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Homomorphisms of Gray-categories as pseudo algebras

Lukas Buhné

*Fachbereich Mathematik, Universität Hamburg,
Bereich Algebra und Zahlentheorie,
Bundesstrae 55, D-20146 Hamburg, Germany*

Abstract. Given $\mathcal{G}ray$-categories $\mathcal{P}$ and $\mathcal{L}$, there is a $\mathcal{G}ray$-category $\mathcal{T}ricat_{ls}(\mathcal{P}, \mathcal{L})$ of locally strict trihomomorphisms with domain $\mathcal{P}$ and codomain $\mathcal{L}$, tritransformations, trimodifications, and perturbations. If the domain $\mathcal{P}$ is small and the codomain $\mathcal{L}$ is cocomplete, we show that this $\mathcal{G}ray$-category is isomorphic as a $\mathcal{G}ray$-category to the $\mathcal{G}ray$-category Ps-T-Alg of pseudo algebras, pseudo functors, transformations, and modifications for a $\mathcal{G}ray$-monad T derived from left Kan extension.

Inspired by a similar situation in two-dimensional monad theory, we apply the coherence theory of three-dimensional monad theory and prove that the the inclusion of the functor category in the enriched sense into this $\mathcal{G}ray$-category of locally strict trihomomorphisms has a left adjoint such that the components of the unit of the adjunction are internal biequivalences. This proves that any locally strict trihomomorphism between $\mathcal{G}ray$-categories with small domain and cocomplete codomain is biequivalent to a $\mathcal{G}ray$-functor. Moreover, the hom $\mathcal{G}ray$-adjunction gives an isomorphism of the hom 2-categories of tritransformations between a locally strict trihomomorphism and a $\mathcal{G}ray$-functor with the corresponding hom 2-categories in the functor $\mathcal{G}ray$-category. A notable example is given by locally strict $\mathcal{G}ray$-valued presheafs with small domain. Our results have applications in three-dimensional descent theory and point into the direction of a Yoneda lemma for tricategories.

1 Introduction

Three-dimensional monad theory is the study of $\mathcal{G}ray$-monads and their different kinds of algebras. While three-dimensional monad theory is by now well-developed, see [7, Part III] and [16], there are only few examples. This paper is based on the insight that one example from two-dimensional monad theory can be transferred to the three-dimensional context. This is in fact nontrivial since the amount of computation is considerably higher than in the two-dimensional context. On the other hand, we have been in need of exactly this three-dimensional result for applications in three-dimensional descent theory.

We here provide the details of the reformulation. We show how under suitable conditions on domain and codomain the locally strict trihomomorphisms between $\mathcal{G}ray$-categories $\mathcal{P}$ and $\mathcal{L}$ correspond to pseudo algebras for a $\mathcal{G}ray$-monad T on the functor $\mathcal{G}ray$-category $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ derived from left Kan extension, where $\mathrm{ob}\mathcal{P}$ is the underlying set of objects of $\mathcal{P}$ considered as a discrete $\mathcal{G}ray$-category. The conditions are that the domain $\mathcal{P}$ is a small and that the codomain $\mathcal{L}$ is a cocomplete $\mathcal{G}ray$-category. In fact, we prove that the $\mathcal{G}ray$-categories Ps-T-Alg and

$\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$ are isomorphic as $\mathcal{G}ray$-categories, which extends the local result mentioned in [16, Ex. 3.5]. On the other hand, the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ for this monad is given by the functor $\mathcal{G}ray$-category $[\mathcal{P},\mathcal{L}]$, and there is an obvious inclusion of $[\mathcal{P},\mathcal{L}]$ into Ps-T-Alg. The relation of these two $\mathcal{G}ray$-categories was studied locally by Power [16] and by Gurski using codescent objects [7]. This mimics the situation one categorical dimension below, which started with Blackwell et al.'s paper [2] and was later refined by Lack using codescent objects [11]. We readily show that a corollary of Gurski's central coherence theorem [7, Th. 15.13] applies to the $\mathcal{G}ray$-monad on $[\mathrm{ob}\mathcal{P},\mathcal{L}]$: The inclusion of the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ into the $\mathcal{G}ray$-category Ps-T-Alg of pseudo T-algebras has a left adjoint such that the components of the unit of this adjunction are internal biequivalences.

This corresponds to the fact from two-dimensional monad theory that given a small 2-category P and a cocomplete 2-category L, the inclusion of the functor category $[P,L]$ (in the sense of $\mathcal{C}at$-enriched category theory) into the 2-category $\mathcal{B}icat(P,L)$ of functors, pseudonatural transformations and modifications has a left adjoint such that the components of the unit of this adjunction are internal equivalences. While this is not explicitly stated, it follows from Lack's coherence theorem [11, Th. 4.10]. Explicit partial results may be found in [15, Ex. 4.2] and [2, Ex. 6.6].

We now go on to expand on the monad T, its properties, and the identification of Ps-T-Alg. For this purpose we will need the unique $\mathcal{G}ray$-functor $H\colon \mathrm{ob}\mathcal{P}\to\mathcal{P}$ which is the identity on objects, and the $\mathcal{G}ray$-functor $[H,1]\colon [\mathcal{P},\mathcal{L}]\to[\mathrm{ob}\mathcal{P},\mathcal{L}]$ from enriched category theory, which is given on objects by precomposition with H. This functor sends any cell of $[\mathcal{P},\mathcal{L}]$ such as a $\mathcal{G}ray$-functor or a $\mathcal{G}ray$-natural transformation to its family of values and components in $\mathcal{L}$ respectively. By the theorem of Kan adjoints, left Kan extension Lan_H along H provides a left adjoint to $[H,1]$, and T is the $\mathcal{G}ray$-monad corresponding to this adjunction. This is all as in the 2-dimensional context, and the story is then usually told as follows: The enriched Beck's monadicity theorem shows that $[H,1]\colon [\mathcal{P},\mathcal{L}]\to[\mathrm{ob}\mathcal{P},\mathcal{L}]$ is strictly monadic. That is, the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ is isomorphic to the functor category $[\mathcal{P},\mathcal{L}]$ such that the forgetful functor factorizes through this isomorphism and $[H,1]$. On the other hand, the monad has an obvious explicit description. In fact, by the description of the left Kan extension in terms of tensor products and coends we must have:

$$(TA)Q=\int^{P\in\mathrm{ob}\mathcal{P}}\mathcal{P}(P,Q)\otimes AP \tag{1}$$

where A is a $\mathcal{G}ray$-functor $\mathrm{ob}\mathcal{P}\to\mathcal{L}$ and where Q is an object of $\mathcal{P}$. Recall that the tensor product is a special indexed colimit. For enrichment in a general symmetric monoidal closed category $\mathcal{V}$, it is characterized by an appropriately natural isomorphism in $\mathcal{V}$:

$$\mathcal{L}(\mathcal{P}(P,Q)\otimes AP,AQ)\cong[\mathcal{P}(P,Q),\mathcal{L}(AP,AQ)]\,, \tag{2}$$

where $[-,-]$ denotes the internal hom of $\mathcal{V}$. Thus, in the case of enrichment in $\mathcal{G}ray$, (2) is an isomorphism of 2-categories. In fact, the tensor product gives rise to a $\mathcal{G}ray$-adjunction, and its hom $\mathcal{G}ray$-adjunction is given by (2).

To achieve the promised identification of Ps-T-Alg with $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$, we have to determine how the data and $\mathcal{G}ray$-category structure of Ps-T-Alg transforms under the adjunction of the tensor product. Indeed, one can also identify the Eilenberg-Moore object with the functor $\mathcal{G}ray$-category in this fashion. For example, an object of $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ is an algebra for the monad T.

The definition of such an algebra is just as for an ordinary monad. Thus, it consists of a 1-cell $a\colon TA \to A$ subject to two algebra axioms. According to equation (1), the $\mathcal{G}ray$-natural transformation a is determined by components $a_{PQ}\colon \mathcal{P}(P,Q)\otimes AP \to AQ$. These are objects in the 2-category $\mathcal{L}(\mathcal{P}(P,Q)\otimes AP, AQ)$. The internal hom of $\mathcal{G}ray$ is given by the 2-category of strict functors, pseudonatural transformations, and modifications. Thus a_{PQ} corresponds under the hom adjunction (2) to a strict functor $A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$, and the axioms of an algebra imply that this gives A the structure of a $\mathcal{G}ray$-functor $\mathcal{P} \to \mathcal{L}$.

Given a $\mathcal{G}ray$-monad on a $\mathcal{G}ray$-category $\mathcal{K}$, the notions of pseudo algebras, pseudo functors, transformations, and modifications are all given by cell data of the $\mathcal{G}ray$-category $\mathcal{K}$. In the case that $\mathcal{K} = [\mathrm{ob}\mathcal{P}, \mathcal{L}]$, this means that the data consists of families of cells in the target $\mathcal{L}$. Parts of these data transform under the adjunction of the tensor product into families of cells in the internal hom, that is, families of strict functors of 2-categories, pseudonatural transformations of those, and modifications of those. This already shows that we only have a chance to recover locally strict trihomomorphisms from pseudo algebras because a general trihomomorphism might consist of nonstrict functors of 2-categories. This is in contrast to the two-dimensional context where pseudo algebras correspond precisely to possibly nonstrict functors of 2-categories.

We now give a short overview of how this paper is organized. In Section 2, we describe the symmetric monoidal closed category $\mathcal{G}ray$ and extend some elementary results on the correspondence of cubical functors and strict functors on Gray products.

In Section 3, we reproduce Gurski's definition of Ps-T-Alg and prove that two lax algebra axioms are redundant for a pseudo algebra.

In Section 4, we introduce the monad T on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ in 4.1 and describe it explicitly in 4.2. In 4.3 we expand on tensor products and derive the two rather involved Lemmata 11 and 12, which play a critical role in the bulk of our technical calculations.

In Section 5, we explicitly identify the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ in the general situation where $\mathcal{V}$ is a complete and cocomplete locally small symmetric monoidal closed category.

In Section 6, we establish the identification of Ps-T-Alg and $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$, on which we will now comment in more detail. To characterize how Ps-T-Alg transforms under the adjunction of the tensor product, in 6.1 we introduce the notion of homomorphisms of $\mathcal{G}ray$-categories, Gray transformations, Gray modifications, and Gray perturbations. With the help of Lemmata 11 and 12 from 4.3, these are seen to be exactly the transforms of pseudo algebras, pseudo functors, transformations, and modifications respectively. This also equips the Gray data with the structure of a $\mathcal{G}ray$-category.

Elementary observations in 6.3 then give that a homomorphism of $\mathcal{G}ray$-categories is the same thing as a locally strict trihomomorphism, much like a $\mathcal{G}ray$-category is the same thing as a strict, cubical tricategory. Similarly, the notion of a Gray transformation corresponds exactly to a tritransformation between locally strict functors, and Gray modifications and perturbations correspond exactly to trimodifications and perturbations of those. This follows from the general correspondence, mediated by Theorem 2 from 2.4, of data for the cubical composition functor and the cartesian product on the hand and data for the composition law of the $\mathcal{G}ray$-category and the Gray product on the other hand. The only thing left to check is that the axioms correspond to each other. Namely, the Gray notions being the transforms of the pseudo notions of three-dimensional monad theory, the axioms are equations of modifications, while the axioms for the tricategorical constructions are equations involving the components of modifications. That these coincide is mostly straightforward, less transparent is only the comparison of interchange cells. Gurski's coherence theorem then gives that the inclusion of the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$ determined by the locally strict functors, denoted by $\mathcal{T}ricat_{ls}(\mathcal{P}, \mathcal{L})$, into the functor $\mathcal{G}ray$-category $[\mathcal{P}, \mathcal{L}]$ has a left adjoint and the components of the unit of this adjunction are internal biequivalences.

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2 Preliminaries

We assume familiarity with enriched category theory. Regarding enriched category theory, we stay notationally close to Kelly's book [9], from which we shall cite freely. We also assume a fair amount of bicategory theory, see for example [1] or the short [12]. Since the notions of tricategory theory are intrinsically involved, and since we consider various slight variations of these, we refrain from supplying all of them. The appropriate references are the original paper by Gordon, Power, and Street [4], Gurski's thesis [6], and his later book [7], which includes much of the material first presented in [6]. Since this is also our primary reference for three-dimensional monad theory, we will usually cite from [7]. In fact, the tricategories considered are all $\mathcal{G}ray$-categories, and we will describe them in terms of enriched notions to the extent possible. We do supply definitions in terms of enriched notions that correspond precisely to locally strict trihomomorphisms, tritransformations, trimodifications, and perturbations, but in describing this correspondence we assume knowledge of the tricategorical definitions. As a matter of fact the sheer amount of notational translation between the $\mathcal{G}ray$-enriched and the tricategorical context can be challenging at times.

Only basic knowledge of the general theory of monads in a 2-category is required cf. [17]. For monads in enriched category theory see also [3].

2.1 Conventions

Horizontal composition in a bicategory is generally denoted by the symbol $*$, while vertical composition is denoted by the symbol $\diamond$. We use the term functor for what is elsewhere called pseudofunctor or weak functor or homomorphism of bicategories and shall indicate whether the functor is strict where it is not clear from context. By an isomorphism we always mean an honest isomorphism, e.g. an isomorphism on objects and hom objects in enriched category theory. The symbol $\otimes$ is reserved both for a monoidal structure and tensor products in the sense of enriched category theory. If not otherwise stated, $\mathcal{V}$ denotes a locally small symmetric monoidal closed category with monoidal structure $\otimes$; associators and unitors a, l, and r; internal hom $[-,-]$; unit d

and counit or evaluation e. We shall usually use the prefix $\mathcal{V}$- to emphasize when the $\mathcal{V}$-enriched notions are meant, although this is occasionally dropped where it would otherwise seem overly redundant. The composition law of a $\mathcal{V}$-category $\mathcal{K}$ is denoted by $M_{\mathcal{K}}$. The unit at the object $K \in \mathcal{K}$ is denoted by j_K or occasionally 1_K, for example when it shall be emphasized that it is also the unit at K in the underlying category $\mathcal{K}_0$. The identification of $\mathcal{V}_0(I, [X, Y])$ and $\mathcal{V}_0(X, Y)$ induced from the closed structure for objects $X, Y \in \mathcal{V}$ of $\mathcal{V}$ is to be understood and usually implicit. We use the terms indexed limit and indexed colimit for what is elsewhere also called weighted limit and weighted colimit. The concepts of ordinary and extraordinary $\mathcal{V}$-naturality cf. [9, Ch. 1] and the corresponding composition calculus are to be understood, and we freely use the underlying ordinary and extraordinary naturality too.

Composition in a monoidal category is generally denoted by juxtaposition. Composition of $\mathcal{V}$-functors is in general also denoted by juxtaposition. For cells of a $\mathcal{G}ray$-category, juxtaposition is used as shorthand for the application of its composition law.

2.2 The category $2\mathcal{C}at$

Let $\mathcal{C}at$ denote the category of small categories and functors. It is well-known that $\mathcal{C}at$ is complete and cocomplete: it clearly has products and equalizers, thus is complete. Coproducts are given by disjoint union, and there is a construction for coequalizers in [5, I,1.3, p. 25] (due to Wolff). In fact, the same strategy applies to the category of $\mathcal{V}$-enriched categories and $\mathcal{V}$-functors in general, where $\mathcal{V}$ is a complete and cocomplete symmetric monoidal closed category. Products are given by the cartesian product of the object sets and the cartesian product of the hom objects. Equalizers of $\mathcal{V}$-functors are given by the equalizer of the maps on objects and the equalizer of the hom morphisms. Coproducts are given by the coproduct of the object sets i.e. the disjoint union and by the coproduct of the hom objects. The construction of coequalizers in [5, I,1.3, p. 25] can easily be transferred to this context. In particular, the category $2\mathcal{C}at$ of small 2-categories and strict functors is complete and cocomplete.

2.3 The symmetric monoidal closed category $\mathcal{G}ray$

We now describe the symmetric monoidal closed category in which we will usually enrich. Its underlying category is $2\mathcal{C}at$, which has a symmetric monoidal closed structure given by the Gray product. We can only provide a brief description of the Gray product here. For details, the reader is referred to [7, 3.1, p. 36ff.] and to [5, I,4.9, p. 73ff.] for a lax variant.

The Gray product of 2-categories X and Y is a 2-category denoted by $X \otimes Y$. Rather than giving a complete explicit description of the Gray product, we mention the following characterization (see [7, Ch. 3]). Considering the sets of objects ob$X$ and ob$Y$ as discrete 2-categories, we denote by $X \square Y$ the pushout in $2\mathcal{C}at$ of the diagram below, where $\times$ denotes the cartesian product of 2-categories, and where the morphisms are given by products of the inclusions ob$X \to X$ and ob$Y \to Y$ and identity functors respectively.

$$\begin{array}{ccc} \mathrm{ob}X \times \mathrm{ob}Y & \longrightarrow & X \times \mathrm{ob}Y \\ \big\downarrow & & \\ \mathrm{ob}X \times Y & & \end{array} \tag{3}$$

By the universal property of the pushout, the products of the inclusions and identity functors, $\mathrm{ob}X \times Y \to X \times Y$ and $X \times \mathrm{ob}Y \to X \times Y$, induce a strict functor $j\colon X \square Y \to X \times Y$.

It is well-known that there is an orthogonal factorization system on $2\mathcal{C}at$ with left class the strict functors which are bijective on objects and 1-cells and right class the strict functors which are locally fully faithful, see for example [7, Corr. 3.20, p. 51]. The Gray product $X \otimes Y$ may be characterized by factorizing j with respect to this factorization system. More precisely, $X \otimes Y$ is uniquely characterized (up to unique isomorphism in $2\mathcal{C}at$) by the fact that there is a strict functor $m\colon X \square Y \to X \otimes Y$ which is an isomorphism on the underlying categories i.e. bijective on objects and 1-cells and a strict functor $i\colon X \otimes Y \to X \times Y$ which is locally fully faithful such that $j = im$.

There is an obvious explicit description of $X \otimes Y$ in terms of generators and relations, which can be used to construct a functor $\otimes\colon 2\mathcal{C}at \times 2\mathcal{C}at \to 2\mathcal{C}at$. Clearly, $X \otimes Y$ has the same objects as $X \times Y$, and we have the images of the 1-cells and 2-cells from $X \times \mathrm{ob}Y$ and $\mathrm{ob}X \times Y$, for which we use the same name in $X \otimes Y$. That is, there are 1-cells $(f, 1)\colon (A, B) \to (A', B)$ for 1-cells $f\colon A \to A'$ in X and objects B in Y, and there are 1-cells $(1, g)\colon (A, B) \to (A, B')$ for objects A in X and 1-cells $g\colon B \to B'$ in Y. All 1-cells in $X \otimes Y$ are up to the obvious relations generated by horizontal strings of those 1-cells, the identity 1-cell being $(1, 1)$. Apart from the obvious 2-cells $(\alpha, 1)\colon (f, 1) \Rightarrow (f', 1)\colon (A, B) \to (A', B)$ and $(1, \beta)\colon (1, g) \Rightarrow (1, g')\colon (A, B) \to (A, B')$, there must be unique invertible interchange 2-cells $\Sigma_{f,g}\colon (f, 1) * (1, g) \Rightarrow (1, g) * (f, 1)$ mapping to the identity of (f, g) under j because the latter is fully faithful—domain and codomain cleary both map to (f, g) under j. In particular, by uniqueness i.e. because j is locally fully faithful, these must be the identity if either f or g is the identity. There are various relations on horizontal and vertical composites of those cells, all rather obvious from the characterization above. We omit those as well as the details how equivalence classes and horizontal and vertical composition are defined.

For functors $F\colon X \to X'$ and $G\colon Y \to Y'$, it is not hard to give a functorial definition of the functor $F \otimes G\colon X \otimes Y \to X' \otimes Y'$. We confine ourselves with the observation that on interchange 2-cells,

$$(F \otimes G)_{(A,B),(A'B')}(\Sigma_{f,g}) = \Sigma_{F_{A,A'}(f),G_{B,B'}(g)} \,. \tag{4}$$

From the characterization above it is then clear how to define associators and unitors for $\otimes$. We only mention here that

$$a(\Sigma_{f,g}, 1) = \Sigma_{f,(g,1)} \tag{5}$$

and

$$a(\Sigma_{(f,1),h}) = \Sigma_{f,(1,h)} \tag{6}$$

and

$$a(\Sigma_{(1,g),h}) = (1, \Sigma_{g,h}) \,. \tag{7}$$

We omit the details that this gives a monoidal structure on $\mathcal{2Cat}$ (pentagon and triangle identity follow from pentagon and triangle identity for the cartesian product and $\Box$).

There is an obvious symmetry c for the Gray product, which on interchange cells is given by

$$c(\Sigma_{f,g}) = \Sigma_{g,f}^{-1} \,. \tag{8}$$

As for any two bicategories (by all means for small domain), there is a functor bicategory $\mathcal{Bicat}(X, Y)$ given by functors of bicategories, pseudonatural transformations, and modifications. As the codomain Y is a 2-category, this is in fact again a 2-category. We denote by $[X, Y]$ the full sub-2-category of $\mathcal{Bicat}(X, Y)$ given by the strict functors. One can show that this gives $\mathcal{2Cat}$ the structure of a symmetric monoidal closed category with internal hom $[X, Y]$:

Theorem 1. *[7, Th. 3.16] The category $\mathcal{2Cat}$ of small 2-categories and strict functors has the structure of a symmetric monoidal closed category. As such, it is referred to as $\mathcal{Gray}$. The monoidal structure is given by the Gray product and the terminal 2-category as the unit object, the internal hom is given by the functor 2-category of strict functors, pseudonatural transformations, and modifications.*

Remark 1. In fact, we will not have to specify the closed structure of $\mathcal{Gray}$ apart from the fact that its evaluation is (partly) given by taking components. This is because our ultimate goal is to compare definitions from three-dimensional i.e. $\mathcal{Gray}$-enriched monad theory to definitions from the theory of tricategories, and we do so in the case where all tricategories are in fact $\mathcal{Gray}$-categories, that is, equivalently, strict, cubical tricategories. These definitions will only formally involve the cubical composition functor, which relates to the composition law of the $\mathcal{Gray}$-category – we will usually not have to specify the composition. Of course, one can explicitly identify the enriched notions, and then there are alternative explicit arguments. However, we think that the formal argumentation is more adequate. The closed structure is worked out in [7, 3.3], and the enriched notions usually turn out to be just as one would expect. We spell out a few explicit prescriptions below the following lemma, but in fact we just need a few consequences of these, for example equation (10) below.

Next recall that a locally small symmetric monoidal closed category $\mathcal{V}$ can be considered as a category enriched in itself i.e. as a $\mathcal{V}$-category. Also recall that if the underlying category $\mathcal{V}_0$ of $\mathcal{V}$ is complete and cocomplete, $\mathcal{V}$ is complete and cocomplete considered as a $\mathcal{V}$-category. This means it has any small indexed limit and any small indexed colimit. For the concept of an indexed limit see [9, Ch. 3]. In fact, completeness follows from the fact that a limit is given by an end, and if the limit is small, this end exists and is given by an equalizer in $\mathcal{V}_0$, see [9, (2.2)]. It is cocomplete because, $\mathcal{V}$ being complete, $\mathcal{V}^{\mathrm{op}}$ is tensored and thus also admits small conical limits because $\mathcal{V}_0$ is cocomplete, hence $\mathcal{V}$ admits small coends because it is also tensored, but then since by [9, (3.70)] any small colimit is given by a small coend over tensor products, it is cocomplete.

Recall that the underlying category $\mathcal{2Cat}$ of $\mathcal{Gray}$ is complete and cocomplete cf. 2.2. Thus in particular, we have the following :

Lemma 1. *The $\mathcal{G}ray$-category $\mathcal{G}ray$ is complete and cocomplete.* □

The composition law of the $\mathcal{G}ray$-category $\mathcal{G}ray$ is given by strict functors $[Y,Z] \otimes [X,Y] \to [X,Y]$, where X, Y and Z are 2-categories . It is given on objects by composition of strict functors. On 1-cells of the form $(\theta,1)\colon (F,G) \to (F',G)$ it is given by the pseudonatural transformation denoted $G^*\theta$ with components θ_{Gx} and naturality 2-cells θ_{Gf}. On 1-cells of the form $(1,\sigma)\colon (F,G) \to (F,G')$ it is given by the pseudonatural transformation denoted $F_*\sigma$ with components $F_{Gx,G'x}(\sigma_x)$ and naturality 2-cells $F_{Gx,G'x'}(\sigma_f)$. Similarly, on 2-cells of the form $(\Gamma,1)\colon (\theta,1) \Rightarrow (\theta',1)\colon (F,G) \to (F',G)$ it is given by the modification denoted $G^*\Gamma$ with components Γ_{Gx}, and on 2-cells of the form $(1,\Delta)\colon (1,\sigma) \Rightarrow (1,\sigma')\colon (F,G) \to (F,G')$ it is given by the modification denoted $F^*\Delta$ with components $F_{Gx,G'x}(\Delta_X)$. Finally, on interchange cells of the form $\Sigma_{\theta,\sigma}$, it is given by the naturality 2-cell θ_{σ_x} of θ at σ_x, hence,

$$(M_{\mathcal{G}ray}(\Sigma_{\theta,\sigma}))_x = \theta_{\sigma_x}\colon\ \theta_{G'x} * F_{Gx,G'x}(\sigma_x) \Rightarrow F'_{Gx,G'x}(\sigma_x) * \theta_{Gx}\ . \tag{9}$$

This follows from the general form of $M_{\mathcal{V}}$ in enriched category theory by inspection of the closed structure of $\mathcal{G}ray$ cf. [7, Prop. 3.10].

Also recall that there is a functor $\mathrm{Ten}\colon\ \mathcal{V}\otimes\mathcal{V} \to \mathcal{V}$ which is given on objects by the monoidal structure. For $\mathcal{V} = \mathcal{G}ray$, its strict hom functor

$$\mathrm{Ten}_{(X,X'),(Y,Y')}\colon\ [X,X'] \otimes [Y,Y'] \to [X \otimes Y, X' \otimes Y']$$

sends an object (F,G) to the functor $F \otimes G$. It sends a transformation $(\theta,1_G)\colon (F,G) \Rightarrow (F',G)$ to the transformation with component the 1-cell $(\theta_x, 1_{Gy})$ in $X' \otimes Y'$ at the object (x,y) in $X \otimes Y$; and naturality 2-cells $(\sigma_f, 1_{Gg})$ and $\Sigma_{\theta_x,Gg}$ at 1-cells $(f,1_y)$ and $(1_x,g)$ respectively. Its effect on a transformation $(1_F,\iota)\colon (F,G) \Rightarrow (F,G')$ is analogous. It sends a modification $(\Gamma, 1_{1_G})\colon (\theta,1_G) \Rrightarrow (\theta',1_G)$ to the modification with component the 2-cell $(\Gamma_X, 1_{1_{Gy}})$ in $X' \otimes Y'$ at (x,y) in $X \otimes Y$. Its effect on a modification $(1_{1_F},\Delta)\colon (1_F,\iota) \Rrightarrow (1_F,\iota')$ is analogous. Finally, it sends the interchange 2-cell $\Sigma_{\theta,\iota}$ to the modification with component the interchange 2-cell $\Sigma_{\theta_x,\iota_y}$, hence,

$$(\mathrm{Ten}_{(X,X'),(Y,Y')}(\Sigma_{\theta,\iota}))_{x,y} = \Sigma_{\theta_x,\iota_y}\ . \tag{10}$$

All of this again follows from inspection of the closed structure of $\mathcal{G}ray$, cf. [7, Prop. 3.10]. See also equation (58) below.

2.4 Cubical functors

Given 2-categories X, Y, Z, recall that a cubical functor in two variables is a functor $\hat{F}\colon\ X \times Y \to Z$ such that for all 1-cells (f,g) in $X \times Y$, the composition constraint

$$\hat{F}_{(1,g),(f,1)}\colon\ \hat{F}(1,g) * \hat{F}(f,1) \Rightarrow \hat{F}(f,g)\ ,$$

is the identity 2-cell, and such that for all composable 1-cells $(f',1)$, $(f,1)$ in $X \times Y$,

$$\hat{F}_{(f',1),(f,1)}\colon\ \hat{F}(f',1) * \hat{F}(f,1) \Rightarrow \hat{F}(f' * f,1)\ ,$$

is the identity 2-cell, and such that for all composable 1-cells $(1, g')$, $(1, g)$ in $X \times Y$,

$$\hat{F}_{(1,g'),(1,g)} \colon \hat{F}(1, g') * \hat{F}(1, g) \Rightarrow \hat{F}(1, g' * g) ,$$

is the identity 2-cell. For composable $(1, g'), (f, g)$ and $(f', g'), (f, 1)$, the constraint cells are then automatically identities by compatibility of $\hat{F}$ with associators i.e. a functor axiom for $\hat{F}$; it also automatically preserves identity 1-cells.

We start with the following elementary result, which extends the natural $\mathcal{S}et$-isomorphism in [7, Th. 3.7] to a $\mathcal{C}at$-isomorphism.

Proposition 1. *Given 2-categories X, Y, Z, there is a universal cubical functor $C \colon X \times Y \to X \otimes Y$ natural in X and Y such that precomposition with C induces a natural isomorphism of 2-categories (i.e. a $\mathcal{C}at$-isomorphism)*

$$[X \otimes Y, Z] \cong \mathcal{B}icat_c(X, Y; Z) ,$$

where $\mathcal{B}icat_c(X, Y; Z)$ denotes the full sub-2-category of $\mathcal{B}icat(X \times Y, Z)$ determined by the cubical functors.

Proof. The functor C is determined by the requirements that it be the identity on objects, that $C(f, 1) = (f, 1)$, $C(\alpha, 1) = (\alpha, 1)$, $C(1, g) = (1, g)$, $C(1,\beta) = (1,\beta)$, and that it be a cubical functor. In particular, observe that this means that $C(f, g) = (1, g) * (f, 1)$ and that the constraint $C_{(f,1),(1,g)}$ is given by the interchange cell $\Sigma_{f,g}$.

As for an arbitrary functor of bicategories, precomposition with C induces a strict functor

$$C^* \colon \mathcal{B}icat(X \otimes Y, Z) \to \mathcal{B}icat(X \times Y, Z) .$$

It sends a functor $G \colon X \otimes Y \to Z$ to the composite functor $GC \colon X \times Y \to C$. In fact, if F is a strict functor $X \otimes Y \to Z$, recalling the definition of the composite of two functors of bicategories, a moment's reflection affirms that $\hat{F} := FC$ is a cubical functor with constraint $\hat{F}_{(f,1),(1,g)} = F(\Sigma_{f,g})$. Thus by restriction, C^* gives rise to a functor $[X \otimes Y, Z] \to \mathcal{B}icat_c(X, Y; Z)$ which we also denote by C^*.

If $\sigma \colon F \Rightarrow G \colon X \otimes Y \to Z$ is a pseudonatural transformation, $C^*\sigma \colon FC \Rightarrow GC$ is the pseudonatural transformation with component

$$(C^*\sigma)_{(A,B)} = \sigma_{C(A,B)} = \sigma_{(A,B)}$$

at an object $(A, B) \in X \times Y$, and naturality 2-cell

$$(C^*\sigma)_{(f,g)} = \sigma_{C(f,g)} = \sigma_{(f,1)*(1,g)} = (\sigma_{(f,1)} * 1) \diamond (1 * \sigma_{(1,g)})$$

at a 1-cell $(f, g) \in X \times Y$, where the last equation is by respect for composition of σ. If σ is the identity pseudonatural transformation, it is immediate that the same applies to $C^*\sigma$. Given another pseudonatural transformation of strict functors $\tau \colon G \Rightarrow H$, we maintain that $(C^*\tau) * (C^*\sigma) = C^*(\tau * \sigma)$. It is manifest that the components coincide: both are given by $\tau_{(A,B)} * \sigma_{(A,B)}$ at the object $(A, B) \in X \times Y$. That the naturality 2-cells at a 1-cell $(f, g) \in X \times Y$ coincide,

$$(\tau_{C(f,g)} * 1) \diamond (1 * \sigma_{C(f,g)}) = (\tau * \sigma)_{C(f,g)}$$

is simply the defining equation for the naturality 2-cell of the horizontal composite $\tau * \sigma$.

If $\Delta\colon \sigma \Rrightarrow \pi$ is a modification of pseudonatural transformations $F \Rightarrow G$ of strict functors $X \otimes Y \to Z$, then $C^*\Delta$ is the modification $C^*\sigma \Rrightarrow C^*\pi$ with component

$$(C^*\Delta)_{(A,B)} = \Delta_{C(A,B)} = \Delta_{(A,B)}$$

at an object $(A, B) \in X \times Y$, and this prescription clearly strictly preserves identities and vertical composition of modifications. Given another modification $\Lambda\colon \tau \Rrightarrow \rho\colon G \Rightarrow H$ where H is strict, one readily checks that $(C^*\Lambda) * (C^*\Delta) = C^*(\Lambda * \Delta)$ both having component $\Lambda_{(A,B)} * \Delta_{(A,B)}$ at an object $(A, B) \in X \times Y$. Thus, we have shown that C^* is indeed a strict functor.

As a side note, we remark that because we only consider 2-categories, C^* is the same as the functor $\mathcal{B}icat(C, Z)$ induced by the composition of the tricategory $\mathcal{T}ricat$ of bicategories, functors, pseudonatural transformations, and modifications.

Let $\hat{F}\colon X \times Y \to Z$ be an arbitrary cubical functor, then the prescriptions $F(f, 1) = \hat{F}(f, 1)$, $F(\alpha, 1) = \hat{F}(\alpha, 1)$, $F(1, g) = \hat{F}(1, g)$, $F(1,\beta) = \hat{F}(1,\beta)$, and $F(\Sigma_{f,g}) = \hat{F}_{(f,1),(1,g)}$, provide a strict functor $F\colon X \otimes Y \to Z$ such that $FC = \hat{F}$. The latter equation and the requirement that F be strict, clearly determine F uniquely. That this is well-defined e.g. that it respects the various relations for the interchange cells is by compatibility of $\hat{F}$ with associators and naturality of

$$\hat{F}_{(A,B),(A',B'),(A'',B'')}\colon \ *_Z (\hat{F}_{(A',B'),(A'',B'')} \times \hat{F}_{(A,B),(A',B')}) \Rightarrow \hat{F}_{(A,B),(A'',B'')} *_{X \times Y}\ , \tag{11}$$

where $*$ denotes the corresponding horizontal composition functors. For example, for the relation

$$\Sigma_{f' * f, g} \sim (\Sigma_{f',g} * (1_f, 1)) \diamond ((1_{f'}, 1) * \Sigma_{f,g}) \tag{12}$$

one has to use that axiom twice giving

$$\hat{F}_{(f'*f,1),(1,g)} = \hat{F}_{(f',1)*(f,1),(1,g)} = \hat{F}_{(f',1),(f,g)} \diamond (\hat{F}(1_{f'}, 1) * \hat{F}_{(f,1),(1,g)})$$

and $\hat{F}_{(f',1),(f,g)} = \hat{F}_{(f',1),(1,g)*(f,1)} = \hat{F}_{(f',1),(g,1)} * \hat{F}(1_f, 1)$. Another way to see this, is to use coherence for the functor $\hat{F}$—then any relation in the Gray product must clearly be mapped to an identity in Z because the constraints in $\mathcal{F}_{\hat{F}}Z$ are mapped to identities in $\mathcal{F}_{2C}Z$, where these are the corresponding free constructions on the underlying category-enriched graphs cf. [7, 2.].

Now let $\hat{\sigma}\colon \hat{F} \Rightarrow \hat{G}$ be an arbitrary pseudonatural transformation of cubical functors. We have already shown that $\hat{F}$ and $\hat{G}$ have the form FC and GC respectively, where F and G were determined above. We maintain that there is a unique pseudonatural transformation $\sigma\colon F \Rightarrow G$ such that $\hat{\sigma} = C^*\sigma$. By the above, the latter equation uniquely determines both the components, $\sigma_{(A,B)} = \hat{\sigma}_{(A,B)}$, and the naturality 2-cells of σ, namely $\sigma_{(f,1)} = \hat{\sigma}_{(f,1)}$ and $\sigma_{(1,g)} = \hat{\sigma}_{(1,g)}$, and thus σ is uniquely determined by respect for composition. That this is compatible with the relations $(f', 1)*(f, 1) \sim (f' * f, 1)$ and $(1, g')*(1, g) \sim (1, g' * g)$ in the Gray product follows from the fact that respect for composition is in this case tantamount to respect for composition of $\hat{\sigma}$ because the constraints are identities here due to the axioms for cubical functors. Hence, what is left to prove is that this is indeed a pseudonatural transformation. First observe that the prescriptions for σ have been determined by the requirement that it respects composition, and respect for units is tantamount to respect for units of $\hat{\sigma}$. Naturality with respect to 2-cells of the form $(\alpha, 1)$ and

$(1,\beta)$ is tantamount to the corresponding naturality condition for $\hat{\sigma}$. Naturality with respect to an interchange cell $\Sigma_{f,g}$, i.e.

$$(G(\Sigma_{f,g}) * 1_{\sigma_{(A,B)}}) \diamond \sigma_{(f,1)*(1,g)} = \sigma_{(1,g)*(f,1)} \diamond (1_{\sigma_{(A',B')}} * F(\Sigma_{f,g}))$$

is—by the requirement that σ respects composition:

$$\sigma_{(f,1)*(1,g)} = (1_{G(f,1)} * \sigma_{(1,g)}) \diamond (\sigma_{(f,1)} * 1_{F(1,g)}) = (1_{\hat{G}(f,1)} * \hat{\sigma}_{(1,g)}) \diamond (\hat{\sigma}_{(f,1)} * 1_{\hat{F}(1,g)})$$

and by respect for composition of $\hat{\sigma}$:

$$\sigma_{(1,g)*(f,1)} = (1_{G(1,g)} * \sigma_{(f,1)}) \diamond (\sigma_{(1,g)} * 1_{F(f,1)}) = (1_{\hat{G}(1,g)} * \hat{\sigma}_{(f,1)}) \diamond (\hat{\sigma}_{(1,g)} * 1_{\hat{F}(f,1)}) = \hat{\sigma}_{(f,g)}$$

(the constraints are trivial here)—tantamount to respect for composition of $\hat{\sigma}$:

$$(\hat{G}_{(f,1),(1,g)} * 1_{\hat{\sigma}_{(A,B)}}) \diamond ((1_{\hat{G}(f,1)} * \hat{\sigma}_{(1,g)}) \diamond (\hat{\sigma}_{(f,1)} * 1_{\hat{F}(1,g)})) = \hat{\sigma}_{(f,g)} \diamond (1_{\hat{\sigma}_{(A',B')}} * \hat{F}_{(f,1),(1,g)}) .$$

Notice that in general, naturality with respect to a vertical composite is implied by naturality with respect to the individual factors. Similarly, naturality with respect to a horizontal composite is implied by functoriality of $F_{,,}$ and $G_{,,}$ (cf. (11)), respect for composition, and naturality with respect to the individual factors.

Finally, let $\hat{\Delta}\colon C^*\sigma \Rrightarrow C^*\pi\colon FC \Rightarrow GC$ be an arbitrary modification. Then we maintain that there is unique modification $\Delta\colon \sigma \Rrightarrow \pi$ such that $\hat{\Delta} = C^*\Delta$. By the above, the latter equation uniquely determines Δ's components, $\Delta_{(A,B)} = \hat{\Delta}_{(A,B)}$ and thus Δ itself, but we have to show that Δ exists i.e. that this gives Δ the structure of a modification. The modification axiom for 1-cells of the form $(f,1)$ is tantamount to the modification axiom for $\hat{\Delta}$ and the corresponding 1-cell in $X \times Y$ of the same name. The same applies to the modification axiom for 1-cells of the form $(1,g)$. This proves that σ is a modification because the modification axiom for a horizontal composite is implied by respect for composition of σ and π, and the modification axiom for the individual factors.

□

Given 2-categories X_1, X_2, X_3, it is an easy observation that

$$a(C(C \times 1)) = C(1 \times C)a_\times\colon\ X_1 \times X_2 \times X_3 \to X_1 \otimes (X_2 \otimes X_3) , \tag{13}$$

where $a_\times$ is the associator of the cartesian product.

It is well-known that a strict, cubical tricategory is the same thing as a $\mathcal{G}ray$-category. To prove this, one has to replace the cubical composition functor by the composition law of a $\mathcal{G}ray$-category. This uses the underlying $\mathcal{S}et$-isomorphism of Proposition 1. In the same fashion, in order to compare locally strict trihomomorphisms between $\mathcal{G}ray$-categories with Gray homomorphisms as it is done in Theorem 6 in 6.3 below, we need the following many-variable version of Proposition 1 to replace adjoint equivalences and modifications of cubical functors by adjoint equivalences and modifications of the corresponding strict functors on Gray products.

Theorem 2. *Given a natural number n and 2-categories $Z, X_1, X_2, ..., X_n$, composition with*

$$C(C(C(...) \times 1_{X_{n-1}}) \times 1_{X_n})\colon\ X_1 \times X_2 \times ... \times X_n \to (...((X_1 \otimes X_2) \otimes X_3) \otimes ...) \otimes X_n ,$$

where C is the universal cubical functor, induces a natural isomorphism of 2-categories (i.e. a $\mathcal{C}at$-isomorphism)

$$[(...((X_1 \otimes X_2) \otimes X_3) \otimes ...) \otimes X_n, Z] \cong \mathcal{B}icat_c(X_1, X_2, ..., X_n; Z) ,$$

where $\mathcal{B}icat_c(X_1, X_2, ..., X_n; Z)$ denotes the full sub-2-category of $\mathcal{B}icat(X_1 \times X_2 \times ... \times X_n, Z)$ determined by the cubical functors in n variables. The same is of course true for any other combination of universal cubical functors (mediated by the unique isomorphism in terms of associators for the Gray product).

Proof. Recall that the composition $(F \circ (F_1 \times ... \times F_k))$ of cubical functors is again a cubical functor. This shows that the restriction of $(C(C(C(...) \times 1_{X_{n-1}}) \times 1_{X_n}))^*$ to $[(...((X_1 \otimes X_2) \otimes X_3) \otimes ...) \otimes X_n, Z]$ does indeed factorize through $\mathcal{B}icat_c(X_1, X_2, ..., X_n; Z)$. The proof that this gives an isomorphism as wanted is then a straightforward extension of the two-variable case. There are Gray product relations on combinations of interchange cells, which correspond to relations for the constraints holding by coherence. Note that indeed any diagram of interchange cells commutes because these map to identities in the cartesian product.

For example, a cubical functor in three variables is determined by compatible partial cubical functors in two variables and a relation on their constraints cf. the diagram in [7, Prop. 3.3, p. 42]. This corresponds to a combination of the Gray product relation

$$\Sigma_{f,g'*g} \sim ((1_{g'}, 1) * \Sigma_{f,g}) \diamond (\Sigma_{f,g'} * (1, 1_g)) \tag{14}$$

for $f = f_1$ and $g' = (f_2, 1)$ and $g = (1, f_3)$ and $g' = (1, f_3)$ and $g = (f_2, 1)$ respectively, and the Gray product relation

$$((1,\beta) * (\alpha, 1)) \diamond \Sigma_{f,g} \sim \Sigma_{f',g'} \diamond ((\alpha, 1) * (1,\beta))$$

for $f = f_1 = f'$, $\alpha = 1_{f_1}$, $g = (f_2, 1) * (1, f_3)$, $g' = (1, f_3) * (f_2, 1)$, and $\beta = \Sigma_{f_2,f_3}$, which reads

$$((1, \Sigma_{f_2,f_3}) * (1_{f_1}, 1)) \diamond ((1_{(f_2,1)}, 1) * \Sigma_{f_1,(1,f_3)}) \diamond (\Sigma_{f_1,(f_2,1)} * (1, 1_{(1,f_3)}))$$
$$\sim((1_{(1,f_3)}, 1) * \Sigma_{f_1,(f_2,1)}) \diamond (\Sigma_{f_1,(1,f_3)} * (1, 1_{(f_2,1)})) \diamond ((1_{f_1}, 1) * (1, \Sigma_{f_2,f_3})) .$$

For pseudonatural transformations and modifications, the arguments are entirely analogous to the two-variable case. □

2.5 $\mathcal{V}$-enriched monad theory

Recall from enriched category theory that there is a $\mathcal{V}$-functor $\mathrm{Hom}_{\mathcal{L}}\colon \mathcal{L}^{\mathrm{op}} \otimes \mathcal{L} \to \mathcal{V}$, which sends and object (M, N) in the tensor product of $\mathcal{V}$-categories $\mathcal{L}^{\mathrm{op}} \otimes \mathcal{L}$ to the hom object $\mathcal{L}(M, N)$ in $\mathcal{V}$. As is common, the corresponding partial $\mathcal{V}$-functors are denoted $\mathcal{L}(M, -)$ and $\mathcal{L}(-, N)$ with hom morphisms determined by the equations

$$e^{\mathcal{L}(M,N)}_{\mathcal{L}(M,N')}(\mathcal{L}(M, -)_{N,N'} \otimes 1) = M_{\mathcal{L}} \tag{15}$$

and

$$e^{\mathcal{L}(M',N)}_{\mathcal{L}(M,N)}(\mathcal{L}(-, N)_{M,M'} \otimes 1) = M_{\mathcal{L}}c \, . \tag{16}$$

For an element $f\colon N \to N'$ in $\mathcal{L}(N,N')$ i.e. a morphism in the underlying category of $\mathcal{L}$, we usually denote by $\mathcal{L}(M,f)\colon \mathcal{L}(M,N) \to \mathcal{L}(M,N')$ the morphism in $\mathcal{V}$ corresponding to the image of f under the underlying functor of $\mathcal{L}(M,-)$ with respect to the identification $\mathcal{V}_0(I,[X,Y]) \cong \mathcal{V}_0(X,Y)$ induced from the closed structure of $\mathcal{V}$ for objects $X, Y \in \mathcal{V}$. Also note that we will occasionally write $\mathcal{L}(1,f)$ instead e.g. if the functoriality of

$$\hom_{\mathcal{L}}\colon \mathcal{L}_0^{\mathrm{op}} \times \mathcal{L}_0 \longrightarrow (\mathcal{L}^{\mathrm{op}} \otimes \mathcal{L})_0 \xrightarrow{(\mathrm{Hom}_{\mathcal{L}})_0} \mathcal{V}_0 \tag{17}$$

is to be emphasized, where the first arrow is the canonical comparison functor [9]. This notation is obviously extended in the case that $\mathcal{V} = \mathcal{G}ray$, e.g. given a 1-cell $\alpha\colon f \to g$ in the 2-category $\mathcal{L}(N,N') \cong [I, \mathcal{L}(N,N')]$, then $\mathcal{L}(M,\alpha)$ denotes the pseudonatural transformation $\mathcal{L}(M,f) \Rightarrow \mathcal{L}(M,g)$ given by $\mathcal{L}(M,-)_{N,N'}(\alpha)$ i.e. the 1-cell in $[\mathcal{L}(M,N),\mathcal{L}(M,N')]$.

Recall that the 2-category $\mathcal{V}\text{-}\mathcal{CAT}$ of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations is a symmetric monoidal 2-category with monoidal structure the tensor product of $\mathcal{V}$-categories and unit object the unit $\mathcal{V}$-category $\mathcal{I}$ with a single object 0 and hom object I. Recall that the 2-functor $(-)_0 = \mathcal{V}\text{-}\mathcal{CAT}(\mathcal{I},-)\colon \mathcal{V}\text{-}\mathcal{CAT} \to \mathcal{CAT}$ sends a $\mathcal{V}$-category to its underlying category, a $\mathcal{V}$-functor to its underlying functor, and a $\mathcal{V}$-natural transformation to its underlying natural transformation.

Let T be a $\mathcal{V}$-monad on a $\mathcal{V}$-category $\mathcal{M}$. Recall that this means that T is a monad in the 2-category $\mathcal{V}\text{-}\mathcal{CAT}$. Thus T is a $\mathcal{V}$-functor $\mathcal{M} \to \mathcal{M}$, and its multiplication and unit are $\mathcal{V}$-natural transformations $\mu\colon TT \Rightarrow T$ and $\eta\colon 1_{\mathcal{M}} \Rightarrow T$ respectively such that

$$\mu(\mu T) = \mu(T\mu) \qquad \text{and} \qquad \mu(\eta T) = 1_T = \mu(T\eta)\,, \tag{18}$$

where μT and $T\mu$ are as usual the $\mathcal{V}$-natural transformations with component

$$\mu_{TM}\colon I \to \mathcal{M}(TTTM, TM) \quad \text{and} \quad T_{TTM,M}\mu_M\colon I \to \mathcal{M}(TTTM, TM)$$

respectively at the object $M \in \mathcal{M}$, and similarly for ηT and $T\eta$.

Under the assumption that $\mathcal{V}$ has equalizers e.g. if $\mathcal{V}$ is complete, the Eilenberg-Moore object $\mathcal{M}^T$ exists and has an explicit description, on which we expand below. For now, recall that the Eilenberg-Moore object is formally characterized by the existence of an isomorphism

$$\mathcal{V}\text{-}\mathcal{C}at(\mathcal{K}, \mathcal{M}^T) \cong \mathcal{V}\text{-}\mathcal{C}at(\mathcal{K}, \mathcal{M})^{T_*} \tag{19}$$

of categories which is $\mathcal{C}at$-natural in $\mathcal{K}$ and where T_* is the ordinary monad induced by composition with T.

In particular, putting $\mathcal{K} = \mathcal{I}$ shows that the underlying category $\mathcal{M}^T$ of the Eilenberg-Moore object in $\mathcal{V}\text{-}\mathcal{CAT}$ is isomorphic to the Eilenberg-Moore object for the underlying monad T_0 on the underlying category $\mathcal{M}_0$ of $\mathcal{M}$.

Thus an object of $\mathcal{M}^T$ i.e. a T-algebra is the same thing as a T_0 algebra. This means, that it is given by a pair (A,a) where A is an object of $\mathcal{M}$ and a is an element $I \to \mathcal{M}(TA,A)$ such that the two algebra axioms hold true:

$$M_{\mathcal{M}}(a, T_{TA,A}a) = M_{\mathcal{M}}(a, \mu_A) \qquad \text{and} \qquad 1_A = M_{\mathcal{M}}(a, \eta_A)\,. \tag{20}$$

Here, the notation is already suggestive for the situation for $\mathcal{V} = \mathcal{G}ray$. Namely, $(a, T_{TA,A}a)$ is considered as an element of the underlying set $V(\mathcal{M}(TA,A) \otimes \mathcal{M}(TTA,TA))$, and we apply the underlying function $VM_{\mathcal{M}}$ to this, where V is usually dropped because for $\mathcal{V} = \mathcal{G}ray$ the equations in (20) make sense as equations of the values of strict functors on objects in the Gray product.

Given T-algebras (A, a) and (B, b), the hom object of $\mathcal{M}^T$ is given by the following equalizer

$$\mathcal{M}^T((A,a),(B,b)) \xrightarrow{(U^T)_{(A,a),(B,b)}} \mathcal{M}(A,B) \underset{\mathcal{M}(1,b)T_{A,B}}{\overset{\mathcal{M}(a,1)}{\rightrightarrows}} \mathcal{M}(TA,B) \ . \tag{21}$$

In fact, it is not hard to show that the composition law $M_{\mathcal{M}}$ and the units j_A of $\mathcal{M}$ induce a $\mathcal{V}$-category structure on $\mathcal{M}^T$ such that U^T is a faithful $\mathcal{V}$-functor $\mathcal{M}^T \to \mathcal{M}$, which we call the forgetful functor. The explicit arguments may be found in [13].

In the case that $\mathcal{V} = \mathcal{G}ray$ and $\mathcal{K}$ is a $\mathcal{G}ray$-category with a $\mathcal{V}$-monad T on it, Gurski identifies $\mathcal{K}^T$ explicitly in [7, 13.1]. This is also what the equalizer description gives when it is spelled out:

Proposition 2. *The $\mathcal{G}ray$-category of algebras for a $\mathcal{G}ray$-monad T on a $\mathcal{G}ray$-category K, i.e. the Eilenberg-Moore object $\mathcal{K}^T$, can be described in the following way. Objects are T-algebras: they are given by an object X in $\mathcal{K}$ and a 1-cell $x\colon TX \to X$ i.e. an object in $\mathcal{K}(TX, X)$ satisfying $M_{\mathcal{K}}(x, Tx) = M_{\mathcal{K}}(x, \mu_X)$ and $1_X = M_{\mathcal{K}}(x, \eta_X)$. These algebra axioms are abbreviated by $xTx = x\mu_X$ and $1_X = x\eta_X$ respectively.*

An algebra 1-cell $f\colon (X, x) \to (Y, y)$ is given by a 1-cell $f\colon X \to Y$ i.e. an object in $\mathcal{K}(X, Y)$ such that $M_{\mathcal{K}}(f, x) = M_{\mathcal{K}}(y, Tf)$, which is abbreviated by $fx = yTf$. An algebra 2-cell $\alpha\colon f \Rightarrow g\colon (X, x) \to (Y, y)$ is given by a 2-cell $\alpha\colon f \to g$ i.e. a 1-cell in $\mathcal{K}(X, Y)$ such that $M_{\mathcal{K}}(1_y, T\alpha) = M_{\mathcal{K}}(\alpha, 1_x)$, which is abbreviated by $1_yT\alpha = \alpha 1_x$. An algebra 3-cell $\Gamma\colon \alpha \Rrightarrow \beta\colon f \Rightarrow g\colon (X, x) \to (Y, y)$ is given by a 3-cell $\Gamma\colon \alpha \Rrightarrow \beta$ i.e. a 2-cell in $\mathcal{K}(X, Y)$ such that $M_{\mathcal{K}}(1_{1_y}, T\Gamma) = M_{\mathcal{K}}(\Gamma, 1_{1_x})$, which is abbreviated $1_{1_y}T\Gamma = \Gamma 1_{1_x}$. The compositions are induced from the $\mathcal{G}ray$-category structure of $\mathcal{K}$. □

Observe here that the common notation $xTx = x\mu_X$ for equations of (composites of) morphisms in the underlying categories has been obviously extended for $\mathcal{V} = \mathcal{G}ray$ to 2-cells and 3-cells i.e. 1- and 2-cells in the hom 2-categories, where juxtaposition now denotes application of the composition law of $\mathcal{K}$, and the axioms for algebra 2- and 3-cells are whiskered equations with respect to this composition on 2-cells and 3-cells in $\mathcal{K}$.

3 The $\mathcal{G}ray$-category Ps-T-Alg of pseudo algebras

Let again T be a $\mathcal{G}ray$-monad on a $\mathcal{G}ray$-category $\mathcal{K}$. Since the underlying category $2\mathcal{C}at$ of the $\mathcal{G}ray$-category $\mathcal{G}ray$ is complete, it has equalizers in particular, so we have a convenient description of the $\mathcal{G}ray$-category $\mathcal{K}^T$ of T-algebras in terms of equalizers as in 2.5.

Recall that for enrichment in $\mathcal{C}at$, there is a pseudo and a lax version of the 2-category of algebras with obvious inclusions of the stricter into the laxer ones respectively. Under suitable conditions on the monad and its (co)domain, there are two coherence results relating those different kinds of algebras. First, each of the inclusions has a left adjoint. Second, each component of the unit of the adjunction is an internal equivalence. The primary references for these results are [2] and [11]. In particular, in the second, Lack provides an analysis of the coherence problem by use of codescent objects. In the case of enrichment in $\mathcal{G}ray$, there are partial results along these lines by Power [16], and a local version of the identification of pseudo notions for the monad (1) from the Introduction with tricategorical structures is mentioned in [16, Ex. 3.5, p. 319]. A perspective similar to Lack's treatment is given by Gurski in [7, Part III].

For a $\mathcal{G}ray$-monad T on a $\mathcal{G}ray$-category $\mathcal{K}$, Gurski gives a definition of lax algebras, lax functors of lax algebras, transformations of lax functors, and modifications of those, and shows that these assemble into a $\mathcal{G}ray$-category Lax-T-Alg. Further, he defines pseudo algebras, pseudo functors of pseudo algebras, and shows that these, together with transformations of pseudo functors and modifications of those, form a $\mathcal{G}ray$-category Ps-T-Alg, which embeds as a locally full sub-$\mathcal{G}ray$-category in the $\mathcal{G}ray$-category Lax-T-Alg of lax algebras. Finally, there is an obvious 2-locally full inclusion of the $\mathcal{G}ray$-category $\mathcal{K}^T$ of algebras into Ps-T-Alg and Lax-T-Alg.

3.1 Definitions and two identities

We reproduce here Gurski's definition of Ps-T-Alg in equational form. In Section 6 we will identify this $\mathcal{G}ray$-category for a particular monad on the functor $\mathcal{G}ray$-category $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ where $\mathcal{P}$ is a small and $\mathcal{L}$ is a cocomplete $\mathcal{G}ray$-category. Namely, we show that it is isomorphic as a $\mathcal{G}ray$-category to the full sub-$\mathcal{G}ray$-category $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ determined by the locally strict trihomomorphisms.

Definition 1. [7, Def. 13.4, Def. 13.8] A pseudo T-algebra consists of

- an object X of $\mathcal{K}$;
- a 1-cell $x\colon TX \to X$ i.e. an object in $\mathcal{K}(TX, X)$;
- 2-cell adjoint equivalences[1] $(m, m^\bullet)\colon M_{\mathcal{K}}(x, Tx) \to M_{\mathcal{K}}(x, \mu_X)$ or abbreviated $(m, m^\bullet)\colon xTx \to x\mu_X$ and $(i, i^\bullet)\colon 1_X \to M_{\mathcal{K}}(x, \eta_X)$ or abbreviated $(i, i^\bullet)\colon 1 \to x\eta_X$ i.e. 1-cells in $\mathcal{K}(T^2X, X)$ and $\mathcal{K}(X, X)$ respectively which are adjoint equivalences;
- and three invertible 3-cells π, λ, μ as in **(PSA1)**-**(PSA3)** subject to the four axioms **(LAA1)**-**(LAA4)** of a lax T-algebra:

(PSA1) An invertible 3-cell π given by an invertible 2-cell in $\mathcal{K}(T^3X, X)$:

$$\pi\colon (m1_{\mu_{TX}}) * (m1_{T^2x}) \Rightarrow (m1_{T\mu_X}) * (1_x Tm)\ ,$$

which is shorthand for

$$\pi\colon M_{\mathcal{K}}(m, 1_{\mu_{TX}}) * M_{\mathcal{K}}(m, 1_{T^2x}) \Rightarrow M_{\mathcal{K}}(m, 1_{T\mu_X}) * M_{\mathcal{K}}(1_x, Tm)\ ;$$

where the horizontal factors on the left compose due to $\mathcal{G}ray$-naturality of μ and the codomains match by the monad axiom $\mu(\mu T) = \mu(T\mu)$.

(PSA2) An invertible 3-cell λ given by an invertible 2-cell in $\mathcal{K}(TX, X)$:

$$\lambda\colon (m1_{\eta_{TX}}) * (i1_x) \Rightarrow 1_x\ ,$$

[1] For adjunctions in a 2-category see [10, §2.].

which is shorthand for

$$\lambda\colon\ M_{\mathcal{K}}(m, 1_{\eta_{TX}}) * M_{\mathcal{K}}(i, 1_x) \Rightarrow 1_x\ ,$$

where the horizontal factors compose due to $\mathcal{G}ray$-naturality of η and the codomains match by the monad axiom $\mu(\eta T) = 1_T$.

(PSA3) An invertible 3-cell ρ given by an invertible 2-cell in $\mathcal{K}(TX, X)$:

$$\rho\colon\ (m1_{T\eta_X}) * (1_x Ti) \Rightarrow 1_x\ ,$$

which is shorthand for

$$\rho\colon\ M_{\mathcal{K}}(m, 1_{T\eta_X}) * M_{\mathcal{K}}(1_x, Ti) \Rightarrow 1_x\ ,$$

where the codomains match by the monad axiom $\mu(T\eta) = 1_T$.

The four lax algebra axioms are:

(LAA1) The following equation in $\mathcal{K}(T^4X, X)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&((\pi 1) * 1_{11T^2m}) \diamond (1_{m11} * \Sigma^{-1}_{m,T^2m}) \diamond ((\pi 1) * 1_{m11})\\ &= (1_{m11} * (1T\pi)) \diamond ((\pi 1) * 1_{1Tm1}) \diamond (1_{m11} * (\pi 1))\ ,\end{aligned}$$

where Σ^{-1}_{m,T^2m} is shorthand for $M_{\mathcal{K}}(\Sigma^{-1}_{m,T^2m})$. A careful inspection shows that the horizontal and vertical factors do indeed compose. Note that any mention of the object X has been omitted, e.g. $T\mu_T$ stands for $T\mu_{TX}$. We refer to this axiom as the *pentagon-like axiom* for π.

(LAA2) The following equation in $\mathcal{K}(T^2X, X)$ of vertical composites of whiskered 3-cells is required:

$$((\rho 1) * 1_{m1}) \diamond (1_{m11} * \Sigma_{m,T^2i}) = (1_{m11} * (1_x T\rho)) \diamond ((\pi 1) * 1_{11T^2i})\ .$$

(LAA3) The following equation in $\mathcal{K}(T^2X, X)$ of vertical composites of whiskered 3-cells is required:

$$1_{m11} * (\lambda 1) = ((\lambda 1) * 1_{1m}) \diamond (1_{m11} * \Sigma^{-1}_{i,m}) \diamond ((\pi 1) * 1_{i11})\ .$$

(LAA4) The following equation in $\mathcal{K}(T^4X, X)$ of vertical composites of whiskered 3-cells is required:

$$(1_{m11} * (1T\lambda)) \diamond ((\pi 1) * 1_{1Ti1}) = 1_{m11} * (\rho 1)\ .$$

We refer to this as the *triangle-like axiom* for λ, ρ, and π. Diagrams for these axioms may be found in Gurski's definition.

Remark 2. In the shorthand notation juxtaposition stands for an application of $M_{\mathcal{K}}$, an instance of a power of T in an index refers to its effect on the object X, any other instance of a power of T is shorthand for a hom 2-functor and only applies to the cell directly following it. Notice that this notation is possible due to the functor axiom for T.

This definition of a pseudo algebra is derived from the definition of a lax algebra by requiring the 2-cells m and i to be adjoint equivalences and the 3-cells π, λ, ρ to be invertible. In fact, under these circumstances we do not need all of the axioms. This is proved in the following proposition, which is central for the comparison of trihomomorphisms of $\mathcal{G}ray$-categories with pseudo algebras. Namely, there are only two axioms for a trihomomorphism, while there are four in the definition of a lax algebra. Proposition 3 shows that, in general, two of the axioms suffice for a pseudo algebra.

Proposition 3. *Given a pseudo T-algebra, the pentagon-like axiom* **(LAA1)** *and the triangle-like axiom* **(LAA4)** *imply the other two axioms* **(LAA2)**-**(LAA3)**, *i.e. these are redundant.*

Proof. We proceed analogous to Kelly's classical proof that the two corresponding axioms in MacLane's original definition of a monoidal category are redundant [8]. The associators and unitors in Kelly's proof here correspond to π, λ, and ρ. Commuting naturality squares for associators have to be replaced by instances of the middle four interchange law, and there is an additional complication due to the appearance of interchange cells – these have no counterpart in Kelly's proof, so that it is gratifying that the strategy of the proof can still be applied. We only show here the proof for the axiom **(LAA3)** involving π and λ, the one for the axiom **(LAA2)** involving π and ρ is entirely analogous.

The general idea of the proof is to transform the equation of the axiom **(LAA3)** into an equivalent form, namely (25) below, which we can manipulate by use of the pentagon-like and triangle-like axiom. This is probably easier to see from the diagrammatic form of the axioms, where it means that we adjoin

$$\begin{array}{ccc}
x1TxT^2x & \xrightarrow{1Ti11} & xTxT\eta TxT^2x \\
{\scriptstyle 11Tm}\Big\downarrow & \Downarrow \Sigma^{-1}_{1Ti,Tm} & \Big\downarrow{\scriptstyle 111Tm} \\
x1TxT\mu & \xrightarrow{1Ti11} & xTxT\eta TxT\mu \\
\Big\| & \Downarrow 1T\lambda 1 & \Big\downarrow{\scriptstyle 1Tm11} \\
xTxT\mu T\eta_T T\mu & = & xTxT\mu T^2\mu T\eta_{T^2}
\end{array}$$

to the right hand side of some image of the pentagon-like axiom **(LAA1)**, and then a diagram equivalent to the right hand side of **(LAA3)** can be identified as a subdiagram of this.

Since $i\colon 1_X \to x\eta_X$ is an equivalence in $\mathcal{L}(X, X)$, and since

$$\mathcal{L}(X', 1_X)\colon\ \mathcal{L}(X', X) \to \mathcal{L}(X', X)$$

is the identity for arbitrary $X' \in \mathrm{ob}\mathcal{L}$, we have that $\mathcal{L}(X', x\eta_X)$ is equivalent to the identity functor. In particular, it is 2-locally fully faithful i.e. a bijection on the sets of 2-cells. On the other hand, by naturality of η we have:

$$\mathcal{L}(X', x\eta_X) = \mathcal{L}(X', x)\mathcal{L}(X', \eta_X) = \mathcal{L}(X', x)\mathcal{L}(\eta_{X'}, TX)T_{,}\ ,$$

where the subscript of T on the right indicates a hom morphism of T. This means that the equation of **(LAA3)** is equivalent to its image under

$$\mathcal{L}(T^2X, x)\mathcal{L}(\eta_{T^2X}, TX)T_{,} = \mathcal{L}(\eta_{T^2X}, X)\mathcal{L}(T^3X, x)T_{,}\ ,$$

where we have used the underlying functoriality of $\hom_{\mathcal{L}}$. We will actually show that the image of the equation under $\mathcal{L}(T^3X, x)T_{,}$ holds, which of course implies that the image of the equation under $\mathcal{L}(\eta_{T^2X}, X)\mathcal{L}(T^3X, x)T_{,}$ holds.

Applying $\mathcal{L}(T^3X, x)T_{,}$ to the lax algebra axiom **(LAA3)** gives

$$1_{1Tm11} * (1T\lambda 1) = ((1T\lambda 1) * 1_{11Tm}) \diamond (1_{1Tm11} * (1T\Sigma^{-1}_{i,m})) \diamond ((1T\pi 1) * 1_{1Ti11}) \,. \qquad (22)$$

Observe that since $\Sigma^{-1}_{i,m}$ is shorthand for $M_{\mathcal{L}}(\Sigma^{-1}_{i,m})$, we have $1T\Sigma^{-1}_{i,m} = 1\Sigma^{-1}_{Ti,Tm}$ by the functor axiom for T and equation (4) from 2.3, which is in fact shorthand for

$$\begin{aligned} M_{\mathcal{L}}((1, M_{\mathcal{L}}(\Sigma^{-1}_{Ti,Tm}))) &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(a^{-1}(1, \Sigma^{-1}_{Ti,Tm})) && \text{(by a } \mathcal{G}ray\text{-category axiom)} \\ &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(\Sigma^{-1}_{(1,Ti),Tm}) && \text{(by eq. (7) from 2.3)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{1Ti,Tm}) \,, && \text{(by eq. (4) from 2.3)} \end{aligned}$$

for which the corresponding shorthand is just $\Sigma^{-1}_{1Ti,Tm}$.

Next, equation (22) is clearly equivalent to the one whiskered with $m111$ on the left because $m111$ is an (adjoint) equivalence by the definition of m, i.e. to

$$\begin{aligned} &1_{m111} * 1_{1Tm11} * (1T\lambda 1) \\ &= (1_{m111} * (1T\lambda 1) * 1_{11Tm}) \diamond (1_{m111} * 1_{1Tm11} * (\Sigma^{-1}_{1Ti,Tm})) \diamond (1_{m111} * (1T\pi 1) * 1_{1Ti11}) \,. \qquad (23) \end{aligned}$$

Here we have used functoriality of $*$ i.e. the middle four interchange law, and it is understood that because horizontal composition is associative, we can drop parentheses.

Now observe that $1_{m111} * (1T\pi 1)$ is the image under $\mathcal{L}(T\eta_{T^2X}, X)$ of the leftmost vertical factor in the right hand side of the pentagon-like axiom **(LAA1)** for π. Namely, the image of **(LAA1)** under $\mathcal{L}(T\eta_{T^2X}, X)$ is

$$\begin{aligned} &((\pi 11) * 1_{11T^2m1}) \diamond (1_{m111} * (\Sigma^{-1}_{m,T^2m}1)) \diamond ((\pi 11) * 1_{m111}) \\ &= (1_{m111} * (1T\pi 1)) \diamond ((\pi 11) * 1_{1Tm11}) \diamond (1_{m111} * (\pi 11)) \,, \end{aligned}$$

where $\Sigma^{-1}_{m,T^2m}1$ is shorthand for

$$\begin{aligned} M_{\mathcal{L}}(M_{\mathcal{L}}(\Sigma^{-1}_{m,T^2m}), 1) &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(\Sigma^{-1}_{m,T^2m}, 1) && \text{(by definition of } (M_{\mathcal{L}} \otimes 1)) \\ &= M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})a(\Sigma^{-1}_{m,T^2m}, 1) && \text{(by a } \mathcal{G}ray\text{-category axiom)} \\ &= M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(\Sigma^{-1}_{m,(T^2m,1)}) && \text{(by eq. (5) from 2.3)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{m,T^2m1}) \,, && \text{(by eq. (4) from 2.3)} \end{aligned}$$

for which the corresponding shorthand is just Σ^{-1}_{m,T^2m1}.

In fact, we have that $T^2m1 = 1Tm$ where on the left the identity is $1_{T\eta_{T^2X}}$ and on the right it is $1_{T\eta_X}$, thus this is simply naturality of $T\eta$. Hence, $\Sigma^{-1}_{m,T^2m1} = \Sigma^{-1}_{m,1Tm}$. In turn, this is shorthand for

$$\begin{aligned} M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(\Sigma^{-1}_{m,(1,Tm)}) &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)a^{-1}(\Sigma^{-1}_{(m,(1,Tm))}) && \text{(by a } \mathcal{G}ray\text{-category axiom)} \\ &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(\Sigma^{-1}_{(m,1),Tm}) && \text{(by eq. (6) from 2.3)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{m1,Tm}) \,, && \text{(by eq. (4) from 2.3)} \end{aligned}$$

for which the corresponding shorthand is just $\Sigma^{-1}_{m1,Tm}$.

Implementing these transformations, the image of the pentagon-like axiom **(LAA1)** then has the form

$$\begin{aligned}&((\pi 11) * 1_{111Tm}) \diamond (1_{m111} * \Sigma^{-1}_{m1,Tm}) \diamond ((\pi 11) * 1_{m111})\\ &= (1_{m111} * (1T\pi 1)) \diamond ((\pi 11) * 1_{1Tm11}) \diamond (1_{m111} * (\pi 11)) \,. \end{aligned} \tag{24}$$

Since they are invertible, composing equation (23) with the other two factors from the pentagon-like axiom for π whiskered with the (adjoint) equivalence $1Ti11$ on the right, gives an equivalent equation. Thus, our goal is now to prove the following equation:

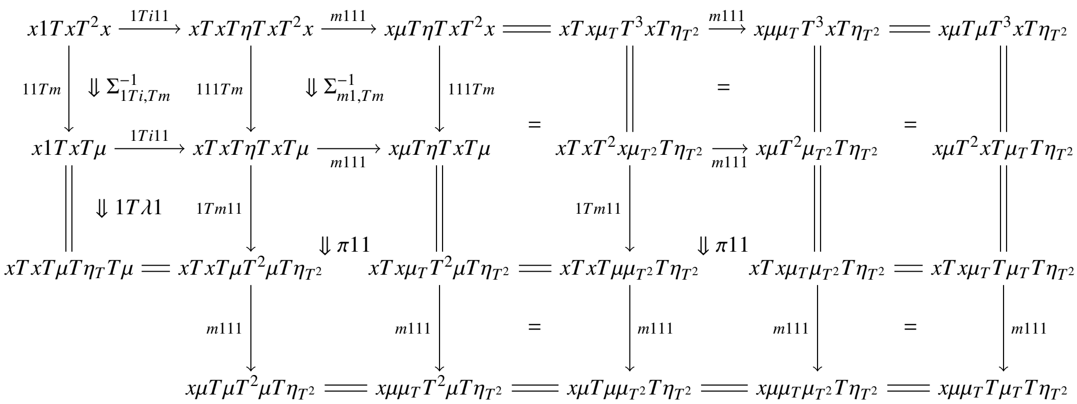
$$\begin{aligned}&(1_{m111} * 1_{1Tm11} * (1T\lambda 1)) \diamond ((\pi 11) * 1_{1Tm11} * 1_{1Ti11}) \diamond (1_{m111} * (\pi 11) * 1_{1Ti11})\\ &= (1_{m111} * (1T\lambda 1) * 1_{11Tm}) \diamond (1_{m111} * 1_{1Tm11} * (\Sigma^{-1}_{1Ti,Tm}))\\ &\quad \diamond \Big(\big((1_{m111} * (1T\pi 1)) \diamond ((\pi 11) * 1_{1Tm11}) \diamond (1_{m111} * (\pi 11))\big) * 1_{1Ti11}\Big) \,. \end{aligned} \tag{25}$$

This is proved by transforming the right hand side by use of the pentagon-like and triangle-like axiom until we finally obtain the left hand side.

Namely, using the image of the pentagon-like axiom for π in the form (24) above, the right hand side of (25) is equal to

$$\begin{aligned}&(1_{m111} * (1T\lambda 1) * 1_{11Tm}) \diamond (1_{m111} * 1_{1Tm11} * (\Sigma^{-1}_{1Ti,Tm}))\\ &\quad \diamond \Big(\big(((\pi 11) * 1_{111Tm}) \diamond (1_{m111} * \Sigma^{-1}_{m1,Tm}) \diamond ((\pi 11) * 1_{m111})\big) * 1_{1Ti11}\Big) \,.\end{aligned}$$

The diagrammatic form of this is drawn below.



The rectangle composed of the two interchange cells is shorthand for

$$M_{\mathcal{L}}((1_{m1}, 1) * \Sigma^{-1}_{1Ti,Tm}) \diamond M_{\mathcal{L}}(\Sigma^{-1}_{m1,Tm} * (1_{1Ti}, 1)) = M_{\mathcal{L}}(\Sigma_{(m1)*(1Ti),Tm}) \,,$$

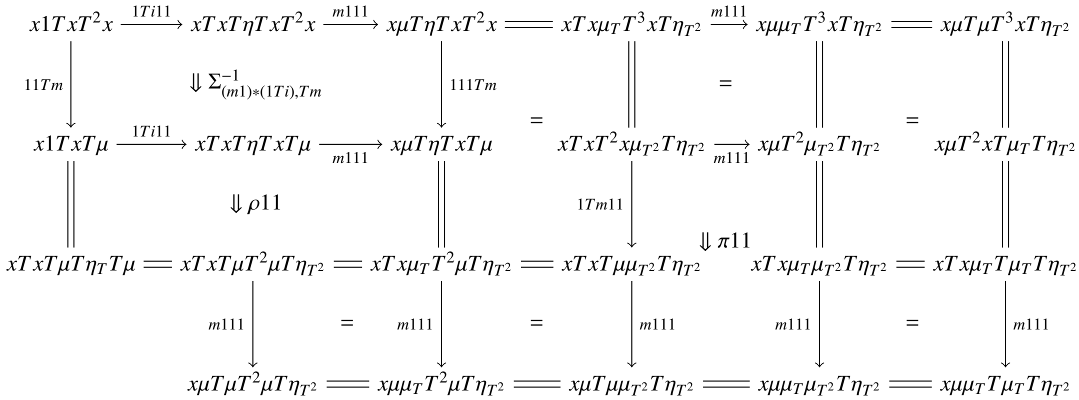
for which the corresponding shorthand is just $\Sigma_{(m1)*(1Ti),Tm}$. Notice that we made use here of the image under $M_{\mathcal{L}}$ of the Gray product relation

$$(\Sigma_{f',g} * (1_f, 1)) \diamond ((1_{f'}, 1) * \Sigma_{f,g}) \sim \Sigma_{f' * f,g} \,.$$

Next, the subdiagram formed by $1T\lambda 1$ and $\pi 11$ may be transformed by use of the image under $\mathcal{L}(T\mu, X)$ of the triangle-like axiom **(LAA3)**:

$$(1_{m111} * (1T\lambda 1)) \diamond ((\pi 11) * 1_{1Ti1}) = 1_{m111} * (\rho 11)\ .$$

Implementing these transformations in the diagram, gives the one drawn below.



The subdiagram formed by the interchange cell and $\rho 11$ is

$$((\rho 11) * 1_{11Tm}) \diamond \Sigma^{-1}_{(m1)*(1Ti),Tm}\ .$$

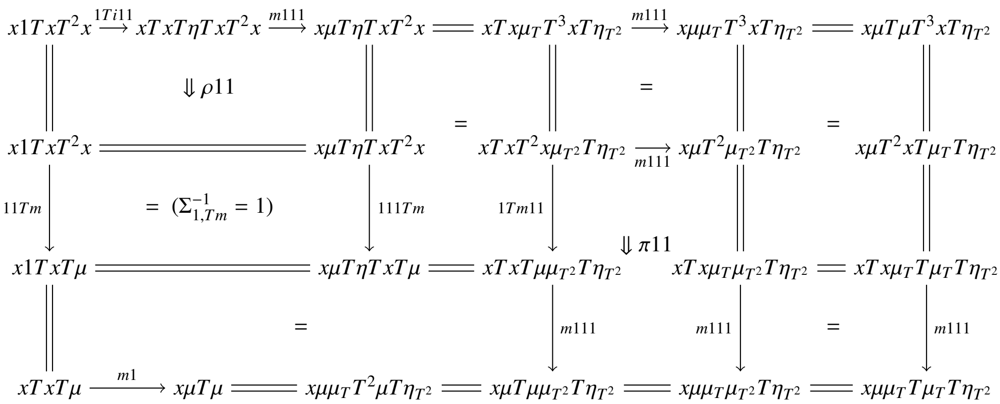
This is shorthand for

$$M_{\mathcal{L}}(((\rho,1) * (1, 1_{Tm})) \diamond \Sigma^{-1}_{(m1)*(1Ti),Tm}) = M_{\mathcal{L}}(\Sigma^{-1}_{1,Tm} \diamond ((1, 1_{Tm}) * (\rho, 1))) = M_{\mathcal{L}}((1, 1_{Tm}) * (\rho, 1))$$

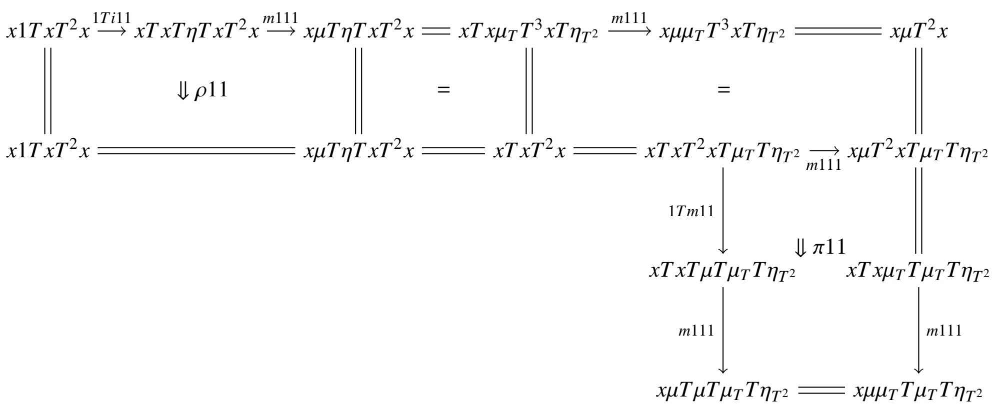
or $1_{111Tm} * (\rho 11)$, where we have used the relation

$$((1,\beta) * (\alpha, 1)) \diamond \Sigma_{f,g} \sim \Sigma_{f',g'} \diamond ((\alpha, 1) * (1,\beta)) \tag{26}$$

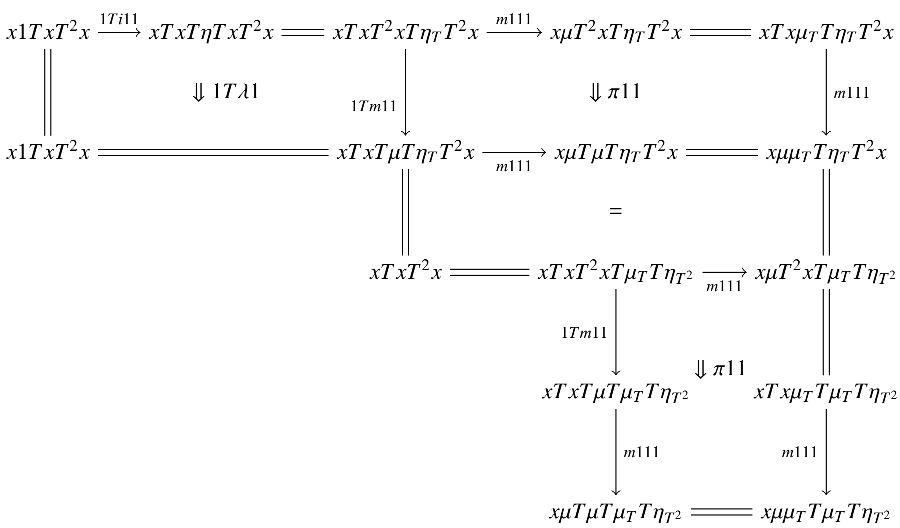
for interchange cells in the Gray product and the fact that $\Sigma^{-1}_{1,Tm}$ is the identity 2-cell cf. 2.3. This means we now have the following diagram.



Slightly rewritten, this is the same as the diagram below.



For the upper right entry we have used another identity to make commutativity obvious. Finally, by another instance of the triangle identity—in fact its image under $\mathcal{L}(T^2x, X)$—we end up with the diagram below. It is easily seen to be the diagrammatic form of the left hand side of (25), so this ends the proof.



□

Definition 2. [7, Def. 13.6 and Def. 13.9] A pseudo T-functor

$$(f, F, \hbar, m)\colon\ (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$$

consists of

- a 1-cell $f\colon X \to Y$ in $\mathcal{K}$ i.e. an object of $\mathcal{K}(X,Y)$;
- a 2-cell adjoint equivalence $(F, F^\bullet)\colon fx \to yTf$ i.e. a 1-cell adjoint equivalence internal to $\mathcal{K}(TX,Y)$;
- and two invertible 3-cells $\mathcal{h}, \mathcal{m}$ as in **(PSF1)**-**(PSF2)** subject to the three axioms **(LFA1)**-**(LFA3)** of a lax T-functor:

(PSF1) An invertible 3-cell $\mathcal{h}$ given by an invertible 2-cell in $\mathcal{K}(X,Y)$:

$$\mathcal{h}\colon (F1_{\eta_X}) * (1_f i^X) \Rightarrow (i^Y 1_f)$$

where the codomains match by $\mathcal{G}ray$-naturality of η.
(PSF2) An invertible 3-cell $\mathcal{m}$ given by an invertible 2-cell in $\mathcal{K}(T^2X,Y)$:

$$\mathcal{m}\colon (m^Y 1_{T^2 f}) * (1_y TF) * (F1_x) \Rightarrow (F1_\mu) * (1_f m)$$

where the codomains match by $\mathcal{G}ray$-naturality of μ.
The three lax T-functor axioms are:
(LFA1) The following equation in $\mathcal{K}(T^3X,Y)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&(1_{F11} * (1\pi^X)) \diamond ((\mathcal{m}1) * 1_{1m^X1}) \diamond (1_{m^Y11} * 1_{11T^2F} * (\mathcal{m}1)) \diamond (1_{m^Y11} * \Sigma_{m^Y,T^2F} * 1_{1TF1} * 1_{F11})\\ &= ((\mathcal{m}1) * 1_{11Tm^X}) \diamond (1_{m^Y11} * 1_{1TF1} * \Sigma^{-1}_{F,Tm^X}) \diamond (1T\mathcal{m}) \diamond ((\pi^Y 1) * 1_{11T^2F} * 1_{TF1} * 1_{F11})\,.\end{aligned}$$

(LFA2) The following equation in $\mathcal{K}(TX,Y)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&((\lambda^Y 1) * 1_{1F}) \diamond (1_{m^Y11} * \Sigma^{-1}_{i^Y,F}) \diamond (1_{m^Y11} * 1_{11F} * (\mathcal{h}1))\\ &= (1_F * (1\lambda^Y)) \diamond ((\mathcal{m}1) * 1_{1i^X1})\,.\end{aligned}$$

(LFA3) The following equation in $\mathcal{K}(TX,Y)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&((\rho^Y 1) * 1_{F1}) \diamond (1_{m^Y11} * (1T\mathcal{h}) * 1_{F1})\\ &= (1_F * (1\rho^X)) \diamond ((\mathcal{m}1) * 1_{1Ti^X}) \diamond (1_{m^Y1} * 1_{1TF1} * \Sigma^{-1}_{F,Ti^X})\,.\end{aligned}$$

A careful inspection shows that the horizontal and vertical factors do indeed compose in all of these axioms. Diagrams may be found in Gurski's definition.

Definition 3. [7, Def. 13.6 and Def. 13.10] A T-transformation

$$(\alpha, A)\colon (f, F, \mathcal{h}^f, \mathcal{m}^f) \Rightarrow (g, G, \mathcal{h}^g, \mathcal{m}^g)\colon (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$$

consists of

- a 2-cell $\alpha\colon f \Rightarrow g$ i.e. an object of $\mathcal{K}(X,Y)$;

- an invertible 3-cell A as in **(T1)** subject to the two axioms **(LTA1)**-**(LTA2)** of a lax T-algebra:

(T1) An invertible 3-cell A given by an invertible 2-cell in $\mathcal{K}(TX, Y)$:

$$A\colon\ (1_y T\alpha) * F \Rightarrow G * (\alpha 1_x)\ .$$

The two lax T-transformation axioms are:
(LTA1) The following equation in $\mathcal{K}(X, Y)$ of vertical composites of whiskered 3-cells is required:

$$(\hbar^g * 1_{\alpha 1}) \diamond (1_{G1} * \Sigma_{\alpha, i^X}) \diamond ((A1) * 1_{1 i^X}) = \Sigma^{-1}_{i^Y, \alpha} \diamond (1_{1T\alpha 1} * \hbar^f)\ .$$

(LTA2) The following equation in $\mathcal{K}(T^2X, X)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&(m^g * 1_{\alpha 1}) \diamond (1_{m^Y 1} * 1_{1TG} * (A1)) \diamond (1_{m^Y 1} * (1TA) * 1_{F1}) \diamond (\Sigma^{-1}_{m^Y, T^2\alpha} * 1_{1TF} * 1_{F1})\\ &= (1_{G1} * \Sigma_{\alpha, m^X}) \diamond ((A1) * 1_{1m^X}) \diamond (1_{1T^2\alpha} * m^f)\ .\end{aligned}$$

A careful inspection shows that the horizontal and vertical factors do indeed compose in the two axioms. Diagrams may be found in Gurski's definition.

Definition 4. A T-modification $\Gamma\colon\ (\alpha, A) \Rrightarrow (\beta, B)$ of T-transformations

$$(f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)\colon\ (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$$

consists of a

- 3-cell $\Gamma\colon\ \alpha \Rrightarrow \beta$ i.e. a 2-cell in $\mathcal{K}(X, Y)$;
- subject to one axiom **(MA1)**:

(MA1) The following equation in $\mathcal{K}(TX, Y)$ of vertical composites of whiskered 3-cells is required:

$$B \diamond ((1T\Gamma) * 1_F) = (1_G * (\Gamma 1)) \diamond A\ .$$

Finally, we provide the $\mathcal{G}ray$-category structure of Ps-T-Alg. We begin with its hom 2-categories.

Definition 5. Given T-algebras $(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X)$ and $(Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$, the prescriptions below give the 2-globular set Ps-T-Alg(X, Y) whose objects are pseudo T-functors from X to Y, whose 1-cells are T-transformations between pseudo T-functors, and whose 2-cells are T-modifications between those, the structure of a 2-category [7, Prop. 13.11].

Given T-modifications $\Gamma\colon\ (\alpha, A) \Rrightarrow (\beta, B)$ and $\Delta\colon\ (\beta, B) \Rrightarrow (\epsilon, E)$ of T-transformations

$$(f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)\colon\ (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)\ ,$$

their vertical composite $\Delta \diamond \Gamma$ is defined by the vertical composite $\Delta \diamond \Gamma$ of 2-cells in $\mathcal{K}(X, Y)$. The identity T-modification of (α, A) as above is defined by the 2-cell 1_α in $\mathcal{K}(X, Y)$.

Given T-transformations $(\alpha, A)\colon (f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)$ and $(\beta, B)\colon (g, G, \hbar^g, m^g) \Rightarrow (h, H, \hbar^h, m^h)$ of pseudo T-functors

$$(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)\ ,$$

their horizontal composite $(\beta, B) * (\alpha, A)$ is defined by

$$(\beta * \alpha, (B * 1_{\alpha 1}) \diamond (1_{1T\beta} A))\ .$$

The identity T-transformation of $(f, F, \hbar^f, m^f)$ is defined by $(1_f, 1_F)$.

Given T-modifications $\Gamma\colon (\alpha, A) \Rrightarrow (\alpha', A')\colon (f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)$ and $\Delta\colon (\beta, B) \Rrightarrow (\beta', B')\colon (g, G, \hbar^g, m^g) \Rightarrow (h, H, \hbar^h, m^h$ of pseudo T-functors

$$(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)\ ,$$

their horizontal composite is defined by the horizontal composite $\Delta * \Gamma$ of 2-cells in $\mathcal{K}(X, Y)$.

We omit the proof that this is indeed a 2-category.

Definition 6. The prescriptions below give the set of pseudo T-algebras with the hom 2-categories from the proposition above, the structure of a $\mathcal{G}ray$-category denoted Ps-T-Alg, see [7, Prop. 13.12 and Th. 13.13].

Given pseudo T-algebras $(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X)$, $(Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$ and $(Z, z, m^Z, i^Z, \pi^Z, \lambda^Z, \rho^Z)$, the composition law is defined by the strict functor

$$\boxtimes\colon \text{Ps-}T\text{-Alg}(Y, Z) \otimes \text{Ps-}T\text{-Alg}(X, Y) \to \text{Ps-}T\text{-Alg}(X, Z)$$

specified as follows.

On an object (g, f) in Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y) i.e. on functors $(g, G, \hbar^g, m^g)$ and $(f, F, \hbar^f, m^f)$, $\boxtimes$ is defined by

$$\Big(gf, (G1_{Tf}) * (1_g F), (\hbar^g 1) \diamond (1_{G11} * (1\hbar^f)),$$
$$(1_{G11} * (1m^f)) \diamond ((m^g 1) * 1_{11TF} * 1_{G11} * 1_{1F1}) \diamond (1_{m^Y 11} * 1_{1TG1} * \Sigma^{-1}_{G,TF} * 1_{1F1})\Big)\ ,$$

which we denote by $g \boxtimes f$.

On a generating 1-cell of the form $((\alpha, A), 1)\colon (g, f) \to (g', f)$ in the Gray product Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y), where (α, A) is a T-transformation

$$(g, G, \hbar^g, m^g) \Rightarrow (g', G', \hbar^{g'}, m^{g'}),$$

and f is as above, $\boxtimes$ is defined by

$$(\alpha 1_f, (1_{G'1} * \Sigma_{\alpha,F}) \diamond ((A1) * 1_{1F}))$$

and denoted $\alpha \boxtimes 1_f$.

Similarly, on a generating 1-cell of the form $(1, (\beta, B))\colon (g, f) \to (g, f')$ in the Gray product Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y), where (β, B) is a T-transformation

$$(f, F, \hbar^f, m^f) \Rightarrow (f', F', \hbar^{f'}, m^{f'}),$$

and g is as above, ⊠ is defined by

$$(1_g\beta, (1_{G1} * (1B)) \diamond (\Sigma^{-1}_{G,\beta} * 1_{1F}))$$

and denoted $\alpha \boxtimes 1_f$.

On a generating 2-cell of the form $(\Gamma, 1)\colon ((\alpha, A), 1) \Rrightarrow ((\alpha', A'), 1)$, ⊠ is defined by the underlying 2-cell $\Gamma 1_f$ in $\mathcal{K}(X, Y)$ and denoted by $\Gamma \boxtimes 1$, and similarly for 2-cells of the form $(1, \Delta)\colon (1, (\beta, B)) \Rrightarrow (1, (\beta', B'))$.

Finally, on an interchange cell $\Sigma_{(\alpha,A),(\beta,B)}$ in Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y), ⊠ is defined by the 2-cell $M_{\mathcal{K}}(\Sigma_{\alpha,\beta})$, the shorthand of which is $\Sigma_{\alpha,\beta}$.

The unit at an object $(X, x, m, i, \pi, \lambda, \rho)$, that is, the functor $j_X\colon I \to$ Ps-T-Alg(X, X) is determined by strictness and the requirement that it sends the unique object $*$ of I to the T-functor $(1_X, 1_x, 1_i, 1_m)$.

We omit the proof that this is well-defined and that Ps-T-Alg is indeed a $\mathcal{G}ray$-category.

3.2 Coherence via codescent

Recall from [9, 3.1] that given a complete and cocomplete locally small symmetric monoidal closed category $\mathcal{V}$, $\mathcal{V}$-categories $\mathcal{K}$ and $\mathcal{B}$, and $\mathcal{V}$-functors $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ and $\mathcal{G}\colon \mathcal{K} \to \mathcal{B}$, the colimit of G indexed by F is a representation $(F * G, \nu)$ of the $\mathcal{V}$-functor $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F-, \mathcal{B}(G-, ?))\colon \mathcal{B} \to \mathcal{V}$ (where this is assumed to exist) with representing object $F*G$ in $\mathcal{B}$ and unit $\nu\colon F \to \mathcal{B}(G-, F*G)$. For the concept of representable functors see [9, 1.10]. In particular, there is a $\mathcal{V}$-natural (in B) isomorphism

$$\mathcal{B}(F * G, B) \xrightarrow{\cong} [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F, \mathcal{B}(G-, B)) , \tag{27}$$

and the unit is obtained by Yoneda when this is composed with the unit j_{F*G} of the $\mathcal{V}$-category $\mathcal{B}$ at the object $F * G$.

Definition 7. A $\mathcal{V}$-functor $T\colon \mathcal{B} \to \mathcal{C}$ preserves the colimit of $G\colon \mathcal{K} \to \mathcal{B}$ indexed by $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ if when $(F * G, \nu)$ exists, the composite

$$T_{G-,B}\nu\colon F \to \mathcal{B}(G-, F * G) \to \mathcal{L}(TG-, T\{F, G\})$$

exhibits $T(F*G)$ as the colimit of TG indexed by F i.e. the composite corresponds under Yoneda to an isomorphism as in (27) with TG instead of G.

Now let again $\mathcal{K}$ be a $\mathcal{G}ray$-category and T be a $\mathcal{G}ray$-monad on it. Below we will make use of the following corollary of the central coherence theorem from three-dimensional monad theory [7, Corr. 15.14].

Theorem 3 (Gurski's coherence theorem)**.** *Assume that $\mathcal{K}$ has codescent objects of codescent diagrams, and that T preserves them. Then the inclusion $i\colon \mathcal{K}^T \hookrightarrow$* Ps-$T$-Alg *has a left adjoint L:* Ps-T-Alg $\to \mathcal{K}^T$ *and each component $\eta_X\colon X \to iLX$ of the unit of this adjunction is a biequivalence in* Ps-T-Alg.

Remark 3. Codescent objects are certain indexed colimits, see [7, 12.3]. In fact, they are built from co-2-inserters, co-3-inserters and coequifiers. These are classes of indexed colimits where each of these classes is determined separately by considering all indexed colimits $F * G$ with a particular fixed $\mathcal{Gray}$-functor $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$. Thus there is no other restriction on G apart from the fact that it must have the same domain as F. In particular, if T is a $\mathcal{Gray}$-monad and T preserves co-2-inserters, co-3-inserters and coequifiers, then also TT preserves co-2-inserters, co-3-inserters and coequifiers because G and TG have the same domain. This is used in the proof of the theorem to show that the Eilenberg-Moore object $\mathcal{K}^T$ has codescent objects and that they are preserved by the forgetful $\mathcal{Gray}$-functor $\mathcal{K}^T \to \mathcal{K}$. Namely, in enriched monad theory one can show that the forgetful functor $\mathcal{K}^T \to \mathcal{K}$ creates any colimit that is preserved by T and TT, just as in ordinary monad theory.

For any of the classes of indexed colimits above, the domain of F is small, so codescent objects are small indexed colimits. Hence, if $\mathcal{K}$ is cocomplete, it has codescent objects of codescent diagrams in particular. This observation gives the following corollary.

Corollary 1. *Let $\mathcal{K}$ be cocomplete and let T be a monad on $\mathcal{K}$ that preserves small indexed colimits. Then the inclusion $i\colon \mathcal{K}^T \hookrightarrow$ Ps-T-Alg has a left adjoint $L\colon$ Ps-T-Alg $\to \mathcal{K}^T$ and each component $\eta_X\colon X \to iLX$ of the unit of the adjunction is an internal biequivalence in* Ps-T-Alg. □

4 The monad of the Kan adjunction

4.1 A $\mathcal{V}$-monad on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$

Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category such that the underlying category $\mathcal{V}_0$ is locally small. By cocompleteness we have an initial object which we denote by $\emptyset$. Let $\mathcal{P}$ be a small $\mathcal{V}$-category and let $\mathcal{L}$ be a cocomplete $\mathcal{V}$-category. In this general situation, we now describe in more detail a $\mathcal{V}$-monad corresponding to the Kan adjunction with left adjoint left Kan extension Lan_H along a particular $\mathcal{V}$-functor H and right adjoint the functor $[H, 1]$ from enriched category theory. For $\mathcal{V} = \mathcal{Gray}$, this is the $\mathcal{Gray}$-monad mentioned in the introduction, for which the pseudo algebras shall be compared to locally strict trihomomorphisms.

First, observe that the set $\mathrm{ob}\mathcal{P}$ of the objects of $\mathcal{P}$ may be considered as a discrete $\mathcal{V}$-category. More precisely, there is a $\mathcal{V}$-category structure on $\mathrm{ob}\mathcal{P}$ such that for objects $P, Q \in \mathcal{P}$ the hom object $(\mathrm{ob}\mathcal{P})(P, Q)$ is given by I if $P = Q$ and by $\emptyset$ otherwise and such that the nontrivial hom morphisms are given by $l_I = r_I$. The $\mathcal{V}$-functor H is defined to be the unique $\mathcal{V}$-functor $\mathrm{ob}\mathcal{P} \to \mathcal{P}$ such that the underlying map on objects is the identity.

Since $\mathcal{P}$ is small and since $\mathcal{V}_0$ is complete, the functor category $[\mathcal{P}, \mathcal{L}]$ exists. For two $\mathcal{V}$-functors $A, B\colon \mathcal{P} \to \mathcal{L}$ the hom object $[\mathcal{P}, \mathcal{L}](A, B)$ is the end

$$\int_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP)\,,$$

which is given by an equalizer

$$\int_{P\in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP,BP) \longrightarrow \prod_{P\in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP,BP) \overset{\rho}{\underset{\sigma}{\rightrightarrows}} \prod_{P,Q\in \mathrm{ob}\mathcal{P}} [\mathcal{P}(P,Q),\mathcal{L}(AP,BQ)] \tag{28}$$

in $\mathcal{V}_0$, see [9, (2.2), p. 27], where—if we denote by π the cartesian projections— ρ and σ are determined by requiring $\pi_{P,Q}\rho$ and $\pi_{P,Q}\sigma$ to be π_P composed with the transform of $\mathcal{L}(AP,B-)_{PQ}$ and π_Q composed with the transform of $\mathcal{L}(A-,BQ)_{QP}$ respectively.

Now let $\mathcal{M}$ be another small $\mathcal{V}$-category and $K\colon\ \mathcal{M}\to\mathcal{P}$ be a $\mathcal{V}$-functor , e.g. $K=H$. The $\mathcal{V}$-functor K induces a $\mathcal{V}$-functor

$$[K,1]\colon\ [\mathcal{P},\mathcal{L}]\to[\mathcal{M},\mathcal{L}]\ ,$$

which sends a $\mathcal{V}$-functor $A\colon\ \mathcal{P}\to\mathcal{L}$ to the composite $\mathcal{V}$-functor AK, cf. [9, (2.26)], and its hom morphisms are determined by the universal property of the end and commutativity of the following diagram

$$\begin{array}{ccc}
[\mathcal{P},\mathcal{L}](A,B) & \xrightarrow{[K,1]_{A,B}} & [\mathcal{M},\mathcal{L}](AK,BK) \\
\Big\downarrow{\scriptstyle E_{KM}} & & \Big\downarrow{\scriptstyle E_M} \\
\mathcal{L}(AKM,BKM) & =\!=\!=\!= & \mathcal{L}(AKM,BKM)\ .
\end{array} \tag{29}$$

Left Kan extension $\mathrm{Lan}_K\colon\ [\mathcal{M},\mathcal{L}]\to[\mathcal{P},\mathcal{L}]$ along K provides a left adjoint to $[K,1]$: this is the usual Theorem of Kan adjoints as given in [9, Th. 4.50, p. 67], and it applies since $\mathcal{M}$ and $\mathcal{P}$ are small and since $\mathcal{L}$ is cocomplete. In particular, we have a hom $\mathcal{V}$-adjunction

$$[\mathcal{P},\mathcal{L}](\mathrm{Lan}_K A,S)\cong[\mathcal{M},\mathcal{L}](A,[K,1](S))\ , \tag{30}$$

cf. [9, (4.39)], which is $\mathcal{V}$-natural in $A\in[\mathcal{M},\mathcal{L}]$ and $S\in[\mathcal{P},\mathcal{L}]$. Thus we have a monad

$$T=[H,1]\mathrm{Lan}_K\colon\ [\mathcal{M},\mathcal{L}]\to[\mathcal{M},\mathcal{L}] \tag{31}$$

on $[\mathcal{M},\mathcal{L}]$, which we call the monad of the Kan adjunction. The unit $\eta\colon\ 1\Rightarrow T$ of T is given by the unit η of the adjunction (30), while the multiplication $\mu\colon\ TT\Rightarrow T$, is given by

$$[H,1]\epsilon\mathrm{Lan}_H\colon\ [H,1]\mathrm{Lan}_H[H,1]\mathrm{Lan}_\mathrm{H}\Rightarrow[H,1]\mathrm{Lan}_H$$

where ϵ is the counit of the adjunction (30).

We now come back to the special case that $\mathcal{M}=\mathrm{ob}\mathcal{P}$ and $K=H$. Since $\mathcal{P}$ is small, we may identify a functor $\mathrm{ob}\mathcal{P}\to\mathcal{L}$ with its family of values in $\mathcal{L}$ i.e. the set of functors is identified with the (small) limit in $\mathcal{S}et$ given by the product $\prod_{\mathrm{ob}\mathcal{P}}\mathrm{ob}\mathcal{L}$.

In fact, the equalizer (28) is trivial for $[\mathrm{ob}\mathcal{P},\mathcal{L}]$, so for two functors $A,B\colon\ \mathrm{ob}\mathcal{P}\to\mathcal{L}$, the hom object $[\mathrm{ob}\mathcal{P},\mathcal{L}](A,B)$ is given by the (small) limit in $\mathcal{V}_0$ given by the product $\prod_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(AP,BP)$.

Namely, ρ and σ are equal in (28): Denoting by π the projections of the cartesian products, $\pi_{P,Q}\rho=\rho_{P,Q}\pi_P$ equals $\pi_{P,Q}\sigma=\sigma_{P,Q}\pi_Q$ because for $P\neq Q$, the two morphisms

$$\prod_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(AP,BP)\to[\emptyset,\mathcal{L}(AP,BQ)]$$

must both be the transform of the unique morphism

$$\emptyset \to [\prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP), \mathcal{L}(AP, BQ)] ;$$

and for $P = Q$, we have $\rho_{P,P} = \sigma_{P,P}$ because these are the transforms of $\mathcal{L}(AP, B-)_{PP}$ and $\mathcal{L}(A-, BP)_{PP}$, which are both equal to

$$j_{\mathcal{L}(AP,BP)} \colon I = (\mathrm{ob}\mathcal{P})(P, P) \to [\mathcal{L}(AP, BP), \mathcal{L}(AP, BP)]$$

by the unit axioms for the $\mathcal{V}$-functors A, B, $\mathcal{L}(AP, -)$, and $\mathcal{L}(-, BP)$. To see this, note that $j_P \colon I \to I$ is the identity functor, so $\mathcal{L}(AP, B-)_{PP} = \mathcal{L}(AP, B-)_{PP} j_P$ and $\mathcal{L}(A-, BP)_{PP} = \mathcal{L}(A-, BP)_{PP} j_P$.

From diagram (29), we see that

$$[H, 1]_{A,B} \colon [\mathcal{P}, \mathcal{L}](A, B) \to [\mathrm{ob}\mathcal{P}, \mathcal{L}](AH, BH)$$

is given by the strict functor of the equalizer (28),

$$\int_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP) \to \prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP) ,$$

that is, the strict functor into the product induced by the family of evaluation functors E_P where P runs through the objects of $\mathcal{P}$.

Lemma 2. *Let $\{F, G\}$ be a pointwise limit, then any representation is pointwise.*

Proof. Let (B, μ) be a pointwise representation and let (B', μ') be any other representation. By Yoneda, μ' has the form $[\mathcal{P}, \mathcal{L}](\alpha, G-)\mu$ for a unique isomorphism $\alpha \colon B' \Rightarrow B$. It follows that

$$E_P\mu' = E_P[\mathcal{P}, \mathcal{L}](\alpha, G-)\mu = \mathcal{L}(\alpha_P, E_P G-)E_P\mu$$

and by extraordinary naturality this induces

$$\mathcal{L}(L, B'P) \xrightarrow{\mathcal{L}(L,\alpha_P)} \mathcal{L}(L, BP) \xrightarrow{\beta} [\mathcal{K}, \mathcal{V}](F, \mathcal{L}(L, (G-)P))$$

where β is the isomorphism induced by $E_P\mu$, and this is an isomorphism that is $\mathcal{V}$-natural in L and P because (B, μ) is pointwise and α_P is an isomorphism that is $\mathcal{V}$-natural in P. This proves that (B', μ') is a pointwise limit. □

The following is the usual non-invariant notion of limit creation as in MacLane's book [14, p. 108] adapted to the enriched context:

Definition 8. A $\mathcal{V}$-functor $T \colon \mathcal{B} \to \mathcal{C}$ creates $F * G$ or creates colimits of $G \colon \mathcal{K} \to \mathcal{B}$ indexed by $F \colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ if (i) for every (C, ν) where $\nu \colon F \to \mathcal{C}(TG-, C)$ exhibits the object $C \in \mathcal{C}$ as the colimit $F * (TG)$, there is a unique (B, ξ) consisting of an object $B \in \mathcal{B}$ with $TB = C$ and a $\mathcal{V}$-natural transformation $\xi \colon F \to \mathcal{B}(G-, B)$ with $T_{G-,B}\xi = \nu$, and if, moreover; (ii) ξ exhibits B as the colimit $F * G$. There is a dual notion for creation of limits.

In particular, a colimit $F * G$ created by the $\mathcal{V}$-functor T is also preserved by T.

Lemma 3. *The functor* $[H, 1]$ *creates arbitrary pointwise (co)limits.*

Proof. We only prove the colimit case, the proof for limits is analogous. If the colimit $(C, \nu) = (F * [H, 1]G, \nu)$ exists pointwise, we have

$$CP = (F * [H, 1]G)P = F * ([H, 1](G-)P) = F * (G-)HP = F * (G-)P ,$$

which means that the value of C at P is a colimit (CP, ν^P) of $(G-)P$ indexed by F. In fact, this determines the $\mathcal{V}$-functor C uniquely since the domain ob$\mathcal{P}$ is discrete. Further, it implies that the colimit $(F * G, \xi)$ exists pointwise because $F * (G-)P$ exists as (CP, ν^P), and this means that the functoriality of $F * G$ is induced from the pointwise representation and that $E_P\xi = \nu^P$. Now since

$$([H, 1](F * G))P = (F * G)HP = (F * G)P = F * (G-)P = CP ,$$

the two functors $[H, 1](F * G)$ and $F * ([H, 1]G)$ coincide pointwise, and this means that they must also coincide as functors ob$\mathcal{P} \to \mathcal{L}$, i.e. $[H, 1](F * G) = F * ([H, 1]G)$. Moreover, since the units coincide pointwise, $E_P\xi = \nu^P$, we must have $[H, 1]_{G-,A}\xi = \nu = \Pi_P \nu^P$.

This proves the existence of a (B, ξ) as in Definition 8. Suppose there would be another (B', ξ') with $[H, 1]B' = C$ and $[H, 1]_{G-,B'}\xi' = \nu$. Then B and B' would coincide pointwise i.e. $BP = BP'$ for any object $P \in \mathcal{P}$, and via ξ and ξ' would both give rise to the same representation isomorphism—by the fact that $[H, 1]_{G-,A}\xi' = \nu = [H, 1]_{G-,A}\xi$ and thus $E_P\xi' = E_p[H, 1]_{G-,A}\xi' = E_P\nu = E_P[H, 1]_{G-,A}\xi = E_P\xi$— and this representation isomomorphism is $\mathcal{V}$-natural in P as well as in L:

$$\mathcal{L}(BP, L) \cong [\mathcal{K}^{\text{op}}, \mathcal{V}](F, \mathcal{L}((G-)P, L)) . \tag{32}$$

But for such a representation isomorphism there is a unique way of making B a $\mathcal{V}$-functor $\mathcal{P} \to \mathcal{L}$ such that the representation isomorphism is $\mathcal{V}$-natural in P as well as in L, see for example [9, 1.10], so B and B' have to coincide as $\mathcal{V}$-functors. Clearly, by Yoneda, also $\xi = \xi'$ then as the representations of (B, ξ) and (B, ξ') coincide because the pointwise representations (32) do, cf. [9, 3.3]. Note here that (B', ξ') must be a pointwise colimit too because by assumption, it is preserved by $[H, 1]$ and (C, ν) is preserved by any E_P, so (B', ξ') is preserved by any E_P and thus it is a pointwise colimit. On the other hand, this is just the general fact that if a colimit exists pointwise, then any representation must in fact be pointwise, see Lemma 2 above. □

Corollary 2. *The functor* $[H, 1]$ *preserves any limit and any pointwise colimit that exists.*

Proof. This follows from the lemma above and the fact that $[H, 1]$ is a right adjoint. □

Remark 4. In case that $[H, 1]$ is also a left adjoint, it in fact preserves any colimit that exists. This is for example the case when the target $\mathcal{L}$ is complete, where the right adjoint is given by right Kan extension Ran_H along H, which exists because $\mathcal{L}$ and ob$\mathcal{P}$ were assumed to be complete and small respectively. In particular, this applies in the situation that $\mathcal{L} = \mathcal{V}$.

Corollary 3. *Let $\mathcal{P}$ be a small and $\mathcal{L}$ be a cocomplete $\mathcal{G}ray$-category, and let T be the monad* $[H, 1]\mathrm{Lan}_H$ *on* $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ *given by the Kan adjunction. Then the inclusion* $i\colon [\mathcal{P}, \mathcal{L}] \hookrightarrow \text{Ps-}T\text{-Alg}$ *has a left adjoint* $L\colon \text{Ps-}T\text{-Alg} \to [\mathcal{P}, \mathcal{L}]$ *and each component* $\eta_A\colon A \to iLA$ *of the unit of the adjunction is an internal biequivalence in* Ps-T-Alg. □

Proof. We aim at applying Corollary 1 of Gurski's coherence theorem. Thus, we have to show that $T = [H, 1]\mathrm{Lan}_H$ preserves small colimits. Since Lan_H is a left adjoint, it preserves any colimit that exists. Since this limit is again a small limit and since, $\mathcal{L}$ being complete, small limits are pointwise limits (cf. Lemma 2), Lemma 3 implies that it is preserved by $[H, 1]$. This proves that any small limit is preserved by T. □

4.2 Explicit description of the monad

In this paragraph, we will give an explicit description of the monad from 4.1 in terms of a coend over tensor products. As a matter of fact, the explicit identification of the monad structure is involved, and an alternative economical strategy adequate for the purpose of this paper, would be to take the description in terms of coends and tensor products as a definition. By functoriality of the colimit it is then readily shown that this gives a monad on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ as required, but one has to show that it preserves pointwise (and thus small) colimits in order to apply Corollary 1 from 3.2. This follows from an appropriate form of the interchange of colimits theorem. For this reason, we will be short on proofs below.

First, we recall the notions of tensor products and coends to present the well-known Kan extension formula (38) below. Then we determine the monad structure $\mu\colon TT \Rightarrow T$ and $\eta\colon 1 \Rightarrow T$ for the monad from 4.1. Given an object $X \in \mathcal{V}$ and an object $L \in \mathcal{L}$, recall that the tensor product $X \otimes L$ is defined as the colimit $X * L$ where X and L are considered as objects i.e. as $\mathcal{V}$-functors in the underlying categories $\mathcal{V}_0 = \mathcal{V}\text{-}\mathcal{CAT}(\mathcal{I}^{\mathrm{op}}, \mathcal{V})$ and $\mathcal{L}_0 = \mathcal{V}\text{-}\mathcal{CAT}(\mathcal{I}, \mathcal{L})$ where $\mathcal{I}$ is the unit $\mathcal{V}$-category. With the identification $[\mathcal{I}, \mathcal{L}] \cong \mathcal{L}$, the corresponding contravariant representation (27) from 3.2 has the form

$$n\colon \mathcal{L}(X \otimes L, M) \cong [X, \mathcal{L}(L, M)]\,, \tag{33}$$

and this is $\mathcal{V}$-natural in all variables by functoriality of the colimit cf. [9, (3.11)]. This means that tensor products are in fact $\mathcal{V}$-adjunctions, and we will dwell on this in the next paragraph 4.3. Because $\mathcal{L}$ is assumed to be cocomplete, tensor products indeed exist.

Next, recall that for a $\mathcal{V}$-functor $G\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{L}$, the coend

$$\int^A G(A, A) \tag{34}$$

is defined as the colimit $\mathrm{Hom}_{\mathcal{A}}^{\mathrm{op}} * G$. The corresponding representation (27) from 3.2 transforms under the extra-variable enriched Yoneda lemma cf. [9, (2.38)] into the following characteristic isomorphism of the coend:

$$\beta\colon \mathcal{L}(\int^A G(A, A), L) \cong \int_A \mathcal{L}(G(A, A), L)\,, \tag{35}$$

which is $\mathcal{V}$-natural in L and where on the right we have an end in the ordinary sense cf. [9, 2.1]. The unit of $\mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}} * G$ corresponds to a $\mathcal{V}$-natural family

$$\kappa_A = \lambda_A \beta j_{\int^A G(A,A)} \colon G(A,A) \to \int^A G(A,A)\,, \tag{36}$$

where λ_A is the counit of the end, and (35) induces the following universal property of κ_P:

$$\mathcal{L}_0(\int^A G(A,A), L) \cong \mathcal{V}\text{-nat}(G(A,A), L)\,. \tag{37}$$

This is a bijection of sets and it is given by precomposition with κ_A, which proves that κ_A is the universal $\mathcal{V}$-natural family with domain $G(A,A)$. Since $\mathcal{L}$ was assumed to be cocomplete, small coends in $\mathcal{L}$ do in fact exist.

We are now ready to present the explicit description of left Kan extension and thus of the monad from 4.1. Since $\mathcal{L}$ admits tensor products and since $\mathcal{P}$ was assumed to be small, left Kan extension along the functor $H\colon \mathrm{ob}\mathcal{P} \to \mathcal{P}$ from 4.1 is given by the following small coend:

$$\mathrm{Lan}_H A \cong \int^P \mathcal{P}(P,-) \otimes AP \tag{38}$$

cf. [9, (4.25)].

Example. In case that $\mathcal{L} = \mathcal{V}$, the coend in (34) is given by a coproduct in $\mathcal{V}_0$: Indeed $\mathrm{ob}\mathcal{P}$ is the free $\mathcal{V}$-category $((\mathrm{ob}\mathcal{P})_0)_{\mathcal{V}}$ where the set of objects of $\mathcal{P}$ is considered as the ordinary discrete category $(\mathrm{ob}\mathcal{P})_0$, and $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{ob}\mathcal{P}} = (\mathrm{Hom}^{\mathrm{op}}_{(\mathrm{ob}\mathcal{P})_0})_{\mathcal{V}}$ is the $\mathcal{V}$-functor corresponding to the ordinary hom functor $\mathrm{Hom}^{\mathrm{op}}_{(\mathrm{ob}\mathcal{P})_0}$ under the identification

$$((\mathrm{ob}\mathcal{P})_0 \times (\mathrm{ob}\mathcal{P})_0)_{\mathcal{V}} \cong \mathrm{ob}\mathcal{P} \otimes \mathrm{ob}\mathcal{P}\,,$$

where we have dropped the superfluous superscript $^{\mathrm{op}}$. Thus, $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{ob}\mathcal{P}} * G(A,A)$ reduces to a conical colimit in $\mathcal{V}$, which, $\mathcal{V}$ being cotensored, coincides with the ordinary colimit, hence the coproduct.

Next one observes that tensor products in $\mathcal{V}$ are given by the monoidal structure as is easily seen from (33) cf. (45) in 4.3. Therefore, (38) reduces to the coproduct

$$\mathrm{Lan}_H A \cong \sum_{P \in \mathrm{ob}\mathcal{P}} \mathcal{P}(P,-) \otimes AP\,. \tag{39}$$

Now let $\mathcal{M}$ be another small $\mathcal{V}$-category and $K\colon \mathcal{M} \to \mathcal{P}$ be a $\mathcal{V}$-functor. Then left Kan extension along K exists in the form of

$$\mathrm{Lan}_K A \cong \int^M \mathcal{P}(KM,-) \otimes AM\,, \tag{40}$$

the relevant functor categories exist, and we again have the Kan adjunction $\mathrm{Lan}_K \dashv [K,1]$.

Lemma 4. *The component at* $A \in [\mathcal{P}, \mathcal{L}]$ *of the counit* $\epsilon\colon \mathrm{Lan}_K[K,1] \Rightarrow 1_{[\mathcal{P},\mathcal{L}]}$ *of the adjunction* $\mathrm{Lan}_K \dashv [K,1]$ *has component*

$$\epsilon_{A,Q}\colon \int^M \mathcal{P}(KM,Q) \otimes AKM \to AQ$$

at $Q \in \mathcal{P}$ *induced from the* $\mathcal{V}$*-natural transform*

$$\mathcal{P}(KM,Q) \otimes AKM \to AQ$$

of the hom morphism

$$A_{KM,Q}\colon \mathcal{P}(KM,Q) \to \mathcal{L}(AKM,AQ)$$

under the adjunction (33) *of the tensor product.*

Proof. The component at A is obtained by composing the unit

$$j_{AK}\colon I \to [\mathrm{ob}\mathcal{P}, \mathcal{L}](AK,AK)$$

with the inverse of the $\mathcal{V}$-natural isomorphism of the Kan adjunction (30). The lemma then follows from inspection of the proof of the theorem of Kan adjoints [9, Th. 4.38]. In particular, the transform of the $\mathcal{V}$-natural $\mathcal{L}(AKM,-)_{AKM,AQ}A_{KM,Q}$, which gives rise to the extra-variable Yoneda isomorphism [9, (2.33)], enters in the inverse of (30), and this is the point where the hom morphism $A_{KM,Q}$ shows up. □

We will show in the next paragraph 4.3 that there are obvious left unitors λ and associators α for the tensor products. These already show up in the following two lemmata, but since we mostly omit the proofs, it seems more stringent to state the lemmata here in order to have the explicit description of T at one place.

Lemma 5. *The component at* $A \in [\mathcal{M}, \mathcal{L}]$ *of the unit of the adjunction* $\mathrm{Lan}_K \dashv [K,1]$ *and the corresponding monad* $T = \mathrm{Lan}_K \dashv [K,1]$ *on* $[\mathcal{M}, \mathcal{L}]$*, i.e. the* $\mathcal{V}$*-natural transformation* $\eta\colon 1_{[\mathcal{M},\mathcal{L}]} \Rightarrow [K,1]\mathrm{Lan}_K$ *has component*

$$\eta_{A,M}\colon AM \xrightarrow{\lambda^{-1}_{AM}} I \otimes AM \xrightarrow{j_{KM}\otimes 1} \mathcal{P}(KM,KM) \otimes AM \xrightarrow{\kappa_{M,KM}} \int^O \mathcal{P}(KO,KM) \otimes AO$$

at M *in* $\mathcal{M}$ *where* λ^{-1}_{AM} *is the unitor of the tensor product cf. 4.3.*

Proof. Note that we have stressed in the statement that the unit of the monad T is exactly given by the unit of the adjunction $\mathrm{Lan}_K \dashv [K,1]$. Hence, its component at $A \in [\mathcal{M}, \mathcal{L}]$ is given by composing the $\mathcal{V}$-natural isomorphism of the Kan adjunction (30) with the unit

$$j_{\mathrm{Lan}_K A}\colon I \to [\mathcal{P}, \mathcal{L}](\mathrm{Lan}_K A, \mathrm{Lan}_K A)\,.$$

This gives an element $I \to [\mathrm{ob}\mathcal{P}, \mathcal{L}](A, [K,1](\mathrm{Lan}_K A))$, that is, a $\mathcal{V}$-natural transformation $A \Rightarrow [K,1](\mathrm{Lan}_K A) = TA$ cf. (30). Since the inverse of the extra-variable enriched Yoneda isomorphism [9, (2.33)] takes part in (30), this is converse to the situation in Lemma 5. Correspondingly, one has to consider the transform of $\mathcal{L}(AO, (\mathrm{Lan}_K A)-)_{O,KM}$, although one does not have to determine $((\mathrm{Lan}_K A)-)_{O,KM}$ in the argument as one only uses the unit axiom for a $\mathcal{V}$-functor. □

Lemma 6. *The hom morphism of* $\mathrm{Lan}_K A = \int^M \mathcal{P}(KM,-)\otimes AM$*,*

$$(\mathrm{Lan}_K A)_{Q,R}\colon\ \mathcal{P}(Q,R)\to\mathcal{L}\Big(\int^M \mathcal{P}(KM,Q)\otimes AM, \int^M \mathcal{P}(KM,R)\otimes AM\Big)$$

corresponds to the $\mathcal{V}$*-natural family (in* $M\in\mathcal{M}$ *and also in* $Q\in\mathcal{P}$ *but* Q *is held constant here)*

$$\kappa_{M,R}(M_{\mathcal{P}}\otimes 1)\alpha^{-1} \tag{41}$$

under (33)*, exchange of the colimits* $\mathcal{P}(Q,R)\otimes -$ *and* $\int^M$*, and* (37)*.*

Proof. A neat way of proving this is by showing that the prescription in the statement of the lemma gives rise to the correct unit of the representation for left Kan adjunction along K via

$$\mathcal{P}(KM,R)\xrightarrow{(\mathrm{Lan}_K)_{KM,R}}\mathcal{L}((\mathrm{Lan}_K A)KM,(\mathrm{Lan}_K A)R)\xrightarrow{\mathcal{L}(\eta_{A,M},1)}\mathcal{L}(AM,(\mathrm{Lan}_K A)R) \tag{42}$$

cf. [9, dual of Th. 4.6 (ii)], where $\eta_{A,M}$ was determined in Lemma 5, and the unit of the representation of the left Kan extension as a colimit in the form of (40) is quickly determined to be $\mathcal{L}(1,\kappa_{M,R})\eta^{AM}_{\mathcal{P}(KM,R)}$. Namely, the unit of (35) is $\kappa_{M,R}$, then n is applied to this, which by (57) in 4.3 below gives

$$n(\kappa_{M,R}) = [\eta^{AM}_{\mathcal{P}(KM,R)},1]\mathcal{L}(AM,-)_{\mathcal{P}(KM,R)\otimes AM,\int^M \mathcal{P}(KM,R)\otimes AM}(\kappa_{M,R})$$

or $[\eta^{AM}_{\mathcal{P}(KM,R)},1]\mathcal{L}(AM,\kappa_{M,R})$, where η is the counit of the adjunction of the tensor product cf. (33). Thus this is the counit in question and it can be identified with $\mathcal{L}(AM,\kappa_Q)\eta^{AM}_{\mathcal{P}(KM,R)}$. One then proves that (42) in fact has exactly this form:

Denoting by x exchange of the tensor product and the coend $\int^M$, the relevant calculation is displayed below.

$$\mathcal{L}(\eta_{A,M},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\eta^{\int^M \mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}$$

$$=\mathcal{L}(\kappa_{M,KM}(j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\eta^{\int^M \mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}$$

(by Lemma 5)

$$=\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\mathcal{L}(\kappa_{M,KM},1)\eta^{\int^M \mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}$$

(functoriality of $\hom_{\mathcal{L}}$)

$$=\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\mathcal{L}(1,1\otimes\kappa_M)\eta^{\mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}$$

(naturality of η)

$$=\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})\kappa_M)\eta^{\mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}$$

(exchange of colimits is induced by an isomorphism of represented functors: $x(1\otimes\kappa_M)=\kappa_M$)

$$=\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})\eta^{\mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(since } (V\beta)=V[\kappa_M,1])$$

$$=\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})a^{-1}(1\otimes(j_{KM}\otimes 1)))\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(functoriality of } \hom_{\mathcal{L}})$$

$$=\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}(1\otimes j_{KM})\otimes 1_{AM})\alpha^{-1})\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(by naturality of } \alpha)$$

$$=\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(r_{\mathcal{P}(KM,R)}\otimes 1_{AM})\alpha^{-1})\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(by a } \mathcal{V}\text{-category axiom)}$$

$$=\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(1\otimes\lambda_{AM}))\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(by the triangle identity (54))}$$

$$=\mathcal{L}(\lambda_{AM}\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R})\eta^{AM}_{\mathcal{P}(KM,R)} \quad \text{(by naturality of } \eta)$$

$$=\mathcal{L}(1,\kappa_{M,R})\eta^{AM}_{\mathcal{P}(KM,R)}=[\eta^{AM}_{\mathcal{P}(KM,R)},1]\mathcal{L}(AM,\kappa_{M,R})$$

Thus the prescription (41) leads to the right counit, but this means that the hom morphism $(\mathrm{Lan}_K)_{Q,R}$ must have precisely the claimed form since $\mathrm{Lan}_K A$ is uniquely functorial such that the representation, which is induced from this counit, is appropriately natural cf. [9, 1.10] (and indeed this is how the functoriality of (40) is defined). □

Corollary 4. *Let $T=[K,1]\mathrm{Lan}_K$ be the monad of the Kan adjunction from 4.1. The component at $A\in[\mathcal{M},\mathcal{L}]$ of the $\mathcal{V}$-natural transformation $\mu\colon TT\Rightarrow T$ has component corresponding to the $\mathcal{V}$-natural (in $M,N\in\mathcal{M}$) family*

$$\kappa_{M,R}(M_{\mathcal{P}}\otimes 1)\alpha^{-1} \tag{43}$$

under exchange of colimits, Fubini, and (37)*, where α is the associator of the tensor product cf. 4.3.*

Proof. The $\mathcal{V}$-natural transformation μ of the monad is determined by the counit ϵ of the adjunction $\mathrm{Lan}_H \dashv [H,1]$. Namely, it is given by the $\mathcal{V}$-natural transformation denoted

$$[H,1]\epsilon\mathrm{Lan}_H\colon\ [H,1]\mathrm{Lan}_H[H,1]\mathrm{Lan}_{\mathrm{H}}\Rightarrow[H,1]\mathrm{Lan}_H$$

with component

$$[H,1]_{\mathrm{Lan}_H((\mathrm{Lan}_H A)H),\mathrm{Lan}_H A}\epsilon_{\mathrm{Lan}_H A}\colon\ I\to[\mathrm{ob}\mathcal{P},\mathcal{L}]((\mathrm{Lan}_H((\mathrm{Lan}_H A)H))H,(\mathrm{Lan}_H A)H)$$

at $A \in [\mathrm{ob}\mathcal{P}, \mathcal{L}]$. Since E_P factorizes through $[H, 1]$ and π_P, the component at $Q \in \mathcal{P}$ of μ_A is simply given by the component of $\epsilon_{\mathrm{Lan}_H A}$ at Q. According to Lemma 4, the component of $\epsilon_{\mathrm{Lan}_H A}$ at $Q \in \mathcal{P}$ is induced from the transform of $(\mathrm{Lan}_H A)_{P,Q}$, and by Lemma 6, this transform is precisely given by (43). □

4.3 Some properties of tensor products

It is clear from the defining representation isomorphism (33) from 4.2 of the tensor product and its naturality in X,L, and M that tensor products, for any object L in a tensored $\mathcal{V}$-category $\mathcal{L}$, give an adjunction of $\mathcal{V}$-categories as below

$$(- \otimes L\colon\ \mathcal{V} \to \mathcal{L}) \quad \dashv \quad (\mathcal{L}(L, -)\colon\ \mathcal{L} \to \mathcal{V})\,. \tag{44}$$

Because the representation isomorphism is also $\mathcal{V}$-natural in L, it is a consequence of the extra-variable Yoneda lemma [9, 1.9] that the unit and counit of this adjunction are also extraordinarily $\mathcal{V}$-natural in L:

Lemma 7. *The unit $\eta^L_X\colon\ X \to \mathcal{L}(L, X \otimes L)$ and counit $\epsilon^L_M\colon\ \mathcal{L}(L, M) \otimes L \to M$ of these adjunctions are extraordinarily $\mathcal{V}$-natural in L (and ordinarily $\mathcal{V}$-natural in X and M).* □

Recall that there is a natural (in $X, Y, Z \in \mathcal{V}$) isomorphism

$$p\colon\ [X \otimes Y, Z] \cong [X, [Y, Z]]\,, \tag{45}$$

which is induced from the closed structure of $\mathcal{V}$ via the ordinary Yoneda lemma cf. [9, 1.5]. From p and the hom $\mathcal{V}$-adjunction (33) of the tensor product, we construct a $\mathcal{V}$-natural isomorphism

$$n^{-1}_{X, Y\otimes L, M}[X, n^{-1}_{Y,L,M}]p n_{X\otimes Y, L, M}\colon\ \mathcal{L}((X \otimes Y) \otimes L, M) \to \mathcal{L}(X \otimes (Y \otimes L), M)\,, \tag{46}$$

which, by Yoneda, must be of the form $\mathcal{L}(\alpha^{-1}_{X,Y,L}, 1)$ for a unique $\mathcal{V}$-natural (in X, Y, L) isomorphism

$$\alpha^{-1}_{X,Y,L}\colon\ X \otimes (Y \otimes L) \cong (X \otimes Y) \otimes L\,. \tag{47}$$

The natural isomorphism (47) is called the associator for the tensor product. Since tensor products reduce to the monoidal structure if $\mathcal{L} = \mathcal{V}$, the natural isomorphism (47) is in this special case, by uniqueness, given by the associator a^{-1} for the monoidal structure of $\mathcal{V}$.

In fact, there is a pentagon identity in terms of associators α and associators a:

Lemma 8. *Given objects W, X, and Y in $\mathcal{V}$, and an object L in a tensored $\mathcal{V}$-category $\mathcal{L}$, the associators α and a satisfy the pentagon identity*

$$\alpha_{W,X,Y\otimes L}\alpha_{W\otimes X,Y,L} = (1_W \otimes \alpha_{X,Y,L})\alpha_{W,X\otimes Y,Z}(a_{W,X,Z} \otimes 1_Z)\,, \tag{48}$$

which is an identity of isomorphisms

$$((W \otimes X) \otimes Y) \otimes L \to W \otimes (X \otimes (Y \otimes L))\,.$$

Proof. The corresponding identity for the inverses is proved by showing that the corresponding $\mathcal{V}$-natural isomorphisms

$$\mathcal{L}(((W\otimes X)\otimes Y)\otimes L, M)\cong \mathcal{L}(W\otimes(X\otimes(Y\otimes L)), M)$$

coincide. The $\mathcal{V}$-natural isomorphism corresponding to the inverse of the left hand side of (48) is readily seen to be given by

$$n^{-1}_{W,X\otimes(Y\otimes L),M}[W, n^{-1}_{X,Y\otimes L,M}[X, n^{-1}_{Y,L,M}]]ppn_{(W\otimes X)\otimes Y,L,M}$$

cf. (46), where we have used naturality of p and cancelled out two factors. Similarly, the $\mathcal{V}$-natural isomorphism corresponding to the inverse of the right hand side of (48) is given by

$$n^{-1}_{W,X\otimes(Y\otimes L),M}[W, n^{-1}_{X,Y\otimes L,M}[X,[n^{-1}_{Y,L,M}]]][W,p]p[\alpha^{-1},\mathcal{L}(L,M)]n_{(W\otimes X)\otimes Y,L,M}\ .$$

Thus, the identity (48) is proved as soon as we show that

$$pp = [W,p]p[a^{-1},\mathcal{L}(L,M)]\ .$$

In fact, this last equation reduces to the pentagon identity for a since p is defined via Yoneda by

$$\mathcal{V}_0(W,p) = \pi\pi\mathcal{V}_0(a,1)\pi^{-1}\ ,$$

where π is the hom $\mathcal{S}et$-adjunction of the closed structure. Namely, one observes that on the one hand,

$$\begin{aligned}\mathcal{V}_0(V,pp) = \mathcal{V}_0(V,p)\mathcal{V}_0(V,p) = \pi\pi\mathcal{V}_0(a,1)\pi^{-1}\pi\pi\mathcal{V}_0(a,1)\pi^{-1} &= \pi\pi\mathcal{V}_0(a,1)\pi\mathcal{V}_0(a,1)\pi^{-1}\\ &= \pi\pi\pi\mathcal{V}_0(a\otimes 1,1)\mathcal{V}_0(a,1)\pi^{-1}\\ &= \pi\pi\pi\mathcal{V}_0(a(a\otimes 1),1)\pi^{-1}\ ,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathcal{V}_0(V,[W,p]p[a^{-1},\mathcal{L}(L,M)]) &= \mathcal{V}_0(V,[W,p])\pi\pi\mathcal{V}_0(a^{-1},1)\pi^{-1}\pi\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \mathcal{V}_0(V,[W,p])\pi\pi\mathcal{V}_0(a,1)\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \pi\mathcal{V}_0(V\otimes W,p)\pi^{-1}\pi\pi\mathcal{V}_0(a,1)\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \pi\pi\pi\mathcal{V}_0(a,1)\mathcal{V}_0(a,1)\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \pi\pi\pi\mathcal{V}_0((1\otimes a^{-1})aa,1)\pi^{-1}\ .\end{aligned}$$

□

Similarly, recall that there is a natural (in $Z\in\mathcal{V}$) isomorphism

$$i\colon\ Z\cong[I,Z] \tag{49}$$

which is defined via Yoneda by

$$[X,i^{-1}] = [r_X^{-1},1]p^{-1}\colon\ [X,[I,Z]]\cong[X\otimes I,Z]\cong[X,Z]\ . \tag{50}$$

Thus from i and the hom $\mathcal{V}$-adjunction n of the tensor product, we construct a $\mathcal{V}$-natural (in L, M) isomorphism

$$n^{-1}_{I,L,M}i\colon\ \mathcal{L}(L,M) \cong [I,\mathcal{L}(L,M)] \cong \mathcal{L}(I\otimes L,M)\ . \tag{51}$$

By Yoneda, (51) must be of the form $\mathcal{L}(\lambda_L, 1)$ for a unique isomorphism

$$\lambda_L\colon\ I\otimes L \to L\ , \tag{52}$$

which must moreover be $\mathcal{V}$-natural in L.

For $M = I\otimes L$, composing the inverse of the isomorphism (51) above with $j_{I\otimes L}$,

$$I \to \mathcal{L}(I\otimes L, I\otimes L) \cong [I,\mathcal{L}(L,I\otimes L)] \cong \mathcal{L}(L,I\otimes L)\ ,$$

must give λ_L^{-1}—since this is how we get hold of the counit of such a natural transformation in general—and it is of course also the map corresponding to η^L_I under the isomorphism $[I,\mathcal{L}(L,I\otimes L)] \cong \mathcal{L}(L,I\otimes L)$ —because this is exactly the inverse of the first isomorphism in (51). Conversely, composing (51) for $M = L$ with j_M must give λ_L.

For $Y = I$, consider the composition of the representation isomorphism (46) i.e.

$$\mathcal{L}(\alpha^{-1}, 1) = n^{-1}_{X,I\otimes L,M}[X, n^{-1}_{I,L,M}]pn_{X\otimes I,L,M}\ , \tag{53}$$

with $\mathcal{L}(r_X\otimes 1, 1)$. By naturality of n, functoriality of $[-,-]$, and the definition of i via Yoneda cf. (50), we have the following chain of equations:

$$\begin{aligned}\mathcal{L}((r_X\otimes 1)\alpha^{-1}, 1) &= n^{-1}_{X,I\otimes L,M}[X, n^{-1}_{I,L,M}]pn_{X\otimes I,L,M}\mathcal{L}(r_X\otimes 1,1)\\ &= n^{-1}_{X,I\otimes L,M}[X, n^{-1}_{I,L,M}]p[r_X,1]n_{X,L,M}\\ &= n^{-1}_{X,I\otimes L,M}[X, n^{-1}_{I,L,M}][X,i]n_{X,L,M}\\ &= n^{-1}_{X,I\otimes L,M}[X,\mathcal{L}(\lambda_L,1)]n_{X,L,M}\\ &= \mathcal{L}(1\otimes\lambda_L,1)\ .\end{aligned}$$

Hence, by Yoneda, we have proved the following lemma.

Lemma 9. *Given an objects X in $\mathcal{V}$ and an object L in a tensored $\mathcal{V}$-category $\mathcal{L}$, there is a triangle identity for r, λ, and α,*

$$(r_X\otimes 1_L)\alpha^{-1}_{X,I,L} = 1_X\otimes\lambda_L\ , \tag{54}$$

which is an identity of isomorphisms

$$X\otimes(I\otimes L)\to X\otimes L\ .$$

□

Lemma 10. *Let L, M, N be objects in a tensored $\mathcal{V}$-category $\mathcal{L}$. Then*

$$M_{\mathcal{L}}\colon\ \mathcal{L}(M,N)\otimes\mathcal{L}(L,M)\to\mathcal{L}(L,N)$$

can be identified in terms of the associator α for tensor products and units η and counits ϵ of the tensor product adjunctions:

$$M_{\mathcal{L}} = \mathcal{L}(L, \epsilon^M_N(1\otimes\epsilon^L_M)\alpha)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)}\ .$$

Proof. First, recall that as for any $\mathcal{V}$-adjunction, Yoneda implies the identity $\mathcal{L}(L,-)_{MN} = n\mathcal{L}(\epsilon^L_M, 1)$ cf. [9, (1.53), p. 24], and thus $M_{\mathcal{L}} = e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)})$ cf. (15) in 2.5.

The lemma is now proved by the following chain of equations

$$\begin{aligned}
M_{\mathcal{L}} &= e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)}) \\
&= \mathcal{L}(1_L, \epsilon^L_N)\eta^L_{\mathcal{L}(L,N)} e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)}) \\
&\quad \text{(by a triangle identity for unit and counit of the adjunction (44))} \\
&= \mathcal{L}(1_L, \epsilon^L_N)\mathcal{L}(1_L, (e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)})) \otimes 1_L)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(by ordinary naturality of } \eta^L_X \text{ in } X) \\
&= \mathcal{L}(1_L, \epsilon^{\mathcal{L}(L,M)\otimes L}_N \alpha((\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)}) \otimes 1_L))\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(see below (*), by functoriality and the identity } \epsilon = \epsilon((e(n \otimes 1)) \otimes 1)\alpha^{-1}) \\
&= \mathcal{L}(1_L, \epsilon^{\mathcal{L}(L,M)\otimes L}_N (\mathcal{L}(\epsilon^L_M, 1) \otimes (1_{\mathcal{L}(L,M)} \otimes 1_L))\alpha)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(by ordinary naturality of } \alpha) \\
&= \mathcal{L}(1_L, \epsilon^M_N(1 \otimes \epsilon^L_M)\alpha)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(by extraordinary naturality of } \epsilon^L_M \text{ in } L)
\end{aligned}$$

To prove the identity used in (*) consider the morphism

$$[\mathcal{L}(\mathcal{L}(L,M) \otimes L, N), \mathcal{L}(\mathcal{L}(L,M) \otimes L, N)] \to \mathcal{L}(\mathcal{L}(\mathcal{L}(L,M) \otimes L, N) \otimes (\mathcal{L}(L,M) \otimes L), N)$$

given by the composite

$$\mathcal{L}(\alpha, N)\mathcal{L}((n_{\mathcal{L}(L,M),L,N} \otimes 1) \otimes 1, N)n^{-1}_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]\otimes\mathcal{L}(L,M),L,N}\,p^{-1}[n^{-1}_{\mathcal{L}(M,N),L,N}, n_{\mathcal{L}(M,N),L,N}] \quad (55)$$

in $\mathcal{V}_0$, where $p\colon [X \otimes Y, Z] \cong [X, [Y, Z]]$ is again the natural isomorphism (45) induced from the closed structure of $\mathcal{V}$, and where we have added subscripts such that the hom $\mathcal{V}$-adjunction (33) from 4.2 is now denoted by $n_{X,L,M}$. If $\mathcal{L}(\alpha, N)$ is spelled out in terms of hom $\mathcal{V}$-adjunctions n and p according to (46), then it is seen by naturality that (55) is in fact the same as

$$n^{-1}_{\mathcal{L}(\mathcal{L}(L,M)\otimes L,N),\mathcal{L}(L,M)\otimes L,N}\,. \quad (56)$$

In particular, it is an isomorphism (although this is also clear because each factor is an isomorphism) and appropriately $\mathcal{V}$-natural (and this follows from the composition calculus respectively). We now want to show that the unit of this natural isomorphism is given by

$$\epsilon^L_N((e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n_{\mathcal{L}(M,N),L,N} \otimes 1)) \otimes 1)\alpha^{-1}$$

because then, by Yoneda, this must be the same as $\epsilon^{\mathcal{L}(L,M)\otimes L}_N$ since this is by definition the unit of (56).

The unit is obtained by applying (55) (or rather V of it) to $1_{\mathcal{L}(\mathcal{L}(L,M)\otimes L,N)}$, and we do this factor-by-factor. First, note that

$$[n^{-1}_{\mathcal{L}(M,N),L,N}, n_{\mathcal{L}(M,N),L,N}](1_{\mathcal{L}(\mathcal{L}(L,M)\otimes L,N)}) = 1_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]}\,,$$

and

$$p(1_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]}) = e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}$$

because p is the hom $\mathcal{V}$-adjunction with underlying adjunction $-\otimes Y \dashv [Y,-]$ given by the closed structure i.e. $Vp = \pi$.

Now note that by ordinary naturality of n, we have

$$[e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}, 1] = n_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]\otimes\mathcal{L}(L,M),L,N}\mathcal{L}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)} \otimes 1_L, 1)n^{-1}_{\mathcal{L}(L,N),L,N} .$$

Thus because $[e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}, 1](1_{\mathcal{L}(L,N)}) = e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}$, we may compute $n_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]\otimes\mathcal{L}(L,M),L,N}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)})$ by applying $\mathcal{L}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)} \otimes 1_L, 1)n^{-1}_{\mathcal{L}(L,N),L,N}$ to $1_{\mathcal{L}(L,N)}$, the result of which is $\epsilon^L_N(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)} \otimes 1)$. Finally, applying the remaining factors $\mathcal{L}((n_{\mathcal{L}(L,M),L,N} \otimes 1) \otimes 1, N)$ and $\mathcal{L}(\alpha, N)$ indeed gives (55). □

Remark 5. A different strategy for the proof of the lemma, is to first observe that the right hand side just as $M_\mathcal{L}$ is ordinarily $\mathcal{V}$-natural in L and N and extraordinarily $\mathcal{V}$-natural in M by Lemma 7, naturality of α, and the composition calculus. Then the identity in the lemma can be proved variable-by-variable by use of the Yoneda lemma where one considers the transforms in the case of the variable M.

For an object $X \in \mathcal{V}$ and objects $L, M \in \mathcal{L}$, recall that the hom $\mathcal{V}$-adjunction (33) from 4.2 of the tensor product has the following description in terms of the unit and the strict hom functor of the right adjoint $\mathcal{L}(L,-)$,

$$n = [\eta^L_X, 1]\mathcal{L}(L,-)_{X\otimes L,M} \colon\ \mathcal{L}(X \otimes L, M) \to [X, \mathcal{L}(L, M)] . \tag{57}$$

With this description of n we are able to derive two important identities for n stated in the two lemmata below. These are in fact the main technical tools that we employ to achieve the promised identification of Ps-T-Alg. Recall that there is a $\mathcal{V}$-functor Ten: $\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ which is given on objects by sending $(X, Y) \in \mathrm{ob}\mathcal{V} \times \mathrm{ob}\mathcal{V}$ to their product $X \otimes Y \in \mathrm{ob}\mathcal{V}$ and whose hom morphism $\mathrm{Ten}_{(X,X'),(Y,Y')} \colon [X, X'] \otimes [Y, Y'] \to [X \otimes Y, X' \otimes Y']$ is such that

$$e^{X\otimes Y}_{X'\otimes Y'}(\mathrm{Ten}_{(X,X'),(Y,Y')} \otimes 1_{X\otimes Y}) = (e^X_{X'} \otimes e^Y_{Y'})m \tag{58}$$

where e denotes evaluation i.e. the counits of the adjunctions comprising the closed structure of $\mathcal{V}$ and where m denotes interchange in $\mathcal{V}$.

The two lemmata specify how n behaves with respect to Ten: $\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ and $M_\mathcal{L}$ in two specific situations that we will constantly face below.

Lemma 11 (First Transformation Lemma)**.** *Given objects X, Y in $\mathcal{V}$, and objects L, M, N in $\mathcal{L}$, the following equality of $\mathcal{V}$-morphisms $\mathcal{L}(X \otimes M, N) \otimes \mathcal{L}(Y \otimes L, M) \to [X \otimes Y, \mathcal{L}(L, N)]$ holds.*

$$\begin{aligned}&[\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{(X\otimes Y)\otimes L,N}\mathcal{L}(\alpha, 1)M_\mathcal{L}(1_{\mathcal{L}(X\otimes M,N)} \otimes (X \otimes -)_{Y\otimes L,M})\\ &= [1, M_\mathcal{L}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(Y\otimes L,M))}(([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N}) \otimes ([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\end{aligned}$$

In terms of the hom $\mathcal{V}$-adjunction (33) *from 4.2, this means that*

$$n\mathcal{L}(\alpha, 1)M_\mathcal{L}(\mathrm{Ten}(\mathcal{L}(X \otimes M, N), -)_{X\otimes(X\otimes L),X\otimes M}(X \otimes -)_{Y\otimes L,M}) = [1, M_\mathcal{L}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}(n \otimes n) .$$

Proof. This is proved by the following chain of equations.

$[\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{(X\otimes Y)\otimes L,N}\mathcal{L}(\alpha,1)M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)}\otimes(X\otimes -)_{Y\otimes L,M})$

$= [\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{X\otimes(Y\otimes L),N}M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)}\otimes(X\otimes -)_{Y\otimes L,M})$

(by the functor axiom for $\mathcal{L}(L,-)$ or ordinary $\mathcal{V}$-naturality of $\mathcal{L}(L,-)_{M,N}$ in M)

$= [\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y}, 1]M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes(\mathcal{L}(L,-)_{X\otimes(Y\otimes L),X\otimes M}(X\otimes -)_{Y\otimes L,M}))$

(by the functor axiom for $\mathcal{L}(L,-)$)

$= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{X\otimes(Y\otimes L),X\otimes M}(X\otimes -)_{Y\otimes L,M}))$

(by ordinary $\mathcal{V}$-naturality of $M_{\mathcal{V}}$ or a $\mathcal{V}$-category axiom if spelled out)

$= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([\mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}(X\otimes\eta^L_Y), 1]\mathcal{L}(L,-)_{X\otimes(Y\otimes L),X\otimes M}(X\otimes -)_{Y\otimes L,M}))$

(see below (A), by a triangle identity and naturality)

$= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([(X\otimes\eta^L_Y), \mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}]\mathrm{Ten}(X,-)_{\mathcal{L}(L,Y\otimes L),\mathcal{L}(L,M)}\mathcal{L}(L,-)_{Y\otimes L,M}))$

(see below (C), by ordinary $\mathcal{V}$-naturality of $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)\eta^L_{X\otimes\mathcal{L}(L,K)}$ in K)

$= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([1, \mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}](\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M})))$

(ordinary $\mathcal{V}$-naturality of $\mathrm{Ten}(X,-)_{V,Z}$ in V)

$= M_{\mathcal{V}}(([\mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}, 1]\mathcal{L}(L,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))$

(extraordinary $\mathcal{V}$-naturality of $M_{\mathcal{V}}$)

$= M_{\mathcal{V}}(([M_{\mathcal{L}}(\eta^M_X\otimes 1_{\mathcal{L}(L,M)}), 1]\mathcal{L}(L,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))$

(see below (B), by a triangle identity, naturality, and Lemma 10)

$= M_{\mathcal{V}}(([(\eta^M_X\otimes 1_{\mathcal{L}(L,M)}), M_{\mathcal{L}}]\mathrm{Ten}(-,\mathcal{L}(L,M))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(M,N)}\mathcal{L}(M,-)_{X\otimes M,N})\otimes$

$(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))$

(see below (D), by ordinary $\mathcal{V}$-naturality of $M_{\mathcal{L}}$)

$= M_{\mathcal{V}}(([1, M_{\mathcal{L}}]\mathrm{Ten}(-,\mathcal{L}(L,M))_{X,\mathcal{L}(M,N)}[\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))$

(ordinary $\mathcal{V}$-naturality of $\mathrm{Ten}(-,Z)_{U,W}$ in U)

$= [1, M_{\mathcal{L}}]M_{\mathcal{V}}((\mathrm{Ten}(-,\mathcal{L}(L,M))_{X,\mathcal{L}(M,N)}[\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))$

(ordinary $\mathcal{V}$-naturality of $M_{\mathcal{V}}$)

$= [1, M_{\mathcal{L}}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}(M_{\mathcal{V}\otimes\mathcal{V}}$

$(((([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes j^{\mathcal{V}}_{\mathcal{L}(L,M)})r^{-1}_{\mathcal{L}(X\otimes M,N)})\otimes((j^{\mathcal{V}}_X\otimes([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))l^{-1}_{\mathcal{L}(Y\otimes L,M)})))$

(functor axiom for Ten (partial functors spelled out))

$= [1, M_{\mathcal{L}}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}$

$((M_{\mathcal{V}}(([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes j^{\mathcal{V}}_X)r^{-1}_{\mathcal{L}(X\otimes M,N)})\otimes(M_{\mathcal{V}}(j^{\mathcal{V}}_{\mathcal{L}(L,M)}\otimes([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))l^{-1}_{\mathcal{L}(Y\otimes L,M)}))$

(by $M_{\mathcal{V}\otimes\mathcal{V}} = (M_{\mathcal{V}}\otimes M_{\mathcal{V}})m$ where m is interchange, naturality of m, and $m(r^{-1}\otimes l^{-1}) = r^{-1}\otimes l^{-1}$)

$= [1, M_{\mathcal{L}}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}(([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))$

($\mathcal{V}$-category axioms for $\mathcal{V}$)

In (A) and (B) we have used the following identities of $\mathcal{V}$-morphisms. For (A), observe that

$$\begin{aligned}\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y} &= \mathcal{L}(L,(1_X\otimes(\epsilon^L_{Y\otimes L}(\eta^L_Y\otimes 1_L)))\alpha)\eta^L_{X\otimes Y}\\ &\quad\text{(by a triangle identity)}\\ &= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})(1_X\otimes(\eta^L_Y\otimes 1_L))\alpha)\eta^L_{X\otimes Y}\\ &\quad\text{(by functoriality of } X\otimes -)\\ &= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha((1_X\otimes\eta^L_Y)\otimes 1_L))\eta^L_{X\otimes Y}\\ &\quad\text{(by ordinary } \mathcal{V}\text{-naturality of } \alpha)\\ &= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}(1_X\otimes\eta^L_Y)\\ &\quad\text{(by ordinary } \mathcal{V}\text{-naturality of } \eta)\\ &= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}(X\otimes\eta^L_Y)\\ &= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}\mathrm{Ten}(X,\eta^L_Y)\,.\end{aligned}$$

Similarly, for (B) observe that

$$\begin{aligned}\mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)} &= \mathcal{L}(L,\epsilon^M_{X\otimes M}(\eta^M_X\otimes 1_M)(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}\\ &\quad\text{(by a triangle identity)}\\ &= \mathcal{L}(L,\epsilon^M_{X\otimes M}(1_{\mathcal{L}(M,X\otimes M)}\otimes\epsilon^L_M)(\eta^M_X\otimes 1_{\mathcal{L}(L,M)\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}\\ &\quad\text{(by underlying functoriality of the functor } \otimes\text{)}\\ &= \mathcal{L}(L,\epsilon^M_{X\otimes M}(1_{\mathcal{L}(M,X\otimes M)}\otimes\epsilon^L_M)\alpha((\eta^M_X\otimes 1_{\mathcal{L}(L,M)})\otimes 1_L))\eta^L_{X\otimes\mathcal{L}(L,M)}\\ &\quad\text{(by underlying functoriality of } \otimes,\ 1_{\mathcal{L}(L,M)\otimes L}=1_{\mathcal{L}(L,M)}\otimes 1_L,\\ &\qquad\text{and by ordinary } \mathcal{V}\text{-naturality of } \alpha\text{)}\\ &= \mathcal{L}(L,\epsilon^M_{X\otimes M}(1_{\mathcal{L}(M,X\otimes M)}\otimes\epsilon^L_M)\alpha)\eta^L_{\mathcal{L}(M,X\otimes M)\otimes\mathcal{L}(L,M)}(\eta^M_X\otimes 1_{\mathcal{L}(L,M)})\\ &\quad\text{(by ordinary } \mathcal{V}\text{-naturality of } \eta)\\ &= M_{\mathcal{L}}(\eta^M_X\otimes 1_{\mathcal{L}(L,M)})\\ &\quad\text{(by the identification of } M_{\mathcal{L}} \text{ in Lemma 10 above)}\\ &= M_{\mathcal{L}}(\eta^M_X\otimes\mathcal{L}(L,M)) = M_{\mathcal{L}}\mathrm{Ten}(\eta^M_X,\mathcal{L}(L,M))\,.\end{aligned}$$

Finally, we comment on the $\mathcal{V}$-naturality used in (C) and (D):

For (C) recall that η^L_J is $\mathcal{V}$-natural in J. Then so is $\eta^L_{X\otimes\mathcal{L}(L,K)}$ because this is η^L_{PK} for $P = \mathcal{L}(L,-)(X\otimes -)$. Next, recall that α is ordinarily $\mathcal{V}$-natural in all of its variables, and that ϵ^L_K is ordinarily $\mathcal{V}$-natural in K. Then so is $1_X\otimes\epsilon^L_K$ because this is $Q_0(\epsilon^L_K)$ for $Q = (X\otimes -)$, and thus the composite $(1_X\otimes\epsilon^L_K)\alpha$ is ordinarily $\mathcal{V}$-natural in K. From this it follows that $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)$ is ordinarily $\mathcal{V}$-natural in K because this is $Q_0((1_X\otimes\epsilon^L_K)\alpha)$ for $Q=\mathcal{L}(L,-)$. Hence, we conclude that the composite family $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)\eta^L_{X\otimes\mathcal{L}(L,K)}$ is ordinarily $\mathcal{V}$-natural in K. That is, $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)\eta^L_{X\otimes\mathcal{L}(L,K)}$ is the component at $K\in\mathcal{L}$ of a $\mathcal{V}$-natural transformation

$$\mathrm{Ten}(X,-)\mathcal{L}(L,-)\Rightarrow\mathcal{L}(L,-)(X\otimes -)$$

where X and L are held constant, and where $X\otimes -\colon\ \mathcal{L}\to\mathcal{L}$ is the partial functor of the tensor product in contrast to the partial functor $\mathrm{Ten}(X,-)\colon\ \mathcal{V}\to\mathcal{V}$ induced by the Gray product.

For (D) recall that $M_{\mathcal{L}}\colon\ \mathcal{L}(M,K)\otimes\mathcal{L}(L,M)\to\mathcal{L}(L,K)$ is $\mathcal{V}$-natural. Since $\mathcal{V}$-naturality may be verified variable-by-variable, $M_{\mathcal{L}}$ is in particular ordinarily $\mathcal{V}$-natural in K. That is, it is the component at $K\in\mathcal{L}$ of a $\mathcal{V}$-natural transformation

$$\mathrm{Ten}(-,\mathcal{L}(L,M))\mathcal{L}(M,-)\Rightarrow\mathcal{L}(L,-)$$

where L and M are held constant. □

Lemma 12 (Second Transformation Lemma)**.** *Given an object X in $\mathcal{V}$, and objects L,M,N in $\mathcal{L}$, the following equality of $\mathcal{V}$-morphisms $\mathcal{L}(X\otimes M,N)\otimes\mathcal{L}(L,M)\to[X,\mathcal{L}(L,N)]$ holds, where c is the symmetry of $\mathcal{V}$.*

$$\begin{aligned}&[\eta^L_X,1]\mathcal{L}(L,-)_{X\otimes L,N}M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)}\otimes(X\otimes -)_{L,M})\\&=M_{\mathcal{V}}((\mathcal{L}(-,N)_{M,N})\otimes([\eta^M_X,1]\mathcal{L}(M,-)_{(X\otimes M,N)})c\end{aligned}$$

In terms of the hom $\mathcal{V}$-adjunction (33) *from 4.2, this means that*

$$nM_{\mathcal{L}}\mathrm{Ten}(\mathcal{L}(X\otimes M,N),-)_{X\otimes L,X\otimes M}(X\otimes -)_{L,M}=M_{\mathcal{V}}(\mathcal{L}(-,N)_{L,M}\otimes n)c$$

Proof. This is proved by the following chain of equations.

$$\begin{aligned}&[\eta^L_X,1]\mathcal{L}(L,-)_{X\otimes L,N}M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)}\otimes(X\otimes -)_{L,M})\\&=[\eta^L_X,1]M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes(\mathcal{L}(L,-)_{X\otimes L,X\otimes M}(X\otimes -)_{L,M}))\\&\quad\text{(functor axiom for }\mathcal{L}(L,-))\\&=M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([\eta^L_X,1]\mathcal{L}(L,-)_{X\otimes L,X\otimes M}(X\otimes -)_{L,M}))\\&\quad\text{(ordinary }\mathcal{V}\text{-naturality of }M_{\mathcal{V}})\\&=M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([\eta^L_X,1]\mathcal{L}(-,X\otimes M)_{L,M}))\\&\quad\text{(extraordinary }\mathcal{V}\text{-naturality of }\eta^L_X\text{ in }L)\\&=[\eta^L_X,1]M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes\mathcal{L}(-,X\otimes M)_{L,M})\\&\quad\text{(by ordinary }\mathcal{V}\text{-naturality of }M_{\mathcal{V}})\\&=[\eta^M_X,1]M_{\mathcal{V}}((\mathcal{L}(-,N)_{L,M})\otimes\mathcal{L}(M,-)_{X\otimes M,N})c\\&\quad\text{(see below, by extraordinary }\mathcal{V}\text{-naturality of }\mathcal{L}(L,-)_{M,N}\text{ in }L\text{)}\\&=M_{\mathcal{V}}((\mathcal{L}(-,N)_{L,M})\otimes([\eta^M_X,1]\mathcal{L}(M,-)_{X\otimes M,N}))c\\&\quad\text{(by ordinary }\mathcal{V}\text{-naturality of }M_{\mathcal{V}})\end{aligned}$$

For the last equation, recall that $\mathcal{L}(L,-)_{M,N}\colon\ \mathcal{L}(M,N)\to[\mathcal{L}(L,M),\mathcal{L}(L,N)]$ is extraordinarily $\mathcal{V}$-natural in L when M and N are held constant. That is, $\mathcal{L}(L,-)_{M,N}$ has the form $I\to[\mathcal{L}(M,N),T(A,A)]$ for the $\mathcal{V}$-functor $T=\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(-,M)^{\mathrm{op}}\otimes\mathcal{L}(-,N))c\colon\ \mathcal{L}^{\mathrm{op}}\otimes\mathcal{L}\to\mathcal{V}$, where $c\colon\ \mathcal{L}^{\mathrm{op}}\otimes\mathcal{L}\cong\mathcal{L}\otimes\mathcal{L}^{\mathrm{op}}$ is the $\mathcal{V}$-functor mediating the symmetry of the 2-category $\mathcal{V}\text{-}\mathcal{CAT}$ of $\mathcal{V}$-categories, which is locally given by c, and the corresponding naturality equation is

$$\begin{aligned}&[\mathcal{L}(L,-)_{X\otimes M,N},1]\mathrm{Hom}_{\mathcal{V}}(-,\mathcal{L}(L,N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}\mathcal{L}(-,X\otimes M)_{L,M}\\&=[\mathcal{L}(M,-)_{X\otimes M,N},1]\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M,X\otimes M),-)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}\mathcal{L}(-,N)_{L,M}\ .\end{aligned}$$

This is an equation of $\mathcal{V}$-morphisms $\mathcal{L}(L, M) \to [\mathcal{L}(X \otimes M, N), [\mathcal{L}(M, X \otimes M), \mathcal{L}(L, N)]]$, and it corresponds to an equation of $\mathcal{V}$-morphisms $\mathcal{L}(L, M) \otimes \mathcal{L}(X \otimes M, N) \to [\mathcal{L}(M, X \otimes M), \mathcal{L}(L, N)]$ under the adjunction of the closed structure, which in turn corresponds to an equation of $\mathcal{V}$-morphisms $\mathcal{L}(X \otimes M, N) \otimes \mathcal{L}(L, M) \to [\mathcal{L}(M, X \otimes M), \mathcal{L}(L, N)]$ by composition with c.

Recall that $\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M, X \otimes M, -), -)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}$ corresponds to $M_{\mathcal{V}}$ under the adjunction, while $\mathrm{Hom}_{\mathcal{V}}(-, \mathcal{L}(L, N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}$ corresponds to $M_{\mathcal{V}}c$. The correspondence is given by application of $- \otimes \mathcal{L}(X \otimes M, N)$ and composition with $e^{\mathcal{L}(X\otimes M,N)}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}$ (and we also compose with c).

Then the transform of the left hand side of the naturality condition is:

$$
\begin{aligned}
& e^{\mathcal{L}(X\otimes M,N)}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]} \\
& \quad (([\mathcal{L}(L, -)_{X\otimes M,N}, 1]\mathrm{Hom}_{\mathcal{V}}(-, \mathcal{L}(L, N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}\mathcal{L}(-, X \otimes M)_{L,M}) \otimes 1_{\mathcal{L}(X\otimes M,N)})c \\
&= e^{[\mathcal{L}(L,X\otimes M),\mathcal{L}(L,N)]}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}((\mathrm{Hom}_{\mathcal{V}}(-, \mathcal{L}(L, N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}\mathcal{L}(-, X \otimes M)_{L,M}) \otimes \mathcal{L}(L, -)_{X\otimes M,N})c \\
& \quad \text{(by extraordinary } \mathcal{V}\text{-naturality of } e) \\
&= M_{\mathcal{V}}c(\mathcal{L}(-, X \otimes M)_{L,M} \otimes \mathcal{L}(L, -)_{X\otimes M,N})c \\
&= M_{\mathcal{V}}(\mathcal{L}(L, -)_{X\otimes M,N} \otimes \mathcal{L}(-, X \otimes M)_{L,M}) \\
& \quad \text{(by naturality of } c \text{ and } c^2 = 1) .
\end{aligned}
$$

Similarly, the transform of the right hand side of the naturality condition is:

$$
\begin{aligned}
& e^{\mathcal{L}(X\otimes M,N)}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]} \\
& \quad (([\mathcal{L}(M, -)_{X\otimes M,N}, 1]\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M, X \otimes M), -)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}\mathcal{L}(-, N)_{L,M}) \otimes 1_{\mathcal{L}(X\otimes M,N)})c \\
&= e^{[\mathcal{L}(M,X\otimes M),\mathcal{L}(M,N)]}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}((\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M, X \otimes M), -)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}\mathcal{L}(-, N)_{L,M}) \otimes \mathcal{L}(M, -)_{X\otimes M,N})c \\
& \quad \text{(by extraordinary } \mathcal{V}\text{-naturality of } e) \\
&= M_{\mathcal{V}}(\mathcal{L}(-, N)_{L,M} \otimes \mathcal{L}(M, -)_{X\otimes M,N})c .
\end{aligned}
$$

This proves the missing equation used in the chain of equations above, and thus ends the proof. □

5 Identification of the $\mathcal{V}$-category of T-algebras

Let $\mathcal{V}$ be complete and cocomplete. Let $\mathcal{P}$ be a small and $\mathcal{L}$ be a cocomplete $\mathcal{V}$-category, and denote again by T the $\mathcal{V}$-monad $\mathrm{Lan}_H[H, 1]\colon [\mathrm{ob}\mathcal{P}, \mathcal{L}] \to [\mathrm{ob}\mathcal{P}, \mathcal{L}]$ from 4.1 given by the Kan adjunction $\mathrm{Lan}_H \dashv [H, 1]$. Recall that we denote by $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ the Eilenberg-Moore object i.e. the $\mathcal{V}$-category of T-algebras, which we described in Proposition 2 from 2.5 explicitly in the special case that $\mathcal{V} = \mathcal{G}ray$.

We are now going to show that T-Alg is isomorphic as a $\mathcal{V}$-category to the functor $\mathcal{V}$-category $[\mathcal{P}, \mathcal{L}]$ i.e. that $[H, 1]$ is strictly monadic, which is the content of Theorem 4 below.

Lemma 13. *Let (A, a) be a T-algebra cf. 2.5. Then $A\colon \mathrm{ob}\mathcal{P} \to \mathcal{L}$ and the transforms $A_{PQ} := n(a_{PQ})\colon \mathcal{P}(P, Q) \to \mathcal{L}$ under the adjunction* (33) *from 4.2 of the components a_{PQ} of a at objects $P, Q \in \mathcal{P}$ have the structure of a $\mathcal{V}$-functor. Conversely, if $A\colon \mathcal{P} \to \mathcal{L}$ is a $\mathcal{V}$-functor, then the* *function on objects considered as a functor $A\colon \operatorname{ob}\mathcal{P} \to \mathcal{L}$ and the transformation a induced by the transforms $n^{-1}(A_{PQ})$ of the strict hom functors are the underlying data of a T-algebra.*

Proof. The 1-cell $a\colon TA \to A$ has component

$$a_Q\colon (TA)Q = \int^{P\in\operatorname{ob}\mathcal{P}} \mathcal{P}(P,Q)\otimes AP \to AQ$$

at the object $Q \in \mathcal{P}$, and this is in turn induced from the $\mathcal{V}$-natural family of components

$$a_{PQ}\colon \mathcal{P}(P,Q)\otimes AP \to AQ\ .$$

These are elements of $\mathcal{L}(\mathcal{P}(P,Q)\otimes AP, AQ)$. Under the hom $\mathcal{V}$-adjunction (33) from 4.2 of the tensor product, these correspond to elements of the internal hom $[\mathcal{P}(P,Q), \mathcal{L}(AP,AQ)]$, i.e.

$$A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP,AQ)\ .$$

We now have to examine how the algebra axiom cf. 2.5

$$M_{[\operatorname{ob}\mathcal{P},\mathcal{L}]}(a, Ta) = M_{[\operatorname{ob}\mathcal{P},\mathcal{L}]}(a, \mu_A)$$

transforms under the adjunction.

This is an equation of morphisms in $[\operatorname{ob}\mathcal{P},\mathcal{L}]_0$, which is equivalent to the equations

$$M_{\mathcal{L}}(a_{QR}, \mathcal{P}(Q,R)\otimes a_{PQ}) = M_{\mathcal{L}}(a_{PR}, (M_{\mathcal{P}}\otimes 1_{AP})\alpha^{-1})$$

of elements in $\mathcal{L}(\mathcal{P}(Q,R)\otimes(\mathcal{P}(P,Q)\otimes AP), AR)$ where P, Q, R run through the objects in $\mathcal{P}$. To apply the hom $\mathcal{V}$-adjunction (33) from 4.2 for $X = \mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)$ and $L = AP$ and $M = AQ$, we consider the equivalent equations

$$M_{\mathcal{L}}(a_{QR}, (\mathcal{P}(Q,R)\otimes a_{PQ})\alpha) = M_{\mathcal{L}}(a_{PR}, M_{\mathcal{P}}\otimes 1_{AP})\ . \tag{59}$$

Applying Lemma 11 from 4.3 to the left hand side shows that its transform is given by[2]

$$M_{\mathcal{L}}(A_{QR}\otimes A_{PQ})\ .$$

On the other hand, the image of the right hand side under (57) from 4.3 is determined by the

[2]In fact, we do not need the full strength of Lemma 11 here, and one could do with more elementary considerations if one was merely concerned with the identification of algebras.

following elementary transformations [3].

$$
\begin{aligned}
&([\eta^{AP}_{\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)}, 1]\mathcal{L}(L,-)_{(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\otimes AP,AR} M_{\mathcal{L}})(a_{PR}, M_{\mathcal{P}} \otimes 1_{AP})\\
&= [\eta^{AP}_{\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)}, 1](M_{\mathcal{V}}(\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR}(a_{PR}),\\
&\qquad \mathcal{L}(AP,-)_{(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\otimes AP,\mathcal{P}(P,R)\otimes AR}(M_{\mathcal{P}} \otimes 1_{AP})))\\
&\quad \text{(by the functor axiom for } \mathcal{L}(AP,-))\\
&= M_{\mathcal{V}}(\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR}(a_{PR}),\\
&\qquad [\eta^{AP}_{\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)}, 1](\mathcal{L}(AP,-)_{(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\otimes AP,\mathcal{P}(P,R)\otimes AR}(M_{\mathcal{P}} \otimes 1_{AP})))\\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\\
&= M_{\mathcal{V}}(\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR}(a_{PR}), [1, \eta^{AP}_{\mathcal{P}(P,R)}](M_{\mathcal{P}}))\\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } \eta)\\
&= M_{\mathcal{V}}(([\eta^{AP}_{\mathcal{P}(P,R)}, 1]\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR})(a_{PR}), M_{\mathcal{P}})\\
&\quad \text{(by extraordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\\
&= M_{\mathcal{V}}(A_{PR}, M_{\mathcal{P}}) = A_{PR}M_{\mathcal{P}}
\end{aligned}
$$

Hence, the algebra axiom is equivalent to the equation

$$M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) = A_{PQ}M_{\mathcal{P}} \;, \tag{60}$$

and this is exactly one of the two axioms for a $\mathcal{V}$-functor.

Now we want to determine the transform of the other axiom of a T-algebra:

$$1_A = M_{[\mathrm{ob}\mathcal{P},\mathcal{L}]}(a, \eta_A) \;.$$

First note that this equation is equivalent to the equations

$$1_{AP} = M_{\mathcal{L}}(a_{PP}, (j_P \otimes 1)\lambda^{-1}_{AP})$$

on objects in $\mathcal{L}(AP, AP)$ where P runs through the objects of $\mathcal{P}$. In turn, these are equivalent to the equations

$$\lambda_{AP} = M_{\mathcal{L}}(a_{PP}, j_P \otimes 1) \;.$$

By definition of the unitor for the tensor product, the transform of the left hand side is given by the unit $j_{AP}\colon I \to \mathcal{L}(AP, AP)$ of the $\mathcal{V}$-category at AP (under the identification of elements of the internal hom and morphisms in $\mathcal{V}$). On the other hand, it is routine to identify the transform of the right hand side as $A_{PP}j_P$. Thus the second axiom of a T-algebra is equivalent to

$$j_{AP} = A_{PP}j_P \;, \tag{61}$$

and this is exactly the other axiom of a $\mathcal{V}$-functor. □

[3]We will be short on such routine transformations below. The computation here should serve as an example for basic naturality transformations of the same kind. One could subsume this into another more elementary lemma.

Recall that the hom object of $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ at algebras (A,a) and (B,b) is given by the equalizer

$$[\mathrm{ob}\mathcal{P},\mathcal{L}]^T(A,B) \longrightarrow [\mathrm{ob}\mathcal{P},\mathcal{L}](A,B) \underset{[\mathrm{ob}\mathcal{P},\mathcal{L}](1,b)T_{A,B}}{\overset{[\mathrm{ob}\mathcal{P},\mathcal{L}](a,1)}{\rightrightarrows}} [\mathrm{ob}\mathcal{P},\mathcal{L}](TA,B)\ .$$

Spelling out the hom objects of $[\mathrm{ob}\mathcal{P},\mathcal{L}]$ as in 4.1, this is the same as the following equalizer

$$[\mathrm{ob}\mathcal{P},\mathcal{L}]^T(A,B) \longrightarrow \Pi_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(AP,BP) \underset{\Pi\mathcal{L}(1,b_P)(T_{A,B})_P}{\overset{\Pi\mathcal{L}(a_P,1)}{\rightrightarrows}} \Pi_{\mathcal{P}\in\mathrm{ob}\mathcal{P}}\mathcal{L}(\int^{R\in\mathrm{ob}\mathcal{P}}\mathcal{P}(R,P)\otimes AR,BP)$$

where $E_PT_{A,B} = (T_{A,B})_PE_P$ for a unique $\mathcal{V}$-morphism $(T_{A,B})_P\colon\ \mathcal{L}(AP,BP)\to\mathcal{L}(TAP,TBP)$. By (35) and the universal property of the end, this equalizer is the same as the equalizer of the compositions with $\Pi_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(\kappa_{RP},1)$ (by definition (36) of κ and by Yoneda), and since

$$n\colon\ \mathcal{L}(\mathcal{P}(R,P)\otimes AR,BP)\cong[\mathcal{P}(R,P),\mathcal{L}(AR,BP)]\,,$$

this is in turn the same as the following equalizer

$$[\mathrm{ob}\mathcal{P},\mathcal{L}]^T(A,B)) \longrightarrow \Pi_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(AP,BP) \underset{\Pi n\mathcal{L}(1,b_{RP})(\mathcal{P}(R,P)\otimes -)_{AR,BR}}{\overset{\Pi n\mathcal{L}(a_{RP},1)}{\rightrightarrows}} \Pi_{R,P\in\mathrm{ob}\mathcal{P}}[\mathcal{P}(R,P),\mathcal{L}(AR,BP)]\ .$$

(where we have used that $a_P\kappa_{R,P} = a_{RP}$ for the first morphism of the equalizer, and where we have used that $\mathcal{L}(\kappa^A_{R,P},1)(E_P)_{TA,TB}T_{A,B} = \mathcal{L}(1,\kappa^B_{R,P})(\mathcal{P}(R,P)\otimes -)_{AR,BR}(E_R)_{A,B}$ by ordinary $\mathcal{V}$-naturality of $\kappa^A_{R,P}$ in A and that $b_P\kappa_{R,P} = b_{RP}$ for the second morphism of the equalizer).

One can now use the Yoneda lemma to show that this is exactly the equalizer (28) from 4.1 which defines the hom object of the functor $\mathcal{V}$-category: one checks that both the first morphism given here and the transform of $\mathcal{L}(A-,AR)_{RP}$ map the identity at AP to A_{RP}, and both the second here and the transform of $\mathcal{L}(AP,B-)_{PR}$ map the identity at BR to B_{RP}.

Finally, note that the composition law of $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ is induced from the composition law of the functor category $[\mathrm{ob}\mathcal{P},\mathcal{L}]$, which in turn, is induced from the composition law of $\mathcal{L}$.

Likewise, the composition law of the functor category $[\mathcal{P},\mathcal{L}]$ is induced via the evaluation functors from the composition law in $\mathcal{L}$, and since the evaluation functors $E_P\colon\ [\mathcal{P},\mathcal{L}]\to\mathcal{L}$ where $P\in\mathrm{ob}\mathcal{P}$ factorize through $[H,1]$ and $E_P\colon\ [\mathrm{ob}\mathcal{P},\mathcal{L}]\to\mathcal{L}$, this means that $[H,1]\colon\ \mathrm{ob}[\mathcal{P},\mathcal{L}]\to\mathrm{ob}[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ and the $\mathcal{V}$-isomorphisms on the hom objects induced from the comparison of equalizers above satisfy the $\mathcal{V}$-functor axiom.

Similarly, this data is shown to satisfy the unit axiom for a $\mathcal{V}$-functor. This proves the following theorem.

Theorem 4. *Given a small $\mathcal{V}$-category $\mathcal{P}$ and a cocomplete $\mathcal{V}$-category $\mathcal{L}$, then the functor* $[H,1]\colon\ [\mathcal{P},\mathcal{L}]\to[\mathrm{ob}\mathcal{P},\mathcal{L}]$ *induced by the inclusion $H\colon\ \mathrm{ob}\mathcal{P}\to\mathcal{L}$ is strictly monadic for the $\mathcal{V}$-monad $T = [H,1]\mathrm{Lan}_H$ given by the Kan adjunction* $\mathrm{Lan}_H\dashv[H,1]$. *In particular, the functor $\mathcal{V}$-category $[\mathcal{P},\mathcal{L}]$ is isomorphic as a $\mathcal{V}$-category to the Eilenberg-Moore object* $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ *in $\mathcal{V}$-$\mathcal{CAT}$.* □

Remark 6. An alternative strategy to achieve this result is to use an enriched version of Beck's monadicity theorem (see for example [3, Th. II.2.1]). The Kan adjunction meets the conditions of such a theorem because $[H, 1]$ creates pointwise colimits, cf. Lemma 3 from 4.1.

Yet another strategy for specific $\mathcal{V}$ where an identification of the functor $\mathcal{V}$-category is known, is to be completely explicit: For example, if $\mathcal{V} = \mathcal{G}ray$, the functor category can be explicitly identified (cf. [7, Prop. 12.2]). With the help of the two Transformation Lemmata 11 and 12 from 4.3 it is then straightforward to identify the algebra 1-cells, algebra 2-cells, and algebra 3-cells from Proposition 2 from 2.5 explicitly as we have done it for algebras above.

6 Identification of the $\mathcal{G}ray$-category Ps-T-Alg of pseudo T-algebras

The $\mathcal{G}ray$-category $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$ of trihomomorphisms $\mathcal{P} \to \mathcal{L}$, tritransformations, trimodifications, and perturbations has been described by Gurski [7, Th. 9.4]. The basic definitions of the objects and the 2-globular data of the local hom 2-categories, i.e. trihomomorphisms, tritransformations, trimodifications, and perturbations may be found in [7, 4.]. These are of course definitions for the general case that domain and codomain are honest tricategories. In our case, they simplify considerably because domain and codomain are always $\mathcal{G}ray$-categories.

Let again $T = [H, 1]\mathrm{Lan}_H$ be the $\mathcal{G}ray$-monad on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ from 4.1 corresponding to the Kan adjunction $\mathrm{Lan}_H \dashv [H, 1]$, where $\mathcal{P}$ is small and $\mathcal{L}$ is cocomplete. The aim of this section is to prove Theorem 6 below, which states that Ps-T-Alg is isomorphic to the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$ determined by the locally strict trihomomorphisms.

The general idea of the proof is to identify how the pseudo data and axioms transform under the adjunction (33) from 4.2 of the tensor product. The main technical tools employed are the two Transformation Lemmata 11 and 12 from 4.3, and elementary identities involving the associators and unitors a, l, r, α, λ, and ρ, which are implied by the pentagon and triangle identity as presented in 4.3.

6.1 Homomorphisms of $\mathcal{G}ray$-categories

To characterize how Ps-T-Alg transforms under the adjunction of the tensor product, we now introduce the notions of Gray homomorphisms between $\mathcal{G}ray$-categories, say $\mathcal{P}$ and $\mathcal{L}$, Gray transformations, Gray modifications, and Gray perturbations in Definitions 9-12. In fact, by reference to Ps-T-Alg, we show that these form a $\mathcal{G}ray$-category Gray$(\mathcal{P}, \mathcal{L})$. On the other hand, we maintain that this is in fact the natural notion of a (locally strict) trihomomorphism when domain and target are $\mathcal{G}ray$-categories, and when the definitions are to be given on Gray products and in terms of their composition laws, say $M_{\mathcal{P}}$ and $M_{\mathcal{L}}$, rather than on cartesian products and in terms of the corresponding cubical composition functors. Thus, the definitions below are easily seen to be mild context-related modifications of the definitions of (locally strict) trihomomorphisms between $\mathcal{G}ray$-categories, tritransformations, trimodifications, and perturbations cf. [7, 4.].

Definition 9. A Gray homomorphism $A\colon \mathcal{P} \to \mathcal{L}$ consists of

- a function on the objects $P \mapsto AP$ denoted by the same letter as the trihomomorphism itself;
- for objects $P, Q \in \mathcal{P}$, a strict functor $A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$;
- for objects $P, Q, R \in \mathcal{P}$, an adjoint equivalence

$$(\chi, \chi^{\bullet})\colon M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) \Rightarrow A_{PR}M_{\mathcal{P}}\colon \mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q) \to \mathcal{L}(AP, AR)\,;$$

- for each object $P \in \mathcal{P}$, an adjoint equivalence

$$(\iota, \iota^{\bullet})\colon j_{AP} \Rightarrow A_{PP}j_P\colon I \to \mathcal{L}(AP, AP)\,;$$

- and three families of invertible modifications **(GHM1)-(GHM3)** which are subject to two axioms **(GHA1)-(GHA2)**:

(GHM1) For objects $P, Q, R, S \in \mathcal{P}$, an invertible modification

$$\begin{aligned} \omega_{PQRS}\colon \quad & [(M_{\mathcal{P}} \otimes 1)a^{-1}, 1](\chi_{PQS}) * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\chi_{QRS}, 1_{A_{PQ}})) \\ \Rrightarrow \quad & [1 \otimes M_{\mathcal{P}}, 1](\chi_{PRS}) * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{A_{RS}}, \chi_{PQR})) \end{aligned}$$

of pseudonatural transformations

$$M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(A_{RS} \otimes (A_{QR} \otimes A_{PQ})) \Rightarrow A_{PS}(1 \otimes M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}}))$$

of strict functors

$$\mathcal{P}(R,S) \otimes (\mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q)) \to \mathcal{L}(AP, AS)\,.$$

(GHM2) For objects $P, Q \in \mathcal{P}$, an invertible modification

$$\gamma_{PQ}\colon [(j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, 1](\chi_{PQQ}) * [l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(\iota_Q, 1_{A_{PQ}})) \Rrightarrow 1_{A_{PQ}}$$

of pseudonatural transformations $A_{PQ} \Rightarrow A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$.
(GHM3) For objects $P, Q \in \mathcal{P}$, an invertible modification

$$\delta_{PQ}\colon 1_{A_{PQ}} \Rrightarrow [(1 \otimes j_P)r^{-1}_{\mathcal{P}(P,Q)}, 1](\chi_{PPQ}) * [r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1_{A_{PQ}}, \iota_P))$$

of pseudonatural transformations $A_{PQ} \Rightarrow A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$.
(GHA1) For objects $P, Q, R, S, T \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned} & [1 \otimes (1 \otimes M_{\mathcal{P}}), 1](\omega) * 1_{[M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})](\mathrm{Ten}(1, \mathrm{Ten}(1, \chi)))} \\ \diamond\ & 1_{[(M_{\mathcal{P}} \otimes M_{\mathcal{P}})a^{-1}, 1](\chi)} * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\Sigma_{\chi,\chi})) \\ \diamond\ & [(M_{\mathcal{P}} \otimes 1)a^{-1}, 1](\omega) * 1_{[a^{-1}a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(\chi, 1), 1))} \\ = \quad & 1_{[1 \otimes (M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})), 1](\chi)} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1, \omega)) \\ \diamond\ & [1 \otimes ((M_{\mathcal{P}} \otimes 1)a^{-1}), 1](\omega) * 1_{[a^{-1}(1 \otimes a^{-1}), M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(1, \chi), 1))} \\ \diamond\ & 1_{[(M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})a) \otimes 1, 1](\chi)} * [(a \otimes 1)a^{-1}a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\omega, 1)) \end{aligned}$$

To save space, we here employed the notation that vertical composition binds less strictly than horizontal composition $*$, which is also indicated by a line break. Also Ten always denotes the corresponding strict hom functor of the $\mathcal{G}ray$-functor Ten: $\mathcal{G}ray \otimes \mathcal{G}ray \to \mathcal{G}ray$ cf. eq. (58) from 4.3 above. It is to be noted that in each vertical factor there appears only one nontrivial horizontal factor, and this applies generally to the following definitions.

The axiom is an equation of 2-cells i.e. modifications

$$\begin{array}{rl} & [((M_{\mathcal{P}}(M_{\mathcal{P}} \otimes 1)) \otimes 1)a^{-1}a^{-1}, 1](\chi_{PQT}) \\ * & [((M_{\mathcal{P}} \otimes 1) \otimes 1)a^{-1}a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\chi_{QRT}, 1)) \\ * & [a^{-1}a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(\chi_{RST}, 1), 1)) \\ \Rrightarrow & [1 \otimes (M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})), 1](\chi_{PST}) \\ * & [1 \otimes (1 \otimes M_{\mathcal{P}}), M_{\mathcal{L}}](\mathrm{Ten}(1, \chi_{PRS})) \\ * & [1, M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})](\mathrm{Ten}(1, \mathrm{Ten}(1, \chi_{PQR}))) \end{array}$$

between 1-cells i.e. pseudonatural transformations

$$M_{\mathcal{L}}(1 \otimes (M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})))(A_{ST} \otimes (A_{RS} \otimes (A_{QR} \otimes A_{PQ}))) \Rightarrow A_{PT} M_{\mathcal{P}}(1 \otimes (M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})))$$

of strict functors

$$\mathcal{P}(S, T) \otimes (\mathcal{P}(R, S) \otimes (\mathcal{P}(Q, R) \otimes \mathcal{P}(P, Q))) \to \mathcal{L}(AP, AT) .$$

We remark that we chose another bracketing than in the definition of a trihomomorphism in the references [7] and [4]. The difference is of course not substantive.

(GHA2) For objects $P, Q, R \in \mathcal{P}$, the following equation of modifications is required:

$$\begin{array}{rl} & 1_{\chi} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{A_{QR}}}, \gamma_{PQ})) \\ \diamond & [1 \otimes ((j_Q \otimes 1) l^{-1}_{\mathcal{P}(P,Q)}), 1](\omega_{PQQR}) * 1_{[r^{-1}_{\mathcal{P}(Q,R)} \otimes 1, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(1,\iota),1))} \\ = & 1_{[((1 \otimes j) r^{-1}) \otimes 1](\chi)} * [1, M_{\mathcal{L}}](\mathrm{Ten}(\delta^{-1}_{QR}, 1_{1_{A_{PQ}}})) \end{array}$$

This is an equation of 2-cells i.e. modifications

$$\chi_{PQR} \Rrightarrow \chi_{PQR} \colon M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) \Rightarrow A_{PR} M_{\mathcal{L}} \colon \mathcal{P}(Q, R) \otimes \mathcal{P}(P, Q) \to \mathcal{L}(AP, AR) .$$

Definition 10. Let $A, B \colon \mathcal{P} \to \mathcal{L}$ be homomorphisms of $\mathcal{G}ray$-categories. A Gray transformation $f \colon A \Rightarrow B$ consists of

- a family $(f_P)_{P \in \mathrm{ob}\mathcal{P}}$ of objects $f_P \colon AP \to BP$ in $\mathcal{L}(AP, BP)$;
- for objects $P, Q \in \mathcal{P}$, an adjoint equivalence

$$(f_{PQ}, f^{\bullet}_{PQ}) \colon \mathcal{L}(AP, f_Q) A_{PQ} \Rightarrow \mathcal{L}(f_P, BQ) B_{PQ} \colon \mathcal{P}(P, Q) \to \mathcal{L}(AP, BQ) ;$$

- and two families of invertible modifications **(GTM1)**-**(GTM2)** which are subject to three axioms **(GTA1)**-**(GTA3)**:

(GTM1) For objects $P, Q, R \in \mathcal{P}$, an invertible modification

$$\begin{aligned} \Pi_{PQR}\colon \quad & [1, \mathcal{L}(f_P, BR)](\chi^B_{PQR}) * [1, M_\mathcal{L}](\mathrm{Ten}(1_{B_{QR}}, f_{PQ})) * [1, M_\mathcal{L}](\mathrm{Ten}(f_{QR}, 1_{A_{PQ}})) \\ \Rrightarrow \quad & [M_\mathcal{P}, 1](f_{PR}) * [1, \mathcal{L}(AP, f_R)](\chi^A_{PQR}) \end{aligned}$$

of pseudonatural transformations $M_\mathcal{L}((\mathcal{L}(AQ, f_R)A_{QR}) \otimes A_{PQ}) \Rightarrow \mathcal{L}(f_P, BR)B_{PR}M_\mathcal{P}$ of strict functors $\mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q) \to \mathcal{L}(AP, BR)$;
(GTM1) For each object $P \in \mathcal{P}$, an invertible modification

$$M_P\colon\ [j_P, 1](f_{PP}) * [1, \mathcal{L}(AP, f_P)](\iota^A_P) \Rrightarrow [1, \mathcal{L}(f_P, BP)](\iota^B_P)$$

of pseudonatural transformations $f_P \Rightarrow \mathcal{L}(f_p, BP)B_{PP}j_P\colon\ I \to \mathcal{L}(AP, BP)$.
(GTA1) For objects $P, Q, R, S \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned} & 1_{[M_\mathcal{P}(1\otimes M_\mathcal{P}),1](f_{PS})} * [1, \mathcal{L}(AP, f_S)](\omega^A) \\ \diamond \quad & [(M_\mathcal{P} \otimes 1)a^{-1}, 1](\Pi) * 1_{[1\otimes a^{-1}, M_\mathcal{L}(1\otimes M_\mathcal{L})](\mathrm{Ten}(\mathrm{Ten}(1,\chi^A),1))} \\ \diamond \quad & 1_{[(M_\mathcal{P}\otimes 1)a^{-1}, \mathcal{L}(f_P, BS)](\chi^B)} * 1_{[(M_\mathcal{P}\otimes 1)a^{-1}, M_\mathcal{L}]\mathrm{Ten}(1, f_{PQ})} * [a^{-1}, M_\mathcal{L}](\mathrm{Ten}(\Pi, 1_{1_{A_{PQ}}})) \\ \diamond \quad & 1_{[(M_\mathcal{P}\otimes 1)a^{-1}, \mathcal{L}(f_P, BS)](\chi^B)} * [a^{-1}, M_\mathcal{L}](\mathrm{Ten}(\Sigma_{\chi^B, f_{PQ}})) \\ & \quad * 1_{[a^{-1}, M_\mathcal{L}(M_\mathcal{L}\otimes 1)]\mathrm{Ten}(\mathrm{Ten}(1, f_{QR}),1)} * 1_{[a^{-1}, M_\mathcal{L}(M_\mathcal{L}\otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS},1),1)} \\ = \quad & [1 \otimes M_\mathcal{P}, 1](\Pi) * 1_{[1, \mathcal{L}(AP, f_S)M_\mathcal{L}]\mathrm{Ten}(1,\chi^A)} \\ \diamond \quad & 1_{[1\otimes M_\mathcal{P}, \mathcal{L}(f_P, BS)](\chi^B)} * 1_{[1\otimes M_P, M_\mathcal{L}]\mathrm{Ten}(1, f_{PR})} * [1, M_\mathcal{L}](\mathrm{Ten}(\Sigma^{-1}_{f_{RS}, \chi^A})) \\ \diamond \quad & 1_{[1\otimes M_\mathcal{P}, \mathcal{L}(f_P, BS)](\chi^B)} * [1, M_\mathcal{L}](\mathrm{Ten}(1_{1_{B_{RS}}}, \Pi)) * 1_{[a^{-1}, M_\mathcal{L}(M_\mathcal{L}\otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS},1),1)} \\ \diamond \quad & [1, \mathcal{L}(f_P, BS)](\omega^B) * 1_{[1, M_\mathcal{L}(M_\mathcal{L}\otimes 1)](\mathrm{Ten}(1, \mathrm{Ten}(1, f_{PQ})))} \\ & \quad * 1_{[a^{-1}, M_\mathcal{L}(M_\mathcal{L}\otimes 1)]\mathrm{Ten}(\mathrm{Ten}(1, f_{QR}),1)} * 1_{[a^{-1}, M_\mathcal{L}(M_\mathcal{L}\otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS},1),1)} \end{aligned}$$

This is an equation of modifications

$$\begin{aligned} * \quad & [(M_\mathcal{P} \otimes 1)a^{-1}, \mathcal{L}(f_P, BS)](\chi^B) \\ * \quad & [a^{-1}, \mathcal{L}(f_P, BS)M_\mathcal{L}](\mathrm{Ten}(\chi^B, 1)) \\ * \quad & [1, M_\mathcal{L}(M_\mathcal{L} \otimes 1)](\mathrm{Ten}(1, \mathrm{Ten}(1, f_{PQ}))) \\ * \quad & [a^{-1}, M_\mathcal{L}(M_\mathcal{L} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(1, f_{QR}), 1) \\ * \quad & [a^{-1}, M_\mathcal{L}(M_\mathcal{L} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS}, 1), 1) \\ \Rrightarrow \quad & [M_\mathcal{P}(1 \otimes M_\mathcal{P}), 1]f_{PS} \\ * \quad & [1 \otimes M_\mathcal{P}, \mathcal{L}(1, f_S)](\chi^A) \\ * \quad & [1, \mathcal{L}(1, f_S)M_\mathcal{L}]\mathrm{Ten}(1, \chi^A) \end{aligned}$$

of pseudonatural transformations

$$\mathcal{L}(AR, f_S)M_\mathcal{L}(1 \otimes M_\mathcal{L})(A_{RS} \otimes (A_{QR} \otimes A_{PQ})) \Rightarrow \mathcal{L}(f_P, BS)B_{PS}M_\mathcal{P}(1 \otimes M_\mathcal{P})$$

of strict functors

$$\mathcal{P}(R,S)\otimes(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\to\mathcal{L}(AP,BS)\ .$$

(GTA2) For objects $P,Q\in\mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
&1_{f_{PQ}} * [1,\mathcal{L}(AP,f_Q)](\gamma^A_{PQ})\\
\diamond\ & [(j_Q\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},1](\Pi_{PQQ}) * 1_{[l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}([1,\mathcal{L}(1,f_Q)](\iota^A_Q),1_{A_{PQ}}))}\\
=\ & [1,\mathcal{L}(f_P,BQ)](\gamma^B_{PQ}) * 1_{f_{PQ}}\\
\diamond\ & 1_{[(j_Q\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(f_P,1)](\chi^B_{PQQ})} * [l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(\Sigma^{-1}_{\iota^B_Q,f_{PQ}}))\\
\diamond\ & 1_{[(j_Q\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(f_P,1)](\chi^B_{PQQ})} * 1_{[((B_{QQ}j_Q)\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(1,f_{PQ}))} * [l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(M_Q,1_{1_{A_{PQ}}}))
\end{aligned}$$

This is an equation of modifications

$$\begin{aligned}
&[(j_Q\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(f_P,BQ)](\chi_{PQQ}) * [((B_{QQ}j_Q)\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(1,f_{PQ}))\\
* \ & [((j_Q\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(f_{QQ},1)) * [l^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(AP,f_Q)](\iota_Q)\\
\Rrightarrow\ & f_{PQ}\ .
\end{aligned}$$

(GTA3) For objects $P,Q\in\mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
&[1,\mathcal{L}(f_P,BQ)]((\delta^B)^{-1}_{PQ}) * 1_{f_{PQ}}\\
\diamond\ & 1_{[(1\otimes j_P)r^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(f_P,BQ)](\chi^B_{PPQ})} * [r^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{B_{PQ}}},M_P)) * 1_{f_{PQ}}\\
=\ & 1_{f_{PQ}} * [1,\mathcal{L}(AP,f_Q)]((\delta^A)^{-1}_{PQ})\\
\diamond\ & [(1\otimes j_P)r^{-1}_{\mathcal{P}(P,Q)},1](\Pi_{PPQ}) * 1_{[r^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(1_{\mathcal{L}(AP,f_Q)A_{PQ}},\iota^A_P))}\\
\diamond\ & 1_{[(1\otimes j_P)r^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(f_P,BQ)](\chi^B_{PPQ})} * 1_{[(1\otimes j_P)r^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{PQ}},f_{PP}))} * [r^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(\Sigma^{-1}_{f_{PQ},\iota^A_P}))
\end{aligned}$$

This is an equation of modifications

$$\begin{aligned}
&[(1\otimes j_P)r^{-1}_{\mathcal{P}(P,Q)},\mathcal{L}(f_P,BQ)](\chi^B_{PPQ}) * [(1\otimes j_P)r^{-1},M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{PQ}},f_{PP}))\\
* \ & [r^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{PQ}},[1,\mathcal{L}(AP,f_P)](\iota_P))) * f_{PQ}\\
\Rrightarrow\ & f_{PQ}\ .
\end{aligned}$$

Definition 11. Let $f,g\colon A\Rightarrow B\colon\mathcal{P}\to\mathcal{L}$ be Gray transformations. A Gray modification $\alpha\colon f\Rrightarrow g$ consists of

- a family $(\alpha_P)_{P\in\mathrm{ob}\mathcal{P}}$ of 1-cells $\alpha_P\colon f_P\to g_P$ in $\mathcal{L}(AP,BP)$;
- and one family of invertible modifications **(GMM1)** which is subject to two axioms **(GMA1)**-**(GMA2)**:

(GMM1) For objects $P, Q \in \mathcal{P}$, an invertible modification

$$\alpha_{PQ}\colon\ [B_{PQ}, 1](\mathcal{L}(\alpha_P, BQ)) * f_{PQ} \Rrightarrow g_{PQ} * [A_{PQ}, 1](\mathcal{L}(AP, \alpha_Q))$$

of pseudonatural transformations

$$\mathcal{L}(AP, f_Q)A_{PQ} \Rightarrow \mathcal{L}(g_P, BQ)B_{PQ}\colon\ \mathcal{P}(P, Q) \to \mathcal{L}(AP, BQ)\ .$$

(GMA1) For objects $P, Q, R \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
& \Pi^g * 1_{[1,M_{\mathcal{L}}](\mathrm{Ten}([A_{QR},1](\mathcal{L}(AQ,\alpha_R)),1))} \\
\diamond\ & 1_{M_{\mathcal{G}ray}(1,\chi^B_{PQR})} * 1_{[1,M_{\mathcal{L}}](\mathrm{Ten}(1,g_{PQ}))} * [1, M_{\mathcal{L}}](\mathrm{Ten}(\alpha_{QR}, 1_{1_{A_{PQ}}})) \\
\diamond\ & 1_{M_{\mathcal{G}ray}(1,\chi^B_{PQR})} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{B_{QR}}}, \alpha_{PQ})) * 1_{[1,M_{\mathcal{L}}](\mathrm{Ten}(f_{QR},1_{A_{PQ}}))} \\
\diamond\ & M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P,BR),\chi^B_{PQR}}) * 1_{[1,M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{QR}},f_{PQ}))} * 1_{[1,M_{\mathcal{L}}](\mathrm{Ten}(f_{QR},1_{A_{PQ}}))} \\
=\ & 1_{[M_{\mathcal{P}},1](\alpha_{PR})} * M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\chi^A_{PQR}}) \\
\diamond\ & [M_{\mathcal{P}}, 1](\alpha_{PR}) * 1_{M_{\mathcal{G}ray}(1,\chi^A_{PQR})} \\
\diamond\ & 1_{[B_{PR}M_{\mathcal{P}},1](\mathcal{L}(\alpha_P,BR))} * \Pi^f
\end{aligned}$$

This is an equation of modifications

$$\begin{aligned}
& [B_{PR}M_{\mathcal{P}}, 1](\mathcal{L}(\alpha_P, BR)) \\
*\ & 1_{M_{\mathcal{G}ray}(1,\chi^B_{PQR})} \\
*\ & [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{QR}}, f_{PQ})) \\
*\ & [1, M_{\mathcal{L}}](\mathrm{Ten}(f_{QR}, 1_{A_{PQ}})) \\
\Rrightarrow\ & [M_{\mathcal{P}}, 1](g_{PR}) \\
*\ & M_{\mathcal{G}ray}(1, \chi^A_{PQR}) \\
*\ & [1, M_{\mathcal{L}}](\mathrm{Ten}([A_{QR}, 1](\mathcal{L}(AQ, \alpha_R)), 1))
\end{aligned}$$

of pseudonatural transformations

$$M_{\mathcal{L}}((\mathcal{L}(AQ, f_R)A_{QR}) \otimes A_{PQ}) \Rightarrow \mathcal{L}(g_P, BR)B_{PR}M_{\mathcal{P}}\ .$$

(GMA2) For an object $P \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
& M^g * 1_{\alpha_P} \\
\diamond\ & 1_{[j_P,1](g_{PP})} * M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota^A_P}) \\
\diamond\ & [j_P, 1](\alpha_{PP}) * 1_{[1,\mathcal{L}(AP,f_P)](\iota^A_P)} \\
=\ & M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota^B_P}) \\
\diamond\ & 1_{[l_I^{-1},M_{\mathcal{L}}](\mathrm{Ten}(1_{G_{PP}j_P},\alpha_P))} * M^f
\end{aligned}$$

This is an equation of modifications

$$[l_I^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(1_{G_{PP}j_P}, \alpha_P)) * f_{PP} * [l_I^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(1_{f_P}, \iota_P^A)) \Rrightarrow [l_I^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\iota_P^G, 1_{g_P})) * \alpha_P$$

of pseudonatural transformations

$$f_P \Rightarrow M_{\mathcal{L}}((G_{PP}j_P) \otimes g_P)l_I^{-1} \colon\ I \to \mathcal{L}(AP, BP)\ .$$

Definition 12. Let $\alpha,\beta\colon\ f \Rrightarrow g\colon\ A \Rightarrow B\colon\ \mathcal{P} \to \mathcal{L}$ be Gray modifications. A Gray perturbation $\Gamma\colon\ \alpha \Rrightarrow \beta$ consists of

- a family of 2-cells $\Gamma_P\colon\ \alpha_P \Rightarrow \beta_P$ in $\mathcal{L}(AP, BP)$;
- subject to one axiom **(GPA1)**:

(GPA1) For objects $P, Q \in \mathcal{P}$ the following equation of vertical composites of whiskered modifications is required:

$$\alpha_{PQ} \diamond ([B_{PQ}, 1](\mathcal{L}(\Gamma_P, BP)) * 1_{f_{PQ}}) = (1_{g_{PQ}} * [A_{PQ}, 1](\mathcal{L}(AP, \Gamma_Q))) \diamond \beta_{PQ}$$

This is an equation of modifications

$$[B_{PQ}, 1](\mathcal{L}(\alpha_P, BP)) * f_{PQ} \Rightarrow [A_{PQ}, 1](\mathcal{L}(AP, \beta_Q))\ .$$

6.2 The correspondence of Gray homomorphisms and pseudo algebras

The following theorem is one of the main results, and forms the first part of the promised correspondence of pseudo algebras and locally strict trihomomorphisms.

Theorem 5. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category and $\mathcal{L}$ be a cocomplete $\mathcal{G}ray$-category, and let T be the monad corresponding to the Kan adjunction. Then the notions of Gray homomorphism, Gray transformation, Gray modification, and Gray perturbation are precisely the transforms of the notions of a pseudo algebra, a pseudo functor, a pseudo transformation, and a pseudo modification respectively for the monad $T = [H, 1]\mathrm{Lan}_H$ on* $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$. □

We only present parts of the proof explicitly. The proof involves the determination of transforms under the hom $\mathcal{G}ray$-adjunction (33) from 4.2. These determinations involve the pentagon identity (48) and triangle identity (54) from 4.3 for associators and unitors both of the tensor products and of the monoidal category $\mathcal{G}ray$. We also need naturality of these associators and unitors as presented in the same paragraph. On the other hand, there are elementary identities due to naturality, which are similar to the one we displayed for the transform of the right hand side of the algebra axiom (59) above in 5., and then there is heavy use of the two technical Transformation Lemmata 11 and 12 from 4.3. In pursuing the proof, one quickly notices that many of the determinations of transforms are similar to each other. While we cannot display all of the computations, it is our aim to at least characterize the arguments needed for these different classes of transforms. Thus, in the lemmata below we provide examples that should serve as a complete guideline for the rest of the proof.

Lemma 14. *Taking transforms under the adjunction* (33) *of the tensor product from 4.2 induces a one-to-one correspondence of Gray homomorphisms $\mathcal{P} \to \mathcal{L}$ and pseudo algebras for the monad $T = [H, 1]\mathrm{Lan}_H$ on* $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$.

Proof. Let $(A, a, m, i, \pi, \lambda, \rho)$ be a pseudo T-algebra. From the identification of algebras we know that the components a_{PQ} of the 1-cell $a\colon TA \to A$ transform into strict functors $A_{PQ}\colon \mathcal{P}(P, Q) \to \mathcal{L}(AP, AQ)$ under the hom $\mathcal{G}ray$-adjunction (33) from 4.2 of the tensor product.

Since the adjoint equivalences m and i replace the two algebra axioms, we define adjoint equivalences $(\chi_{PQR}, \chi^{\bullet}_{PQR}) := (n\mathcal{L}(\alpha, 1))(m_{PQR}, m^{\bullet}_{PQR})$ for objects $P, Q, R \in \mathcal{P}$ and $(\iota_P, \iota^{\bullet}_P) := (n\mathcal{L}(\lambda, 1))(i_P, i^{\bullet}_P)$ for an object $P \in \mathcal{P}$. We have already determined domain and codomain of these transforms in the identification of the algebra axiom, and the adjoint equivalences in Definition 9 from 6.1 do indeed replace the axioms of a $\mathcal{G}ray$-functor cf. (60) and (61) in 5. above.

Next we have to show that the transforms of the components of the invertible 3-cells π, λ, and ρ correspond to the invertible modifications ω_{PQRS}, γ_{PQ}, and δ_{PQ} in the definition of a Gray homomorphism.

The components of π at objects $P, Q, R, S \in \mathcal{P}$ are invertible 3-cells in $\mathcal{L}$. We apply the invertible strict functor $\mathcal{L}((1 \otimes \alpha)\alpha, 1)$ to bring them into a form where we can apply the hom $\mathcal{G}ray$-adjunction (33) from 4.2. We then obtain invertible 3-cells $\mathcal{L}((1 \otimes \alpha)\alpha, 1)(\pi_{PQRS})$ of the form

$$\begin{aligned} & M_{\mathcal{L}}(m_{PQS}, 1_{\alpha(((M_{\mathcal{P}}\otimes 1)a^{-1})\otimes 1)}) * M_{\mathcal{L}}(m_{QRS}, 1_{(1_{\mathcal{P}(R,S)\otimes\mathcal{P}(Q,R)}\otimes a_{PQ})\alpha(a^{-1}\otimes 1)}) \\ \Rrightarrow\quad & M_{\mathcal{L}}(m_{PRS}, 1_{(\mathcal{P}(R,S)\otimes((M_{\mathcal{P}}\otimes 1_{AP})a^{-1}))}) * M_{\mathcal{L}}(1_{a_{RS}}, \mathcal{L}(\alpha, 1)(\mathcal{P}(R, S) \otimes m_{PQR}))\,, \end{aligned}$$

where we have already used the pentagon identity (48) from 4.3 for α and a.

The computations below then determine the transforms of the horizontal factors and show that these coincide precisely with the horizontal factors in the domain and codomain of the invertible modification ω_{PQRS} in Definition 9 from 6.1. For brevity we have suppressed many indices e.g. those of hom morphisms where we leave a comma as a subscript to indicate that they are hom morphisms.

Transform of the right hand factor of the domain:

$$\begin{aligned} & ([\eta, 1]\mathcal{L}(AP, -)_{,})(M_{\mathcal{L}}(m_{QRS}, 1_{(1_{\mathcal{P}(R,S)\otimes\mathcal{P}(Q,R)}\otimes a_{PQ})\alpha(a^{-1}\otimes 1)})) \\ =& ([a^{-1}, 1][\eta, 1]\mathcal{L}(AP, -)_{,}\mathcal{L}(\alpha, 1)M_{\mathcal{L}}(1 \otimes ((\mathcal{P}(R, S) \otimes \mathcal{P}(Q, R)) \otimes -)_{,}))(m_{QRS}, 1_{a_{PQ}}) \\ & \text{(by naturality)} \\ =& [a^{-1}, M_{\mathcal{L}}]\mathrm{Ten}_{,}(\chi_{QRS}, 1_{A_{PQ}}) \\ & \text{(by Lemma 11 from 4.3)} \end{aligned}$$

Transform of the left hand factor of the domain:

$$\begin{aligned} & ([\eta, 1]\mathcal{L}(AP, -)_{,})(M_{\mathcal{L}}(m_{PQS}, 1_{\alpha(((M_{\mathcal{P}}\otimes 1)a^{-1})\otimes 1)})) \\ =& ([(M_{\mathcal{P}} \otimes 1)a^{-1}, 1][\eta, 1]\mathcal{L}(AP, -)_{,}\mathcal{L}(\alpha, 1))(m_{PQS}) \\ & \text{(by naturality)} \\ =& [(M_{\mathcal{P}} \otimes 1)a^{-1}, 1](\chi_{PQS}) \end{aligned}$$

Transform of the right hand factor of the codomain:

$$
\begin{aligned}
&([\eta, 1]\mathcal{L}(AP, -)_,)(M_{\mathcal{L}}(1_{a_{RS}}, \mathcal{L}(a, 1)(\mathcal{P}(R, S) \otimes m_{PQR})))\\
=&[1, M_{\mathcal{L}}]\mathrm{Ten}_,(1_{A_{RS}}, \chi_{PQR})\\
&\text{(by Lemma 11 from 4.3)}
\end{aligned}
$$

Transform of the left hand factor of the codomain:

$$
\begin{aligned}
&([\eta, 1]\mathcal{L}(AP, -)_,)(M_{\mathcal{L}}(m_{PRS}, 1_{(\mathcal{P}(R,S)\otimes((M_{\mathcal{P}}\otimes 1_{AP})\alpha^{-1}))}))\\
=&[1 \otimes M_{\mathcal{P}}, 1](\chi_{PRS})\\
&\text{(by naturality)}
\end{aligned}
$$

Thus we may define ω_{PQRS} as the transform $(n\mathcal{L}((1 \otimes \alpha)\alpha, 1)(\pi_{PQRS})$.

Similarly, it is shown that γ_{PQ} and δ_{PQ} may be defined as the transforms of λ_{PQ} and ρ_{PQ} (where one has to use the first Transformation Lemma and the triangle identity).

Finally we have to show that the axioms of a Gray homomorphism are precisely the transforms of the axioms of a pseudo algebra. Observe that because there are only two axioms in the definition of a pseudo algebra, it is crucial that by Proposition 3 from 3.1 two of the lax algebra axioms are redundant for a pseudo algebra[4].

Now consider the pentagon-like axiom. The corresponding Gray homomorphism axiom and the pseudo algebra axiom are both composed out of three vertical factors on each side of the axiom. Each of the vertical factors is the horizontal composition of a nontrivial 2-cell and an identity 2-cell. Since the hom $\mathcal{G}ray$-adjunction (33) of the tensor product from 4.2 is given by strict functors, it preserves vertical and horizontal composition and it preserves identity 2-cells. It follows that we only have to show that the nontrivial 2-cells in each vertical factor match. In diagrammatic language this means that we only have to compare the nontrivial subdiagrams.

In fact, the determination of the transforms of the nontrivial 2-cells of the pseudo algebra axiom is perfectly straightforward and similar to the identification of the transforms of π_{PQRS}'s domain and codomain above. For example, the transform of the interchange cell is:

$$
\begin{aligned}
&([\eta, 1]\mathcal{L}(AP, -)_, M_{\mathcal{L}})(\Sigma_{m_{RST}, \mathcal{L}(\alpha(a^{-1}\otimes 1), 1)(\mathcal{P}(R,S)\otimes -)_,(m_{PQR})})\\
=&([\eta, 1]\mathcal{L}(AP, -)_, M_{\mathcal{L}}(1 \otimes (\mathcal{L}(\alpha(a^{-1} \otimes 1), 1)(\mathcal{P}(R, S) \otimes -)_,)))(\Sigma_{m_{RST}, m_{PQR}})\\
&\text{(by equation (4) from 2.3)}\\
=&([a^{-1}, 1][\eta, 1]\mathcal{L}(AP, -)_, \mathcal{L}(\alpha, 1)M_{\mathcal{L}}(1 \otimes (\mathcal{P}(R, S) \otimes -)_,))(\Sigma_{m_{RST}, m_{PQR}})\\
&\text{(by naturality of } M_{\mathcal{L}},\ \mathcal{L}(AP, -)_,, \text{ and } \eta)\\
=&[a^{-1}, M_{\mathcal{L}}]\mathrm{Ten}_,(\Sigma_{\chi_{RST}, \chi_{PQR}})\\
&\text{(by Lemma 11 from 4.3)}
\end{aligned}
$$

This is exactly the interchange cell appearing in the pentagon-like axiom of a Gray homomorphism. □

[4]We show below that Gray homomorphisms correspond to locally strict trihomomorphisms. The two redundant axioms correspond to two equations for a trihomomorphism that hold generally: This can be shown because they hold for strict trihomomorphisms by the left and right normalization axiom of a tricategory, and then they hold for a general trihomomorphism by coherence.

Notice that the proof above only involved the first Transformation Lemma from 4.3, namely Lemma 11. To show how the second Transformation Lemma i.e. Lemma 12 from 4.3 enters in the proof of Theorem 5, we provide the following lemma regarding the first axiom of a T-transformation. In fact, one has to employ Lemma 12 already in the proof of the correspondence for pseudo T-functors, but only for the axioms of a T-transformation, there appears a new class of interchange cells. Thus we skip the proof for the correspondence of pseudo T-functors and Gray transformations, and for the correspondence of the data of a T-transformation and the data of a Gray modification.

Lemma 15. *The transform of the first axiom* **(LTA1)** *of a T-transformation $\alpha\colon f \to g\colon A \to B$ is precisely the second axiom* **(GMA2)** *of a Gray modification.*

Proof. First note that the T-transformation axiom **(LTA1)** cf. Definition 3 from 3.1 is equivalent to the equations

$$\begin{aligned}&(\hbar^g_P * 1) \diamond (1 * M_{\mathcal{L}}(\Sigma_{\alpha_P, i^A_P})) \diamond (M_{\mathcal{L}}(A_{PP}, 1_{(j_P\otimes 1)\lambda^{-1}_{AP}}) * 1)\\ &= M_{\mathcal{L}}(\Sigma^{-1}_{i^B_P, \alpha_P}) \diamond (1 * \hbar^f_P)\,.\end{aligned}$$

where P runs through the objects of $\mathcal{P}$. We apply the invertible strict functor $\mathcal{L}(\lambda_{AP}, 1)$ to these equations, which gives the following equivalent equations:

$$\begin{aligned}&(\mathcal{L}(\lambda_{AP}, 1)(\hbar^g_P) * 1) \diamond (1 * M_{\mathcal{L}}(\Sigma_{\alpha_P, \mathcal{L}(\lambda_{AP},1)(i^A_P)})) \diamond (M_{\mathcal{L}}(A_{PP}, 1_{j_P\otimes 1}) * 1)\\ &= M_{\mathcal{L}}(\Sigma^{-1}_{i^B_P, \mathcal{L}(\lambda_{AP},1)(\alpha_P)}) \diamond (1 * \mathcal{L}(\lambda_{AP}, 1)(\hbar^f_P))\,.\end{aligned}$$

Here we have used naturality of $M_{\mathcal{L}}$ and equation (4) from 2.3 for the manipulation of the interchange cells. As above we only have to compare the transforms of the nontrivial 2-cells. The transforms of $\mathcal{L}(\lambda_{AP}, 1)(\hbar^g_P)$ and $\mathcal{L}(\lambda_{AP}, 1)(\hbar^f_P)$ are by definition the modifications M^g and M^f of the Gray transformation corresponding to the pseudo T-functors f and g. The transform of $M_{\mathcal{L}}(A_{PP}, 1_{j_P\otimes 1})$ is by naturality $[j_P, 1](\alpha_{PP})$.
The transform of the interchange cell $M_{\mathcal{L}}(\Sigma_{\alpha_P, \mathcal{L}(\lambda_{AP},1)i^A_P})$ is determined as follows:

$$\begin{aligned}&([\eta^{AP}_I, 1]\mathcal{L}(AP, -)_{I\otimes AP, BP})(M_{\mathcal{L}}(\Sigma_{\alpha_P, \mathcal{L}(\lambda_{AP},1)i^A_P}))\\ =&M_{\mathcal{G}ray}(\mathcal{L}(AP, -)_{AP,BP} \otimes ([\eta^{AP}_I, 1]\mathcal{L}(AP, -)_{I\otimes AP, AP}))(\Sigma_{\alpha_P, (\mathcal{L}(\lambda_{AP},1)(i^A_P))})\\ &\text{(by the functor axiom for } \mathcal{L}(AP, -) \text{ and naturality of } M_{\mathcal{G}ray})\\ =&M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P), [\eta^{AP}_I, 1]\mathcal{L}(AP,-)_{I\otimes AP, AP}(\mathcal{L}(\lambda_{AP},1)(i^A_P))})\\ &\text{(by equation (4) from 2.3)}\\ =&M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P), \iota^A_P})\end{aligned}$$

The transform of the interchange cell left is:

$$([\eta_I^{AP},1]\mathcal{L}(AP,-)_{I\otimes AP,BP})(M_{\mathcal{L}}(\Sigma^{-1}_{i_P^B,\mathcal{L}(\lambda_{AP},1)(\alpha_P)}))$$
$$=([\eta_I^{AP},1]\mathcal{L}(AP,-)_{I\otimes AP,BP})((M_{\mathcal{L}}(1\otimes(I\otimes -),))(\Sigma^{-1}_{\mathcal{L}(\lambda_{AP},1)(i_P^B),\alpha_P}))$$

(by naturality of λ, equation (4) from 2.3, and extraordinary naturality of $M_{\mathcal{Gray}}$)

$$=(M_{\mathcal{Gray}}((\mathcal{L}(-,BP)_{AP,BP})\otimes([\eta_I^{AP},1]\mathcal{L}(AP,-)_{(I\otimes AP,AP)})c)(\Sigma^{-1}_{\mathcal{L}(\lambda_{AP},1)(i_P^B),\alpha_P}))$$

(by Lemma 12 from 4.3)

$$=(M_{\mathcal{Gray}}((\mathcal{L}(-,BP)_{AP,BP})\otimes([\eta_I^{AP},1]\mathcal{L}(AP,-)_{(I\otimes AP,AP)}))(\Sigma_{\alpha_P,\mathcal{L}(\lambda_{AP},1)(i_P^B)}))$$

(by equation (8) from 2.3)

$$=M_{\mathcal{Gray}}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota_P^B})$$

(by equation (4) from 2.3)

□

This finishes our exhibition of the proof of Theorem 5, and we end this paragraph with the following trivial corollary of Theorem 5:

Corollary 5. *Given $\mathcal{Gray}$-categories $\mathcal{P}$ and $\mathcal{L}$, there is a $\mathcal{Gray}$-category* Gray$(\mathcal{P},\mathcal{L})$ *with objects Gray homomorphisms, 1-cells Gray transformations, 2-cells Gray modifications, and 3-cells Gray perturbations. If $\mathcal{P}$ is small and $\mathcal{L}$ is cocomplete,* Gray$(\mathcal{P},\mathcal{L})$ *is uniquely characterized by the requirement that the correspondence from Theorem 5 induces an isomorphism*

$$\text{Ps-}T\text{-Alg}\cong \text{Gray}(\mathcal{P},\mathcal{L})$$

of $\mathcal{Gray}$-categories for $T=[H,1]\mathrm{Lan}_H\colon\ [\mathrm{ob}\mathcal{P},\mathcal{L}]\to[\mathrm{ob}\mathcal{P},\mathcal{L}]$. □

Remark 7. Strictly speaking, Gray$(\mathcal{P},\mathcal{L})$ can of course only inherit the $\mathcal{Gray}$-category structure from Ps-T-Alg in the situation that the left Kan extension Lan_H along $H\colon\ \mathrm{ob}\mathcal{P}\to\mathcal{P}$ exists and has the explicit description from 4.2, e.g. if $\mathcal{P}$ is small and $\mathcal{L}$ is cocomplete, but in fact the prescriptions obtained in this case for the local 2-category structure and the $\mathcal{Gray}$-category structure of Gray$(\mathcal{P},\mathcal{L})$ are also valid if we are not in this situation i.e. if there are no restrictions on $\mathcal{P}$ and $\mathcal{L}$ cf. Theorem 6 below.

6.3 The correspondence with locally strict trihomomorphisms

The next theorem forms the second part of the promised correspondence of pseudo algebras and locally strict trihomomorphisms, which is then proved in Theorem 7 below.

Theorem 6. *Given $\mathcal{Gray}$-categories $\mathcal{P}$ and $\mathcal{L}$, the $\mathcal{Gray}$-category* Gray$(\mathcal{P},\mathcal{L})$ *is isomorphic as a $\mathcal{Gray}$-category to the full sub-$\mathcal{Gray}$-category $\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$ of $\mathcal{Tricat}(\mathcal{P},\mathcal{L})$ determined by the locally strict trihomomorphisms.* □

Again, we just indicate how the proof works, but we want to stress that the steps of the proof not displayed have been explicitly checked and they are indeed entirely analogous to the situations we discuss in the lemmata below.

Lemma 16. *Given $\mathcal{G}ray$-categories $\mathcal{P}$ and $\mathcal{L}$, there is a one-to-one correspondence between locally strict trihomomorphisms $\mathcal{P} \to \mathcal{L}$ and Gray homomorphisms $\mathcal{P} \to \mathcal{L}$.*

Proof. Comparing the definitions, the first thing to be noticed is that Definitions 9-12 from 6.1 involve considerably less cell data than the tricategorical definitions cf. [7, 4.3]. Since $\mathcal{P}$ and $\mathcal{L}$ are $\mathcal{G}ray$-categories, these supernumerary cells are all trivial.

Consider, for example, a locally strict trihomomorphism $A\colon \mathcal{P} \to \mathcal{L}$. Recall that this is given by (i) a function on the objects $P \mapsto AP$; (ii) for objects $P, Q \in \mathcal{P}$, a strict functor $A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP,AQ)$; (iii) for objects $P,Q,R \in \mathcal{P}$, an adjoint equivalence

$$(\chi_{PQR}, \chi^{\bullet}_{PQR})\colon\ M_{\mathcal{L}}C(A_{QR} \times A_{PQ}) \Rightarrow A_{PR}M_{\mathcal{P}}C\ ,$$

where C is again the universal cubical functor, and an adjoint equivalence

$$(\iota_P, \iota^{\bullet}_P)\colon\ j_{AP} \Rightarrow A_{PP}j_P$$

if $P = Q = R$; (iv) and three families ω, γ, δ of invertible modifications subject to two axioms. Up to this point, this looks very similar to Definition 9 from 6.1, the difference being in the form of domain and codomain of the adjoint equivalence $(\chi_{PQR}, \chi^{\bullet}_{PQR})$. However, observing that $C(A_{QR}\times A_{PQ}) = (A_{QR}\otimes A_{PQ})C$ by naturality of C, it is clear from Proposition 1 from 2.4 that this corresponds to an adjoint equivalence $(\hat{\chi}_{PQR}, \hat{\chi}^{\bullet}_{PQR})$ as in the definition of a Gray homomorphism such that $C^*(\hat{\chi}_{PQR}, \hat{\chi}^{\bullet}_{PQR}) = (\chi_{PQR}, \chi^{\bullet}_{PQR})$.

Next, given objects $P, Q, R, S \in \mathcal{P}$, the modification ω_{PQRS} has the form[5]

$$\begin{aligned} \omega_{PQRS}\colon \quad & (((M_{\mathcal{P}}C) \times 1)a_{\times}^{-1})^*(\chi_{PQS}) * (a_{\times}^{-1})^*(M_{\mathcal{L}}C)_*(\chi_{QRS} \times 1_{A_{PQ}}) \\ \Rrightarrow \quad & (1 \times (M_{\mathcal{P}}C))^*(\chi_{PRS}) * (M_{\mathcal{L}}C)_*(1_{A_{RS}} \times \chi_{PQR}) \end{aligned}$$

where we used strictness of the local functors, and where we made the monoidal structure of the cartesian product explicit, e.g. $a_{\times}$ denotes the corresponding associator.

This is the same as

$$\begin{aligned} \omega_{PQRS}\colon \quad & (C(1\times C))^*([(M_{\mathcal{P}} \otimes 1)a^{-1}, 1](\hat{\chi}_{PQS}) * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\hat{\chi}_{QRS}, 1_{A_{PQ}}))) \\ \Rrightarrow \quad & (C(1\times C))^*([1 \otimes M_{\mathcal{P}}, 1](\hat{\chi}_{PRS}) * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{A_{RS}}, \hat{\chi}_{PQR})))\ . \end{aligned}$$

Here we have used that $aC(C \times 1)a_{\times}^{-1} = C(1 \times C)$ on the left hand side, that C commutes with the hom functors of Ten and $\times$ i.e.

$$C_*\times_{(X,Y),(X',Y')} = C^*\mathrm{Ten}_{(X,Y),(X',Y')}C\colon\ [X,X'] \times [Y,Y'] \to [X \times Y, X' \otimes Y']$$

where $\times_{(X,Y),(X',Y')}\colon\ [X,X'] \times [Y,Y'] \to [X \times Y, X' \times Y']$ (on objects, this is naturality of C), that $(FG)_* = F_*G_*$ and $(FG)^* = G^*F^*$, that $(C(1 \times C))^*$ is strict, that $(-)^*$ and $(-)_*$ coincide with the partial functors of $[-,-]$ for strict functors, and that apart from the cubical functor C, all functors are strict.

By Theorem 2 from 2.4, ω_{PQRS} corresponds to an invertible modification $\hat{\omega}_{PQRS}$ as in the definition of a Gray homomorphism such that $(C(1 \times C))^*\hat{\omega}_{PQRS} = \omega_{PQRS}$.

[5] We again remark that we here and in fact always use a different bracketing than the one employed in [7]. Thus, specifying a modification as the one displayed is equivalent to specifying a modification as in [7].

Given objects $P, Q \in \mathcal{P}$, the modifications γ_{PQ} and δ_{PQ} are of the same form as in the definition of a Gray homomorphism: Using strictness of the local functors, γ_{PQ} is seen to be of the form

$$\gamma_{PQ}\colon\ ((j_Q \times 1)l_\times^{-1})^*(\chi_{PQQ}) * (l_\times^{-1})^*(M_\mathcal{L}C)_*(\iota_Q \times 1_{A_{PQ}}) \Rrightarrow 1_{A_{PQ}}\ ,$$

and this is clearly the same as

$$\gamma_{PQ}\colon\ [(j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, 1](\hat{\chi}_{PQQ}) * [l^{-1}_{\mathcal{P}(P,Q)}, M_\mathcal{L}](\mathrm{Ten}(\iota_Q, 1_{A_{PQ}})) \Rrightarrow 1_{A_{PQ}}\ .$$

Similarly, it is shown that δ_{PQ} is of the form required in the definition of a Gray homomorphism.

Finally, we have to compare the axioms. By Theorem 2 from 2.4, the axioms of a trihomomorphism correspond to equations involving the components of the modifications $\hat{\omega}_{PQRS}$, γ_{PQ}, and δ_{PQ}. On the other hand, the axioms of a Gray homomorphism are equations for the modifications $\hat{\omega}_{PQRS}$, γ_{PQ}, and δ_{PQ} themselves (involving an interchange modification in the case of the pentagon-like axiom). In fact, apart from the interchange cell in the pentagon-like axiom, it is obvious that the components of the nontrivial modifications in the Gray homomorphism axioms are precisely the nontrivial 2-cells in the axioms of the corresponding trihomomorphism axioms. Note here that the correspondence of Theorem 2 from 2.4 is trivial on components.

Recall that the interchange cell in the pentagon-like axiom of a Gray-homomorphism is given by $[a^{-1}, M_\mathcal{L}]\mathrm{Ten}_,(\Sigma_{\chi_{RST},\chi_{PQR}})$. We maintain that at the object

$$(g,(h,(i,j))) \in \mathcal{P}(S,T) \otimes (\mathcal{P}(R,S) \otimes (\mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q)))\ ,$$

the component $([a^{-1}, M_\mathcal{L}]\mathrm{Ten}_,(\Sigma_{\chi_{RST},\chi_{PQR}}))_{ghij}$ is given by $M_\mathcal{L}(\Sigma^{-1}_{\chi_{gh},\chi_{ij}})$. This is because evaluation in $\mathcal{G}ray$ is in this case given by taking components and now equation (58) from 4.3 for the strict hom functor $\mathrm{Ten}_,$ implies that

$$([a^{-1}, M_\mathcal{L}]\mathrm{Ten}_,(\Sigma_{\chi_{RST},\chi_{PQR}}))_{g(h(ij))} = M_\mathcal{L}(\mathrm{Ten}_,(\Sigma_{\chi_{RST},\chi_{PQR}}))_{(gh)(ij)} = M_\mathcal{L}(\Sigma_{\chi_{gh},\chi_{ij}})\ .$$

This is exactly the interchange cell on the right hand side of the corresponding axiom for a trihomomorphism, cf. [7, p. 68]. □

As noted above, in the proof of Theorem 6, there appear additional classes of interchange cells of which the components have to be compared to the interchange cells appearing in the definitions of the data and the $\mathcal{G}ray$-category structure of $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$. To give examples for these classes, we skip the proof for the correspondence of Gray transformations and tritransformations and for the correspondence of the data of Gray modifications and trimodifications, and we come back to the second axiom of a Gray modification cf. Lemma 15 from 6.2:

Lemma 17. *The components of the interchange cells in the second axiom of a Gray modification correspond precisely to the interchange cells appearing in the second axiom of a trimodification.*

Proof. The interchange cell appearing on the left hand side of the Gray modification axiom **(GMA2)** is $M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota^A_P})$. Note that $\Sigma_{\mathcal{L}(AP,\alpha_P),\iota^A_P}$ is an interchange 2-cell in the Gray product

$$[\mathcal{L}(AP, AP), \mathcal{L}(AP, BP)] \otimes [I, \mathcal{L}(AP, AP)]\ ,$$

and now we have to determine how $M_{\mathcal{G}ray}$ acts on such an interchange cell. Recall that $M_{\mathcal{G}ray}\colon\ [Y,Z]\otimes[X,Y]\to[X,Z]$ is defined by

$$e^X_Z(M_{\mathcal{G}ray}\otimes 1_X) = e^Y_Z(1\otimes e^X_Y)a^{-1}\ .$$

For the component of $M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota^A_P})$ at the single object $*\in I$ this implies that

$$\begin{aligned}(M_{\mathcal{G}ray}\Sigma_{\mathcal{L}(AP,\alpha_P),\iota^A_P})_* &= (e^{\mathcal{L}(AP,AP)}_{\mathcal{L}(AP,BP)}(\mathcal{L}(AP,-)_{AP,BP}\otimes 1))(\Sigma_{\alpha_P,(\iota^A_P)_*})\\ &\quad\text{(by equation (4) from 2.3)}\\ &= M_{\mathcal{L}}(\Sigma_{\alpha_P,(\iota^A_P)_*})\\ &\quad\text{(by definition of } \mathcal{L}(AP,-) \text{ see (15) from 2.5)}\end{aligned}$$

In fact, this is exactly the interchange cell for the left hand side of the axiom in [7, p. 77].

Similarly, the component of the interchange 2-cell $M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota^B_P})$ at the single object $*\in I$, is given by

$$\begin{aligned}(M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota^B_P}))_* &= e^{\mathcal{L}(BP,BP)}_{\mathcal{L}(AP,BP)}(\mathcal{L}(-,BP)_{AP,BP}\otimes 1)(\Sigma_{\alpha_P,(\iota^B_P)_*})\\ &= M_{\mathcal{L}}c(\Sigma_{\alpha_P,(\iota^B_P)_*})\\ &\quad\text{(by definition of } \mathcal{L}(-BP) \text{ see (16) from 2.5)}\\ &= M_{\mathcal{L}}(\Sigma^{-1}_{(\iota^B_P)_*,\alpha_P})\\ &\quad\text{(by equation (8) from 2.3)}\ .\end{aligned}$$

In fact, this is exactly the interchange cell for the right hand side of the axiom in [7, p. 77]. □

Remark 8. It is entirely analogous to show that the composition laws as given in [7, Th. 9.1 and 9.3] and Definitions 5 and 6 from 3.1 of the two $\mathcal{G}ray$-categories coincide under the correspondence. This concludes our exhibition of the critical ingredients of the proof of Theorem 6.

Combining Corollary 5 from 6.2 and Theorem 6, we have proved our main theorem:

Theorem 7. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category and $\mathcal{L}$ be a cocomplete $\mathcal{G}ray$-category, and let T be the monad corresponding to the Kan adjunction. Then the $\mathcal{G}ray$-category* Ps-T-Alg *is isomorphic to the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{P},\mathcal{L})$ determined by the locally strict trihomomorphisms.* □

The identification of the functor category $[\mathcal{P},\mathcal{L}]$ with $[\text{ob}\mathcal{P},\mathcal{L}]^T$ in Theorem 4 from 5., and the coherence result for Ps-T-Alg given in Corollary 3 from 4.1, then prove the following coherence theorem for $\mathcal{T}ricat_{\text{ls}}(\mathcal{P},\mathcal{L})$:

Theorem 8. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category and $\mathcal{L}$ be a cocomplete $\mathcal{G}ray$-category. Then the inclusion $i\colon\ [\mathcal{P},\mathcal{L}]\to\mathcal{T}ricat_{\text{ls}}(\mathcal{P},\mathcal{L})$ of the functor $\mathcal{G}ray$-category $[\mathcal{P},\mathcal{L}]$ into the $\mathcal{G}ray$-category $\mathcal{T}ricat_{\text{ls}}(\mathcal{P},\mathcal{L})$ of locally strict trihomomorphisms has a left adjoint such that the components $\eta_A\colon\ A\to iLA$ for objects $A\in\mathcal{T}ricat_{\text{ls}}(\mathcal{P},\mathcal{L})$ of the unit of this adjunction are internal biequivalences.* □

Example. Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category. Recall that $\mathcal{G}ray$ considered as a $\mathcal{G}ray$-category is complete and cocomplete cf. Lemma 1 from 2.3. Thus Theorem 8 applies for $\mathcal{L} = \mathcal{G}ray$. As a consequence, a locally strict trihomomorphism $\mathcal{P} \to \mathcal{G}ray$ which is nothing else than a locally strict $\mathcal{G}ray$-valued presheaf is biequivalent to a $\mathcal{G}ray$-functor $\mathcal{P} \to \mathcal{L}$.

In particular, let $\mathcal{P}$ be a category C considered as a discrete $\mathcal{G}ray$-category. Then locally strict trihomomorphisms $C \to \mathcal{L}$ are the homomorphisms of interest, and we have proved that any such homomorphism is biequivalent to a $\mathcal{G}ray$-functor $C \to \mathcal{L}$.

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