ZMP-HH/14-09
Hamburger Beiträge zur Mathematik Nr. 509

April 2014

On the Brauer groups of symmetries of abelian Dijkgraaf-Witten theories

Jürgen Fuchs [a], Jan Priel [b], Christoph Schweigert [b] and Alessandro Valentino [b]

[a] *Teoretisk fysik, Karlstads Universitet*
Universitetsgatan 21, S – 651 88 Karlstad

[b] *Fachbereich Mathematik, Universität Hamburg*
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D – 20 146 Hamburg

Abstract

Symmetries of three-dimensional topological field theories are naturally defined in terms of invertible topological surface defects. Symmetry groups are thus Brauer-Picard groups. We present a gauge theoretic realization of all symmetries of abelian Dijkgraaf-Witten theories. The symmetry group for a Dijkgraaf-Witten theory with gauge group a finite abelian group A, and with vanishing 3-cocycle, is generated by group automorphisms of A, by automorphisms of the trivial Chern-Simons 2-gerbe on the stack of A-bundles, and by partial e-m dualities.
We show that transmission functors naturally extracted from extended topological field theories with surface defects give a physical realization of the bijection between invertible bimodule categories of a fusion category $\mathcal{A}$ and braided auto-equivalences of its Drinfeld center $\mathcal{Z}(\mathcal{A})$. The latter provides the labels for bulk Wilson lines; it follows that a symmetry is completely characterized by its action on bulk Wilson lines.

1 Symmetries of abelian Dijkgraaf-Witten theories

Dijkgraaf-Witten theories are extended topological field theories that have a mathematically precise gauge theoretic formulation with finite gauge group. In that setting, the fields of the Dijkgraaf-Witten theory with gauge group G are obtained by first considering G-bundles, to which then in a second step a linearization procedure is applied (see [Mo] for a recent description). In the present note we investigate the notion of symmetries of three-dimensional Dijkgraaf-Witten theories, regarded as extended 1-2-3-dimensional topological field theories. To keep the presentation simple we restrict ourselves to the case that the gauge group is an abelian group, which we denote by A.

Braided auto-equivalences of bulk Wilson lines. The task of understanding symmetries in Dijkgraaf-Witten theories can be approached from two different angles, either algebraically or gauge theoretically. From a purely algebraic point of view, one would consider the modular category of bulk Wilson lines, which is the representation category $\mathcal{D}(A)$-mod of the Drinfeld double of A. Symmetries should then in particular induce braided auto-equivalences of $\mathcal{D}(A)$-mod.

The group of braided auto-equivalences (up to monoidal natural equivalence) can be described as follows. Denote by A^* the group of complex characters of A. The group $A \oplus A^*$ comes with a natural quadratic form $q\colon A \oplus A^* \to \mathbb{C}^\times$, given by $q(g+\chi) = \chi(g)$ for $g + \chi \in A \oplus A^*$. The automorphism group of $A \oplus A^*$ then has a subgroup, denoted by $\mathrm{O}_q(A\oplus A^*)$, consisting of those group automorphisms φ which preserve this form, i.e. satisfy $q(\varphi(z)) = q(z)$ for all $z \in A \oplus A^*$. Now the group of braided auto-equivalences is isomorphic to this group $\mathrm{O}_q(A\oplus A^*)$ [ENOM]. Simple objects of $\mathcal{D}(A)$-mod, and thus simple labels for bulk Wilson lines of the Dijkgraaf-Witten theory with gauge group A, are in bijection with elements of $A \oplus A^*$; a braided auto-equivalence induces the natural action of the corresponding element of $\mathrm{O}_q(A\oplus A^*)$ on the group $A \oplus A^*$.

In this approach the auto-equivalences of $\mathcal{D}(A)$-mod are not intrinsically realized in the Dijkgraaf-Witten theory as a gauge theory. It is therefore not clear whether every braided auto-equivalence of the category of bulk Wilson lines preserves all aspects of the three-dimensional topological field theory so that it can indeed be regarded as a full-fledged symmetry of the theory. It is not clear either whether a braided auto-equivalence would then describe a symmetry uniquely. There might be several different realizations, or also none at all, of the auto-equivalences on other field theoretic quantities, such as boundary conditions.

Universal kinematical symmetries. It is thus important to find a field theoretic realization of the auto-equivalences, relating to the fact that Dijkgraaf-Witten theories can be formulated as gauge theories. At the same time we then get additional insight into the structure of the group $\mathrm{O}_q(A\oplus A^*)$. From a gauge theoretic point of view it is natural to expect that the symmetries of the stack $\mathrm{Bun}(G)$ of G-bundles are symmetries of both classical and quantum Dijkgraaf-Witten theories.[1] One might call these symmetries universal kinematical symmetries – kinematical, because they are symmetries of the kinematical setting, i.e. G-bundles; and universal, because the manifold on which the G-bundles are defined does not enter. The

[1] Actually, a general Dijkgraaf-Witten theory involves a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. Here we only consider the case of trivial ω, and hence do not expect any compatibility relations between the automorphism and ω.

symmetries of $\mathrm{Bun}(G)$ form the 2-group $\mathrm{AUT}(G)$, i.e. the category whose objects are group automorphisms $\varphi\colon G \to G$ and whose morphisms $\varphi_1 \to \varphi_2$ are given by group elements $h \in G$ that satisfy $\varphi_2(g) = h\,\varphi_1(g)\,h^{-1}$ for all $g \in G$. Since in the case of our interest the group A is abelian, we can safely ignore the morphisms in the category $\mathrm{AUT}(A)$ and work with the ordinary automorphism group $\mathrm{Aut}(A)$ of the group A.

The group $\mathrm{Aut}(A)$ of symmetries of $\mathrm{Bun}(A)$ can be identified in a natural way with a subgroup of the group $\mathrm{O}_q(A \oplus A^*)$ of braided auto-equivalences. Indeed, for any $\alpha \in \mathrm{Aut}(A)$, the automorphism $\alpha \oplus (\alpha^{-1})^*$ of $A \oplus A^*$ belongs to $\mathrm{O}_q(A \oplus A^*)$, where $(\alpha)^* \in \mathrm{Aut}(A^*)$ is defined by $[\alpha^*\chi](a) := \chi(\alpha(a))$ for all $\chi \in A^*$ and all $a \in A$. But this argument is purely group theoretical, and it is not clear at this point whether the embedding has any physical relevance and relates symmetries of bundles to braided auto-equivalences of bulk Wilson lines.

Universal dynamical symmetries. The realization of Dijkgraaf-Witten theories as gauge theories leads to even more symmetries. Apart from a finite group G, a three-cocycle $\omega \in Z^3(G, U(1))$ is another ingredient of a Dijkgraaf-Witten theory. Geometrically this cocycle is interpreted [Wi] as a (Chern-Simons) 2-gerbe on the stack $\mathrm{Bun}(G)$ of G-bundles, and we may think of ω heuristically as a topological Lagrangian. In the present note we restrict ourselves to the case of vanishing cocycle ω, corresponding to a trivial 2-gerbe. Still, the automorphism group of the trivial 2-gerbe is a non-trivial 3-group: it is the 3-group of 1-gerbes on G. It is thus again natural to expect that this 3-group provides us with symmetries of the Dijkgraaf-Witten theory with gauge group G. We call these symmetries dynamical universal symmetries, as they involve symmetries of the topological Lagrangian.

By the results of [Wi], the objects of the 3-group of 1-gerbes on G are 2-cocycles on G; isomorphism classes are described by elements of the group cohomology $H^2(G, \mathbb{C}^\times)$. The group generated by classical kinematical and dynamical symmetries has the structure of a semi-direct product, $H^2(G, \mathbb{C}^\times) \rtimes \mathrm{Aut}(G)$. By [NR, Prop. 4.1] this group is isomorphic to the automorphism group of the fusion category G-vect of G-graded vector spaces. Indeed, this fusion category enters in the construction of Dijkgraaf-Witten theories as topological field theories of Turaev-Viro type.

If $G = A$ is abelian, cohomology classes in $H^2(A, \mathbb{C}^\times)$ are in bijection with alternating bicharacters. (An alternating bicharacter is a map $\beta\colon A \times A \to \mathbb{C}^\times$ that is a group homomorphism in each argument and satisfies $\beta(a, a) = 1$ for all $a \in A$, and thus $\beta(a_1, a_2) = \beta(a_2, a_1)^{-1}$ for all $a_1, a_2 \in A$.) Again, there is a natural embedding $H^2(A, \mathbb{C}^\times) \hookrightarrow \mathrm{O}_q(A \oplus A^*)$ of finite groups: to a class in $H^2(A, \mathbb{C}^\times)$ described by an alternating bicharacter β, we associate a map $\phi_\beta\colon A \oplus A^* \to A \oplus A^*$ defined by $\phi_\beta(a + \chi) := (a + \beta(a, -) + \chi(-))$. One immediately verifies that ϕ_β is an element of $\mathrm{O}_q(A \oplus A^*)$. Again it remains to be shown, though, that this embedding is of physical relevance in the sense that it relates symmetries of the topological Lagrangian to braided auto-equivalences.

Electric-magnetic dualities. The universal kinematical and dynamical symmetries cannot, however, exhaust the symmetries of Dijkgraaf-Witten models – the subgroup of $\mathrm{O}_q(A \oplus A^*)$ generated by them is a proper subgroup. As an illustration, consider the case that A is the cyclic group $\mathbb{Z}_2$. This group does not admit any non-trivial automorphisms, i.e. $\mathrm{Aut}(A) = 1$. It does not admit any non-trivial alternating group homomorphism either, and hence the group generated by the universal dynamical and kinematical symmetries is trivial. On the other hand, the group $\mathrm{Aut}(A \oplus A^*)$ is the symmetric group S_3 that permutes the three order-two elements of $A \oplus A^*$. Its subgroup $\mathrm{O}_q(A \oplus A^*)$ is the subgroup $S_2 \cong \mathbb{Z}_2$ of S_3 whose non-trivial element exchanges the generator of A with the one of A^*; in physics terminology, a transformation of this type is called an electric-magnetic duality, or e-m duality. The presence of such electric-magnetic dualities is a central feature of gauge theories in various dimensions (see e.g. [KaW, KaBSS] for a general discussion). Electric-magnetic dualities have a particularly explicit description in theories that can be realized as lattice models, compare [DW, Ki, BuCKA] and references therein.

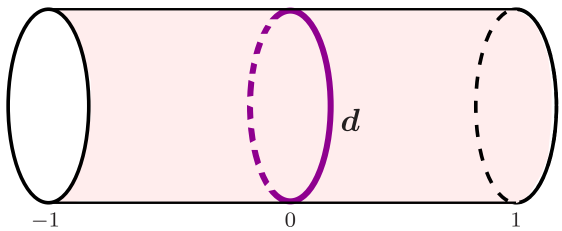
Topological surface defects and bimodule categories. A proper understanding of the situation, including a physical realization of the subgroups described above, calls for a unified field theoretic perspective. In this note we explain that in the present situation, for which no rigorous definition of symmetry for an extended topological field theory has been fully tested out so far, topological surface defects provide such a perspective. In fact, the relation between symmetries and classes of invertible topological codimension-one defects has been established long ago [FFRS1, FFRS2] for the case of *two*-dimensional field theories. But the mechanism that implements symmetries via topological defects is not restricted to the two-dimensional case. One of the virtues of realizing symmetries in terms of topological defects of codimension one is that this realization immediately determines how the symmetries act on all kinds of aspects of the field theory, including in particular labels of boundaries and defects.

Topological surface defects in 3d TFTs have recently attracted increasing interest, see [Bo, KK, KaS, BaMS, EKRS, KhTH, GGP, FSV1] for a selection of recent contributions. The case of three-dimensional topological field theories of Turaev-Viro type is particularly well understood. In particular, it is by now well-established that topological surface defects in Dijkgraaf-Witten theories with gauge group A correspond to bimodule categories over the fusion category $\mathcal{A} = A$-vect of finite-dimensional A-graded vector spaces. Those defects which describe symmetries correspond to invertible bimodule categories: accordingly we call them invertible defects. Their fusion product with the opposite defect is the monoidal unit for fusion, which is also called the invisible or transparent defect. Invertible defects can alternatively be characterized by the fact that the only bulk Wilson lines that 'condense' on them are the invisible bulk Wilson lines.

The group of (equivalence classes of) invertible bimodule categories, the so-called Brauer-Picard group of $\mathcal{A}$, has been described in [ENOM, NN]. In particular, a bijection has been established [ENOM, Thm 1.1] between invertible bimodule categories of a fusion category – in our case A-vect – and braided auto-equivalences of its center – in our case the category $\mathcal{D}(A)$-mod of bulk Wilson lines. As a consequence, also (equivalence classes of) invertible bimodule categories are described by the group $\mathrm{O}_q(A \oplus A^*)$.

The transmission functor. The results of [ENOM] are of purely representation theoretic nature. The purpose of the present note is to investigate their consequences and counterparts in Dijkgraaf-Witten theories as gauge theories. The bijection between equivalence classes of invertible bimodule categories and braided auto-equivalences in [ENOM] leads us to consider braided auto-equivalences F_d of $\mathcal{D}(A)$-mod labeled by invertible bimodule categories d over $\mathcal{D}(A)$-mod. Thus to any invertible topological surface defect d we have to associate such a braided auto-equivalence.

Now in an extended three-dimensional topological field theory, functors are obtained from surfaces with boundaries, and there is indeed a natural candidate for the relevant two-dimensional cobordism with defect. Namely, to yield an endofunctor of the category of bulk Wilson lines, the cobordism should have one ingoing and one outgoing boundary; and it should not induce any additional topological information; hence we have to consider a cylinder. The cylinder can be thought of as coming from a cut-and-paste boundary in a three-dimensional topological field theory. Such boundaries have to intersect surface defects transversally. Hence a surface defect results in a line embedded in the cobordism. We are thus lead to consider a cylinder $Z = S^1 \times [-1,1]$ with a defect line along the circle $D = S^1 \times \{0\} \subset Z$, as shown in the following picture:



(1.1)

In the sequel we regard the circle $S^1 \times \{-1\} \subset Z$ as incoming and the circle $S^1 \times \{1\} \subset Z$ as outgoing. We denote the functor described by the cobordism (1.1) by F_d and call it the *transmission functor* for the defect d. We will show in section 2.3 that for an invertible defect in a general three-dimensional extended topological field theory, the transmission functor F_d is a braided auto-equivalence of the category of bulk Wilson lines. The transmission functor describes what happens to the type of a bulk Wilson line when it passes through the surface defect d.

We note that in some physical applications Wilson lines can be interpreted as world lines of quasi-particles, with the type of the quasi-particle specified by the type of the Wilson line. When such a quasi-particle crosses an invertible topological surface defect of type d, then the type of quasi-particle is changed according to the transmission functor F_d. In field-theoretic terms, this change is brought about by a so-called Alice string [Sc, ABCMW, DB]. Let us illustrate this interpretation with the situation that the surface defect is a half-plane $\mathbb{R}^2_{x\geq 0} \times \mathbb{R}$ in three-dimensional space $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$. The boundary of the half-plane consists of a Wilson line that separates the surface defect d from the transparent defect (such Wilson lines always exist). The intersection of the defect d with a plane $\mathbb{R}^2 \times \{t_0\}$ of fixed time is a half-line labeled by d; this half-line constitutes the Alice string. Since the surface defect is topological, the precise position of the half-line does not matter.

In the case of Dijkgraaf-Witten theories, transmission functors are explicitly accessible: there is a gauge theoretic realization of topological surface defects in Dijkgraaf-Witten theories based on relative bundles [FSV2]. As a consequence, topological defects are classified by a subgroup $H \leq A \oplus A$ together with a cohomology class in $H^2(H, \mathbb{C}^\times)$. The formalism developed in [Mo] then allows one to compute the transmission functor.

In this note we provide a set of generators for the group $\mathrm{O}_q(A \oplus A^*)$ of braided auto-equivalences, which implies that universal kinematical and dynamical symmetries together with electric-magnetic dualities generate all symmetries. We then give, for each of these generators of $\mathrm{O}_q(A \oplus A^*)$, a topological defect, compute the resulting transmission functor and show that it acts on simple labels for bulk Wilson lines by the natural action of $\mathrm{O}_q(A \oplus A^*)$ on $A \oplus A^*$.

This provides a field theoretic realization in terms of topological surface defects for all braided auto-equivalences. At the same time it establishes that the embeddings of the subgroups of dynamical and kinematical universal symmetries described above are indeed physical. When combined with the results of [ENOM], it also follows that the braided equivalences of bulk Wilson lines are in bijection with field-theoretic symmetries. [2] Hereby we realize all elements of the Brauer-Picard group as gauge-theoretic dualities.

Plan of the paper. The rest of this note is organized as follows. Section 2 collects some background about topological surface defects in Dijkgraaf-Witten theories and provides information about transmission functors arising from invertible defects. In Section 3 we construct these defects explicitly for various classes of generators and compute their transmission functors. Finally we show in Section 4 that the group of invertible defects is generated by kinematical and dynamical symmetries together with e-m dualities. Technically, this is proven as the group-theoretical statement that a certain set of elements of the group $\mathrm{O}_q(A{\oplus}A^*)$ generates this group.

2 Surface defects in DW theories and the transmission functor

2.1 Surface defects in Dijkgraaf-Witten theories

A model independent analysis of topological surface defects between topological field theories of Reshetikin-Turaev type has been presented in [FSV1]. We summarize the pertinent aspects of that analysis: For $\mathcal{C}$ and $\mathcal{C}'$ modular tensor categories, a topological surface defect separating the Reshetikhin-Turaev theories with bulk Wilson lines labeled by $\mathcal{C}$ and by $\mathcal{C}'$, respectively, exists if and only if the modular category $\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}}$ is braided equivalent to the Drinfeld center of some fusion category $\mathcal{A}$; [3] here $(\mathcal{C}')^{\mathrm{rev}}$ is the same monoidal category as $\mathcal{C}'$, but with opposite braiding. We call the corresponding braided equivalence functor $\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A})$ a trivialization of $\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}}$. If such a trivialization exists, then the bicategory of defects is equivalent to the bicategory of module categories over the fusion category $\mathcal{A}$.

In the present paper we are interested in the case of defects that separate a Dijkgraaf-Witten theory based on the abelian group A from itself. Thus the category of bulk Wilson lines is already a Drinfeld center, $\mathcal{C}=\mathcal{C}'=\mathcal{Z}(A\text{-vect})$, and accordingly there is a distinguished trivialization

$$\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(A{\oplus}A\text{-vect})\,. \tag{2.1}$$

The defects of our interest are thus classified by module categories over the category of $A\oplus A$-graded vector spaces.

Indecomposable $A{\oplus}A$-vect-module categories have been classified in [Os]: they correspond to subgroups $H \leq A\oplus A$, together with a two-cocycle on H. That they describe surface defects of Dijkgraaf-Witten theories has been explicitly demonstrated in [FSV2].

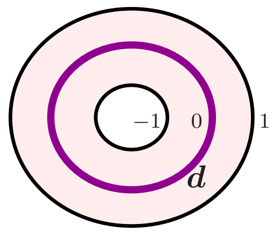
[2] This is reminiscent of the situation in two-dimensional rational conformal field theories, where the action of topological line defects on bulk fields characterizes isomorphism classes of defects, so that the action of topological line defects on bulk fields has been used in the classification of defects.

[3] Then the classes of $\mathcal{C}$ and $\mathcal{C}'$ in the Witt group [DaMNO] of modular tensor categories coincide.

2.2 The transmission functor

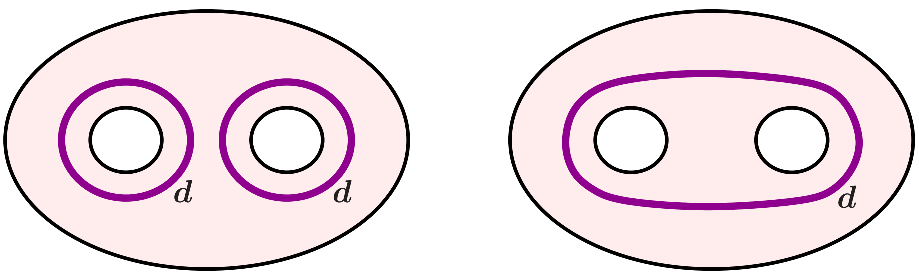
We want to determine the transmission functor $F_d\colon \mathcal{C} \to \mathcal{C}$ for an invertible topological surface defect d described by an indecomposable module category over $\mathcal{C}$. The physical interpretation of the transmission functor F_d for an invertible defect is as follows. When a bulk Wilson line labeled by an object $U \in \mathcal{C}$ passes through the surface defect d, its label changes to $F_d(U) \in \mathcal{C}$. (Recall that no bulk Wilson lines condense on the defect.)

We now explain why the transmission functor for an *invertible* surface defect in an extended three-dimensional topological field theory has a natural structure of a braided auto-equivalence. First of all, by composing the transmission functor F_d for a surface defect d with the transmission functor $F_{\overline{d}}$ for the opposite defect $\overline{d}$ and invoking fusion of defects, we conclude that $F_d \circ F_{\overline{d}} = Id_{\mathcal{C}} = F_{\overline{d}} \circ F_d$, so that F_d is indeed an auto-equivalence. To proceed, it will be convenient to draw the cylinder (1.1) as an annulus with an embedded defect line, according to



(2.2)

To discuss monoidality we then have to compare the functors corresponding to the two 'trinion' surfaces shown in the following picture:

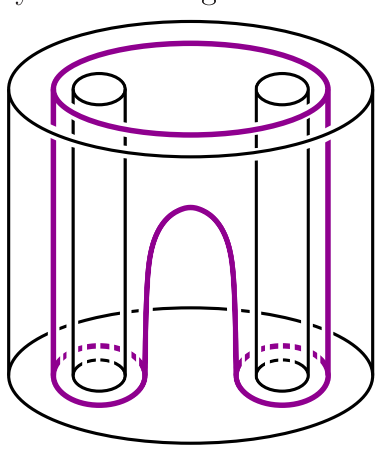


(2.3)

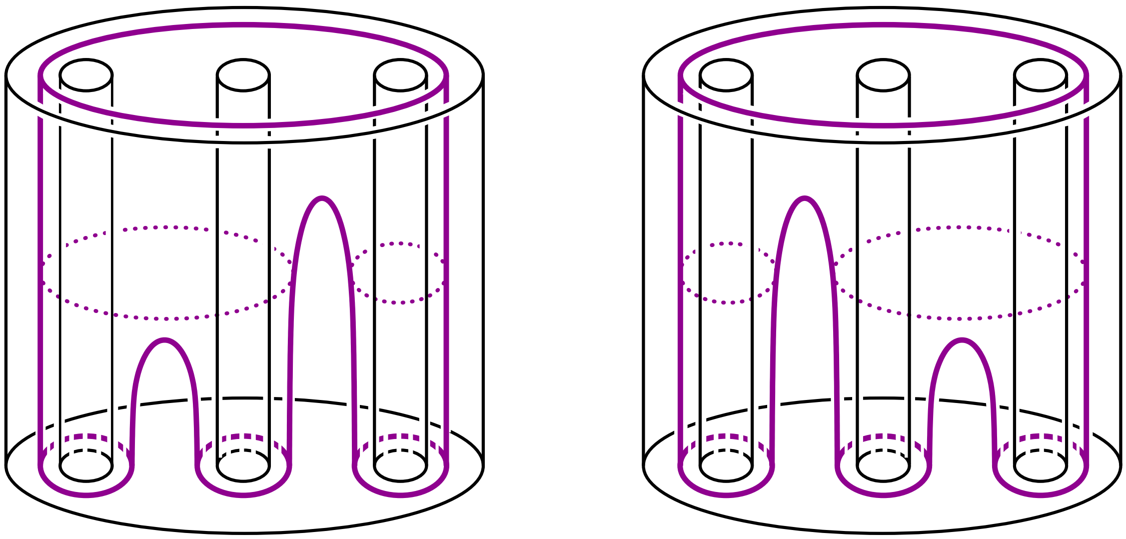
For a general defect, the functors associated to these two trinions are not isomorphic and the transmission functor is not monoidal; one rather obtains monoidal functors between categories of local modules over braided-commutative algebras in $\mathcal{C}$. But if the defect is invertible, then the functors corresponding to the two trinions are isomorphic. In fact, a natural isomorphism

$$\otimes \circ (F_d \times F_d) \Longrightarrow F_d \circ \otimes \tag{2.4}$$

of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is furnished by the following three-manifold with corners and defects:

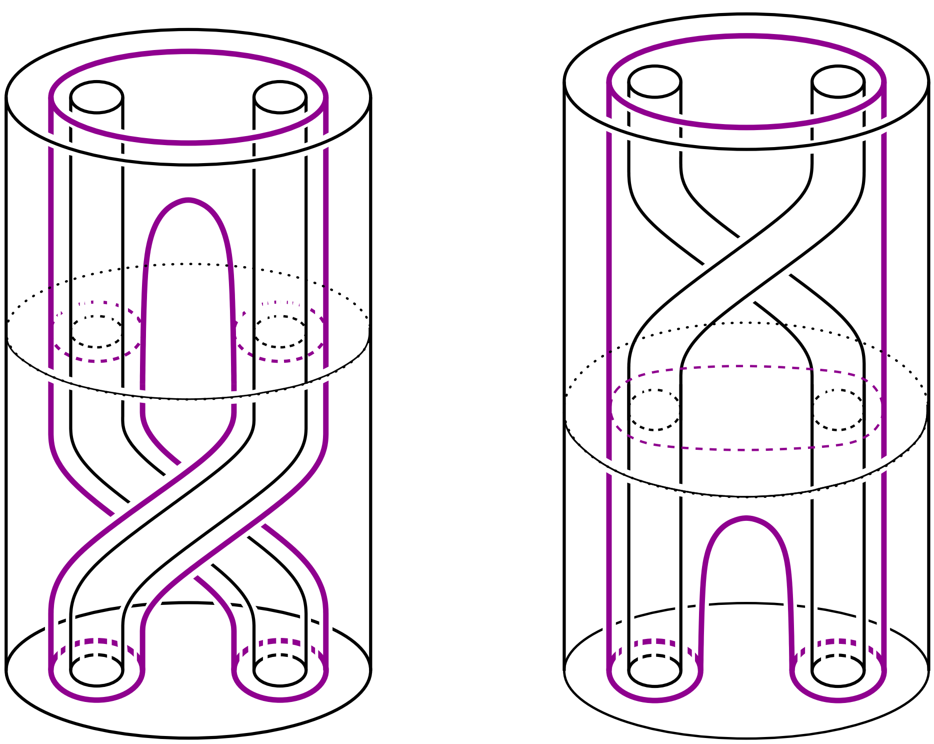

(2.5)

Such a three-manifold with corners is to be read as a span of manifolds from the bottom lid to the top lid. To show that this natural transformation provides a monoidal structure on the functor F_d, one needs to check an identity of natural transformations. This identity follows from the fact that the following two three-manifolds with corners and defects are related by a homotopy relative to the boundary:



(2.6)

(This homotopy, restricted to the surface defect, looks like the homotopy used in two-dimensional topological field theories to show associativity of the algebras assigned to circles, but its role is rather different.) In a similar manner the property that the monoidal structure on F_d is braided can be deduced from the fact that the following two three-manifolds with corners and defects are homotopic as well:



(2.7)

Remark 2.1. In passing, we mention another physical application: According to [KaS], surface defects provide an interpretation of the so-called TFT construction (see [SFR] for a review) of correlators of two-dimensional rational conformal field theories associated with the category $\mathcal{C}$. Thereby a surface defect d in particular determines a modular invariant torus partition function Z^d of the conformal field theory. For an invertible defect d with transmission functor F_d, the resulting torus partition function is of automorphism type; its coefficient matrix reads

$$\mathrm{Z}^d_{ij} = \delta_{[U_i],[F_d(U_j^\vee)]}\,, \tag{2.8}$$

where $\{U_i\}$ is a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}$.

2.3 Transmission functors for Dijkgraaf-Witten theories

In the case of our interest the modular tensor category $\mathcal{C}$ is the representation category of the (untwisted) Drinfeld double $\mathcal{D}(A)$ of a finite abelian group A, and a topological surface defect is described by a subgroup H of $A\oplus A$ and a two-cocycle on H. To obtain the relevant groupoids of bundles we follow the prescription of [FSV2] to find the appropriate relative bundles: For a defect associated to the subgroup $H \le A\oplus A$ with two-cocycle $\theta \in Z^2(H,\mathbb{C}^\times)$, the objects of the category of relative bundles consist of an A-bundle $P_A^\pm$ on each of the two cylinders $Z_-: S^1\times[-1,0]$ and $Z_+ := S^1\times[0,1]$, an H-bundle P_H on D and an isomorphism

$$\alpha: \quad \mathrm{Ind}_H^{A\oplus A} P_H \xrightarrow{\simeq} (P_A^+)\big|_D \times (P_A^-)\big|_D \tag{2.9}$$

of $A\oplus A$-bundles on D. Using that the cylinders $Z_\pm$ are homotopic to the circle D, one can describe all bundles appearing in (2.9) by bundles on a circle. And since α is an isomorphism, one can work with an equivalent groupoid in which only the H-bundles appear as data. As a consequence the category of relative bundles can be replaced by the action groupoid $H/\!\!/_{\mathrm{ad}}H$ for the adjoint action of H on itself. The objects of this groupoid are group elements $h\in H$, which can be thought of as holonomies of the H-bundle on the defect circle with respect to some fixed base point; the morphisms of the groupoid correspond to gauge transformations.

According to the general picture of Dijkgraaf-Witten theories [Mo], for the cylinder we thus get a span of groupoids. For each boundary circle, we have the category of A-bundles on S^1, which we replace by the equivalent action groupoid $A/\!\!/_{\mathrm{ad}}A$. The relevant functor is restriction of bundles to the boundary components. To describe it, consider the group homomorphisms obtained from the canonical projections $p_{1,2}$ for $A\oplus A$ to its two summands,

$$\pi_i: \quad H \hookrightarrow A\oplus A \xrightarrow{p_i} A\,. \tag{2.10}$$

These give rise to functors

$$\hat\pi_i: \quad H/\!\!/_{\mathrm{ad}}H \to A/\!\!/_{\mathrm{ad}}A \tag{2.11}$$

on action groupoids, acting both on objects and morphisms like π_i. We thus can replace the span of groupoids of categories of bundles and relative bundles by the equivalent span

$$\begin{array}{ccc} & H/\!\!/_{\mathrm{ad}}H & \\ {}^{\hat\pi_1}\swarrow & & \searrow^{\hat\pi_2} \\ A/\!\!/_{\mathrm{ad}}A & & A/\!\!/_{\mathrm{ad}}A \end{array} \tag{2.12}$$

of finite groupoids. Next we linearize, i.e. for each groupoid consider the category of functors from the groupoid to vect. This gives us two pullbacks

$$\hat{\pi}_i^* : \quad [A/\!/_{\rm ad} A, \mathrm{vect}] \to [H/\!/_{\rm ad} H, \mathrm{vect}] \,, \tag{2.13}$$

as well as pushforwards

$$\hat{\pi}_{i\,*} : \quad [H/\!/_{\rm ad} H, \mathrm{vect}] \to [A/\!/_{\rm ad} A, \mathrm{vect}] \tag{2.14}$$

as their two-sided adjoints. Note that the category $[A/\!/_{\rm ad} A, \mathrm{vect}] \cong \mathcal{D}(A)$-mod $\cong \mathcal{C}$ is the category of labels for bulk Wilson lines of the Dijkgraaf-Witten theory.

We finally construct a functor $[H/\!/_{\rm ad} H, \mathrm{vect}] \to [H/\!/_{\rm ad} H, \mathrm{vect}]$ from the two-cocycle θ, following [Mo, Sect. 5.4]. To this end we first transgress $\theta \in Z^2(H, \mathbb{C}^\times)$ to $\omega_\theta \in Z^1(H/\!/_{\rm ad} H, \mathbb{C}^\times)$, a one-cocycle for the loop groupoid $H/\!/_{\rm ad} H \cong [*/\!/\mathbb{Z}, */\!/G]$. According to [Wi, Thm. 3] this is the commutator

$$\omega_\theta(h_1; h_2) = \frac{\theta(h_1, h_2)}{\theta(h_2, h_1)} \,, \tag{2.15}$$

which is an alternating bicharacter on the abelian group A. (As is well known, alternating bicharacters for an abelian group A are in bijection with the group cohomology $H^2(A, \mathbb{C}^\times)$.) The groupoid algebra $\mathbb{C}[H/\!/_{\rm ad} H]$ has as a basis the morphisms $b_{\gamma;h} : \gamma \xrightarrow{h} \gamma$ in $H/\!/_{\rm ad} H$; its product is composition of morphisms, wherever this is defined, and zero else. We can canonically identify

$$\mathbb{C}[H/\!/_{\rm ad} H]\text{-mod} \simeq [H/\!/_{\rm ad} H, \mathrm{vect}] \,. \tag{2.16}$$

The two-cocycle ω_θ gives an algebra automorphism

$$\begin{aligned} \varphi_\theta : \quad \mathbb{C}[H/\!/_{\rm ad} H] \;&\to\; \mathbb{C}[H/\!/_{\rm ad} H] \,, \\ b_{\gamma;h} \;&\mapsto\; \omega_\theta(\gamma : h)\, b_{\gamma;h} \,, \end{aligned} \tag{2.17}$$

which in turn provides us with the desired functor

$$\varphi_\theta^* : \quad \mathbb{C}[H/\!/_{\rm ad} H]\text{-mod} \to \mathbb{C}[H/\!/_{\rm ad} H]\text{-mod} \,. \tag{2.18}$$

The transmission functor $F_{H,\theta}$ is now obtained [Mo, Sect. 5.4] by pre- and post-composing this functor with the pullback and pushforward functors obtained above:

$$F_{H,\theta} : \quad [A/\!/_{\rm ad} A, \mathrm{vect}] \xrightarrow{\hat{\pi}_1^*} [H/\!/_{\rm ad} H, \mathrm{vect}] \xrightarrow{\varphi_\theta^*} [H/\!/_{\rm ad} H, \mathrm{vect}] \xrightarrow{(\hat{\pi}_2)_*} [A/\!/_{\rm ad} A, \mathrm{vect}] \,. \tag{2.19}$$

In particular the transmission functor is explicitly computable. Thus for any given invertible surface defect $(H \le A \oplus A, \theta)$ of the Dijkgraaf-Witten theory with gauge group A we can find the corresponding braided equivalence $F_{H,\theta}$ explicitly. From these explicit expressions, it is clear that the transmission functor only depends on the cohomology class of θ.

2.4 Action of the transmission functor on simple objects

Let us determine the action of the transmission functor on the isomorphism classes of simple objects. For the double of a general finite group G these classes are in bijection with pairs consisting of a conjugacy class c of G and an irreducible representation χ of the centralizer of a representative of c. If $G = A$ is abelian, this reduces to pairs (a, χ) consisting of a group element $a \in A$ and an irreducible character $\chi \in A^*$; thus the isomorphism classes of simple objects are

$$\pi_0([A/\!/_{\rm ad} A, \text{vect}]) \cong A \oplus A^* . \tag{2.20}$$

The group structure on $\pi_0([A/\!/_{\rm ad} A, \text{vect}])$ coming from the monoidal structure on the category $[A/\!/_{\rm ad} A, \text{vect}]$ coincides with the natural group structure on $A \oplus A^*$.

It is straightforward to determine the action of each of the three functors (2.13), (2.14) and (2.18) on such pairs. First, for the pullback along $\hat\pi_1$ maps we find

$$\begin{aligned} [\hat\pi_1^*]: \quad \pi_0([A/\!/_{\rm ad} A, \text{vect}]) &\rightarrow \pi_0([H/\!/_{\rm ad} H, \text{vect}])\,, \\ (a\,, \chi) &\mapsto \bigoplus_{h \in p_1^{-1}(a)} (h\,,\, p_1^* \chi) \end{aligned} \tag{2.21}$$

with p_1^* defined by $[p_1^*\chi](h) := \chi(p_1(h))$. Second, the functor φ_θ^* acts as

$$\begin{aligned} [\varphi_\theta^*]: \quad \pi_0([H/\!/_{\rm ad} H, \text{vect}]) &\rightarrow \pi_0([H/\!/_{\rm ad} H, \text{vect}])\,, \\ (h\,, \psi) &\mapsto (h\,,\, \psi + \omega_\theta(h; -)) \end{aligned} \tag{2.22}$$

with ω_θ as in (2.15). And third, the pushforward along $\hat\pi_2$ maps

$$\begin{aligned} [(\hat\pi_2)_*]: \quad \pi_0([H/\!/_{\rm ad} H, \text{vect}]) &\rightarrow \pi_0([A/\!/_{\rm ad} A, \text{vect}])\,, \\ (h\,, \psi) &\mapsto \bigoplus_{\chi_2 \in A^*} (p_2(h)\,, \chi_2)\, \delta_{p_2^*\chi_2, \psi}\,. \end{aligned} \tag{2.23}$$

3 Realizing the symmetries

As discussed in detail in Section 4, the group $\mathrm{O}_q(A \oplus A^*)$ is generated by the following elements:

1. The *kinematical universal symmetries*, which come from automorphisms of the stack of A-bundles. They are given by the subgroup $S_{\rm kin} := \{\alpha \oplus \alpha^{-1\,*} \,|\, \alpha \in \mathrm{Aut}(A)\}$, which is isomorphic to $\mathrm{Aut}(A)$.

2. The *dynamical universal symmetries*, which can be identified with the group of (equivalence classes of) 1-gerbes on the stack of A-bundles. They are given by the group of alternating bicharacters on A. In the terminology of quantum field theory [SW], the connection on a 1-gerbe is called a B-field. Accordingly we refer to the subgroup of alternating bicharacters as B-fields and denote it by $S_{\rm B}$.

3. *Partial electric-magnetic* (or e-m, for short) *dualities*. Such a symmetry is given by the exchange, in $A \oplus A^*$, of a cyclic summand C of A with its character group C^*. More explicitly, for A written in the form $A = A' \oplus C$ with C a cyclic subgroup, it acts on $A \oplus A^* = A' \oplus C \oplus (A')^* \oplus C^*$ as $id_{A'} \oplus \delta \oplus id_{(A')^*} \oplus \delta^{-1}$, with $\delta_C\colon C \to C^*$ any isomorphism from C to C^*.
If one fixes a decomposition of A into a direct sum of cyclic groups C_i, together with an isomorphism $\delta_i\colon C_i \to C_i^*$ for each cyclic summand, then the corresponding partial e-m dualities generate a subgroup of $\mathrm{O}_q(A \oplus A^*)$, which we denote by $S_{\text{e-m}}$.

For each type of generator, we will now specify the subgroup H of $A \oplus A$ and cocycle θ that label the corresponding invertible surface defect.

Remark 3.1. In principle, for any element of the group $\mathrm{O}_q(A\oplus A^*)$ the subgroup $H \leq A \oplus A$ and the cocycle θ can be computed from the results in [ENOM, Sect. 10.2]. However, Theorem 1.1. of [ENOM] ensures that there is a bijection between equivalence classes of invertible topological surface defects and equivalence classes of braided equivalences. Hence it is sufficient to verify that a given defect described by a pair (H, θ) reproduces the correct braided equivalence.

We will make use of the following fact, which is the specialization to abelian groups of Proposition 5.2 of [NR]:

Corollary 3.2. *The A-vect-bimodule category associated with the pair (H, θ) is invertible iff*

$$H \cdot (A \oplus \{0\}) = A \oplus A = (A \oplus \{0\}) \cdot H \tag{3.1}$$

and the restriction of the commutator cocycle ω_θ (2.15) to

$$H_\cap := \big(H \cap (A \oplus \{0\})\big) \times \big((A \oplus \{0\}) \cap H\big) \tag{3.2}$$

is non-degenerate.

3.1 Kinematical symmetries: group automorphisms

The automorphisms in S_{kin} are the symmetries of the stack $\mathrm{Bun}(A)$ and are thus symmetries of the classical configurations.

A group automorphism $\alpha\colon A \to A$ induces a group automorphism $\alpha^*\colon A^* \to A^*$ acting on $\chi \in A^*$ as

$$[\alpha^*\chi](a) := \chi(\alpha(a)) \tag{3.3}$$

for all $a \in A$. The combined group automorphism $\widetilde{\alpha} := \alpha \oplus (\alpha^{-1})^*\colon A \oplus A^* \to A \oplus A^*$ satisfies

$$\begin{aligned} q\big(\widetilde{\alpha}(a{+}\chi)\big) &= q\big(\alpha(a) + \alpha^{-1*}(\chi)\big) \\ &= [\alpha^{-1*}(\chi)](\alpha(a)) = \chi(\alpha^{-1}\alpha(a)) = \chi(a) = q(a{+}\chi)\,, \end{aligned} \tag{3.4}$$

i.e. preserves the quadratic form q and is thus an element of $\mathrm{O}_q(A\oplus A^*)$.

We claim that the surface defect whose transmission functor corresponds to the automorphism $\widetilde{\alpha}$ is the following: For the subgroup, we take the graph of α, i.e.

$$H_\alpha := \{(a, \alpha(a)) \,|\, a \in A\} \,<\, A \oplus A\,, \tag{3.5}$$

and for two-cocycle on H_α the trivial two-cocycle $\theta_\circ$. (We could actually take any exact two-cocycle; for the transmission functor only the cohomology class matters.) For instance, for $\alpha = id$, H is the diagonal subgroup of $A \oplus A$, which describes the invisible defect, while for the 'charge conjugation' $a \mapsto a^{-1}$ it is the antidiagonal subgroup.

Let us first check that the pair $(H_\alpha, \theta_\circ)$ defines an invertible surface defect. We have

$$H_\alpha \cdot (A \oplus \{0\}) = \{(ab\,, \alpha(a) \,|\, a, b \in A\} = A \oplus A \tag{3.6}$$

and analogously $(A \oplus \{0\}) \cdot H_\alpha = A \oplus A$. Moreover,

$$\big(H_\alpha \cap (A \oplus \{0\})\big) = \{0\} = \big((A \oplus \{0\}) \cap H_\alpha\big)\,, \tag{3.7}$$

so that trivially the restriction of $\omega_{\theta_\circ}$ to $(H_\alpha)_\cap$ is non-degenerate. Thus both conditions in Corollary 3.2 are satisfied, and hence the defect labeled by $(H_\alpha, \theta_\circ)$ is indeed invertible.

Next we compute the action of the transmission functor $F_{H_\alpha,\theta_\circ}$ on isomorphism classes of simple objects. The functor $\varphi^*_{\theta_\circ}$ is the identity, so that the transmission functor is the composition

$$(a;\chi) \xmapsto{[\hat\pi_1^*]} (a,\alpha(a);\tilde\chi) \xmapsto{[\hat\pi_{2*}]} (\alpha(a),\alpha^{-1*}(\chi)) \tag{3.8}$$

where $\tilde\chi \in H_\alpha^*$ is defined by $\tilde\chi(a,\alpha(a)) = \chi(a)$. Thus indeed $F_{H_\alpha,\theta_\circ}$ acts on isomorphism classes of simple objects by $\widetilde\alpha \in \mathrm{O}_q(A \oplus A^*)$.

3.2 Dynamical symmetries: B-fields

These symmetries come from automorphisms of the trivial 2-gerbe on $\mathrm{Bun}(A)$. They are thus symmetries of the classical action. A dynamical symmetry is described by an alternating bicharacter $\beta\colon A \times A \to \mathbb{C}^\times$. To such a bicharacter β we associate the group homomorphism $\xi_\beta\colon A \to A^*$ that acts as $[\xi_\beta(a)](b) = \beta(a,b)$ for $a,b \in A$. The automorphism $\widetilde\beta$ for a dynamical symmetry is then given by

$$\widetilde\beta(a+\chi) = a + \xi_\beta(a) + \chi\,. \tag{3.9}$$

This is an automorphism because ξ_β is a group homomorphism, and it is in $\mathrm{O}_q(A \oplus A^*)$ because β is in addition antisymmetric:

$$[\xi_\beta(a)](a) = \beta(a,a) = 1 \tag{3.10}$$

for all $a \in A$, which implies $\beta(a,b) = (\beta(b,a))^{-1}$ for $a,b \in A$.

We claim that the surface defect whose transmission functor reproduces $\widetilde\beta \in \mathrm{O}_q(A \oplus A^*)$ looks as follows: The relevant subgroup is the diagonal subgroup $A_{\mathrm{diag}} \le A \oplus A$ (independently of the particular choice of β), and the relevant two-cocycle θ_β on $A_{\mathrm{diag}} \cong A$ is characterized by the fact that its commutator cocycle ω_{θ_β} is β. (Recall that for the transmission functor only the cohomology class of the two-cocycle matters; the alternating bicharacters are in bijection to these classes.)

Now notice that we have $A_{\mathrm{diag}} = H_{\alpha=id}$ with H_α as in (3.5), so that as a special case of (3.6) and of (3.7) we see that

$$A_{\mathrm{diag}} \cdot (A \oplus \{0\}) = A \oplus A = (A \oplus \{0\}) \cdot A_{\mathrm{diag}} \tag{3.11}$$

and

$$\big(A_{\mathrm{diag}} \cap (A \oplus \{0\})\big) = \{0\} = \big((A \oplus \{0\}) \cap A_{\mathrm{diag}}\big)\,, \tag{3.12}$$

respectively. Thus precisely as in Section 3.1 we can conclude that the surface defect labeled by the pair $(A_{\mathrm{diag}}, \theta_\beta)$ is invertible.

The action of the transmission functor $F_{A_{\mathrm{diag}},\theta_\beta}$ on isomorphism classes of simple objects is obtained as follows:

$$(a;\chi) \xmapsto{[\hat\pi_1^*]} (a,a;\tilde\chi) \xmapsto{[\varphi^*_{\theta_\delta}]} (a,a;\tilde\chi+\xi_\beta(a)) \xmapsto{[\hat\pi_{2*}]} (a;\chi+\xi_\beta(a)) = \widetilde\beta(a+\chi)\,, \tag{3.13}$$

where $\tilde\chi \in A_{\mathrm{diag}}$ is defined by $\tilde\chi(a,a) = \chi(a)$. Thus $F_{A_{\mathrm{diag}},\theta_\beta}$ acts on isomorphism classes precisely by $\widetilde\beta \in \mathrm{O}_q(A \oplus A^*)$.

3.3 Partial e-m dualities

The partial e-m dualities appear as symmetries of quantized Dijkgraaf-Witten theories. Every partial e-m duality can be obtained in the following manner. Suppose that A is written as a direct sum $A \cong A' \oplus C$ with C a cyclic subgroup (allowing for the case that A' is the trivial subgroup). This induces a similar decomposition of the character group A^*: denoting by C^* the subgroup of A^* of characters that vanish on A', and by $(A')^*$ the subgroup of characters vanishing on C, we have $A^* \cong (A')^* \oplus C^*$.

As abstract groups, C and C^* are isomorphic. Fix an isomorphism $\delta_C\colon C \to C^*$ and define the automorphism δ of the group $A \oplus A^* \cong A' \oplus C \oplus (A')^* \oplus C^*$ as follows: δ is the identity on the summands A' and $(A')^*$, while on $C \oplus C^*$ it acts as

$$(c \,, \psi\,) \longmapsto \big(\, \delta_C^{-1}(\psi) \,, \delta_C(c) \,\big) \,. \tag{3.14}$$

That δ preserves the quadratic form is seen by calculating

$$\begin{aligned} q\big(\delta(a'{+}c{+}\chi'{+}\psi)\big) &= q\big(a' + \delta_C^{-1}(\psi) + \chi' + \delta_C(c)\big) \\ &= \chi'(a') \cdot [\delta_C(c)](\delta_C^{-1}(\psi)) = \chi'(a') \cdot \psi(c) = q(a'{+}c{+}\chi'{+}\psi) \,. \end{aligned} \tag{3.15}$$

We claim that the surface defect whose transmission functor corresponds to $\delta \in \mathrm{O}_q(A{\oplus}A^*)$ is as follows: The relevant subgroup of $A \oplus A$ is the group $H_\delta := A'_{\mathrm{diag}} \oplus C \oplus C$, where A'_{diag} is embedded diagonally into the summand $A' \oplus A'$ of $A \oplus A$, while the cocycle θ_δ on H_δ is characterized by its commutator cocycle, which is defined to act as

$$\omega_{\theta_\delta}((a', c_1, c_2), (\tilde a', \tilde c_1, \tilde c_2)) := [\delta_C(c_1)](\tilde c_2) \cdot ([\delta_C(c_2)](\tilde c_1))^{-1} \tag{3.16}$$

(this is obviously an alternating bicharacter on H_δ). For determining the transmission functor, it again suffices to know this bicharacter.

To verify invertibility, note that

$$A'_{\mathrm{diag}} \cdot (A' \oplus \{0\}) = \{(a'b', b') \,|\, a', b' \in A'\} = A' \oplus A' \tag{3.17}$$

and analogously $(A' \oplus \{0\}) \cdot A'_{\mathrm{diag}} = A' \oplus A'$, which implies that $(A \oplus \{0\}) \cdot H_\delta = A \oplus A \,=\, H_\delta \cdot (A' \oplus \{0\})$. Moreover, we have

$$A'_{\mathrm{diag}} \cap (A' \oplus \{0\}) = \{0\} = (A' \oplus \{0\}) \cap A'_{\mathrm{diag}} \,, \tag{3.18}$$

which implies that $(H_\delta)_\cap = (C \oplus C) \times (C \oplus C)$. To see that ω_{θ_δ} restricted to $(H_\delta)_\cap$ is non-degenerate, we fix a generator a of C and denote by $\psi \in C^*$ the character with value $\psi(a) = \mathrm{e}^{2\pi\mathrm{i}/N}$, with $N = |C|$. Then $\delta_C(a) = \psi^l$ with l such that $(l, N) = 1$, and we find

$$\omega_{\theta_\delta}(a^{p_1}; a^{q_1}, a^{p_2}, a^{q_2}) = \mathrm{e}^{2\pi\mathrm{i}l(p_1q_2 - q_1p_2)/N}. \tag{3.19}$$

Now $\mathrm{e}^{2\pi\mathrm{i}l/N}$ is a primitive N-th root of unity, so that for any pair (p_1, q_1) we can find $(p_2, q_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $p_1q_2 - q_1p_2 \neq 0 \bmod N$. Hence ω_{θ_δ} is non-degenerate. We can thus again invoke Corollary 3.2 to conclude that the defect labeled by $(H_\delta, \theta_\delta)$ is invertible.

To compute the action of the transmission functor on simple objects, we note that the problem splits into a part involving only the subgroup A' and another part involving only the cyclic group C. The first problem reduces to the computation of the transmission functor for the defect associated with the identity automorphism, which was treated in section 3.1. Thus we can restrict ourselves to the case that $A = C$ is cyclic. In this case the action of the pullback functor on the simple object (c,χ) with $b \in C$ and $\chi \in C^*$ reads

$$(c;\chi) \xmapsto{[\hat\pi_1^*]} \bigoplus_{\tilde c \in C} \big(c,\tilde c;\chi^{[1]}\big) \tag{3.20}$$

with the character $\chi^{[1]} \in (C \oplus C)^*$ taking the values $\chi^{[1]}(d,\tilde d) = \chi(d)$ for $d,\tilde d \in C$. Next we note that the functor $\varphi^*_{\theta_\delta}$ acts on simple objects of $\mathcal{D}(C{\oplus}C)$-mod as

$$(c,\tilde c,\chi^{[1]}) \xmapsto{[\varphi^*_{\theta_\delta}]} (c,\tilde c,\chi^{[2]}) \tag{3.21}$$

with the character $\chi^{[2]} \in (C \oplus C)^*$ taking the values $\chi^{[2]}(d,\tilde d) = \chi(d)\,[\delta_C(c)](\tilde d)/[\delta_C(\tilde c)](d)$ for $d,\tilde d \in C$. This is, in turn, mapped by the pushforward functor $[\hat\pi_{2*}]$ to those characters $\chi^{[3]} \in C^*$ for which $p_2^*\chi^{[3]} = \chi^{[2]}$. This condition amounts to the identity

$$\chi^{[3]}(\tilde d) = \chi^{[2]}(d,\tilde d) = \chi(d)\,\frac{[\delta_C(c)](\tilde d)}{[\delta_C(\tilde c)](d)} \tag{3.22}$$

for all $d,\tilde d \in C$. Considering the dependence of both sides of this equality on $\tilde d$ determines $\chi^{[3]} = \delta_C(c)$, while the fact that the dependence on d on the right hand must be trivial shows that we need $\chi(d) = [\delta_C(\tilde c)](d)$ for all $d \in C$. This means that in the summation over $\tilde c$ in (3.20) only the term with $\delta_C(\tilde c) = \chi$ survives the pushforward. We conclude that the composition of the three functors maps the simple object (c,χ) to a single simple object, as befits an equivalence. Concretely,

$$(c,\chi) \longmapsto (\delta_C^{-1}(\chi), \delta_C(c))\,, \tag{3.23}$$

and thus the defect realizes an e-m duality.

4 Generators of $\mathrm{O}_q(A{\oplus}A^*)$

It remains to be shown that the three types of group elements discussed in the preceding section – corresponding to kinematical and dynamical classical symmetries and to partial e-m dualities – indeed constitute a set of generators for the Brauer-Picard group $\mathrm{O}_q(A{\oplus}A^*)$. To this end we have to show that an arbitrary element of $\mathrm{O}_q(A{\oplus}A^*)$ can be expressed as a product of elements in a suitable explicitly specified set of generators. This description turns out to be similar to the description of symplectic or orthogonal groups over the integers (see e.g. [HuR, SW]) and the proof involves a variant of the Euclidean algorithm similar as in the proof of Bruhat decompositions (see e.g. [Re]). Technical complications arise from the need to respect the divisibility properties of the orders of the generators.

We start by introducing pertinent notation. Any finite abelian group A can be presented as $A = \bigoplus_p A^{(p)}$ with the sum being over all primes and $A^{(p)}$ a direct sum of cyclic groups of order a power of p. To analyze the group A we present it in terms of some arbitrary, but fixed, ordered family $(a_i \mid i = 1,2,...,r)$ of generators such that (writing the group product additively)

$$A^{(p)} = \bigoplus_{i=1}^{r} \langle a_i \mid p^{\ell_i} a_i = 0\rangle = \bigoplus_{i=1}^{r} \mathbb{Z}_{p^{\ell_i}} = \bigoplus_{s} \big(\mathbb{Z}_{p^s}\big)^{\oplus n_s} \tag{4.1}$$

with non-negative integers n_s, $r = \sum_s n_s$ and ℓ_i. It will be convenient to order the generators such that the powers of p appear in ascending order, i.e. $\ell_i \le \ell_j$ for $i < j$. It is easy to see that

$$\mathrm{Aut}(A) = \times_{p\ \mathrm{prime}} \mathrm{Aut}(A^{(p)}) \quad \text{as well as} \quad \mathrm{O}_q(A \oplus A^*) = \times_{p\ \mathrm{prime}} \mathrm{O}_q(A^{(p)} \oplus A^{(p)\,*}) \,. \tag{4.2}$$

As a consequence we can, and will, restrict our attention to a single prime p. By a slight abuse of notation, in the sequel we will just write A for $A^{(p)}$.

In terms of the generators, a general group element is a linear combination $\sum_{i=1}^r \bar\gamma_i\, a_i$ with $\bar\gamma_i \in \mathbb{Z} \bmod p^{\ell_i}$. In the sequel we freely replace such classes $\bar\gamma_i$ by representatives $\gamma_i \in \mathbb{Z}$; also, we denote by $\gamma_i^{-1} \in \mathbb{Z}$ a representative of the inverse of γ_i modulo p^{ℓ_i}. For the character group A^* we choose generators x_i in such a way that $x_i(a_i)$ is a primitive p^{ℓ_i}th root of unity while $x_i(a_j) = 1$ for $i \ne j$, so that the quadratic form q is given by

$$q\big(\textstyle\sum_{i=1}^r (\gamma_i\, a_i + \zeta_i\, x_i)\big) = \exp\big(2\pi\mathrm{i} \displaystyle\sum_{i=1}^r p^{-\ell_i}\, \gamma_i\, \zeta_i\big) \,, \tag{4.3}$$

and in particular $\mathrm{ord}(x_i) = \mathrm{ord}(a_i)$. With these conventions, an element g of $\mathrm{O}_q(A \oplus A^*)$ (or, for that matter, of $\mathrm{End}(A \oplus A^*)$) is determined by the expressions

$$g(a_i) = \sum_{j=1}^r \big(\alpha^g_{i,j}\, a_j + \xi^g_{i,j}\, x_j\big) \qquad \text{and} \qquad g(x_i) = \sum_{j=1}^r \big(\beta^g_{i,j}\, a_j + \eta^g_{i,j}\, x_j\big) \tag{4.4}$$

for $i = 1, 2, \ldots, r$, with suitable constraints on the coefficients $\alpha^g_{i,j}$, $\xi^g_{i,j}$, $\beta^g_{i,j}$, $\eta^g_{i,j} \in \mathbb{Z}$ which, however, we do not need to spell out.

We introduce three subgroups of $\mathrm{Aut}(A \oplus A^*)$:

$$\begin{aligned} &S_{\mathrm{kin}} := \big\{\alpha \oplus \alpha^{-1\,*} \,|\, \alpha \in \mathrm{Aut}(A)\big\} \;\cong \mathrm{Aut}(A) \,, \\ &S_{\mathrm{B}} := \big\langle b_{i,j} \mid 1 \le i < j \le r \big\rangle \qquad \text{and} \qquad S_{\text{e-m}} := \textstyle\bigoplus_{i=1}^r D_i \;\cong \mathbb{Z}_2^{\oplus r} . \end{aligned} \tag{4.5}$$

Here $D_i \cong \mathbb{Z}_2$ is generated by the automorphism d_i that exchanges a_i and x_i and leaves all other generators fixed, while $b_{i,j}$ is given by

$$b_{i,j}: \quad \begin{cases} a_i \mapsto a_i + x_j \,, \\ a_j \mapsto a_j - x_i \,, \\ a_k \mapsto a_k \quad \text{for } k \notin \{i,j\} \,, \\ x_k \mapsto x_k \,. \end{cases} \tag{4.6}$$

It is not hard to check that the groups (4.5) are actually subgroups of $\mathrm{O}_q(A \oplus A^*) < \mathrm{Aut}(A \oplus A^*)$. The groups (4.5) describe kinematical universal symmetries (S_{kin}), dynamical symmetries or B-fields (S_{B}), and partial e-m-dualities associated to the direct sum decomposition (4.1) of A ($S_{\text{e-m}}$), respectively.

We will also be interested in two particular types of elements of S_{kin}: for $i \ne j$ satisfying $\mathrm{ord}(a_i) = \mathrm{ord}(a_j)$ we set

$$t_{i,j}: \quad \begin{cases} a_j \mapsto a_j - a_i \,, \\ a_k \mapsto a_k \quad \text{for } k \ne j \,, \\ x_i \mapsto x_i + x_j \,, \\ x_k \mapsto x_k \quad \text{for } k \ne i \,, \end{cases} \tag{4.7}$$

and for $\gamma\neq 0 \bmod p$ and any j

$$\omega_{j;\gamma}: \quad \begin{cases} a_j \mapsto \gamma^{-1} a_j\,, \\ a_k \mapsto a_k \quad \text{for } k\neq j\,, \\ x_j \mapsto \gamma\, x_j\,, \\ x_k \mapsto x_k \quad \text{for } k\neq j\,. \end{cases} \tag{4.8}$$

We further introduce a separate notation for those elements of S_{kin} that act as a transposition on pairs of generators of some fixed order and leave all other generators fixed, according to

$$\tau_{i,j}: \quad \begin{cases} a_i \leftrightarrow a_j\,, \\ a_k \mapsto a_k \quad \text{for } k\notin\{i,j\}\,, \\ x_i \leftrightarrow x_j\,, \\ x_k \mapsto x_k \quad \text{for } k\notin\{i,j\} \end{cases} \tag{4.9}$$

with $\text{ord}(a_j)=\text{ord}(a_i)$. These generate a subgroup $\mathfrak{S}=\bigoplus_s \mathfrak{S}_{n_s}\leq S_{\text{kin}}$ consisting of elements that permute pairs (a_i,x_i) of generators of the same order. Below, for convenience we allow for $i=j$ in (4.9), i.e. for any i, $\tau_{i,i}$ is just the unit element of $\text{O}_q(A\oplus A^*)$.

We now establish the following

Fact 4.1. $\text{O}_q(A\oplus A^*)$ *is generated by the subgroups* (4.5).

Proof.

Step 1: Given $g\in\text{O}_q(A\oplus A^*)$ we show that multiplying g with suitable elements of the subgroups (4.5) yields a group element that leaves the last generator x_r invariant.

Step 1a: Describe g as in (4.4). We first consider the case that $\text{ord}(\eta^g_{r,i}\,x_i)<\text{ord}(x_r)$ for all i. Then, since g must preserve the order of x_r, there exists at least one value of i such that $\text{ord}(\beta^g_{r,i}\,a_i)=\text{ord}(x_r)$. Take one such value (say, the largest one satisfying the equality) and denote it by $k(r)$ or, for brevity, just by k. Then the group element $g':=d_k\circ g$ acts as

$$g'(x_r)\equiv d_k(g(x_r)) = g(x_r)+(\beta^g_{r,k}-\eta^g_{r,k})\,(x_r-a_r)\,, \tag{4.10}$$

so that in particular $\text{ord}(\eta^{g'}_{r,k}\,x_k)=\text{ord}(\beta^{g'}_{r,k}\,x_k)=\text{ord}(x_r)$.

Step 1b: By step 1a we can assume that g satisfies $\text{ord}(\eta^g_{r,k}\,x_k)=\text{ord}(x_r)$, i.e. $\text{ord}(x_k)=\text{ord}(x_r)$ and $\eta^g_{r,k}\neq 0 \bmod p$. It follows that $\tau_{k,r}\in\mathfrak{S}\leq S_{\text{kin}}$ and that there exists a $\gamma\in\mathbb{Z}$ such that $\gamma\,\eta^g_{r,k}=1 \bmod p$. Then the group element $g':=\omega_{r;\gamma}\circ\tau_{r,k}\circ g$ acts as in (4.4) with $\eta^{g'}_{r,r}=1$.

Step 1c: Invoking step 1b we assume from now on that g satisfies $\eta^g_{r,r}=1$. Further, for $i\neq r$ the element $g':=(b_{i,r})^{\beta^g_{r,i}}$ satisfies $\beta^{g'}_{r,i}=\beta^g_{r,i}-\beta^g_{r,i}=0$. Hence by composing g successively, for all $i=1,2,\dots,r{-}1$, with the group element $b_{i,r}$ raised to the power $\beta^g_{r,i}$ one obtains a group element $\tilde g$ satisfying

$$\tilde g(x_r) = \beta^g_{r,r}\,a_r + x_r + \sum_{i=1}^{r-1}\xi^g_{r,i}\,x_r\,. \tag{4.11}$$

Now by construction, $\tilde g\in\text{O}_q(A\oplus A^*)$, while on the other hand $q(\tilde g(x_r))=\exp(2\pi\text{i}\,p^{-\ell_r}\beta^g_{r,r})$. Thus in fact we must have $\beta^g_{r,r}=0 \bmod p^{\ell_r}$.

Step 1d: By step 1c we can assume that g satisfies $\beta^g_{r,i}=0$ for all $i=1,2,,\dots,r$. Further, for $i\neq r$ the element $g':=(t_{r,i})^{\eta^g_{r,i}}$ satisfies $\eta^{g'}_{r,i}=\eta^g_{r,i}-\eta^g_{r,i}=0$. Hence by composing g successively, for all $i=1,2,\dots,r{-}1$, with the group element $t_{r,i}$ raised to the power $\eta^g_{r,i}$ one obtains a group element $\tilde g$ satisfying $\tilde g(x_r)=x_r$.

Step 2: By step 1 we can assume that $g(x_r)=x_r$. Now consider the image $g(x_{r-1})$ of the group element x_{r-1}. We manipulate it in full analogy with what we did with $g(x_r)$ in step 1, just replacing $r\mapsto r{-}1$ everywhere, but with the following amendment: In case that in the analogue of step 1a the label $k=k(r{-}1)$ should turn out to take the value r, before proceeding to replacing $g\mapsto d_k\circ g$ we consider instead of g the group element

$$g' := t_{r-1,r}\circ g\,. \tag{4.12}$$

After this replacement we can assume that $k\leq r{-}1$. As a consequence, afterwards one never will have to compose with elements from (4.5) of the form $\omega_{r;\gamma}$ or $b_{r,j}\circ d_r$ which would potentially alter the input relation $g(x_r)=x_r$. Thus by further proceeding along the lines of step 1 we end up with a group element $\tilde g$ satisfying both $\tilde g(x_r)=x_r$ and $\tilde g(x_{r-1})=x_{r-1}$.

Steps 3, 4, ... , r: Proceed iteratively for $g(x_{r-j})$ for $j=2,3,\dots,r{-}1$, where in the jth iteration the role of $t_{r-1,r}$ in (4.12) is taken over by $t_{r-j,r-l}$ for suitable $l<j$.
The result is a group element $\tilde g$ satisfying $\tilde g(x_i)=x_i$ for all $i=1,2,\dots,r$.

Step $r{+}1$: By step r we can assume that $g(x_i)=x_i$ for all $i=1,2,\dots,r$. We show that this in fact implies that $g(a_i)=a_i+\sum_{j=1}^r\xi^g_{i,j}\,x_j$ for all $i=1,2,\dots,r$. Indeed, from [HiR] we know that in order for g to belong to $\mathrm{Aut}(A\oplus A^*)$, the matrix $M(g)=\begin{pmatrix}\alpha^g & \xi^g\\ \beta^g & \eta^g\end{pmatrix}$ with block matrices α^g, ξ^g, β^g, η^g consisting of the coefficients in (4.4), must satisfy $\det(M(g) \bmod p)\neq 0$.
Now for g of the form considered here we have $\eta^g=\mathbb{1}_{r\times r}$ and $\beta^g=0$; this implies in particular that $0\neq\det(M(g) \bmod p)=\det(\alpha^g \bmod p)$, and thus that $\alpha^g\in\mathrm{Aut}(A)$. As a consequence, together with g also the product $g':=g\circ\big((\alpha^g)^{-1}\oplus(\alpha^g)^*\big)$ is an element of $\mathrm{O}_q(A\oplus A^*)$. On the other hand, we have explicitly

$$g'(a_i) = a_i+\sum_j \xi^{g'}_{i,j}\,x_j \qquad\text{and}\qquad g(x_i) = \sum_j \eta^{g'}_{i,j}\,x_j\,. \tag{4.13}$$

Hence the fact that g' belongs to $\mathrm{O}_q(A\oplus A^*)$ amounts in particular to the following restrictions, which together are also sufficient:

$$\begin{aligned}
&q(g'(a_i)) = q(a_i) \quad\Longrightarrow\quad \xi^{g'}_{i,i}=0\,,\\
&q(g'(a_i{+}a_j)) = q(a_i{+}a_j) \quad\Longrightarrow\quad \xi^{g'}_{i,j}+\xi^{g'}_{j,i}=0 \ \text{ for } \ i\neq j\,,\\
&q(g'(a_i{+}x_i)) = q(a_i{+}x_i) \quad\Longrightarrow\quad \eta^{g'}_{i,i}=1\,,\\
&q(g'(a_j{+}x_i)) = q(a_j{+}x_i) \quad\Longrightarrow\quad \eta^{g'}_{i,j}=0 \ \text{ for } \ i\neq j\,.
\end{aligned} \tag{4.14}$$

Together, these restrictions just say that $g'\in S_{\mathrm{B}}$.
This concludes the proof. □

In the following example we illustrate how the result follows from an explicit analysis in a particularly simple case.

Example 4.2. $A=\mathbb{Z}_p$.

It is not hard to see that in order for (4.4) to be in $\mathrm{O}_q(A\oplus A^*)$, it is necessary and sufficient that the numbers α, β, ξ, η satisfy $\alpha\xi=0=\beta\eta$ and $\alpha\eta+\beta\xi=1$ modulo p. These constraints are solved by

$$\xi = 0 = \beta\,, \quad \eta = \alpha^{-1} \qquad \text{and by} \qquad \alpha = 0 = \eta\,, \quad \beta = \xi^{-1}. \tag{4.15}$$

Among the solutions of the second type is in particular the case $\xi=\beta=1$, which gives the (unique) e-m duality, while all other solutions of this type are obtained from one of the first type by composing with the e-m duality. In short, we have

$$\mathrm{O}_q(\mathbb{Z}_p\oplus\mathbb{Z}_p^*) = S_{\mathrm{kin}} \rtimes S_{\text{e-m}} \tag{4.16}$$

with

$$S_{\mathrm{kin}} = \mathrm{Aut}(\mathbb{Z}_p) = GL_1(\mathbb{F}_p) = \mathbb{F}_p^{\times} \cong \mathbb{Z}_{p-1} \qquad \text{and} \qquad S_{\text{e-m}} = \mathbb{Z}_2\,. \tag{4.17}$$

In particular, $|\mathrm{O}_q(\mathbb{Z}_p\oplus\mathbb{Z}_p^*)|=2(p-1)$.

Acknowledgments: JF is supported by VR under project no. 621-2013-4207. CS is partially supported by the Collaborative Research Centre 676 "Particles, Strings and the Early Universe - the Structure of Matter and Space-Time" and by the DFG Priority Programme 1388 "Representation Theory". JF is grateful to Hamburg University, and in particular to CS, Astrid Dörhöfer and Eva Kuhlmann, for their hospitality when part of this work was done. JF, CS and AV are grateful to the Erwin-Schrödinger-Institute (ESI) for the hospitality during the program "Modern trends in topological field theory" while part of this work was done; this was also supported by the network "Interactions of Low-Dimensional Topology and Geometry with Mathematical Physics" (ITGP) of the European Science Foundation.

References

[ABCMW] M.G. Alford, K. Benson, S. Coleman, J. March-Russell, and F. Wilczek, *Interactions and excitations of non-abelian vortices*, Phys. Rev. Lett. 64 (1990) 1632–1635

[BaMS] J.W. Barrett, C. Meusburger, and G. Schaumann, *Gray categories with duals and their diagrams*, preprint `math.QA/1211.0529`

[Bo] H. Bombin, *Topological order with a twist: Ising anyons from an abelian model*, Phys. Rev. Lett. 105 (2010) 030403_1–4 `[cond-mat/1004.1838]`

[BuCKA] O. Buerschaper, M. Christandl, L. Kong, and M. Aguado, *Electric-magnetic duality of lattice systems with topological order*, Nucl. Phys. B 876 (2013) –636 `[cond-mat/1006.5823]`

[DaMNO] A.A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, J. reine angew. Math. 677 (2013) 135–177 `[math.QA/1009.2117]`

[DB] M. de Wild Propitius M and F.A. Bais, *Discrete gauge theories*, in: *Particles and Fields*, G.W. Semenoff, ed. (Springer Verlag, Berlin 1998), p. 353–440 `[hep-th/9511201]`

[DW] K. Drühl and H. Wagner, *Algebraic formulation of duality transformations for abelian lattice models*, Ann. Phys. 141 (1982) 225–253

[EKRS] S. Elitzur, B. Karni, E. Rabinovici, and G. Sarkissian, *Defects, super-Poincar line bundle and fermionic T-duality*, J. High Energy Phys. 04 (2013) 088_1–32 `[hep-th/1301.6639]`

[ENOM] P.I. Etingof, D. Nikshych, V. Ostrik, and E. Meir, *Fusion categories and homotopy theory*, Quantum Topology 1 (2010) 209–273 `[math.QA/0909.3140]`

[FFRS1] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Kramers-Wannier duality from conformal defects*, Phys. Rev. Lett. 93 (2004) 070601_1–4 `[cond-mat/0404051]`

[FFRS2] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. B 763 (2007) 354–430 `[hep-th/0607247]`

[FSV1] J. Fuchs, C. Schweigert, and A. Valentino, *Bicategories for boundary conditions and for surface defects in 3-d TFT*, Commun. Math. Phys. 321 (2013) 543–575 `[hep-th/1203.4568]`

[FSV2] J. Fuchs, C. Schweigert, and A. Valentino, *A geometric approach to boundaries and surface defects in Dijkgraaf-Witten theories*, preprint `hep-th/1307.3632`

[GGP] A. Gadde, S. Gukov, and P. Putrov, *Duality defects*, preprint `hep-th/1404.2929`

[HiR] C.J. Hillar and D. Rhea, *Automorphisms of finite Abelian groups*, Amer. Math. Monthly 114 (2007) 917–923 `[math.GR/0605185]`

[HuR] L.K. Hua and I. Reiner, *On the generators of the symplectic group*, Trans. Amer. Math. Soc. 65 (1949) 415–426

[KaBSS] L. Kampmeijer, F.A. Bais, B.J. Schroers, and J.K. Slingerland, *Towards a non-abelian electric-magnetic symmetry: the skeleton group*, J. High Energy Phys. 1001 (2010) 095_1–32 `[0812.1230]`

[KaS] A. Kapustin and N. Saulina, *Surface operators in 3d topological field theory and 2d rational conformal field theory*, in: *Mathematical Foundations of Quantum Field and Perturbative String Theory*, H. Sati and U. Schreiber, eds. (American Mathematical Society, Providence 2011), p. 175–198 `[hep-th/1012.0911]`

[KaW] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. 1 (2007) 1–236 `[hep-th/0604151]`

[KhTH] M.N. Khan, J.C.Y. Teo, and T.L. Hughes, *Anyonic symmetries and topological defects in abelian topological phases: an application to the* ADE *classification*, preprint `cond-mat/1403.6478`

[Ki] A.Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. 303 (2003) 2–30 `[quant-ph/9707021]`

[KK] A. Kitaev and L. Kong, *Models for gapped boundaries and domain walls*, Commun. Math. Phys. 313 (2012) 351–373 `[cond-mat/1104.5047]`

[Mo] J.C. Morton, *Cohomological twisting of 2-linearization and extended TQFT*, J. Homotopy Relat. Struct. (2014) in press `[math.QA/1003.5603]`

[NN] D. Naidu and D. Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, Commun. Math. Phys. 279 (2008) 845–872 `[math.QA/0705.0665]`

[NR] D. Nikshych, and, B. Riepel, *Categorical Lagrangian Grassmannians and Brauer-Picard groups of pointed fusion categories*, preprint `math.QA/1208.0840`

[Os] V. Ostrik, *Module categories over the Drinfeld double of a finite group*, Int. Math. Res. Notices No. 27 (2003) 1507–1520 `[math.QA/0202130]`

[Re] M. Reeder, *Notes on group theory* `[www2.bc.edu/~reederma/Groups.pdf]`

[Sc] A.S. Schwarz, *Field theories with no local conservation of the electric charge*, Nucl. Phys. B 208 (1982) 141–158

[SFR] C. Schweigert, J. Fuchs, and I. Runkel, *Categorification and correlation functions in conformal field theory*, in: Proceedings of the *ICM 2006*, M. Sanz-Solé, J. Soria, J.L. Varona, and J. Verdera, eds. (European Math. Society, Zürich 2006), p. 443–458 `[math.CT/0602079]`

[SW] A. Shapere and F. Wilczek, *Self-dual models with theta terms*, Nucl. Phys. B 320 (1989) 669–695

[Wi] S. Willerton, *The twisted Drinfeld double of a finite group via gerbes and finite groupoids*, Alg. & Geom. Topol. 8 (2008) 1419–1457 `[math.QA/0503266]`