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Topological infinite gammoids, and a new Menger-type theorem for infinite graphs

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Abstract

Answering a question of Diestel, we develop a topological notion of gammoids in infinite graphs which, unlike traditional infinite gammoids, always define a matroid.

As our main tool, we prove for any infinite graph G with vertex sets A and B that if every finite subset of A is linked to B by disjoint paths, then the whole of A can be linked to the closure of B by disjoint paths or rays in a natural topology on G and its ends.

This latter theorem re-proves and strengthens the infinite Menger theorem of Aharoni and Berger for 'well-separated' sets A and B. It also implies the topological Menger theorem of Diestel for locally finite graphs.

1 Introduction

Unlike finite gammoids, traditional infinite gammoids do not necessarily define a matroid. Diestel [9] asked whether a suitable topological notion of infinite gammoid might mend this, so that gammoids always give rise to a matroid. We answer this in the positive by developing such a topological notion of infinite gammoid. Our main tool is a new topological variant of Menger's theorem for infinite graphs, which is also interesting in its own right.

Given a directed graph G with a set $B \subseteq V(G)$ of vertices, the set $\mathcal{L}(G, B)$ contains all vertex sets I that can be linked by vertex-disjoint directed paths[1] to B. If G is finite, $\mathcal{L}(G, B)$ is the set of independent sets

[1] In this paper, *paths* are always finite.

of a matroid, called the *gammoid of G with respect to B*. If G is infinite, $\mathcal{L}(G,B)$ does not always define a matroid [1].

In 1968, Perfect [11] looked at the question of when $\mathcal{L}(G,B)$ is a matroid. As usual at that time, she restricted her attention to matroids with every circuit finite, now called *finitary matroids*. In [5], Bruhn et al found a more general notion of infinite matroids, which are closed under duality and need not be finitary. Afzali, Law and Müller [1] study infinite gammoids in this more general setting and find conditions under which $\mathcal{L}(G,B)$ is a matroid. In this paper, we introduce a topological notion of gammoids in infinite graphs that always define a matroid.

These gammoids can be defined formally without any reference to topology, as follows. A ray R in G *dominates* B if G contains infinitely many vertex-disjoint directed paths from R to B. A vertex v *dominates* B if there are infinitely many directed paths from v to B that are vertex-disjoint except in v. A path *dominates* B if its last vertex dominates B. A *domination linkage* from A to B is a family of vertex-disjoint directed paths or rays $(Q_a \mid a \in A)$ where Q_a starts in a and either ends in some vertex of B or else dominates B. A vertex set I is in $\mathcal{L}_T(G,B)$ if there is a domination linkage from I to B. We offer the following solution to Diestel's question:

Theorem 1.1. *$\mathcal{L}_T(G,B)$ is a finitary matroid.*

When G is undirected[2], Theorem 1.1 has the following topological interpretation. On G and its ends consider the topology whose basic open sets are the components C of $G \setminus X$ where X is a finite set of inner points of edges, together with the ends that have rays in C. The closure of $B \subseteq V(G)$ consists of B, the vertices dominating B, and the ends ω whose rays $R \in \omega$ dominate B. Thus $I \in \mathcal{L}_T(G,B)$ if and only if the whole of I can be linked to the closure of B by vertex-disjoint paths or rays.[3] We shall not need this topological interpretation:

Theorem 1.1 can be used to prove that under certain conditions the naive, non-topological, gammoid $\mathcal{L}(G,B)$ is a matroid, too:

Corollary 1.2. *Let G be a digraph with a set B of vertices such that there are neither infinitely many vertex-disjoint rays dominating B nor infinitely many vertices dominating B. Then $\mathcal{L}(G,B)$ is a matroid.*

[2]Formally, we consider those directed graphs G obtained from an undirected graph by replacing each edge by two parallel edges directed both ways.

[3]Instead of just taking paths and rays, one might want to take all 'topological arcs'. However, this would result in a weaker theorem.

Corollary 1.2 does not follow from the existence criterion of Afzali, Law and Müller for non-topological gammoids. Also its converse is not true, see Section 5 for details.

The main tool in our proof of Theorem 1.1 is a purely graph-theoretic Menger-type theorem, which seems to be interesting in its own right. It is not difficult to show that if there is a domination linkage from A to B, then there is a linkage from every finite subset of A to B. Our theorem says that the converse is also true:

Theorem 1.3. (i) *In any infinite digraph with vertex sets A and B, there is a domination linkage from A to B if and only if every finite subset of A can be linked to B by vertex-disjoint directed paths.*

(ii) *In any infinite undirected graph G, a set A of vertices can be linked by disjoint paths and rays to the closure of another vertex set B if and only if every finite subset can be linked to B by vertex-disjoint paths.*

We remark that the proof of Theorem 2.1 is non-trivial and not merely a compactness result. Applying compactness, one would get a topological linkage from A to the closure of B by arbitrary topological arcs, not necessarily paths and rays. Our graph-theoretical version of Theorem 2.1 is considerably stronger than this purely topological variant.

In Section 4 we study the relationship between Theorem 1.3 and existing Menger-type theorems for infinite graphs: the Aharoni-Berger theorem [3] and the topological Menger theorem for arbitrary infinite graphs. The latter was proved by Bruhn, Diestel and Stein [6], extending an earlier result of Diestel [8] for countable graphs. For infinite graphs with 'well-separated' sets A and B (defined in Section 4), Theorem 1.3 implies and strengthens the Aharoni-Berger theorem. This in turn allows us to give a proof of the topological Menger theorem for locally finite graphs which, unlike the earlier proofs, does not rely on the (countable) Aharoni-Berger theorem (which was proved earlier by Aharoni [2]).

The paper is organised as follows. After a short preliminary section we prove in Section 3 the directed edge version of Theorem 1.3. In Section 4 we sketch how this variant implies Theorem 1.3, and how Theorem 1.3 implies the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs. In Section 5 we summarise some basics about infinite matroids, and prove Theorem 1.1 and Corollary 1.2.

2 Preliminaries

Throughout, notation and terminology for graphs are that of [7]. In this paper, we will mainly be concerned with sets of edge-disjoint directed paths. Thus, we abbreviate edge-disjoint by disjoint, edge-separator by separator and directed path by path. Given a digraph G and $A, B \subseteq V(G)$, a *linkage from A to B* is a set of disjoint paths from the whole of A to B. We update the definitions of when a ray dominates B, a vertex dominates B, a path dominates B, and what a domination linkage is: these are the definitions made in the Introduction with "vertex-disjoint" replaced by "edge-disjoint". The proof of the following theorem takes the whole of Section 3.

Theorem 2.1. *Let G be a digraph and $b \in V(G)$, and $I \subseteq V(G) - b$. There is a domination linkage from I to $\{b\}$ if and only if every finite subset of I has a linkage into b.*

We delay the proof that Theorem 2.1 implies Theorem 1.3 until Section 4. In a slight abuse of notion, we shall suppress the set brackets of $\{b\}$ and just talk about "domination linkages from I to b". One implication of Theorem 2.1 is indeed easy:

Lemma 2.2. *If there is a domination linkage from I to b, then every finite subset S of I has a linkage into b.*

Proof. For $s \in S$, let P_s be the path or ray from the domination linkage starting in s. Suppose for a contradiction, there is no linkage from S into b. Then by Menger's theorem, there is a set F of at most $|S| - 1$ edges such that after its removal there is no directed path from S to b.

Suppose for a contradiction that there is some P_s not containing an edge of F. Then P_s cannot end at b. So P_s dominates b, and thus there is some P_s-b-path avoiding F, contradicting the fact that F was an edge-separator. Thus each P_s contains an edge of F. As the P_s are edge-disjoint, $|F| \geq |S|$, which is the desired contradiction. □

3 Proof of Theorem 2.1

The proof of Theorem 2.1 takes the whole of this section.

3.1 Exact graphs

The core of the proof of Theorem 2.1 is the special case where G is exact (defined below). In this subsection, we show that the special case of Theorem 2.1 where G is exact implies the general theorem. More precisely, we prove that the Lemma 3.1 below implies Theorem 2.1.

Given a vertex set D, an edge is *D-crossing (or crossing for D)* if its starting vertex is in D and the endvertex is outside. We abbreviate $V(G)\setminus D$ by $D^{\complement}$. The *order of* D is the number of D-crossing edges. The vertex set D is *exact* (for some set $I \subseteq V(G)$ and $b \in V(G)$) if $b \notin D$ and the order of D is finite and equal to $|D \cap I|$. A graph is *exact* (for b and I) if for every $v \in V(G) - b$, there is an exact set D containing v.

Lemma 3.1. *Let G be an exact digraph and $b \in V(G)$. Let $I \subseteq V(G) - b$ such that every finite subset of I has a linkage into b. Then there is a domination linkage from I to b.*

First we need some preparation. Let G be a graph and let $b \in V(G)$. Let $\mathcal{I}$ be the set of all sets $I \subseteq V(G) - b$ such that every finite subset of I has a linkage into b. The following is an easy consequence of Zorn's lemma.

Remark 3.2. *Let $I \in \mathcal{I}$, and $X \subseteq E(G)$, then there is $J \in \mathcal{I}$ maximal with $I \subseteq J \subseteq X$.*

Lemma 3.3. *Let G be a directed graph, and let $I \subseteq V(G) - b$ be maximal with the property that every finite subset of I has a linkage into b. Let $v \in (V(G) - b) \setminus I$. Then there is an exact D containing v.* □

Proof. By the maximality of I, there is a finite subset I' of I such that $I' + v$ cannot be linked to b. By Menger's theorem, there is a set D of order at most $|I'|$ not containing b but containing $I' + v \subseteq D$. The order must be precisely $|I'|$ since I' can be linked to b. Thus D is exact, which completes the proof. □

Proof that Lemma 3.1 implies Theorem 2.1. By Lemma 2.2, it suffices to prove the "if"-implication. Let G, b, I be as in Theorem 2.1. We obtain the graph H_1 from G by identifying b with all vertices v such that there are infinitely many edge-disjoint v-b-paths. Note that in H_1 every vertex $v \neq b$ can be separated from b by a finite separator.

It suffices to prove the theorem for H_1 since then the set of dominating paths and rays we get for H_1 extends to a set of dominating paths and rays for G by adding a singleton path for every vertex in I that is identified with b in H_1.

We build an exact graph H_2 that has H_1 as a subgraph. Let $v \in V(H_1) - b$. Let k_v be the smallest order of some vertex set D containing D and not containing b. By construction of H_1, the number k_v is finite.

We obtain H_2 from H_1 by for each $v \in V(H_1) - b$ adding k_v-many vertices whose forward neighbourhood is that of v and that do not have any incoming edges. We shall refer to these newly added vertices for the vertex v as the clones of v.

Now we extend I to a maximal set $I_2 \subseteq V(H_2) - b$ such that every finite subset of I_2 has a linkage into b. This is possible by Remark 3.2.

Next we show that H_2 is exact with respect to I_2. Suppose for a contradiction that there is some $v \in I_2$ such that there is no exact D containing v. First we consider the case that $v \in V(H_1)$. Since v together with all its clones cannot be linked to b, there is a clone w of v that is not in I_2. Since $w \notin I_2$, there must be some exact D' containing w by Lemma 3.3. If $v \in D'$, we are done, otherwise we get a contradiction since there is no linkage from $((I \cap D') + v)$ to b. The case that $v \notin V(H_1)$ is similar.

Having shown that H_2 is exact, we now use the assumption that Lemma 3.1 is true for H_2 and I_2: We get for each $v \in I$ some path or ray that dominates b in H_2. This path or ray also dominates in H_1 because a clone-vertex cannot be an interior vertex of any directed path or ray. And it also dominates in G, which completes the proof that Lemma 3.1 implies Theorem 2.1. □

3.2 Exact vertex sets

In this subsection, we prove some lemmas needed in the proof of Lemma 3.1.

Until the end of the proof of Theorem 2.1, we shall fix a graph G that is exact with respect to a fixed vertex b and some set $I \subseteq V(G) - b$. We further assume that every finite subset of I has a linkage into b. First we shall prove some lemmas about exact vertex sets.

Lemma 3.4. *Let D be exact and let $P_1, \ldots P_n$ be a linkage from $I \cap D$ to b. Then each P_i contains precisely one D-crossing edge, and each D-crossing edge is contained in one P_i.*

Proof. Clearly, each P_i contains an D-crossing edge. Since the P_i are edge-disjoint no two of them contain the same crossing edge. Since D is exact, there are precisely n D-crossing edges, and thus there is precisely one on each P_i. □

Lemma 3.5. *Let $D, D' \subseteq V(G)$ such that $D' \subseteq D$, and D' is exact. Let $\mathcal{L}$ be a linkage from $(I \cap D)$ to b. If some $P \in \mathcal{L}$ starts at a vertex in $D \setminus D'$, then no vertex of P lies in D'.*

Proof. Since D' is exact, each D'-crossing edge lies on some path of $\mathcal{L}$. On the other hand $|I \cap D'|$ of the paths start in D', and thus contain an D'-crossing edge. So P cannot contain any D'-crossing edge. If P meets D', then it would meet D' in a last vertex, and the edge pointing away from this vertex would be an D'-crossing edge. Hence P does not meet D', which completes the proof. □

Lemma 3.6. *Let D and D' be exact.*

(I) Then $D \cup D'$ is exact.

(II) Then $D \cap D'$ is exact.

(III) Then there does not exist an edge from $D \setminus D'$ to $D' \setminus D$.

Proof. Let $\mathcal{L}$ be a linkage from $I \cap (D \cup D')$ to b. For $X \subseteq D \cup D'$, let $\mathcal{L}(X)$ denote the set of those paths in $\mathcal{L}$ that have their starting vertex in X. For $X \subseteq V(G)$, let $\mathcal{C}(X)$ denote the set of X-crossing edges. It is immediate that.

$$|\mathcal{L}(D \cap D')| + |\mathcal{L}(D \cup D')| = |\mathcal{L}(D)| + |\mathcal{L}(D')| \tag{1}$$

Since D and D' are exact, (1) gives the following:

$$|\mathcal{L}(D \cap D')| + |\mathcal{L}(D \cup D')| = |\mathcal{C}(D)| + |\mathcal{C}(D')| \tag{2}$$

Next, we prove the following.

$$|\mathcal{C}(D \cap D')| + |\mathcal{C}(D \cup D')| \leq |\mathcal{C}(D)| + |\mathcal{C}(D')| \tag{3}$$

Each edge in both $\mathcal{C}(D \cap D')$ and $\mathcal{C}(D \cup D')$ points from $D \cap D'$ to $D^{\complement} \cap D'^{\complement}$, and hence is in both $\mathcal{C}(D)$ and $\mathcal{C}(D')$. Each edge in $\mathcal{C}(D \cap D')$ is in either $\mathcal{C}(D)$ or $\mathcal{C}(D')$. Similarly, each edge in $\mathcal{C}(D \cup D')$ is in either $\mathcal{C}(D)$ or $\mathcal{C}(D')$. This proves inequation (3). Note that if we have equality, we cannot have an edge from $D \setminus D'$ to $D' \setminus D$.

In order to prove (I) and (II), it suffices to show that $|\mathcal{L}(D \cup D')| = |\mathcal{C}(D \cup D')|$ and that $|\mathcal{L}(D \cap D')| = |\mathcal{C}(D \cap D')|$. Since $\mathcal{L}(D \cup D')$ and $\mathcal{L}(D \cap D')$ are sets of edge-disjoint paths, that each contain at least one crossing edge, it must be that $|\mathcal{L}(D \cup D')| \leq |\mathcal{C}(D \cup D')|$ and that $|\mathcal{L}(D \cap D')| \leq |\mathcal{C}(D \cap D')|$.

By equations (3) and (2), we get that

$$|\mathcal{C}(D \cap D')| + |\mathcal{C}(D \cup D')| \leq |\mathcal{L}(D \cap D')| + |\mathcal{L}(D \cup D')| \tag{4}$$

Combining this with the two inequalities before, we must have that $|\mathcal{L}(D \cup D')| = |\mathcal{C}(D \cup D')|$ and $|\mathcal{L}(D \cap D')| = |\mathcal{C}(D \cap D')|$, which proves (I) and (II).

Now it must be that we must have equality in (3). So there cannot be an edge from $D \setminus D'$ to $D' \setminus D$, which proves (III). This completes the proof. $\square$

Lemma 3.7. *Let G be an exact graph, and F be a finite set of vertices. Then there is an exact D with $F - b \subseteq D$.*

Proof. For each $v \in F - b$, there is an exact D_v containing v by exactness of G. Then $\bigcup_{v \in F-b} D_v$ is exact, which can easily be proved by induction over $|F - b|$, using (I) of Lemma 3.6 in the induction step. $\square$

Let D be exact and let $\mathcal{L}$ be a linkage from $D \cap I$ to b. Then D' is called a *forwarder of D with respect to $\mathcal{L}$* if D' is exact and $\bigcup \mathcal{L} - b \subseteq D'$ and $D \subseteq D'$.

Lemma 3.8. *Let G be an exact graph. Then each exact D has an forwarder with respect to each linkage $\mathcal{L}$ from some subset of $D \cap I$ to b.*

Proof. Apply Lemma 3.7 to the set of all vertices in $\bigcup \mathcal{L}$ to get a D' with all those vertices in $D' + b$. The desired forwarder is then $D \cup D'$. $\square$

The *hull* $\hat{D}$ of a vertex set D consists of those vertices that are separated by the D-crossing edges from b. Note that $D \subseteq \hat{D}$ and that $D^{\complement}$ consists of those vertices v such that there is a v-b-path all of whose internal vertices are outside D. Since every vertex on such a path is in $\hat{D}^{\complement}$, the hull of any hull $\hat{D}$ is $\hat{D}$ itself.

We say that two vertex sets D and D' are *equivalent* if they have the same hull. This clearly defines an equivalence relation, which we shall call $\sim$. Note that $D \sim D'$ if and only if D and D' have the same crossing edges.

Remark 3.9. *Let F, F', $\tilde{F}$ and $\tilde{F}'$ be exact with $\tilde{F} \sim F$ and $\tilde{F}' \sim F'$. Then $F \cup F' \sim \tilde{F} \cup \tilde{F}'$.*

Proof. Clearly, the set of $(F \cup F')$-crossing edges is equal to the set of $(\tilde{F} \cup \tilde{F}')$-crossing edges, which gives the desired result. $\square$

Lemma 3.10. *For any exact D, the hull $\hat{D}$ is exact.*

Proof. It suffices to show that $I \cap \hat{D} = I \cap D$. Since $\hat{D} \supseteq D$, clearly $I \cap \hat{D} \supseteq I \cap D$. In order to prove the other inclusion, suppose for a contradiction that there is some $v \in I \cap (\hat{D} \setminus D)$. Since $(I \cap D) + v$ is finite, there is some linkage $\mathcal{L}$ from $((I \cap D) + v)$ to b. Let P be the path from that linkage that starts in v. By Lemma 3.5, the path P avoids D. So P witnesses that $v \notin \hat{D}$. This is a contradiction, thus $I \cap \hat{D} = I \cap D$. $\square$

3.3 Good functions

We define what a good function is and prove that the existence of a good function in every exact graph implies Lemma 3.1.

First, we fix some notation. Let $\mathcal{E}$ be the set of exact vertex sets D, and let $\bar{\mathcal{L}}$ be the set of linkages from finite subsets of I to b. For a vertex set D, the set $N(D)$ consists of D together with all endvertices of D-crossing edges.

For $v \in I$ and some linkage $\mathcal{L}$, let $Q_v(\mathcal{L})$ denote the path in $\mathcal{L}$ starting from v. For every exact D, the edges of $Q_v(\mathcal{L})$ contained in $G[N(D)]$ are the edges of some initial path of $Q_v(\mathcal{L})$. We call this initial path $P_v(D; \mathcal{L})$. We follow the convention that $P_v(D; \mathcal{L})$ is empty if $v \notin D$.

A function $f : \mathcal{E} \to \bar{\mathcal{L}}$ is *good* if it satisfies the following:

(i) $f(F)$ is a linkage from $I \cap F$ to b.

(ii) If $v \in I$ and $F, F' \in dom(f)$ with $F' \subseteq F$, then $P_v(F'; f(F')) = P_v(F'; f(F))$.

(iii) If $\bigcup P_v(F; f(F))$ is a ray, then it dominates b. Here the union ranges over all exact F.

Before proving that there is a good function, we first show how to deduce Theorem 2.1 from that. Let us abbreviate $P_v(D; f(D))$ by $P_v(D; f)$. If it is clear by the context, which function f we mean, we even just write $P_v(D)$.

Lemma 3.11. *Let $f : \mathcal{E} \to \bar{\mathcal{L}}$ be a partial function satisfying (i) and (ii). Further assume that for any two exact F and F' with $F' \subseteq F$ and $F \in dom(f)$, also $F' \in dom(f)$.*

Let $v \in I$, and let $D, D' \in dom(f)$ be exact with $v \in D \cap D'$. Then $P_v(D) \subseteq P_v(D')$, or $P_v(D') \subseteq P_v(D)$.

Proof. $D \cap D'$ is exact by Lemma 3.6 and in the domain of f. Since f satisfies (ii), we get that $P_v(D \cap D')$ is a subpath of both $P_v(D)$ and $P_v(D')$. Let e be the last edge of $P_v(D \cap D')$, and x be its endpoint in $D^\complement \cup D'^\complement$.

Now we distinguish three cases. If $x \in D^\complement \cap D'^\complement$, the edge e is crossing for both D and D', and thus is the last edge of both $P_v(D)$ and $P_v(D')$. So $P_v(D) = P_v(D')$, so the lemma is true in this case.

If $x \in D^\complement \cap D'$, then e is the last edge of $P_v(D)$. So $P_v(D) = P_v(D \cap D') \subseteq P_v(D')$, so the lemma is true in this case.

The case $x \in D'^\complement \cap D$ is similar to the last case. This completes the proof. $\square$

For the remainder of this subsection, let us fix a good function f. The last Lemma motivates the following definition. For $v \in I$, let P_v be the union of all the paths $P_v(D)$ over all exact D containing v. By the last Lemma P_v is either a path or a ray.

Lemma 3.12. *If P_v and P_w share an edge, then $v = w$.*

Proof. Let e be an edge in both P_v and P_w. Let D_v be exact with $e \in P_v(D_v)$. Similarly, let D_w be exact with $e \in P_w(D_w)$.

By (I) of Lemma 3.6, we get that $D_v \cup D_w$ is exact. Since f is good, we have that $P_v(D_v \cup D_w)$ includes $P_v(D_v)$, and that $P_w(D_v \cup D_w)$ includes $P_w(D_w)$. Since $P_v(D_v \cup D_w)$ and $P_w(D_v \cup D_w)$ share the edge e, we must have that $v = w$, which completes the proof. $\square$

Lemma 3.13. *If P_v is a path, then it ends at b.*

Proof. Suppose for a contradiction that P_v does not end at b. Then P_v does not contain b.

Then by Lemma 3.7, there is an exact D with $P_v \subseteq D$. Then $P_v(D)$ contains some D-crossing edge whose endvertex does not lie on P_v, which gives a contradiction to the construction of P_v. $\square$

The following lemma tells us that to prove Theorem 2.1, it remains to show that every exact graph has a good function.

Lemma 3.14. *Let G be an exact digraph that has a good function. Let $b \in V(G)$. Let $I \subseteq V(G) - b$ such that every finite subset of I has a linkage into b. Then there is a domination linkage from I to b.*

Proof. Each P_v dominates b: If P_v is a path, this is shown in Lemma 3.13. If P_v is a ray, this follows from the fact that f is good. By Lemma 3.12 all the P_v are edge-disjoint, which completes the proof. $\square$

3.4 Intermezzo: The countable case

The purpose of this subsection is to prove that there is a good function under the assumption that G is countable. This case is easier than the general case and some of the ideas can already be seen in this special case. However, in the general case we do not rely on the countable case. At the end of this subsection, we tell why this proof does not extend to the general case. We think that this helps to get a better understanding of the general case.

Lemma 3.15. *Let G be an exact graph with $V = \{v_0 = b, v_1, v_2, \ldots\}$ countable. Then there is a sequence of exact hulls D_n and linkages $\mathcal{L}_n$ from $I \cap D_n$ to b satisfying the following.*

1. *$D_n \subseteq D_{n+1}$;*
2. *$\{v_1, \ldots, v_n\} \subseteq D_n$;*
3. *$P_v(D_n; \mathcal{L}_n) = P_v(D_n; \mathcal{L}_{n+1})$ for any $v \in I$;*
4. *D_{n+1} is a forwarder of D_n with respect to $\mathcal{L}_n$.*

Proof. Assume that for all $i \leq n$, we already constructed exact hulls D_i and linkages $\mathcal{L}_i$ satisfying 1-4.

Next, we define D_{n+1}. By Lemma 3.7, there is an exact F_n containing v_{n+1}. By Lemma 3.6, $D_n \cup F_n$ is exact. Let D'_{n+1} be a forwarder of $D_n \cup F_n$ with respect to the linkage $\mathcal{L}_n$, which exists by Lemma 3.8. Let D_{n+1} be the hull of D'_{n+1}, which is exact by Lemma 3.10.

It remains to construct $\mathcal{L}_{n+1}$ so as to make 3 true. Let $\mathcal{L}$ be some linkage from $I \cap D_{n+1}$ to b. By Lemma 3.4, for each D_n-crossing edge e there is precisely one $P_e \in \mathcal{L}_n$ that contains e, and precisely one $Q_e \in \mathcal{L}$ that contains e. Let $R_e = P_e e Q_e b$. Since $P_e e \subseteq D_n + e$ and $e Q_e b \subseteq D_n^{\complement} + e$, the R_e are edge-disjoint. For $\mathcal{L}_{n+1}$ we pick the set of the R_e together with all $Q \in \mathcal{L}$ that do not contain any D_n-crossing edge. Clearly $\mathcal{L}_{n+1}$ is an linkage from $I \cap D_{n+1}$ to b. And 3 is true by construction. This completes the construction. $\square$

Lemma 3.16. *Every countable exact graph G has a good function f.*

Proof. Let D_n and $\mathcal{L}_n$ as in Lemma 3.15. We let $f(D_n) = \mathcal{L}_n$. Next, we define f at all other exact D. Since there are only finitely many D-crossing edges, there is a large number m such that all these crossing edges are in $N(D_m)$. Then $D \subseteq D_m$ as for each $v \notin D_m$ there is a v-b-path included avoiding D_m. Now we let $f(D)$ consist of those paths in $f(D_m)$ that start in D. We remark that this definition does not depend on the choice of m.

Having defined f, it remains to check that it is good: clearly it satisfies (i) and (ii), and it just remains to verify (iii). So assume that for some $v \in I$, the union $R = \bigcup_{F \in \mathcal{E}} P_v(F; f(F))$ is a ray. Then $R = \bigcup_{n \in \mathbb{N}} P_v(D_n; \mathcal{L}_n)$. Let e_v^n be the unique D_n-crossing edge on $Q_v(\mathcal{L}_n)$. Since D_{n+1} is a forwarder of D_n, the path $R_v^n = e_v^n Q_v(\mathcal{L}_n)$ is contained in $D_n + b$ and avoids D_{n+1}. Thus the paths R_v^n are edge-disjoint and witness that R dominates b. So f is good, which completes the proof. $\square$

Remark 3.17. *Our proof above heavily relies on the fact that we can find a nested set of exact vertex sets D_n indexed with the natural numbers that exhaust the graph (compare 2 in Lemma 3.15). However if we can find such a nested set, then I must be countable since each D_n contains only finitely many vertices of I. Thus this proof does not extend to the general case.*

3.5 Infinite sequences of exact vertex sets

The purpose of this subsection is to prove some lemmas that help proving that there is a good function in every exact graph. These lemmas are about infinite sequences of exact vertex sets.

Lemma 3.18. *There does not exist a sequence $(D_n|n \in \mathbb{N})$ with $D_n \subsetneq D_{n+1}$ of exact hulls that all have bounded order.*

Proof. Suppose for a contradiction that there is a such sequence $(D_n|n \in \mathbb{N})$. By taking a subsequence if necessary, we may assume that all D_n have the same order. Since any two D_n are exact and have the same order, we must have $I \cap D_1 = I \cap D_n$ for every n. Let $\mathcal{L}$ be some linkage from $(D_1 \cap I)$ to b. Any $P \in \mathcal{L}$ contains a unique D_n-crossing edge for every n by Lemma 3.4. Since $D_n \subseteq D_{n+1}$, there is a large number n_P such that for all $n \geq n_P$ it is the same crossing edge. Let m be the maximum of the numbers n_P over all $P \in \mathcal{L}$. Then for all $n \geq m$, the D_n have the same crossing edges and thus are equivalent. This is a contradiction, completing the proof. $\square$

Lemma 3.19. *Let D be exact and let $\mathcal{X}$ be a nonempty set of exact $D' \subseteq D$ that is closed under $\sim$ and taking unions. Then there is some $D'' \in \mathcal{X}$ including all $D' \in \mathcal{X}$.*

Proof. Suppose for a contradiction that there is no such $D'' \in \mathcal{X}$. We shall construct an infinite sequence $(D_n|n \in \mathbb{N})$ as in Lemma 3.18.

Let $D_1 \in \mathcal{X}$ be arbitrary. Since $\mathcal{X}$ is $\sim$-closed, we may assume that D_1 is its hull. Now assume that D_n is already constructed. By assumption, there is $D'_n \in \mathcal{X}$ with $D'_n \not\subseteq D_n$. Let $D''_n = D_n \cup D'_n$. Let D_{n+1} be the hull of D''_n. Then $D_{n+1} \in \mathcal{X}$, and $D_n \subsetneq D_{n+1}$. This completes the construction of the infinite sequence $(D_n|n \in \mathbb{N})$, which contradicts Lemma 3.18 and hence completes the proof. $\square$

3.6 Existence of good functions

The purpose of this subsection is to prove that every exact graph has a good function, which implies Theorem 2.1 by Lemma 3.14. Next we shall define when a partial function is good for the following reason. In order to construct a good function f defined on the whole of $\mathcal{E}$ we shall construct an ordinal indexed family of good partial functions f_α such that if $\alpha > \beta$, then the domain of f_α includes that of f_β and agrees with f_β on the domain of f_β. Eventually some f_α will be defined on the whole of $\mathcal{E}$ and will be the desired good function.

The domain of a partial function f is denoted by $dom(f)$. A partial function $f : \mathcal{E} \to \bar{\mathcal{L}}$ is *good* if it satisfies the following:

(i) $f(F)$ is a linkage from $I \cap F$ to b.

(ii) If $v \in I$ and $F, F' \in dom(f)$ with $F' \subseteq F$, then $P_v(F'; f(F')) = P_v(F'; f(F))$.

(iii) If $\bigcup P_v(F; f(F))$ is a ray, then it dominates b. Here the union ranges over all $F \in dom(f)$.

(iv) Let F and F' be exact with $F' \subseteq F$. If F is in the domain of f, then so is F'.

(v) If F and F' are in the domain of f, then so is $F \cup F'$.

(vi) $dom(f)$ is closed under $\sim$.

Note that if $F, F' \in dom(f)$, then so is $F \cap F'$ by (iv). Note that each good partial function defined on the whole of $\mathcal{E}$ is a good function.

Lemma 3.20. *Let f be a partial function with domain X that satisfies (i)-(v). Then there is a good partial function $\hat{f}$ whose domain is the $\sim$-closure $\hat{X}$ of X such that $\hat{f}\restriction_X = f$.*

Proof. For each $F \in \hat{X}$, there is some $\tilde{F} \in X$ such that $F \sim \tilde{F}$. We let $\hat{f}(F) = f(\tilde{F})$. Clearly, $\hat{f}$ satisfies (i), (iii) and (vi). Since f satisfies (v) and by Remark 3.9, $\hat{f}$ satisfies (v).

To see that $\hat{f}$ satisfies (ii), let $v \in I$ and $F, F' \in dom(\hat{f})$ with $F' \subseteq F$. Then $P_v(F'; \hat{f}(F')) = P_v(F'; \hat{f}(F))$ as $P_v(\tilde{F}'; f(\tilde{F}') = P_v(\tilde{F}'; f(\tilde{F})$.

To see that $\hat{f}$ satisfies (iv), let F and F' be exact with $F' \subseteq F$ and $F \in \hat{X}$. Then $F' \cap \tilde{F}$ is exact, and since f satisfies (iv), it must be in X. Since $F' \cap \tilde{F}$ and F' have the same crossing edges, they are equivalent. So $F' \in \hat{X}$. So $\hat{f}$ satisfies (iv). This completes the proof. $\square$

For $S \subseteq \mathcal{E}$, let $S(iv) \subseteq \mathcal{E}$ denote the smallest set including S that satisfies (iv). Similarly, let $S(v) \subseteq \mathcal{E}$ denote the smallest set including S that satisfies (v).

Lemma 3.21. $[S(iv)](v) = [S(v)](iv)$ *for any set* S.
In particular, $[S(iv)](v)$ *is the smallest set included in* $\mathcal{E}$ *satisfying (iv) and (v).*

Proof. First let $D \in [S(iv)](v)$. Then there are $F_1, F_2 \in S(iv)$ such that $D = F_1 \cup F_2$. Then there are $F_1', F_2' \in S$ such that $F_1 \subseteq F_1'$ and $F_2 \subseteq F_2'$. Then $F_1' \cup F_2' \in S(v)$ by (I) of Lemma 3.6. Since $F_1 \cup F_2 \subseteq F_1' \cup F_2'$, we deduce that $D \in [S(v)](iv)$. So $[S(iv)](v) \subseteq [S(v)](iv)$.

The proof that $[S(v)](iv) \subseteq [S(iv)](v)$ is similar: Now let $D \in [S(v)](iv)$. Then there is $D' \in S(v)$ with $D \subseteq D'$. Then there are $F_1, F_2 \in S$ such that $D' = F_1 \cup F_2$. Then $F_i \cap D \in S(iv)$ for $i = 1, 2$ by (II) of Lemma 3.6. Since $D \subseteq F_1 \cup F_2$, we deduce that $D \in [S(iv)](v)$. This completes the proof. □

Let $X \subseteq \mathcal{E}$, and D be exact. Then $X[D]$ denotes the smallest set including $X + D$ that satisfies (iv) and (v).

3.6.1 Extending good partial functions

The aim of this subsubsection is to prove the following lemma that helps us building a good function in that it allows us to extend a good partial function a little bit.

Lemma 3.22. *Let* f *be a good partial function, and let* D *be exact. Then there is a good partial function* g *whose domain consists of the* $\sim$*-closure of* $dom(f)[D]$*, and that agrees with* f *at each point in* $dom(f)$*.*

If $D \in dom(f)$, then we just take $g = f$. So we may assume that $D \notin dom(f)$. Before we define g, we define auxiliary functions g_1, g_2 and g_3 with domains X_1, X_2 and X_3, respectively, such that $dom(f) \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq dom(g)$, and g will be defined such that $g\restriction_{X_1} = g_1$, $g\restriction_{X_2} = g_2$, and $g\restriction_{X_3} = g_3$. We let $X_1 = dom(f) + D$.

For all $D' \in dom(f)$, we let $g_1(D') = f(D')$. Next we define $g_1(D)$. Since $I \cap D$ is finite, there is some linkage from $I \cap D$ into b. Let $P_1, P_2, \ldots P_n$ be such a linkage. By Lemma 3.19, there is some $D'' \in dom(f)$ with $D'' \subseteq D$ such that $D' \subseteq D''$ for all $D' \in dom(f)$ with $D' \subseteq D$. Since D'' is exact, each D''-crossing edge lies on one of the P_i by Lemma 3.4.

We define $g_1(D)$ as follows. If no D''-crossing edge lies on P_i, then we put P_i into $g_1(D)$. If some D''-crossing edge, say e_i, lies on P_i, we take the path Q_i from the linkage $f(D'')$ that contains e_i, and put the path $Q_i e_i P_i$ into $g_1(D)$. This completes the definition of $g_1(D)$, and so of g_1.

Sublemma 3.23. g_1 *satisfies (ii).*

Proof. Let $F, F' \in X_1$ with $F' \subseteq F$. If F is not D, then $P_v(F'; f(F')) = P_v(F'; f(F))$ since f satisfies (ii) and (iv).

So we may assume that $F = D$. Then $F' \subseteq D'' \subseteq D$. So $P_v(F'; f(F')) = P_v(F'; f(D'')) = P_v(F'; f(D))$, which completes the proof. $\square$

Having defined X_1 and g_1, we now define X_2 and g_2. We let $X_2 = X_1(iv)$. For each $F \in X_2$ there is some $F' \in X_1$ such that $F \subseteq F'$. We let $g_2(F)$ to consists of those paths from $g_1(F')$ that start in F. By construction, g_2 satisfies (i) and (iv). By Lemma 3.6(III), $P_v(F, g_2(F)) = P_v(F, g_1(F'))$ for all $v \in I$. Thus g_2 satisfies (ii) as g_1 does.

Having defined g_2, we now define g_3. We let X_3 be $X_2(v)$, which is equal to $dom(f)[D]$. We let $P_v = P_v(F; g_2) \cup P_v(F'; g_2)$. By Lemma 3.11, it must be that $P_v = P_v(F; g_2)$ or $P_v = P_v(F'; g_2)$. Since g_2 satisfies (ii), no vertex of $P_v(F; g_2)$ that is not on $P_v(F'; g_2)$ can be in $F \cap F'$. By (III) of Lemma 3.6, it must be that every vertex of $P_v(F; g_2)$ that is not on $P_v(F'; g_2)$ is in $F \setminus F'$. Hence the P_v are edge-disjoint.

By (III) of Lemma 3.6, each P_v contains some $(F \cup F')$-crossing edge e_v. Let $\mathcal{L}$ be some linkage from $I \cap (F \cup F')$ to b. Let Q_v be the path in $\mathcal{L}$ that contains e_v. We define $g_3(F \cup F')$ to consist of the paths $P_v e_v Q_v$. Clearly, $g_3(F \cup F')$ is a linkage from $I \cap (F \cup F')$ to b.

By Lemma 3.21, g_3 satisfies not only (v) but also (iv).

Sublemma 3.24. *g_3 satisfies (ii).*

Proof. Let $v \in I$ and $F, F' \in dom(f)$ with $F' \subseteq F$. Our aim is to prove that $P_v(F'; g_3(F')) = P_v(F'; g_3(F))$. In the definition of g_3 at F, we have picked F_1 and F_2 in X_2 such that $F = F_1 \cup F_2$ in order to define $g_3(F)$. Similarly, we have picked F'_1 and F'_2 in X_2 such that with $F' = F'_1 \cup F'_2$ to define $g_3(F')$. It suffices to show that $P_v(X_{ij}; g_3(F')) = P_v(X_{ij}; g_3(F))$ where $X_{ij} = F'_j \cap F_i$ and $(i, j) \in \{1, 2\} \times \{1, 2\}$.

By the definition of g_3, we get the following two equations.

$$P_v(F'; g_3(F')) = P_v(F'_1; g_2) \cup P_v(F'_2; g_2) \tag{5}$$

$$P_v(F'; g_3(F)) = P_v(F' \cap F_1; g_2(F_1)) \cup P_v(F' \cap F_2; g_2(F_2)) \tag{6}$$

Since g_2 satisfies (ii), these two equations give the desired result when restricted to X_{ij}. $\square$

Sublemma 3.25. *g_3 satisfies (iii).*

Proof. For each $v \in I$, we compare the sets $\bigcup P_v(F; f(F))$ where first the union ranges over all $F \in dom(f)$ and second it ranges over all $F \in dom(g_3)$. The second set is a superset of the first and all its additional elements are in $P_v(D, g_3)$, which is finite. In particular, if the second set is a ray, then so is the first set by Lemma 3.11. In this case, the first set dominates b since f satisfies (iii), so the second set also dominates b. This completes the proof. □

Having defined g_3, we let $g = \hat{g}_3$ as in Lemma 3.20. Since g_3 satisfies (i) -(v), g is good by Lemma 3.20. This completes the proof of Lemma 3.22.

3.6.2 Construction of a good function

In this subsubsection, we construct a good function in every exact graph, which is the last step in the proof of Theorem 2.1. Each ordinal α has a unique representation $\alpha = \beta + n$ where β is the largest limit ordinal smaller than α, and n is a natural number. We say that α is *odd* if n is odd. Otherwise it is *even*.

Lemma 3.26. *Let G be an exact graph. Then there is a good function f defined on the whole of $\mathcal{E}$.*

Proof. In order to construct f we shall construct an ordinal indexed family of good partial functions f_α such that if $\alpha > \beta$, then the domain of f_α includes that of f_β and agrees with f_β on the domain of f_β. Eventually some f_α will be defined on the whole of $\mathcal{E}$ and will be the desired good function.

Assume that f_β is already defined for all $\beta < \alpha$. First we consider the case that $\alpha = \beta + 1$ is a successor ordinal. If f_β is defined on the whole of $\mathcal{E}$, we stop. Otherwise, we shall find some exact F_α. Then we let f_α be the partial function g given to us from Lemma 3.22 applied to f_β and F_α.

How we find F_α depends on whether α is an odd or an even successor ordinal. If α is odd, then we pick some $D \in \mathcal{E} \setminus dom(f_\beta)$, and let $F_\alpha = D$.

If α is even, say $\alpha = \delta + 2n$, where δ is the largest limit ordinal less than α, then for F_α we pick the forwarder of $F_{\delta+n}$ with respect to the linkage $f(F_{\delta+n})$, which exists by Lemma 3.8.

Having considered the case where α is a successor ordinal, we now consider the case where α is a limit ordinal. For the domain of f_α we take the union of the domains of all f_β with $\beta < \alpha$, and we let $f_\alpha(D) = f_\beta(D)$ for some β where this is defined. It is clear that f_α satisfies (i),(ii),(iv),(v),(vi), so it remains to show that f satisfies (iii). So let $v \in I$ such that $R = \bigcup P_v(D; f_\alpha)$ is a ray. Here the union ranges over all D in the domain of f_α.

Let $\mathcal{O}$ be the set of ordinals $\beta < \alpha$ such that there is some $D \in dom(f_\beta)$ with $v \in D$. $\mathcal{O}$ is nonempty, so it must contain a smallest ordinal ϵ. Note that ϵ is a successor ordinal. Let ϵ^- be such that $\epsilon = \epsilon^- + 1$.

We shall prove that $v \in F_\epsilon$. Suppose not for a contradiction, then $v \notin D$ for all $D \in dom(f_{\epsilon^-}) + F_\epsilon$. So $v \notin D$ for all $D \in [dom(f_{\epsilon^-}) + F_\epsilon](iv)$, and hence also $v \notin D$ for all $D \in dom(f_{\epsilon^-})[F_\epsilon]$ by Lemma 3.21. By Lemma 3.10 and since $v \in I$, also $v \notin D$ for all D in the $\sim$-closure of $dom(f_{\epsilon^-})[F_\epsilon]$. This contradicts the choice of ϵ. Hence $v \in F_\epsilon$. Let x be the unique F_ϵ-crossing edge contained in $P_v(F_\epsilon; f_\alpha)$.

We have a representation $\epsilon = \delta + k$ where δ is the largest limit ordinal less than ϵ. By construction, the $F_{\epsilon(l)}$ with $\epsilon(l) = \delta + 2^l \cdot k$ are nested with each other. To prove that R dominates b, it will suffice just to investigate the $F_{\epsilon(l)}$.

The paths $P_v(l) = Q_v(f_\alpha(F_{\epsilon(l)}))$ are contained in $F_{\epsilon(l+1)}+b$. By Lemma 3.4, there is a unique $F_{\epsilon(l)}$-crossing edge a_l on $P_v(l)$. The paths $a_l P_v(l)$ meet $F_{\epsilon(l)}$ only in their starting vertex. Thus the paths $a_l P_v(l)$ are edge disjoint. Since a_l is on $P_v(F_{\epsilon(l)}; f_\alpha)$, it is on R. Hence the paths $a_l P_v(l)$ witness that R dominates b. So f_α is good.

There must be some successor step $\alpha = \beta + 1$ at which we stop. Then f_β is a good function defined on the whole of $\mathcal{E}$. This completes the proof. □

Proof of Theorem 2.1. Recall that the easy implication is already proved in Lemma 2.2. For the other implication, combine Lemma 3.26 with Lemma 3.14 to get a proof of Lemma 3.1. Then remember that Lemma 3.1 implies Theorem 2.1. □

4 Graph-theoretic applications of Theorem 2.1

In this section, we show how Theorem 2.1 implies Theorem 1.3 and how Theorem 1.3 implies the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs.

4.1 Variants of Theorem 2.1

In this subsection, we explain how Theorem 2.1 implies Theorem 1.3. Theorem 2.1 is equivalent to the following.

Theorem 4.1. *There is a domination linkage from A to B if and only if every finite subset of A can be linked to B.*

Menger's theorem comes in four different versions: the directed edge version, the undirected edge version, the directed vertex version and the undirected vertex version. Depending on the version, we have different notions of path, separator and disjointness. Taking these different notions instead, we know in each of these 4 versions what it means that a ray dominates B, a vertex dominates B, a path dominates B, and what a domination linkage is, and what a linkage is.

The purpose of this subsection is to explain how Theorem 4.1 implies its undirected-edge-version, directed-vertex-version and undirected-vertex-version. These versions are like Theorem 4.1 but with the appropriate notions of domination linkage and linkage. The proof is done in the same way how one shows that the directed-edge-version of Menger's theorem for finite graphs implies all the other versions.

Starting with the sketch, one first shows that the directed-edge-version implies the directed-vertex-version for every graph G. For this one considers the auxiliary digraph H of G with $V(H) = V(G) \times \{in, out\}$. The edges of H are of two types: For each $v \in V(G)$, we add an edge pointing from (v, in) to (v, out). For each edge of H pointing from v to w, we add an edge pointing from (v, out) to (w, in). Then the directed-vertex-version for G is equivalent to the directed-edge-version for H.

Next one shows that the directed-vertex-version implies the undirected-vertex-version for every graph G. For this, one considers the directed graph H obtained from G by replacing each edge by two edges in parallel pointing in different directions. Then the undirected-vertex-version for G is equivalent to the directed-vertex-version for H.

Finally, one shows that the undirected-vertex-version implies the undirected-edge-version for every graph G. For this, one considers the line graph H of G. Then the undirected-edge-version for G is equivalent to the undirected-vertex-version for H.

It is clear that the directed vertex version of Theorem 4.1 is just Theorem 1.3(i). We call domination linkages in the undirected vertex version *vertex-domination linkages*. Similarly, we define *vertex-linkages*. The undirected vertex version of Theorem 4.1 is the following.

Corollary 4.2. *Let G be a graph and $A, B \subseteq V(G)$. There is a vertex-domination linkage from A to B if and only if every finite subset of A has a vertex-linkage into B.* $\square$

Corollary 4.2 is a reformulation of Theorem 1.3(ii).

4.2 Well-separatedness

In this subsection, we prove Corollary 4.3 below, which is used to deduce the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs.

A pair (A, B) of vertex sets is *well-separated* if every vertex or end can be separated from one of A or B by removing finitely many vertices.

Corollary 4.3. *(undirected vertex version) Let (A, B) be a well-separated pair of vertex sets. Then there is a vertex-linkage from the whole of A to B if and only if every finite subset of A has a vertex-linkage to B.*

Our next aim is to deduce Corollary 4.3 from Corollary 4.2. First we need some lemmas. For this, we fix a graph G and a well-separated pair (A, B) of vertex sets. Let $(P_a|a \in A)$ be a vertex-domination linkage from A to B. Let ω be an end that cannot be separated from B by removing finitely many vertices. Let A_ω be the set of those $a \in A$ such that P_a is a ray and belongs to ω.

Lemma 4.4. *There is a vertex-linkage $(Q_a|a \in A_\omega)$ from A_ω to B such that Q_a and P_x are vertex-disjoint for all $a \in A_\omega$ and all $x \in A \setminus A_\omega$.*

Proof. For a finite vertex set S, we denote by $C(S, \omega)$ the component of $G \setminus S$ that contains ω.

As (A, B) is well-separated, there is a finite set of vertices S that separates ω from A. So the set Z of those $a \in A$ such that P_a meets $C(S, \omega) \cup S$ is finite. As $A_\omega \subseteq Z$, the set A_ω must be finite. Furthermore there is a finite set T such that $C(T, \omega)$ meets precisely those P_a with $a \in A_\omega$. For $a \in A_\omega$, let t_a be first vertex on P_a such that $t_a P_a$ is contained in $C(T, \omega)$, which exist as these P_a are eventually contained in $C(T, \omega)$. The $(t_a P_a|a \in A_\omega)$ form a vertex-domination linkage from $(t_a|a \in A_\omega)$ to B in G', where G' is obtained from $G[C(T, \omega)]$ by deleting all edges on the paths $P_a t_a$ with $a \in A_\omega$. By the easy implication of Corollary 4.2 applied to G', we get a vertex-linkage $(K_a|a \in A_\omega)$ from $(t_a|a \in A_\omega)$ to B. Each walk $P_a t_a K_a$ includes a path Q_a from a to B. From the construction, it is clear, that the Q_a form a vertex-linkage from A_ω to B and that Q_a and P_x are vertex-disjoint for all $a \in A_\omega$ and all $x \in A \setminus A_\omega$. □

Lemma 4.5. *There is a vertex-domination linkage $(R_a|a \in A)$ from A to B such that each R_a is a path.*

Proof. We shall construct $(R_a|a \in A)$ by transfinite recursion. First we well-order the set Ω of ends: $\Omega = \{\omega_\alpha|\alpha \in \kappa\}$ for $\kappa = |\Omega|$. At each step β we have a current set of vertex-disjoint A-B-paths $\mathcal{Q}_\beta$. The set A_β of start vertices of paths in $\mathcal{Q}_\beta$ consists of those $a \in A$ such that P_a is a ray and belongs to some end ω_α with $\alpha < \beta$. We shall also ensure that $\mathcal{R}_\beta = \mathcal{Q}_\beta \cup \{P_a | a \notin A_\beta\}$ is a vertex-domination linkage from A to B.

If β is a limit ordinal, we just set $\mathcal{Q}_\beta = \bigcup_{\alpha<\beta} \mathcal{Q}_\alpha$. It is immediate that $\mathcal{Q}_\beta$ has the desired property assuming that the $\mathcal{Q}_\alpha$ with $\alpha < \beta$ have the property. If $\beta = \alpha + 1$ is a successor ordinal, we apply Lemma 4.4 to the vertex-domination linkage $\mathcal{R}_\alpha$. Then we let $\mathcal{Q}_{\alpha+1} = \mathcal{Q}_\alpha \cup \{Q_a | a \in A_{\omega_{\alpha+1}}\}$. It is clear from that lemma that $\mathcal{Q}_{\alpha+1}$ has the desired property.

This completes the recursive construction. It is clear that $\mathcal{R}_\kappa = \mathcal{Q}_\kappa \cup \{P_a | a \notin A_\kappa\}$ is the desired vertex-domination linkage. □

Proof that Corollary 4.2 implies Corollary 4.3. Let (A, B) be well-separated such that from every finite subset of A there is a vertex-linkage to B. By Corollary 4.2, there is a vertex-domination linkage $(P_a | a \in A)$ from the whole of A to B. By Lemma 4.5, we may assume that each P_a is a path. However, $(P_a | a \in A)$ may still contain a path P_u that does not end in B. Then P_u has to contain a vertex ω that cannot be separated from B by removing finitely many vertices. An argument as in the proof of Lemma 4.4, shows that there is a path Q_u from u to some vertex in B such that $(P_a | a \in A - u)$ together with Q_u is a vertex-domination linkage from A to B. Similar as in the proof of Lemma 4.5, we can now apply transfinite induction to replace each P_u one by one by such a path Q_u. The final vertex-domination linkage is then a vertex-linkage, which completes the proof. □

4.3 Existing Menger-type theorems

In this subsection, we show how Corollary 4.3 implies the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs.

The Aharoni-Berger theorem [3] says that for every graph G with vertex sets A and B, there is a set of vertex-disjoint A-B-paths together with an A-B-separator consisting of precisely one vertex from each of these paths.

At first glance, it might seem that the Aharoni-Berger theorem does not tell under which conditions there is a linkage from A to B - but actually it does. To explain this, we need a definition. A *wave* is a set of vertex-disjoint paths from a subset of A to some A-B-separator C. It is not difficult to show that the Aharoni-Berger theorem is equivalent to the following: The whole of A can be linked to B if and only if for every wave there is a linkage from A to its separator set C. Thus Corollary 4.3 implies the Aharoni-Berger theorem for well-separated sets A and B. We remark that neither Corollary 4.3 nor Theorem 2.1 follows from the Aharoni-Berger theorem.

Using this implication, we get the first proof of the topological Menger-Theorem of Diestel [8] for locally finite graphs that does not rely on the Aharoni-Berger theorem. Indeed, the argument of Diestel only relies on the Aharoni-Berger theorem for vertex sets A and B that have disjoint closure in $|G|$, which is equivalent to being well-separated if G is locally finite.

5 Infinite gammoids

In this section, we use Theorem 1.3 to prove Theorem 1.1 and Corollary 1.2. Throughout, notation and terminology for matroids are that of [10, 5]. M always denotes a matroid and $E(M)$ and $\mathcal{I}(M)$ denote its ground set and its sets of independent sets, respectively.

Recall that the set $\mathcal{I}(M)$ is required to satisfy the following *independence axioms* [5]:

(I1) $\emptyset \in \mathcal{I}(M)$.

(I2) $\mathcal{I}(M)$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}(M)$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}(M)$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}(M)$, the set $\{I' \in \mathcal{I}(M) \mid I \subseteq I' \subseteq X\}$ has a maximal element.

An *$\mathcal{I}$-circuit* is a set minimal with the property that it is not in $\mathcal{I}$. The following is true in any matroid.

(+) For any two finite $\mathcal{I}$-circuits o_1 and o_2 and any $x \in o_1 \cap o_2$, there is some $\mathcal{I}$-circuit included in $(o_1 \cup o_2) - x$.

Given $\mathcal{I} \subseteq \mathcal{P}(E)$, its *finitarization* $\mathcal{I}^{fin}$ consists of those sets J all of whose finite subsets are in $\mathcal{I}$. Usually, it is made a requirement that $\mathcal{I}$ is the set of independent sets of a matroid [4]. Then $\mathcal{I}^{fin}$ is the set of independent sets of a finitary matroid, called M^{fin} [4]. We shall need the following slight strengthening of this fact.

Lemma 5.1. *If $\mathcal{I}$ satisfies (I1), (I2) and (+), then $\mathcal{I}^{fin}$ is the set of independent sets of a finitary matroid.*

Proof. Clearly $\mathcal{I}^{fin}$ satisfies (I1) and (I2), and it satisfies (IM) by Zorn's Lemma. Thus it remains to check (I3). So let $I, I' \in \mathcal{I}^{fin}$ with I' maximal and I not maximal. So there is some $y \notin I$ with $I + y \in \mathcal{I}$. We may assume that $y \notin I'$ since otherwise we are done. Thus there is some finite $\mathcal{I}$-circuit o with $y \in o \subseteq I' + y$. Suppose for a contradiction that for each $x \in o \setminus (I+y)$, there is some finite $\mathcal{I}$-circuit o_x with $x \in o_x \subseteq I+x$. Applying (+) successively to o and the o_x, we obtain a finite $\mathcal{I}$-circuit o' included in $I + y$, which contradicts the assumption that $I + y \in \mathcal{I}^{fin}$. Thus there is some $x \in o \setminus (I + y)$ such that $I + x \in \mathcal{I}^{fin}$, which completes the proof. □

We shall also need the following slight variation of ($I3$).

(*) For all $I, J \in \mathcal{I}$ and all $y \in I \setminus J$ with $J + y \notin \mathcal{I}$ there exists $x \in J \setminus I$ such that $(J + y) - x \in \mathcal{I}$.

A matroid N is *nearly finitary* if for every base B of N there is a base B' of N^{fin} such that $B \subseteq B'$ and $|B' \setminus B|$ is finite. It is not difficult to show that N is nearly finitary if and only if for every base B' of N^{fin} there is a base B of N such that $B \subseteq B'$ and $|B' \setminus B|$ is finite. The proof of Lemma 4.15 in [4] actually proves the following strengthening of itself.

Lemma 5.2. *Let $M = (E, \mathcal{J})$ be a matroid with ground set E. Let $\mathcal{I} \subseteq \mathcal{J}$ satisfying ($I1$), ($I2$) ($I3$), ($*$) such that for any $J \in \mathcal{J}$ there is some $I \in \mathcal{I}$ such that $|J \setminus I|$ is finite. Then $N = (\mathcal{I}, E)$ is a matroid.*

In the special case where M is finitary, N is nearly finitary.

Next, we shall summarise the results from [1] that are relevant to this paper.

Lemma 5.3 (Afzali, Law, Müller [1, Lemma 2.2])**.** *For any digraph G and $B \subseteq V(G)$, the system $\mathcal{L}(G, B)$ satisfies (I3).*

Lemma 5.4 (Afzali, Law, Müller [1, Lemma 2.7])**.** *For any digraph G and $B \subseteq V(G)$, the system $\mathcal{L}(G, B)$ satisfies* (*).

Let $B_{AC} = \{b_0, b_1, \ldots\}$. Let $V_{AC} = B_{AC} \cup V^1 \cup V^2$, where $V^i = \{v_0^i, v_1^i, \ldots\}$. The digraph G_{AC} has vertex set V_{AC} and three types of edges: For $j \in \mathbb{N}$ it has an edge from v_j^1 to b_j. For each $j \in \mathbb{N}$, it has two edges, both start at v_j^2, and end at v_j^1 and v_{j+1}^1. The pair (G_{AC}, B_{AC}) is called an *alternating comb (AC)*. A subdivision of AC is drawn in Figure 1. Formally, a *subdivision of AC* is a pair (H_{AC}, B_{AC}) where H_{AC} is obtained from G_{AC} by replacing each directed edge xy by a directed path from x to y that is internally disjoint from all other such paths. Here edges from V_2 to V_1 are not allowed to be replaced by a trivial path[4] but the edges $v_j^1 b_j$ are allowed to be replaced by a trivial path. A pair (G,B) *has a subdivision of* AC if there is a subgraph H_{AC} of G and $B_{AC} \subseteq B \cap V(H_{AC})$ such that (H_{AC}, B_{AC}) is isomorphic to a subdivision of AC.

Theorem 5.5 (Afzali, Law, Müller [1, Theorem 2.6]). *Let G be a digraph and $B \subseteq V(G)$ such that (G,B) has no a subdivision of AC. Then $\mathcal{L}(G,B)$ is a matroid.*

For the remainder of this section, let G denote a digraph and $B \subseteq V(G)$. In the following, we shall explore for which digraphs G and sets B the system $\mathcal{L}_T(G,B)$ is the set of independent sets of a matroid, and how $\mathcal{L}_T(G,B)$ relates to $\mathcal{L}(G,B)$. If G is finite $\mathcal{L}(G,B)$ is a matroid and thus satisfies (+). The latter easily extends to infinite graphs G.

Lemma 5.6. *$\mathcal{L}(G,B)$ satisfies (+).*

Sketch of the proof. Given two finite $\mathcal{L}(G,B)$-circuits o_1 and o_2 intersecting in some vertex x, there are separations (A_i, B_i) with $o_i \subseteq A_i$ and $B \subseteq B_i$ of order at most $|o_i| - 1$. Then with a lemma like Lemma 3.6, one shows that either $(A_1 \cup A_2, B_1 \cap B_2)$ or $(A_1 \cap A_2, B_1 \cup B_2)$ separates some $\mathcal{L}(G,B)$-circuit $o \subseteq (o_1 \cup o_2) - x$ from B. $\square$

Using Theorem 1.3, we can prove the following slight extension of Theorem 1.1.

Corollary 5.7. *$\mathcal{L}_T(G,B) = \mathcal{L}(G,B)^{fin}$ for any digraph G and $B \subseteq V(G)$. Moreover, $\mathcal{L}_T(G,B)$ is a finitary matroid.*

Proof. By Theorem 1.3, $\mathcal{L}_T(G,B)$ consists of those sets I all of whose finite subsets can be linked to B by vertex-disjoint directed paths, and thus $\mathcal{L}_T(G,B) = \mathcal{L}(G,B)^{fin}$. As $\mathcal{L}(G,B)$ satisfies (I1), (I2) and (+), $\mathcal{L}_T(G,B)$ is a finitary matroid by Lemma 5.1. $\square$

Next we prove the following slight strengthening of Corollary 1.2 from the Introduction. Below we shall refer to the definition of dominating as defined in the Introduction.

Corollary 5.8. *Let G be a digraph with a set B of vertices. Then $\mathcal{L}(G,B)$ is a nearly finitary matroid if and only if there are neither infinitely many vertex-disjoint rays dominating B nor infinitely many vertices dominating B.*

[4] A *trivial path* consists of a single vertex only.

Proof. Clearly, if $\mathcal{L}(G, B)$ is nearly finitary, there are neither infinitely many vertex-disjoint rays dominating B nor infinitely many vertices dominating B. Conversely, assume that there are neither infinitely many vertex-disjoint rays dominating B nor infinitely many vertices dominating B. $\mathcal{L}(G, B)$ clearly satisfies (I1) and (I2), and it satisfies (I3) and (*) by Lemma 5.3 and Lemma 5.4. Let $J \in \mathcal{L}_T(G, B)$. By Theorem 1.3, we get for each $v \in J$ a ray or path P_v starting at v such that all these P_v are vertex-disjoint. Moreover each such P_v either ends in B or is a ray dominating B or its last vertex dominates B. Let I be the set of those v such that P_v ends in B. By assumption $J \setminus I$ is finite. So by Corollary 5.7, we can apply Lemma 5.2 with $\mathcal{J} = \mathcal{L}_T(G, B)$ to deduce that $\mathcal{L}(G, B)$ satisfies (IM), and thus is a nearly-finitary matroid. $\square$

A natural question that comes up is to ask how Theorem 5.5 and Corollary 5.8 relate to each other. In [1], Afzali, Law and Müller construct a pair (G, B) without AC such that $\mathcal{L}(G, B)$ is not nearly finitary. They also do it in a way to make $\mathcal{L}(G, B)$ 3-connected. Thus Corollary 5.8 does not imply Theorem 5.5.

To see that Theorem 5.5 does not imply Corollary 5.8, let G be the 3 by $\mathbb{Z}$ grid, formally: $V(G) = \{1, 2, 3\} \times \mathbb{Z}$, see Figure 1. In G, there is a directed

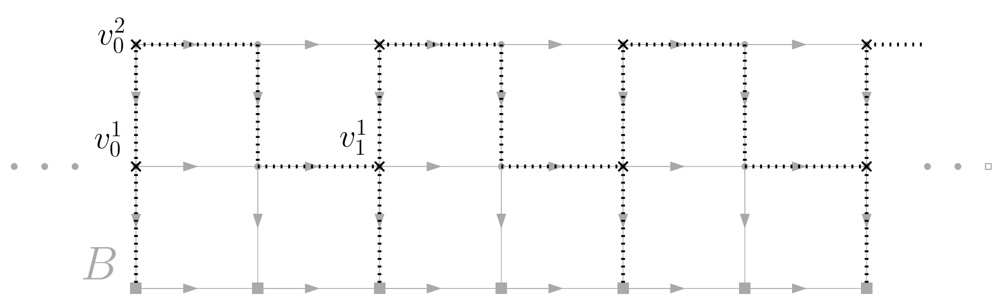

Figure 1: The graph G is depicted in gray. The vertices of B are squares. (G, B) has a subdivision of AC. One is indicated in this figure: The vertices of V^1 and V^2 are black crosses and the subdivided edges are drawn dotted.

edge from (x, y) to (x', y') if and only if either $x = x'$ and $y' = y+1$ or $y = y'$ and $x' = x + 1$. Let: $B = \{3\} \times \mathbb{Z}$. Then it is easy to see that no vertex of G dominates B and there are not infinitely many vertex-disjoint rays dominating B. However (G, B) has a subdivision of AC, which is indicated in Figure 1. Thus, there arises the question if there is a nontrivial common generalization of Corollary 5.8 and Theorem 5.5.

During this whole section, we have only considered the directed-vertex-version. Of course, similar results are true if we consider the undirected-vertex-version, the directed-edge-version or the undirected-edge-version instead.

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