let $X$ and $Y$ be two $A$-comodules. A morphism $f : X \to Y$ is called $A$-colinear, if*
$$\delta_Y \circ f = (\mathrm{id}_A \otimes f) \circ \delta_X.$$

**Remark 2.7.**

(1) *The left $A$-modules over a Hopf algebra $A$, together with $A$-linear maps, form a monoidal category $A$-$\mathrm{mod}_{\mathcal{C}}$. The tensor product of an $A$-module $(X, \rho_X)$ and an $A$-module $(Y, \rho_Y)$ is given by the usual action of $A$ on $X \otimes Y$, i.e. $\rho_{X \otimes Y} := (\rho_X \otimes \rho_Y) \circ (\mathrm{id}_A \otimes c_{A,X} \otimes \mathrm{id}_Y) \circ (\Delta \otimes \mathrm{id}_{X \otimes Y})$. The monoidal unit is the $A$-module $(\mathbf{1}, \varepsilon)$.*

(2) *Similarly, left $A$-comodules over a Hopf algebra $A$ form a monoidal category $A$-$\mathrm{comod}_{\mathcal{C}}$. Given an $A$-comodule $(X, \delta_X)$ and an $A$-comodule $(Y, \delta_Y)$ the coaction of $A$ on $X \otimes Y$ is given by $\delta_{X \otimes Y} := (\mu \otimes \mathrm{id}_{X \otimes Y}) \circ (\mathrm{id}_A \otimes c_{X,A} \otimes \mathrm{id}_Y) \circ (\delta_X \otimes \delta_Y)$. The monoidal unit is the $A$-comodule $(\mathbf{1}, \eta)$.*

(3) *The monoidal categories of right $A$-modules and right $A$-comodules are denoted by $\mathrm{mod}_{\mathcal{C}}$-$A$ and $\mathrm{comod}_{\mathcal{C}}$-$A$, respectively.*

(4) *Figure 3 lists our graphical notation for structure morphism of Hopf algebras and left and right modules and comodules. Different colours have the only purpose of improving the readability of the diagrams.*

**Lemma 2.8.** *For any Hopf algebra homomorphism $\varphi : A \to B$, the* restriction *is the strict monoidal functor $\_ \circ (\varphi \otimes \mathrm{id}) : B$-$\mathrm{mod}_{\mathcal{C}} \to A$-$\mathrm{mod}_{\mathcal{C}}$, sending the $B$-module $(Y, \rho)$ to the $A$-module $(Y, \rho \circ (\varphi \otimes \mathrm{id}_Y))$.*
*Corestriction is strict monoidal functor $(\varphi \otimes \mathrm{id}) \circ \_ : A$-$\mathrm{comod}_{\mathcal{C}} \to B$-$\mathrm{comod}_{\mathcal{C}}$, sending the $A$-comodule $(X, \delta)$ to the $B$-comodule $(X, (\varphi \otimes \mathrm{id}_X) \circ \delta)$.*

The side switch functor uses the braiding on $\mathcal{C}$ to turn a left comodule into a right comodule:

$$\mu = \begin{array}{c} A \\ \sqcap \\ A\ A \end{array}, \quad \eta = \begin{array}{c} A \\ \mid \\ \circ \end{array}, \quad \Delta = \begin{array}{c} A\ A \\ \sqcup \\ A \end{array}, \quad \varepsilon = \begin{array}{c} A \\ \circ \\ A \end{array}, \quad S = \begin{array}{c} A \\ \circ \\ A \end{array}, \quad S^{-1} = \begin{array}{c} A \\ \bullet \\ A \end{array},$$

$$\rho = \begin{array}{c} X \\ \sqcap \\ A\ X \end{array}, \quad \delta = \begin{array}{c} A\ X \\ \sqcup \\ X \end{array}, \quad \rho^r = \begin{array}{c} X \\ \mid \\ X\ A \end{array}, \quad \delta^r = \begin{array}{c} X\ A \\ \sqcup \\ X \end{array}.$$
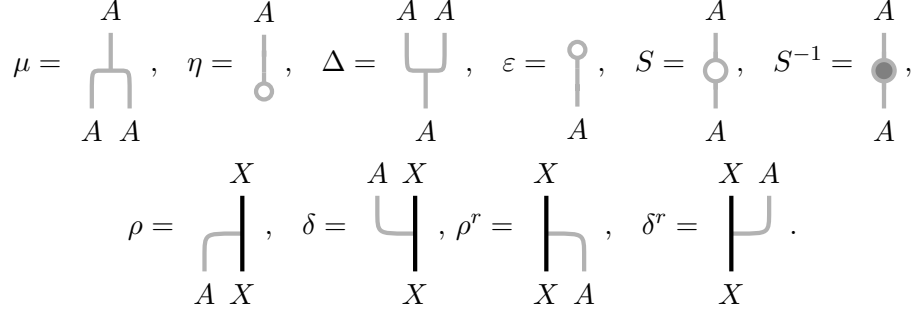
FIGURE 3. Standard notation for structure morphisms

**Lemma 2.9.** *Let $A$ be a Hopf algebra in $\mathcal{C}$ and $(X, \delta)$ and $A$-comodule. The pair*

$$^A\mathsf{T}(X, \delta) := (X, c_{X,A}^{-1} \circ \delta)$$

*is a right $A^{\mathrm{cop}}$-comodule; the assignment $^A\mathsf{T}$ defines a strict monoidal functor $^A\mathsf{T} : A\text{-comod}_{\mathcal{C}} \to \text{comod}_{\overline{\mathcal{C}}}\text{-}A^{\mathrm{cop}}$.*

**Remark 2.10.** *In the same way, the braiding induces strict monoidal functors*

$$_A\mathsf{T} : A\text{-mod}_{\mathcal{C}} \to \text{mod}_{\overline{\mathcal{C}}}\text{-}A^{\mathrm{op}},$$
$$\mathsf{T}^A : \text{comod}_{\mathcal{C}}\text{-}A \to A^{\mathrm{cop}}\text{-comod}_{\overline{\mathcal{C}}},$$
$$\mathsf{T}_A : \text{mod}_{\mathcal{C}}\text{-}A \to A^{\mathrm{op}}\text{-mod}_{\overline{\mathcal{C}}}.$$

*Note that the functor $\mathsf{T}^{A^{\mathrm{cop}}} : \text{comod}_{\overline{\mathcal{C}}}\text{-}A^{\mathrm{cop}} \to (A^{\mathrm{cop}})^{\mathrm{cop}}\text{-comod}_{\overline{\overline{\mathcal{C}}}} = A\text{-comod}_{\mathcal{C}}$ is inverse to $^A\mathsf{T}$. Similarly, the functors $_A\mathsf{T}, \mathsf{T}^A$ and $\mathsf{T}_A$ are invertible.*

2.3. **Hopf pairings.** We finally turn to the definition of a Hopf pairing between two Hopf algebras in a braided category $\mathcal{C}$.

**Definition 2.11.** *Let $A$ and $B$ be Hopf algebras in $\mathcal{C}$. A morphism $\omega : A \otimes B \to \mathbf{1}$ in $\mathcal{C}$ is called a* Hopf pairing, *if the following identities hold*

$$\omega \circ (\mu_A \otimes \mathrm{id}_B) = \omega \circ (\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_B) \circ (\mathrm{id}_{A \otimes A} \otimes \Delta_B),$$
$$\omega \circ (\eta_A \otimes \mathrm{id}_B) = \varepsilon_B,$$
$$\omega \circ (\mathrm{id}_A \otimes \mu_B) = \omega \circ (\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_B) \circ (\Delta_A \otimes \mathrm{id}_{B \otimes B}),$$
$$\omega \circ (\mathrm{id}_A \otimes \eta_B) = \varepsilon_A.$$

**Remark 2.12.** (1) *If $A$ has a right-dual object $A^\vee$ (see [K95, Ch. XIV] for a definition), then $A^\vee$ has a natural structure of a Hopf algebra in $\mathcal{C}$ such that the evaluation morphism $\mathrm{ev} : A \otimes A^\vee \to \mathbf{1}$ is a Hopf pairing.*

(2) *A Hopf pairing $\omega$ relates antipodes,*

$$\omega \circ (S_A \otimes \mathrm{id}_B) = \omega \circ (\mathrm{id}_A \otimes S_B).$$

A Hopf pairing $\omega : A \otimes B \to \mathbf{1}$ gives rise to a *dualization functor* $^\omega\mathsf{D}$ which relates modules and comodules of Hopf algebras in different categories:

**Lemma 2.13.** *Let $\omega : A \otimes B \to \mathbf{1}$ be a Hopf pairing and $(X, \delta)$ a left $B$-comodule and set*

$$^\omega\mathsf{D}(X, \delta) := (X, (\omega \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \delta)).$$

*The assignment $^\omega\mathsf{D}(X, \delta)$ defines a strict monoidal functor $^\omega\mathsf{D} : B\text{-comod}_\mathcal{C} \to A^{\mathrm{cop}}\text{-mod}_{\overline{\mathcal{C}}}$.*

*Proof.* Keeping in mind that $A$ and $A^{\mathrm{cop}}$ are equal as algebras, the first two conditions on a Hopf pairing imply that $^\omega\mathsf{D}(X, \delta)$ is an $A^{\mathrm{cop}}$-module. To see that the functor $^\omega\mathsf{D}$ is strict monoidal, we note the equality of comodules $^\omega\mathsf{D}(X) \otimes {^\omega\mathsf{D}}(Y) = {^\omega\mathsf{D}}(X \otimes Y)$ in $\overline{\mathcal{C}}$ for any two $B$-comodules $X$ and $Y$, which immediately follows from the naturality of the braiding and the third condition on a Hopf pairing. Finally, the equality $^\omega\mathsf{D}(X) \otimes {^\omega\mathsf{D}}(\mathbf{1}) = {^\omega\mathsf{D}}(X) = {^\omega\mathsf{D}}(\mathbf{1}) \otimes {^\omega\mathsf{D}}(X)$ follows from the fourth condition for a Hopf pairing. $\quad\square$

**Remark 2.14.** *If $\omega : A \otimes B \to \mathbf{1}$ is a Hopf pairing, then also*

$$\omega^+ := \omega \circ c_{B,A} \circ (S_B \otimes S_A) : B \otimes A \to \mathbf{1}$$

*and*

$$\omega^- := \omega \circ c_{A,B}^{-1} \circ (S_B^{-1} \otimes S_A^{-1}) : B \otimes A \to \mathbf{1}$$

*are Hopf pairings.*

**Definition 2.15.** *A Hopf pairing $\omega : A \otimes B \to \mathbf{1}$ is called* non-degenerate, *if there is a morphism $\omega' : \mathbf{1} \to B \otimes A$, such that*

$$(\omega \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \omega') = \mathrm{id}_A \quad and \quad (\mathrm{id}_B \otimes \omega) \circ (\omega' \otimes \mathrm{id}_B) = \mathrm{id}_B.$$

**Remark 2.16.**    (1) *If $A$ and $B$ are Hopf algebras over a field $\Bbbk$ that are related by a non-degenerate Hopf pairing, then both $A$ and $B$ are finite-dimensional.*
  (2) *If $\omega : A \otimes B \to \mathbf{1}$ is non-degenerate, the morphism $\omega' : \mathbf{1} \to B \otimes A$ is unique. We call $\omega'$ the* inverse copairing *of $\omega$.*
     *The inverse copairing of a Hopf pairing is a* Hopf copairing*, i.e. the following axioms are fulfilled:*

$$(\Delta_B \otimes \mathrm{id}_A) \circ \omega' = (\mathrm{id}_{B \otimes B} \otimes \mu_A) \circ (\mathrm{id}_B \otimes \omega' \otimes \mathrm{id}_A) \circ \omega',$$
$$(\varepsilon_B \otimes \mathrm{id}_A) \circ \omega' = \eta_A,$$
$$(\mathrm{id}_B \otimes \Delta_A) \circ \omega' = (\mu_B \otimes \mathrm{id}_{A \otimes A}) \circ (\mathrm{id}_B \otimes \omega' \otimes \mathrm{id}_A) \circ \omega',$$
$$(\mathrm{id}_B \otimes \varepsilon_A) \circ \omega' = \eta_B.$$

*If the Hopf pairing $\omega : A \otimes B \to \mathbf{1}$ is non-degenerate, then the Hopf pairings $\omega^+, \omega^- : B \otimes A \to \mathbf{1}$ are non-degenerate as well. The inverse copairing of $\omega^+$ is the morphism $(S^{-1} \otimes S^{-1}) \circ c_{A,B}^{-1} \circ \omega'$, the inverse copairing of $\omega^-$ is the morphism $(S \otimes S) \circ c_{B,A} \circ \omega'$.*

**Lemma 2.17.** *If $\omega : A \otimes B \to \mathbf{1}$ is a non-degenerate Hopf pairing, the strict monoidal functor*

$$^\omega\mathsf{D} : B\text{-comod}_\mathcal{C} \to A^{\mathrm{cop}}\text{-mod}_{\overline{\mathcal{C}}}$$

*from Lemma 2.13 is an isomorphism of categories.*

*Proof.* Use the copairing $\omega'$ to define a functor $_{\omega'}\mathsf{D} : A^{\mathrm{cop}}\text{-mod}_{\overline{\mathcal{C}}} \to B\text{-comod}_{\mathcal{C}}$ by sending the module $(X, \rho)$ to the $B$-comodule

$$(X, (\mathrm{id}_B \otimes \rho) \circ (\omega' \otimes \mathrm{id}_X)).$$

The properties of a Hopf copairing imply that $_{\omega'}\mathsf{D}$ is a functor. The relation between $\omega$ and $\omega'$ imply that the functors $_{\omega'}\mathsf{D}$ and $^{\omega}\mathsf{D}$ are inverses. $\qquad\square$

## 3. Yetter-Drinfel'd modules in braided categories

Yetter-Drinfel'd modules or crossed modules [Mo93, K95] for Hopf algebras over a field have been generalized in [B95] for Hopf algebras in a braided category $\mathcal{C}$. In this section, we show that the dualization functor $^{\omega}\mathsf{D}$ from Lemma 2.17 associated to a non-degenerate Hopf pairing extends to a strict monoidal functor between categories of Yetter-Drinfel'd modules. Moreover, we combine the side switch functors $\mathsf{T}^A$ and $\mathsf{T}_A$ for modules and comodules from Lemma 2.10 into a (non-strict) side switch functor $\mathsf{T}$ for Yetter-Drinfel'd modules.

3.1. **The Yetter-Drinfel'd condition.** A Yetter-Drinfel'd module is a module and a comodule, subject to a compatibility condition. Actions and coactions can be on the left or right; thus there are four different types of Yetter-Drinfel'd modules. Our main result can be understood in terms of Yetter-Drinfel'd modules with left action and left coaction; the other categories only serve as a tool in the proofs.

**Definition 3.1.** *Let $A$ be a Hopf algebra in a braided category $\mathcal{C}$; suppose that $X$ is a left or right module and comodule over $A$. The corresponding Yetter-Drinfel'd conditions are depicted in Figure 4.*
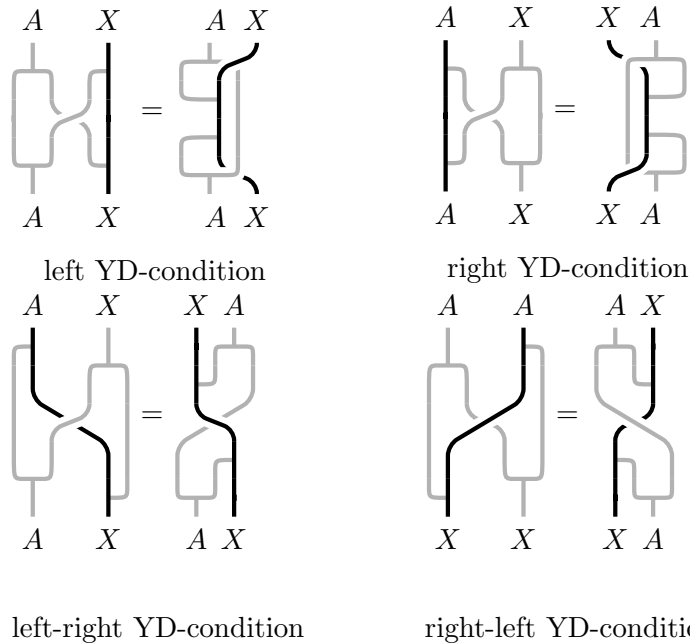


left YD-condition

right YD-condition

left-right YD-condition

right-left YD-condition

FIGURE 4. Yetter-Drinfel'd conditions

The (left) Yetter-Drinfel'd modules over $A$ are objects of a category $^A_A\mathcal{YD}(\mathcal{C})$; morphisms in $^A_A\mathcal{YD}(\mathcal{C})$ are morphisms in $\mathcal{C}$ that are $A$-linear and $A$-colinear.

The tensor product of a Yetter-Drinfel'd module $X$ and a Yetter-Drinfel'd module $Y$ is given by the object $X \otimes Y$ with the obvious action and coaction. The unit object is the monoidal unit $\mathbf{1}$ of $\mathcal{C}$, together with trivial action given by the counit and trivial coaction given by the unit of $A$.

The braiding isomorphism

$$c^{\mathcal{YD}}_{X,Y} : X \otimes Y \to Y \otimes X$$

is given by

$$c^{\mathcal{YD}}_{X,Y} := (\rho_Y \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes c_{X,Y}) \circ (\delta_X \otimes \mathrm{id}_Y) \ ; \tag{1}$$

its inverse is

$$(c^{\mathcal{YD}}_{X,Y})^{-1} := c^{-1}_{X,Y} \circ (\rho_Y \otimes \mathrm{id}_X) \circ (c^{-1}_{A,Y} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes S^{-1} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes \delta_X).$$

The structure is summarized in the following proposition whose proof can be found in [B95].

**Proposition 3.2.** *Let $A$ be a Hopf algebra in $\mathcal{C}$. The left Yetter-Drinfel'd modules over $A$ in $\mathcal{C}$ have a natural structure of a braided monoidal category $^A_A\mathcal{YD}(\mathcal{C})$.*

**Remark 3.3.** *The definition of Yetter-Drinfel'd module does not require the existence of an antipode so that Yetter-Drinfel'd modules can be defined over a bialgebra as well.*

*If $A$ is a Hopf algebra, the antipode allows us to reformulate the Yetter-Drinfel'd condition: a graphical calculation shows that a module and comodule $X$ is a left Yetter-Drinfel'd module, iff*
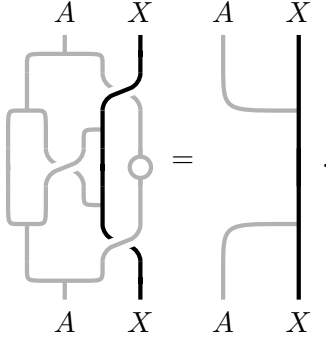


*This reformulation is useful to prove the following lemma which is proven by straightforward calculations:*

**Lemma 3.4.** *Let $A$ be a Hopf algebra in $\mathcal{C}$. For a left Yetter-Drinfel'd $X$ consider*

$$\theta_X := \rho_X \circ (S \otimes \mathrm{id}) \circ \delta_X \in \mathrm{End}_\mathcal{C}(X).$$

*The following holds*

(1) $\theta_X \circ \rho_X = \rho_X \circ c_{X,A} \circ c_{A,X} \circ (S^2 \otimes \mathrm{id})$.
(2) $\delta_X \circ \theta_X = (S^2 \otimes \mathrm{id}) \circ c_{X,A} \circ c_{A,X} \circ \delta_X$.

(3) *If $Y$ is another Yetter-Drinfel'd module, we have*

$$c_{Y,X}^{\mathcal{YD}} \circ \theta_{Y \otimes X} \circ c_{X,Y}^{\mathcal{YD}} = c_{Y,X} \circ (\theta_Y \otimes \theta_X) \circ c_{X,Y} \ .$$

**Remark 3.5.** *The right Yetter-Drinfel'd modules also form a braided monoidal category, which is denoted by $\mathcal{YD}_A^A(\mathcal{C})$. The braiding is given by*

$$c_{X,Y}^{\mathcal{YD}} := (\mathrm{id}_Y \otimes \rho_X^r) \circ (c_{X,Y} \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes \delta_Y^r).$$

*If $\mathcal{C}$ is the category of vector spaces over a field $\Bbbk$, we also write ${}_A^A \mathcal{YD}_\Bbbk$ or ${}_A^A \mathcal{YD}$ for the category of Yetter-Drinfel'd modules.*

**Remark 3.6.** *The collection of left-right Yetter-Drinfel'd modules forms a category ${}_A \mathcal{YD}^A(\mathcal{C})$. In contrast to the categories ${}_A^A \mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}_A^A(\mathcal{C})$, the category ${}_A \mathcal{YD}^A(\mathcal{C})$ can be endowed with two different tensor products (leading to monoidally equivalent categories)*

(1) *For the first product, $A$ acts as usual on the product of two Yetter-Drinfel'd modules, while the coaction is given by*

$$(\mathrm{id}_{X \otimes Y} \otimes \mu) \circ (\mathrm{id}_X \otimes \delta_Y^r \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes c_{Y,A}^{-1}) \circ (\delta_X^r \otimes \mathrm{id}_Y).$$

*Since this is the diagonal coaction of the Hopf algebra $A^{\mathrm{op}}$ in $\overline{\mathcal{C}}$, we denote left-right Yetter-Drinfel'd modules with this tensor product by ${}_A \mathcal{YD}^{A^{\mathrm{op}}}(\mathcal{C})$. The monoidal category ${}_A \mathcal{YD}^{A^{\mathrm{op}}}(\mathcal{C})$ admits the braiding*

$$c_{X,Y}^{\mathcal{YD}} := (\mathrm{id}_Y \otimes \rho_X) \circ (\delta_Y^r \otimes \mathrm{id}_X) \circ c_{Y,X}^{-1}.$$

(2) *A second tensor product on ${}_A \mathcal{YD}^A(\mathcal{C})$ is given by the diagonal action of $A^{\mathrm{cop}}$ and the diagonal coaction of $A$ on tensor products of Yetter-Drinfel'd modules. We denote this monoidal category by ${}_{A^{\mathrm{cop}}} \mathcal{YD}^A(\mathcal{C})$; it admits the braiding*

$$c_{X,Y}^{\mathcal{YD}} := c_{Y,X}^{-1} \circ (\mathrm{id}_X \otimes \rho_Y) \circ (\delta_X^r \otimes \mathrm{id}_Y).$$

*The two braided categories ${}^A \mathcal{YD}_{A^{\mathrm{cop}}}(\mathcal{C})$ and ${}^{A^{\mathrm{op}}} \mathcal{YD}_A(\mathcal{C})$ are defined in complete analogy.*

3.2. **Radford biproduct and projection theorem.** The following situation is standard: let $A$ be a Hopf algebra over a field $\Bbbk$. Let $K$ be a Hopf algebra in the braided category ${}_A^A \mathcal{YD}$ of $A$-Yetter-Drinfel'd modules.

The category of Yetter-Drinfel'd modules over $K$ in ${}_A^A \mathcal{YD}$ can be described as the category of Yetter-Drinfel'd modules over a Hopf algebra $K \rtimes A$ over the field $\Bbbk$. The Hopf algebra $K \rtimes A$ is called *Majid bosonization* or *Radford's biproduct*. The definition of the biproduct $K \rtimes A$ directly generalizes to the description of Yetter-Drinfel'd modules over a Hopf algebra $K$ in the braided category ${}_A^A \mathcal{YD}(\mathcal{C})$, where $\mathcal{C}$ is now an arbitrary braided category. We collect in this subsection results from [B95] that we will be needed in the construction of the partially dualized Hopf algebra in Section 4.

**Definition 3.7** (Radford Biproduct)**.** *Let $\mathcal{C}$ be a braided category and let $A \in \mathcal{C}$ and $K \in {}_A^A \mathcal{YD}(\mathcal{C})$ be Hopf algebras. The* Radford biproduct $K \rtimes A$ *is*

*defined as the object $K \otimes A$ in $\mathcal{C}$ together with the following morphisms:*

$$\mu_{K \rtimes A} := \qquad , \quad \Delta_{K \rtimes A} := \qquad , \quad S_{K \rtimes A} \qquad .$$

$$K \quad A \quad K \quad A \qquad\qquad K \qquad A \qquad\qquad K \quad A$$

**Proposition 3.8.** *The Radford biproduct $K \rtimes A$ is a Hopf algebra in $\mathcal{C}$.*

Definition 3.7 and Proposition 3.8 are found in [B95, Subsection 4.1].

**Remark 3.9.** *If $K$ is a Hopf algebra in the category $^A_A\mathcal{YD}$ of Yetter-Drinfel'd modules over a $\Bbbk$-Hopf algebra $A$, the Radford biproduct is given by the following formulas for multiplication and comultiplication, cf. [Mo93, Section 10.6]:*

$$(h \otimes a) \cdot (k \otimes b) = h \cdot (a_{(1)}.k) \otimes a_{(2)} \cdot b$$
$$\Delta(h \otimes a) = h_{(1)} \otimes (h_{(2)})_{(-1)} \cdot a_{(1)} \otimes (h_{(2)})_{(0)} \otimes a_{(2)}.$$

*This is a special case of the formulas expressed graphically in Definition 3.7. If $K$ has only the structure of an algebra in $^A_A\mathcal{YD}$, the vector space $K \otimes A$ with the multiplication $\mu_{K \rtimes A}$ is called a smash-product. If $K$ has only the structure of a coalgebra in $^A_A\mathcal{YD}$, $K \otimes A$ with the comultiplication $\Delta_{K \rtimes A}$ is called a cosmash-product.*

**Theorem 3.10** (Radford projection theorem)**.** *Let $H$ and $A$ be Hopf algebras in a braided category $\mathcal{C}$. Let $\pi : H \to A$ and $\iota : A \to H$ be Hopf algebra morphisms such that $\pi \circ \iota = \mathrm{id}_A$. If $\mathcal{C}$ has equalizers and $A \otimes \_$ preserves equalizers, there is a Hopf algebra $K$ in the braided category $^A_A\mathcal{YD}(\mathcal{C})$, such that*

$$H = K \rtimes A.$$

*Proof.* For a complete proof we refer to [AF00]. $\qquad\square$

**Remark 3.11.** *To illustrate the situation, we discuss the case when $\mathcal{C}$ is the braided category of $\Bbbk$-vector spaces and $\pi : H \to A$ is a projection to a Hopf subalgebra $A \subset H$:*
*The vector space underlying the Hopf algebra $K$ in $^A_A\mathcal{YD}$ is then the space of coinvariants of $H$:*

$$K := H^{\mathrm{coin}(\pi)} := \left\{ r \in H \mid r_{(1)} \otimes \pi(r_{(2)}) = r \otimes 1 \right\} .$$

*One easily checks that $K$ is a subalgebra of $H$ and $K$ is invariant under the left adjoint action of $A$ on $H$.*
*The subspace $K$ is also a left $A$-comodule with coaction $\delta_K(k) := \pi(k_{(1)}) \otimes k_{(2)}$. The fact that $H$ is a left $H$-Yetter-Drinfel'd module with the adjoint action and regular coaction implies that $K$ is even an $A$-Yetter-Drinfel'd module. The comultiplication of $K$ is given by the formula*

$$\Delta_K(k) := k_{(1)}\pi(S_H(k_{(2)})) \otimes k_{(3)}$$

*and the antipode is $S_K(k) = \pi(k_{(1)})S_H(k_{(2)})$.*

**Theorem 3.12** (Bosonization Theorem). *Let $A$ be a Hopf algebra in $\mathcal{C}$ and $K$ a Hopf algebra in $_A^A\mathcal{YD}(\mathcal{C})$. There is an obvious isomorphism of braided categories*

$$_{K \rtimes A}^{K \rtimes A}\mathcal{YD}(\mathcal{C}) \cong {}_K^K\mathcal{YD}(_A^A\mathcal{YD}(\mathcal{C})).$$

For a proof, we refer to [B95, Proposition 4.2.3].

3.3. **Equivalence of categories of left and right modules.** In this subsection we discuss the side switch functor $\mathsf{T} : \mathcal{YD}_A^A(\mathcal{C}) \to {}_A^A\mathcal{YD}(\mathcal{C})$ for Yetter-Drinfel'd modules. It turns out that, for our purposes, a non-trivial monoidal structure $\mathsf{T}_2 : \mathsf{T} \otimes \mathsf{T} \to \mathsf{T} \circ \otimes$ has to be chosen for the switch functor, even in those cases (for $\mathcal{C}$ symmetric) where the identities provide a monoidal structure on $\mathsf{T}$.

**Lemma 3.13.** *The isomorphism $^A\mathsf{T} : A\text{-comod}_\mathcal{C} \to \text{comod}_{A^{\mathrm{cop}}}\text{-}\overline{\mathcal{C}}$ of categories from Lemma 2.9 extends to an isomorphism of categories*

$$^A\mathsf{T} : {}_A^A\mathcal{YD}(\mathcal{C}) \to {}_{A^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}).$$

*The functor $^A\mathsf{T}$ is braided and strict monoidal, considered as a functor between the following monoidal categories:*

$$^A\mathsf{T} : {}_A^A\mathcal{YD}(\mathcal{C}) \to {}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}).$$

**Remark 3.14.** *The equality $(A^{\mathrm{cop}})^{\mathrm{cop}} = A$ of Hopf algebras from Remark 2.5 might suggest the notation*

$$_A\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}) := {}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}})$$

*which is not in conflict with other notation used in this article. To avoid confusion with the different monoidal category $_A\mathcal{YD}^{A^{\mathrm{op}}}(\mathcal{C})$, we refrain from using this notation.*

*Proof.* Let $X = (X, \rho, \delta)$ be in $_A^A\mathcal{YD}(\mathcal{C})$. It follows from Lemma 2.9 that $^A\mathsf{T}(X) = (X, \rho, c_{X,A}^{-1} \circ \delta)$ is an $A^{\mathrm{cop}}$-comodule and $A^{\mathrm{cop}}$-module in $\overline{\mathcal{C}}$. It remains to be shown that $^A\mathsf{T}(X)$ obeys the condition of a left-right $A^{\mathrm{cop}}$-Yetter-Drinfel'd module in $\overline{\mathcal{C}}$:



One finally verifies that the braiding isomorphisms in the categories $_A^A\mathcal{YD}(\mathcal{C})$ and $_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}})$ of Yetter-Drinfel'd modules coincide as morphisms in the underlying category $\mathcal{C}$. $\square$

**Remark 3.15.** *One can show by similar arguments that the isomorphisms $_A\mathsf{T}, \mathsf{T}^A$ and $\mathsf{T}_A$ extend to braided and strict monoidal functors*

$$_A\mathsf{T} : {}_A^A\mathcal{YD}(\mathcal{C}) \to {}^{(A^{\mathrm{op}})^{\mathrm{op}}}\mathcal{YD}_{A^{\mathrm{op}}}(\overline{\mathcal{C}}),$$

$$\mathsf{T}^A : \mathcal{YD}_A^A(\mathcal{C}) \to {}^{A^{\mathrm{cop}}}\mathcal{YD}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}(\overline{\mathcal{C}}),$$

$$\mathsf{T}_A : \mathcal{YD}_A^A(\mathcal{C}) \to {}_{A^{\mathrm{op}}}\mathcal{YD}^{(A^{\mathrm{op}})^{\mathrm{op}}}(\overline{\mathcal{C}}).$$

**Theorem 3.16.** *Let $A$ be a Hopf algebra in a braided category $\mathcal{C}$ and $(X, \rho^r, \delta^r)$ a right Yetter-Drinfel'd module over $A$. Consider*

$$\mathsf{T}(X, \rho^r, \delta^r) = (X, \rho^- \circ (S^{-1} \otimes \mathrm{id}_X), (S \otimes \mathrm{id}_X) \circ \delta^+),$$

*with $\rho^- := \rho^r \circ c_{X,A}^{-1}$ and $\delta^+ := c_{X,A} \circ \delta^r$. The functor*

$$\mathsf{T} = (\mathsf{T}_A)^A : \mathcal{YD}_A^A(\mathcal{C}) \to {}_A^A\mathcal{YD}(\mathcal{C})$$

*has a monoidal structure $\mathsf{T}_2(X, Y) : \mathsf{T}(X) \otimes \mathsf{T}(Y) \to \mathsf{T}(X \otimes Y)$ given by*

$$\mathsf{T}_2(X, Y) := (\mathrm{id}_X \otimes \rho_Y^r) \circ (\mathrm{id}_X \otimes c_{Y,A}^{-1}) \circ (\delta_X^r \otimes \mathrm{id}_Y).$$

*The monoidal functor $(\mathsf{T}, \mathsf{T}_2)$ is braided.*

*Proof.* The functor $\mathsf{T} : \mathcal{YD}_A^A(\mathcal{C}) \to {}_A^A\mathcal{YD}(\mathcal{C})$ is defined as the composition of the functors in the diagram

$$
\begin{array}{ccc}
\mathcal{YD}_A^A(\mathcal{C}) & \dashrightarrow{\ \ \mathsf{T}\ \ } & {}_A^A\mathcal{YD}(\mathcal{C}) \\
\Big\downarrow{\mathsf{T}_A} & & \Big\uparrow{({}^A\mathsf{T})^{-1}} \\
{}_{A^{\mathrm{op}}}\mathcal{YD}^{A^{\mathrm{op}}}(\overline{\mathcal{C}}) & \xrightarrow{\ \ \mathsf{S}\ \ } & {}_{A^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}})
\end{array}
$$

Here $\mathsf{S}$ denotes the functor of restriction along $S^{-1} : A^{\mathrm{cop}} \to A^{\mathrm{op}}$ and corestriction along $S : A^{\mathrm{op}} \to A^{\mathrm{cop}}$. Thus, $\mathsf{T}$ is a functor. Expressing the monoidal structure in terms of braidings,

$$T_2(X, Y) = c_{Y,X}^{\mathcal{YD}} \circ c_{Y,X}^{-1},$$

and noting that the isomorphism $c_{Y,X}^{\mathcal{YD}}$ is $A$-linear and $A$-colinear, we see that the morphism $T_2(X, Y)$ is $A$-(co)linear, iff $c_{Y,X}^{-1}$ is $A$-(co)linear as a morphism $\mathsf{T}X \otimes \mathsf{T}Y \to \mathsf{T}(Y \otimes X)$; this is easily checked.

The inverse of $\mathsf{T}_2(X, Y)$ is given by

$$\mathsf{T}_2(X, Y)^{-1} = (\mathrm{id}_X \otimes \rho_Y^r) \circ (\mathrm{id}_X \otimes c_{Y,A}^{-1}) \circ (\mathrm{id}_X \otimes S^{-1} \otimes \mathrm{id}_Y) \circ (\delta_X^r \otimes \mathrm{id}_Y).$$

This follows by using that $S^{-1}$ is the antipode of $A^{\mathrm{op}}$. We leave it to the reader to show that the equality

$$\mathsf{T}_2(X \otimes Y, Z) \circ (\mathsf{T}_2(X, Y) \otimes \mathrm{id}_{\mathsf{T}(Z)}) = \mathsf{T}_2(X, Y \otimes Z) \circ (\mathrm{id}_{\mathsf{T}(X)} \otimes \mathsf{T}_2(Y, Z))$$

is a direct consequence of the Yetter-Drinfel'd condition. We conclude that $(\mathsf{T}, \mathsf{T}_2)$ is a monoidal functor.

Finally we show that $(\mathsf{T}, \mathsf{T}_2)$ is a braided monoidal functor, i.e. that the

diagram

$$\begin{array}{ccc}
\mathsf{T}(X) \otimes \mathsf{T}(Y) & \xrightarrow{\;\;\mathsf{T}_2(X,Y)\;\;} & \mathsf{T}(X \otimes Y) \\
c^{\mathcal{YD}}_{\mathsf{T}(X),\mathsf{T}(Y)} \downarrow & & \downarrow \mathsf{T}\big(c^{\mathcal{YD}}_{X,Y}\big) \\
\mathsf{T}(Y) \otimes \mathsf{T}(X) & \xrightarrow{\;\;\mathsf{T}_2(Y,X)\;\;} & \mathsf{T}(Y \otimes X)
\end{array}$$

commutes. One easily sees by drawing the corresponding braid diagrams, that $c^{\mathcal{YD}}_{\mathsf{T}(X),\mathsf{T}(Y)}$ is equal to $c_{X,Y} \circ \mathsf{T}_2(X,Y)$ and $\mathsf{T}(c^{\mathcal{YD}}_{X,Y})$ is equal to $\mathsf{T}_2(Y,X) \circ c_{X,Y}$. Thus we have

$$\mathsf{T}_2(Y,X) \circ c^{\mathcal{YD}}_{\mathsf{T}(X),\mathsf{T}(Y)} \circ \mathsf{T}_2(X,Y)^{-1} = \mathsf{T}_2(Y,X) \circ c_{X,Y} = \mathsf{T}(c^{\mathcal{YD}}_{X,Y}).$$

$\square$

**Remark 3.17.** *There is another braided equivalence between the same braided categories of left/right Yetter-Drinfel'd modules*

$$\mathsf{T}' = (\mathsf{T}^A)_A : \mathcal{YD}^A_A(\mathcal{C}) \to {}^A_A\mathcal{YD}(\mathcal{C}).$$

*The functor $\mathsf{T}'$ is given on objects by*

$$\mathsf{T}'(X, \rho^r, \delta^r) = (X, \rho^+ \circ (S \otimes \mathrm{id}_X), (S^{-1} \otimes \mathrm{id}_X) \circ \delta^-).$$

*The monoidal structure $T'_2(X,Y) : \mathsf{T}'(X) \otimes \mathsf{T}'(Y) \to \mathsf{T}'(X \otimes Y)$ on $\mathsf{T}'$ is given by*

$$\mathsf{T}'_2(X,Y) := (\rho^r_X \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes S^{-1} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c^{-1}_{A,Y}) \circ (\mathrm{id}_X \otimes \delta^r_Y).$$

*The two monoidal functors $\mathsf{T}, \mathsf{T}' : \mathcal{YD}^A_A(\mathcal{C}) \to {}^A_A\mathcal{YD}(\mathcal{C})$ are isomorphic as monoidal functors. An isomorphism is given by the family of morphisms*

$$\theta_{\mathsf{T}X} := \rho_{\mathsf{T}X} \circ (S \otimes \mathrm{id}_X) \circ \delta_{\mathsf{T}X}.$$

*The inverse is*

$$\theta^{-1}_{\mathsf{T}X} = \rho_{\mathsf{T}X} \circ c^{-1}_{A,X} \circ (\mathrm{id}_X \otimes S^{-2}) \circ c^{-1}_{X,A} \circ \delta_{\mathsf{T}X}.$$

*A lemma for right Yetter-Drinfel'd modules that is analogous to Lemma 3.4 implies that $\theta$ is indeed a monoidal isomorphism.*

3.4. **Equivalence of categories from Hopf pairings.** In this subsection, we prove that for Hopf algebras $A$ and $B$ that are related by a non-degenerate Hopf pairing, there is a braided monoidal equivalence between the categories ${}^A_A\mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}^B_B(\mathcal{C})$. This equivalence is a strict monoidal functor.

**Lemma 3.18.** *Let $\omega : A \otimes B \to \mathbf{1}$ be a non-degenerate Hopf pairing with inverse copairing $\omega' : \mathbf{1} \to B \otimes A$. Then*

$$\begin{aligned}
{}_{\omega'}\mathsf{D}^\omega : {}_{A^{\mathrm{cop}}}\mathcal{YD}^A(\mathcal{C}) &\to {}^B\mathcal{YD}_{B^{\mathrm{cop}}}(\mathcal{C}) \\
(X, \rho, \delta) &\mapsto (X, (\mathrm{id} \otimes \omega) \circ (\delta \otimes \mathrm{id}), (\mathrm{id} \otimes \rho) \circ (\omega' \otimes \mathrm{id})),
\end{aligned}$$

*is a strict monoidal braided functor. In particular, the two categories ${}_{A^{\mathrm{cop}}}\mathcal{YD}^A(\mathcal{C})$ and ${}^B\mathcal{YD}_{B^{\mathrm{cop}}}(\mathcal{C})$ are equivalent as braided monoidal categories.*

*Proof.* Let $(X, \rho, \delta)$ be an $A$-Yetter-Drinfel'd module. From Lemma 2.13 it is clear that $\mathsf{D}(X, \rho, \delta)$ is a $B$-module and $B$-comodule. We have to check the Yetter-Drinfel'd condition. Since $X$ is an $A$-Yetter-Drinfel'd module, we have the equality



Using that $\omega$ is a Hopf pairing, $\omega'$ is a Hopf copairing and $(\mathrm{id}_B \otimes \omega) \circ (\omega' \otimes \mathrm{id}_B) = \mathrm{id}_B$ we get the equality



which is the Yetter-Drinfel'd condition for the $B$-module and $B$-comodule structure on $\mathsf{D}(X)$. The functor $\mathsf{D}$ is strict monoidal, since the functors $_{\omega'}\mathsf{D} : A^{\mathrm{cop}}\text{-mod}_{\mathcal{C}} \to B\text{-comod}_{\mathcal{C}}$ and $\mathsf{D}^{\omega} : \mathrm{comod}_{\mathcal{C}}\text{-}A \to \mathrm{mod}_{\mathcal{C}}\text{-}B^{\mathrm{cop}}$ are strict monoidal. Finally, the braiding is preserved:

$$c_{\mathsf{D}(X),\mathsf{D}(Y)}^{\mathcal{YD}} = c_{X,Y}^{\mathcal{YD}} = \mathsf{D}(c_{X,Y}^{\mathcal{YD}}) \, .$$

This follows from $(\omega \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \omega') = \mathrm{id}_A$. $\qquad\square$

**Corollary 3.19.** *Let $\omega : A \otimes B \to \mathbf{1}$ be a non-degenerate Hopf pairing with inverse copairing $\omega' : \mathbf{1} \to B \otimes A$. Then*

$$_{\omega'}^{\omega}\mathsf{D} : {}_A^A\mathcal{YD}(\mathcal{C}) \to \mathcal{YD}_B^B(\mathcal{C})$$
$$(X, \rho_X, \delta_X) \mapsto (X, \rho_{\mathsf{D}(X)}, \delta_{\mathsf{D}(Y)})$$

*with*

$$\rho_{\mathsf{D}(X)} = (\mathrm{id} \otimes \omega) \circ (c_{A,X}^{-1} \otimes S^{-1}) \circ (\delta \otimes \mathrm{id})$$
$$\delta_{\mathsf{D}(X)} = c_{B,X} \circ (S \otimes \rho) \circ (\omega' \otimes \mathrm{id})$$

*defines a braided, strict monoidal functor.*

*In particular, the categories $_A^A\mathcal{YD}(\mathcal{C})$ and $\mathcal{YD}_B^B(\mathcal{C})$ are equivalent as braided monoidal categories.*

*Proof.* Note that $\omega : A \otimes B \to \mathbf{1}$ is a Hopf pairing of the two Hopf algebras $A^{\mathrm{cop}}$ and $B^{\mathrm{op}}$ in $\overline{\mathcal{C}}$. So we have the following composite of braided, strict monoidal functors

$$
\begin{array}{ccc}
{}^A_A\mathcal{YD}(\mathcal{C}) \dashrightarrow \mathcal{YD}^{(B^{\mathrm{op}})^{\mathrm{cop}}}_{(B^{\mathrm{op}})^{\mathrm{cop}}}(\mathcal{C}) & \xrightarrow{\;\mathsf{S}\;} & \mathcal{YD}^B_B(\mathcal{C}) \\
\Big\downarrow {}^A\mathsf{T} & & \Big\uparrow {}^{B^{\mathrm{op}}}\mathsf{T} \\
{}_{(A^{\mathrm{cop}})^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}(\overline{\mathcal{C}}) & \xrightarrow{\;{}_{\omega'}\mathsf{D}^{\omega}\;} & {}^{(B^{\mathrm{op}})^{\mathrm{cop}}}\mathcal{YD}_{B^{\mathrm{op}}}(\overline{\mathcal{C}})
\end{array}
$$

Here $\mathsf{S}$ denotes the functor of restriction along $S^{-1} : B \to (B^{\mathrm{op}})^{\mathrm{cop}}$ and corestriction along $S : (B^{\mathrm{op}})^{\mathrm{cop}} \to B$. The top line of the above diagram is the functor ${}^{\omega}_{\omega'}\mathsf{D}$. $\qquad\square$

Combining Theorem 3.16, Remark 3.17 and Corollary 3.19, we are now in a position to exhibit explicitly two braided equivalences

$$
\Omega, \Omega' : {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^B_B\mathcal{YD}(\mathcal{C}).
$$

The first functor is the composition $\Omega := \mathsf{T} \circ \mathsf{D}$ with monoidal structure

$$
\Omega_2(X,Y) = \mathsf{T}(\mathsf{D}_2(X,Y)) \circ \mathsf{T}_2(\mathsf{D}X, \mathsf{D}Y) = \mathrm{id}_{X \otimes Y} \circ (c^{\mathcal{YD}}_{\mathsf{D}Y, \mathsf{D}X} \circ c^{-1}_{Y,X})
$$

$$
= c^{\mathcal{YD}}_{Y,X} \circ c^{-1}_{Y,X}.
$$

The second to last equal sign uses that $\mathsf{D}$ is a strict braided functor. The other functor is $\Omega' := \mathsf{T}' \circ \mathsf{D}$ with monoidal structure

$$
\Omega'_2(X,Y) = \mathsf{T}'(\mathsf{D}_2(X,Y)) \circ \mathsf{T}'_2(\mathsf{D}X, \mathsf{D}Y) = \left( c^{\mathcal{YD}}_{X,Y} \right)^{-1} \circ c_{X,Y}.
$$

Graphically the functors and the monoidal structures look as follows:



We summarize our findings:

**Theorem 3.20.** *Let $\omega : A \otimes B \to \mathbf{1}$ be a non-degenerate Hopf pairing. The categories ${}^A_A\mathcal{YD}(\mathcal{C})$ and ${}^B_B\mathcal{YD}(\mathcal{C})$ are braided equivalent via the monoidal functors $\Omega$ and $\Omega'$ above.*

We end this subsection by relating the equivalence $\Omega$ to the equivalence $\Omega^{\mathrm{HS}}$ of rational modules over $\Bbbk$-Hopf algebras discussed in [HS13].

**Remark 3.21.**   (1) *Let $\Bbbk$ be a field and $\mathcal{L}_{\Bbbk}$ the category of linearly topologized vector spaces over $\Bbbk$. Fix a Hopf algebra $h$ in $\mathcal{L}_{\Bbbk}$ and two Hopf algebras $(R, R^{\vee})$ in $_h^h\mathcal{YD}(\mathcal{L}_{\Bbbk})$ that are related by a non-degenerate Hopf pairing. It is then shown in* [HS13] *that the categories $_{R \rtimes h}^{R \rtimes h}\mathcal{YD}_{\mathrm{rat}}$ and $_{R^{\vee} \rtimes h}^{R^{\vee} \rtimes h}\mathcal{YD}_{\mathrm{rat}}$ are equivalent as braided categories. Here, the subscript* rat *denotes the subcategory of rational modules.*

*The non-degenerate pairing $\langle,\rangle : R^{\vee} \otimes R \to \Bbbk$ and the structural morphisms of the bosonized Hopf algebra $R \rtimes h$ are used in* [HS13, Theorem 7.1] *to construct a functor*

$$(\Omega^{\mathrm{HS}}, \Omega_2^{\mathrm{HS}}) : {}_{R \rtimes h}^{R \rtimes h}\mathcal{YD}_{\mathrm{rat}} \to {}_{R^{\vee} \rtimes h}^{R^{\vee} \rtimes h}\mathcal{YD}_{\mathrm{rat}} \ .$$

*In detail, the functor $\Omega^{\mathrm{HS}}$ is constructed as follows: Let $M$ be a rational $(R \rtimes h)$-Yetter-Drinfel'd modules and denote the left $R$-coaction by $\delta(m) = m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle}$.*

*The $(R^{\vee} \rtimes h)$-Yetter-Drinfel'd module $\Omega^{\mathrm{HS}}(M)$ is equal to $M$ as an $h$-Yetter-Drinfel'd module and has the following $R^{\vee}$-Yetter-Drinfel'd structure*

*action:* $\qquad\qquad \xi m = \langle \xi, m_{\langle -1 \rangle} \rangle m_{\langle 0 \rangle}$

*coaction:* $\qquad\quad \delta_{\Omega^{\mathrm{HS}}(M)} = \left( c_{M,R^{\vee}}^{\mathcal{YD}} \circ c_{R^{\vee},M}^{\mathcal{YD}} \right)(m_{[-1]} \otimes m_{[0]}),$

*where $m_{[-1]} \otimes m_{[0]}$ is the unique element of $R^{\vee} \otimes M$ such that for all $r \in R$ and $m \in M$ we have*

$$rm = \left\langle m_{[-1]}, \theta_R(r) \right\rangle m_{[0]}.$$

*The monoidal structure of $\Omega^{\mathrm{HS}}$ is given by the family of morphisms*
$\Omega_2^{\mathrm{HS}}(M, N) : \Omega^{\mathrm{HS}}(M) \otimes \Omega^{\mathrm{HS}}(N) \to \Omega^{\mathrm{HS}}(M \otimes N)$

$$m \otimes n \mapsto S_{R \rtimes h}^{-1} S_R(n_{\langle -1 \rangle}) m \otimes n_{\langle 0 \rangle}.$$

(2) *In this paper, we started with a non-degenerate Hopf pairing $\omega : A \otimes B \to \mathbf{1}$ and constructed an equivalence*

$$\Omega^{\omega} : {}_A^A\mathcal{YD}(\mathcal{C}) \to {}_B^B\mathcal{YD}(\mathcal{C}).$$

*Let $\mathcal{C}$ be the category of finite dimensional Yetter-Drinfel'd modules over the finite dimensional Hopf algebra $h$. Set $A = R$ and $B = R^{\vee}$ and $\omega : A \otimes B \to \Bbbk$, such that $\omega^{-}(b \otimes a) = \langle b, a \rangle$, cf. Example 2.14. One can show by straight-forward computations, that our functor $\Omega^{\omega^{-}}$ coincides with the functor $\Omega^{\mathrm{HS}}$ on the full subcategory $_{R \rtimes h}^{R \rtimes h}\mathcal{YD}_{\mathrm{fin}} \subset {}_{R \rtimes h}^{R \rtimes h}\mathcal{YD}_{\mathrm{rat}}$ of finite dimensional $(R \rtimes h)$-Yetter-Drinfel'd modules.*

3.5. **The square of $\Omega$.** From a non-degenerate Hopf pairing $\omega : A \otimes B \to \mathbf{1}$, we obtained an equivalence $\Omega^{\omega} : {}_A^A\mathcal{YD}(\mathcal{C}) \to {}_B^B\mathcal{YD}(\mathcal{C})$. As noted in Example 2.14, we also have a non-degenerate Hopf pairing $\omega^{-} : B \otimes A \to \mathbf{1}$ from which we obtain an equivalence $\Omega^{\omega^{-}} : {}_B^B\mathcal{YD}(\mathcal{C}) \to {}_A^A\mathcal{YD}(\mathcal{C})$.

**Proposition 3.22.** *The braided monoidal functor*

$$\Omega^{\omega^{-}} \circ \Omega^{\omega} : {}_A^A\mathcal{YD}(\mathcal{C}) \to {}_A^A\mathcal{YD}(\mathcal{C})$$

*is isomorphic to the identity functor.*

*Proof.* A direct computation shows that the monoidal functors

$$(\Omega^\omega, \Omega_2^\omega) \circ \left((\Omega')^{\omega^-}, (\Omega')_2^{\omega^-}\right) \quad \text{and} \quad \left((\Omega')^{\omega^-}, (\Omega')_2^{\omega^-}\right) \circ (\Omega^\omega, \Omega_2^\omega)$$

are both equal to the identity functor with identity monoidal structure. Remark 3.17 implies that $(\Omega')^{\omega^-}$ is monoidally isomorphic to $\Omega^{\omega^-}$.

Alternatively, a concrete calculation shows that $\Omega^{\omega^-} \circ \Omega^\omega$ is equal to the monoidal functor that sends the Yetter-Drinfel'd module $(X, \rho, \delta)$ to the Yetter-Drinfel'd module

$$(X, \rho \circ (S^{-2} \otimes \mathrm{id}_X) \circ c_{A,X}^{-1} \circ c_{X,A}^{-1}, c_{X,A} \circ c_{A,X} \circ (S^2 \otimes \mathrm{id}_X) \circ \delta).$$

The monoidal structure of $\Omega^{\omega^-} \circ \Omega^\omega$ is given by the family of isomorphisms

$$c_{Y,X}^{\mathcal{YD}} \circ c_{X,Y}^{\mathcal{YD}} \circ c_{X,Y}^{-1} \circ c_{Y,X}^{-1}.$$

From the this and Lemma 3.4 it is clear that $\theta_X := \rho_X \circ (S \otimes \mathrm{id}_X) \circ \delta_X$ defines a monoidal isomorphism

$$\theta : \Omega^{\omega^-} \circ \Omega^\omega \to \mathrm{Id}.$$

$\square$

## 4. Partial dualization of a Hopf algebra

We now present the main construction of this article: Let $H$ be a Hopf algebra in a braided category $\mathcal{C}$, $A$ be a Hopf subalgebra and $\pi : H \to A$ a Hopf algebra projection. Moreover, let $B$ be a Hopf algebra in $\mathcal{C}$ and $\omega : A \otimes B \to \mathbf{1}$ a non-degenerate Hopf pairing. These data constitute a *partial dualization datum* $\mathcal{A}$ for the Hopf algebra $H$ to which we associate a *partial dualization* $r_\mathcal{A}(H)$, a new Hopf algebra in the braided category $\mathcal{C}$. The construction makes use of the functors introduced in Section 3 that relate various categories of Yetter-Drinfel'd modules. We show that the assignment $H \mapsto r_\mathcal{A}(H)$ is involutive up to an isomorphism. We also prove a fundamental equivalence of braided categories

$$_H^H\mathcal{YD}(\mathcal{C}) \cong {}_{r_\mathcal{A}(H)}^{r_\mathcal{A}(H)}\mathcal{YD}(\mathcal{C}).$$

This equivalence implies that the Drinfel'd doubles of $H$ and $r_\mathcal{A}(H)$ are Morita-equivalent Hopf algebras.

4.1. **Main construction.** We start with some definitions:

**Definition 4.1.** *Let $\mathcal{C}$ be a braided monoidal category. A* partial dualization datum $\mathcal{A} = (H \xrightarrow{\pi} A, B, \omega)$ *for a Hopf algebra $H$ in $\mathcal{C}$ consists of*

- *a Hopf algebra projection $\pi : H \to A$ to a Hopf subalgebra $A \subset H$,*
- *a Hopf algebra $B$ with a non-degenerate Hopf pairing $\omega : A \otimes B \to \mathbf{1}_\mathcal{C}$.*

Given a partial dualization datum $\mathcal{A}$ for a Hopf algebra $H$ in $\mathcal{C}$, the *partial dualization* $r_\mathcal{A}(H)$ is the following Hopf algebra in $\mathcal{C}$:

- By the Radford projection theorem 3.10, the projection $\pi : H \to A$ induces a Radford biproduct decomposition of $H$

$$H \cong K \rtimes A ,$$

  where $K := H^{\mathrm{coin}(\pi)}$ is a Hopf algebra in the braided category $_A^A\mathcal{YD}(\mathcal{C})$.

- The non-degenerate Hopf pairing $\omega : A \otimes B \to \mathbf{1}$ induces by Theorem 3.20 a braided equivalence:

$$\Omega : \, {}^A_A\mathcal{YD}(\mathcal{C}) \stackrel{\sim}{\to} {}^B_B\mathcal{YD}(\mathcal{C}) \,.$$

  Thus, the image of the Hopf algebra $K$ in ${}^A_A\mathcal{YD}(\mathcal{C})$ under the braided functor $\Omega$ is a Hopf algebra $L := \Omega(K)$ in the braided category ${}^B_B\mathcal{YD}(\mathcal{C})$.
- The Radford biproduct from Definition 3.7 of $L$ over $B$ allows us to introduce the partially dualized Hopf algebra,

$$r_{\mathcal{A}}(H) := L \rtimes B \,,$$

  which is a Hopf algebra in $\mathcal{C}$. As a Radford biproduct, it comes with a projection $\pi' : r_{\mathcal{A}}(H) \to B$.

We summarize:

**Definition 4.2.** *For a partial dualization datum $\mathcal{A} = (H \xrightarrow{\pi} A, B, \omega)$, we call the Hopf algebra $r_{\mathcal{A}}(H)$ in $\mathcal{C}$ the* partial dual *of $H$ with respect to $\mathcal{A}$.*

Our construction is inspired by the calculations in [HS13] using smash-products. In Section 5.3, we explain the relation of these calculations to our general construction.

4.2. **Involutiveness of partial dualizations.** The Hopf algebra $r_{\mathcal{A}}(H)$ comes with a projection to the subalgebra $B$. The two Hopf pairings $\omega^{\pm} : B \otimes A \to \mathbf{1}_{\mathcal{C}}$ from Example 2.14 (2) yield two possible partial dualization data for $r_{\mathcal{A}}(H)$:

$$\mathcal{A}^+ = (r_{\mathcal{A}}(H) \xrightarrow{\pi'} B, A, \omega^+)$$

$$\mathcal{A}^- = (r_{\mathcal{A}}(H) \xrightarrow{\pi'} B, A, \omega^-) \,.$$

Recall from Subsection 3.5 the natural isomorphism

$$\theta : \, \Omega^{\omega^-} \circ \Omega^{\omega} \cong \mathrm{Id}_{{}^A_A\mathcal{YD}(\mathcal{C})} \,.$$

In a similar way, one has a natural isomorphism

$$\tilde{\theta} : \, \Omega^{\omega} \circ \Omega^{\omega^+} \cong \mathrm{Id}_{{}^B_B\mathcal{YD}(\mathcal{C})}.$$

**Corollary 4.3.** *The two-fold partial dualization $r_{\mathcal{A}^-}(r_{\mathcal{A}}(H))$ is isomorphic to $H$, as Hopf algebra in the braided category $\mathcal{C}$. A non-trivial isomorphism of Hopf algebras is*

$$r_{\mathcal{A}^-}(r_{\mathcal{A}}(H)) = \Omega^{\omega^-}(\Omega^{\omega}(K)) \rtimes A \xrightarrow{\theta_K \otimes id_A} K \rtimes A = H,$$

*with $\theta_K = \rho_K \circ (S_A \otimes id_K) \circ \delta_K$ as in Lemma 3.4.*

4.3. **Relations between the representation categories.** It is natural to look for relations between categories of representations of a Hopf algebra $H$ in $\mathcal{C}$ and its partial dualization $r_{\mathcal{A}}(H)$:

**Theorem 4.4.** *Let $H$ be a Hopf algebra in a braided category $\mathcal{C}$, let $\mathcal{A} = (H \xrightarrow{\pi} A, B, \omega)$ be a partial dualization datum and $r_{\mathcal{A}}(H)$ the partially dualized Hopf algebra. Then the equivalence of braided categories*

$$\Omega : \, {}^A_A\mathcal{YD}(\mathcal{C}) \to {}^B_B\mathcal{YD}(\mathcal{C})$$

*from Theorem 3.20 induces an* equivalence of braided categories:

$$ {}^H_H\mathcal{YD}\left(\mathcal{C}\right) \cong {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}\left(\mathcal{C}\right)\right) \xrightarrow{\tilde{\Omega}} {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}\left(\mathcal{C}\right)\right) \cong {}^{r_\mathcal{A}(H)}_{r_\mathcal{A}(H)}\mathcal{YD}\left(\mathcal{C}\right). $$

*Proof.* The Hopf algebra $L \in {}^B_B\mathcal{YD}(\mathcal{C})$ was defined as the image of $K \in {}^A_A\mathcal{YD}(\mathcal{C})$ under the functor $\Omega$, i.e. $L = \Omega(K)$. The braided equivalence $\Omega$ induces an equivalence $\tilde{\Omega}$ of Yetter-Drinfel'd modules over the Hopf algebra $K$ in the braided category ${}^A_A\mathcal{YD}(\mathcal{C})$ to Yetter-Drinfel'd modules over the Hopf algebra $L = \Omega(K)$ in ${}^B_B\mathcal{YD}(\mathcal{C})$

$$ {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}\left(\mathcal{C}\right)\right) \xrightarrow{\tilde{\Omega}} {}^{\Omega(K)}_{\Omega(K)}\mathcal{YD}\left({}^B_B\mathcal{YD}\left(\mathcal{C}\right)\right) $$
$$ =: {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}\left(\mathcal{C}\right)\right). $$

By Theorem 3.10, the source category of $\tilde{\Omega}$ is

$$ {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}\left(\mathcal{C}\right)\right) \cong {}^{K\rtimes A}_{K\rtimes A}\mathcal{YD}\left(\mathcal{C}\right) = {}^H_H\mathcal{YD}\left(\mathcal{C}\right). $$

Similarly, we have for the target category of $\tilde{\Omega}$

$$ {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}\left(\mathcal{C}\right)\right) \cong {}^{L\rtimes B}_{L\rtimes B}\mathcal{YD}\left(\mathcal{C}\right) = {}^{r_\mathcal{A}(H)}_{r_\mathcal{A}(H)}\mathcal{YD}\left(\mathcal{C}\right). $$

Altogether, we obtain a braided equivalence

$$ {}^H_H\mathcal{YD}\left(\mathcal{C}\right) \cong {}^K_K\mathcal{YD}\left({}^A_A\mathcal{YD}\left(\mathcal{C}\right)\right) \xrightarrow{\tilde{\Omega}} {}^L_L\mathcal{YD}\left({}^B_B\mathcal{YD}\left(\mathcal{C}\right)\right) \cong {}^{r_\mathcal{A}(H)}_{r_\mathcal{A}(H)}\mathcal{YD}\left(\mathcal{C}\right). $$

$\square$

If $\mathcal{C}$ is the category of vector spaces over a field $\Bbbk$, Yetter-Drinfeld modules over a Hopf algebra $H$ can be described as modules over the Drinfel'd double $\mathcal{D}(H)$. For Hopf algebra $H$ in a general braided category $\mathcal{C}$, the appropriate notion of a Drinfel'$f$d double $\mathcal{D}(H)$ has been introduced in [BV12] such that a braided equivalence $\mathcal{D}(H)\text{-mod}_\mathcal{C} \cong {}^H_H\mathcal{YD}\left(\mathcal{C}\right)$ holds. Hence Theorem 4.4 implies

**Corollary 4.5.** *The categories of left modules over the Drinfeld double $\mathcal{D}(H)$ of a Hopf algebra $H$ and over the Drinfel'd double $\mathcal{D}(r_\mathcal{A}(H))$ of its partial dualization $r_\mathcal{A}(H)$ are braided equivalent.*

## 5. Examples

We illustrate our general construction in three different cases:

5.1. **The complex group algebra of a semi-direct product.** For the complex Hopf algebra associated to a finite group $G$, we take

$$ \mathcal{C} = \text{vect}_\mathbb{C} \qquad H = \mathbb{C}[G]. $$

To get a partial dualization datum for $H$, suppose that there is a split extension $N \to G \to Q$, which allows us to identify $Q$ with a subgroup of $G$, i.e. $G = N \rtimes Q$. We then get a split Hopf algebra projection to $A := \mathbb{C}[Q]$:

$$ \pi: \ \mathbb{C}[G] \to \mathbb{C}[Q]. $$

The coinvariants of $H$ with respect to $\pi$, which by Theorem 3.10 have the structure of a Hopf algebra $K \in {}^A_A\mathcal{YD}(\mathcal{C})$, turn out to be

$$ K := H^{\text{coin}(\pi)} = \mathbb{C}[N]. $$

The $A$-coaction on the $A$-Yetter-Drinfel'd module $K$ is trivial, since the Hopf algebra $H$ is cocommutative. The $A$-action on $K$ is non-trivial; it is given by the action of $Q \subset G$ on the normal subgroup $N$. Because of the trivial $A$-coaction, the self-braiding of $K$ in ${}_A^A \mathcal{YD}$ is trivial; thus $K$ is even a complex Hopf algebra. Writing $H$ as in Theorem 3.10 as a Radford biproduct, we recover

$$H = K \rtimes A = \mathbb{C}[N] \rtimes \mathbb{C}[Q].$$

Since the $A$-coaction on $K$ is trivial, the coalgebra structure is just given by the tensor product of the coalgebra structures on the group algebras.

As the dual of $A$, we take the commutative Hopf algebra of functions on $Q$, $B := \mathbb{C}^Q$; we denote its canonical basis by $(e_q)_{q \in Q}$; the Hopf pairing $\omega$ is the canonical evaluation. This gives the partial dualization datum

$$\mathcal{A} = (\mathbb{C}[G] \xrightarrow{\pi} \mathbb{C}[Q], \mathbb{C}^Q, \omega) \ .$$

Since the coaction of $A$ on $K$ is trivial, the morphism $\Omega_2(K, K)$ from the monoidal structure on $\Omega$ is trivial. Hence the functor $\Omega^\omega$ maps $K$ to the same complex Hopf algebra

$$L := \Omega^\omega(K) \cong \mathbb{C}[N] \ ,$$

which however has now to be seen as a Yetter-Drinfel'd module over $\mathbb{C}^Q$, i.e. $L \in {}_{\mathbb{C}^Q}^{\mathbb{C}^Q} \mathcal{YD}$: $L$ has trivial action of $B = \mathbb{C}^Q$ and the coaction is given by the dualized action of $Q$ on $N$

$$n \longmapsto \sum_{q \in Q} e_q \otimes q^{-1} n q.$$

The partial dualization $r_\mathcal{A}(H)$ is, by definition, the Radford biproduct

$$r_\mathcal{A}(H) = L \rtimes B = \mathbb{C}[N] \rtimes \mathbb{C}^Q.$$

In this biproduct, the algebra structure is given by the tensor product of algebras.

An $H$-module is a complex $G$-representation. To give an alternative description of the category $r_\mathcal{A}(H)$-mod, we make the definition of $r_\mathcal{A}(H)$-modules explicit: An $r_\mathcal{A}(H)$-module $V$, with $r_\mathcal{A}(H) = \mathbb{C}[N] \rtimes \mathbb{C}^Q$ has the structure of a $\mathbb{C}^Q$-module and thus of a $Q$-graded vector space: $V = \bigoplus_{q \in Q} V_q$. Moreover, it comes with an action of $N$ denoted by $n.v$ for $n \in N$ and $v \in V$. Since the algebra structure is given by the tensor product of algebras, the $N$-action preserves the $Q$-grading. The tensor product of two $r_\mathcal{A}(H)$-modules $V$ and $W$ is graded in the obvious way,

$$(V \otimes W)_q = \bigoplus_{q_1 q_2 = q} V_{q_1} \otimes W_{q_2}.$$

The non-trivial comultiplication

$$\Delta_{\mathbb{C}[N] \rtimes \mathbb{C}^Q}(n) = \sum_{q \in Q} (n \otimes e_q) \otimes (q^{-1} n q \otimes 1)$$

for the Radford biproduct implies a non-trivial $N$-action on the tensor product: on homogeneous components $V_{q_1}$ and $W_{q_2}$, with $q_1, q_2 \in Q$, we have for $n \in N$

$$n.(V_{q_1} \otimes W_{q_2}) = (n.V_{q_1}) \otimes ((q_1^{-1} n q_1).V_{q_2}).$$

We are now in a position to give the alternative description of the category $r_{\mathcal{A}}(H)$-mod. We denote by $\text{vect}_G$ the monoidal category of $G$-graded finite-dimensional complex vector spaces, with the monoidal structure inherited from the category of vector spaces. Representatives of the isomorphism classes of simple objects are given by the one-dimensional vector spaces $\mathbb{C}_g$ in degree $g \in G$. Given a subgroup $N \leq G$, the object $\mathbb{C}[N] := \oplus_{n \in N} \mathbb{C}_n$ has a natural structure of an associative, unital algebra in $\text{vect}_G$. It is thus possible to consider $\mathbb{C}[N]$-bimodules in the monoidal category $\text{vect}_G$; together with the tensor product $\otimes_{\mathbb{C}[N]}$, these bimodules form a monoidal category $\mathbb{C}[N]$-$\text{bimod}_{\text{vect}_G}$. In this setting, we have the following description of the category $r_{\mathcal{A}}(H)$-mod:

**Lemma 5.1.** *The monoidal category $r_{\mathcal{A}}(H)$-mod is monoidally equivalent to the category of $\mathbb{C}[N]$-$\text{bimod}_{\text{vect}_G}$.*

The braided equivalence of Yetter-Drinfel'd modules over $H$ and $r_{\mathcal{A}}(H)$ established in Theorem 4.4, more precisely the braided equivalence of the categories of modules over their Drinfel'd doubles from Corollary 4.5, implies the braided equivalence

$$\mathcal{Z}(\mathbb{C}[G]\text{-mod}) \cong \mathcal{Z}(\text{vect}_G) \cong \mathcal{Z}(\mathbb{C}[N]\text{-bimod}_{\text{vect}_G})$$

which has been shown in [S01, Theorem 3.3] in a more general context.

*Proof.* It suffices to specify a monoidal functor

$$\Phi : \mathbb{C}[N]\text{-bimod}_{\text{vect}_G} \to r_{\mathcal{A}}(H)\text{-mod}$$

that is bijective on the spaces of morphisms and to give a preimage for every object $D \in r_{\mathcal{A}}(H)$-mod. Suppose that $B$ is a $\mathbb{C}[N]$-bimodule in the category $\text{vect}_G$, i.e. $B = \bigoplus_{g \in G} B_g$, with $\mathbb{C}[N]$-actions denoted by arrows $\rightharpoonup, \leftharpoonup$.

To define the functor $\Phi$ on objects, consider for a bimodule $B$ the $Q$-graded vector space $\Phi(B) := \oplus_{q \in Q} B_q \subset B$, obtained by retaining only the homogeneous components with degree in $Q \subset G$. A left $N$-action is defined for any homogeneous vector $v_q \in \Phi(B)_q$ by

$$n.v_q := n \rightharpoonup v_q \leftharpoonup (q^{-1}n^{-1}q).$$

Moreover,

$$n.v_q = n \rightharpoonup v_q \leftharpoonup (q^{-1}n^{-1}q) \ \in \Phi(B)_{nq(q^{-1}n^{-1}q)} = \Phi(B)_q \ ,$$

since $\rightharpoonup, \leftharpoonup$ are morphisms in $\text{vect}_G$. Thus the $N$-action preserves the $Q$-grading; we conclude that $\Phi(B)$ is an object in $r_{\mathcal{A}}(H)$-mod.

On the morphism spaces, the functor $\Phi$ acts by restriction to the vector subspace $\Phi(B) \subset B$. We show that this gives a bijection on morphisms: Suppose $\Phi(f) = 0$, then $f(v_q) = 0$ for all $v_q$ with grade $q \in Q$. For an arbitrary $v_g \in B$ with grade $g \in G$, we may write $g = nq$ with $n \cdot q \in N \rtimes Q$ and get an element $n^{-1} \rightharpoonup v_g$ of degree $q$. Using that $f$ is a morphism of $\mathbb{C}[N]$-bimodules, we find $f(v_g) = n \rightharpoonup f(n^{-1} \rightharpoonup v_g) = 0$. Thus $\Phi$ is injective on morphisms. To show surjectivity, we take a morphism $f_\Phi : \Phi(B) \to \Phi(C)$; writing again $g = nq$, we define a linear map $f : B \to C$ on $v_g \in V_g$ by $f(v_g) := n \rightharpoonup f_\Phi(n^{-1} \rightharpoonup v_g)$. This linear map is, by construction, a morphism of left $\mathbb{C}[N]$-modules in $\text{vect}_G$. It remains to verify that $f$ is also a morphism of right $\mathbb{C}[N]$-modules. We note that for $g \in G$, the decomposition

$g = nq$ with $n \in N$ and $q \in Q$ implies $gm = (nqmq^{-1})q$ with $nqmq^{-1} \in N$ for all $m \in N$. We thus find:

$$
\begin{aligned}
f(v_g \leftharpoonup m) &= (nqmq^{-1}) \rightharpoonup f_\Phi((nqmq^{-1})^{-1} \rightharpoonup v_g \leftharpoonup m) \\
&= (nqmq^{-1}) \rightharpoonup f_\Phi((qm^{-1}q^{-1}) \rightharpoonup (n^{-1} \rightharpoonup v_g) \leftharpoonup m) \\
&= (nqmq^{-1}) \rightharpoonup f_\Phi((qm^{-1}q^{-1}).(n^{-1} \rightharpoonup v_g)) \\
&= (nqmq^{-1}) \rightharpoonup (qm^{-1}q^{-1}).f_\Phi((n^{-1} \rightharpoonup v_g)) \\
&= (nqmq^{-1}) \rightharpoonup (qm^{-1}q^{-1}) \rightharpoonup f_\Phi(n^{-1} \rightharpoonup v_g) \leftharpoonup m \\
&= n \rightharpoonup f_\Phi(n^{-1} \rightharpoonup v_g) \leftharpoonup m \\
&= f(v_g) \leftharpoonup m
\end{aligned}
$$

In the forth identity, we used that $f_\Phi$ is $r_\mathcal{A}(H)$-linear.

Next we show that $\Phi$ has a natural structure of a monoidal functor. Recall that the tensor product $V \otimes W$ in $\mathrm{vect}_G$ (resp. $\mathrm{vect}_Q$) is defined as the tensor product of vector spaces with diagonal grading $V_g \otimes W_h \subset (V \otimes W)_{gh}$. Furthermore, the tensor product in $\mathbb{C}[N]$-bimod is defined by $\otimes_{\mathbb{C}[N]}$. On the other side, the tensor product $\otimes$ in $r_\mathcal{A}(H)$-mod is the tensor product of modules over the Hopf algebra $r_\mathcal{A}(H) = \mathbb{C}[N] \rtimes \mathbb{C}^Q$ with diagonal grading and action

$$
n.(\Phi(B)_{q_1} \otimes \Phi(B)_{q_2}) = (n.\Phi(B)_{q_1}) \otimes ((qnq^{-1}).\Phi(B)_{q_2}).
$$

We now show that the canonical projection of vector spaces $B \otimes C \to B \otimes_{\mathbb{C}[N]} C$ gives rise to a monoidal structure on $\Phi$:

$$
\Phi_2 : \Phi(B) \otimes \Phi(C) \to \Phi(B \otimes_{\mathbb{C}[N]} C).
$$

It is clear that this map is compatible with the $Q$-grading. The compatibility with the $N$-action is calculated as follows: for $n \in N$, $b \in B_{q_1}$ $c \in C_{q_2}$:

$$
\begin{aligned}
(n.(b \otimes c)) &= n.b \otimes (q_1^{-1}nq_1).c \\
&\xmapsto{\phi} (n.b) \otimes_{\mathbb{C}[N]} (q_1^{-1}nq_1).c \\
&= (n \rightharpoonup b \leftharpoonup (q_1^{-1}n^{-1}q_1)) \otimes_{\mathbb{C}[N]} ((q_1^{-1}nq_1) \rightharpoonup c \leftharpoonup (q_2^{-1}q_1^{-1}nq_1q_2)) \\
&= (n \rightharpoonup b \leftharpoonup (q_1^{-1}n^{-1}q_1q_1^{-1}nq_1)) \otimes_{\mathbb{C}[N]} (c \leftharpoonup (q_2^{-1}q_1^{-1}nq_1q_2)) \\
&= (n \rightharpoonup b) \otimes_{\mathbb{C}[N]} (c \leftharpoonup ((q_1q_2)^{-1}n(q_1q_2)) \\
&= n.\left(b \otimes_{\mathbb{C}[N]} c\right).
\end{aligned}
$$

Moreover, $\Phi_2$ is clearly compatible with the associativity constraint. We now show $\Phi_2$ is bijective by giving an explicit inverse: Consider an element $v \otimes w \in B \otimes_{\mathbb{C}[N]} C$ which is in $\Phi(B \otimes_{\mathbb{C}[N]} C) \subset B \otimes_{\mathbb{C}[N]} C$. Restricting to homogeneous elements, we take $v \otimes w \in (B \otimes_{\mathbb{C}[N]} C)$ with $v$ of degree $g \in G$ and $w$ of degree $h \in G$. Since $v \otimes w$ is even in the subspace $\Phi(B \otimes_{\mathbb{C}[N]} C)$, we have $q := gh \in Q$. Writing $h = n'q'$ with $n' \in N$ and $q' \in Q$, have in the tensor product over $\mathbb{C}[N]$ the identity $v \otimes w = (v \leftharpoonup n') \otimes ((n')^{-1} \rightharpoonup w)$ with tensor factors both graded in $Q$, hence in $\Phi(B) \otimes \Phi(C)$. We may now define the inverse $\Phi_2^{-1}(v \otimes w) := (v \leftharpoonup n) \otimes (n^{-1} \rightharpoonup w)$, which is a left- and right-inverse of $\Phi_2$. Finally the monoidal units in the categories are $\mathbb{C}_1$, $1 \in Q$

resp. $\mathbb{C}[N]$; then $N \cap Q = \{1\}$ implies that there is an obvious isomorphism $\mathbb{C}_1 \cong \Phi(\mathbb{C}[N])$. Hence $\Phi$ is a monoidal functor.

To verify that $\Phi$ indeed defines an equivalence of tensor categories, it remains to construct for each $V \in r_{\mathcal{A}}(H)\text{-mod}$ an object $D \in \mathbb{C}[N]\text{-bimod}_{\text{vect}_G}$ such that $\Phi(D) \cong V$.

The following construction could be understood as an induced corepresentation via the cotensor product, but we prefer to keep the calculation explicit: For $V = \bigoplus V_q$ consider the vector space

$$D := \bigoplus_{q \in Q} \mathbb{C}[N] \otimes V_q.$$

Since $G = NQ$, the vector space $D$ is naturally endowed with a $G$-grading. Left multiplication on $\mathbb{C}[N]$ gives a natural left $N$-action $\rightharpoonup$ via left-multiplication on $\mathbb{C}[N]$, which is clearly a morphism in $\text{vect}_G$. We define a right $N$-action on $D$ by

$$(n \otimes v_q) \leftharpoonup m := n(qmq^{-1}) \otimes (qn^{-1}q^{-1}).v_q.$$

Since the left action preserves the $Q$-grading, the vector $(n \otimes v_q) \leftharpoonup m$ has degree $n(qmq^{-1})q = (nq)m$; thus also the right action $\leftharpoonup$ is a morphism in $\text{vect}_G$.

We finally verify that $\Phi(D) \cong V$: the homogeneous components of $D$ with degree in the subgroup $Q$ only are spanned by elements $1 \otimes v_q$, hence we can identify $\Phi(D)$ with $V$. We check that the $N$-action defined on $\Phi(D)$ coincides with the one on $V$ we started with:

$$\begin{aligned}
n.(1 \otimes v_q) &= n \rightharpoonup (1 \otimes v_q) \leftharpoonup (q^{-1}n^{-1}q) \\
&= n(q(q^{-1}n^{-1}q)q^{-1}) \otimes (q(q^{-1}n^{-1}q)^{-1}q^{-1}).v_q \\
&= 1 \otimes n.v_q.
\end{aligned}$$

$\square$

5.2. **The Taft algebra.** Fix a natural number $d$ and let $\zeta \in \mathbb{C}$ be a primitive $d$-th root of unity. We consider the Taft algebra $T_\zeta$ which is a complex Hopf algebra. As an algebra, $T_\zeta$ is generated by two elements $g$ and $x$ modulo the relations

$$g^d = 1, \ x^d = 0 \quad \text{and} \quad gx = \zeta x g.$$

A coassociative comultiplication on $T_\zeta$ is defined by the unique algebra homomorphism $\Delta : T_\zeta \to T_\zeta \otimes T_\zeta$ with

$$\Delta(g) = g \otimes g \quad \text{and} \quad \Delta(x) = g \otimes x + x \otimes 1.$$

**Lemma 5.2.** *Let $\zeta$ and $\xi$ be primitive $d$-th roots of unity. If there exists an isomorphism $\psi : T_\zeta \to T_\xi$ of Hopf algebras, then $\zeta = \xi$.*

*Proof.* The set $\{x^n g^m \mid 0 \leq n, m < d\}$ is a $\mathbb{C}$-basis of $T_\xi$ consisting of eigenvectors for the automorphisms

$$\text{ad}_h : T_\xi \to T_\xi, \quad a \mapsto hah^{-1},$$

with $h = g^c$ for $c \in \{1, 2, \ldots, N-1\}$.

Suppose that $\psi : T_\zeta \to T_\xi$ is a Hopf algebra isomorphism. Then the image $h := \psi(g)$ of the generator $g$ of $T_\zeta$ is equal to $g^c \in T_\xi$ for some

$c \in \{1, 2, \ldots, d-1\}$. The generator $x$ of $T_\zeta$ is mapped by the algebra homomorphism $\psi$ to an eigenvector $y := \psi(x)$ of $\mathrm{ad}_h$ to the eigenvalue $\zeta$:

$$hyh^{-1} = \psi(gxg^{-1}) = \zeta y.$$

Since $\xi$ is a primitive root of unit, we find $0 < n < d$ such that $\zeta = \xi^n$. Thus $y$ is an element of the $\mathbb{C}$-linear subspace $\langle x^n g^m \mid 0 \le m < d \rangle_\mathbb{C}$ of $T_\xi$. This implies that $y^k = 0$ for $k$ the smallest number such that $kn \ge d$. Since $\psi$ is an isomorphism, $n$ has to be 1 and hence $\zeta = \xi$. $\qquad\square$

Denote by $A$ the Hopf subalgebra of $T_\zeta$ generated by $g$. We will deduce from Proposition 5.5 that the partial dual of $T_\zeta$ with respect to $A$ is isomorphic to $T_\zeta$. Hence we have to enlarge the class of complex Hopf algebras beyond Taft algebras to get a non-trivial example.

Let $N$ be a natural number and $d$ a divisor of $N$. Now let $\zeta$ be a primitive $d$-th root of unity and $q$ a primitive $N$-th root of unity. Let $c + N\mathbb{Z}$ be the unique residue class such that $\zeta = q^c$. Define $\hat{T}_{\zeta,q}$ as the $\mathbb{C}$-algebra

$$\hat{T}_{\zeta,q} := \langle x, g \mid g^N = 1, x^d = 0, gx = \zeta xg \rangle$$

and define $\check{T}_{\zeta,q}$ as the $\mathbb{C}$-algebra

$$\check{T}_{\zeta,q} := \langle x, g \mid g^N = 1, x^d = 0, gx = qxg \rangle.$$

Both algebras are finite-dimensional of dimension $Nd$.

One checks the following

**Lemma 5.3.** *Let $\hat{T}_{\zeta,q}$ and $\check{T}_{\zeta,q}$ be the algebras from above. The unique algebra homomorphisms $\hat{\Delta} : \hat{T}_{\zeta,q} \to \hat{T}_{\zeta,q} \otimes \hat{T}_{\zeta,q}$ and $\check{\Delta} : \check{T}_{\zeta,q} \to \check{T}_{\zeta,q} \otimes \check{T}_{\zeta,q}$ defined on the generators by*

$$\hat{\Delta}(g) := g \otimes g \qquad\qquad \hat{\Delta}(x) := g \otimes x + x \otimes 1$$
$$\check{\Delta}(g) := g \otimes g \qquad\qquad \check{\Delta}(x) := g^c \otimes x + x \otimes 1$$

*give the structure of an coassociative counital Hopf algebra on $\hat{T}_{\zeta,q}$ and $\check{T}_{\zeta,q}$, respectively.*

*Furthermore, we have exact sequences of Hopf algebras, with $k := \frac{N}{d}$*

$$\mathbb{C}[\mathbb{Z}_k] \longrightarrow \hat{T}_{\zeta,q} \longrightarrow T_\zeta$$
$$T_\zeta \longrightarrow \check{T}_{\zeta,q} \longrightarrow \mathbb{C}[\mathbb{Z}_k].$$

The Hopf subalgebra $A \subset \hat{T}_{\zeta,q}$ generated by the grouplike element $g$ and the Hopf subalgebra $B \subset \check{T}_{\zeta,q}$ generated by $g$ are both isomorphic to the complex group Hopf algebra $\mathbb{C}[\mathbb{Z}_N]$. To apply a partial dualization, we need a Hopf pairing; it is given by the following lemma whose proof we leave to the reader:

**Lemma 5.4.** *Let $q$ be an $N$-th primitive root of unity and let $g \in \mathbb{C}[\mathbb{Z}_N]$ be a generator of the cyclic group $\mathbb{Z}_N$.*

  (1) *The bilinear form $\omega : \mathbb{C}[\mathbb{Z}_N] \times \mathbb{C}[\mathbb{Z}_N] \to \Bbbk$ given by $\omega(g^n, g^m) = q^{nm}$ is a Hopf pairing.*

(2) *The linear map $\omega' : \Bbbk \to \mathbb{C}[\mathbb{Z}_N] \otimes \mathbb{C}[\mathbb{Z}_N]$ with*

$$\omega'(1_{\Bbbk}) = \frac{1}{N} \sum_{k,\ell=1}^{N} q^{-k\ell} g^k \otimes g^{\ell}$$

*is the inverse copairing of $\omega$.*

The partial dual of $\hat{T}_{\zeta,q}$ with respect to $A$ and $\omega$ is isomorphic to $\check{T}_{\zeta,q}$:

**Proposition 5.5.** *Let $N$ be a natural number and $d$ be a divisor of $N$. Let $\zeta$ be a primitive $d$-th root of unity and $q$ a primitive $N$-th root of unity with $q^c = \zeta$. Let $A \subset \hat{T}_{\zeta,q}$ and $B \subset \check{T}_{\zeta,q}$ be as above and $\omega : A \otimes B \to \Bbbk$ the non-degenerate Hopf pairing from Lemma 5.4.*

1. *The algebra homomorphism $\pi : \hat{T}_{\zeta,q} \to A$ which sends $g$ to $g$ and $x$ to $0$ is a Hopf algebra projection.*
2. *The partial dualization of $\hat{T}_{\zeta,q}$ with respect to the partial dualization datum $(\hat{T}_{\zeta,q} \xrightarrow{\pi} A, B, \omega)$ is isomorphic to $\check{T}_{\zeta,q}$.*

In particular, for $N = d$, we have $\hat{T}_{\zeta,q} = \check{T}_{\zeta,q}$.

*Proof.* The space of coinvariants $K := \hat{T}_{\zeta,q}^{\mathrm{coin}(\pi)} = \{a \in \hat{T}_{\zeta,q} \mid \hat{\Delta}(a) = a \otimes 1\}$ equals the $\mathbb{C}$-linear span of $\{1, x, x^2, \ldots, x^{d-1}\}$. Remark 3.11 implies that $K$ is a Yetter-Drinfel'd module with $A$-action $\rho : A \otimes K \to K$ and $A$-coaction $\delta : K \to A \otimes K$ given by

$$\rho : g \otimes x \mapsto gxg^{-1} = \zeta x,$$
$$\delta : x \mapsto \pi(g) \otimes x = g \otimes x = x_{(-1)} \otimes x_{(0)}.$$

Moreover, $K$ has the structure of a Hopf algebra in $^A_A\mathcal{YD}$ with multiplication and comultiplication given by

$$\mu : x \otimes x \mapsto x^2,$$
$$\Delta : x \mapsto 1 \otimes x + x \otimes 1 .$$

The dualization functor $(\Omega, \Omega_2)$ from Section 3.4 for the Hopf pairing $\omega : A \otimes A \to \Bbbk$ yields the $B$-Yetter-Drinfel'd module $L = \langle 1, x, x^2, \ldots, x^{d-1} \rangle_{\mathbb{C}}$ with action $\rho' : B \otimes L \to L$ and coaction $\delta' : L \to B \otimes L$ given by

$$\rho' : g \otimes x \mapsto \omega(x_{(-1)}, g) x_{(0)} = qx,$$
$$\delta' : x \mapsto \frac{1}{N} \sum_{k,\ell=1}^{N} q^{-k \cdot \ell} g^k \otimes \rho(g^{\ell} \otimes x) = \frac{1}{N} \sum_{k,\ell=1}^{N} (q^{-k} q^c)^{\ell} g^k \otimes x = g^c \otimes x.$$

The Yetter-Drinfel'd module $L$ has a natural structure of a Hopf algebra in $^B_B\mathcal{YD}$ with multiplication $\mu' = \mu \circ \Omega_2(K, K)$ and comultiplication $\Delta' = \Omega_2^{-1}(K, K) \circ \Delta$

$$\mu' : x \otimes x \mapsto \zeta x^2,$$
$$\Delta' : x \mapsto 1 \otimes x + x \otimes 1 .$$

As an algebra, $L$ is generated by $x$, so the biproduct $r_{\mathcal{A}}(\hat{T}_{\zeta,q}) = L \rtimes B$ is generated by $x \cong x \otimes 1$ and $g \cong 1 \otimes g$. In the biproduct $r_{\mathcal{A}}(\hat{T}_{\zeta,q})$, the relations

$$g^N = 1, x^d = 0 \text{ and } gx = \rho'(g \otimes x)g = qxg$$

hold. This gives a surjective algebra homomorphism $\psi : r_{\mathcal{A}}(\hat{T}_{\zeta,q}) \to \check{T}_{\zeta,q}$; since $\check{T}_{\zeta,q}$ and $r_{\mathcal{A}}(\hat{T}_{\zeta,q})$ have the same complex dimension, $\psi$ is an isomorphism.

The map $\psi$ also respects the coalgebra structures, since

$$\Delta_{r_{\mathcal{A}}(\hat{T}_{\zeta,q})}(x) = 1 \cdot x_{[-1]} \otimes x_{[0]} + x \otimes 1 = g^c \otimes x + x \otimes 1 .$$

$\square$

5.3. **Reflection on simple roots in a Nichols algebra.** We finally discuss the example of Nichols algebras [HS13]. We take for $\mathcal{C}$ the category of finite-dimensional Yetter-Drinfel'd modules over a complex Hopf algebra $h$, e.g. the complex group algebra of a finite group $G$. Let $M \in \mathcal{C}$ be a finite direct sum of simple objects $(M_i)_{i \in I}$,

$$M = \bigoplus_{i \in I} M_i .$$

Thus, $M$ is a complex braided vector space. The Nichols algebra $\mathcal{B}(M)$ of $M$ is defined as a quotient by the kernels of the quantum symmetrizer maps $Q_n$

$$\mathcal{B}(M) := \bigoplus_{n \geq 0} M^{\otimes n} / \ker(Q_n).$$

The Nichols algebra $\mathcal{B}(M)$ is a Hopf algebra in the braided category $\mathcal{C}$. If $M$ is a direct sum of $n$ simple objects in $\mathcal{C}$, the Nichols algebra is said to be of rank $n$.

Each simple subobject $M_i$ of $M$ provides a partial dualization datum: Denote by $M_i^*$ the braided vector space dual to $M_i$. Denote by $\mathcal{B}(M_i)$ the Nichols algebra for $M_i$. The fact that $M_i$ is a subobject and a quotient of $M$ implies that $\mathcal{B}(M_i)$ is a Hopf subalgebra of $\mathcal{B}(M)$ and that there is a natural projection $\mathcal{B}(M) \xrightarrow{\pi_i} \mathcal{B}(M_i)$ of Hopf algebras. Similarly, the evaluation and coevaluation for $M$ induce a non-degenerate Hopf pairing $\omega_i : \mathcal{B}(M_i) \otimes \mathcal{B}(M_i^*) \to \mathbb{C}$ on the Nichols algebras. We thus have for each $i \in I$ a partial dualization datum

$$\mathcal{A}_i := (\mathcal{B}(M) \xrightarrow{\pi_i} \mathcal{B}(M_i), \mathcal{B}(M_i^*), \omega_i)$$

We denote by $r_i(\mathcal{B}(M)) := r_{\mathcal{A}_i}(\mathcal{B}(M))$ the partial dualization of $\mathcal{B}(M)$ with respect to $\mathcal{A}_i$. As usual, we denote by $K_i$ the coinvariants for the the projection $\pi_i$; $K_i$ is a Hopf algebra in the braided category of $\mathcal{B}(M_i)$-Yetter-Drinfel'd modules.

We summarize some results of [AHS10],[HS10] and [HS13]; for simplicity, we assume that the Nichols $\mathcal{B}(M)$ algebra is finite-dimensional. To make contact with our results, we note that the $i$-th partial dualization

$$r_i(\mathcal{B}(M)) := \Omega(K_i) \rtimes \mathcal{B}(M_i^*) ,$$

as introduced in the present paper, coincides by with the $i$-th reflection of $\mathcal{B}(M)$ in the terminology of [AHS10].

**Theorem 5.6.** *Let $h$ be a complex Hopf algebra. Let $M_i$ be a finite collection of simple $h$-Yetter-Drinfel'd modules. Consider $M := \bigoplus_{i=1}^{n} M_i \in {}^h_h\mathcal{YD}$ and assume that the associated Nichols algebra $H := \mathcal{B}(M)$ is finite-dimensional. Then the following assertions hold:*

- *By construction, the Nichols algebras $\mathcal{B}(M), r_i(\mathcal{B}(M))$ have the same dimension as complex vector spaces.*
- *For $i \in I$, denote by $\hat{M}_i$ the braided subspace*

$$\hat{M}_i = M_1 \oplus \ldots \oplus M_{i-1} \oplus M_{i+1} \oplus \ldots .$$

  *of $M$. Denote by $\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$ the braided vector space obtained as the image of $\hat{M}_i \subset \mathcal{B}(M)$ under the adjoint action of the Hopf subalgebra $\mathcal{B}(M_i) \subset \mathcal{B}(M)$. Then, there is a unique isomorphism [HS13, Prop. 8.6] of Hopf algebras in the braided category ${}_{\mathcal{B}(M_i)}^{\mathcal{B}(M_i)}\mathcal{YD} \left( {}_h^h\mathcal{YD} \right):$*

$$K_i \cong \mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i))$$

  *which is the identity on $\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$.*
- *Define, with the usual convention for the sign,*
  $a_{ij} := -\max\{m \mid \mathrm{ad}_{M_i}^m(M_j) \neq 0\}$. *Fix $i \in I$ and denote for $j \neq i$*

$$V_j := \mathrm{ad}_{M_i}^{-a_{ij}}(M_j) \subset \mathcal{B}(M) .$$

  *The braided vector space*

$$R_i(M) = V_1 \oplus \cdots M_i^* \cdots \oplus V_n \in {}_h^h\mathcal{YD}$$

  *is called the the $i$-th reflection of the braided vector space $M$. Then there is a unique isomorphism [HS13, Thm. 8.9] of Hopf algebras in ${}_h^h\mathcal{YD}$*

$$r_i(\mathcal{B}(M_1 \oplus \cdots \oplus M_n)) \cong \mathcal{B}(V_1 \oplus \cdots M_i^* \cdots \oplus V_n)$$

  *which is the identity on $M$.*
- *With the same definition for $a_{ij}$ for $i \neq j$ and $a_{ii} := 2$, the matrix $(a_{ij})_{i,j=1,\ldots n}$ is a generalized Cartan matrix [AHS10, Thm. 3.12]. Moreover, one has $r_i^2(\mathcal{B}(M)) \cong \mathcal{B}(M)$, as a special instance of Corollary 4.3, and the Cartan matrices coincide, $a_{ij}^M = a_{ij}^{r_i(M)}$. In the terminology of [HS10, Thm. 6.10], one obtains a Cartan scheme.*
- *The maps $r_i$ give rise to a Weyl groupoid which controls the structure of the Nichols algebra $\mathcal{B}(M)$. For details, we refer to [AHS10, Sect. 3.5] and [HS10, Sect. 5].*

We finally give examples that illustrate the appearance of Nichols algebras as Borel algebras in quantum groups. We end with an example in which a reflected Nichols algebra is not isomorphic to the original Nichols algebra.

The first example serves to fix notation:

**Example 5.7.** *Let $n > 1$ be a natural number and $q$ be a primitive $n$-th root of unity in $\mathbb{C}$. Let $M$ be the one-dimensional complex braided vector space with basis $x_1$ and braiding matrix $q_{11} = q$. As a quotient of the tensor algebra, the associated Nichols algebra $\mathcal{B}(M)$ inherits a grading, $\mathcal{B}(M) = \oplus_{k \in \mathbb{N}} \mathcal{B}(M)_{(k)}$. As a graded vector space, it is isomorphic to*

$$\mathcal{B}(M) \cong \mathbb{C}[x_1]/(x_1^n)$$

*and thus of complex dimension $n$. The Hilbert series is*

$$\mathcal{H}(t) := \sum_{k \geq 1} t^k \dim \left( \mathcal{B}(M)_{(k)} \right) = 1 + t + \cdots t^{n-1}.$$

The next example exhibits the role of Nichols algebras as quantum Borel parts.

**Example 5.8.** *Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra of rank $n$ with Cartan matrix $(a_{ij})_{i,j=1\ldots n}$. Let $(\alpha_i)_{i=1,\ldots,n}$ be a set of simple roots for $\mathfrak{g}$ and let $d_i := \langle \alpha_i, \alpha_i \rangle / 2$.*

*We construct a braided vector space $M$ with diagonal braiding as a Yetter-Drinfel'd module over an abelian group: fix a root $q \neq 1$ of unity; find a diagonal braiding matrix $q_{ij}$ with*

$$q_{ii} = q^{d_i} \qquad q_{ij} q_{ji} = q_{ii}^{a_{ij}}.$$

*The associated Nichols algebra $\mathcal{B}(M)$ is then the quantum Borel part of the Frobenius-Lusztig kernel $u_q(\mathfrak{g})$. In this case, all Nichols algebras $r_i(\mathcal{B}(M))$ obtained by reflections are isomorphic. The isomorphisms give rise to the Lusztig automorphisms $T_{s_i}$ of the algebra $u_q(\mathfrak{g})$ for the simple root $\alpha_i$. These automorphisms enter e.g. in the construction of a PBW-basis for $U(\mathfrak{g})$.*

In the following example [H09], the two Nichols algebras describe two possible Borel parts of the Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(2|1)$; they also appear in the description [ST13] of logarithmic conformal field theories. In this example, non-isomorphic Nichols algebras are related by reflections.

**Example 5.9.** *Let $q \neq \pm 1$ be a primitive $n$-th root of unity. Find two two-dimensional diagonally braided vector spaces $M, N$, with bases $(x_1^{(M)}, x_2^{(M)})$ $(x_1^{(N)}, x_2^{(N)})$ respectively, such that*

$$q_{11}^{(M)} = q_{22}^{(M)} = -1 \qquad q_{12}^{(M)} q_{21}^{(M)} = q^{-1}$$
$$q_{11}^{(N)} = -1 \quad q_{22}^{(N)} = q \qquad q_{12}^{(N)} q_{21}^{(N)} = q^{-1}.$$

*We describe a PBW-basis of the Nichols algebras $\mathcal{B}(M)$ and $\mathcal{B}(N)$ by isomorphisms of graded vector spaces to symmetric algebras. To this end, denote for a basis element $x_i^{(M)}$ of $M$ the corresponding Nichols subalgebra by $\mathcal{B}(x_i^{(M)})$, and similarly for $N$. (We will drop superscripts from now on, wherever they are evident.) A PBW-basis for the Nichols algebra $\mathcal{B}(x_i^{(M)})$ has been discussed in Example 5.7. Moreover, we need the shorthand $x_{12} := x_1 x_2 - q_{12} x_2 x_1$. One can show that the multiplication in the Nichols algebras leads to isomorphisms of graded vector spaces:*

$$\mathcal{B}(M) \overset{\sim}{\leftarrow} \mathcal{B}(x_1) \otimes \mathcal{B}(x_2) \otimes \mathcal{B}(x_1 x_2 - q_{12} x_2 x_1)$$
$$\cong \mathbb{C}[x_1]/(x_1^2) \otimes \mathbb{C}[x_2]/(x_2^2) \otimes \mathbb{C}[x_{12}]/(x_{12}^n),$$
$$\mathcal{B}(N) \overset{\sim}{\leftarrow} \mathcal{B}(x_1) \otimes \mathcal{B}(x_2) \otimes \mathcal{B}(x_1 x_2 - q_{12} x_2 x_1)$$
$$\cong \mathbb{C}[x_1]/(x_1^2) \otimes \mathbb{C}[x_2]/(x_2^n) \otimes \mathbb{C}[x_{12}]/(x_{12}^2).$$

*Both Nichols algebras $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are of dimension $4n$ and have a Cartan matrix of type $A_2$. Their Hilbert series can be read off from the PBW-basis:*

$$\mathcal{H}_{\mathcal{B}(M)}(t) = (1+t)(1+t)(1+t^2+t^4 \cdots t^{2(n-1)}),$$
$$\mathcal{H}_{\mathcal{B}(N)}(t) = (1+t)(1+t+t^2+\cdots t^{n-1})(1+t^2).$$

*The two Hilbert series are different; thus the two Nichols algebras $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are* not *isomorphic. The Nichols algebras are, however, related by partial dualizations:*

$$r_1(\mathcal{B}(M)) = \mathcal{B}(N) \qquad r_2(\mathcal{B}(M)) \cong \mathcal{B}(N)$$
$$r_1(\mathcal{B}(N)) = \mathcal{B}(M) \qquad r_2(\mathcal{B}(N)) = \mathcal{B}(N)$$

*where $r_i$ is the partial dualization with respect to the subalgebra $\mathcal{B}(\mathbb{C}x_i)$. For the isomorphism indicated by $\cong$, the generators $x_1$ and $x_2$ have to be interchanged.*

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