A Frobenius homomorphism for Lusztig's quantum groups for arbitrary roots of unity

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ABSTRACT. For a finite dimensional semisimple Lie algebra and a root of unity, Lusztig defined an infinite dimensional quantum group of divided powers. Under certain restrictions on the order of the root of unity, he constructed a Frobenius homomorphism with finite dimensional Hopf kernel and with image the universal enveloping algebra.

In this article we define and completely describe the Frobenius homomorphism for arbitrary roots of unity by systematically using the theory of Nichols algebras. In several new exceptional cases the Frobenius-Lusztig kernel is associated to a different Lie algebra than the initial Lie algebra. Moreover, the Frobenius homomorphism often switches short and long roots and/or maps to a braided category.

CONTENTS

1. Introduction

Fix a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ and a primitive ℓ-th root of unity q. For this data, Lusztig defined in 1989 an infinite-dimensional complex Hopf algebra $U_q^{\mathcal{L}}(\mathfrak{g})$ called *restricted specialization* [Lusz90a][Lusz90b]. He conjectured that for ℓ prime the representation theory of $U_q^{\mathcal{L}}(\mathfrak{g})$ is deeply connected to the one of the respective affine Lie algebra as well as to the respective adjoint Lie group over $\bar{\mathbb{F}}_\ell$. The former statement has been proven in a certain form by Kazhdan and Lusztig in a series of papers, the latter statement has been proven in 1994 by Andersen, Jantzen & Soergel [AJS94].

For ℓ odd (and in case $\mathfrak{g} = G_2$ not divisible by 3) Lusztig had in the cited papers obtained a remarkable Hopf algebra homomorphism to the classical universal enveloping algebra $U(\mathfrak{g})$, which was for ℓ prime related to the Frobenius homomorphism over the finite field $\mathbb{F}_\ell$. The Hopf algebra kernel (more precisely the coinvariants) of this map turned out to be a finite-dimensional Hopf algebra, called the *small quantum group* or *Frobenius-Lusztig-kernel*, yielding an exact sequence of Hopf algebras:

$$u_q^{\mathcal{L}}(\mathfrak{g}) \xrightarrow{\subset} U_q^{\mathcal{L}}(\mathfrak{g}) \xrightarrow{Frob} U(\mathfrak{g})$$

The discovery of this finite-dimensional Hopf algebra $u_q^{\mathcal{L}}(\mathfrak{g})$ triggered among others the development of the theory of finite-dimensional *pointed Hopf algebras* that culminated in the classification results by Andruskiewitsch & Schneider [AS10] (for small prime divisors see [AnI11]) and the more general classification of possible quantum Borel parts, so-called *Nichols algebras* by Heckenberger [Heck09] using Weyl groupoids in general.

The aim of this article is to consider a more general short exact sequence of Hopf algebras without restrictions on the root of unity. Our approach somewhat differs from Lusztig's explicit approach, as discussed below, and uses crucially the theory of Nichols algebras. On the other hand we will restrict ourselves in this article to the positive Borel part.

Our results are as follows: The cases with $2, 3 \mid \ell$ exhibit in some cases a Frobenius-homomorphism to the Lie algebra with the dual root system $(B_n \leftrightarrow C_n)$ as has already been observed in [Lusz94]. As we shall see, moreover for small roots of unity Lusztig's implicit definition of $u_q^{\mathcal{L}}(\mathfrak{g})$ does **not** coincide with the common definition of $u_q(\mathfrak{g})$ by generators and relations. Altogether we shall treat in this article arbitrary q and find in all cases a Frobenius homomorphism with finite-dimensional kernel

$$u_q(\mathfrak{g}^{(0)})^+ \cong u_q^{\mathcal{L}}(\mathfrak{g})^+ \xrightarrow{\subset} U_q^{\mathcal{L}}(\mathfrak{g})^+ \xrightarrow{Frob} U(\mathfrak{g}^{(\ell)})^+$$

with $\mathfrak{g}^{(0)}, \mathfrak{g}, \mathfrak{g}^{(\ell)}$ quite different Lie algebras, some in braided symmetric tensor categories. An exotic case is $\mathfrak{g} = G_2, q = \pm i$, where among others $\mathfrak{g}^{(0)} = A_3$ has even larger rank. The author is very interested in a similar list for affine Lie algebras (see [Len14b] and Problem 7.2) as well as for other Nichols algebra extensions (see Problem 7.3).

As one application of our results, let us mention that [FGST05][FT10] have conjectured remarkable connections of $u_q(\mathfrak{g}), U_q^{\mathcal{L}}(\mathfrak{g})$ to certain vertex algebras (see [FHST04]). As the author has recently observed with I. Runkel and A. Gainutdinov, the case $\mathfrak{g} = B_n, q = \pm i$ in the present article ($\mathfrak{g}^{(0)} = A_1^{\times n}, \mathfrak{g}^{(\ell)} = C_n$) seems to be intimately related to the vertex algebra of n symplectic fermions with global symmetry $C_n = sp_{2n}$ (see [Ru12]). One might conjecture that in this case the category of $u_q(\mathfrak{g}^{(0)})$-modules is equivalent to the representation category of this vertex algebra, and similarly for $U_q^{\mathcal{L}}(\mathfrak{g})$ to the vertex algebra WB_n at a certain small level. The $n = 1$ case was treated in [FGST05] as abelian categories and in [GR14] as tensor categories, which requires an additional 3-cocycle!

We now review the results of this article in more detail:

In Section 2 we fix the Lie-theoretic notation and prove some technical preliminaries. We also introduce Nichols algebras in the special cases relevant to this article.

In Section 3 we review the construction of the Lusztig quantum group $U_q^{\mathcal{L}}(\mathfrak{g})$ via rational and integral forms and some basic properties.

In Section 4 we slightly improve some results in [Lusz90a][Lusz90b] to account for arbitrary rational forms and arbitrary roots of unity and start to target the algebra structure.

In Section 5 we obtain the first main result: We use Nichols algebras to explicitly describe the assumed kernel $u_q^{\mathcal{L}}(\mathfrak{g})$ and its root system without restrictions on q:

Theorem (5.4). *For $\mathrm{ord}(q^2) > d_\alpha$ for all $\alpha \in \Phi^+$ we have $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda) \cong u_q(\mathfrak{g}, \Lambda)$. If some $\mathrm{ord}(q^2) \leq d_\alpha$ we can express $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ in terms of some ordinary $u_q(\mathfrak{g}^{(0)}, \Lambda)^+$ as follows:*

q	$\mathfrak{g}$	$u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$	dim	*primitive generators*	*Comment*
± 1	*all*	$\mathbb{C}$	1	*none*	*trivial*
$\pm i$	B_n	$u_q(A_1^{\times n})^+$	2^n	$E_{\alpha_n}, E_{\alpha_n+\alpha_{n-1}}, E_{\alpha_n+\alpha_{n-1}+\alpha_{n-2}}, \ldots$	*short roots*
$\pm i$	C_n	$u_q(D_n)^+$	$2^{n(n-1)}$	$E_{\alpha_1}, \ldots E_{\alpha_{n-1}}, E_{\alpha_n+\alpha_{n-1}}$	*short roots*
$\pm i$	F_4	$u_q(D_4)^+$	2^{12}	$E_{\alpha_4}, E_{\alpha_3}, E_{\alpha_3+\alpha_2}, E_{\alpha_3+\alpha_2+\alpha_1}$	*short roots*
$\sqrt[3]{1}, \sqrt[6]{1}$	G_2	$u_q(A_2)^+$	3^3	$E_{\alpha_1}, E_{\alpha_1+\alpha_2}$	*short roots*
$\pm i$	G_2	$u_{\bar{q}}(A_3)^+$	2^6	$E_{\alpha_2}, E_{\alpha_1}, E_{2\alpha_1+\alpha_2}$	*exotic*

In Section 6 most of the work is done. The strategy to obtain a Frobenius homomorphism is quite conceptual, uses the previously obtained kernels and works for arbitrary q:

a) In Theorem 6.3 we extend a trick used by Lusztig in the simply-laced case: We prove that all pairs of roots can be simultaneously reflected into rank 2 parabolic subsystems; we also add a complete classification of orbits. Hence it often suffices to verify statements only in rank 2.

b) In Lemma 6.6 we prove that $u_q^{\mathcal{L},+}$ is a normal Hopf subalgebra of $U_q^{\mathcal{L},+}$. This is done using the explicit description of $u_q^{\mathcal{L},+}$ in the previous section together with the trick a). Our proof actually returns the adjoint action quite explicitly.

c) We then consider abstractly the quotient H of $U_q^{\mathcal{L},+}$ by the normal Hopf subalgebra $u_q^{\mathcal{L},+}$ (in the category of Λ-Yetter-Drinfel'd modules). Using again trick a) we prove it is generated by primitive elements and has the expected commutator structure; it is hence isomorphic to some explicit $U(\mathfrak{g}^{(\ell)})^+$. Note that the identification sometimes switches long and short root and picks up additional factors. Except if certain lattices are even we prove H is an ordinary Hopf algebra, in the other cases H is in a symmetrically braided category (so calling $\mathfrak{g}^{(\ell)}$ a Lie algebras still makes sense).

Combining these results we finally achieve our main theorem:

Theorem (6.1). *Depending on $\mathfrak{g}$ and ℓ we have the following exact sequences of Hopf algebras in the category of Λ-Yetter-Drinfel'd modules:*

$$u_q(\mathfrak{g}^{(0)}, \Lambda)^+ \xrightarrow{\subseteq} U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+ \xrightarrow{Frob} U(\mathfrak{g}^{(\ell)})^+$$

	$\mathfrak{g}$	$\ell = ord(q)$	$\mathfrak{g}^{(0)}$	$\mathfrak{g}^{(\ell)}$	*is braided for*
Trivial cases:	*all*	$\ell = 1$	0	$\mathfrak{g}$	*no*
	all	$\ell = 2$	0	$\mathfrak{g}$	$ADE_{n \geq 2}, C_{n \geq 3}, F_4, G_2$
	ADE	$\ell \neq 1, 2$	$\mathfrak{g}$	$\mathfrak{g}$	$\ell = 2 \bmod 4, n \geq 2$
Generic cases:	B_n	$4 \nmid \ell \neq 1, 2$	B_n	B_n	*no*
	C_n	$4 \nmid \ell \neq 1, 2$	C_n	C_n	$\ell = 2 \bmod 4, n \geq 3$
	F_4	$4 \nmid \ell \neq 1, 2$	F_4	F_4	$\ell = 2 \bmod 4$
	G_2	$3 \nmid \ell \neq 1, 2, 4$	G_2	G_2	$\ell = 2 \bmod 4$
	B_n	$4 \mid \ell \neq 4$	B_n	C_n	$\ell = 4 \bmod 8, n \geq 3$
		$\ell = 4$	$A_1^{\times n}$	C_n	$n \geq 3$
	C_n	$4 \mid \ell \neq 4$	C_n	B_n	*no*
		$\ell = 4$	D_n	B_n	*no*
Duality cases:	F_4	$4 \mid \ell \neq 4$	F_4	F_4	$\ell = 4 \bmod 8$
		$\ell = 4$	D_4	F_4	*yes*
	G_2	$3 \mid \ell \neq 3, 6$	G_2	G_2	$\ell = 2 \bmod 4$
		$\ell = 3, 6$	A_2	G_2	$\ell = 6$
Exotic case:	G_2	$\ell = 4$	A_3	G_2	*no*

In the "duality cases" the Frobenius homomorphism interchanges short and long roots. For small values of ℓ the Frobenius-Lusztig kernel $u_q(\mathfrak{g}^{(0)})$ usually degenerates, up to the point where it vanishes in the "trivial case" $q = \pm 1$. Several cases are "braided", meaning $\mathfrak{g}^{(\ell)}$ is a Lie algebras in a braided symmetric category (precisely the even lattices $\Lambda_R^{(\ell)}$ in Lemma 2.5). The "exotic case" will exhibit strange phenomena throughout this article.

In Section 7 we state some open problems in the context of this article.

2. Preliminaries

2.1. **Lie Theory.** Let $\mathfrak{g}$ be a finite-dimensional, semisimple complex Lie algebra with simple roots α_i indexed by $i \in I$ and a set of positive roots Φ^+. Denote the Killing form by $(,)$, normalized such that $(\alpha, \alpha) = 2$ for the short roots. The Cartan matrix is

$$a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

Be warned that there are different conventions for the index order of a, here we use the convention usual in the theory of quantum groups.

It is custom to call $d_\alpha := (\alpha, \alpha)/2$ with $d_\alpha \in \{1, 2, 3\}$, especially $d_i := d_{\alpha_i}$, which only depends on the orbit of α under the Weyl group. In this notation $(\alpha_i, \alpha_j) = d_i a_{ij}$.

Definition 2.1. *The* root lattice *$\Lambda_R = \Lambda_R(\mathfrak{g})$ is the free abelian group with $rank(\Lambda_R) = rank(\mathfrak{g}) = |I|$ and is generated by K_{α_i} for each simple root α_i. We denote general group elements in Λ_R by K_α for elements α in the root lattice of $\mathfrak{g}$. The Killing form induces an integral pairing of abelian groups, turning Λ_R into an* integral lattice*:*

$$(\_,\_): \ \Lambda_R \times \Lambda_R \to \mathbb{Z}$$

$$(K_\alpha, K_\beta) := (\alpha, \beta)$$

Definition 2.2. *The* weight lattice *$\Lambda_W = \Lambda_W(\mathfrak{g})$ is the free abelian group with $rank(\Lambda_W) = rank(\mathfrak{g})$ generated by K_{λ_i} for each fundamental dominant weight λ_i. We denote general group elements in Λ_W by K_λ with λ in the weight lattice of $\mathfrak{g}$. It is a standard fact of Lie theory (cf. [Hum72], Section 13.1) that the root lattice is contained in the weight lattice and we shall in what follows tacitly identify $\Lambda_R \subset \Lambda_W$. Moreover it is known that the pairing on Λ_R can be extended to a* integral pairing*:*

$$(\_,\_): \ \Lambda_W \times \Lambda_R \to \mathbb{Z}$$

$$(K_\lambda, K_\beta) := (\lambda, \beta)$$

Note that for multiply-laced $\mathfrak{g}$ the group Λ_W is no integral lattice.

For later use, we also define the following sublattice of the root lattice Λ_R:

Definition 2.3. *The ℓ-lattice $\Lambda_R^{(\ell)} \subset \Lambda_R$ for any positive integer ℓ is defined as follows*

$$\Lambda_R^{(\ell)} := \langle K_{\alpha_i}^{\ell_i}, \ i \in I \rangle$$

where $\ell_i = \ell/gcd(\ell, 2d_i)$ is the order of q^{2d_i} for q a primitive ℓ-th root of unity. More generally we define for any root $\ell_\alpha = \ell/gcd(\ell, 2d_\alpha)$, which only depends on the orbit of α under the Weyl group.

Example 2.4. *In the case where $\mathfrak{g}$ is simply-laced (hence all $d_\alpha = 1$) we have*

$$\ell_i = \begin{cases} \ell, & \ell \text{ odd} \\ \frac{\ell}{2}, & \ell \text{ even} \end{cases} \qquad \Lambda_R^{(\ell)} = \begin{cases} \ell \cdot \Lambda_R, & \ell \text{ odd} \\ \frac{\ell}{2} \cdot \Lambda_R, & \ell \text{ even} \end{cases}$$

Frequently, later statements can be simplified if all $\ell_i = \ell$, which is equivalent to the "generic case" $2 \nmid \ell$ (and $3 \nmid \ell$ for $\mathfrak{g} = G_2$). Moreover for small ℓ the set of roots with $\ell_\alpha = 1$ will be important. For later use we prove

Lemma 2.5. *For all $\alpha, \beta \in \Lambda_R^{(\ell)}$ we have*

$$(\alpha, \alpha) \in \ell\mathbb{Z} \qquad (\alpha, \beta) \in \frac{\ell}{2}\mathbb{Z}$$

Moreover we have $(\alpha, \beta) \in \ell\mathbb{Z}$ except in the following cases:

$\mathfrak{g}$	Exceptions
$A_n, D_n, E_6, E_7, E_8, G_2$	$\ell = 2 \bmod 4$
$B_n, n \geq 3$	$\ell = 4 \bmod 8$
$C_n, n \geq 3$	$\ell = 2 \bmod 4$
F_4	$\ell = 2, 4, 6 \bmod 8$

The exceptions will correspond to braided cases of the short exact sequence in the Main Theorem 6.1.

Proof. It is sufficient to check the condition $\ell | (\alpha, \beta)$ on the lattice basis $\ell_i \alpha_i, i \in I$. We check for each i, j whether the quotient X is an integer:

$$X := \frac{(\ell_i \alpha_i, \ell_j \alpha_j)}{\ell} = \frac{\ell_i \ell_j (\alpha_i, \alpha_j)}{\ell} = \frac{\ell \cdot (\alpha_i, \alpha_j)}{gcd(\ell, 2d_i) \cdot gcd(\ell, 2d_j)}$$

We start by checking the cases $i = j$ where we find indeed $X = \frac{\ell}{gcd(\ell, 2d_i)} \cdot \frac{2d_i}{gcd(\ell, 2d_i)} \in \mathbb{Z}$. To check the cases $i \neq j$ we can restrict ourselves to Lie algebras of rank 2, where we check the claim case by case:

- For type $A_1 \times A_1$ we have $(\alpha_i, \alpha_j) = 0$.
- For type A_2 we have $d_i = d_j = d \in \{1, 2\}$ and $(\alpha_i, \alpha_j) = -d$, hence

$$X = \frac{\ell \cdot (-d)}{gcd(\ell, 2d) \cdot gcd(\ell, 2d)}$$

  If $2 \nmid \ell$ we have $gcd(\ell, 2d) = gcd(\ell, d)$ and hence $X = \frac{\ell}{gcd(\ell,d)} \cdot \frac{-d}{gcd(\ell,d)} \in \mathbb{Z}$. If $d | \ell$ but $2d \nmid \ell$ we have $gcd(\ell, 2d) = d$ and hence $X = \frac{\ell}{d} \cdot \frac{-d}{d} \in \mathbb{Z}$. If $4d | \ell$ we have $X = \frac{\ell}{2d} \cdot \frac{-d}{2d} = \frac{\ell}{4d} \cdot \frac{-d}{d} \in \mathbb{Z}$. If however $2d | \ell$ but $4d \nmid \ell$ we have $X = \frac{\ell}{2d} \cdot \frac{-d}{2d} \in \mathbb{Z} + \frac{1}{2}$.
- For type B_2 we have $d_i = 1, d_j = 2$ and $(\alpha_i, \alpha_j) = -2$. Hence

$$X = \frac{\ell \cdot (-2)}{gcd(\ell, 2) \cdot gcd(\ell, 4)} = \frac{-2}{gcd(\ell, 2)} \cdot \frac{\ell}{gcd(\ell, 4)} \in \mathbb{Z}$$

- For type G_2 we have $d_i = 1, d_j = 3$ and $(\alpha_i, \alpha_j) = -3$. Hence

$$X = \frac{\ell \cdot (-3)}{gcd(\ell, 2) \cdot gcd(\ell, 6)}$$

If $2 \nmid \ell$ or $4 | \ell$ we have as for A_2 that $X \in \mathbb{Z}$, while for $2 | \ell, 4 \nmid \ell$ we have $X \in \mathbb{Z} + \frac{1}{2}$.

The assertion follows now from considering all pairs of simple roots (i, j):

- For $\mathfrak{g}$ simply-laced, all (i, j) are either $A_1 \times A_1$ or A_2 for short roots $d = 1$. The exceptional cases are hence $\ell = 2$ mod 4 whenever an edge exists i.e. $n \geq 2$.
- For $\mathfrak{g} = C_n$, all (i, j) are either $A_1 \times A_1$ or A_2 for short roots $d = 1$ or $B_2 = C_2$. The exceptional cases are hence $\ell = 2$ mod 4 for $n \geq 3$.
- For $\mathfrak{g} = B_n$, all (i, j) are either $A_1 \times A_1$ or A_2 for long roots $d = 2$ or $B_2 = C_2$. The exceptional cases are hence $\ell = 4$ mod 8 for $n \geq 3$.
- For $\mathfrak{g} = F_4$, all (i, j) are either $A_1 \times A_1$ or A_2 for short roots $d = 1$ or long roots $d = 2$ or $B_2 = C_2$. The exceptional cases are hence $\ell = 2$ mod 4 as well as $\ell = 4$ mod 8.
- For G_2 we already calculated the exceptional cases to be $\ell = 2$ mod 4.

$\square$

2.2. **Nichols algebras.** Nichols algebras generalize the Borel parts of quantum groups in the classification of pointed Hopf algebras, see [AS10] Sec. 5.1. In this article we only use the Nichols algebras appearing in ordinary quantum groups, as briefly introduced in the following, but their use makes the later constructions more transparent. For a detailed account on Nichols algebras see e.g. [HLecture08].

Definition 2.6. *Assume we are over the base field* $\mathbb{C}$*. A* Yetter-Drinfel'd module M *over a finite abelian group* Γ *is a* Γ*-graded vector space,* $M = \bigoplus_{g \in \Gamma} M_g$ *with a* Γ*-action on* M *such that* $g.M_h = M_h$*.*

The category of Yetter-Drinfel'd modules form a braided category. Let M be an n-dimensional Yetter-Drinfel'd module over the field $\mathbb{C}$, then we may choose a homogeneous vector space basis v_i with grading some $g_i \in \Gamma$ and express the action via $g_i.v_j = q_{ij} v_j$ for some $q_{ij} \in \mathbb{C}^\times$. Then the braiding of M has the form

$$v_i \otimes v_j \mapsto q_{ij} \, v_j \otimes v_i$$

Definition 2.7. *Consider the tensor algebra* $T(M)$*, which can be identified with the algebra of words in the letters* v_i *and is again a* Γ*-Yetter-Drinfel'd module. We can uniquely obtain* skew derivations $\partial_i : T(M) \to T(M)$ *by*

$$\partial_k(1) = 0 \qquad \partial_k(v_l) = \delta_{kl} 1 \qquad \partial_k(x \cdot y) = \partial_k(x) \cdot (g_k.y) + x \cdot \partial_k(y)$$

The Nichols algebra $\mathcal{B}(M)$ *is the quotient of* $T(M)$ *by the largest homogeneous ideal* $\mathfrak{I}$ *in degree* ≥ 2, *invariant under all* ∂_k. *It is a Hopf algebra in the braided category of* Γ-*Yetter-Drinfel'd module.*

Heckenberger classified all finite-dimensional Nichols algebras over finite abelian groups Γ in [Heck09]. In the present article, we only need the following examples:

Let Φ^+ be the set of positive roots for a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ of rank n and normalized Killing form $(,)$ as in the Lie theory preliminaries. Let q be a primitive ℓ-th root of unity. Then the Yetter-Drinfel'd module defined by the braiding matrix $q_{ij} := q^{(\alpha_i, \alpha_j)}$ has a finite-dimensional Nichols algebra $\mathcal{B}(M)$ iff all $q_{ii} = q^{d_{\alpha_i}}$ are $\neq 1$. More precisely, $\mathcal{B}(M)$ has a PBW-like basis associated to Φ and especially the dimension is

$$\dim(\mathcal{B}(M)) = \prod_{\alpha \in \Phi^+} \mathrm{ord}(q^{(\alpha,\alpha)}) = \prod_{\alpha \in \Phi^+} \mathrm{ord}(q^{d_\alpha})$$

unless the case $\mathfrak{g} = G_2, \ell = 4$, which is excluded in Heckenberger's list entry for G_2 ([Heck06] Figure 1 Row 11). Indeed, the braiding matrix is in this case equal to the braiding matrix for A_2, namely $q_{ij} = \begin{pmatrix} -1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix}$ and $\dim(\mathcal{B}(M)) = 2 \cdot 2 \cdot 2$ is less than expected for G_2.

The condition $q_{ii} \neq 1$ and the exceptional case $\mathfrak{g} = G_2, \ell = 4$ will play a prominent role in the present article. It will be the direct cause why the Borel part of the small quantum group $u_q(\mathfrak{g})^+$ is for small ℓ *not* isomorphic to the corresponding Nichols algebra $\mathcal{B}(M)$ as one might expect.

2.3. **Coradical extensions.** We introduce the following tool without referring to quantum groups. It will later allow us to quickly transport results about the adjoint rational form $(\Lambda = \Lambda_R)$ in literature to arbitrary Λ.

Suppose H a Hopf algebra over a commutative ring $\Bbbk$ with group of grouplikes $G(H)$ and fix some subgroup the group of grouplikes $\Lambda_R \subset G(H)$. Let $\Lambda \triangleright \Lambda_R$ be a group containing Λ_R normally and let $\rho : \Bbbk[\Lambda] \otimes H \to H$ be an action, such that

- The action ρ turns H into a $\Bbbk[\Lambda]$-module Hopf algebra.
- The action ρ restricts on $\Bbbk[\Lambda_R] \subset \Bbbk[\Lambda]$ to the adjoint representation ρ_R of the Hopf subalgebra $\Bbbk[\Lambda_R] \subset H$.
- The action ρ restricts on $\Bbbk[\Lambda_R] \subset H$ to the adjoint representation ρ_Λ of $\Bbbk[\Lambda]$ on the Hopf subalgebra $\Bbbk[\Lambda_R]$, given by conjugacy action of the group Λ on the normal subgroup Λ_R.

Theorem 2.8. *The Hopf algebra structure on the smash-product $\Bbbk[\Lambda] \ltimes H$ factorizes to a Hopf algebra structure on the vector space $\Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} H$*

$$H_\Lambda := \Bbbk[\Lambda] \ltimes_{\Bbbk[\Lambda_R]} H$$

where the left-/right $\Bbbk[\Lambda_R]$-module structures are the multiplication with respect to the inclusions $\Bbbk[\Lambda_R] \subset \Bbbk[\Lambda]$ and $\Bbbk[\Lambda_R] \subset G(H) \subset H$

Especially the choice $\Lambda = \Lambda_R$ recovers $H_{\Lambda_R} := \Bbbk[\Lambda_R] \otimes_{\Bbbk[\Lambda_R]} H = H$.

Proof. The smash-product of two Hopf algebras $H'_\Lambda := \Bbbk[\Lambda] \ltimes H$ with respect to an action ρ on the Hopf algebra H is the vector spaces $H'_\Lambda := \Bbbk[\Lambda] \otimes_\Bbbk H$ with the coalgebra structure of the tensor product and the multiplication $\mu_{H'_\Lambda}$ given for $g, h \in \Lambda, x, y \in H$ by:

$$\mu_{H'_\Lambda}((g \otimes x) \otimes (h \otimes y)) = gh^{(1)} \otimes (\rho(S(h^{(2)}) \otimes x) \cdot y) = gh \otimes (\rho(h^{-1} \otimes x) \cdot y)$$

We have to show that the structures $1_{H'_\Lambda}, \mu_{H'_\Lambda}, \Delta_{H'_\Lambda}, \epsilon_{H'_\Lambda}$ factorize over the surjection

$$\phi : \ H'_\Lambda := \Bbbk[\Lambda] \otimes_\Bbbk H \longrightarrow \Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} H =: H_\Lambda$$

- The multiplication $\mu_{H'_\Lambda}$ factorizes as follows: For all $g, h \in \Lambda, t \in \Lambda_R, x, y \in H$ we have

$$
\begin{aligned}
(\phi \circ \mu_{H'_\Lambda})((gt \otimes x) \cdot (h \otimes y)) &= gth \otimes_{\Bbbk[\Lambda_R]} \rho(h^{-1} \otimes x)y \\
&= gh \otimes_{\Bbbk[\Lambda_R]} \rho(h^{-1}th \otimes \rho(h^{-1} \otimes x))(h^{-1}th)y \\
&= gh \otimes_{\Bbbk[\Lambda_R]} \rho(h^{-1} \otimes \rho_R(t \otimes x))\rho(h^{-1} \otimes t)y \\
&= gh \otimes_{\Bbbk[\Lambda_R]} \rho(h^{-1} \otimes tx)y \\
&= (\phi \circ \mu_{H'_\Lambda})((g \otimes tx) \cdot (h \otimes y))
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
(\phi \circ \mu_{H'_\Lambda})((g \otimes x) \cdot (ht \otimes y)) &= ght \otimes_{\Bbbk[\Lambda_R]} \rho((ht)^{-1} \otimes x)y \\
&= ght \otimes_{\Bbbk[\Lambda_R]} t^{-1}\rho(h^{-1} \otimes x)ty \\
&= gh \otimes_{\Bbbk[\Lambda_R]} \rho(h^{-1} \otimes x)ty \\
&= (\phi \circ \mu_{H'_\Lambda})((g \otimes x) \cdot (h \otimes ty))
\end{aligned}
$$

- The unit $1_{H'_\Lambda}$ maps to $\phi(1_{H'_\Lambda}) \in H_\Lambda$.

- The comultiplication $\Delta_{H'_\lambda}$ factorizes as follows: For all $g, h \in \Lambda, t \in \Lambda_R, x, y \in H$ we have t grouplike and hence

$$
\begin{aligned}
(\phi \circ \Delta_{H'_\lambda})(gt \otimes x) &= \left(gt \otimes_{\Bbbk[\Lambda_R]} x^{(1)}\right) \otimes \left(gt \otimes_{\Bbbk[\Lambda_R]} x^{(2)}\right) \\
&= \left(g \otimes_{\Bbbk[\Lambda_R]} tx^{(1)}\right) \otimes \left(g \otimes_{\Bbbk[\Lambda_R]} tx^{(2)}\right) \\
&= \left(g \otimes_{\Bbbk[\Lambda_R]} (tx)^{(1)}\right) \otimes \left(g \otimes_{\Bbbk[\Lambda_R]} (tx)^{(2)}\right) \\
&= (\phi \circ \Delta_{H'_\lambda})(g \otimes tx)
\end{aligned}
$$

- The counit $\epsilon_{H'_\lambda}$ factorizes as follows: For all $g \in \Lambda, x \in H$ we have

$$
\begin{aligned}
(\phi \circ \epsilon_{H'_\lambda})(gt \otimes x) &= \epsilon_{H'_\lambda}(gt) \cdot \epsilon_{H'_\lambda}(x) \\
&= \epsilon_{H'_\lambda}(g) \cdot \epsilon_{H'_\lambda}(tx) \\
&= (\phi \circ \epsilon_{H'_\lambda})(gt \otimes x)
\end{aligned}
$$

□

3. Different forms of quantum groups

We recall several Hopf algebras associated to $\mathfrak{g}$ over various commutative rings $\Bbbk$.

Remark 3.1. *The following notion is added for completeness and not used in the sequel: There is a so-called* topological Hopf algebra $U_q^{\mathbb{C}[[q]]}(\mathfrak{g})$ *over the ring of formal power series* $\Bbbk = \mathbb{C}[[q]]$ *cf.* [CP95] *6.5.1. It was defined by Drinfel'd (1987) and Jimbo (1985).*

3.1. **The rational forms.** We next define the *rational form* $U_q^{\mathbb{Q}(q)}(\mathfrak{g})$. There are in fact several rational forms $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ associated to the $U_q^{\mathbb{C}[[q]]}(\mathfrak{g})$ that differ by a choice of a subgroup $\Lambda_R \subset \Lambda \subset \Lambda_W$ resp. a choice of a subgroup in the *fundamental group* $\pi_1 := \Lambda_W / \Lambda_R$. This corresponds to choosing a Lie group associated to the Lie algebra $\mathfrak{g}$; we call the two extreme cases $\Lambda = \Lambda_W$ the *simply-connected form* and $\Lambda = \Lambda_R$ the usual *adjoint form* (e.g. SL_2 vs. PSL_2), see e.g. [CP95] Sec. 9.1 or [Lusz94].

Definition 3.2. *For each abelian group Λ with $\Lambda_R \subset \Lambda \subset \Lambda_W$ we define the* rational form $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ *over the ring of rational functions $\Bbbk = \mathbb{Q}(q)$ as follows:*

As algebra, let $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ be generated by the group ring $\Bbbk[\Lambda]$ spanned by $K_\lambda, \lambda \in \Lambda$ and additional generators $E_{\alpha_i}, F_{\alpha_i}$ for each simple root $\alpha_i, i \in I$ with relations:

$$
\begin{aligned}
K_\lambda E_{\alpha_i} K_\lambda^{-1} &= q^{(\lambda, \alpha_i)} E_{\alpha_i}, \ \forall \lambda \in \Lambda & \text{(group action)} \\
K_\lambda F_{\alpha_i} K_\lambda^{-1} &= \bar{q}^{(\lambda, \alpha_i)} F_{\alpha_i}, \ \forall \lambda \in \Lambda & \text{(group action)} \\
[E_{\alpha_i}, F_{\alpha_j}] &= \delta_{i,j} \cdot \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_{\alpha_i} - q_{\alpha_i}^{-1}} & \text{(linking)}
\end{aligned}
$$

and two sets of Serre-relations *for any* $i \neq j \in I$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q^{d_i}} E_{\alpha_i}^{1-a_{ij}-r} E_{\alpha_j} E_{\alpha_i}^r = 0$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{\bar{q}^{d_i}} F_{\alpha_i}^{1-a_{ij}-r} F_{\alpha_j} F_{\alpha_i}^r = 0$$

where $\bar{q} := q^{-1}$, *the* $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{d_i}}$ *are the quantum binomial coefficients (see* [Lusz94] *Sec. 1.3) and by definition* $q^{(\alpha_i, \alpha_j)} = (q^{d_i})^{a_{ij}}$. *As a coalgebra, let the* coproduct Δ, *the* counit ϵ *and the* antipode S *be defined on the group-Hopf-algebra* $\Bbbk[\Lambda]$ *as usual*

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda \qquad \epsilon(K_\lambda) = 1 \qquad S(K_\lambda) = K_\lambda^{-1} = K_{-\lambda}$$

and on the additional generators $E_{\alpha_i}, F_{\alpha_i}$ *for each simple root* $\alpha_i, i \in I$ *as follows:*

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i} \qquad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i}$$

$$S(E_{\alpha_i}) = -E_{\alpha_i} K_{\alpha_i}^{-1} \qquad S(F_{\alpha_i}) = -K_{\alpha_i} F_{\alpha_i}$$

$$\epsilon(E_{\alpha_i}) = 0 \qquad \epsilon(F_{\alpha_i}) = 0$$

Theorem 3.3 (Rational Form). $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ *is a Hopf algebra over the field* $\Bbbk = \mathbb{Q}(q)$. *For arbitrary* Λ *using the construction in Theorem 2.8 we have*

$$U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda) = \Bbbk[\Lambda] \ltimes_{\Bbbk[\Lambda_R]} U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda_R)$$

Moreover, we have a triangular decomposition*: Consider the subalgebras* $U_q^{\mathbb{Q}(q),+}$ *generated by the* E_{α_i} *and* $U_q^{\mathbb{Q}(q),-}$ *generated by the* F_{α_i} *and* $U_q^{\mathbb{Q}(q),0} = \Bbbk[\Lambda]$ *spanned by the* K_λ. *Then multiplication in* $U_q^{\mathbb{Q}(q)}$ *induces an isomorphism of vector spaces:*

$$U_q^{\mathbb{Q}(q),+} \otimes U_q^{\mathbb{Q}(q),0} \otimes U_q^{\mathbb{Q}(q),-} \xrightarrow{\cong} U_q^{\mathbb{Q}(q)}$$

Proof. The case of the adjoint form $\Lambda = \Lambda_R$ is classical, see e.g. [Jan03] II, H.2 & H.3. In principle, this and later proofs work totally analogous for arbitrary $\Lambda_R \subset \Lambda \subset \Lambda_W$, but to connect them directly to results in literature without repeating everything, we deduce the case of arbitrary Λ from $\Lambda = \Lambda_R$ and the construction in Section 2.3:

Let $\Bbbk = \mathbb{Q}(q)$, take $\Lambda_R \subset \Lambda \subset \Lambda_W$ an abelian group and let $H = U_q^{\mathbb{Q}(q)}(\mathfrak{g}) := U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda_R)$ be the adjoint form with smallest $\Lambda = \Lambda_R$. Define an action ρ of $\Bbbk[\Lambda]$ on H given by

$$\rho(K_\lambda \otimes E_{\alpha_i}) = q^{(\lambda, \alpha_i)} E_{\alpha_i}$$

$$\rho(K_\lambda \otimes F_{\alpha_i}) = \bar{q}^{(\lambda, \alpha_i)} F_{\alpha_i}$$

Then certainly the restriction of this action to $\Bbbk[\Lambda_R] \subset \Bbbk[\Lambda]$ is the adjoint action in Definition 3.2 for $H = U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda_R)$ and the restriction to $\Bbbk[\Lambda_R] \subset H$ is trivial (Λ is an abelian group). Hence we can apply extension of scalars by an abelian group in Theorem 2.8 and yield a Hopf algebra

$$H_\Lambda := \Bbbk[\Lambda] \ltimes_{\Bbbk[\Lambda_R]} H$$

Denote the elements $K_\lambda \otimes_{\Bbbk[\Lambda_R]} 1$ by K_λ, especially for $\alpha \in \Lambda_R$ we have $K_\alpha = 1 \otimes_{\Bbbk[\Lambda_R]} K_\alpha$ with $K_\alpha \in U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda_R)$. Then it is clear that these elements fulfill the relations given in the previous Definition of $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ for general Λ. It follows from the triangular decomposition of H that this is an isomorphism of vector spaces as

$$\Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} \Bbbk[\Lambda_R] \cong \Bbbk[\Lambda]$$

Especially, $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ defined above is a Hopf algebra with a triangular decomposition as vector spaces

$$U_q^{\mathbb{Q}(q),+} \otimes U_q^{\mathbb{Q}(q),0} \otimes U_q^{\mathbb{Q}(q),-} \xrightarrow{\cong} U_q^{\mathbb{Q}(q)}$$

with $U_q^{\mathbb{Q}(q),0} \cong \Bbbk[\Lambda]$ and $U_q^{\mathbb{Q}(q),\pm}$ independent of the choice of Λ. $\qquad\square$

A tool of utmost importance has been introduced by Lusztig, see [Jan03] H.4:

Definition 3.4. *Fix a reduced expression $s_{i_1} \cdots s_{i_t}$ of the longest element in the Weyl group $\mathcal{W}(\mathfrak{g})$ in terms of reflections s_i on simple roots α_i.*

(1) There exist algebra automorphisms $T_i : U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda) \to U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$, such that the action restricted to $K_\lambda \in \Lambda \subset \Lambda_W$ is the reflection of the weight λ on α_i.

(2) Every positive root β has a unique expression $\beta = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ for some index k. This defines a total ordering on the set of positive roots Φ^+ and the reversed ordering on Φ^-. Define the root vectors for a root $\beta \in \Phi^+$ by

$$E_\beta := T_{i_1} \cdots T_{i_{k-1}} E_{\alpha_{i_k}}$$

$$F_\beta := T_{i_1} \cdots T_{i_{k-1}} F_{\alpha_{i_k}}$$

With these definitions, Lusztig establishes a PBW vector space basis:

Theorem 3.5 (PBW-basis). *Multiplication in $U_q^{\mathbb{Q}(q)}$ induces an isomorphism of $\Bbbk$-vector spaces for the field $\Bbbk = \mathbb{Q}(q)$*

$$\Bbbk[\Lambda] \bigotimes_{\alpha \in \Phi^+} \Bbbk[E_\alpha] \bigotimes_{-\alpha \in \Phi^-} \Bbbk[F_\alpha] \xrightarrow{\cong} U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$$

where the orderings on Φ^+, Φ^- are as above.

Proof. The adjoint case $\Lambda = \Lambda_R$ is classical and in [Jan03] H.4. Note that using the relations between K_α, E_α all K's can be sorted to the left side. The case of arbitrary Λ could be derived totally analogously, but it also follows directly from the presentation as extension in Theorem 3.3. Namely, we have by construction isomorphisms of vector spaces

$$
\begin{aligned}
U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda) &\cong \Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda_R) \\
&\cong \Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} \Bbbk[\Lambda_R] \bigotimes_{\alpha \in \Phi^+} \Bbbk[E_\alpha] \bigotimes_{-\alpha \in \Phi^-} \Bbbk[F_\alpha] \\
&\cong \Bbbk[\Lambda] \bigotimes_{\alpha \in \Phi^+} \Bbbk[E_\alpha] \bigotimes_{-\alpha \in \Phi^-} \Bbbk[F_\alpha]
\end{aligned}
$$

$\square$

3.2. **Two integral forms.** Next we define two distinct *integral forms* $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda)$ and $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g}, \Lambda)$. These are $\mathbb{Z}[q, q^{-1}]$-subalgebras of $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ which are after extension of scalars $\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{Q}(q)$ isomorphic to $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ as $\mathbb{Q}(q)$-algebras.

Definition 3.6. *(cf.* [CP95] *Sec. 9.2 and 9.3) Recall* $q_\alpha := q^{d_\alpha} = q^{(\alpha,\alpha)/2}$.

- *The so-called* unrestricted integral form $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda)$ *is generated as a* $\mathbb{Z}[q, q^{-1}]$-*algebra by* Λ *and the following elements in* $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)^{+,-,0}$

$$
E_\alpha, \ F_\alpha, \ \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_{\alpha_i} - q_{\alpha_i}^{-1}} \qquad \forall \alpha \in \Phi^+, i \in I
$$

  We use the superscript $\mathcal{K}$ *in honor of Victor Kac, who has defined and studied it in characteristic p in 1967 with Weisfeiler and the present form in 1990–1992 with DeConcini and Procesi.*

- *The so-called* restricted integral form $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g}, \Lambda)$ *is generated as a* $\mathbb{Z}[q, q^{-1}]$-*algebra by* Λ *and the following elements in* $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)^{\pm}$ *called* divided powers

$$
E_\alpha^{(r)} := \frac{E_\alpha^r}{\prod_{s=1}^r \frac{q_\alpha^s - q_\alpha^{-s}}{q_\alpha - q_\alpha^{-1}}}, \ F_\alpha^{(r)} := \frac{F_\alpha^r}{\prod_{s=1}^r \frac{\bar{q}_\alpha^s - \bar{q}_\alpha^{-s}}{\bar{q}_\alpha - \bar{q}_\alpha^{-1}}} \qquad \forall \alpha \in \Phi^+, r \geq 0
$$

  and by the following elements in $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)^0$:

$$
K_{\alpha_i}^{(r)} = \begin{bmatrix} K_{\alpha_i}; 0 \\ r \end{bmatrix} := \prod_{s=1}^r \frac{K_{\alpha_i} q_{\alpha_i}^{1-s} - K_{\alpha_i}^{-1} q_{\alpha_i}^{s-1}}{q_{\alpha_i}^s - q_{\alpha_i}^{-s}} \qquad i \in I
$$

  We use the superscript $\mathcal{L}$ *in honor of Georg Lusztig, who has defined and studied it in 1988–1990.*

Theorem 3.7. *The Lusztig quantum group* $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g},\Lambda)$ *is a Hopf algebra over the ring* $\Bbbk = \mathbb{Z}[q,q^{-1}]$ *and is an integral forms for* $U_q^{\mathbb{Q}(q)}(\mathfrak{g},\Lambda)$. *Hereby for arbitrary* Λ *we have again by Theorem 2.8*

$$U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g},\Lambda) \cong \Bbbk[\Lambda] \ltimes_{\Bbbk[\Lambda_R]} U_q^{\mathbb{Q}(q)}(\mathfrak{g},\Lambda_R)$$

A similar result holds for the Kac integral form, see Chari [CP95] Sec. 9.2. Generators and relations for simply-laced $\mathfrak{g}$ are discussed in [CP95] Thm. 9.3.4. The proof, that the $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{K}}, U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$ are integral forms for $U_q^{\mathbb{Q}(q)}$ follows immediately from the remarkable knowledge of a PBW-basis:

Theorem 3.8 (PBW-Basis)**.** *For the Lusztig integral form* $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$ *over the commutative integral domain* $\Bbbk = \mathbb{Z}[q,q^{-1}]$, *multiplication induces an isomorphism of* $\Bbbk$-*modules*

$$\Bbbk[\Lambda/2\Lambda_R] \bigotimes_{i\in I}\left(\bigoplus_{r\geq 0} K_{\alpha_i}^{(r)}\Bbbk\right) \bigotimes_{\alpha\in\Phi^+}\left(\bigoplus_{r\geq 0} E_\alpha^{(r)}\Bbbk\right) \bigotimes_{-\alpha\in\Phi^-}\left(\bigoplus_{r\geq 0} F_\alpha^{(r)}\Bbbk\right) \xrightarrow{\cong} U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g},\Lambda)$$

Especially, the Lusztig integral form is free as $\Bbbk$-*module. Note that the group algebra* $\Bbbk[\Lambda/2\Lambda_R]$ *is not contained in* $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$ *as an algebra, just as* $\Bbbk$-*module!*

Proof. For $\Lambda = \Lambda_R$ it is proven by Lusztig (see [Jan03] H.5) that the sorted monomials in the root vectors E_α resp. F_α for $\alpha \in \Phi^+$ form a basis of $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L},\pm}$ as a module over the commutative ring $\Bbbk = \mathbb{Z}[q,q^{-1}]$ and that the products $\prod_{i\in I} K_{\alpha_i}^{\delta_i} K_{\alpha_i}^{(r_i)}$ with $\delta_i \in \{0,1\}, r_i \geq 0$ form a $\Bbbk$-basis of $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L},0}$. The latter statement is by the commutativity equivalent to the statement $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L},0} \cong \Bbbk[\Lambda/2\Lambda_R]\bigotimes_{i\in I}(\bigoplus_{r\geq 0} K_{\alpha_i}^{(r)}\Bbbk)$. Note that the PBW-basis theorem does not follows from the PBW-basis of the rational form. Rather, the proof proceeds parallel and roughly uses that the T_i preserve the chosen generator set of $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$.

The case of arbitrary Λ could be derived totally analogously, but it also follows directly from the presentation in Theorem 3.7, and is proven as in the proof of Theorem 3.5:

$$U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g},\Lambda)$$

$$\cong \Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}(\mathfrak{g},\Lambda_R)$$

$$\cong \Bbbk[\Lambda] \otimes_{\Bbbk[\Lambda_R]} \Bbbk[\Lambda_R/2\Lambda_R] \bigotimes_{i\in I}\left(\bigoplus_{r\geq 0} K_{\alpha_i}^{(r)}\Bbbk\right) \bigotimes_{\alpha\in\Phi^+}\left(\bigoplus_{r\geq 0} E_\alpha^{(r)}\Bbbk\right) \bigotimes_{-\alpha\in\Phi^-}\left(\bigoplus_{r\geq 0} F_\alpha^{(r)}\Bbbk\right)$$

$$\cong \Bbbk[\Lambda/2\Lambda_R] \bigotimes_{i\in I}\left(\bigoplus_{r\geq 0} K_{\alpha_i}^{(r)}\Bbbk\right) \bigotimes_{\alpha\in\Phi^+}\left(\bigoplus_{r\geq 0} E_\alpha^{(r)}\Bbbk\right) \bigotimes_{-\alpha\in\Phi^-}\left(\bigoplus_{r\geq 0} F_\alpha^{(r)}\Bbbk\right)$$

$\square$

3.3. **Specialization to roots of unity.** Next we define the *restricted specialization* $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$. It is a complex Hopf algebra depending on a specific choice $q \in \mathbb{C}^{\times}$.

Definition 3.9. *(cf.* [CP95] *Sec. 9.2 and 9.3) The infinite-dimensional complex Hopf algebra $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ is defined by*

$$U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda) := U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}_q$$

where (by slight abuse of notation) $\mathbb{C}_q = \mathbb{C}$ with the $\mathbb{Z}[q,q^{-1}]$-module structure defined by the specific value $q \in \mathbb{C}^{\times}$.

Note that we have a PBW-basis in Theorem 3.8, which especially shows $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$ is free as a $\mathbb{Z}[q,q^{-1}]$-module. Hence the specialization has an induced vector space basis, the impact of the specialization is to severely modify the algebra structure, such that e.g. former powers may become new algebra generators.

Corollary 3.10. *For the Lusztig quantum group $U_q^{\mathcal{L}}$ over $\mathbb{C}$, multiplication induces an isomorphism of $\mathbb{C}$-vector spaces:*

$$\mathbb{C}[\Lambda/2\Lambda_R] \bigotimes_{i \in I} \left(\bigoplus_{r \geq 0} K_{\alpha_i}^{(r)} \mathbb{C} \right) \bigotimes_{\alpha \in \Phi^+} \left(\bigoplus_{r \geq 0} E_{\alpha}^{(r)} \mathbb{C} \right) \bigotimes_{-\alpha \in \Phi^-} \left(\bigoplus_{r \geq 0} F_{\alpha}^{(r)} \mathbb{C} \right) \xrightarrow{\cong} U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$$

This PBW-basis will we refined in Lemma 4.3.

Example 3.11. *For $q = 1$ we have a cosmash-product*

$$U_1^{\mathcal{L}}(\mathfrak{g}, \Lambda) \cong \mathbb{C}[\Lambda/2\Lambda] \ltimes U(\mathfrak{g})$$

4. First properties of the specialization

For the rest of the article we assume $q \in \mathbb{C}^{\times}$ a primitive ℓ-th root of unity without restrictions on ℓ. We study the infinite-dimensional Lusztig quantum group $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ from Definition 3.9, which is a Hopf algebra over $\mathbb{C}$. It was defined as a specialization of the Lusztig integral form in Definition 3.6 and hence shares the explicit vector space basis given by Theorem 3.8.

4.1. **The zero-part.** The zero-part $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^0$ in the triangular decomposition uses different arguments than the quantum Borel parts. Recall from Corollary 3.10, that multiplication in $U_q^{\mathcal{L}}$ induces an isomorphism of vector spaces

$$\bigotimes_{i \in I} \mathbb{C}[K_{\alpha_i}]/(K_{\alpha_i}^{2\ell_i}) \bigotimes_{i \in I} \left(\bigoplus_{r \geq 0} K_{\alpha_i}^{(r)} \mathbb{C} \right) = \mathbb{C}[\Lambda/2\Lambda_R] \bigotimes_{i \in I} \left(\bigoplus_{r \geq 0} K_{\alpha_i}^{(r)} \mathbb{k} \right) \xrightarrow{\cong} U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^0$$

We want to determine the algebra structure of $U_q^{\mathcal{L},0}$. It is clear from the definition, that $U_q^{\mathcal{L},0}$ is a commutative, cocommutative complex Hopf algebra. Note that by the theorem

of Kostant-Cartier (see e.g. [Mont93] Sec. 5.6) this already implies it is of the form $\mathbb{C}[G] \otimes U(\mathfrak{h})$ with group of grouplikes $G = G(U_q^{\mathcal{L},0})$ and $\mathfrak{h}$ an abelian Lie algebra.

Theorem 4.1. *With $\ell_i = ord(q_\alpha^2)$ as always we have an isomorphism of Hopf algebras*

$$U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^0 \cong \mathbb{C}[\Lambda/2\Lambda^{(\ell)}] \otimes U(\mathfrak{h}) \qquad \mathfrak{h} = \bigoplus_{i \in I} K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} \mathbb{C}$$

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda \qquad \Delta(K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)}) = 1 \otimes K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} + K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} \otimes 1$$

(the term $K_{\alpha_i}^{-r} K_{\alpha_i}^{(r)}$ is denoted $K_{i,r}$ in [Lusz90a] *Section 6)*

Proof. The elements $K_{\alpha_i}, K_{\alpha_i}^{(\ell_i)}$ lay in $U_q^{\mathcal{L},0}$. It is proven in [Lusz90b] Thm. 8.3 (without restrictions on ℓ) that in $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda_R)$ the elements K_{α_i} generate the group $\Lambda_R/\langle K_{\alpha_i}^{2\ell_i}, i \in I\rangle = \Lambda_R/2\Lambda_R^{(\ell)}$. The presentation in Theorem 3.7 easily extends this result to arbitrary Λ.

We show first that the coproduct of $K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)}$ for each $i \in I$ is as prescribed. This follows from the following equation in $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$

$$K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} = K_{\alpha_i}^{-\ell_i} \prod_{s=1}^{\ell_i} \frac{K_{\alpha_i} q_{\alpha_i}^{1-s} - K_{\alpha_i}^{-1} q_{\alpha_i}^{s-1}}{q_{\alpha_i}^s - q_{\alpha_i}^{-s}} = \frac{\prod_{s=1}^{\ell_i} q_{\alpha_i}^{1-s} - K_{\alpha_i}^{-2} q_{\alpha_i}^{s-1}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}}$$

$$= \prod_{s=1}^{\ell_i} q_{\alpha_i}^{s-1} \cdot \frac{\prod_{s=1}^{\ell_i} q_{\alpha_i}^{2(1-s)} - K_{\alpha_i}^{-2}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}} = q_{\alpha_i}^{\frac{\ell_i(\ell_i-1)}{2}} \cdot \frac{1 - K_{\alpha_i}^{-2\ell_i}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}}$$

$$\Delta(K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)}) = q_{\alpha_i}^{\frac{\ell_i(\ell_i-1)}{2}} \cdot \frac{1 \otimes 1 - K_{\alpha_i}^{-2\ell_i} \otimes K_{\alpha_i}^{-2\ell_i}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}}$$

$$= q_{\alpha_i}^{\frac{\ell_i(\ell_i-1)}{2}} \cdot \frac{1 \otimes 1 - K_{\alpha_i}^{-2\ell_i} \otimes 1 + K_{\alpha_i}^{-2\ell_i} \otimes 1 - K_{\alpha_i}^{-2\ell_i} \otimes K_{\alpha_i}^{-2\ell_i}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}}$$

$$= \left(q_{\alpha_i}^{\frac{\ell_i(\ell_i-1)}{2}} \cdot \frac{1 - K_{\alpha_i}^{-2\ell_i}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}} \right) \otimes 1 + K_{\alpha_i}^{-2\ell_i} \otimes \left(q_{\alpha_i}^{\frac{\ell_i(\ell_i-1)}{2}} \cdot \frac{1 - K_{\alpha_i}^{-2\ell_i}}{\prod_{s=1}^{\ell_i} q_{\alpha_i}^s - q_{\alpha_i}^{-s}} \right)$$

$$= K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} \otimes 1 + K_{\alpha_i}^{-2\ell_i} \otimes K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)}$$

In the third line we used that by definition $\ell_i = ord(q_{\alpha_i}^2)$. The asserted coproduct follows from the previously shown relation $K_{\alpha_i}^{2\ell_i} = 1$ in the specialization $U_q^{\mathcal{L}}$.

It hence remains to show, that without restrictions on ℓ these elements generate all of $U_q^{\mathcal{L},0}$. Following Lusztig, we do so by reducing a generator $K_{\alpha_i}^{(r)}$ to these generators for any fixed $i \in I$. Note however that the factorization formula [Lusz90a] Lemma 6.4 does *not* hold for arbitrary ℓ and a direct expression would be significantly more complicated!

We hence prove the claim $K_{\alpha_i}^{(r)} \in \Bbbk[\Lambda]\Bbbk[K_{\alpha_i}^{(\ell_i)}]$ by induction: For $r < \ell_i$ we have certainly that $K_{\alpha_i}^{(r)} \in \Bbbk[\Lambda]$ and for $r = \ell_i$ we recover the generator $K_{\alpha_i}^{(\ell_i)}$. Suppose inductively we have shown the claim for all $r' < r$. We have two cases:

- Let $\ell_i \nmid r$, then we have by definition and induction

$$K_{\alpha_i}^{(r)} = K_{\alpha_i}^{(r-1)} \frac{K_{\alpha_i} q_{\alpha_i}^{1-r} - K_{\alpha_i}^{-1} q_{\alpha_i}^{r-1}}{q_{\alpha_i}^r - q_{\alpha_i}^{-r}} \in \Bbbk[\Lambda]\Bbbk[K_{\alpha_i}^{(\ell_i)}]$$

  since the denominator $q_{\alpha_i}^r - q_{\alpha_i}^{-r} \neq 0$.
- Let $k\ell_i = r$, then we have in $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$

$$[k]_{q_{\alpha_i}^{\ell_i}} K_{\alpha_i}^{(\ell_i-1)} \cdot K_{\alpha_i}^{(r)}$$

$$= \frac{q_{\alpha_i}^{k\ell_i} - q_{\alpha_i}^{-k\ell_i}}{q_{\alpha_i}^{\ell_i} - q_{\alpha_i}^{-\ell_i}} K_{\alpha_i}^{(\ell_i-1)} \cdot K_{\alpha_i}^{(r-1)} \frac{K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1}}{q_{\alpha_i}^{k\ell_i} - q_{\alpha_i}^{-k\ell_i}}$$

$$= K_{\alpha_i}^{(r-1)} \frac{K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{-1+\ell_i-(k-1)\ell_i} + K_{\alpha_i}^{-1} q_{\alpha_i}^{-1+\ell_i-(k-1)\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1}}{q_{\alpha_i}^{\ell_i} - q_{\alpha_i}^{-\ell_i}}$$

$$= K_{\alpha_i}^{(\ell_i-1)} K_{\alpha_i}^{(r-1)} \left(q^{-(k-1)\ell_i} \frac{K_{\alpha_i} q_{\alpha_i}^{1-\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{-1+\ell_i}}{q_{\alpha_i}^{\ell_i} - q_{\alpha_i}^{-\ell_i}} + q^{\ell_i-1} \frac{K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{(k-1)\ell_i}}{q_{\alpha_i}^{\ell_i} - q_{\alpha_i}^{-\ell_i}} \right)$$

$$= K_{\alpha_i}^{(r-1)} \left(q^{-(k-1)\ell_i} K_{\alpha_i}^{(\ell_i-1)} \frac{K_{\alpha_i} q_{\alpha_i}^{1-\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{-1+\ell_i}}{q_{\alpha_i}^{\ell_i} - q_{\alpha_i}^{-\ell_i}} + q^{\ell_i-1} K_{\alpha_i}^{(\ell_i-1)} \frac{K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{(k-1)\ell_i}}{q_{\alpha_i}^{\ell_i} - q_{\alpha_i}^{-\ell_i}} \right)$$

$$= K_{\alpha_i}^{(r-1)} \left(q^{-(k-1)\ell_i} K_{\alpha_i}^{(\ell_i)} + q^{\ell_i-1} K_{\alpha_i}^{(\ell_i-1)} K_{\alpha_i}^{-1} [k-1]_{q_{\alpha_i}^{\ell_i}} \right)$$

In the specialization $U_q^{\mathcal{L}}$ we have $[k]_{q_{\alpha_i}^{\ell_i}} = \pm k$ invertible, moreover $K_{\alpha_i}^{(\ell_i-1)} \in \Bbbk[\Lambda]$ invertible, hence $K_{\alpha_i}^{(r)} \in \Bbbk[\Lambda]\Bbbk[K_{\alpha_i}^{(\ell_i)}]$ as claimed.

$\square$

4.2. **The coradical.** The following assertion is known under various restrictions and follows from a standard argument, see e.g. [Mont93] Lemma 5.5.5. We include it for completeness in the case of arbitrary ℓ. It would be interesting to determine the full coradical filtration.

Lemma 4.2. *The coradical of the infinite-dimensional Hopf algebra $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ is $\mathbb{C}[\Lambda]$. Especially the Hopf algebra is pointed with group of grouplikes Λ.*

Proof. Consider the (very coarse) coalgebra $\mathbb{N}$-grading induced by setting $deg(E_\alpha^{(r)}) = deg(F_\alpha^{(r)}) = r$ and $deg(x) = 0$ for $x \in U_q^{\mathcal{L},0}(\mathfrak{g}, \Lambda)$. By [Sw69] Prop. 11.1.1 this already

implies that the coradical is contained in $U_q^{\mathcal{L},0}(\mathfrak{g}, \Lambda)$. We have shown in Theorem 4.1 that

$$U_q^{\mathcal{L},0}(\mathfrak{g}, \Lambda) \cong \mathbb{C}[\Lambda] \otimes U(\mathfrak{h})$$

Hence the coradical is indeed the group algebra $\mathbb{C}[\Lambda]$. This especially shows that there are no other grouplikes than Λ. $\qquad\square$

4.3. **The positive part.**

A curious aspect of this article is, that via Lusztig's PBW-basis of root vectors, we have complete control over the vector space $U_q^{\mathcal{L},+}$, also in degenerate cases. The involved question we addressed is the algebra and Hopf algebra structure. We start in this section by some preliminary observations in this direction.

From the PBW-basis in $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$ we have already determined in Corollary 3.10 that multiplication induces an isomorphism of vector spaces

$$\mathbb{C}[\Lambda/2\Lambda_R]\bigotimes_{i\in I}\left(\bigoplus_{r\geq 0}K_{\alpha_i}^{(r)}\mathbb{C}\right)\bigotimes_{\alpha\in\Phi^+}\left(\bigoplus_{r\geq 0}E_\alpha^{(r)}\mathbb{C}\right)\bigotimes_{-\alpha\in\Phi^-}\left(\bigoplus_{r\geq 0}F_\alpha^{(r)}\mathbb{C}\right)\xrightarrow{\cong}U_q^{\mathcal{L}}(\mathfrak{g},\Lambda)$$

The aim of the next theorem is to use the knowledge of the zero-part in the previous section and a straight-forward-calculation to incorporate at least part of the algebra relations that hold specifically in the specialization, without any restrictions on ℓ:

Lemma 4.3. *Let q be a primitive ℓ-th root of unity and again $\mathfrak{h} = \bigoplus_{i\in I} K_{\alpha_i}^{-\ell_i}K_{\alpha_i}^{(\ell_i)}\mathbb{C}$. Then multiplication in $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ induces an isomorphism of vector spaces, which restricts on each tensor factor to an injection of algebras:*

$$\mathbb{C}[\Lambda/2\Lambda_R^{(\ell)}]\otimes U(\mathfrak{h})\bigotimes_{\alpha\in\Phi^+}\left(\mathbb{C}[E_\alpha]/(E_\alpha^{\ell_\alpha})\otimes\mathbb{C}[E_\alpha^{(\ell_\alpha)}]\right)\bigotimes_{-\alpha\in\Phi^-}\left(\mathbb{C}[F_\alpha]/(F_\alpha^{\ell_\alpha})\otimes\mathbb{C}[F_\alpha^{(\ell_\alpha)}]\right)\xrightarrow{\cong}U_q^{\mathcal{L}}(\mathfrak{g},\Lambda)$$

Proof. By Corollary 3.10, multiplication in $U_q^{\mathcal{L}}$ induces an isomorphism of vector spaces

$$\mathbb{C}[\Lambda/2\Lambda_R]\bigotimes_{i\in I}\left(\bigoplus_{r\geq 0}K_{\alpha_i}^{(r)}\mathbb{C}\right)\bigotimes_{\alpha\in\Phi^+}\left(\bigoplus_{r\geq 0}E_\alpha^{(r)}\mathbb{C}\right)\bigotimes_{-\alpha\in\Phi^-}\left(\bigoplus_{r\geq 0}F_\alpha^{(r)}\mathbb{C}\right)\xrightarrow{\cong}U_q^{\mathcal{L}}(\mathfrak{g},\Lambda)$$

We clarified in Theorem 4.1 the zero-part $U_q^{\mathcal{L},0}$, so we get an isomorphism

$$\mathbb{C}[\Lambda/2\Lambda^{(\ell)}]\otimes U(\mathfrak{h})\bigotimes_{\alpha\in\Phi^+}\left(\bigoplus_{r\geq 0}E_\alpha^{(r)}\mathbb{C}\right)\bigotimes_{-\alpha\in\Phi^-}\left(\bigoplus_{r\geq 0}F_\alpha^{(r)}\mathbb{C}\right)\xrightarrow{\cong}U_q^{\mathcal{L}}(\mathfrak{g},\Lambda)$$

with the abelian Lie algebra $\mathfrak{h} = \bigoplus_{i\in I} K_{\alpha_i}^{-\ell_i}K_{\alpha_i}^{(\ell_i)}\mathbb{C}$.

We next turn our attention to the algebra generated for a fixed root $\alpha \in \Phi^+$ by all elements $E_\alpha^{(r)} = E^r/[r]_{q_\alpha}$ in the specialization to a primitive ℓ-th root of unity (respectively

for F). Since $[r]_{q_\alpha} = 0$ iff $\ell_\alpha := ord(q_\alpha^2)|r$, it is clearly isomorphic to

$$\bigoplus_{r \geq 0} E_\alpha^{(r)} \mathbb{C} \cong \begin{cases} \mathbb{C}[E_\alpha]/(E_\alpha^{\ell_i}) \otimes \mathbb{C}[E_\alpha^{(\ell_\alpha)}] & \ell_\alpha > 1 \\ \mathbb{C}[E_\alpha] & \ell_\alpha = 1 \end{cases}$$

This yields the asserted isomorphism. $\qquad\square$

For later use we observe:

Lemma 4.4. *Assume for some $\alpha, \beta \in \Phi^+$ holds $E_\alpha E_\beta = q^{(\alpha,\beta)} E_\beta E_\alpha$ already in $U_q^{\mathbb{Q}(q)}$,*
then $E_\alpha^{(k)} E_\beta^{(k')} = q^{(\alpha,\beta)kk'} E_\beta^{(k')} E_\alpha^{(k)}$. Assume $\mathfrak{g}$ of rank 2 and $\alpha = \alpha_i$ simple and $\alpha+\beta \notin \Phi^+$
then the assumption holds except for the following cases:

$$B_2 : (\alpha_{112}, \alpha_2)$$
$$G_2 : (\alpha_{11122}, \alpha_2), (\alpha_{112}, \alpha_2), (\alpha_1, \alpha_{11122})$$

Note that $(\alpha_1, \alpha_{11122})$ is the reflection of (α_{112}, α_2) on α_{12}.

Proof. The first assertion is trivially checked in $U_q^{\mathbb{Q}(q)}$, where divided powers can be written as powers. The second assertion follows by inspecting [Lusz90b] Sec. 5.2. Note that the other exceptional cases with α not simple would follow easily by Weyl reflection. Precisely these exceptions will generate the dual root system in Lemma 6.10. $\qquad\square$

5. The small quantum group for arbitrary q

Lusztig has in [Lusz90b] Thm 8.10. discovered a remarkable homomorphism from $U_q^{\mathcal{L}}(\mathfrak{g})$ to the ordinary universal enveloping Hopf algebra $U(\mathfrak{g})$

$$U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda) \xrightarrow{Frob} U(\mathfrak{g})$$

whenever ℓ is odd and for $\mathfrak{g} = G_2$ not divisible by 3. He described the kernel in terms of an even more remarkable finite-dimensional Hopf algebra $u_q(\mathfrak{g})$. This Hopf algebra has under the name *Frobenius-Lusztig kernel* triggered the development of the theory and several far-reaching classification results on finite-dimensional pointed Hopf algebras and Nichols algebras in the past $\sim$20 years.

Lusztig's definition and PBW-basis for $u_q^{\mathcal{L}}(\mathfrak{g})$ is without restrictions on the order of q. It does however not describe the structure of $u_q^{\mathcal{L}}(\mathfrak{g})$ as an algebra. After reviewing Lusztig's definition and theorem, we will describe the algebra in terms of Nichols algebras and clarify its structure. Especially $u_q^{\mathcal{L}}(\mathfrak{g})$ does for small order of q **not** coincide with the usual description in terms of generators and relations, which we will denote $u_q(\mathfrak{g})$ – in the *exotic case* $\mathfrak{g} = G_2, q = \pm i$ it will be even of larger rank, namely A_3.

Definition 5.1 ([Lusz90b] Sec. 8.2). *Let $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda) \subset U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ be the subalgebra generated by Λ and all E_α, F_α with $\alpha \in \Phi^+$ such that $\ell_\alpha > 1$.*

Note this implicit definition does not give the algebra structure, but the vector space is well understood using Lusztig reflection operators:

Theorem 5.2 ([Lusz90b] Thm. 8.3). *$u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ is a Hopf subalgebra and multiplication in $u_q^{\mathcal{L}}$ defines an isomorphism of vector spaces:*

$$\mathbb{C}[\Lambda/2\Lambda_R^{(\ell)}] \bigotimes_{\alpha \in \Phi^+, \ell_\alpha > 1} \mathbb{C}[E_\alpha]/(E_\alpha^{\ell_\alpha}) \bigotimes_{-\alpha \in \Phi^-, \ell_\alpha > 1} \mathbb{C}[F_\alpha]/(F_\alpha^{(\ell_\alpha)}) \xrightarrow{\cong} u_q^{\mathcal{L}}$$

Especially $u_q^{\mathcal{L}}$ is of finite dimension $|\Lambda| \cdot \prod_{\alpha \in \Phi^+} \ell_\alpha^2$

Proof. In [Lusz90b] Thm 8.3 iii) Lusztig proved for $\Lambda = \Lambda_R$ and without restrictions on ℓ that $u_q^{\mathcal{L}}$ has a PBW-basis consisting of Λ_R and all $E_\alpha^{(r)}, F_\alpha^{(r)}$ with $r < \ell_\alpha$. Note that this set is empty for $\ell_\alpha = 1$. We've proven as part of Lemma 4.3 the simple fact that $\mathbb{C}[E_\alpha]$ consists precisely of all $E_\alpha^{(r)}$ with $r < \ell_\alpha$. This shows the claim for $\Lambda = \Lambda_R$. The case of arbitrary Λ via $\mathbb{C}[\Lambda] \otimes_{\mathbb{C}[\Lambda_R]}$ follows again from the presentation in Theorem 3.7. $\square$

Definition 5.3. *Assume that $\operatorname{ord}(q^2) > d_\alpha$ for all $\alpha \in \Phi$. Let $u_q(\mathfrak{g}, \Lambda)$ to be the following Hopf algebra defined by generators and relations: As an algebra, $u_q(\mathfrak{g}, \Lambda)$ is generated by the group ring $\Bbbk[\Lambda]$ spanned by $K_\lambda, \lambda \in \Lambda$ and additional generators $E_{\alpha_i}, F_{\alpha_i}$ for each simple root $\alpha_i, i \in I$ with relations:*

$$K_\lambda E_{\alpha_i} K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_{\alpha_i}, \ \forall \lambda \in \Lambda \qquad \text{(group action)}$$

$$K_\lambda F_{\alpha_i} K_\beta^{-1} = \bar{q}^{(\lambda, \alpha_i)} F_{\alpha_i}, \ \forall \lambda \in \Lambda \qquad \text{(group action)}$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{i,j} \cdot \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_{\alpha_i} - q_{\alpha_i}^{-1}} \qquad \text{(linking)}$$

and two sets of Serre-relations for any $i \neq j \in I$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q^{d_i}} E_{\alpha_i}^{1-a_{ij}-r} E_{\alpha_j} E_{\alpha_i}^r = 0$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{\bar{q}^{d_i}} F_{\alpha_i}^{1-a_{ij}-r} F_{\alpha_j} F_{\alpha_i}^r = 0$$

where $\bar{q} := q^{-1}$ and by definition $q^{(\alpha_i, \alpha_j)} = (q^{d_i})^{a_{ij}}$. In contrast to $U^{\mathcal{L}}$ we additionally impose for each root vector $E_\alpha, \alpha \in \Phi^+$:

$$E_\alpha^{\ell_\alpha} = F_\alpha^{\ell_\alpha} = 0 \qquad \text{(root relation)}$$

As a coalgebra, let again be K_λ grouplikes and the simple root vectors $E_{\alpha_i}, F_{\alpha_i}$ skew-primitives: $\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i}, \ \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i}.$

The following theorem shows that for sufficiently large order of q the two definitions $u_q^{\mathcal{L}}, u_q$ coincide (which seems to be well-known, but we prove it nevertheless). More importantly it also shows that this is not true otherwise and gives an explicit description in terms of a different u_q and thereby determine its root system:

Theorem 5.4. *For $\mathrm{ord}(q^2) > d_\alpha$ for all $\alpha \in \Phi^+$ we have $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda) \cong u_q(\mathfrak{g}, \Lambda)$. If some $\mathrm{ord}(q^2) \leq d_\alpha$ we can express $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ in terms of some ordinary $u_q(\mathfrak{g}^{(0)}, \Lambda)^+$ as follows:*

q	$\mathfrak{g}$	$u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$	dim	*primitive generators*	*Comment*
± 1	*all*	$\mathbb{C}$	1	*none*	*trivial*
$\pm i$	B_n	$u_q(A_1^{\times n})^+$	2^n	$E_{\alpha_n}, E_{\alpha_n + \alpha_{n-1}}, E_{\alpha_n + \alpha_{n-1} + \alpha_{n-2}}, \dots$	*short roots*
$\pm i$	C_n	$u_q(D_n)^+$	$2^{n(n-1)}$	$E_{\alpha_1}, \dots E_{\alpha_{n-1}}, E_{\alpha_n + \alpha_{n-1}}$	*short roots*
$\pm i$	F_4	$u_q(D_4)^+$	2^{12}	$E_{\alpha_4}, E_{\alpha_3}, E_{\alpha_3 + \alpha_2}, E_{\alpha_3 + \alpha_2 + \alpha_1}$	*short roots*
$\sqrt[3]{1}, \sqrt[6]{1}$	G_2	$u_q(A_2)^+$	3^3	$E_{\alpha_1}, E_{\alpha_1 + \alpha_2}$	*short roots*
$\pm i$	G_2	$u_{\bar{q}}(A_3)^+$	2^6	$E_{\alpha_2}, E_{\alpha_1}, E_{2\alpha_1 + \alpha_2}$	*exotic*

The **exotic case** is special in several instances: It is the only non-trivial case with $\mathrm{ord}(q^2) \lneq d_\alpha$. Since all $\ell_\alpha = 2$ all root vectors are contained in $u_q^{\mathcal{L},+}$ and the rank even increases because an additional "premature" relation $\mathrm{ad}_{E_{\alpha_1}}^2(E_{\alpha_2}) = 0$ requires a new algebra generator $E_{2\alpha_1 + \alpha_2}$, yielding an A_3-root system (which has also 6 positive roots). Moreover, the braiding matrix of $u_q^{\mathcal{L},+}$ is not the braiding matrix of $u_q(A_3)^+$, rather of the complex conjugate root of unity $u_{\bar{q}}(A_3)^+$, which is the other choice of a primitive fourth root of unity. This case will exhibit strange phenomena throughout this article.

The author has compiled in the recent preprint [Len14b] a similar list for affine Lie algebras, where more exotic cases appear, see also Problem 7.2.

Proof of Theorem 5.4. The algebra $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ is a Hopf algebra in the category of Λ-Yetter-Drinfel'd modules. The strategy of this proof is the following

a) Prove that for a certain subset $X \subset \Phi^+$ the root vectors E_α, $\alpha \in X$ are in $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ and consist of primitive elements. These elements X are "guessed" for each root system at this point of the proof and appear in the fifth column of the table.

b) Determine the braiding matrix of the vector space V spanned by E_α, $\alpha \in X$ and determine the Nichols algebra $\mathcal{B}(V)$ using [Heck09] and especially $\dim(\mathcal{B}(V))$.

c) Now by the universal property of the Nichols algebra we have a surjection from the Hopf subalgebra $H \subset u_q^{\mathcal{L},+}$ generated by the E_α, $\alpha \in X$ to the Nichols algebra $\mathcal{B}(V)$. Note that $u_q^{\mathcal{L},+}(\mathfrak{g})$ will be a graded algebra for trivial reasons except for the exotic case, whrere we show this explicitly. Using Lusztig's PBW-basis of $u_q^{\mathcal{L},+}$ in the previous theorem we show in each case $\dim(u_q^{\mathcal{L},+}) = \dim(\mathcal{B}(V))$ and hence $\mathcal{B}(V) = H = u_q^{\mathcal{L},+}$.

We now proceed to the proof according to the steps outlined above. The trivial case is if all $\ell_\alpha = 1$ i.e. $q^2 = 1$, the generic case is if all $\mathrm{ord}(q^2) > d_\alpha$. For simply-laced $\mathfrak{g}$ all $d_\alpha = 1$, so these two cases exhaust all possibilities. For the non-simply-laced $\mathfrak{g} = B_n, C_n, F_4$ we have $d_\alpha = 2$ for long roots, hence the condition is also violated for $\mathrm{ord}(q^2) = 2$; for the non-simply-laced $\mathfrak{g} = G_2$ we have $d_\alpha = 3$ for long roots, hence the condition is also violated for $\mathrm{ord}(q^2) = 2, 3$. Thus, we have to check precisely the cases in the table of the theorem.

a) We show that the E_α, $\alpha \in X$ given in the last column of the theorem are primitive. We first show the well-known general fact that if x, y are primitive elements with braidings $(x \otimes y) \mapsto q_{12}(y \otimes x) \mapsto q_{12}q_{21}(x \otimes y)$, then $q_{12}q_{21} = 1$ implies the braided commutators $[x, y] := xy - q_{12}yx$ resp. $[y, x] := yx - q_{21}xy$ are primitive as well (possibly $= 0$):

$$\Delta([x, y]) = \Delta(xy) - q_{12}\Delta(yx)$$
$$= (1 \otimes xy + q_{12}y \otimes x + x \otimes y + xy \otimes 1)$$
$$- q_{12}(1 \otimes yx + q_{21}x \otimes y + y \otimes x + yx \otimes 1)$$
$$= 1 \otimes [x, y] + (1 - q_{12}q_{21})(x \otimes y) + [x, y] \otimes 1$$

Now we check in each case that the E_α, $\alpha \in X$ given in the last column of the theorem fulfill $\ell_\alpha > 1$, hence $E_\alpha \in u_q^{\mathcal{L},+}$, and can be in $U_q^{\mathbb{Q}(q)}$ obtained (inductively) as braided commutators with $q_{12}q_{21} = 1$, hence they are primitive (the exotic case works different).

   i) In the trivial case $\ell_\alpha = 1$ there are no root vectors with $\ell_\alpha > 1$, hence we chose $X := \{\}$. In the generic case where all $\ell_\alpha > d_\alpha$ then all $\ell_\alpha > 1$ and all root vectors are in $u_q^{\mathcal{L},+}$, we hence choose $X := \{\alpha_1, \dots \alpha_n\}$ (simple root vectors are by definition primitive).

   ii) For $\mathfrak{g} = B_n, q = \pm i$ we have for short roots $\ell_\alpha = \mathrm{ord}(q^2) = 2$; the elements $X := \{\alpha_n, \alpha_n + \alpha_{n-1}, \dots\}$ are (in fact all) short roots. Moreover we can in in $U_q^{\mathbb{Q}(q)}$ inductively obtain $E_{\alpha_n + \alpha_{n-1} + \dots + \alpha_k}$ by braided commutators of the primitive $E_{\alpha_n} \in u_q^{\mathcal{L},+}$ with the primitive $E_{\alpha_{k \neq n}} \notin u_q^{\mathcal{L},+}$. We verify the condition $q_{12}q_{21} = 1$ in each step:

$$q^{(\alpha_n + \alpha_{n-1} + \dots + \alpha_k, \alpha_{k-1})}q^{(\alpha_{k-1}, \alpha_n + \alpha_{n-1} + \dots + \alpha_k)}$$
$$= q^{2(\alpha_k, \alpha_{k-1})} = q^{-4} = 1$$

Note that we have to convince ourselves that the commutator is not accidentally zero in the specialization: Reflection easily reduces this case to B_2, where we check the commutator explicitly to be $q^2 E_{12} \neq 0$ by [Lusz90b] Sec. 5.2.

iii) For $\mathfrak{g} = C_n, q = \pm i$ we have for short roots $\ell_\alpha = \mathrm{ord}(q^2) = 2$; the elements $X := \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1} + \alpha_n\}$ are short roots. Moreover all E_{α_k} are primitive and $E_{\alpha_{n-1}+\alpha_n}$ is primitive by applying the case B_2 to the subsystem generated by α_{n-1}, α_n.

iv) For $\mathfrak{g} = F_4, q = \pm i$ we have for short roots $\ell_\alpha = \mathrm{ord}(q^2) = 2$; the elements $X := \{\alpha_4, \alpha_3, \alpha_3+\alpha_2, \alpha_3+\alpha_2+\alpha_1\}$ are short roots. Moreover E_{α_4} is primitive and $E_{\alpha_3}, E_{\alpha_3+\alpha_2}, E_{\alpha_3+\alpha_2+\alpha_1}$ are primitive by applying the case B_3 to the subsystem generated by $\alpha_3, \alpha_2, \alpha_1$.

v) For $\mathfrak{g} = G_2, q = \sqrt[3]{1}, \sqrt[6]{1}$ we have for short roots $\ell_\alpha = \mathrm{ord}(q^2) = 3$; the elements $X := \{\alpha_1, \alpha_1 + \alpha_2\}$ are short roots. Moreover $E_{\alpha_1+\alpha_2}$ is primitive since we get it in $U_q^{\mathbb{Q}(q)}$ as a braided commutator of $E_{\alpha_1}, E_{\alpha_2}$ and

$$q^{(\alpha_1,\alpha_2)}q^{(\alpha_2,\alpha_1)} = q^{-6} = 1$$

We also check the commutator explicitly to be $q^3 E_{12} \neq 0$ by [Lusz90b] Sec. 5.2.

vi) For $\mathfrak{g} = G_2, q = \pm i$ we have for all roots $\ell_\alpha = \mathrm{ord}(q^2) = \mathrm{ord}(q^6) = 2$. We choose $X := \{\alpha_2, \alpha_1, 2\alpha_1 + \alpha_2\}$. Moreover the elements $E_{\alpha_2}, E_{\alpha_1}$ are primitive. Checking primitivity of E_{112} is more involved. We could in principle express E_{112} by definition via reflections in terms of $E_1, E_2, F_1, E_1^{(2)}, E_2^{(3)}$ but using the relation [Lusz90b] Sec. 5.4 (a6) for $k = 2$ is more convenient as follows (note that in this exotic case there will be no way of expressing E_{112} by E_1, E_2 as will become clear later in the proof. E.g. Lusztig's relations (a3) returns zero for $q = \pm i$):

$$E_{112} = -q^2(E_2 E_1^{(2)} - q^{-6} E_1^{(2)} E_2) - q E_{12} E_1$$

With this formula we can calculate directly:

$$\Delta(E_{12}) = \Delta(T_2(E_1)) = \Delta(-E_2 E_1 + q^{-3} E_1 E_2) = -\Delta([E_2, E_1])$$
$$= 1 \otimes E_{12} - (1 - q^{-6})E_2 \otimes E_1 + E_{12} \otimes 1$$

$$\begin{aligned}
\Delta(E_{112}) &= -q^2 \Big((1 \otimes E_2 + E_2 \otimes 1)(1 \otimes E_1^{(2)} + q E_1 \otimes E_1 + E_1^{(2)} \otimes 1) \\
&\quad -q^{-6}(1 \otimes E_1^{(2)} + q E_1 \otimes E_1 + E_1^{(2)} \otimes 1)(1 \otimes E_2 + E_2 \otimes 1)\Big) \\
&\quad - q \Big((1 \otimes E_{12} - (1 - q^{-6})E_2 \otimes E_1 + E_{12} \otimes 1)(1 \otimes E_1 + E_1 \otimes 1)\Big) \\
&= -q^2 \Big(1 \otimes (E_2 E_1^{(2)} - q^{-6} E_1^{(2)} E_2) + q^{-2} E_1 \otimes (E_2 E_1 - q^{-3} E_1 E_2) \\
&\quad +(1 - q^{-12})E_2 \otimes E_1^{(2)} + q(E_2 E_1 - q^{-9} E_1 E_2) \otimes E_1 + (E_2 E_1^{(2)} - q^{-6} E_1^{(2)} E_2) \otimes 1\Big) \\
&\quad - q \Big(1 \otimes E_{12} E_1 + q^{-1} E_1 \otimes E_{12} - (1 - q^{-6})E_2 \otimes E_1^2 \\
&\quad -q^2(1 - q^{-6})E_2 E_1 \otimes E_1 + E_{12} \otimes E_1 + E_{12} E_1 \otimes E_1\Big) \\
&= 1 \otimes E_{112} + E_{112} \otimes 1
\end{aligned}$$

b) We next determine the braiding matrix and hence from [Heck09] the root system and dimension of the Nichols algebra $\mathcal{B}(V)$ of the vector space V generated by all E_α, $\alpha \in X$ defined above. This yields the information in the third and fourth row of the table. We again proceed case-by-case:

  i) For the trivial case we defined $X := \{\}$ so the braiding matrix on $V = 0$ is trivial, hence the Nichols algebra is 1-dimensional. For the generic case we defined $X := \{\alpha_1, \ldots \alpha_n\}$ so the braiding matrix is $q_{ij} = q^{(\alpha_i, \alpha_j)} = q^{d_i a_{ij}}$. All ℓ_α coincides and the Nichols algebra $\mathcal{B}(V)$ is the Nichols algebra associated to the Lie algebra $\mathfrak{g}$ and hence of dimension $\prod_{\alpha \in \Phi^+} \ell_\alpha^{|\Phi^+|}$ (note that we inspected every row in [Heck09] and provided $\mathrm{ord}(q^2) > d_\alpha$ the Nichols algebras in question have indeed the claimed root system and hence the claimed dimension).

  ii) For $\mathfrak{g} = B_n, q = \pm i$ we defined $X = \{\alpha_n, \alpha_n + \alpha_{n-1}, \ldots\}$. An easy calculation shows
$$q^{(\alpha_n + \alpha_{n-1} + \cdots \alpha_k, \, \alpha_n + \alpha_{n-1} + \cdots \alpha_l)} = \begin{cases} q^2, & k = l \\ 1, & k \neq l \end{cases}$$
Hence the Nichols algebra has a root system of type $A_1^{\times n}$ and dimension 2^n.

  iii) For $\mathfrak{g} = C_n, q = \pm i$ we defined $X := \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1} + \alpha_n\}$. The braiding matrix of the first $n-1$ root vectors is that of $A_n \subset C_n$ and for the braiding with the last root vector $E_{\alpha_{n-1} + \alpha_n}$ we easily obtain
$$q^{(\alpha_{n-2}, \alpha_n + \alpha_{n-1})} = q^{-1} \qquad q^{(\alpha_{n-1}, \alpha_{n-1} + \alpha_n)} = q^0 \qquad q^{(\alpha_{n-1} + \alpha_n, \alpha_{n-1} + \alpha_n)} = q^2$$
Hence the Nichols algebra has a root system of type D_n (with α_{n-2} the center node) and dimension $2^{|\Phi^+|} = 2^{n(n-1)}$.

  iv) For $\mathfrak{g} = F_4, q = \pm i$ we defined $X = \{\alpha_4, \alpha_3, \alpha_3 + \alpha_2, \alpha_3 + \alpha_2 + \alpha_1\}$. We explicitly calculate the braiding matrix of these root vectors:
$$\begin{pmatrix} q^2 & q^{-1} & q^{-1} & q^{-1} \\ q^{-1} & q^2 & 1 & 1 \\ q^{-1} & 1 & q^2 & 1 \\ q^{-1} & 1 & 1 & q^2 \end{pmatrix}$$
Hence the Nichols algebra has a root system of type D_4 (with α_4 the center node) and dimension 2^{12}. It extends the root system $A_1^{\times 3}$ for B_3 (generated by $\alpha_3, \alpha_3 + \alpha_2, \alpha_3 + \alpha_2 + \alpha_1$) as well as the root system A_3 for C_3 (generated by $\alpha_4, \alpha_3, \alpha_3 + \alpha_2$).

  v) For $\mathfrak{g} = G_2, q = \sqrt[3]{1}, \sqrt[6]{1}$ we defined $X = \{\alpha_1, \alpha_1 + \alpha_2\}$. We explicitly calculate the braiding matrix of these root vectors:
$$\begin{pmatrix} q^2 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix}$$

Hence the Nichols algebra has a root system A_2 and dimension 3^3.

vi) For the exotic case $\mathfrak{g} = G_2, q = \pm i$ we defined $X = \{\alpha_2, \alpha_1, 2\alpha_1 + \alpha_2\}$. We explicitly calculate the braiding matrix of these elements:

$$\begin{pmatrix} q^6 & q^{-3} & 1 \\ q^{-3} & q^2 & q \\ 1 & q & q^2 \end{pmatrix} \overset{q^4=1}{=} \begin{pmatrix} q^2 & -q^{-1} & 1 \\ -q^{-1} & q^2 & -q^{-1} \\ 1 & -q^{-1} & q^2 \end{pmatrix} = \begin{pmatrix} \bar{q}^2 & \bar{q}^{-1} & 1 \\ \bar{q}^{-1} & \bar{q}^2 & \bar{q}^{-1} \\ 1 & \bar{q}^{-1} & \bar{q}^2 \end{pmatrix}$$

Hence the Nichols algebra has a root system A_3 and dimension 2^3. However, the braiding matrix is not the standard braiding matrix from $u_q(A_3)$, rather for then small quantum group $u_{\bar{q}}(A_3)$ associated to the respective other choice of a primitive fourth root of unity, which is complex conjugate.

c) Let H denote the subalgebra of $u_q^{\mathcal{L}}$ generated by the chosen primitive root vectors $E_\alpha, \alpha \in X$, which span a vector space V. In all cases except the exotic case the $\mathbb{N}^n$-grading of these generators is linearly independent, hence H is a $\mathbb{N}$-graded algebra. By the universal property of the Nichols algebra we thus have a surjection $H \to \mathcal{B}(V)$. To finally show equality $u_q^{\mathcal{L}} \cong \mathcal{B}(V)$ we use Lusztig's PBW-basis of $u_q^{\mathcal{L},+}$ in Theorem 5.2 to show in each case $\dim(u_q^{\mathcal{L},+}) = \dim(\mathcal{B}(V))$:

i) In the trivial case the set of all roots with $\ell_\alpha > 1$ is empty, hence the dimension of $u_q^{\mathcal{L},+}$ is 1. In the generic case all roots have coinciding $\ell_\alpha > 1$ hence the dimension of $u_q^{\mathcal{L},+}$ is $\ell_\alpha^{|\Phi^+|}$.

ii) For all other cases except the exotic case $G_2, q = \pm i$ the set of all roots with $\ell_\alpha > 1$ is precisely the set of short roots and all short roots fulfill $\ell_\alpha = 2$ (resp. $= 3$ for $G_2, q = \sqrt[3]{1}, \sqrt[6]{2}$). Hence the dimension of $u_q^{\mathcal{L},+}$ is 2^N (resp. 3^N) with N the number of positive short roots, i.e. $n, n(n-1), 12, 3$ for B_n, C_n, F_4, G_2.

iii) For the exotic case $G_2, q = \pm i$ we have for long and short roots $\ell_\alpha = 2$. Since G_2 has 6 positive roots, the dimension of $u_q^{\mathcal{L},+}$ is 2^6. It is quite remarkable that this coincides with the dimension 2^6 of the Nichols algebra of type $A_3, q = \pm i$. Note that a-priori it is not clear $u_q^{\mathcal{L},+}$ is a graded Hopf algebra. This only follows after inspecting the relations between E_2, E_1 and E_1, E_{112} and E_{1112}, E_2 and E_{12}, E_{112} in [Lusz90b] Sec. 5.2, which are all graded with respect to E_2, E_1, E_{112} having degree 1 (except the $[2]E_{112}$-term for E_1, E_{12}, which is zero for $q = \pm i$).

We notice that in each case the dimension of $\mathcal{B}(V)$ calculated in b) coincides with the dimension of $u_q^{\mathcal{L},+}$ by Lusztig's PBW-basis obtained in c). Hence the two are isomorphic which concludes the proof.

$\square$

6. A SHORT EXACT SEQUENCE

Lusztig has in [Lusz90b] Thm 8.10. discovered a remarkable homomorphism from $U_q^{\mathcal{L}}(\mathfrak{g})$ to the ordinary universal enveloping Hopf algebra $U(\mathfrak{g})$

$$U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda) \xrightarrow{Frob} U(\mathfrak{g})$$

whenever ℓ is odd and for $\mathfrak{g} = G_2$ not divisible by 3. He called it *Frobenius homomorphism* to emphasize it should be viewed as a "lift" of the Frobenius homomorphism in characteristic ℓ to the quantum group in characteristic 0.

The following is a more systematic construction, using the techniques of Nichols algebras and generalizes to the situation of small prime divisors (which has been excluded by Lusztig and throughout the following literature, note however Lusztig's book [Lusz94] Thm. 35.1.9 for large order but small prime divisors). First, we prove that $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ is a normal Hopf subalgebra in $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$, this relies crucially on the explicit description of the assumed kernel $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ by Theorem 5.4. Then we form the quotient Hopf algebra, the quotient is then the quantum Frobenius homomorphism. Finally we inspect the quotient and prove it is (close to) a universal enveloping of some Lie algebra $\mathfrak{g}^{(\ell)}$.

Theorem 6.1. *Depending on $\mathfrak{g}$ and ℓ we have the following exact sequences of Hopf algebras in the category of Λ-Yetter-Drinfel'd modules:*

$$u_q(\mathfrak{g}^{(0)}, \Lambda)^+ \xrightarrow{\subseteq} U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+ \xrightarrow{Frob} U(\mathfrak{g}^{(\ell)})^+$$

	$\mathfrak{g}$	$\ell = ord(q)$	$\mathfrak{g}^{(0)}$	$\mathfrak{g}^{(\ell)}$	*is braided for*
Trivial cases:	*all*	$\ell = 1$	0	$\mathfrak{g}$	*no*
	all	$\ell = 2$	0	$\mathfrak{g}$	$ADE_{n \geq 2}, C_{n \geq 3}, F_4, G_2$
Generic cases:	ADE	$\ell \neq 1, 2$	$\mathfrak{g}$	$\mathfrak{g}$	$\ell = 2 \bmod 4, n \geq 2$
	B_n	$4 \nmid \ell \neq 1, 2$	B_n	B_n	*no*
	C_n	$4 \nmid \ell \neq 1, 2$	C_n	C_n	$\ell = 2 \bmod 4, n \geq 3$
	F_4	$4 \nmid \ell \neq 1, 2$	F_4	F_4	$\ell = 2 \bmod 4$
	G_2	$3 \nmid \ell \neq 1, 2, 4$	G_2	G_2	$\ell = 2 \bmod 4$
Duality cases:	B_n	$4 | \ell \neq 4$	B_n	C_n	$\ell = 4 \bmod 8, n \geq 3$
		$\ell = 4$	$A_1^{\times n}$	C_n	$n \geq 3$
	C_n	$4 | \ell \neq 4$	C_n	B_n	*no*
		$\ell = 4$	D_n	B_n	*no*
	F_4	$4 | \ell \neq 4$	F_4	F_4	$\ell = 4 \bmod 8$
		$\ell = 4$	D_4	F_4	*yes*
	G_2	$3 | \ell \neq 3, 6$	G_2	G_2	$\ell = 2 \bmod 4$
		$\ell = 3, 6$	A_2	G_2	$\ell = 6$
Exotic case:	G_2	$\ell = 4$	A_3	G_2	*no*

The author would be very interested to obtain a similar list for affine Lie algebras as well other Nichols algebra extensions (see Problems 7.2 and 7.3).

Proof. The rest of this article is devoted to prove this theorem as follows:

- The structure of $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ in the table column $\mathfrak{g}^{(0)}$ was already determined in Theorem 5.4.
- By Theorem 6.3 for the root systems in question all pairs of roots can be reflected simultaniously into some parabolic subgroup of rank 2. This will be excessively used in the following two steps to reduce all calculations to rank 2.
- In Lemma 6.6 we prove $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$ is a normal Hopf subalgebra of $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ in the category of Λ-Yetter-Drinfel'd modules. By simultaneous reflection we will only have to check rank 2, then we use the convenient generator set for $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$. Note that for the exotic case $G_2, q = \pm 1$ is would not suffice to check the action on simple root vectors (as one might do), since there is an additional algebra generator E_{112}.
- Then we will then consider the quotient of $U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ by the normal Hopf subalgebra $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)$. We show in Lemma 6.10 that the quotient is generated by primitive elements $E_{\alpha_i}^{(\ell_{\alpha_i})}$ with the (possibly braided) commutator structure as prescribed in the table column $\mathfrak{g}^{(\ell)}$. We will do so again by trick a) to reduce to rank 2 and check the isomorphisms explicitly. Note that the braiding corresponds to even lattices in the Lie algebra Lemma 2.5 and that (independently) the dual root system is formed from the exceptions in Lemma 4.4.

Note that the cases with $\mathrm{ord}(q^2) > d_\alpha$ have been verified in [Lusz94]. $\square$

Remark 6.2. *Recently Angiono has in [An14] characterized (dually) the Borel part of the Kac-Procesi-DeConcini-Form $U_q^{\mathcal{K}}(\mathfrak{g})^+$ purely in terms of so-called distinguished Pre-Nichols algebra in the braided category of Λ-Yetter-Drinfel'd Modules. These algebras are much more general and all come with a version of a Frobenius homomorphism.*

6.1. **Orbits of pairs in root systems.** We will start with a technical theorem (which may be known) that should be in general helpful for quantum groups of high rank $\mathfrak{g}$ by reducing issues to rank 2. It has been already observed by Lusztig in [Lusz90b] Sec. 3.6 for the simply-laced case as part of an explicit description of all roots by diagrams Γ_i.

Theorem 6.3. [1]

a) *For any pair of roots $\alpha \neq \pm\beta$ there is a Weyl group element mapping α, β simultaneously into a rank 2 parabolic subsystem $\langle \alpha_i, \alpha_j \rangle$.*

---

[1]I am very thankful to the referee for pointing out the much shorter and conceptual proof given here for a), which also works for larger n-tuples of roots.

More generally, for any set A of roots of order $|A| < \mathrm{rank}(\mathfrak{g})$ there is a Weyl group element mapping A simultaneously into a parabolic subsystem of rank $|A|$.

b) *The type of the rank 2 subsystem $\langle \alpha_i, \alpha_j \rangle$ (including lengths) is uniquely determined by α, β in a) and all such subsystems are in a single Weyl group orbit, except three parabolic $A_1 \times A_1$ in D_4 interchanged by the triality diagram automorphism as well as two orbits for $D_n, n \geq 5$, namely the parabolic $A_1 \times A_1$ consisting of the two tiny legs α_n, α_{n-1} and all remaining parabolic $A_1 \times A_1$.*

c) *Unordered pairs of roots in a Lie algebra of rank 2 are classified by length and angle up to action of the Weyl group.*

Altogether, Weyl orbits of unordered pairs of roots are in bijective correspondence to types of rank 2 parabolic subsystems, lengths and angle with the mentioned exceptions.

Before we proceed to the proof we give examples how this theorem works and fails:

Example 6.4. *For $B_n, n \geq 4$ the possible types of parabolic subsystems of rank 2 are $B_2, A_2^{long}, A_1^{long} \times A_1^{long}, A_1^{short} \times A_1^{long}$. In the subsystem B_2 there are four orbits (classified by their lengths and angle), in A_2 are two orbits (classified by their lengths and angle) and in the others each one orbit. Hence the Theorem returns 8 orbits of pairs, each with an explicit representative inside a rank 2 parabolic subgroup.*

Example 6.5. *We show that the theorem fails for the exception D_4: The Weyl group acts transitively, so the one-point stabilizer (fixing α) has order $|W|/|\Phi| = 8$. The number of β in an orbit of pairs (α, β) has to divide this order, the quotient being the order of the stabilizer of the (ordered) pair. For $\mathfrak{g} = D_4$ the positive roots orthogonal to a given α (say the highest root ω) are three simple roots not in the center $\alpha_1, \alpha_2, \alpha_3$. The three pairs $\{\omega, \alpha_i\}$ are interchanged by the triality diagram automorphism, but since $3 \nmid 8$ they have to belong to different Weyl group orbits. After reflection, these three orbits can be recognized as the three parabolic subsystems $\{\alpha_i, \alpha_j\}$ of type $A_1 \times A_1$, hence statement a) holds, but not b).*

Proof of Theorem 6.3. For $\mathfrak{g}$ of rank 2 the statements a) and b) are trivial, the statement c) follows by explicit inspection. Note that in A_2 there are two orbits of *ordered* pairs interchanged by the diagram automorphism.

a) The following proof shows in fact that any set of roots $A \subset \Phi^+$ of order $k < n =:$ rank$(\mathfrak{g})$ can be simultaneously reflected into a parabolic subsystem of rank k. By induction it suffices to show that we can reflect A into a parabolic subsystem of rank $< n$, then we my proceed until $k = n$. Let v be a vector in $\Phi \otimes \mathbb{R}$, which is of dimension n, such that $v \perp A$. Consider the set

$$M_v := \{\gamma \in \Phi^+ \mid (v, \gamma) < 0\}$$

If $M_v \neq \varnothing$ then there exists at least one simple root $\alpha_i \in M_v$ (otherwise, being positive linear combinations, no positive root could be in M_v). If we apply a reflection s_i, then since $(s_i\alpha, s_i\beta) = (\alpha, \beta)$:

$$M_{s_iv} = \{\gamma \in \Phi^+ \mid (s_iv, \gamma) < 0\} = \Phi^+ \cap s_i(M_v) = s_i\left(M_v \backslash \{\alpha_i\}\right)$$

So in terms of cardinality $|M_{s_iv}| = |M_v| - 1$. Hence by successive reflection one can find $v \perp A$ with $M_v = \varnothing$, especially all $(v, \alpha_i) \geq 0$. But this implies that for any $v \perp \alpha = \sum_i n_i\alpha_i$ we have $v \perp \alpha_i$ for all $n_i \neq 0$. Hence $v^\perp \cap \Phi$ is a parabolic subsystem of rank $< n$ and by construction it contains A.

b) It is clear that the type of the rank 2 parabolic subsystem (including lengths) is uniquely determined by projecting to the vector subspace (note the statement is more trivially true in most cases by angle and length, but e.g. orthogonal long roots in B_2 are distinguished from orthogonal roots in $A_1^{long} \times A_1^{long}$, since in the former there exists a $2\gamma = \alpha + \beta$).

We wish to show that all rank 2 parabolic subsystems are in a single Weyl orbit with the exception that the three resp. two parabolic $A_1 \times A_1$ in D_4 resp. D_n interchanged by diagram automorphism are in different orbits. This is done by inspecting every case of $\mathfrak{g}$ explicitly (and mostly reduce to previous cases):

i) Let $\mathfrak{g} = A_n, n \geq 3$, then we have rank 2 parabolic subsystems of type A_2 and $A_1 \times A_1$. It is easy to shift a subsystem $\{\alpha_i, \alpha_{i+1}\}$ to $\{\alpha_{i-1}, \alpha_i\}$ by reflecting on $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$. Similarly, one can easily give direct expressions that shift any $\alpha_i, \alpha_j, |i - k| > 1$ to any other such pairs by shifting each one separately (more abstractly spoken, by using the transitivity of the Weyl group of a suitable subsystem).

ii) Let $\mathfrak{g} = B_n, n \geq 3$ (resp C_n by duality) then we have rank 2 parabolic subsystems of type B_2 and A_2^{long} and for $n \geq 4$ of type $A_1^{long} \times A_1^{long}$. But the parabolic subsystem B_2 is unique and all the other subsystems lay in the parabolic subsystem of type A_{n-1}^{long}, for which the claim has been shown in a).

iii) Let $\mathfrak{g} = D_n, \ n \geq 4$. Any two parabolic subsystems of type A_2 lay in a common subsystem of type A_{n-1}, hence by i) we are finished. For the parabolic case of type $A_1 \times A_1$ we have already clarified the situation for D_4 in Example 6.5, so let's assume $n \geq 5$: By transitivity of the Weyl group, fix $\alpha = \alpha_1$ the outmost vertex, then the roots orthogonal to α_1 are of the form $k \cdot \alpha_1 + 2k \cdot \alpha_2 + \cdots$. The roots in the class $k = 0$ are by definition the parabolic subsystem D_{n-2} and explicit inspection of the root system shows the only element in the class $k = 1$ is the highest root and there are no roots for $k > 1$. All roots in the first class can by the mapped to each other by the Weyl group of D_{n-2} without affecting α_1.

We only have to show that this is not possibly for the highest root as well and we argue as follows: The Weyl group of D_n has order $2^{n-1}n!$ and the root system has $2n(n-1)$ roots. Hence the one-point-stabilizer has double order as the contained Weyl group of D_{n-2} and additionally contains the reflection on the highest root ω of D_n, which accounts for the entire stabilizer. But none of these elements can map ω to something else than $\pm\omega$, which shows it forms a single orbit. It is easy to see that the pair α_1, ω has to be in the same orbit as the simple roots α_n, α_{n-1} at the tiny legs of D_n.

iv) Let $\mathfrak{g} = E_n$, $n = 6, 7, 8$. Any two parabolic subsystems are in a parabolic subsystem of type A_k hence by i) we are finished. For a parabolic subsystem α_i, α_j of type $A_1 \times A_1$ we have three cases: Either both α_i, α_j are not the tiny leg α_2, or say α_i is the tiny leg and α_j is on either of the other legs. All subsystems in each class are in a parabolic subsystem of type A_{n-1}, A_{n-2}, A_4, hence each case for an orbit by i). It remains to map a representative of each case to another case (which was not possible for D_n): We move α_j (not the tiny leg) to an outmost vertex without affecting α_i using a parabolic subsystem of type A_k. Then we may change α_i to the other case by using the A_3 subsystem around the center node.

v) Let $\mathfrak{g} = F_4$, then we have unique rank 2 parabolic subsystems of type A_2^{long}, A_2^{short}, $B_2 = C_2$ and three parabolic subsystems of type $A_1^{long} \times A_1^{short}$, namely $\{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_4\}, \{\alpha_2, \alpha_4\}$. These three parabolic subsystems can be reflected to each other exactly as in the A_n case a), explicitly:

$$\{\alpha_1, \alpha_3\} \overset{\alpha_4}{\mapsto} \{\alpha_1, \alpha_3 + \alpha_4\} \overset{\alpha_3}{\mapsto} \{\alpha_1, \alpha_4\}$$
$$\{\alpha_1, \alpha_4\} \overset{\alpha_2}{\mapsto} \{\alpha_1 + \alpha_2, \alpha_4\} \overset{\alpha_1}{\mapsto} \{\alpha_2, \alpha_4\}$$

c) We finally convince ourselves for A_2, B_2, G_2 that lengths and angle of $\{\alpha, \beta\}$ classify the pairs: Fix one root by transitivity, then there is a unique choice $\pm\beta$, for B_2, G_2 we can reflect on a root orthogonal to α, for A_2 we can reflect $\{\alpha, -\beta\}$ to $\{\beta, \alpha\}$. Note that the *ordered* pairs α_1, α_2 and α_2, α_1 are not in the same orbit.

$\square$

6.2. Adjoint action on the small quantum group.

Lemma 6.6. *The Hopf subalgebra $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+ \subset U_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ from Definition 5.1 is a normal Hopf subalgebra, i.e. stable under the adjoint action. We explicitly give the respective (skew-)derivations in the degenerate cases of Theorem 5.4.*

Proof. For $\mathrm{ord}(q^2) > d_\alpha$ Lusztig obtained $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ in [Lusz94] Thm. 35.1.9 as kernel of his explicit Frobenius homomorphism, so it remains to check the degenerate cases in Theorem 5.4. Note that our proof works in other cases as well.

To calculate $\mathrm{ad}_{E_\alpha^{(\ell_\alpha)}}(E_\beta)$ we may invoke Theorem 6.3 to find a Weyl group element that maps α, β to a rank 2 parabolic subsystem. It hence suffices to check the cases of rank 2. Using another reflection we may assume α to be a simple root. Moreover it suffices to check normality on a set of generators for $u_q^{\mathcal{L}}(\mathfrak{g}, \Lambda)^+$ as explicitly given in Theorem 5.4. From

$$\Delta(E_{\alpha_i}^{(k)}) = \sum_{b=0}^{k} q^{d_{\alpha_i} b(k-b)} E_{\alpha_i}^{(N-b)} \otimes E_{\alpha_i}^{(b)}$$

we see that the adjoint action of any $E_{\alpha_i}^{(\ell_i)}$ is the sum of the braided commutator $\delta_i := [E_{\alpha_i}^{(\ell_i)}, \_]$ and lower terms $E_{\alpha_i}^{(k)}, k < \ell_i$ that are by definition completely in $u_q^{\mathcal{L},+}$. These δ_i are (sometimes symmetrically braided) derivations, and have been introduced by Lusztig, who states in [Lusz90b] Lm. 8.5 that they preserve $u_q^{\mathcal{L}}$ (for $2 \nmid \ell$ and $3 \nmid \ell$ for $\mathfrak{g} = G_2$). We will restrict ourselves to computing δ_i in each degenerate case, since they capture the nontrivial (non-inner) part of the adjoint action on $u_q^{\mathcal{L},+}$ and are invariant under reflection (which is an algebra map). Thus we can prove normality:

We calculate $\delta_i = [E_{\alpha_i}^{(\ell_i)}, \_]$ on each algebra generator of $u_q^{\mathcal{L},+}$ as given in Theorem 5.4. Whenever $\alpha_i + \beta \notin \Phi^+$ we have $\delta_i(E_\beta) = 0$ except possibly the four cases in Lemma 4.4

$$B_2 : (\alpha_{112}, \alpha_2) \qquad G_2 : (\alpha_{11122}, \alpha_2), (\alpha_{112}, \alpha_2), (\alpha_1, \alpha_{11122})$$

where the last two are in a Weyl orbit. For the remaining generators we proceed case-by-case using the commutation formulae in [Lusz90b] Sec. 5:

a) For $\mathfrak{g} = A_1 \times A_1$ there is no pair $\alpha + \beta \in \Phi$. For $q = -1$ we have $[E_\alpha, E_\beta] = -[E_\alpha, E_\beta]$ hence both δ_1, δ_2 vanish.

b) For $\mathfrak{g} = A_2, q = \pm i$ (parabolic in $C_n, F_4, q = \pm i$) we have $\ell_1 = \ell_2 = 2$ and we check the two pairs with $\alpha_i + \beta \in \Phi$:

$$\begin{aligned}
\delta_2(E_1) &= E_2^{(\ell_2)} E_2 - q^{-\ell_2} E_2 E_1^{(\ell_2)} \\
&= E_2^{(\ell_2)} E_2 - q^{-\ell_2} \sum_{\substack{r+s=\ell_1 \\ s+t=1}} q^{tr+s} E_2^{(r)} E_{12}^{(s)} E_1^{(t)} \\
&= E_2^{(\ell_2)} E_2 - q^{-\ell_2+\ell_1} E_2^{(\ell_2)} E_2 - q^{-\ell_2+1} E_2^{(\ell_2-1)} E_{12} \overset{\ell_2=2}{=} q E_2 E_{12} \\
\delta_1(E_2) &= E_1^{(\ell_1)} E_2 - q^{-\ell_1} E_2 E_1^{(\ell_1)} \\
&= \sum_{\substack{r+s=1 \\ s+t=\ell_1}} q^{tr+s} E_2^{(r)} E_{12}^{(s)} E_1^{(t)} - q^{-\ell_1} E_2 E_1^{(\ell_1)} \\
&= (q^{\ell_1} - q^{-\ell_1}) E_1^{(\ell_1)} + q E_{12} E_1^{(\ell_1-1)} \overset{\ell_1=2}{=} q E_{12} E_1
\end{aligned}$$

c) For $\mathfrak{g} = B_2, q = \pm i$ the subalgebra $u_q^{\mathcal{L},+}$ (in fact of type $A_1 \times A_1$) is generated by the short root vectors E_1, E_{12} and $\ell_1 = 2, \ell_2 = 1$ so we have to check the following two pairs with $\alpha_i + \beta \in \Phi$:

$$\delta_2(E_1) = E_2 E_1 - q^{-2} E_1 E_2$$
$$= E_2 E_1 - q^{-2} \left(q^2 E_2 E_1 + q^2 E_{12} \right) = -E_{12}$$
$$\delta_1(E_{12}) = E_1^{(2)} E_{12} - E_{12} E_1^{(2)}$$
$$= \sum_{\substack{r,s,t \geq 0 \\ r+s=1 \\ s+t=2}} q^{-sr-st+s} \left(\prod_{i=1}^{s} \left(q^{2i} + 1 \right) \right) E_{12}^{(r)} E_{112}^{(s)} E_1^{(t)} - E_{12} E_1^{(2)}$$
$$= (q^2 + 1) E_{112} E_1 \overset{q=\pm i}{=} 0$$

Note that E_{112} would not have been in $u_q^{\mathcal{L},+}$.

d) For $\mathfrak{g} = G_2, q = \sqrt[3]{1}, \sqrt[6]{1}$ the subalgebra $u_q^{\mathcal{L},+}$ (in fact of type A_2) is generated by the short root vectors $E_{\alpha_1}, E_{\alpha_{12}}$ and we have $\ell_1 = \ell_{12} = \ell_{112} = 3, \ell_2 = \ell_{1112} = \ell_{11122} = 1$. We denote $\epsilon := q^3 = \pm 1$. We have to check two pairs $\delta_2(E_1)$ and $\delta_2(E_{12})$. Reflection on α_2 maps $\alpha_2 \leftrightarrow \alpha_{12}$ hence we may alternatively check $\delta_{12}(E_2)$:

$$\delta_2(E_1) = E_2 E_1 - q^{-3} E_1 E_2 \overset{(a2)}{=} E_2 E_1 - q^{-3}(q^3 E_2 E_1 + q^3 E_{12}) = -E_{12}$$
$$\delta_{12}(E_2) = E_{12}^{(3)} E_2 - q^{-9} E_2 E_{12}^{(3)}$$
$$\overset{(a3)}{=} E_{12}^{(3)} E_2 - q^{-9}(q^3 E_{12}^{(3)} E_1 + q(q + q^{-1}) E_{12}^{(2)} E_{112} + q^{-1}(q^2 + 1 + q^{-2}) E_{12} E_{11122})$$
$$\overset{ord(q^2)=3}{=} -\epsilon q E_{12}^{(2)} E_{112}$$

This is a a product of short root vectors $E_\gamma^{(k)}, k < 3$, so again in $u_q^{\mathcal{L},+}$. Note that E_{11122} would not have been in $u_q^{\mathcal{L},+}$.

e) For $\mathfrak{g} = G_2, q = \pm i$ we have all $\ell_\alpha = 2$ so all $E_\alpha^{(k)}, k < 2$ are in $u_q^{\mathcal{L},+}$. However the subalgebra $u_q^{\mathcal{L},+}$ (in fact of type A_3) is generated by the short root vectors E_1, E_2, E_{112}; this is why having Theorem 5.4 is crucial for the proof of this theorem. Thus we have to check the four pairs $\delta_1(E_2), \delta_2(E_1), \delta_1(E_{112})$ as well as the exception in Lemma 4.4 $\delta_2(E_{112})$. The first two pairs are easy:

$$\delta_1(E_2) = E_1^{(2)} E_2 - q^{-6} E_2 E_1^{(2)}$$
$$\overset{(a6)}{=} q^6 E_2 E_1^{(2)} + q^5 E_{12} E_1 + q^4 E_{112} - q^{-6} E_2 E_1^{(2)} \overset{q=\pm i}{=} q E_{12} E_1 + E_{112}$$
$$\delta_2(E_1) = E_2^{(2)} E_1 - q^{-6} E_1 E_2^{(2)}$$

$$\overset{(a2)}{=} E_2^{(2)}E_1 - q^{-6}\left(q^6 E_2^{(2)}E_1 + q^3 E_2 E_{12}\right) = -q^{-3} E_2 E_{12}$$

For the other two pair we have to proceed as follows: The Weyl group elements $T_2 T_1 T_2$ and $T_2 T_1$ both map $\alpha_{112} \mapsto \alpha_1$, the first one maps $\alpha_1 \mapsto \alpha_{112}$ and the second one maps $\alpha_2 \mapsto \alpha_{11122}$. We may hence alternatively calculate

$$\delta_{112}(E_1) := E_{112}^{(2)}E_1 - q^2 E_1 E_{112}^{(2)}$$

$$\overset{(a4)}{=} E_{112}^{(2)}E_1 - q^2\left(q^{-2}E_{112}^{(2)}E_1 + q^{-3}(q^2 + 1 + q^{-2})E_{112}E_{1112}\right) \overset{q=\pm i}{=} q^{-1}E_{112}E_{1112}$$

$$\delta_{11122}(E_1) := E_{11122}^{(2)}E_1 - E_1 E_{11122}^{(2)}$$

$$\overset{(a7)}{=} E_{11122}^{(2)}E_1 - E_{11122}^{(2)}E_1 - q^{-4}(1 - q^4)E_{11122}E_{112}^{(2)} \overset{q=\pm i}{=} 0$$

Note that $E_{112}^{(2)}$ would not have been in $u_q^{\mathcal{L},+}$.

$\square$

6.3. **Structure of the quotient.** We have proven in Lemma 6.6 that the Hopf sub-algebra $u_q^{\mathcal{L},+} \subset U_q^{\mathcal{L},+}$ described in Theorem 5.4 is normal. We may hence consider the left ideal and two-sided coideal $U_q^{\mathcal{L},+} \ker_\epsilon(u_q^{\mathcal{L},+})$ which is by normality a two-sided Hopf ideal. We now form the Hopf algebra quotient:

Definition 6.7. *Define the following Hopf algebra in the category of Λ-Yetter-Drinfel'd modules:*

$$H := U_q^{\mathcal{L},+}/U_q^{\mathcal{L},+}\ker_\epsilon(u_q^{\mathcal{L},+})$$

The goal of this section is to analyze H and prove it is isomorphic to the a Hopf algebra $U(\mathfrak{g}^{(\ell)})^+$ with $\mathfrak{g}^{(\ell)}$ the Lie algebra given for each case in the statement of Theorem 6.1.

Lemma 6.8. *H has as vector space a PBW-basis consisting of monomials in $E_\alpha^{\ell_\alpha}$ for each positive root α.*

Proof. This follows from the PBW-basis given in Lemma 4.3 and the fact that by construction $u_q^{\mathcal{L},+}$ has a PBW-basis consisting of monomials in $E_\alpha^k, k < \ell_\alpha$. Note that by using Lusztig's PBW-basis of root vectors, we have complete control over the vector space $U_q^{\mathcal{L},+}$, also in degenerate cases. The involved question addressed in this article is the Hopf algebra structure. $\square$

We next address the question when H is actually an ordinary Hopf algebra. This is precisely the use of the lattice calculations in Lemma 2.5 – even lattices will correspond to properly (but symmetrically) braided Hopf algebras and are marked as such for each case in the statement of Theorem 6.1.

Lemma 6.9. *In the following cases is H an ordinary complex Hopf algebra:*

$$
\begin{array}{ll}
A_n, D_n, E_6, E_7, E_8, G_2, n \geq 2 & \ell \neq 2 \bmod 4 \\
B_n, n \geq 3 & \ell \neq 4 \bmod 8 \\
C_n, n \geq 3 & \ell \neq 2 \bmod 4 \\
F_4 & \ell \neq 2, 4, 6 \bmod 8
\end{array}
$$

In the other cases H is a Hopf algebra in a symmetrically braided category explicitly described in the proof.

Proof. The braiding between the generators of the PBW-basis is

$$
c(E_\alpha^{(\ell_\alpha)} \otimes E_\beta^{(\ell_\beta)}) = q^{(\ell_\alpha \alpha, \ell_\beta \beta)} \cdot E_\beta^{(\ell_\beta)} \otimes E_\alpha^{(\ell_\alpha)}
$$

We have proven in Lemma 2.5 that except in the excluded cases we have $(\ell_\alpha \alpha, \ell_\beta \beta) \in \ell\mathbb{Z}$, hence the braiding is trivial. Note that in the excluded cases we have $(\ell_\alpha \alpha, \ell_\beta \beta) \in \frac{\ell}{2}\mathbb{Z}$, hence $c^2 = 1$ and the braiding is still symmetric. Moreover we've generally shown that $(\ell_\alpha \alpha, \ell_\alpha \alpha) \in \ell\mathbb{Z}$, hence the self-braiding is trivial and U^+ is a domain (i.e. no truncations). $\qquad\qquad\square$

The most tedious part is now to verify that H is indeed the asserted universal enveloping algebra. Note that the theorem of Kostant-Cartier could easily be applied, but there are two downsides: For one there seems to be no apparent reason, why the simple root vectors generate the entire algebra (compare the case $u_q^{\mathcal{L}}$), especially since we do not have F's (see Problem 7.1). Moreover, we do not know a-priori whether some $H = U(\mathfrak{g}^{(\ell),+})$ would actually be the positive part of some semisimple Lie algebra $\mathfrak{g}^{(\ell)}$, especially in the braided cases), hence calculating the Cartan matrix would not suffice.

Rather, we shall in the following explicitly check all braided commutators and verify they actually lead to the positive part of the Lie algebra $\mathfrak{g}^{(\ell)}$) given for each case in the statement of the Main Theorem 6.1. This becomes again feasible through the trick staged in Theorem 6.3, namely reflecting the relevant cases to a rank 2 parabolic subsystem. There we calculate by hand, which is quite tedious for G_2:

Lemma 6.10. *There is an algebra isomorphism $H \cong U(\mathfrak{g}^{(\ell,+)})$ for some $\mathfrak{g}^{(\ell,+)}$ as follows:*

- *Generic case: For $\ell_\alpha = \ell_\beta$ for long and short roots we have $\mathfrak{g}^{(\ell,+)} \cong \mathfrak{g}^+$. Note that the isomorphism typically picks up scalar factors for non-simple root vectors.*
- *Duality case: For $\ell_\alpha \neq \ell_\beta$ for long and short roots we have $\mathfrak{g}^{(\ell,+)} \cong (\mathfrak{g}^\vee)^+$ for the dual root system. More precisely we wish to prove:*
  - *For $4|\ell$ we have $B_n^{(\ell,+)} \cong C_n^+$ and $C_n^{(\ell,+)} \cong B_n^+$ and a nontrivial automorphism $F_4^{(\ell,+)} \cong F_4^+$. All three maps double short roots and hence interchange short and long roots.*
  - *For $3|\ell$ we have a nontrivial automorphism $G_2^{(\ell,+)} \cong G_2^+$. The map triples short roots and hence interchanges short and long roots.*

Especially this shows that H is generated as an algebra by simple root vectors $E_\alpha^{(\ell_\alpha)}$. Since these are primitive up to lower terms (which vanish in H), this also proves we have a Hopf algebra isomorphism.

Proof. The case $\mathrm{ord}(q^2) > d_\alpha$ has been treated in [Lusz94] Thm. 35.1.9. Hence we again restrict ourselves to the degenerate cases, but note that our proof works in general. The degenerate cases with small orders of q were already given in Theorem 5.4 and are of type B_n, C_n, F_4, G_2. To calculate (symmetrically) braided commutators $[E_\alpha^{(\ell_\alpha)}, E_\alpha^{(\ell_\alpha)}]$ (and especially verify they are nonzero at the respective cases) we may hence invoke Theorem 6.3, which states that α, β can be mapped simultaneously to a rank 2 parabolic subsystem of $\mathfrak{g}$ and we may demand that one root is even a simple root.

We hence have to verify for the potential rank 2 parabolic subsystems $A_1 \times A_1, A_2, B_2, G_2$ that the commutators $[E_{\alpha_i}^{(\ell_\alpha)}, E_\beta^{(\ell_\beta)}]$ for roots $\alpha_i, \beta \in \Phi(\mathfrak{g})^+$ are zero and nonzero in $\mathfrak{g}^{(\ell)}$ in agreement with the assumed isomorphism $H \cong U(\mathfrak{g}^\ell)^+$. We again excessively use Lusztig's commutation formulae [Lusz90b] Sec. 5 and since we calculate in the quotient H, all terms $E_\alpha^{(k)}, k < \ell_\alpha$ all zero.

a) First recall from Lemma 4.4 that whenever $\alpha_i + \beta \notin \Phi(\mathfrak{g})$ then $E_{\alpha_i}^{(k)}, E_\beta^{(k')}$ braided commute with the following exceptions

$$B_2 : \ (\alpha_{112}, \alpha_2) \qquad G_2 : \ (\alpha_{11122}, \alpha_2), (\alpha_{112}, \alpha_2), (\alpha_1, \alpha_{11122})$$

(where the last two are in a Weyl orbit). These exceptions will be extremely crucial for the formation of the dual root system.

b) $A_1 \times A_1$ is trivial by a).

c) For A_2 (especially $\ell = 2$) we wish to verify $H \cong U(\mathfrak{g}^{(\ell)})^+$ for $\mathfrak{g}^{(\ell)} = \mathfrak{g}$. In view of a) we only have to check that the nontrivial commutator is indeed nonzero:

$$E_1^{(k)} E_2^{(k')} = \sum_{\substack{r+s=k' \\ s+t=k}} q^{tr+s} E_2^{(r)} E_{12}^{(s)} E_1^{(t)}$$

Since A_2 is simply-laced, all $\ell_\alpha = \mathrm{ord}(q^2)$ coincide. If we apply the commutation formula to $k = k' = \ell_\alpha$ all terms on the right hand side vanish in the quotient except $r = t = \ell_\alpha, s = 0$ and $r = t = 0, s = \ell_\alpha$, hence in the quotient:

$$E_1^{(\ell_\alpha)} E_2^{(\ell_\alpha)} = q^{\ell_\alpha^2} E_2^{(\ell_\alpha)} E_1^{(\ell_\alpha)} + q^{\ell_\alpha} E_{12}^{(\ell_\alpha)}$$

$$\Rightarrow \quad \left[E_1^{(\ell_\alpha)} E_2^{(\ell_\alpha)} \right] = q^{\ell_\alpha} E_{12}^{(\ell_\alpha)} \quad \text{braided for } \ell = 2 \bmod 4$$

As proven already in Lemma 6.9, the commutator is *braided* whenever $\ell = 2 \bmod 4$, since then $q^{\ell_\alpha^2} = q^{\ell \frac{\ell}{4}} = -1$. Note that the commutator calculation above is rather general and would in fact is able to treat all cases with equal d_α (simply-laced or not) – this is roughly how Lusztig argues in [Lusz90b] Lm. 8.5. for $2 \nmid \ell$.

d) For $\mathfrak{g} = B_2, 4|\ell$ (especially $\ell = 4$) we wish to verify $H \cong U(\mathfrak{g}^{(\ell)})^+$ for $\mathfrak{g}^{(\ell)} = B_2^\vee = C_2 \cong B_2$ with the isomorphism doubling short roots and hence switching short and long roots. We have for short roots $\ell_1 = \ell_{12} = \frac{\ell}{2}$ and for long roots $\ell_2 = \ell_{112} = \frac{\ell}{4}$. We have to check the commutators in the quotient H for α_1, α_2 and α_1, α_{12} with $\alpha_i + \beta \in \Phi(\mathfrak{g})$ as well as the exception α_2, α_{112} in a):

$$E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{4})} = \sum_{\substack{r,s,t,u \geq 0 \\ r+s+t=\frac{\ell}{4} \\ s+2t+u=\frac{\ell}{2}}} q^{2ru+2rt+us+2s+2t} E_2^{(r)} E_{12}^{(s)} E_{112}^{(t)} E_1^{(u)}$$

$$= \underbrace{q^{2\frac{\ell}{4}\frac{\ell}{2}} E_2^{(\frac{\ell}{4})} E_1^{(\frac{\ell}{2})}}_{r=\frac{\ell}{4},s=0,t=0,u=\frac{\ell}{2}} + \underbrace{q^{2\frac{\ell}{4}} E_{112}^{(\frac{\ell}{4})}}_{r=0,s=0,t=\frac{\ell}{4},u=0}$$

$$\Rightarrow \quad \left[E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{4})} \right] = -E_{112}^{(\frac{\ell}{4})}$$

$$E_{112}^{(\frac{\ell}{4})} E_2^{(\frac{\ell}{4})} = \sum_{\substack{r,s,t \geq 0 \\ r+s=\frac{\ell}{4} \\ s+t=\frac{\ell}{4}}} q^{-2sr-2st+2s} \left(\prod_{i=1}^{s} (q^{2-4i} - 1) \right) E_2^{(r)} E_{12}^{(2s)} E_{112}^{(t)}$$

$$= \underbrace{E_2^{(\frac{\ell}{4})} E_{112}^{(\frac{\ell}{4})}}_{r=\frac{\ell}{4},s=0,t=\frac{\ell}{2}} + \underbrace{\left(\prod_{i=1}^{\frac{\ell}{4}} (q^{2-4i} - 1) \right) E_{12}^{(\frac{\ell}{2})}}_{r=0,s=\frac{\ell}{4},t=0}$$

$$\Rightarrow \quad \left[E_{112}^{(\frac{\ell}{4})}, E_2^{(\frac{\ell}{4})} \right] = \left(\prod_{i=1}^{\frac{\ell}{4}} (q^{2-4i} - 1) \right) \cdot E_{12}^{(\frac{\ell}{2})} \quad \neq 0 \qquad \text{since } -2 > 2 - 4i > 2 - \ell$$

$$E_1^{(\frac{\ell}{2})} E_{12}^{(\frac{\ell}{2})} = \sum_{\substack{r,s,t \geq 0 \\ r+s=\frac{\ell}{2} \\ s+t=\frac{\ell}{2}}} q^{-sr-st+s} \left(\prod_{i=1}^{s} q^{2i} + 1 \right) E_{12}^{(r)} E_{112}^{(s)} E_1^{(t)}$$

$$= \underbrace{E_{12}^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})}}_{r=\frac{\ell}{2},s=0,t=\frac{\ell}{2}} + + q^{\frac{\ell}{2}} \underbrace{\left(\prod_{i=1}^{\frac{\ell}{2}} (q^{2i} + 1) \right) E_{112}^{(\frac{\ell}{2})}}_{r=0,s=\frac{\ell}{2},t=0}$$

$$\Rightarrow \quad \left[E_1^{(\frac{\ell}{2})}, E_{12}^{(\frac{\ell}{2})} \right] = 0 \qquad \text{since for } 2|\ell \text{ the product vanishes for } i = \frac{\ell}{4}$$

We convince ourselves that this result agrees with the assumed root system $\mathfrak{g}^{(\ell)} = B_2^\vee$ with $\alpha_1' := \frac{\ell}{2}\alpha_1$ now the long root and $\alpha_2' := \frac{\ell}{4}\alpha_2$ now the short root:

- The commutator for α_1', α_2' is nonzero and proportional to the root vector

$$\alpha_1' + \alpha_2' = \frac{\ell}{2}\alpha_1 + \frac{\ell}{4}\alpha_2 = \frac{\ell}{4}\alpha_{112}$$

- The commutator for $\alpha_1' + \alpha_2', \alpha_2'$ involving the exceptional pair $\alpha_2 + \alpha_{112} \notin \Phi(\mathfrak{g})$ is nonzero and proportional to the root vector (with higher divided power)

$$(\alpha_1' + \alpha_2') + \alpha_1' = 2\frac{\ell}{2}\alpha_1 + \frac{\ell}{4}\alpha_2 = \frac{\ell}{2}\alpha_{12}$$

- The commutator for $\alpha_1', \alpha_1' + 2\alpha_2'$ is exceptionally zero in the case $4|\ell$ (even though $\alpha_1 + \alpha_{112} \in \Phi$), which is in agreement with the assumed root system $B_2^\vee$.
- All other commutators are trivial by a) in agreement with $B_2^\vee$.

We have hence verified $H \cong U(\mathfrak{g}^{(\ell)})^+$ with $\mathfrak{g}^{(\ell)} = B_2^\vee \cong B_2$ for $4|\ell$.

For $\mathfrak{g} = G_2$ we do not have the luxury of [Lusz90b] Sec. 5.3 and instead have to use Sec. 5.4. We restrict ourselves to the relevant cases $\ell = 3, 6$ (duality case, the most tedious) and $\ell = 4$ (exotic case). We again use the convention $\epsilon := q^3 = \pm 1$. To reduce the number of commutator calculations we can by reflection restrict ourselves to one representative per Weyl group orbit of pairs (α, β). For G_2 these are classified by angle and lengths.

e) For $\ell = 3, 6$ we wish to verify $H \cong U(\mathfrak{g}^{(\ell)})^+$ for $\mathfrak{g}^{(\ell)} = G_2^\vee \cong G_2$ with the isomorphism tripling short roots and hence switching short and long roots. We have for short roots $\ell_1 = \ell_{12} = \ell_{112} = 3$ and for long roots $\ell_2 = \ell_{1112} = \ell_{11122} = 1$. Hence in the quotient H all (left- or rightmost) $E_\alpha^{(k)} = 0$ when α short and $0 < k < 3$. In view of a) we have to check all pairs $\alpha_i + \beta \in \Phi^+(\mathfrak{g})$ as well as the three exceptions (of which one is a reflection of the other): We start with all pairs including a long root:

$$\left[E_1^{(3)}, E_2\right] = E_1^{(3)}E_2 - q^{-9}E_2 E_1^{(3)}$$
$$\stackrel{(a6)}{=} q^9 E_2 E_1^{(3)} + q^7 E_{12}E_1^{(2)} + q^5 E_{112}E_1 + q^3 E_{1112} - q^{-9}E_2 E_1^{(3)}$$
$$= (\epsilon^3 - \epsilon^{-3})E_2 E_1^{(3)} + \epsilon E_{1112} = \epsilon E_{1112}$$

$$[E_{1112}, E_2] = E_{1112}E_2 - q^{-3}E_2 E_{1112}$$
$$\stackrel{(a9)}{=} q^3 E_2 E_{1112} + (-q^4 - q^2 + 1)E_{11122} + (q^2 - q^4)E_{12}E_{112} - q^{-3}E_2 E_{1112}$$
$$= (\epsilon - \epsilon^{-1})E_{1112}E_2 + (-q^4 - q^2 + 1)E_{11122} = 2E_{11122}$$

$$\left[E_{112}^{(3)}, E_2\right] = E_{112}^{(3)}E_2 - E_2 E_{112}^{(3)}$$
$$\stackrel{(a8)}{=} E_2 E_{112}^{(3)} + q^{-4}(q^{-3} - q^3)E_{12}E_{11122}E_{112} + q^{-3}(q^{-6} - q^6)E_{11122}^{(2)}$$
$$+ q^{-1}(q^{-2} - q^2)E_{12}^{(2)}E_{112}^{(2)} - E_2 E_{112}^{(3)}$$
$$= \epsilon(\epsilon^{-2} - \epsilon^2)E_{11122}^{(2)} = 0$$

$$[E_{11122}, E_2] = E_{11122}E_2 - q^3 E_2 E_{11122}$$

$$\overset{(a7)}{=} q^{-3}E_2E_{11122} + q^{-3}(q^2-1)(q^4-1)E_{12}^{(3)} - q^3E_2E_{11122}$$

$$= (\epsilon^{-1}-\epsilon)E_2E_{11122} + \epsilon^{-1}(q^6-q^4-q^2+1)E_{12}^{(3)} = 3\epsilon E_{12}^{(3)}$$

The remaining pair of short roots are $\alpha_1 + \alpha_{12} \in \Phi$ is more work, due to $\ell_1 = \ell_{12} = 3$. We shall from now on calculate in $U_q^{\mathbb{Q}(q)}$ (to get rid of the divided power of E_1) and successively apply the commutation rule $(a3)$ for single powers $E_{12}^{(k)}E_1$:

$$E_1^3 E_{12}^{(3)} = E_1^2 \left(q^3 E_{12}^{(3)} E_1 + [2]q E_{12}^{(2)} E_{112} + [3]q^{-1} E_{12} E_{11122} \right)$$

$$= q^3 E_1 \left(q^3 E_{12}^{(3)} E_1 + [2]q E_{12}^{(2)} E_{112} + [3]q^{-1} E_{12} E_{11122} \right) E_1$$

$$+ [2]q E_1 \left(q^2 E_{12}^{(2)} E_1 + [2]q E_{12} E_{112} + [3] E_{11122} \right) E_{112}$$

$$+ [3]q^{-1} E_1^2 E_{12} E_{11122}$$

$$= q^6 \left(q^3 E_{12}^{(3)} E_1 + [2]q E_{12}^{(2)} E_{112} + [3]q^{-1} E_{12} E_{11122} \right) E_1^2$$

$$+ [2]q^4 \left(q^2 E_{12}^{(2)} E_1 + [2]q E_{12} E_{112} + [3] E_{11122} \right) E_{112} E_1$$

$$+ [2]q^3 \left(q^2 E_{12}^{(2)} E_1 + [2]q E_{12} E_{112} + [3] E_{11122} \right) E_1 E_{112}$$

$$+ [2]^2 q^2 \left(q E_{12} E_1 + [2]q E_{112} \right) E_{112}^2$$

$$+ [3]q^{-1} E_1^2 E_{12} E_{11122} + [3][2]q E_1 E_{11122} E_{112}$$

After multiplying out we have four types of summands: The leading terms $q^9 E_{12}^{(3)} E_1^3$ and $q^3 [2]^3 E_{112}^3$, several terms involving $[2][3]$ (say X_1), two terms involving $[3]E_1^2$ (say X_2) and other terms involving E_{12} or $E_{12}^{(2)}$, say Y_1 and Y_2. We shall further simplify the Y_i and use $E_1 E_{112} = q^{-1} E_{112} E_1$:

$$Y_1 = [2]^2 E_{12} \left(q^5 E_{112}^2 E_1 + q^4 E_{112} E_1 E_{112} + q^3 E_1 E_{112}^2 \right)$$

$$= [3][2]^2 q^3 E_{12} E_{112}^2 E_1$$

$$Y_2 = [2] E_{12}^{(2)} \left(q^7 E_{112} E_1^2 + q^6 E_1 E_{112} E_1 + q^5 E_1^2 E_{112} \right)$$

$$= [3][2]q^5 E_{12}^{(2)} E_{112} E_1^2$$

so these terms Y_1, Y_2 can also be brought to a form involving $[3][2]$. If we now multiply the overall expression we derived for $E_1^3 E_{12}^{(3)}$ by $\frac{1}{[3]!}$ and reinstate integral powers, we indeed find that all summands above are in the Lusztig integral form $U_q^{\mathbb{Z}[q,q^{-1}],\mathcal{L}}$ by themselves: We have the leading terms $q^9 E_{12}^{(3)} E_1^{(3)}$ and $q^3 [2]^3 E_{112}^{(3)}$, terms X_1', Y_1', Y_2' where the present $[3][2] = [3]!$ cancels and X_2' (involving E_1^2) where $[3]$ cancels and we get $E_1^{(2)}$ from E_1^2.

We may hence consider the above decomposition in the specialization $\mathcal{U}_q^{\mathcal{L}}$ and then in the quotient H:

$$E_1^{(3)} E_{12}^{(3)} = q^9 E_{12}^{(3)} E_1^{(3)} + q^3(q + q^{-1})^3 E_{112}^{(3)} + X_1' + X_2' + Y_1' + Y_2'$$

But in the quotient all monomials involving powers $E_\alpha^{(k)}, E_\alpha^k, k < 3$ for the short root vectors E_1, E_{12}, E_{112} as either leftmost or rightmost factor vanish. We convince ourselves that such powers appear in every summand of X_0', X_1', Y_0', Y_1', which are hence zero in H. Hence we have finally proven for $q = \sqrt[3]{-1}, \sqrt[6]{-1}$:

$$\left[E_1^{(3)}, E_{12}^{(3)} \right] = q^3 [2]^3 E_{112}^{(3)} = E_{112}^{(3)} \neq 0, \qquad \text{note } \ell_{112} = 3$$

We convince ourselves that our result agrees with the assumed root system $\mathfrak{g}^{(\ell)} = G_2^\vee$ with $\alpha_1' := 3\alpha_1$ now the long root and $\alpha_2' := \alpha_2$ now the short root:

- The commutator for α_1', α_2' is nonzero and proportional to the root vector

$$\alpha_1' + \alpha_2' = 3\alpha_1 + \alpha_2 = \alpha_{1112}$$

- The commutator for $\alpha_1' + \alpha_2', \alpha_2'$ is nonzero and proportional to the root vector

$$(\alpha_1' + \alpha_2') + \alpha_2' = 3\alpha_1 + 2\alpha_2 = \alpha_{11122}$$

- The commutator for $\alpha_1' + 2\alpha_2', \alpha_2'$ involving the exceptional pair $\alpha_{11122} + \alpha_2 \notin \Phi(\mathfrak{g})$ is nonzero and proportional to the root vector (with higher divided power)

$$(\alpha_1' + 2\alpha_2') + \alpha_2' = 3\alpha_1 + 3\alpha_2 = 3\alpha_{12}$$

- The commutator for $\alpha_1, \alpha_1' + 3\alpha_2'$ is nonzero and proportional to the root vector

$$\alpha_1' + (\alpha_1' + 3\alpha_2') = 6\alpha_1 + 3\alpha_2 = 3\alpha_{112}$$

- The commutator for $2\alpha_1' + 3\alpha_2', \alpha_2'$ involving the exceptional pair $\alpha_{112} + \alpha_2 \notin \Phi(\mathfrak{g})$ is zero.
- All other commutators are trivial by a) in agreement with $G_2^\vee$.

We have hence verified $H \cong U(\mathfrak{g}^{(\ell)})^+$ with $\mathfrak{g}^{(\ell)} = G_2^\vee \cong G_2$ for $\ell = 3, 6$.

f) For $\ell = 4$ we we wish to verify $H \cong U(\mathfrak{g}^{(\ell)})^+$ for $\mathfrak{g}^{(\ell)} = G_2$. We have $\ell_\alpha = 2$ for all roots, so all $E_\alpha = 0$ in the quotient H, which implies $E_\alpha^{(k)} = 0$ for all $2 \nmid k$. This present case hence works analogously to b). However we have to exclusively calculate in $U_q^{\mathbb{Q}(q)}$ as in the last case of e). We only spell out one non-trivial case and the

exceptional pair (α_2, α_{112}) in a):

$$E_1^2 E_{112}^{(2)} \overset{(a4)}{=} q^{-2} E_1 E_{112}^{(2)} E_1 + [3]q^{-3} E_1 E_{112} E_{1112}$$

$$\overset{(a4)}{=} q^{-4} E_{112}^{(2)} E_1^2 + [3]q^{-5} E_{112} E_{1112} E_1 + [3]q^{-4} E_{112} E_1 E_{1112} + [3]^2 q^{-4} E_{1112}^2$$

$$= q^{-4} E_{112}^{(2)} E_1^2 + [2][3]q^{-6} E_{112} E_{1112} E_1 + [3]^2 q^{-4} E_{1112}^2$$

$$\Rightarrow \quad \left[E_1^{(2)}, E_{112}^{(2)} \right] = [3]q^{-6} E_{112} E_{1112} E_1 + [3]^2 q^{-4} E_{1112}^{(2)} \overset{H}{=} [3]^2 q^{-4} E_{1112}^{(2)} \overset{q=\pm i}{=} 4 E_{1112}^{(2)} \neq 0$$

$$E_{112}^{(2)} E_2^2 \overset{(a8)}{=} E_2 E_{112}^{(2)} E_2 + q^{-2}(q^{-3} - q^3) E_{12} E_{11122} E_2 - [2](q^{-1} - q) E_{12}^{(2)} E_{112} E_2$$

$$\overset{(a8)}{=} E_2^2 E_{112}^{(2)} + q^{-2}(q^{-3} - q^3) E_2 E_{12} E_{11122} - [2](q^{-1} - q) E_2 E_{12}^{(2)} E_{112}$$

$$+ q^{-5}(q^{-3} - q^3) E_{12} E_2 E_{11122} + [2]q^{-4}(q^{-3} - q^3)(q^2 - 1)^2 E_{12} E_{12}^{(3)}$$

$$- [2](q^{-1} - q) E_{12}^{(2)} E_2 E_{112} - q[2]^2 (q^{-1} - q)^2 E_{12}^{(2)} E_{12}^{(2)}$$

$$= E_2^2 E_{112}^{(2)} + [2]q^{-7}(q^2 - q+)(q^{-3} - q^3) E_2 E_{12} E_{11122} - [2](q^{-1} - q) E_2 E_{12}^{(2)} E_{112}$$

$$+ [2]q^{-4}(q^{-3} - q^3)(q^2 - 1)^2 E_{12} E_{12}^{(3)} - [2](q^{-1} - q) E_{12}^{(2)} E_2 E_{112}$$

$$- q[2]^2 (q^{-1} - q)^2 E_{12}^{(2)} E_{12}^{(2)}$$

$$\Rightarrow \quad \left[E_{112}^{(2)}, E_2^{(2)} \right] \overset{H}{=} 0$$

These calculations become quicker, if summands in the integral form are eliminated already during the calculation.

$$\square$$

7. OPEN QUESTIONS

We finally give some open questions that the author would find interesting:

In view of the boundaries of the present article:

Problem 7.1. *It would be desirable to have a short exact sequence for the full quantum group $U_q^{\mathcal{L}}(\mathfrak{g})$ instead of just the Borel part $U_q^{\mathcal{L}}(\mathfrak{g})^+$. The author has however stumbled precisely in the braided cases (Lemma 6.9) over a relation*

$$[K_i^{(1)}, E_j] = -2E_j \left(K_i^{(1)} + \frac{1}{2} \right)$$

that seems to imply $u_q^{\mathcal{L}}$ is not normal in these cases. Does it encode a nontrivial behaviour related to the braiding or should one rather try to slightly refine the integral form on the grouplikes?

In view of our first Main Theorem 5.4 on the structure of $u_q^{\mathcal{L}}(\mathfrak{g})$ for small q:

Problem 7.2. *One should calculate this table for affine quantum groups for q of small order (compare the explicit Frobenius homomorphism for large order in* [CP97]*). Depending on the case, one might expect a different affine root system of an infinite union of finite root systems. During the publication of this article, the author has indeed calculated the respective Nichols algebras and root systems in* [Len14b]*, but many questions are open, in particular regarding the "'shifting"' of the isotropic roots.*

In view of our second Main Theorem 6.1 on the short exact sequence for $U_q^{\mathcal{L}}(\mathfrak{g})$

Problem 7.3. *The author has already asked in Oberwolfach (*[MFO14] *Question 5) for more examples or even a classification of infinite-dimensional Hopf algebra extensions H of a finite-dimensional pointed Hopf algebra h by a universal enveloping algebra U. There seem to be several sources of interesting examples:*

a) *The canonical examples is $H = U_q^{\mathcal{L}}(\mathfrak{g}), h = u_q(\mathfrak{g}), U = U(\mathfrak{g})$.*
b) *By the results in this paper, for small roots of unity $H = U_q^{\mathcal{L}}(\mathfrak{g}), h = u_q(\mathfrak{g}^{(0)}), U = U(\mathfrak{g}^{(\ell)})$ are examples with $\mathfrak{g}^{(0)} \neq \mathfrak{g} \neq \mathfrak{g}^{(\ell)}$.*
c) *The graded dual of Angiono's Prenichols algebras* [An14] *(which corresponds to a Kac-Procesi-DeConcini form). Here U consists of Cartan-type simple roots, but h is a larger Nichols algebra. They conjecture several intriguing universal properties of H.*
d) *The Hopf algebra in* [Good09] *Construction 1.2., where (implicitly and again dual) h is of type $A_1^{\times n}$ and U is of type A_1. This should work much more general by joining suitable elements in $U(\mathfrak{g})$ for $U_q^{\mathcal{L}}(\mathfrak{g})$.*

e) *The families of large-rank Nichols algebras h over nonabelian groups constructed by the author in* [Len14a] *using a diagram automorphism σ on a Lie algebra $\mathfrak{g}$ should yield examples with $U = U(\mathfrak{g})$ and h with root system $\mathfrak{g}^\sigma$, e.g. $\mathfrak{g} = E_6, \mathfrak{g}^\sigma = F_4$.*

A good general classification approach should be to consider lifting data for Nichols algebras as in [AS10]*, but instead of coradical elements introduce new primitives (forming U) and then take the graded dual.*

Besides their theoretical charm, these extensions should have interesting applications to conformal field theory. E.g. the author's example b) for $\mathfrak{g} = B_n, q = \pm i$ should correspond to n symplectic fermions and the example d) is precisely the Hopf algebra considered by Gainutdinov, Tipunin for $W(p, p')$-models (unpublished).

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