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Some considerations on amoeba forcing notions

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Abstract

In this paper we analyse some notions of amoeba for tree forcings. In particular we introduce an amoeba-Silver and prove that it satisfies quasi pure decision but not pure decision. Further we define an amoeba-Sacks and prove that it satisfies the Laver property. We also show some application to regularity properties. We finally present a generalized version of amoeba and discuss some interesting associated questions.

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1 Introduction

The amoeba forcings play an important role when dealing with questions concerning the real line, such as cardinal invariants and regularity properties. As an intriguing example, one may consider the difference between the amoeba for measure and category in Shelah's proof regarding the use of the inaccessible cardinal to build models for regularity properties, presented in [7] and [8]; in fact, since the amoeba for category is *sweet* (a strengthening of σ-centeredness), one can construct, via amalgamation, a Boolean algebra as limit of length ω_1 (without any need of the inaccessible), in order to get an extension where all projective sets have the Baire property. On the contrary, for Lebesgue measurability, Shelah proved that if we assume all $\mathbf{\Sigma}^1_3$ sets to be Lebesgue measurable, we obtain, for all $x \in \omega^\omega$, $\mathrm{L}[x] \models$ "ω_1^{V} is inaccessible". If one then goes deeply into Shelah's construction of the model satisfying projective Baire property just mentioned, one can realize that the unique difference with Lebesgue measurability consists of the associated amoeba forcing, which is not sweet for measure. Such an example is probably one of the oldest and most significant ones to underline the importance of the amoeba forcing notions in set theory of the real line. In other cases, it is interesting to define amoeba forcings satisfying certain particular features, like not adding specific types of generic reals, not collapsing ω_1 and so on; these kinds of constructions are particularly important when one tries to separate regularity properties of projective sets, or when one tries to blow up certain cardinal invariants without affecting other ones. For a general and detailed approach to regularity properties, one may see [4]. The main aim of the present paper is precisely to study two versions of amoeba, for Sacks and Silver forcing, respectively.

Definition 1. Let $\mathbf{P}$ be either Sacks or Silver forcing. We say that $\mathbf{AP}$ is an amoeba-$\mathbf{P}$ iff for any ZFC-model $\mathrm{M} \supseteq \mathrm{N}^{\mathbf{AP}}$, we have

$$\mathrm{M} \models \forall T \in \mathbf{P} \cap \mathrm{N} \quad \exists T' \in \mathrm{M} \cap \mathbf{P} \quad (T' \subseteq T \wedge \forall x \in [T'](x \text{ is } \mathbf{P}\text{-generic over N})).$$

Note that this definition works even when $\mathbf{P}$ is any other tree forcing notions (Laver, Miller, Mathias, and so on). We would like to mention that a similar work for Laver and Miller forcing is developed in detail by Spinas in [10] and [11].

Let us now recall some basic notions and standard notation. Given $t, t' \in 2^{<\omega}$, we write $t' \trianglelefteq t$ iff t' is an initial segment of t. A *tree* T is a subset of $2^{<\omega}$ closed under initial segments, i.e., for every $t \in T$, for every $k < |t|$, $t{\restriction}k \in T$, where $|t|$ represents the length of t. Given $s, t \in T$, we say that s and t are *incompatible* (and we write $s \not\parallel t$) iff neither $s \trianglelefteq t$ nor $t \trianglelefteq s$; otherwise one says that s and t are compatible ($s \parallel t$). We denote with $\textsc{Stem}(T)$ the longest element $t \in T$ compatible with every node of T. For every $t \in T$, we say that t is a *splitting node* whenever both $t^\frown 0 \in T$ and $t^\frown 1 \in T$, and we denote with $\textsc{Split}(T)$ the set of all splitting nodes. Moreover, for $n \geq 1$, we say $t \in T$ is an *nth splitting node* iff $t \in \textsc{Split}(T)$ and there exists $n \in \omega$ maximal such that there are natural numbers $k_0 < \dots < k_{n-1}$ with $t \restriction k_j \in \textsc{Split}(T)$, for every $j \leq n-1$. We denote with $\textsc{Split}_n(T)$ the set consisting of the nth splitting nodes. For a finite tree T, the *height* of T is defined by $\mathrm{ht}(T) := \max\{n : \exists t \in T, |t| = n\}$, while $\textsc{Term}(T)$ denotes the set of terminal nodes of T, i.e, those nodes having no proper extensions in T. Finally, for every $t \in T$, the set $\{s \in T : s \parallel t\}$ is denoted by T_t, the *body of* T is defined as $[T] := \{x \in 2^\omega : \forall n \in \omega (x \restriction n \in T)\}$, and $T|n := \{t \in T : |t| \leq n\}$.

Further, given a tree T and a finite subtree $p \subset T$, we define:

- $T {\downarrow} p := \{t \in T : \exists s \in \textsc{Term}(p)(s \parallel t)\}$;
- $p \sqsubseteq T \Leftrightarrow \forall t \in T \setminus p \quad \exists s \in \textsc{Term}(p)(s \trianglelefteq t)$, and we will say that p *is an initial segment of* T, or equivalently T *end-extends* p.

Our attention is particularly focused on the following two types of infinite trees of $2^{<\omega}$:

- $T \subseteq 2^{<\omega}$ is a *perfect* (or *Sacks*) tree iff each node can be extended to a splitting node.
- $T \subseteq 2^{<\omega}$ is a *Silver tree* (or *uniform tree*) iff T is perfect and for every $s, t \in T$, such that $|s| = |t|$, one has $s^\frown 0 \in T \Leftrightarrow t^\frown 0 \in T$ and $s^\frown 1 \in T \Leftrightarrow t^\frown 1 \in T$.

Sacks forcing $\mathbf{S}$ is defined as the poset consisting of Sacks trees, ordered by inclusion, and Silver forcing $\mathbf{V}$ is analogously defined by using Silver trees. Further, if G is the $\mathbf{S}$-generic filter over N, we call the generic branch $z_G = \bigcup\{\textsc{Stem}(T) : T \in G\}$ a Sacks real (and analogously for Silver). Other common posets that will appear in the paper will be the Cohen forcing $\mathbf{C}$, consisting of finite sequences of 0's and 1's, ordered by extension, and the random forcing **B**, consisting of perfect trees T with strictly positive measure, ordered by inclusion. We recall the notion of *axiom A*, which is a strengthening of properness.

Definition 2. A forcing P satisfies *Axiom A* if and only if there exists a sequence $\{\leq_n : n \in \omega\}$ of orders of P such that:

1. for every $a, b \in P$, for every $n \in \omega$, $b \leq_{n+1} a$ implies both $b \leq_n a$ and $b \leq a$;
2. for every sequence $\langle a_n : n \in \omega \rangle$ of conditions in P such that for every $n \in \omega, a_{n+1} \leq_n a_n$, there exists $b \in P$ such that for every $n \in \omega$, $b \leq_n a_n$;
3. for every maximal antichain $A \subseteq P$, $b \in P$, $n \in \omega$, there exists $b' \leq_n b$ such that $\{a \in A : a \text{ is compatible with } b'\}$ is countable.

Notational convention. In the literature, the Silver forcing is usually denoted by **V**, and we keep such a convention. As a consequence, to avoid possible confusion, the ground model will be denoted by the letter N, instead of the more common V.

The paper is organized as follows: in section 2, we show that the natural amoeba-Silver satisfies axiom A, and so in particular does not collapse ω_1. In section 3, we introduce a version of amoeba-Sacks and prove that it satisfies the Laver property. We remark that our construction is very different from the one presented by Louveau, Shelah and Velickovic in [6], and in particular we do not use any strong partition theorem (like Halpern-Laüchli theorem). Finally, a last section is devoted to discuss some difficulties when trying to kill Cohen reals added by the amoeba-Silver, and we discuss a generalized notion of amoeba together with some possible further developments concerning regularity properties. At the end of section 3, we also present an application of amoeba-Sacks, to separate Sacks measurability from Baire property at some projective level.

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2 Amoeba-Silver

In this section we discuss some properties of the amoeba-Silver **AV** defined by:

$$(p, T) \in \mathbf{AV} \text{ iff } T \text{ is a Silver tree and } \exists n \in \omega \text{ such that } p = T|n,$$

ordered by $(p', T') \leq (p, T) \Leftrightarrow T' \subseteq T \wedge p' \restriction \mathrm{ht}(p) = p$.

For a proof that this is a well-defined notion of amoeba for Silver forcing, i.e, it satisfies Definition 1, one can see [5, Lemma 18, Corollary 20].

In order to show that **AV** satisfies axiom A, we define the sequence of orders on **AV** as follows:

$$\begin{aligned} (p', T') \leq_n (p, T) \quad \Leftrightarrow \quad & (p', T') \leq (p, T) \\ & p' = p \wedge \forall k \leq n (\textsc{Split}_k(T') = \textsc{Split}_k(T)). \end{aligned}$$

Clearly, conditions 1 and 2 of Definition 2 are satisfied. To obtain condition 3, we first check Lemma 4, i.e., **AV** satisfies quasi pure decision.

Definition 3. Given $D \subseteq \mathbf{AV}$ open dense, $(p, T) \in \mathbf{AV}$ and q finite subtree of T, we say that q is *deciding* iff there exists $S \subseteq T$ such that $(q, S) \in D$.

Lemma 4. *Let $D \subseteq \mathbf{AV}$ be open dense, $(p, T) \in \mathbf{AV}$ and $m \in \omega$. Then there exists $T^* \subseteq T$ such that $(p, T^*) \leq_m (p, T)$ and*

$$q \textit{ is deciding } \Rightarrow (q, T^* {\downarrow} q) \in D.$$

Proof. For every tree T, let $\mathtt{SL}_T(n) := |t|$, where $t \in T$ is an nth splitting node. Let $D \subseteq \mathbf{AV}$ be open dense and fix $(p, T) \in \mathbf{AV}$ arbitrarily. Let $p^0 = T|(\mathtt{SL}_T(h_0) + 1)$, where the h_0-th splitting nodes are the first splitting nodes occurring above p, i.e., if $t \in \textsc{Split}_{h_0}(T)$, then $|t| > \mathrm{ht}(p)$ and there are no splitting nodes t' such that $\mathrm{ht}(p) < |t'| < |t|$. We assume $m = h_0$ and leave the general case to the reader.

We use the following notation: given T infinite tree and p finite tree, put

$$T \otimes p := \{t \in 2^{<\omega} : \exists t' \in T\, \exists t'' \in \textsc{Term}(p) \text{ s.t. } \forall n < |t''|(t(n) = t''(n)) \\ \wedge\ \forall n \geq |t''|(t(n) = t'(n))\}.$$

(Intuitively, $T \otimes p$ is the translation of T over p).

Let $\{p^0_j : j < 3\}$ enumerate the uniform finite trees such that $p \subseteq p^0_j \subseteq p^0$, $\mathrm{ht}(p^0_j) = \mathrm{ht}(p^0)$ and $p^0_j \restriction \mathrm{ht}(p) = p$. Starting from such p^0, one develops the following construction for $i \geq h_0$ and $j < 3^{i-h_0+1}$.

- Start from $i = h_0$:
  - Substep $j = 0$: if there exists $S \subseteq T$ such that $(p^0_0, S) \in D$, then put $T^0_0 = S$; otherwise put $T^0_0 = T$;
  - Substep $j+1$: if there exists $S \subseteq T^0_j \otimes p^0_{j+1}$ such that $(p^0_{j+1}, S) \in D$, then put $T^0_{j+1} = S$; otherwise let $T^0_{j+1} = T^0_j$;
  - when the operation is done for every $j < 3$, put $T^1_* = T^0_2 \otimes p^0$ and $p^1 = T^1_*|(\mathtt{SL}_{T^1_*}(h_0 + 1) + 1)$; furthermore, let $\{p^1_j : j < 3^2\}$ be the enumeration of all the uniform finite trees such that $p^1_j \subseteq p^1$, $\mathrm{ht}(p^1_j) = \mathrm{ht}(p^1)$ and $p^1_j \restriction \mathrm{ht}(p) = p$;
- Step $i = h_0 + k$:
  - Substep $j = 0$: if there exists $S \subseteq T^k_*$ such that $(p^k_0, S) \in D$, then put $T^k_0 = S$; otherwise let $T^k_0 = T^k_*$;

– Substep $j+1$: if there exists $S \subseteq T^k_j \otimes p^k_{j+1}$ such that $(p^k_{j+1}, S) \in D$, then put $T^k_{j+1} = S$; otherwise let $T^k_{j+1} = T^k_j$;

– when the operation is done for every $j < 3^{k+1}$, put $T^{k+1}_* = T^k_{3^{k+1}-1} \otimes p^k$ and $p^{k+1} = T^{k+1}_* | (\mathtt{SL}_{T^{k+1}_*}(i+1)+1)$; furthermore, let $\{p^{k+1}_j : j < 3^{k+2}\}$ be the enumeration of all the uniform finite trees such that $p^{k+1}_j \subseteq p^{k+1}$, $\mathrm{ht}(p^{k+1}_j) = \mathrm{ht}(p^{k+1})$ and $p^{k+1}_j \restriction \mathrm{ht}(p) = p$.

Once such a construction is finished, one obtains a sequence $\langle T^k_* : k \in \omega \rangle$ such that $T^{k+1}_* \leq_{h_0+k} T^k_*$ (where $T^0_* = T$). Hence, the tree T^* obtained by fusion, i.e., $T^* = \bigcap_{k \in \omega} T^k_*$, is a Silver tree, and so the pair (p, T^*) belongs to $\mathbf{AV}$ and $(p, T^*) \leq_{h_0} (p, T)$.

By construction, one gets

$$\forall (q, S) \leq (p, T^*), \text{ if } (q, S) \in D \text{ then } (q, T^* \downarrow q) \in D,$$

which completes the proof. □

Lemma 4 is the core of the next key result.

Lemma 5. *Let $A \subseteq P$ be a maximal antichain, $(p, T) \in \mathbf{AV}$ and $m \in \omega$. Then there exists $T^* \subseteq T$ such that $(p, T^*) \leq_m (p, T)$ and (p, T^*) has only countably many compatible elements in A.*

Proof. Fix a condition $(p, T) \in \mathbf{AV}$. Let D_A be the open dense subset associated with A, i.e., $D_A = \{(q, S) \in \mathbf{AV} : \exists (q', S') \in A((q, S) \leq (q', S'))\}$. Let T^* be as in Lemma 4. To reach a contradiction, assume there are uncountably many elements in A compatible with (p, T^*), i.e., there is a sequence $\langle (p_\alpha, T_\alpha) : \alpha < \omega_1 \rangle$ of distinct elements of A and there are (q_α, S_α)'s such that, for every $\alpha < \omega_1$,

$$(q_\alpha, S_\alpha) \leq (p_\alpha, T_\alpha), (p, T^*).$$

Note that $(q_\alpha, S_\alpha) \in D_A$. Thus, by Lemma 4, one obtains $(q_\alpha, T^* \downarrow q_\alpha) \in D_A$, and therefore

$$(q_\alpha, T^* \downarrow q_\alpha) \leq (p_\alpha, T_\alpha), (p, T^*).$$

Note that there are only countably many different q_α's and therefore there exist $\alpha_0, \alpha_1 < \omega_1$ such that $(q_{\alpha_0}, T^*{\downarrow}q_{\alpha_0}) = (q_{\alpha_1}, T^*{\downarrow}q_{\alpha_1})$, and this contradicts $(p_{\alpha_0}, T_{\alpha_0}) \perp (p_{\alpha_1}, T_{\alpha_1})$. □

Corollary 6. **AV** *satisfies axiom A.*

Proof. Conditions 1 and 2 of Definition 2 are straightforward, while condition 3 follows from Lemma 5. □

Remark 7. Consider the following notation:

- for every $p \subset 2^{<\omega}$ finite and uniform, let $\mathsf{ns}(p)$ = number of splitting levels of p;
- let $\Delta_p = \langle \mathsf{SL}_p(0), \mathsf{SL}_p(1), \ldots, \mathsf{SL}_p(\mathsf{ns}(p) - 1) \rangle$.

Finally, if G is **AV**-generic over N, let $h = \bigcup \{\Delta_p : (p, T) \in G\}$.

Claim: $\Vdash_{\mathbf{AV}}$ " $\dot{h}$ is dominating over N".

Proof. Fix an increasing $x \in \omega^\omega \cap \mathrm{N}$ and $(p, T) \in \mathbf{AV}$, arbitrarily. Pick $T' \subseteq T$, $T' \restriction \mathrm{ht}(p) = p$ such that for every $n \geq \mathsf{ns}(p)$, $\mathsf{SL}_{T'}(n) > x(n)$. It is clear that $(p, T') \Vdash \forall n \geq \mathsf{ns}(p)(\dot{h}(n) > x(n))$. □

Amoeba-Silver does not have pure decision, as pointed out by the following observation.

Remark 8. Let T_G be the generic tree added by **AV** and define the following $c \in 2^\omega$: for every $n \in \omega$,

$$c(n) = \begin{cases} 0 & \text{if } \{j \in \omega : \mathsf{SL}_{T_G}(n+1) + 2 < j \leq \mathsf{SL}_{T_G}(n+2) + 1 \wedge T_G(j) = 0\} \\ & \text{ is even;} \\ 1 & \text{otherwise.} \end{cases}$$

(Intuitively, $c(n)$ represents the parity of 0s between the $n+1$st and $n+2$nd splitting level.)

Claim: $\Vdash_{\mathbf{AV}}$ " $\dot{c}$ is Cohen over N".

Proof. Fix a closed nowhere dense set F of the ground model. Given $(p, T) \in$ **AV**, let c_0 be the part of c already decided by such a condition. Denote with s the sequence in $2^{<\omega}$ such that $[c_0{}^\frown s] \cap F = \emptyset$. Now, it is clear that one can remove some splitting nodes and choose 0 if needed, according to what s tells us; more precisely, for every n, $|c_0| \leq n < |c_0{}^\frown s|$, if $c_0{}^\frown s(n) = 0$ and there is an even number of 0s between the $n+1$st and the $n+2$nd splitting level then we do nothing, otherwise, we remove the $n+2$nd splitting level, and we freely choose 0 or 1 in order to have an even number of 0s between the $n+1$st and the "new" $n+2$nd splitting level. An analogous argument works when $c_0{}^\frown s(n) = 1$. $\square$

3 Amoeba-Sacks

The standard amoeba-Sacks consists of the set of pairs (p, T), where T is a perfect tree and $p = T|n$, for some $n \in \omega$, ordered by $(p', T') \leq (p, T)$ iff $T' \subseteq T$ and p' end-extends p. However such a forcing has the bad feature of adding a Cohen real: let $T_G = \bigcup\{p : \exists T((p, T) \in G)\}$, where G is the generic over N, we define, for every $n \in \omega$,

$c(n) = 0$ iff the shortest $n+2$nd splitnode above the leftmost $n+1$st splitnode $t \in T_G$ extends $t^\frown 0$, or if the two $n+2$nd splitting nodes extending t have the same length;

$c(n) = 1$ otherwise.

Claim 9. *c is Cohen over the ground model* N*.*

Proof. Let $B \in$ N be closed nowhere dense and (p, T) an amoeba condition. We aim at finding a condition $(p', T') \leq (p, T)$ such that $(p', T') \Vdash \dot{c} \notin B$. Let $t_0 \in 2^{<\omega}$ such that $(p, T) \Vdash t_0 \lhd \dot{c}$, and pick $s_0 \in 2^{<\omega}$ such that $[t_0{}^\frown s_0] \cap B = \emptyset$. We can then extend p to p' in order to *follow* s_0, since we can freely choose the subsequent splitting nodes extending the leftmost branch. Hence, $(p', T') \Vdash t_0{}^\frown s_0 \lhd \dot{c} \notin B$.

$\square$

We are therefore interested in introducing a finer version of amoeba-Sacks which does not add Cohen reals. Actually we will do more, by showing that our forcing satisfies the Laver property.

Before going on we need to introduce some notation:

- given a perfect tree T, consider the natural $\trianglelefteq$-isomorphism $e : \textsc{Split}(T) \to 2^{<\omega}$ and put on $\textsc{Split}(T)$ the following order:

$$s \preccurlyeq t \text{ iff } |e(s)| < |e(t)| \vee (|e(s)| = |e(t)| \wedge e(s) \leq_{\text{lex}} e(t)).$$

$$s \prec t \text{ iff } s \preccurlyeq t \wedge s \neq t.$$

  We will say that $t \in \textsc{Split}(T)$ has *depth* n (and we will write $d(t,T) = n$) iff there is a maximal $n \in \omega$ such that there are $t_0, \dots, t_{n-1} \in \textsc{Split}(T)$ with $t_0 \prec \dots t_{n-1} \prec t$ (in case there are no such t_j's we say that t has depth 0, i.e., t is the $\textsc{Stem}(T)$.)

- $T{\restriction}^* n := \{t \in T : \exists k \leq n \exists s \in \textsc{Split}(T) \exists i \in \{0,1\}(s \text{ has depth } k \wedge (t \trianglelefteq s^\frown i)\}$.

- Given T, T' perfect trees, we define

$$T' \subseteq_n T \Leftrightarrow T' \subseteq T \wedge T'{\restriction}^* n = T{\restriction}^* n.$$

Definition 10. We say that a tree T is *good* iff for every $s, t \in \textsc{Split}(T)$, one has $s \preccurlyeq t \Rightarrow |s| \leq |t|$.

We then define our version of amoeba-Sacks **AS** as follows: a pair $(p,T) \in \mathbf{AS}$ iff T is a good perfect tree and $p \sqsubset T$. The order is given by $(p',T') \leq (p,T)$ iff $T' \subseteq T$ and p' end-extends p.

Remark 11. Given a perfect tree T there exists a good perfect tree $T' \subseteq T$. In fact, we can build a sequence $\{T_n : n \in \omega\}$ such that for every $n \in \omega$, $T_{n+1} \subseteq_n T_n$ and $T_n{\restriction}^* n$ is good, by using the following recursive pruning-argument:

- start from $T_0 := T$;
- assume T_n already defined and pick the node $t \in T_n$ with $d(t, T_n) = n$. If $T_n \restriction^* n$ is good, then put $T_{n+1} := T_n$; otherwise, *cut* the splitting node, by removing the part of T_n above $t^\frown 1$, go to the next splitting node and keep cutting as far as one finds a tree S so that $S \restriction^* n$ be good. Let $T_{n+1} := S$.
- Put $T' := \bigcap_{n \in \omega} T_n$.

Throughout this section, we will use the symbol $T' \overset{g}{\ll} T$ for denoting the good perfect subtree T' of T, built via this pruning-argument. Note that such T' is uniquely determined.

First of all, we check that the name amoeba-Sacks be justified.

Lemma 12. *Let G be $\mathbf{AS}$-generic over N and let $\mathrm{M} \supseteq \mathrm{N}[G]$ be a ZFC-model. Then*

$$\mathrm{M} \models \forall T \in \mathrm{N} \cap \mathbf{S} \quad \exists T' \in \mathrm{M} \cap \mathbf{S} \quad ([T'] \subseteq [T] \wedge [T'] \subseteq \mathbf{S}(\mathrm{N})),$$

where $\mathbf{S}(\mathrm{N})$ is the set of Sacks generic reals over N.

Proof. It is analogous to the argument used for other notions of amoeba, see [5] for Silver and [10] for Laver and Miller. Since such an argument is not widely known, we give it here for completeness. We first check that $T_G := \bigcup\{p : \exists T((p,T) \in G)\} \subseteq \mathbf{S}(\mathrm{N})$ in M, i.e., every $x \in [T_G] \cap \mathrm{M}$ is Sacks generic over N, and we then see how to find a *copy* of T_G inside any perfect tree $T \in \mathrm{N}$.

Given $(p, T) \in \mathbf{AS}$ and $D \subseteq \mathbf{S}$ open dense, we build $T^* \subseteq T$ as follows: let $\{t_0, \dots, t_n\}$ enumerate all terminal nodes of p, and, for every $j \leq n$, pick $T_j \subseteq T_{t_j}$ such that $T_j \in D$; then put $T^* \overset{g}{\ll} \bigcup\{T_j : j \leq n\}$. By construction, we obtain $(p, T^*) \Vdash \forall z \in [T_G](H_z \cap D \neq \emptyset)$, where H_z is defined by $H_z = \{S \in \mathbf{S} \cap \mathrm{N} : z \in [S]\}$.

We have just shown H_z to be generic. It is then left to show that it is a filter: towards contradiction, assume there are $T_1, T_2 \in H_z$ incompatible (note that by absoluteness they are incompatible in N as well). Hence, $[T_1]\cap[T_2]$ is countable, i.e., $[T_1]\cap[T_2] = \{x_i : i \leq \omega\}$. Then $E := \{T \in \mathbf{S} : \forall i \in \omega(x_i \notin [T])\}$ is an open dense set in the ground model N, and so, by genericity, there is $T \in E$ and $z \in [T]$, which is a contradiction.

We remark that the argument we used so far works not only for $z \in \mathrm{N}[G]$, but even for all $z \in \mathrm{M}$. In fact, the above argument shows that we can find a front $F \subseteq T_G$, i.e., a set such that for every $t \in F$ we have $(T_G)_t \in D$, and so, since being a front is Π^1_1, it follows that F remains a front in any ZFC-model $\mathrm{M} \supseteq \mathrm{N}[G]$, and so for every $z \in [T_G] \cap \mathrm{M}$, $\mathrm{M} \models H_z \cap D \neq \emptyset$.

It is then left to show that we can find a tree T' only consisting of Sacks generic reals, inside any perfect T of the ground model. To this aim, it is enough to note that, for any $T \in \mathbf{S} \cap \mathrm{N}$, the forcing $\mathbf{AS}_T$ defined as $\mathbf{AS}_T := \{(p,S) \in \mathbf{AS} : S \subseteq T\}$, with the analogous order, is isomorphic to $\mathbf{AS}$.

□

Remark 13. Let $(p_0, T) \in \mathbf{AS}$. By goodness, there exists $p \sqsupseteq p_0$ maximal (w.r.t. $\sqsubseteq$) such that for every $T' \subseteq T$ with $(p_0, T') \in \mathbf{AS}$ one has $p \sqsubset T'$ (in particular, every $(q, S) \leq (p_0, T)$ is compatible with (p, T) and so the two conditions are forcing equivalent). Note that such p is of the form $T{\restriction}^* n$, for some n, *but* with every terminal node of the latter extended to the corresponding subsequent splitting node.

To show that $\mathbf{AS}$ satisfies the Laver property, we first have to introduce a notion of $\leq_n$:

$$(p_0', T') \leq_n (p_0, T) \Leftrightarrow (p_0', T') \leq (p_0, T) \wedge p_0 = p_0' \wedge T' \subseteq_{n+N} T,$$

where $N := \max\{k \in \omega : \exists t \in \textsc{Split}(p)(d(t,p) = k)\}$, with $p \sqsupseteq p_0$ as in Remark 13. $\mathbf{AS}$ satisfies axiom A, and the proof works similarly to the one for amoeba-Silver $\mathbf{AV}$ viewed in the previous section. In fact, $\mathbf{AS}$ satifies quasi pure decision, together with an akin version of Lemma 5.

Lemma 14. **AS** *has quasi pure decision, i.e., given* $D \subseteq \mathbf{AS}$ *open dense,* $(p, T) \in \mathbf{AS}$ *and* $m \in \omega$*, there exists* $T^+ \subseteq T$ *such that* $(p, T^+) \leq_m (p, T)$ *and*

$$q \text{ is deciding} \Rightarrow (q, T^+{\downarrow}q) \in D.$$

Sketch of the proof. It is analogous to that of Lemma 4 for **AV**. Given $D \subseteq \mathbf{AS}$ open dense and $(p, T) \in \mathbf{AS}$, we can build $T^+ \subseteq_m T$ with the desired property, for some arbitrary fixed $m \in \omega$, by using the following inductive argument: start with $q^0 = T{\restriction}^* m$ and $T^0 = T$. for $j > 0$, let $q_j = T^{j-1}{\restriction}^*(m + j)$. Then use an analogous *shrinking-argument* as in the proof of Lemma 4 in order to get $T^j \subseteq_{m+j} T^{j-1}$ so that

$$\forall q(p \sqsubseteq q \subseteq q_j \wedge q \text{ is deciding } \Rightarrow (q, T^j{\downarrow}q) \in D).$$

Finally put $T^+ = \bigcap_{j\in\omega} T_j$. We then get $(p, T^+) \leq_m (p, T)$ with the required property. □

Note that even the standard amoeba-Sacks satisfies quasi pure decision, and the argument for proving that is analogous.

Lemma 15. **AS** *has pure decision, i.e., given a formula* φ *and a condition* $(p_0, T) \in \mathbf{AS}$*, there exists* $(p_0, T') \leq_0 (p_0, T)$ *such that* $(p_0, T') \in D$*, with* $D = \{(q, S) \in \mathbf{AS} : (q, S) \Vdash \varphi \vee (q, S) \Vdash \neg\varphi\}$.

Proof. First of all, let $p \sqsupseteq p_0$ be as in Remark 13. The idea of the proof by contradiction is the following. Assume there is no $T' \subseteq T$ such that $(p, T') \in D$, and so also no $(p_0, T') \in D$. We will construct $T^* \subseteq T$ such that $(p, T^*) \in \mathbf{AS}$ and for every $(q, S) \leq (p_0, T^*)$ one has:

($\star_1$) if q is deciding then $(q, T^*{\downarrow}q) \in D$ (this can be done by virtue of Lemma 14);

($\star_2$) there exists q' such that $q \sqsubseteq q' \sqsubset S$, $q' \sqsupseteq p$ and $(q', T^*{\downarrow}q') \notin D$.

This two facts obviously contradict D being dense.

We use the following notation: for every $s \in \textsc{Split}(T)$, p finite tree,

$$p \oplus s := \{t : t \in p \vee \exists i \in \{0,1\}(t \trianglelefteq s^\frown i)\}.$$

Let $t_0 \in p$ be such that $t_0 = r^\frown 0$, with $r \in \textsc{Split}(p)$ satisfying:

(i) there is no $v \triangleright r$ such that $v \in \textsc{Split}(p)$, and

(ii) r has smallest depth with property (i), i.e., for every $u \in \textsc{Split}(p)$, if $d(u,p) < d(r,p)$ then there exists $u' \triangleright u$ such that $u' \in \textsc{Split}(p)$.

(In case $\textsc{Split}(p) = \emptyset$, let $t_0 = \textsc{Stem}(T)^\frown 0$.)

We can assume T to be as the T^+ of Lemma 14, so that $(\star_1)$ be satisfied. The rest of the proof is devoted to building $T^* \subseteq T$ satisfying $(\star_2)$ as well. We split it into three claims.

Claim 16. *There are perfectly many s_j's in* $\textsc{Split}(T)$ *extending t_0 such that $p \oplus s_j$ is not deciding.*

Proof. Assume, towards contradiction, that such a set were not perfect. Then one could find a perfect P consisting of all $t \trianglerighteq t_0$ in $\textsc{Split}(T)$ such that $(p \oplus t, T{\downarrow}(p \oplus t)) \in D$ and moreover

(i) either for all $t \in P$, $(p \oplus t, T{\downarrow}(p \oplus t)) \Vdash \varphi$,

(ii) or for all $t \in P$, $(p \oplus t, T{\downarrow}(p \oplus t)) \Vdash \neg\varphi$,

Hence, by letting $T^- \overset{g}{\ll} \bigcup_{t \in P}(p \oplus t) \cup \bigcup\{T_r : r \in T \wedge r \not\parallel t_0\}$ we would have

$$\text{(i)} \Rightarrow \forall (q,S) \leq (p,T^-)\exists (q',S') \leq (q,S)((q',S') \Vdash \varphi) \Rightarrow (p,T^-) \Vdash \varphi$$

$$\text{(ii)} \Rightarrow \forall (q,S) \leq (p,T^-)\exists (q',S') \leq (q,S)((q',S') \Vdash \neg\varphi) \Rightarrow (p,T^-) \Vdash \neg\varphi,$$

and so in both cases $(p,T^-) \in D$, contradicting our initial assumption. □

Let $S^1 := T^-$. Furthermore, note that $(p,S^1) \leq_0 (p,T)$.

Claim 17. *Let* $t_1 \in \textsc{Split}(S^1)$ *such that* $t_1 = r^\frown 1$, *where* r *is the same as in the definition of* t_0 *above. There exists* $W \subseteq S^1_{t_1}$ *perfect and good such that for every* $u \in \textsc{Split}(S^1)$ *extending* t_0, *for every* $s \in \textsc{Split}(W)$, *we have* $p \oplus u \oplus s$ *is not deciding.*

Proof. Let u be the first splitting node of S^1 extending t_0. By an analogous argument as in the above lemma, we find perfectly many $s_j \in \textsc{Split}(S^1)$ extending t_1 such that, for every $j \in \omega$, $p \oplus u \oplus s_j$ is not deciding, otherwise $p \oplus u$ would be deciding, contradicting Claim 16. Let $R^0 := \{s_j : j \in \omega\}$, $S^1_0 \overset{g}{\lll} \bigcup_{s \in R^0}(p \oplus s) \cup \bigcup\{(S^1_0)_t : t \in S^1_0 \wedge t \not\parallel t_1\}$ and let w be the shortest node in S^1_0 extending t_1.

Then let u_0 be the first splitting node of S^1_0 extending $u^\frown 0$ and analogously u_1 the one extending $u^\frown 1$. By the usual argument, we find perfectly many s's in $\textsc{Split}(S^1_0)$ extending $w^\frown 0$ such that

$$p \oplus u_0 \oplus s \text{ is not deciding.} \tag{1}$$

Let $P^0_0 \subseteq R^0$ be the set of such perfectly many nodes. Moreover, we also find perfectly many $s \in P^0_0$ such that

$$p \oplus u_1 \oplus s \text{ is not deciding.} \tag{2}$$

Let $P^0_1 \subseteq P^0_0$ be the set of such nodes.

A specular argument can be done also for $w^\frown 1$ in order to find $P^1_1 \subseteq R^0$ such that every $s \in P^1_1$ extends $w^\frown 1$ and satisfies both (1) and (2). Finally put $R^1 = \{w\} \cup P^0_1 \cup P^1_1$ (note that R^1 and R^0 have the same first node, namely w). Then put

$$S^1_1 \overset{g}{\lll} \bigcup\{p \oplus u \oplus s : u \in \textsc{Split}((S^1_0)_{t_0}), s \in R^1\} \cup$$
$$\bigcup\{(S^1_0)_t : t \in S^1_0 \wedge t \not\parallel u \wedge t \not\parallel w\}.$$

Furthermore let, for $i, j, k \in \{0, 1\}$,

- $w_i \trianglerighteq w^\frown i$ be the first splitting node occurring in R^1;

- $u_{kj} \trianglerighteq u_k{}^\frown j$ be the first splitting node occurring in S^1_1 (note that, by goodness, for each $i \in \{0,1\}$, one has $|u_{kj}| > |w_i|$).

Note that $u_0, u_1 \in \textsc{Split}(S^1_1)$, since $|u_0|, |u_1| < |w_0|$ by goodness.

By repeating this procedure, we obtain, for $n \in \omega$, $R^n \subseteq R^{n-1}$ such that for every $s \in R^n$, for every $\sigma \in 2^{\leq n}$, $p \oplus u_\sigma \oplus s$ is not deciding, where we identify u with $u_\emptyset$. Moreover, put

$$S^1_n \overset{g}{\ll} \bigcup\{p \oplus u \oplus s : u \in \textsc{Split}((S^1_{n-1})_{t_0}), s \in R^n\} \cup$$
$$\bigcup\{(S^1_{n-1})_t : t \in S^1_{n-1} \wedge t \not\parallel u \wedge t \not\parallel w\}.$$

Note, for every $\sigma \in 2^n$, we have $u_\sigma \in \textsc{Split}(S^1_n)$. Finally, put $R = \bigcap_{n\in\omega} R^n$ and $W = \bigcup\{t : \exists s \in R(t \trianglelefteq s)\}$. Note that the definition of R makes sense, since for every $n \in \omega$, $R^{n+1} \cap R^n \supseteq \{w_\sigma : \sigma \in 2^{\leq n}\}$, and so the construction is obtained by a kind of standard fusion argument (note that we identify w with $w_\emptyset$). By construction, such W has the required properties. $\square$

Then define $S^2 := \bigcap_{n\in\omega} S^1_n$. Note that $u^\frown 0, u^\frown 1 \in S^2 \cap S^1$ and therefore $(p, S^2) \leq_1 (p, S^1)$.

Claim 18. *Let t_n be as follows: if t_{n-1} was of the form $r^\frown 0$, then let $t_n = r^\frown 1$; if t_{n-1} was of the form $r^\frown 1$, then let $t_n = z^\frown 0$, where z is the splitting node of S^{n-1} such that $d(z, S^{n-1}) = d(r, S^{n-1}) + 1$. There exists $W \subseteq (S^{n-1})_{t_n}$ perfect and good such that for every $A := (s_0, \dots, s_{n-1}) \in (\textsc{Split}(S^{n-1}))^n$, for every $w \in \textsc{Split}(W)$, we have $p(A, w)$ is not deciding, where $p(A, w) := p \oplus s_0 \oplus \cdots \oplus s_{n-1} \oplus w$.*

Proof. The proof of Claim 18 is a generalization of the one of Claim 17.

Use the following notation: for $w \in \textsc{Split}(S^{n-1})$, let

$$\mathfrak{A}(w, S^{n-1}) = \{(s_0, \dots, s_{n-1}) \in (\textsc{Split}(S^{n-1}))^n :$$
$$p \oplus s_0 \oplus \cdots \oplus s_{n-1} \oplus w \text{ is good}\}.$$

Note that $\mathfrak{A}(w, S^{n-1})$ is always finite. For any $A \in \mathfrak{A}(w, S^{n-1})$, say $A = (s_0, \dots s_{n-1})$, we will use the notation $p(A, w) = p \oplus s_0 \oplus \cdots \oplus s_{n-1} \oplus w$ (for $w \in \textsc{Split}(S^{n-1})$).

We define the set S^n as the limit of the following inductive construction:

Step 0 : Let $p^+ = p \oplus u_0 \oplus \cdots \oplus u_{n-1}$, where each u_j is the first splitting node occurring in S^{n-1} extending t_j. By the usual argument, one can find perfectly many s_j's extending t_n such that, $p^+ \oplus s_j$ is not deciding, otherwise p^+ would be deciding. Let $P_\emptyset$ be the set of such perfectly many s_j's and $w_\emptyset$ its least element. Moreover, let

$$S_0^{n-1} \overset{g}{\lll} \bigcup\{(S^{n-1})_{u_j} : j < n\} \cup \bigcup\{p^+ \oplus s : s \in P_\emptyset\} \cup \bigcup\{(S^{n-1})_t : t \in S^{n-1} \wedge \forall j < n(t \not\parallel u_j) \wedge t \not\parallel w_\emptyset\}.$$

For every $j \leq n-1$, $i \in \{0,1\}$, pick $u_{j,i} \trianglerighteq u_j{}^\frown i$ to be the first splitting node of S_0^{n-1}such that $|u_{j,i}| > |w_\emptyset|$. Finally let A_0 be the set of all such $u_{j,i}$'s and all u_j's.

Step $l+1$: Assume P_σ, w_σ and $u_{j,\sigma^\frown i}$ already constructed, for every $\sigma \in 2^l$, $i \in \{0,1\}$. Remind that A_l is the set of these $u_{j,\tau}$'s, for $\tau \in 2^{\leq l+1}$. For $i \in \{0,1\}$, $\sigma \in 2^l$, find a perfect $P_{\sigma^\frown i} \subseteq P_\sigma{\downarrow}w_\sigma{}^\frown i$ such that, for all $s \in P_{\sigma^\frown i}$, for all $A \in \mathfrak{A}(s, S_l^{n-1})$ we have $p(A, s)$ is not deciding. Let

$$S_{l+1}^{n-1} \overset{g}{\lll} \bigcup\{(S_l^{n-1})_{u_j}, j < n\} \cup \bigcup\{p^+ \oplus s : s \in P_\tau, \tau \in 2^{l+1}\} \cup \bigcup\{(S_l^{n-1})_t : t \in S_l^{n-1} \wedge \forall j < n(t \not\parallel u_j) \wedge t \not\parallel w_\emptyset\}.$$

Then, for every $\sigma \in 2^l$, $\tau \in 2^{l+1}$, $j < n$, $i, k \in \{0,1\}$, let:

- $w_{\sigma^\frown i} \trianglerighteq w_\sigma{}^\frown i$ be the first splitting node in $P_{\sigma^\frown i}$;
- $u_{j,\tau^\frown k} \trianglerighteq u_{j,\tau}{}^\frown k$ be the first splitting node in S_l^{n-1} such that, for all $\varsigma \in 2^{l+1}$, $|u_{j,\tau^\frown k}| > |w_\varsigma|$.

Finally let A_{l+1} be the set of such $u_{j,\nu}$'s, for $\nu \in 2^{\leq l+2}$.

We keep on the construction for every $l \in \omega$ and we finally put $R = \bigcap_{\sigma \in 2^{<\omega}} P_\sigma$ and $W = \{t : \exists s \in R(t \trianglelefteq s)\}$. It follows from the construction that W has the required properties.

Let $S^n := \bigcap_{l \in \omega} S_l^{n-1}$. Note that, for all $j < n$, $u_j{}^\frown 0, u_j{}^\frown 1 \in S^n \cap S^{n-1}$, and hence $S^n \leq_n S^{n-1}$. □

By applying iteratively Claim 18 for every $n \in \omega$, we end up with a perfect tree $T^* := \bigcap_{n \in \omega} S^n$ (we identify S^0 with the tree T which we started from). It follows from the construction that T^* satisfies $(\star_2)$, and so the proof is completed. □

Next lemma shows that **AS** satisfies the L_f-property, with $f(n) = 4^n$ ([1, Definition 7.2.1]). Such a property, together with axiom A, implies that **AS** satisfies the Laver property, and so it does not add Cohen reals (see [1, Lemma 7.2.2-7.2.3]).

Lemma 19. *Let A be a finite subset of ω and $f(n) = 4^n$. For every $n \in \omega$, $(p_0, T) \in$ **AS** the following holds:*
if $(p_0, T) \Vdash \dot{a} \in A$ then there exists $(p_0, T') \leq_n (p_0, T)$ and $B \subseteq A$ of size $\leq f(n)$ such that $(p_0, T') \Vdash \dot{a} \in B$.

Proof. Let $(p_0, T) \in$ **AS**, $n \in \omega$, $A \subseteq \omega$ finite and $\dot{a}$ name for an element of A. We aim at finding $T' \subseteq T$ such that $(p_0, T') \leq_n (p_0, T)$ and B of size $\leq 4^n$ such that $(p_0, T') \Vdash \dot{a} \in B$. First of all, pick $p \sqsupseteq p_0$ as in Remark 13.
Let $q = T{\restriction}^* l + n$, where $l := \max\{j \in \omega : \exists t \in \textsc{Split}(p)(d(t, p) = j)\}$. We call q^* a *master* subtree of q iff it satisfies the following property:

(*i*) $p \sqsubseteq q^* \subseteq q$, with $q^* \setminus p \neq \emptyset$ and q^* good;

(*ii*) $\forall t \in q^* \exists t' \trianglerighteq t(t' \in \textsc{Term}(q) \cap q^*)$.

Let $\Gamma := \{q_j : j \leq N\}$ be the set consisting of all master subtrees of q. Note that $N \leq 4^n$; in fact, a master subtree q^* is uniquely determined by what we do on the splitting nodes of q, and so we have four choices for each $t \in \textsc{Split}(q)$:

1. $t \in \textsc{Split}(q^*)$;
2. $t \notin \textsc{Split}(q^*)$ and $t^\frown 0 \in q^*$;
3. $t \notin \textsc{Split}(q^*)$ and $t^\frown 1 \in q^*$;
4. $t \notin q^*$.

We also remark that the upper-bound 4^n is not optimal, since many combinations are forbidden, by goodness. Then consider the following recursive construction, for $j \leq N$:

- by pure decision, pick $T_0 \subseteq T{\downarrow}q_0$ and $b_0 \in \omega$ such that $(p, T_0) \Vdash \dot{a} = b_0$. Then put $S_1 \overset{g}{\ll} \bigcup\{T_t : t \in q \setminus q_0\} \cup T_0$.
- for $j+1$, by pure decision, pick $T_{j+1} \subseteq S_j{\downarrow}q_{j+1}$ and $b_{j+1} \in \omega$ such that $(p, T_{j+1}) \Vdash \dot{a} = b_{j+1}$. Then put $S_{j+1} \overset{g}{\ll} \bigcup\{(S_j)_t : t \in q \setminus q_{j+1}\} \cup T_{j+1}$.

Finally, put

$$T' := T_N \text{ and } B := \{b_j : j \leq N\}.$$

Note that, since q is good, whenever we use $\overset{g}{\ll}$, we certainly do not remove any splitting node of q, and so $(p_0, T') \leq_n (p_0, T)$.

Given any $(q', S) \leq (p_0, T')$ there exists $j \leq k$ such that $(q_j, T'{\downarrow}q_j)$ is compatible with (q', S), and therefore either (q', S) does not decide $\dot{a}$ or $(q', S) \Vdash \dot{a} = b_j \in B$. Hence, we obtain $(p_0, T') \Vdash \dot{a} \in B$ and $|B| \leq f(n)$. $\square$

We conclude with an application of our amoeba-Sacks to separate regularity properties, and then with an observation concerning finite product of amoeba-Sacks. We recall some standard definitions.

1. We say $X \subseteq 2^\omega$ to be *Sacks measurable* iff

$$\forall T \in \mathbf{S} \exists T' \subseteq T, T' \in \mathbf{S}([T'] \subseteq X \vee [T'] \cap X = \emptyset).$$

2. Let Γ be a certain family of sets of reals. $\Gamma(\textsc{Sacks})$ is the statement asserting that all sets in Γ are Sacks measurable. Analogously, $\Gamma(\textsc{Baire})$ stands for all sets in Γ have the Baire property.

3. for $z \in 2^\omega$, $X \subseteq 2^\omega$ is said to be *provable* $\Delta^1_n(z)$ iff there are $\Sigma^1_n(z)$ formulae φ_0 and φ_1 such that $X = \{x \in 2^\omega : \varphi_0(x)\} = \{x \in 2^\omega : \neg\varphi_1(x)\}$ and ZFC $\vdash \forall x \in 2^\omega(\varphi_0(x) \Leftrightarrow \neg\varphi_1(x))$. The corresponding family of provable $\mathbf{\Delta}^1_n$ sets is denoted by $\mathbf{p\Delta}^1_n$.

Lemma 20. *Let G be $\mathbf{AS}_{\omega_1}$-generic over* L*, where $\mathbf{AS}_{\omega_1}$ is the iteration of length ω_1 of $\mathbf{AS}$ with countable support. Then*

$$\mathrm{L}[G] \models \mathbf{p\Delta}^1_3(\textsc{sacks}) \wedge \neg\mathbf{\Delta}^1_2(\textsc{Baire})$$

Proof. Let $X \subseteq 2^\omega$ be defined via the Σ^1_3-formulae φ_0 and φ_1 with parameter $z \in 2^\omega$. Further let $\alpha < \omega_1$ such that $z \in \mathrm{L}[G_\alpha]$, possible by properness. Let $\dot{x}$ be a name for a Sacks real over $\mathrm{L}[G_\alpha]$. Since X is provable $\Delta^1_3(z)$, it follows

$$\mathrm{L}[G_\alpha] \models \text{“}\exists T \in \mathbf{S}(T \Vdash \varphi_0(\dot{x}) \vee T \Vdash \varphi_1(\dot{x}))\text{”}.$$

First assume $T \Vdash \varphi_0(\dot{x})$, which means, for every Sacks real over $\mathrm{L}[G_\alpha]$ through T, $\mathrm{L}[G_\alpha][x] \models \varphi_0(x)$.

Let us now argue within $\mathrm{L}[G]$. Since $\mathbf{AS}$ adds a perfect set of Sacks reals inside any perfect set from the ground model, we have a perfect tree $T' \subseteq T$ such that any $x \in [T']$ is Sacks over $\mathrm{L}[G_\alpha]$. Hence, for every $x \in [T']$, we get $\mathrm{L}[G_\alpha][x] \models \varphi_0(x)$, which gives $\varphi_0(x)$, by Σ^1_3-upward absoluteness. We have therefore shown that

$$\mathrm{L}[G] \models \exists T' \in \mathbf{S}([T'] \subseteq X).$$

Analogously, if $T \Vdash \varphi_1(\dot{x})$ we obtain $\mathrm{L}[G] \models \exists T' \in \mathbf{S}([T'] \cap X = \emptyset)$. This concludes the proof concerning $\mathbf{p\Delta}^1_3(\textsc{sacks})$. To show that $\mathbf{\Delta}^1_2(\textsc{Baire})$ fails it is sufficient to note that no Cohen reals are added by $\mathbf{AS}_{\omega_1}$ because it satisfies the Laver property, and so $L[G] \models \neg\mathbf{\Delta}^1_2(\textsc{Baire})$, by well-known result proved in [9]. $\square$

We remark that some very interesting results about $\mathbf{\Delta}^1_3$-measurability related to tree-forcings have been recently found by Fischer, Friedman and Khomskii in [3].

Remark 21. Let $\mathbf{AS}^*$ be the natural amoeba-Sacks adding Cohen reals. Consider the following map $\phi : \mathbf{AS}^* \times \mathbf{AS}^* \to \mathbf{AS}^*$ such that $\langle (p_0, T_0), (p_1, T_1) \rangle$ is mapped to $(0^\frown p_0 \cup 1^\frown p_1, 0^\frown T_0 \cup 1^\frown T_1)$, where $i^\frown T := \{s : \exists t \in T(s = i^\frown t)\}$, for every (possibly finite) tree in $2^{<\omega}$. It is straightforward to check that such ϕ is an isomorphism between $\mathbf{AS}^* \times \mathbf{AS}^*$ and $\mathbf{AS}^*$ below the condition $(\{\langle 0\rangle, \langle 1\rangle\}, 2^{<\omega})$. Hence, $\mathbf{AS}^* \times \mathbf{AS}^*$ completly embeds into $\mathbf{AS}^*$, and so $\mathbf{S} \times \mathbf{S}$ complete embeds into $\mathbf{AS}^*$ as well. In particular, we indirectly get that $\mathbf{S} \times \mathbf{S}$ is proper.

Finally, note that such an argument holds for any finite product $(\mathbf{AS}^*)^n$. In fact, fixed $n \in \omega$, let $t_0, t_1, \dots t_{n-1}$ be a list of n-many sequences of $2^{<\omega}$ which are pairwise incompatible. Then let $\phi : (\mathbf{AS}^*)^n \to \mathbf{AS}^*$ be such that

$$\phi(\langle (p_j, T_j) : j < n \rangle) = \Big(\bigcup_{j<n} t_j^\frown p_j, \bigcup_{j<n} t_j^\frown T_j \Big).$$

As above, ϕ is an isomorphism between $(\mathbf{AS}^*)^n$ and $\mathbf{AS}^*$ below the condition $(p^*, 2^{<\omega})$, with p^* the finite tree generated by $t_0, \dots t_{n-1}$, i.e., the set of initial segments of sequences in $\bigcup_{j<n} t_j$.

Note that this argument is no longer true for our amoeba-Sacks $\mathbf{AS}$ analyzed in this paper. In fact, it is easy to see that $\mathbf{AS}$ adds a dominating real. Now consider the product $\mathbf{AS} \times \mathbf{AS}$ and let d_0, d_1 be a pair of mutually dominating reals added by $\mathbf{AS} \times \mathbf{AS}$. Define the real c as follows: $c(n) = 0$ iff $d_0(n) \leq d_1(n)$, $c(n) = 1$ otherwise. Such c is obviously Cohen, since we can freely make either $d_0(n) > d_1(n)$ or $d_0(n) < d_1(n)$, and hence $\mathbf{AS} \times \mathbf{AS}$ does not completely embed into $\mathbf{AS}$.

4 Concluding remarks

Many difficulties come out when trying to remove the pathology of Remark 8 about amoeba-Silver, as we did for amoeba-Sacks. A first idea to remove Cohen reals could be to oblige the Silver tree T of the pair (p, T) to have always an even number of 0s between two subsequent splitting nodes. Nevertheless, even if this modification formally removes the Cohen real defined as in Remark 8, it cannot suppress *any* Cohen real; in fact, putting $\Gamma_n = \{j \in \omega : \mathtt{SL}_{T_G}(n+1) + 2 < j \leq \mathtt{SL}_{T_G}(n+2) + 1 \wedge T_G(j) = 0\}$, one can similarly define a Cohen real by putting $c(n) = 0$ iff $|\Gamma_n| = 0$ modulo 3 (and $c(n) = 1$ otherwise). More generally, one could fix a new condition saying that the number of 0s between two splitting levels of T has to be a multiple of a given sequence of natural numbers $n_0, \dots n_k$; in any case, this will not settle the problem, since one could define a *new* Cohen real such that $c(n) = 0$ iff $|\Gamma_n| = 0$ modulo $n_0 \cdot n_1 \cdot \dots \cdot n_k + 1$. Furthermore, if we look at the construction of the amoeba-Sacks, one can realize that it does not work for the amoeba-Silver; in fact, we cannot freely remove splitting nodes as in claim 16, since we have to respect the uniformity of the Silver tree.

As we said in the introduction, the notion of "amoeba" is meant as a "forcing adding a *large* set of *generic* reals", where the words "large" and "generic" depend on the forcing we are dealing with. In the examples we have mentioned and studied in the previous sections, "large" and "generic" were connected to the same forcing notion; in fact, we have considered an amoeba-Sacks adding a *Sacks* tree of *Sacks* branches and an amoeba-Silver adding a *Silver* tree of *Silver* branches. Furthermore, the usual amoeba for measure and category add a *measure one* set of *random* reals and a *comeager* set of *Cohen* reals, respectively. What can also be done is to consider amoeba for which the notion of "large" and the one of "generic" are not necessarily connected. As a simple example, one can consider the Cohen forcing, viewed as a forcing adding a perfect tree of Cohen branches. Or otherwise, one could pick the forcing $\mathbf{RT}$ consisting of pairs (p, T), where $T \subseteq 2^\omega$ is a perfect tree with positive measure and $p \subset T$ is a finite subtree. It is clear that such a forcing adds a perfect tree of random reals. These two examples are particular cases of a more general definition.

Definition 22. Let $\mathbf{P}_0$ and $\mathbf{P}_1$ be tree-forcing notions. We say that a forcing Q is a $(\mathbf{P}_0, \mathbf{P}_1)$-amoeba iff for every $p \in \mathbf{P}_1 \cap \mathrm{N}$ there is $p' \in \mathbf{P}_0 \cap \mathrm{N}^Q$ such that $p' \subseteq p$ and

$$\mathrm{M} \models \text{“ every branch } x \in [p'] \text{ is } \mathbf{P}_1\text{-generic over N ”},$$

where M is *any* model of ZFC containing as a subset the extension of N via Q.

Hence, the forcing $\mathbf{RT}$ mentioned above is an $(\mathbf{S}, \mathbf{B})$-amoeba, while Cohen forcing can be seen as an $(\mathbf{S}, \mathbf{C})$-amoeba. Note that this general version of amoeba can be useful to obtain some results regarding regularity properties, such as that in lemma 20. In fact, a similar proof shows that an ω_1-iteration of $\mathbf{RT}$ provides a model for $\mathbf{p}\mathbf{\Delta}^1_3(\textsc{sacks})$ as well. However, the two iterations are different. In fact, $\mathbf{RT}$ adds Cohen reals but not dominating reals. The latter is proven in [1, lemma 3.2.24, 6.5.10 and theorem 6.5.11], whereas the former can be shown as follows: pick an interval partition $\{I_n : n \in \omega\}$ of ω such that, for all but finitely many n, any perfect tree of positive measure has at least one splitting node of length occurring in I_n. Then define the real $x \in 2^\omega$ such that $x(n) =$ the parity of splitting nodes of T_G occurring in I_n, where T_G is the $\mathbf{RT}$-generic tree given by $\bigcup\{p : \exists T((p, T) \in G)\}$. It is straightforward to check that x is a Cohen real. Hence if we pick $\mathbf{RT}_{\omega_1}$ to be the ω_1-iteration of $\mathbf{RT}$ with finite support (this to make sure that no dominating reals are added by the iteration), we obtain a model for $\mathbf{p}\mathbf{\Delta}^1_3(\textsc{sacks}) \wedge \neg\mathbf{\Delta}^1_2(\textsc{Laver}) \wedge \mathbf{\Delta}^1_2(\textsc{Baire})$, where Laver measurability is defined analogously as Sacks measurability, and we use [2, Theorem 4.1] to obtain $\neg\mathbf{\Delta}^1_2(\textsc{Laver})$. Hence, such a model is different from the one presented in lemma 20 satisfying $\mathbf{p}\mathbf{\Delta}^1_3(\textsc{sacks}) \wedge \mathbf{\Delta}^1_2(\textsc{Laver}) \wedge \neg\mathbf{\Delta}^1_2(\textsc{Baire})$.

These observations, together with Remark 8, give rise to the following interesting questions:

(Q1) Can one define an amoeba-Sacks not adding either Cohen or dominating reals?

(Q2) Can one define an amoeba-Silver not adding Cohen reals? And/or not adding either Cohen or dominating reals?

(Q3) Does "adding a perfect tree of random branches" imply "adding Cohen reals"?

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