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The Lie group of bisections of a Lie groupoid

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Abstract

In this article we endow the group of bisections of a Lie groupoid with compact base with a natural locally convex Lie group structure. Moreover, we develop thoroughly the connection to the algebra of sections of the associated Lie algebroid and show for a large class of Lie groupoids that their groups of bisections are regular in the sense of Milnor.

Keywords: global analysis, Lie groupoid, infinite-dimensional Lie group, mapping space, local addition, bisection, regularity of Lie groups

MSC2010: 22E65 (primary); 58H05, 46T10, 58D05 (secondary)

Contents

Introduction

Infinite-dimensional and higher structures are amongst the important concepts in modern Lie theory. This comprises homotopical and higher Lie algebras (L_∞-algebras) and Lie groups (group stacks and Kan simplicial manifolds), Lie algebroids, Lie groupoids and generalisations thereof (e.g., Courant algebroids) and infinite-dimensional locally convex Lie groups and Lie algebras. This paper is a contribution to the heart of Lie theory in the sense that is connects two regimes, namely the theory of Lie groupoids, Lie algebroids and infinite-dimensional Lie groups and Lie algebras. This connection is established by associating to a Lie groupoid its group of bisections and establishing a locally convex Lie group structure on this group.

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The underlying idea is not new per se, statements like "...the group of (local) bisections is a (local) Lie group whose Lie algebra is given by the sections of the associated Lie algebroid..." can be found at many places in the literature. In fact it depends on the setting of generalised manifolds that one uses, whether or not this statement is a triviality or a theorem. For instance, if the category of smooth spaces in which one works is cartesian closed and has finite limits, then the bisections are automatically a group object in this category and the only difficulty might be to calculate its Lie algebra. This applies for instance to diffeological spaces, where it follows from elementary theory that the bisections of a diffeological groupoid are naturally a diffeological group[1]. Another such setting comes from (higher) smooth topoi. See for instance [Sch13, FRS13a] for a generalisation of bisections to higher groupoids and [FRS13b] for a construction of the corresponding infinitesimal L_∞-algebra. Moreover, in synthetic differential geometry the derivation of the Lie algebra of the group of bisections can also be done formally [Nis06].

What we aim for in this paper is a natural locally convex Lie group structure on the group of bisections, which is not covered by the settings and approaches mentioned above. What comes closest to this aim are the results from [Ryb02], where a group structure in the "convenient setting of global analysis" is established. However, the results of the present paper are much stronger and more general than the ones from [Ryb02] in various respects, which we now line out. First of all, we work throughout in the locally convex setting [Nee06, Glö02, Mil84] for infinite-dimensional manifolds. The locally convex setting has the advantage that it is compatible with the underlying topological framework. In particular, smooth maps and differentials of those are automatically continuous. This will become important in the geometric applications that we have in mind (work in progress). Secondly, we not only construct a Lie group structure on the bisections, but also relate it to (and in fact derive it from) the canonical smooth structure on manifolds of mappings. Thus one is able to identify many naturally occurring maps as smooth maps. For instance, the natural action of the bisections on the arrow manifold is smooth, which allows for an elegant identification of the Lie bracket on the associated Lie algebra. The latter then gives rise to a *natural* isomorphism between the functors that naturally arise in this context, namely the (bi)section functors and the Lie functors. This is the third important feature of this paper. The last contribution of this paper is that we prove that the bisections are in fact a regular Lie group for all Banach-Lie groupoids.

On the debit side, one should say that the exhaustive usage of smooth structures on mapping spaces forces us to work throughout with locally metrisable manifolds and over compact bases, although parts of our results should be valid in greater generality. Moreover, the proof of regularity is quite technical, which is the reason for deferring several details of it to a separate section. To say it once more, the results are not surprising in any respect, it is the coherence of all these concepts that is the biggest value of the paper.

We now go into some more detail and explain the main results. Suppose $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid. This means that G, M are smooth manifolds, equipped with submersions $\alpha, \beta\colon G \to M$ and an associative and smooth multiplication $G \times_{\alpha,\beta} G \to G$ that admits a smooth identity map $1\colon M \to G$ and a smooth inversion $\iota\colon G \to G$. Then the bisections $\mathrm{Bis}(\mathcal{G})$ of $\mathcal{G}$ are the sections $\sigma\colon M \to G$ of α such that $\beta \circ \sigma$ is a diffeomorphism of M. This becomes a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M.$$

Our main tool to construct a Lie group structure on the group of bisections are certain local additions on the space of arrows G. This is generally the tool one needs on the target manifold to understand smooth structures on mapping spaces (see [Mic80, KM97, Woc13] or Appendix A). We require that the local addition on G is adapted to the source projection α, i.e. it restricts to a local addition on each fibre $\alpha^{-1}(x)$ for $x \in M$. If the groupoid $\mathcal{G}$ admits such an addition, we deduce the following (Theorem 2.8):

Theorem A. *Suppose $\mathcal{G} = (G \rightrightarrows M)$ is a locally convex and locally metrisable Lie groupoid with M compact. If G admits an adapted local addition, then the group $\mathrm{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$. With this structure, $\mathrm{Bis}(\mathcal{G})$ is a locally convex Lie group modelled on a metrisable space.* ■

[1] A natural diffeology on the bisections of a diffeological groupoid would be the subspace diffeology of the functional diffeology on the space of smooth maps from the objects to the arrows.

After having constructed the Lie group structure on $\mathrm{Bis}(\mathcal{G})$ we show that a large variety of Lie groupoids admit adapted local additions, including all finite-dimensional Lie groupoids, all Banach Lie-groupoids with smoothly paracompact M and all locally trivial Lie groupoids with locally exponential vertex group.

We then determine the Lie algebra associated to $\mathrm{Bis}(\mathcal{G})$. Our investigation shows that the Lie algebra is closely connected to the Lie algebroid associated to the Lie groupoid. Explicitly, Theorem 3.4 may be subsumed as follows.

Theorem B. *Suppose $\mathcal{G}$ is a Lie groupoid which satisfies the assumptions of Theorem A. Then the Lie algebra of the Lie group $\mathrm{Bis}(\mathcal{G})$ is naturally isomorphic (as a topological Lie algebra) to the Lie algebra of sections of the Lie algebroid associated to $\mathcal{G}$, endowed with the negative of the usual bracket.* ■

After this we briefly discuss some perspectives for further research. We then investigate regularity properties of the Lie group $\mathrm{Bis}(\mathcal{G})$. To this end, recall the notion of regularity for Lie groups:

Let H be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the right translation $\rho_h \colon H \to H$, $x \mapsto xh$ by $h \in H$ to define $v.h := T_{\mathbf{1}}\rho_h(v) \in T_h H$ for $v \in T_{\mathbf{1}}(H) =: L(H)$. Following [Dah12] and [Glö12], H is called C^r*-regular* if for each C^r-curve $\gamma \colon [0,1] \to L(H)$ the initial value problem

$$\begin{cases} \eta'(t) &= \gamma(t).\eta(t) \\ \eta(0) &= \mathbf{1} \end{cases} \tag{1}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(\gamma) := \eta \colon [0,1] \to H$, and the map

$$\mathrm{evol} \colon C^r([0,1], L(H)) \to H, \quad \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

is smooth. If H is C^r-regular and $r \leq s$, then H is also C^s-regular. A C^∞-regular Lie group H is called *regular (in the sense of Milnor)* – a property first defined in [Mil84]. Every finite dimensional Lie group is C^0-regular (cf. [Nee06]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [Mil84], [Nee06], [Glö12], cf. also [KM97] and the references therein). We prove the following result (Theorems 4.1 and 4.5):

Theorem C. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid that admits a local addition and has compact space of objects M. Suppose either that G is a Banach-manifold or that $\mathcal{G}$ is locally trivial with locally exponential and C^k-regular vertex group. Then the Lie group $\mathrm{Bis}(\mathcal{G})$ is C^k-regular for each $k \in \mathbb{N}_0 \cup \{\infty\}$. In particular, the group $\mathrm{Bis}(\mathcal{G})$ is regular in the sense of Milnor.* ■

Note that all assumptions that we will impose throughout this paper are satisfied for finite-dimensional Lie groupoids over compact manifolds. For this case, the above theorems may be subsumed as follows:

Theorem D. *If $\mathcal{G} = (G \rightrightarrows M)$ is a finite-dimensional Lie groupoid with compact M, then $\mathrm{Bis}(\mathcal{G})$ is a regular Fréchet-Lie group modelled on the space of sections $\Gamma(\mathbf{L}(\mathcal{G}))$ of the Lie algebroid $\mathbf{L}(\mathcal{G})$. Moreover, the Lie bracket on $\Gamma(\mathbf{L}(\mathcal{G}))$ induced from the Lie group structure on $\mathrm{Bis}(\mathcal{G})$ is the negative of the Lie bracket underlying $\mathbf{L}(\mathcal{G})$.* ■

1 Locally convex Lie groupoids and Lie groups

In this section we recall the Lie theoretic notions and conventions that we are using in this paper. We refer to [Mac05] for an introduction to (finite-dimensional) Lie groupoids and the associated group of bisections. The notation for Lie groupoids and their structural maps also follows [Mac05]. However, we do not restrict our attention to finite dimensional Lie groupoids. Hence, we have to augment the usual definitions with several comments. Note that we will work all the time over a fixed base manifold M.

1.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a groupoid over M with source projection $\alpha \colon G \to M$ and target projection $\beta \colon G \to M$. Then $\mathcal{G}$ is a *(locally convex and locally metrisable) Lie groupoid over M*[2] if

[2] See Appendix A for references on differential calculus in locally convex spaces.

- the objects M and the arrows G are locally convex and locally metrisable manifolds,
- the smooth structure turns α and β into surjective submersions, i.e., they are locally projections[3]
- the partial multiplication $m\colon G \times_{\alpha,\beta} G \to G$, the object inclusion $1\colon M \to G$ and the inversion $\iota\colon G \to G$ are smooth. ■

The *group of bisections* $\operatorname{Bis}(\mathcal{G})$ of $\mathcal{G}$ is given as the set of sections $\sigma\colon M \to G$ of α such that $\beta \circ \sigma\colon M \to M$ is a diffeomorphism. This is a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M. \tag{2}$$

The object inclusion $1\colon M \to G$ is then the neutral element and the inverse element of σ is

$$\sigma^{-1}(x) := \iota(\sigma((\beta \circ \sigma)^{-1}(x))) \text{ for } x \in M. \tag{3}$$

Remark 1.2. a) The definition of bisection is not symmetric with respect to α and β. This lack of symmetry can be avoided by defining a bisection as a set (see [Mac05, p. 23]). This point of view is important for instance in Poisson geometry, where one wants to restrict the image of bisection to be Lagrangian submanifolds in a symplectic groupoid [Ryb01, Xu97]. However, we will not need this point of view in the present article.

b) Each bisection σ gives rise to a *left-translation* $L_\sigma\colon G \to G, g \mapsto \sigma(\beta(g))g$. The map

$$(\operatorname{Bis}(\mathcal{G}), \star) \to (\operatorname{Diff}(G), \circ), \sigma \mapsto L_\sigma$$

induces a group isomorphism onto the subgroup of all left translations (cf. [Mac05, p. 22]). Similarly we could identify the bisections with right translations on G

c) The group of bisections naturally acts on the arrows by

$$\gamma\colon \operatorname{Bis}(\mathcal{G}) \times G \to G, (\psi, g) \mapsto L_\psi(g) = \psi(\beta(g))g$$

(in fact, the group structure on $\operatorname{Bis}(\mathcal{G})$ is derived from this action, see [Mac05, §1.4] or [SW99, §15.3]). This action will play an important rôle when computing the Lie algebra of the Lie group of bisections in Section 3.

d) The group $\operatorname{Bis}(\mathcal{G})$ depends functorially on $\mathcal{G}$ in the following way. Suppose $\mathcal{H} = (H \rightrightarrows M)$ is another Lie groupoid over M and that $f\colon \mathcal{G} \to \mathcal{H}$ is a morphism of Lie groupoids over M, i.e., it is a smooth functor $f\colon G \to H$ which is the identity on the objects $f \circ 1_{\mathcal{G}} = 1_{\mathcal{H}}$. Then there is an induced morphism of the groups of bisections

$$\operatorname{Bis}(f)\colon \operatorname{Bis}(\mathcal{G}) \to \operatorname{Bis}(\mathcal{H}), \quad \sigma \mapsto f \circ \sigma.$$

If $\mathcal{K}$ is another Lie groupoid over M and $g\colon \mathcal{H} \to \mathcal{K}$ another morphism, then we clearly have $\operatorname{Bis}(g \circ f) = \operatorname{Bis}(g) \circ \operatorname{Bis}(f)$. Lie groupoids over M, together with their morphisms form a category $\mathsf{LieGroupoids}_M$. We can thus interpret Bis as a functor

$$\operatorname{Bis}\colon \mathsf{LieGroupoids}_M \to \mathsf{Groups}. \tag{4}$$

e) From $\mathcal{G} = (G \rightrightarrows M)$ we can construct a new Lie groupoid $T\mathcal{G} := (TG \rightrightarrows TM)$. This has the surjective submersions $T\alpha$ and $T\beta$ as source and target projections, $T1$ as object inclusion and $T\iota$ as inversion.

In order to define the multiplication we first have to identify $T(G \times_{\alpha,\beta} G)$ with $TG \times_{T\alpha,T\beta} G$. To this end we first recall that $G \times_{\alpha,\beta} G$ is the submanifold $\{(a,b) \in G \times G \mid \alpha(a) = \beta(b)\}$ of $G \times G$. We may

[3] This implies in particular that the occurring fibre-products are submanifolds of the direct products, see [Woc13, Appendix C].

thus identify $T(G \times_{\alpha,\beta} G)$ via the isomorphism $T(G\times G) \cong TG \times TG$ with a subset of $TG \times TG$. Now we claim that

$$T(G \times_{\alpha,\beta} G) = \{(x,y) \in TG \times TG \mid T\alpha(x) = T\beta(y)\} = TG \times_{T\alpha,T\beta} TG \tag{5}$$

as subsets (and thus as submanifolds) of $TG \times TG$. Note that the statement is a local one (we just have to find representing smooth curves), meaning that we may assume G and M to be diffeomorphic to open subsets $U \Subset X$ and $V \Subset Y$ for locally convex spaces X and Y. Since α and β are submersions we can also assume that $X = Z \times Y$, that $U = W \times V$, that $\alpha = \beta = \mathrm{pr}_2$ and that there are diffeomorphisms $\varphi\colon W \times V \to W \times V$ and $\psi\colon V \to V$ that make

$$\begin{array}{ccccccc}
G & \xrightarrow{\cong} & W \times V & \xrightarrow{\varphi} & W \times V & \xleftarrow{\cong} & G \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \beta} \\
M & \xrightarrow{\cong} & V & \xrightarrow{\psi} & V & \xleftarrow{\cong} & M
\end{array}$$

commute. In particular, we have $\varphi(w,v) = (\varphi_1(w,v), \psi(v))$. By composing φ with the diffeomorphism $(x,y) \mapsto (x, \psi^{-1}(y))$ we may assume that ψ is the identity. For the inner square we then have

$$T(W \times V \times_{\mathrm{pr}_2,\mathrm{pr}_2} W \times V) = \{((w,z,v,y),(w',z',v',y')) \in (W \times Z \times V \times Y)^2 \mid v = v' \text{ and } y = y'\}.$$

Since $T\,\mathrm{pr}_2 = \mathrm{pr}_2 \times \mathrm{pr}_2$ one sees that for the inner square we also have

$$T(W\times V) \times_{T\,\mathrm{pr}_2, T\,\mathrm{pr}_2} T(W \times V) = \{((w,z,v,y),(w',z',v',y')) \in (W \times Z \times V \times Y)^2 \mid v = v' \text{ and } y = y'\}.$$

This shows that both sides in (5) are actually the same. We thus may set

$$T\mu\colon TG \times_{T\alpha,T\beta} TG \to G$$

with respect this identification. One can easily check that yields in fact a new Lie groupoid $T\mathcal{G}$. ∎

We now recall the construction of the Lie algebroid associated to a Lie groupoid.

1.3. We consider the subset $T^\alpha G = \bigcup_{g\in G} T_g\alpha^{-1}\alpha(g)$ of TG. Note that for all $x \in T^\alpha_g G$ the definition implies $T\alpha(x) = 0_{\alpha(g)} \in T_{\alpha(g)}M$, i.e. fibre-wise we have $T^\alpha_g G = \ker T_g\alpha$. Since α is a submersion, the same is true for $T\alpha$. Computing in submersion charts, the kernel of $T_g\alpha$ is a direct summand of the model space of TG. Furthermore, the submersion charts of $T\alpha$ yield submanifold charts for $T^\alpha G$ whence $T^\alpha G$ becomes a split submanifold of TG. Restricting the projection of TG, we thus obtain a subbundle $\pi_\alpha\colon T^\alpha G \to G$ of the tangent bundle TG. ∎

1.4. We now recall the construction of the Lie algebroid $\mathbf{L}(\mathcal{G})$ associated to a Lie groupoid $\mathcal{G}$. The vector bundle underlying $\mathbf{L}(\mathcal{G})$ is the pullback $1^*T^\alpha G$ of the bundle $T^\alpha G$ via the embedding $1\colon M \to G$. We denote this bundle also by $\mathbf{L}(G) \to M$. The anchor $a_{\mathbf{L}(\mathcal{G})}\colon \mathbf{L}(G) \to TM$ is the composite of the morphisms

$$\mathbf{L}(G) \to T^\alpha G \xrightarrow{\subseteq} TG \xrightarrow{T\beta} TM$$

To describe the Lie bracket on $\Gamma(\mathbf{L}(G))$ we need some notation: Let g be an element of G. We define the smooth map $R_g\colon \alpha^{-1}(\beta(g)) \to G$, $h \mapsto hg$. A vertical vector field $Y \in \Gamma(T^\alpha G)$ is called *right-invariant* if for all $(h,g) \in G \times_{\alpha,\beta} G$ the equation $Y(hg) = T_h(R_g)(Y(h))$ holds. We denote the Lie subalgebra of all right invariant vector fields on G by $\Gamma^\rho(T^\alpha G)$. Then [Mac05, Corollary 3.5.4] shows that the assignment

$$\Gamma(\mathbf{L}(\mathcal{G})) \to \Gamma^\rho(T^\alpha G), \quad X \mapsto \overrightarrow{X}, \quad \text{with} \quad \overrightarrow{X}(g) = T(R_g)(X(\beta(g))) \tag{6}$$

is an isomorphism of $C^\infty(G)$-modules. Its inverse is given by $\Gamma^\rho(T^\alpha G) \to \Gamma(\mathbf{L}(\mathcal{G}))$, $X \mapsto X \circ 1$. Now we define the Lie bracket on $\Gamma(\mathbf{L}(\mathcal{G}))$ via

$$[X,Y] := \left[\overrightarrow{X}, \overrightarrow{Y}\right] \circ 1. \tag{7}$$

Then the *Lie algebroid* $\mathbf{L}(\mathcal{G})$ *of* $\mathcal{G}$ is the vector bundle $\mathbf{L}(G) \to M$ together with the bracket $[\cdot,\cdot]$ from (7) and the anchor $a_{\mathbf{L}(\mathcal{G})}$. ∎

1.5. To fully describe the Lie functor on Lie groupoids, suppose that $\mathcal{H} = (H \rightrightarrows M)$ is another Lie groupoid over M and that $f\colon G \to H$ is a smooth functor satisfying $f \circ 1_{\mathcal{G}} = 1_{\mathcal{H}}$. Then $Tf(T^\alpha G) \subseteq T^\alpha H$ and from $f \circ 1_{\mathcal{G}} = 1_{\mathcal{H}}$ it follows that Tf induces a morphism $1_{\mathcal{G}}^* T^\alpha G \to 1_{\mathcal{H}}^* T^\alpha H$ of vector bundles. This morphism is in fact a morphism of Lie algebroids [Mac05, Proposition 3.5.10], which we denote by $\mathbf{L}(f)\colon \mathbf{L}(\mathcal{G}) \to \mathbf{L}(\mathcal{H})$. In total, this defines the Lie functor

$$\mathbf{L}\colon \mathsf{LieGroupoids}_M \to \mathsf{LieAlgebroids}_M. \qquad \blacksquare$$

We now turn to the Lie functor defined on the category of locally convex Lie groups (cf. [Nee06, Mil84]).

1.6. Let H be a locally convex Lie group, i.e., a locally convex manifold which is a group such that the group operations are smooth. The Lie algebra $\mathbf{L}(H)$ of H is the tangent space $T_1 H$ endowed with a suitable Lie bracket $[\cdot,\cdot]$ (cf. [Nee06, Definition II1.5], [Mil84, §5]). To obtain the bracket, we identify $T_1 H$ with the Lie algebra of left invariant vector fields $\Gamma^\lambda(H)$. Each element $X \in T_1 H$ extends to a (unique) left invariant vector field

$$X^\lambda \in \Gamma(TH) \quad \text{via} \quad X^\lambda(h) = T_1\lambda_h(X).$$

Here λ_h is the left translation in H by the element h. Similarly, to X there corresponds a unique right invariant vector field X^ρ. Since the bracket of left invariant vector fields is left invariant and the bracket of right invariant vector fields is right invariant there are now two ways of endowing $T_1 H$ with a Lie bracket. The convention here is to define the bracket $T_1 H$ via *left* invariant vector field. Thus $X \mapsto X^\lambda$ becomes an isomorphism of Lie algebras and $X \mapsto X^\rho$ becomes an anti-isomorphism of Lie algebras, i.e., we have $-[X,Y]^\rho = [X^\rho, Y^\rho]$ ([Mil84, Assertion 5.6]). ■

1.7. Suppose H, H' are locally convex Lie groups and $f\colon H \to H'$ is a smooth group homomorphism. Then $T_1 f\colon T_1 H \to T_1 H'$ is a continuous and linear map which preserves the Lie bracket. This defines the morphism $\mathbf{L}(f)\colon \mathbf{L}(H) \to \mathbf{L}(H')$ of topological Lie algebras associated to f. In total, we obtain this way the Lie functor

$$\mathbf{L}\colon \mathsf{LieGroups} \to \mathsf{LieAlgebras}. \qquad \blacksquare$$

Warning. *Each Lie group H gives rise to a Lie groupoid $(H \rightrightarrows *)$ over the point $*$ and each Lie algebra $\mathfrak{h}$ gives rise to a Lie algebroid $\mathfrak{h} \to *$. However, with the above convention the Lie algebroid $\mathbf{L}(H) \to *$ is* not *isomorphic to the Lie algebroid $\mathbf{L}(H \rightrightarrows *)$. It rather is anti isomorphic. This is an annoying but unavoidable fact if one wants to stick to the usual and natural conventions.* ■

We will now line out one main example that will be developed throughout the text to illustrate our results.

Example 1.8. Let $\pi\colon P \to M$ be a principal H-bundle. Then the gauge groupoid $\frac{P\times P}{H} \rightrightarrows M$ is defined as follows. The manifold of objects is M and the manifold of arrows is the quotient of $P \times P$ by the diagonal action of H. We denote by $\langle p, q\rangle$ the equivalence class of (p, q) in $(P \times P)/H$.

For later reference we recall the construction of charts for $\frac{P\times P}{H}$. In order to obtain manifold charts for $(P \times P)/H$, let $(U_i)_{i\in I}$ be an open cover of M such that there exist smooth local sections $\sigma_i\colon U_i \to P$ of π. This yields an atlas $(U_i, \kappa_i)_{i\in I}$ of local trivialisations of the bundle $\pi\colon P \to M$ which are given by

$$\kappa_i\colon \pi^{-1}(U_i) \to U_i \times H, \quad p \mapsto (\pi(p), \delta(\sigma_i(\pi(p)), p))$$

with $\delta\colon P \times_\pi P \to H$, $(p, q) \mapsto p^{-1} \cdot q$. Here we use $p^{-1} \cdot q$ as the suggestive notation for the element in $h \in H$ that satisfies $p.h = q$ (whereas p^{-1} alone has in general no meaning).

The local trivialisations commute with the right H-action on P since

$$\kappa_i(p.h) = (\pi(p.h), \delta(\sigma_i(\pi(p.h)), p.h) = (\pi(p), \delta(\sigma_i(\pi(p)), p) \cdot h.$$

In particular, the trivialisations descent to manifold charts for the arrow manifold of the gauge groupoid:

$$K_{ij}\colon \frac{\pi^{-1}(U_i) \times \pi^{-1}(U_j)}{H} \to U_i \times U_j \times H, \quad \langle p_1, p_2\rangle \mapsto (\pi(p_1), \pi(p_2), \delta(\sigma_i(\pi(p_1)), p_1)\delta(\sigma_j(\pi(p_2)), p_2)^{-1}).$$

One then easily checks that $\alpha(\langle p,q\rangle) := \pi(p)$, $\beta(\langle p,q\rangle) := \pi(q)$, $1(x) = \langle \sigma_i(x), \sigma_i(x)\rangle$ if $x \in U_i$ and

$$m(\langle p,q\rangle, \langle v,w\rangle) := \langle p, w \cdot \delta(v,q)\rangle$$

complete the definition of a Lie groupoid.

The associated Lie algebroid is naturally isomorphic to the Atiyah algebroid $TP/H \to M$. If we identify sections of $TP/H \to M$ with right-invariant vector fields on P, then the bracket is the usual bracket of vector fields on P and the anchor is induced by $T\pi$. To see that this is naturally isomorphic to $\mathbf{L}(\frac{P\times P}{H} \rightrightarrows M)$ it is a little more convenient to identify M with the submanifold $\frac{\Delta P}{H}$ via the diffeomorphism $\frac{\Delta P}{H} \cong \frac{P}{H} \cong M$. Here, ΔP denotes the diagonal $\Delta P \subseteq P \times P$. Then we have $\alpha(\langle p,q\rangle) = \langle p,p\rangle$, $\beta(\langle p,q\rangle) = \langle q,q\rangle$. Consequently, $T^\alpha \frac{P\times P}{H} = \frac{0_P \times TP}{H}$ and thus we have the natural isomorphism

$$\mathbf{L}\left(\frac{P\times P}{H}\right) = T^\alpha \left.\frac{P\times P}{H}\right|_{\frac{\Delta P}{H}} = \left.\frac{0_P \times TP}{H}\right|_{\frac{\Delta P}{H}} \cong TP/H$$

of vector bundles over M. One easily checks that this is in fact a morphism of Lie algebroids. ■

2 The Lie group structure on the bisections

It is the task of this section to lift the functor from (4) to a functor that takes values in locally convex Lie groups. Our main technical tool for understanding the Lie group structure on $\mathrm{Bis}(\mathcal{G})$ will be local additions (cf. Definition A.4) which respect to the fibres of a submersion. This is an adaptation of the construction of manifold structures on mapping spaces [Woc13, KM97, Mic80] (see also Appendix A). In particular, special cases of our constructions are covered by [Mic80, Chapter 10], but we aim here at a greater generality. In the end, we will have to restrict the functor from (4) to those Lie groupoids that admit such a local addition.

Definition 2.1. (cf. [Mic80, 10.6]) Let $s\colon Q \to N$ be a surjective submersion. Then a *local addition adapted to* s is a local addition $\Sigma\colon U \Subset TQ \to Q$ such that the fibres of s are additively closed with respect to Σ, i.e. $\Sigma(v_q) \in s^{-1}(s(q))$ for all $q \in Q$ and $v_q \in T_q s^{-1}(s(q))$ (note that $s^{-1}(s(q))$ is a submanifold of Q). ■

We will mostly be interested in local addition which respect the source projection of a Lie groupoid.

Lemma 2.2. *If $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid with a local addition adapted to the source projection α, then there exists a local addition adapted to the target projection β.*

Proof. Let $\Sigma\colon U \Subset TG \to G$ be a local addition adapted to α. Recall that the inversion map $\iota\colon G \to G$ is a diffeomorphism. Hence the tangent $T\iota$ is a diffeomorphism mapping the zero-section in TG to itself. In particular, $T\iota(U)$ is an open neighbourhood of the zero-section in TG. Define $\Sigma^{\mathrm{op}}\colon T\iota(U) \Subset TG \to G$ via $\Sigma^{\mathrm{op}} = \iota \circ \Sigma \circ T\iota$ and observe that $\Sigma^{\mathrm{op}}(0_g) = g$ holds for all $g \in G$. Now Σ being a local addition implies that $(\pi|_{T\iota(U)}, \Sigma^{\mathrm{op}})\colon T\iota(U) \Subset TG \to G \times G$ induces a diffeomorphism onto an open neighbourhood of the diagonal, whence Σ^{op} is a local addition. To prove that Σ^{op} is adapted to β we use that ι intertwines α and β, i.e. $\beta = \alpha \circ \iota$. Thus ι maps each submanifold $\beta^{-1}(\beta(g))$ to $\alpha^{-1}(\beta(g))$ and thus $T\iota(T_g\beta^{-1}(\beta(g))) = T_{g^{-1}}\alpha^{-1}(\beta(g))$. As Σ is adapted to α, we can deduce easily from these facts that Σ^{op} is adapted to β. ■

Remark 2.3. Exchanging the rôle of α and β in the proof of Lemma 2.2, we see that a local addition adapted to the source projection exists if and only if a local addition adapted to the target projection exists. ■

Definition 2.4. We say that that a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ admits an *adapted local addition* if G admits a local addition which is adapted to the source projection α (or, equivalently, to the target projection β). We denote the full subcategory of $\mathsf{LieGroupoids}_M$ of Lie groupoids over M that admit an adapted local addition by $\mathsf{LieGroupoids}_M^\Sigma$. ■

In the following, all manifolds of smooth mappings are endowed with the smooth compact-open topology from A.6 and the manifold structure from Theorem A.8. If a Lie groupoid admits an adapted local addition, then we can prove the following result. Note that if the manifold $G \times G$ admits tubular neighbourhoods for embedded submanifolds, then the assertion of the next lemma is a special case of [Mic80, Proposition 10.8].

Lemma 2.5. *Suppose K is a compact manifold and $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid which admits an adapted local addition Σ. Then $C^\infty(K, G \times_{\alpha,\beta} G)$ is a submanifold of $C^\infty(K, G \times G)$.*

Proof. Since α is a submersion the fibre product $G \times_{\alpha,\beta} G$ is a split submanifold of $G \times G$. By Lemma 2.2 G admits a local addition Σ adapted to α and a local addition Σ^{op} adapted to β. We will identify $T(G \times_{\alpha,\beta} G)$ with $TG \times_{T\alpha,T\beta} TG$ as in Remark 1.2 e). Then we obtain a local addition $\Sigma^{\mathrm{prod}} := \Sigma \times \Sigma^{\mathrm{op}}$ on $T(G \times G)$. Since Σ is adapted to α and Σ^{op} is adapted to β, the local addition Σ^{prod} restricts to a local addition on the submanifold $G \times_{\alpha,\beta} G$ (i.e. the submanifold is additively closed).

Now let $g \in C^\infty(K, G \times_{\alpha,\beta} G) \subseteq C^\infty(K, G \times G)$. We consider the chart (O_g, φ_g) for g on $C^\infty(K, G \times G)$ which is induced by Σ^{prod}. As $G \times_{\alpha,\beta} G$ is additively closed with respect to Σ^{prod} we derive the condition:

$$f \in C^\infty(K, G \times_{\alpha,\beta} G) \cap O_g \Leftrightarrow \varphi_g(f) = (\pi_{T(G\times G)}, \Sigma^{\mathrm{prod}})^{-1}(g, f) \in \Gamma(g^*T(G \times_{\alpha,\beta} G))$$

Notice that the linear subspace $\Gamma(g^*T(G \times_{\alpha,\beta} G))$ is closed. This follows from the continuity of the point evaluations $\mathrm{ev}_x \colon \Gamma(g^*T(G \times G)) \to T(G \times G), f \mapsto f(x)$ (cf. [AS12, Proposition 3.20]): As the vector subspace $T_y(G \times_{\alpha,\beta} G)$ is complemented (thus closed) for all $y \in G \times_{\alpha,\beta} G$, we can write $\Gamma(g^*T(G \times_{\alpha,\beta} G))$ as an intersection of closed subspaces $\bigcap_{x\in K}((\pi^*_{TG\times G} g) \circ \mathrm{ev}_x)^{-1}(T_{g(x)} G \times_{\alpha,\beta} G)$. Thus the canonical chart restricts to a submanifold chart $\varphi_g \colon O_g \cap C^\infty(K, G \times_{\alpha,\beta} G) \to \Gamma(g^*T(G \times_{\alpha,\beta} G)) \subseteq \Gamma(g^*T(G \times G))$. ∎

We now use adapted local additions to endow the sections of a submersion with a smooth manifold structure.

Proposition 2.6. *Let N be a locally convex and locally metrisable manifold and K be a compact manifold. Furthermore, let $s \colon N \to K$ be a submersion. If there exists a local addition $\Sigma \colon U \Subset TN \to N$ adapted to s, then the set*

$$\Gamma(K \xleftarrow{s} N) := \{\sigma \in C^\infty(K, N) \mid s \circ \sigma = \mathrm{id}_K\}$$

is a submanifold of $C^\infty(K, N)$. Furthermore, the model space of an open neighbourhood of $\sigma \in \Gamma(K \xleftarrow{s} N)$ is the closed subspace

$$E_\sigma := \{\gamma \in \Gamma(\sigma^*TN) \mid \forall x \in K,\ \gamma(x) \in T_{\sigma(x)} s^{-1}(x)\}$$

of all vertical sections in $\Gamma(\sigma^*TN)$.

Proof. Endow $C^\infty(K, N)$ with the manifold structure from Theorem A.8.b) constructed with respect to the local addition Σ that is adapted to s. We claim that for a section σ of the submersion s the canonical charts $(\varphi_\sigma, O_\sigma)$ of $C^\infty(K, N)$ from A.8.a) define submanifold charts for $\Gamma(K \xleftarrow{s} N)$. To see this, consider $g \in O_\sigma$ and recall $\varphi_\sigma(g) = (\pi_{TN}, \Sigma)^{-1} \circ (\sigma, g)$. Since the local addition Σ is adapted to s, the formula for φ_σ shows that

$$g \in \Gamma(K \xleftarrow{s} N) \cap O_\sigma \Leftrightarrow \varphi_\sigma(g) \in E_\sigma \cap C^\infty(K, U),$$

where $U \Subset TN$ is as is Theorem A.8. For $x \in K$ we define the evaluation map $\mathrm{ev}_x \colon \Gamma(\sigma^*TN) \to TN, f \mapsto f(x)$. It is easy to see that the evaluation maps are continuous, since they are continuous in each chart (cf. [AS12, Proposition 3.20]). The vector subspace $E_\sigma \subseteq \Gamma(\sigma^*TN)$ is thus closed as an intersection of closed subspaces $E_\sigma = \bigcap_{x\in K}((\pi^*_{TN}\sigma) \circ \mathrm{ev}_x)^{-1}(T_{f(x)} s^{-1}(x))$. In particular, φ_σ is a submanifold chart and the assertion follows. ∎

Remark 2.7. In certain cases the submanifold $\Gamma(K \xleftarrow{s} N)$ constructed in Proposition 2.6 will be a split submanifold of $C^\infty(K, N)$. For example this will happen if N is a finite dimensional manifold (see [Mic80, Proposition 10.10]). The same proof carries over to the following slightly more general setting: If N is a manifold such that for each embedded submanifold $Y \subseteq N$ there exists a tubular neighbourhood, then $\Gamma(K \xleftarrow{s} N)$ is a split submanifold of $C^\infty(K, N)$. ∎

Using this manifold structure we can finally prove the first main result of this paper.

Theorem 2.8. *Suppose M is compact and $\mathcal{G} = (G \rightrightarrows M)$ is a locally convex and locally metrisable Lie groupoid over M which admits an adapted local addition. Then $\mathrm{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M,G)$ (with the manifold structure from Theorem A.8). Moreover, the induced manifold structure and the group multiplication*

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M$$

turn $\mathrm{Bis}(\mathcal{G})$ into a Lie group modelled on

$$E_1 := \{\gamma \in C^\infty(M, TG) \mid \forall x \in M,\ \gamma(x) \in T_{1_x} s^{-1}(x)\}. \qquad \blacksquare$$

Note that E_1 is in fact isomorphic to the space of sections of the Lie algebroid $\mathbf{L}(\mathcal{G})$ associated to $\mathcal{G}$. It will be the content of Section 3 to analyse this isomorphism and show that it is natural and respects the Lie bracket (up to a sign).

Proof (of Theorem 2.8). In Proposition 2.6 we endowed the space of sections $\Gamma(M \overset{\alpha}{\leftarrow} G)$ with the structure of a submanifold of $C^\infty(M,G)$. Observe that by Theorem A.8 e) the map

$$C^\infty(M,G) \to C^\infty(M,M), \quad f \mapsto \beta \circ f$$

is smooth, and so is its restriction β_* to $\Gamma(M \overset{\alpha}{\leftarrow} G)$. Recall from [Mic80, Corollary 5.7] that the subset of all diffeomorphisms $\mathrm{Diff}(M) \subseteq C^\infty(M,M)$ is open. By construction $\mathrm{Bis}(\mathcal{G}) = (\beta_*)^{-1}(\mathrm{Diff}(M))$ is thus an open submanifold of $\Gamma(M \overset{\alpha}{\leftarrow} G)$, and thus also a submanifold of $C^\infty(M,G)$.

As $\Gamma(M \overset{\alpha}{\leftarrow} G)$ is locally metrisable by Proposition 2.6, so is $\mathrm{Bis}(\mathcal{G})$. Thus all we have to show is that the group operations (2) and (3) of the group of bisections are smooth with respect to the submanifold structure. We begin with the group multiplication.

By Theorem A.8 e) the maps $\beta_* \colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$ and $m_* \colon C^\infty(M, G \times_{\alpha,\beta} G) \to C^\infty(M,G)$ are smooth. Furthermore, by Theorem A.8 f) the composition $\mathrm{Comp} \colon C^\infty(M,G) \times C^\infty(M,M) \to C^\infty(M,G)$ is smooth. Hence for $\sigma, \tau \in \mathrm{Bis}(\mathcal{G})$ the map

$$\mu \colon \mathrm{Bis}(\mathcal{G})^2 \to C^\infty(M,G)^2, (\sigma,\tau) \mapsto (\sigma \circ \beta \circ \tau, \tau) = (\mathrm{Comp}(\sigma, \beta_*(\tau)), \tau)$$

is smooth. Let $\Delta \colon M \to M \times M$ be the diagonal map. The canonical identification

$$h \colon C^\infty(M,G)^2 \to C^\infty(M, G \times G), \quad (f,g) \mapsto (f,g) \circ \Delta.$$

is a diffeomorphism by an argument analogous to [Mic80, Proposition 10.5]. Indeed, the proof carries over verbatim to our setting of infinite dimensional locally convex manifolds since G admits a local addition. Observe that the map $h \circ \mu$ takes its image in the submanifold $C^\infty(M, G \times_{\alpha,\beta} G)$ (cf. Lemma 2.5). Hence we can rewrite the multiplication formula (2) for $\sigma, \tau \in \mathrm{Bis}(\mathcal{G})$ as a composition of smooth maps:

$$\sigma \star \tau = (\sigma \circ \beta \circ \tau) \cdot \tau = m_* \circ h \circ \mu(\sigma, \tau).$$

In conclusion the group multiplication is smooth with respect to the manifold structure on $\mathrm{Bis}(\mathcal{G})$.

We are left to prove that inversion in $\mathrm{Bis}(\mathcal{G})$ is smooth. To this end let us recall the formula (3) for the inverse of $\sigma \in \mathrm{Bis}(\mathcal{G})$:

$$\sigma^{-1} = (\sigma \circ (\beta \circ \sigma)^{-1})^{-1} = \iota_* \, \mathrm{Comp}(\sigma, (\beta_*(\sigma))^{-1})$$

Here $\iota \colon G \to G$ is the inversion in the groupoid, whence $\iota_* \colon C^\infty(M,G) \to C^\infty(M,G)$ is smooth by Theorem A.8 e). Furthermore, β_* maps $\mathrm{Bis}(\mathcal{G})$ into the open submanifold $\mathrm{Diff}(M)$. Inversion of $\beta_*(\sigma)$ in (3) is thus inversion in the group $\mathrm{Diff}(M)$. The group $\mathrm{Diff}(M)$ is a Lie group with respect to the open submanifold structure induced by $C^\infty(M,M)$ (see [Mic80, Theorem 11.11]). We conclude from (3) that inversion in the group $\mathrm{Bis}(\mathcal{G})$ is smooth and thus $\mathrm{Bis}(\mathcal{G})$ is a Lie group. $\blacksquare$

Proposition 2.9. *Suppose $\mathcal{G} = (G \rightrightarrows M)$ and $\mathcal{H} = (H \rightrightarrows M)$ are Lie groupoids over the compact manifold M and that $\mathcal{G}$ and $\mathcal{H}$ admit an adapted local addition. If then $f \colon \mathcal{G} \to \mathcal{H}$ is a morphism of Lie groupoids over M, then*

$$\mathrm{Bis}(f) \colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{H}), \quad \sigma \mapsto f \circ \sigma$$

is a smooth morphism of Lie groups.

Proof. The map $f_*\colon C^\infty(M,G) \to C^\infty(M,H)$, $\gamma \mapsto f \circ \gamma$ is smooth by Theorem A.8 e). Thus its restriction to the submanifold $\Gamma(M \overset{\alpha}{\leftarrow} G)$ and in there to the open subset $\mathrm{Bis}(\mathcal{G})$ is smooth. That it is a group homomorphism follows directly from the definition. ■

Remark 2.10. Suppose M is a compact manifold. Then we consider the full subcategory $\mathsf{LieGroupoids}_M^\Sigma$ of $\mathsf{LieGroupoids}_M$ whose objects are locally convex and locally metrisable Lie groupoids with object space M that admit an adapted local addition. Then the results of this section show that Bis may be regarded as a functor

$$\mathrm{Bis}\colon \mathsf{LieGroupoids}_M^\Sigma \to \mathsf{LieGroups},$$

where $\mathsf{LieGroups}$ denotes the category of locally convex Lie groups. ■

Proposition 2.11. *Under the assumptions from Theorem 2.8, the natural action*

$$\gamma\colon \mathrm{Bis}(\mathcal{G}) \times G \to G, \quad (\psi, g) \mapsto \psi(\beta(g))g.$$

is smooth, as well as the restriction of this action to the α-fibre

$$\gamma_m\colon \mathrm{Bis}(\mathcal{G}) \times \alpha^{-1}(m) \to \alpha^{-1}(m), \quad (\psi, g) \mapsto \psi(\beta(g))g$$

for each $m \in M$.

Proof. The action γ is given as the composition $\gamma(\psi, g) = m(\mathrm{ev}(\psi, \beta(g)), g)$ for $(\psi, g) \in \mathrm{Bis}(\mathcal{G}) \times G$. Here $\mathrm{ev}\colon \mathrm{Bis}(\mathcal{G}) \times M \to G$ is the canonical evaluation map, which is smooth by A.8 d). Thus the action γ is smooth as a composition of smooth maps.

From $\alpha(\psi(\beta(g))g) = \alpha(g)$ it follows that the action γ preserves the α-fibres. Since α is a submersion, $\alpha^{-1}(m)$ is a submanifold. Consequently, the action restricted to $\alpha^{-1}(m)$ is also smooth. ■

We will now give examples for Lie groupoids which admit an adapted local addition. Hence the following classes of Lie groupoids we can apply the previous results of this section to.

Proposition 2.12. *Suppose $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid such that G, M are Banach manifolds and M admits smooth partitions of unity. Then $\mathcal{G}$ admits an adapted local addition.*

The statement suggest similarities to [Ryb02, Proposition 3.2]. However, the difference between [Ryb02, Proposition 3.2] and the previous proposition is that we will construct an adapted local addition on all of G, whereas in [Ryb02, Proposition 3.2] a local addition is only constructed on a neighbourhood of M in G. This allows us to view $\mathrm{Bis}(\mathcal{G})$ as a *submanifold* of $C^\infty(M, G)$ and to use the known results on mapping spaces, rather than constructing an auxiliary manifold structure on $\mathrm{Bis}(\mathcal{G})$.

Proof (of Proposition 2.12). We first recall some concepts from [Lan95, §IV.3]. Suppose X is an arbitrary Banach manifold and write $\pi_{TX}\colon TX \to X$ for the projection of the tangent bundle. Then a vector field $TX \to T(TX)$ on TX is said to be of *second order* if

$$T(\pi_{TX})(F(v)) = v \tag{8}$$

holds for all $v \in TX$. For each $s \in \mathbb{R}$ we denote by $s_{TX}\colon TX \to TX$ the vector bundle morphism which is given in each fibre by multiplication with s. With this notion fixed we define a second order vector field $F\colon TX \to T(TX)$ to be a *spray* if

$$F(s \cdot v) = T(s_{TX})(s \cdot F(v)) \tag{9}$$

holds for all $s \in \mathbb{R}$ and all $v \in TX$. For each $v \in TX$ there exists a unique integral curve β_v of F with initial condition v. The subset $W \subseteq TX$ such that β_v is defined on $[0,1]$ is an open neighbourhood of the zero section in TX. Then the *exponential map* of F is defined to be

$$\exp_F\colon W \to X, \quad v \mapsto \pi_{TX}(\beta_v(1)).$$

The restriction of $\exp_F$ to T_xX is for each $x \in X$ a local diffeomorphism at 0_x. By the Inverse Function Theorem there exists $U_x \Subset TX \cap W$ with $0_x \in U_x$ such that $(\pi \times \exp_F)|_{U_x}$ is a diffeomorphism onto its image. Consequently, we obtain a local addition

$$\Sigma_F \colon U := \cup_{x\in X} U_x \to X, \quad v \mapsto \exp_F(v).$$

Thus sprays are the key to constructing local additions on Banach manifolds. We now assume that this manifold is the manifold of arrows of the Lie groupoid $(G \rightrightarrows M)$. For the rest of this proof we identify M with the submanifold $1(M)$ of G and TM with the submanifold $T1(TM)$ of TG. There exists a spray $F\colon TM \to T(TM)$ on M by [Lan95, Theorem IV.3.1]. With respect to the mentioned identifications, we may interpret F as a section of $T^{T\alpha}TG\big|_{TM} \to TM$ that satisfies (8) and (9) for all $v \in TM$ and all $s \in \mathbb{R}$.

We now want to extend this spray by right translation. To this end, recall from Remark 1.2 e) that $T\mathcal{G} = (TG \rightrightarrows TM)$ is also a Lie groupoid, where we take the tangent maps at all levels.

Claim: The vertical right-invariant extension $\overrightarrow{F}\colon TG \to T^{T\alpha}TG$ of F is a spray satisfying $\overrightarrow{F}(v) \in T^{T\alpha}TG$ for all $v \in TG$.

We first establish (8) for all $v \in TG$. To this end we compute

$$\begin{aligned} T(\pi_{TG})(\overrightarrow{F}(v)) &= T(\pi_{TG})(TR_v(F(T\beta(v)))) = T(\pi_{TG}\circ Tm)(F(T\beta(v)),0_v) \\ &= T(m \circ \pi_{TG\times TG})(F(T\beta(v)),0_v) = Tm(T\pi_{TG}(F(T\beta(v))),v) = Tm(T\beta(v),v) = v, \end{aligned}$$

where we have used $T\pi_{TG}(F(T\beta(v))) = T\beta(v)$ since $T\beta(v) \in TM$, that $T\beta(v)$ is an identity in $(TG \rightrightarrows TM)$ and that

$$\begin{array}{ccc} TG \times_{T\alpha,T\beta} TG & \xrightarrow{Tm} & TG \\ \downarrow{\scriptstyle \pi_{TG\times TG}} & & \downarrow{\scriptstyle \pi_{TG}} \\ G \times_{\alpha,\beta} G & \xrightarrow{m} & G \end{array}$$

commutes. That (9) holds for all $s \in \mathbb{R}$ and all $v \in TG$ follows from the linearity of the tangent maps on each tangent space and the equality $R_{sv} \circ s_{TG} = s_{TG} \circ R_v$, which imply

$$\begin{aligned} \overrightarrow{F}(s\cdot v) &= TR_{sv}(F(T\beta(s\cdot v))) = TR_{sv}(F(s\cdot T\beta(v))) = TR_{sv}(Ts_{TG}(s\cdot F(T\beta(v)))) \\ &= Ts_{TG}(TR_v(s\cdot F(T\beta(v)))) = Ts_{TG}(s\cdot TR_v(F(T\beta(v)))) = Ts_{TG}(s\cdot \overrightarrow{F}(v)). \end{aligned}$$

Claim: The local addition $\Sigma_{\overrightarrow{F}}$ constructed from $\overrightarrow{F}$ is adapted to the source projection α.

Suppose $\beta_v\colon [0,1] \to TG$ is an integral curve for $\overrightarrow{F}$. By definition we have

$$\alpha(\Sigma_{\overrightarrow{F}}(v)) = \alpha(\pi_{TG}(\beta_v(1))) = \pi_{TM}(T\alpha(\beta_v(1))).$$

Thus it suffices by [Lan95, Proposition 2.11] to check that $T\alpha \circ \beta_v\colon [0,1] \to TM$ is an integral curve for the zero vector field. The latter is in fact the case since we have

$$(T\alpha\circ\beta_v)'(t) = T\left(T\alpha(\beta_v'(t))\right) = T\left(T\alpha(\underbrace{\overrightarrow{F}(\beta_v(t))}_{\in \ker T_{\beta_v(t)}T\alpha})\right) = 0.$$

Putting these two proven claims together establishes the proof of the proposition. ■

Remark 2.13. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Lie groupoid, i.e., one for which $(\beta,\alpha)\colon G \to M\times M$ is a surjective submersion. Then $\mathcal{G}$ is equivalent over M to the gauge groupoid of a principal bundle (the argument from [Mac05, §1.3] carries verbatim over to our more general setting). Thus the following proposition implies that each locally trivial Lie groupoid with locally exponential vertex group and finite-dimensional space of objects admits a local addition. ■

Proposition 2.14. *Let $\pi\colon P \to M$ be a principal H-bundle with finite-dimensional base M and locally exponential structure group H. Then the associated gauge groupoid $\frac{P\times P}{H} \rightrightarrows M$ admits an adapted local addition.*

Proof. We begin with the construction of a local addition for the Lie group H. As H is locally exponential, the exponential map $\exp_H\colon L(H) \to H$ restricts to a diffeomorphism on a zero-neighbourhood. Fix a convex zero-neighbourhood $W \Subset L(H) = T_eH$ (the tangent space at the identity $e \in H$) together with an identity-neighbourhood $V \Subset H$ such that $\psi := (\exp_H|_W^V)^{-1}\colon V \to W$ becomes a manifold chart for H. By construction this chart satisfies the following properties:

a) $\psi(e) = 0$

b) If $x \in V$ and for $k \in H$ the product kxk^{-1} is also contained in V, then $\psi(kxk^{-1}) = \mathrm{Ad}(k)(\psi(x))$ holds. Here $\mathrm{Ad}\colon H \to \mathrm{Aut}(L(H))$ is the adjoint representation of H on its Lie-algebra.

Let m_H be the group multiplication in H and $\lambda_h = m_H(h,\cdot)$, $\rho_h = m_H(\cdot\,,h)$ for $h \in H$. The tangent bundle TH of the Lie group H admits the following trivialisation $\Phi\colon H\times T_eH \to TH, (h,V) \mapsto h.V = 0_h\cdot V = T\lambda_h(V)$. Hence $\tilde{W} := \Phi(H\times W)$ is an open neighbourhood of the zero-section in TH. We define a smooth map

$$\Sigma_H\colon \tilde{W} \to H, \quad \Sigma_H(h.V) := h\cdot \exp_H(V),$$

which obviously satisfies $\Sigma_H(0_h) = h$ for all $h \in H$. The inverse to $(h.V) \mapsto (h, \Sigma_H(h.V))$ is given by $(h,h') \mapsto h.\psi(h^{-1}h')$ and thus Σ_H is a local addition. Moreover, the local addition Σ_H is left-invariant and right-invariant, i.e. for $h \in H$, $V_1 \in \tilde{W}$ and $V_2 \in T\rho_{h^{-1}}(\tilde{W})$ we have

$$\Sigma_H(h.V_1) = h\cdot\Sigma_H(V_1) \tag{10}$$

by definition and, if $V_2 \in T_{h'}H$,

$$\begin{aligned}\Sigma_H(V_2.h) &= \Sigma_H((h'h(h'h)^{-1}).V_2.h) = h'h\cdot\exp_H(\mathrm{Ad}(h^{-1})(h'^{-1}.V_2)) = h'hh^{-1}\exp_H(h'^{-1}.V_2)\cdot h\\ &= \Sigma_H(V_2)\cdot h.\end{aligned} \tag{11}$$

We use the local addition on H to construct the desired local addition on the gauge groupoid $\frac{P\times P}{H} \rightrightarrows M$.

We will use the notation introduced in Example 1.8 for the gauge groupoid. To simplify the notation, define $\gamma_{p_1,p_2} := \delta(\sigma_i(\pi(p_1),p_1))\delta(\sigma_i(\pi(p_2)),p_2)^{-1}$, set $U_{ij} := U_i \cap U_j$ and denote by $k_{ji}\colon U_{ij} \to H$ the smooth map $x \mapsto \delta(\sigma_j(x),\sigma_i(x))$.

Construction of the local addition in charts: Fix $i \in I$ and let $\Sigma_M\colon TM \to M$ be a local addition for M. Since M is finite dimensional a globally defined local addition always exists by [KM97, p. 441]. For $i \in I$ we set $W_i := \Sigma_M^{-1}(U_i) \cap TU_i$ and $V_i := \pi_{TU_i}(W_i)$. Then V_i is open in U_i and W_i is an open neighbourhood of the zero-section in TV_i. By making the indexing set I larger and shrinking U_i if necessary we may assume that $(V_i)_{i\in I}$ is still an open cover of M.

We now now want to use similar trivialisations as in Example 1.8 for the gauge groupoid of the principal TH-bundle $T\pi\colon TP \to TM$. In order to obtain trivialisations with a slightly more specialised property, we proceed as follows. Since $V_i \subseteq U_i$, for the open cover $(TV_i)_{i\in I}$ there exist the local sections $T\sigma_i\colon TV_i \to TP$ of $T\pi$. Now choose a connection on TP, which we interpret as a decomposition $TP = T^vP \oplus \mathfrak{H}$ of the tangent bundle of P into the vertical bundle $T^vP := \ker(T\pi)$ and an H-equivariant horizontal complement. The existence of such a connection is ensured by the smooth paracompactness of the base TM by constructing a connection on $TP|_{TV_i}$ and patching them together with a partition of unity. From this we obtain the projection $\pi^{\mathfrak{H}}\colon TP \to \mathfrak{H}$, which is a morphism of vector bundles over TM. Consequently, $\widetilde{\sigma}_i := \pi^{\mathfrak{H}} \circ T\sigma_i\colon TV_i \to TP$ is another system of sections of $T\pi$. From this we deduce for $v_x \in TV_i \cap TV_j$ the formula

$$\widetilde{\sigma}_i(v_x) = \widetilde{\sigma}_j(v_x)\cdot k_{ji}(x),$$

since $\widetilde{\sigma}_i(v_x)$ and $\widetilde{\sigma}_j(v_x)$ are both *horizontal* tangent vectors in $T_{\sigma_i(x)}P$ and $T_{\sigma_j(x)}P$ respectively. If we denote as in Example 1.8 the trivialisations associated to the sections $\widetilde{\sigma}_i$ by

$$\widetilde{TK_{ij}}\colon \frac{T\pi^{-1}(TV_i)\times T\pi^{-1}(TV_j)}{TH} \to TV_i\times TV_j\times TH,$$

then the associated chart changes are given by

$$\widetilde{TK_{ij}} \circ \widetilde{TK_{mn}}^{-1} : TV_i \times TV_j \times TH \to TV_m \times TV_n \times TH, \quad (V_x, V_y, V_h) \mapsto (V_x, V_y, k_{mi}(x) \cdot V_h \cdot k_{nj}(y)^{-1}).$$

Here the product in the third component is the product in the tangent group TH. Now we can define a smooth map

$$\Sigma_{ij} \colon \widetilde{TK_{ij}}^{-1}(V_i \times V_j \times \tilde{W}) \to K_{ij}(U_i \times U_j \times H), \quad \Sigma_{ij} := K_{ij}^{-1} \circ (\Sigma_M \times \Sigma_M \times \Sigma_H) \circ \widetilde{TK_{ij}}. \tag{12}$$

By construction $\widetilde{TK_{ij}}(0_{\langle p,q\rangle}) = (0_{\pi(p)}, 0_{\pi(q)}, 0_{\gamma_{u_2,u_1}})$ holds for $\langle p,q\rangle$ in the domain of K_{ij}. Since Σ_M and Σ_H are local additions, for all such $\langle p,q\rangle$ we obtain $\Sigma_{ij}(0_{\langle p,q\rangle}) = \langle p,q\rangle$.

Claim: For $i,j,m,n \in I$ the maps Σ_{ij} and Σ_{mn} coincide on the intersection of their domains. Assume that the intersection $\frac{P_{im} \times P_{jn}}{H}$ of the domains of K_{ij} and K_{mn} is non-empty. Let $\langle p,q\rangle$ be an element of $\frac{P_{im} \times P_{jn}}{H}$ with $x := \pi(p)$ and $y := \pi(q)$. We will show that for $V_{\langle p,q\rangle} \in T_{\langle p,q\rangle}\frac{P \times P}{H} \cap \operatorname{dom}\Sigma_{ij} \cap \operatorname{dom}\Sigma_{mn}$ the mappings Σ_{ij} and Σ_{mn} yield the same image. The image $\widetilde{TK_{ij}}(V_{\langle p,q\rangle}) = (V_x, V_y, V_{\gamma_{p,q}}) \in T_xV_{im} \times T_yV_{jn} \times T_{\gamma_{p,q}}H$ is related to the image under $\widetilde{TK_{mn}}$ via the formula

$$\widetilde{TK_{mn}}(V_{\langle p,q\rangle}) = (V_x, V_y, k_{mi}(x) \cdot V_{\gamma_{p,q}} \cdot k_{nj}(y)^{-1}) \tag{13}$$

The change of charts formula (13) shows that the first two components of the image are invariant under change of charts. By definition of the maps Σ_{ij} and Σ_{mn}, the two maps coincide in these components since they are just a restriction of the map Σ_M. We compute now a formula for the third component of $K_{mn} \circ \Sigma_{mn}(V_{\langle p,q\rangle})$, which is given by (13), (12), (11) and (10) by

$$\Sigma_H(k_{mi}(x) \cdot V_{\gamma_{p,q}} \cdot k_{nj}(y)^{-1}) = k_{mi}(x)\, \Sigma_H(V_{\gamma_{p,q}}) k_{nj}(y)^{-1}.$$

We can thus conclude $\Sigma_{ij}(V_{\langle p,q\rangle}) = \Sigma_{mn}(V_{\langle p,q\rangle})$. As the smooth maps $\Sigma_{ij}, (i,j) \in I^2$ coincide on the intersection of their domains, we obtain a well-defined smooth map

$$\Sigma_{\mathrm{Gau}} \colon \bigcup_{(i,j)\in I^2} \operatorname{dom}\Sigma_{ij} \mathrel{\underset{o}{\subseteq}} T\left(\frac{P \times P}{H}\right) \to G, \quad V \mapsto \Sigma_{ij}(V) \text{ if } V \in \operatorname{dom}\Sigma_{ij}\,.$$

Claim: Σ_{Gau} is a local addition adapted to the source projection: By construction Σ_{Gau} is defined on an open neighbourhood of the zero-section. The local additions Σ_M and Σ_H do not depend on the chart K_{ij}. Using this fact, an easy computation in charts shows that $(\pi_{T\frac{P\times P}{H}}|_{\bigcup_{(i,j)\in I^2} \operatorname{dom}\Sigma_{ij}}, \Sigma_{\mathrm{Gau}})$ is a diffeomorphism onto an open neighbourhood of the diagonal in $\frac{P\times P}{H} \times \frac{P\times P}{H}$. We conclude that Σ_{Gau} is a local addition.

Recall that the source projection of the gauge groupoid $\frac{P\times P}{H}$ is the mapping $\alpha \colon \frac{P\times P}{H} \to M, \langle p,q\rangle \mapsto \pi(q)$. Computing in the chart K_{ij} we see that α is just the projection onto the second factor of the product. Hence the kernel of $T\alpha$ at a point $\langle p,q\rangle$ in this chart is the subspace of elements $V_{\langle p,q\rangle} \in T_{\langle p,q\rangle}\frac{P\times P}{H}$ with $\mathrm{pr}_2 \circ \widetilde{TK_{ij}}(V_{\langle p,q\rangle}) = 0_{\pi(q)}$. The local addition Σ_M satisfies $\Sigma_M(0_{\pi(q)}) = \pi(q)$. By construction, this implies $\Sigma_{\mathrm{Gau}}(\ker T_{\langle p,q\rangle}T\alpha) \subseteq \pi^{-1}(\pi(q))$. Summing up, the local addition Σ_{Gau} is adapted to the source projection. ■

Remark 2.15. The same argument as in the previous proof shows that under the same assumptions the associated Lie group bundle $\frac{P\times H}{H} \rightrightarrows M$ has an adapted local addition. In fact, $\frac{P\times H}{H} \rightrightarrows M$ may be considered as the arrow manifold of the subgroupoid $\frac{P\times_M P}{H} \rightrightarrows M$ (with equal source and target projection) of $\frac{P\times P}{H} \rightrightarrows M$, and the local addition constructed in the previous proof restricts to a local addition on this submanifold with the desired properties. ■

Example 2.16. For a principal H-bundle $\pi \colon P \to M$ with locally exponential structure group H and compact M we thus obtain a Lie group structure on $\mathrm{Bis}(\frac{P\times P}{H})$. Moreover, the natural map

$$\beta_* \colon \mathrm{Bis}\left(\frac{P\times P}{H}\right) \to \mathrm{Diff}(M)$$

is smooth by Theorem A.8 e). Its kernel is the group of bisections $\mathrm{Bis}(\frac{P\times_M P}{H})$ of the associated Lie group bundle. The latter is a submanifold of $\mathrm{Bis}(\frac{P\times P}{H})$ since the adapted local addition on $\frac{P\times P}{H}$ used in the construction of the manifold structure on $\mathrm{Bis}(\frac{P\times P}{H})$ restricts to an adapted local addition on $\frac{P\times_M P}{H}$, and thus the corresponding carts for the manifold structure on $\mathrm{Bis}(\frac{P\times P}{H})$ yield submanifold charts for $\mathrm{Bis}(\frac{P\times_M P}{H})$. In total, we have a sequence of Lie groups

$$\mathrm{Bis}\left(\frac{P\times_M P}{H}\right)\hookrightarrow \mathrm{Bis}\left(\frac{P\times P}{H}\right)\twoheadrightarrow \mathrm{im}(\beta_*). \tag{14}$$

In fact, we get this sequence if we start with an arbitrary locally trivial Lie groupoid [Mac05, p. 130] (or in finite dimensions, equivalently with a Lie groupoid whose base is connected and whose Lie algebroid is transitive [Mac05, Corollary 3.5.18]).

We will now explain how to turn this sequence into an extension of Lie groups, i.e., into a locally trivial bundle. To this end it suffices to construct a smooth section of β_* on some identity neighbourhood of $\mathrm{Diff}(M)$. This then implies in particular that $\mathrm{im}(\beta_*)$ is an open subgroup of $\mathrm{Diff}(M)$.

Recall the notation from Example 1.8. We choose a finite subcover $U_1,...,U_n$ of the trivialising cover $(U_i)_{i\in I}$. From Theorem A.8 a) recall the chart

$$\varphi_{\mathrm{id}}\colon O_{\mathrm{id}}\to \Gamma(M\leftarrow TM)\cap C^\infty(M,U)$$

of $\mathrm{Diff}(M)$, where U denotes the domain of a local addition $\Sigma\colon U\Subset TM\to M$. Observe that $\varphi_{\mathrm{id}}^{-1}(h)(x)=\varphi_{\mathrm{id}}^{-1}(h')(x)$ if $h(x)=h'(x)$ follows from the construction of φ_{id}. We now choose a partition of unity $\lambda_i\colon M\to\mathbb{R}$ with $\mathrm{supp}(\lambda_i)\subseteq U_i$. For $f\in O_{\mathrm{id}}$ we then have that

$$s_i(f):=\varphi_{\mathrm{id}}^{-1}((\lambda_1+...+\lambda_{i-1})\cdot\varphi_{\mathrm{id}}(f))^{-1}\circ\varphi_{\mathrm{id}}^{-1}((\lambda_1+...+\lambda_i)\cdot\varphi_{\mathrm{id}}(f))$$

defines a smooth map $s_i\colon O_{\mathrm{id}}\to\mathrm{Diff}(M)$. Moreover, we have $s_i(f)(x)=x$ if $x\notin\mathrm{supp}(\lambda_i)$, since

$$((\lambda_1+...+\lambda_{i-1})\cdot\varphi_{\mathrm{id}}(f))(x)=((\lambda_1+...+\lambda_i)\cdot\varphi_{\mathrm{id}}(f))(x)\quad\text{for}\quad x\notin\mathrm{supp}(\lambda_i).$$

In addition, $s_1(f)\circ...\circ s_n(f)=f$ follows directly from the definition (see also [HT03, Proposition 1]).

With the aid of the chart

$$K_{ij}\colon\frac{\pi^{-1}(U_i)\times\pi^{-1}(U_j)}{H}\to U_i\times U_j\times H$$

we then define the bisection

$$\widetilde{s}_{ij}(f)\colon M\times M\to\frac{P\times P}{H},\quad (x,y)\mapsto\begin{cases}K_{ij}^{-1}((x,s_j(f)(y),e)) & \text{if } (x,y)\in U_i\times U_j\\ 1_{(x,y)}=K_{ij}^{-1}(x,y,e) & \text{else.}\end{cases}$$

This defines a smooth map since $s_j(f)(x)=x$ if $x\notin\mathrm{supp}(\lambda_j)$ and $M\times(M\setminus\mathrm{supp}(\lambda_j))$ is open in $M\times M$. Moreover,

$$\widetilde{s}_{ij}\colon O_{\mathrm{id}}\to\mathrm{Bis}\left(\frac{P\times P}{H}\right),\quad f\mapsto\widetilde{s}_{ij}(f)$$

is smooth by Theorem A.8 e). From $\beta\circ K_{ij}^{-1}=\mathrm{pr}_2$ we infer $\beta_*(\widetilde{s}_{ij}(f))=s_j(f)$ and thus

$$O\to\mathrm{Bis}\left(\frac{P\times P}{H}\right),\quad f\mapsto\widetilde{s}_{11}(f)\star...\star\widetilde{s}_{nn}(f)$$

is a smooth section of β_*, where $\star$ is the product (2). This turns (14) into an extension of Lie groups.

We now identify $\mathrm{Bis}\left(\frac{P\times P}{H}\right)$ with the group of bundle automorphism of P via the group isomorphism

$$\mathrm{Aut}(P)\to\mathrm{Bis}\left(\frac{P\times P}{H}\right),\quad f\mapsto(m\mapsto\langle\sigma_i(m),f(\sigma_i(m))\rangle\text{ if }x\in U_i).$$

Under this isomorphism the subgroup $\mathrm{Bis}(\frac{P\times_M P}{H})$ maps to the group of gauge transformations $\mathrm{Gau}(P)$. If we denote by $\mathrm{Diff}(M)_{[P]} = \mathrm{im}(\beta_*)$ the open subgroup of diffeomorphisms of M that lift to bundle automorphisms, then we obtain the well-known extension of Lie groups

$$\mathrm{Gau}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M)_{[P]}$$

from [Woc07]. Moreover, the natural action

$$\mathrm{Aut}(P) \times P \to P, \quad (f,p) \mapsto f(p)$$

is smooth with respect to this identification, since it can be identified (non-canonically) with the action of $\mathrm{Bis}(\frac{P\times P}{H})$ on the α-fibre $\alpha^{-1}(m) = \frac{P_m \times P}{H} \cong P$ for each $m \in M$. ■

3 The Lie algebra of the group of bisections

In this section the Lie algebra of the Lie group $\mathrm{Bis}(\mathcal{G})$ is computed for a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$. Throughout this section $\mathcal{G}$ denotes a locally convex and locally metrisable Lie groupoid with compact space of objects M that admits an adapted local addition.

It will turn out that the Lie algebra of the group of bisections is naturally (anti-) isomorphic to the Lie algebra of sections of the Lie algebroid $L(\mathcal{G})$ associated to $\mathcal{G}$ (see Section 1 for the corresponding notions). Before we compute the bracket, let us identify the tangent space $T_1 \mathrm{Bis}(\mathcal{G})$.

Remark 3.1. By construction $\mathrm{Bis}(\mathcal{G})$ is an open submanifold of $\Gamma(M \overset{\alpha}{\leftarrow} G)$. We first analyse the space $T_1 C^\infty(M,G)$. This is by Theorem A.9 isomorphic to the space $\Gamma(1^*TG)$, the isomorphism given by restricting the vector bundle isomorphism

$$\Phi_{M,G} \colon TC^\infty(M,G) \to C^\infty(M,TG), \quad [t \mapsto \eta(t)] \mapsto (m \mapsto [t \mapsto \eta^\wedge(t,m)])$$

to $T_1 C^\infty(M,G)$. Here we have identified tangent vectors in $C^\infty(M,G)$ with equivalence classes $[\eta]$ of smooth curves $\eta \colon]-\varepsilon,\varepsilon[\to C^\infty(M,G)$ for some $\varepsilon > 0$ [Nee06, Definition I.3.3]. This isomorphism maps $T_1 C^\infty(M,G)$ onto

$$\{f \in C^\infty(M,TG) \mid f(m) \in T_{1_m} G \text{ for all } m \in M\},$$

and the latter space is naturally isomorphic to $\Gamma(1^*TG)$. If we restrict in $C^\infty(M,G)$ to the submanifold $\Gamma(M \overset{\alpha}{\leftarrow} G)$, then this isomorphism maps $T_1(\Gamma(M \overset{\alpha}{\leftarrow} G))$ onto

$$\left\{f \in C^\infty(M,TG) \mid f(m) \in T^\alpha_{1_m} G \text{ for all } m \in M\right\},$$

which in turn is naturally isomorphic to $\Gamma(\mathbf{L}(\mathcal{G}))$. In the sequel we will denote by

$$\varphi_{\mathcal{G}} \colon T_1 \mathrm{Bis}(\mathcal{G}) = \mathbf{L}(\mathrm{Bis}(\mathcal{G})) \to \Gamma(1^*T^\alpha G) = \Gamma(\mathbf{L}(\mathcal{G})). \tag{15}$$

the resulting isomorphism. ■

Following some preparations, we will prove in Theorem 3.4 that the Lie algebra bracket $[\cdot,\cdot]$ on $T_1 \mathrm{Bis}(\mathcal{G})$ is, with respect to the isomorphism $\varphi_{\mathcal{G}}$, the negative of the bracket of the Lie algebroid associated to $\mathcal{G}$. To compute the Lie bracket on $T_1 \mathrm{Bis}(\mathcal{G})$ we adapt an idea of Milnor. In [Mil84, p. 1041] a natural action of the diffeomorphism group was used to compute the Lie bracket of its Lie algebra. In the present context we exploit the natural action of the group of bisections via left-translations on the manifold of arrows G from Remark 1.2 c).

Proposition 3.2. *Let X be an element of $T_1 \operatorname{Bis}(\mathcal{G})$ and denote by 0 the zero-section in $\Gamma(TG)$. Then the vector fields $\overrightarrow{\varphi_{\mathcal{G}}(X)}$ (cf. (6) and (15)) and $X^\rho \times 0$ are γ-related, i.e. the diagram*

$$\begin{array}{ccc} T\operatorname{Bis}(\mathcal{G}) \times TG & \xrightarrow{T\gamma} & TG \\ \Big\uparrow {\scriptstyle X^\rho \times 0} & & \Big\uparrow {\scriptstyle \overrightarrow{\varphi_{\mathcal{G}}(X)}} \\ \operatorname{Bis}(\mathcal{G}) \times G & \xrightarrow{\gamma} & G \end{array} \tag{16}$$

commutes.

Proof. To simplify computations we identify X with the equivalence class $[\eta]$ of a smooth curve $\eta\colon]-\varepsilon,\varepsilon[\to \operatorname{Bis}(\mathcal{G})$ satisfying $\eta(0) = 1$ and $\eta'(0) = X$. From Theorem A.8 d) we infer that $\eta^\wedge\colon]-\varepsilon,\varepsilon[\times M \to G$, $(t,m) \mapsto \eta(t)(m)$ is smooth. Thus for each $\psi \in \operatorname{Bis}(\mathcal{G})$ we obtain the smooth map $(\rho_\psi \circ \eta)^\wedge$. Evaluating in $(t,x) \in]-\varepsilon,\varepsilon[\times M$ we obtain the formula

$$(\rho_\psi \circ \eta)^\wedge(t,x) = (\eta^\wedge(t,\cdot) \star \psi)(x) = m(\eta^\wedge(t,\beta(\psi(x))),\psi(x)). \tag{17}$$

Moreover, by definition of right invariant vector fields $X^\rho(\psi) = [t \mapsto \rho_\psi \circ \eta^\wedge(t,\cdot)]$ holds for each $\psi \in \operatorname{Bis}(\mathcal{G})$. We use the above facts to compute for $(\psi, g) \in \operatorname{Bis}(\mathcal{G}) \times G$

$$\begin{aligned} \overrightarrow{\varphi_{\mathcal{G}}(X)} \circ \gamma(\psi, g) &= \overrightarrow{\varphi_{\mathcal{G}}(X)}(m(\psi(\beta(g)),g)) \overset{(6)}{=} TR_{m(\psi(\beta(g)),g)}\varphi_{\mathcal{G}}(X)(\beta(m(\psi(\beta(g)),g))) \\ &= TR_{m(\psi(\beta(g)),g)}\varphi_{\mathcal{G}}(X)(\beta(\psi(\beta(g)))) = [t \mapsto m(m(\eta^\wedge(t,\beta(\psi(\beta(g)))),\psi(\beta(g))),g)] \\ &\overset{(17)}{=} [t \mapsto m(\rho_\psi \circ \eta^\wedge(t,\beta(g)),g)] = [t \mapsto \gamma(\rho_\psi \circ \eta^\wedge(t,\cdot),g)] \\ &= T\gamma(X^\rho(\psi),0_g) = T\gamma \circ (X^\rho \times 0)(\psi,g). \end{aligned}$$

Hence (16) commutes and the assertion follows. ■

3.3. Before phrasing the main result of this section we introduce the following notation. If we fix a manifold M and consider the category $\mathsf{LieAlgebroids}_M$ of locally convex Lie algebroids over M, then taking sections gives rise to a functor

$$\Gamma\colon \mathsf{LieAlgebroids}_M \to \mathsf{LieAlgebras}.$$

Likewise, there is the functor

$$-\Gamma\colon \mathsf{LieAlgebroids}_M \to \mathsf{LieAlgebras}$$

which assigns to a Lie algebroid its Lie algebra of sections, but with the negative Lie bracket on it. ■

Theorem 3.4. *Let M be a compact manifold and $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid admitting an adapted local addition. Then the morphism of topological vector spaces*

$$\varphi_{\mathcal{G}}\colon \mathbf{L}(\operatorname{Bis}(\mathcal{G})) \to \Gamma(\mathbf{L}(\mathcal{G}))$$

from (15) is actually an anti-isomorphism of Lie algebras. Moreover, $\varphi_{\mathcal{G}}$ constitutes a natural isomorphism fitting into the diagram

$$\begin{array}{ccc} \mathsf{LieGroupoids}_M^\Sigma & \xrightarrow{\mathbf{L}} & \mathsf{LieAlgebroids}_M \\ \Big\downarrow {\scriptstyle \operatorname{Bis}} & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \Big\downarrow {\scriptstyle -\Gamma} \\ \mathsf{LieGroups} & \xrightarrow{\mathbf{L}} & \mathsf{LieAlgebras}, \end{array}$$

of functors.

Proof. Recall that the bracket on $\Gamma(\mathbf{L}(\mathcal{G}))$ is induced from the isomorphism (6) with the right invariant vector fields on G. In Proposition 3.2 we have seen that for $X, Y \in T_1 \operatorname{Bis}(\mathcal{G})$ the right-invariant vector fields $\overrightarrow{\varphi_{\mathcal{G}}(X)}$ and $\overrightarrow{\varphi_{\mathcal{G}}(Y)}$ are γ-related to $X^\rho \times 0$ and $Y^\rho \times 0$, respectively. Hence the Lie bracket $[X^\rho \times 0, Y^\rho \times 0] = [X^\rho, Y^\rho] \times 0$ is γ-related to the Lie bracket $\left[\overrightarrow{\varphi_{\mathcal{G}}(X)}, \overrightarrow{\varphi_{\mathcal{G}}(Y)}\right]$, i.e., we have

$$T\gamma \circ ([X^\rho, Y^\rho] \times 0) = [\overrightarrow{\varphi_{\mathcal{G}}(X)}, \overrightarrow{\varphi_{\mathcal{G}}(Y)}] \circ \gamma.$$

From this we deduce

$$\begin{aligned}
-\varphi_{\mathcal{G}}([X,Y])(x) &= -\varphi_{\mathcal{G}}(\overrightarrow{[X,Y]})(1_x) = -\varphi_{\mathcal{G}}(\overrightarrow{[X,Y]})(\gamma(1,1_x)) = T\gamma((-[X,Y]^\rho \times 0)(1,1_x)) \\
&\overset{1.6}{=} T\gamma(([X^\rho, Y^\rho] \times 0)(1,1_x)) = [\overrightarrow{\varphi_{\mathcal{G}}(X)}, \overrightarrow{\varphi_{\mathcal{G}}(Y)}](\gamma(1,1_x)) = [\overrightarrow{\varphi_{\mathcal{G}(}X)}, \overrightarrow{\varphi_{\mathcal{G}}(Y}](1_x) \\
&= [\overrightarrow{\varphi_{\mathcal{G}}(X)}, \overrightarrow{\varphi_{\mathcal{G}}(Y)}](x).
\end{aligned}$$

Hence (7) implies that $\varphi_{\mathcal{G}}$ is an anti-isomorphism.

To check that $\varphi_{\mathcal{G}}$ is natural, suppose $\mathcal{H} = (H \rightrightarrows M)$ is another Lie groupoid over M admitting an adapted local addition and $f\colon \mathcal{G} \to \mathcal{H}$ is a morphism. Then $-\Gamma(\mathbf{L}(f))$ is the induced map on sections $\Gamma(\mathbf{L}(\mathcal{G})) \to \Gamma(\mathbf{L}(\mathcal{H}))$, $\xi \mapsto Tf \circ \xi$. On the other hand, $\operatorname{Bis}(f)$ is the map $\operatorname{Bis}(\mathcal{G}) \to \operatorname{Bis}(\mathcal{H})$, $\sigma \mapsto f \circ \sigma$. The tangent map of $f_*\colon C^\infty(M,G) \to C^\infty(M,H)$, $\sigma \mapsto f \circ \sigma$ at 1 is given by Theorem A.9 by

$$T_1(f_*)\colon \Gamma(1^*TG) \to \Gamma(1^*TH), \quad \xi \mapsto Tf \circ \xi \tag{18}$$

with respect to the identifications $T_1 C^\infty(M,G) \cong \Gamma(1^*TG)$ and $T_1 C^\infty(M,H) \cong \Gamma(1^*TH)$. Restricting the latter isomorphism to vertical sections gives exactly the isomorphism $\varphi_{\mathcal{G}}$ and (18) gives the above formula for $-\Gamma(\mathbf{L}(f))$. ■

Example 3.5. For the gauge groupoid $\frac{P \times P}{H} \rightrightarrows M$ of the principal H-bundle $\pi\colon P \to M$ with locally exponential structure group H we have the natural isomorphisms $\mathbf{L}(\frac{P\times P}{H} \rightrightarrows M) \cong TP/H \to M$ from Example 1.8. Thus the extension

$$\operatorname{Bis}\left(\frac{P \times_M P}{H}\right) \hookrightarrow \operatorname{Bis}\left(\frac{P \times P}{H}\right) \twoheadrightarrow \operatorname{im}(\beta_*) \tag{19}$$

of locally convex Lie groups from (14) gives rise via the latter isomorphism and the isomorphism from (15) to the extension

$$\Gamma(M \leftarrow (T^v P)/H) \to \Gamma(M \leftarrow TP/H) \to \Gamma(M \leftarrow TM)$$

of topological Lie algebras, where $T^v P := \ker(T\pi)$ denotes the vertical subalgebroid of TP/H. This is of course the extension of Lie algebras which is naturally associated to the Atiyah sequence $T^v P/H \to TP/H \to TM$.■

3.6. For the following corollary, recall that a Lie algebroid $\mathcal{A}$ is called *integrable* if there is a Lie groupoid $\mathcal{G}$ such that $\mathbf{L}(\mathcal{G})$ is isomorphic (over M) to $\mathcal{A}$. In the same way, a Lie algebra $\mathfrak{h}$ is called *integrable* if there is a Lie group H such that $\mathbf{L}(H)$ is isomorphic to $\mathfrak{h}$. ■

Corollary 3.7. *Suppose M is a compact manifold and $\mathcal{A} = (A \to M, a, [\cdot,\cdot])$ is a finite-dimensional Lie algebroid over M. If $\mathcal{A}$ is integrable, then so is its algebra of sections $(\Gamma(M \leftarrow A), [\cdot,\cdot])$.*

Question 3.8. The previous result is not a surprise. What is more interesting is the question about the converse statement: suppose that a finite-dimensional Lie algebroid is *not* integrable, is then its algebra of sections also not integrable? ■

Remark 3.9. Note that the Lie algebras that arise here as the Lie algebras of Lie groups of bisections carry more information that just the structure of a Lie algebra. In fact, the geometric structure that they have is subsumed in the notion of a Lie-Rinehart algebra [Hue90, KSM90]. Thus a way to solve the above question could be to create a theory of objects that are integrating Lie-Rinehart algebras on an algebraic level (something that one might call Lie-Rinehart groups). To our best knowledge such a theory does not exist at the moment.■

4 Regularity properties of the group of bisections

This section contains an investigation of regularity properties for the Lie group of bisections. Throughout this section $\mathcal{G}$ denotes a locally convex and locally metrisable Lie groupoid with compact space of objects M that admits an adapted local addition. Moreover, we identify throughout this section the Lie algebra $\mathbf{L}(\operatorname{Bis}(\mathcal{G}))$ with $-\Gamma(\mathbf{L}(\mathcal{G}))$ via the isomorphism $\varphi_{\mathcal{G}}$ from Theorem 3.4.

We will give two completely different proofs of the C^k-regularity of $\operatorname{Bis}(\mathcal{G})$ in the case of a locally trivial Lie groupoid and in the case of a Banach-Lie groupoid. While the argument in the locally continuous case is geometric in nature (and rather elementary), the argument in the case of Banach-Lie groupoids is analytical.

Theorem 4.1. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $k \in \mathbb{N}_0 \cup \{\infty\}$. Assume that G is a locally trivial Lie groupoid with locally exponential and C^k-regular vertex group and compact M. Then the Lie group $\operatorname{Bis}(\mathcal{G})$ is C^k-regular. In particular, the Lie group $\operatorname{Bis}(\mathcal{G})$ is regular in the sense of Milnor.*

Proof. First note that $\mathcal{G}$ is isomorphic over M to the gauge groupoid of a principal K-bundle $P \to M$, where K is the vertex group of $\mathcal{G}$. So we may assume without loss of generality that $\mathcal{G} = (\frac{P\times P}{H} \rightrightarrows M)$. We consider the extension

$$\operatorname{Bis}\left(\frac{P \times_M P}{H}\right) \hookrightarrow \operatorname{Bis}\left(\frac{P \times P}{H}\right) \twoheadrightarrow \operatorname{im}(\beta_*)$$

of Lie groups from Example 2.16. As explained in Example 2.16 the Lie group $\operatorname{Bis}\left(\frac{P\times_M P}{H}\right)$ is isomorphic to the gauge group $\operatorname{Gau}(P)$. This isomorphism is even an isomorphism of locally metrisable Lie groups since it maps smooth curves to smooth curves. From [Glö13] it now follows that $\operatorname{Bis}\left(\frac{P\times_M P}{H}\right)$ is C^k-regular, as well as $\operatorname{im}(\beta_*)$ (the latter is just an open subgroup of $\operatorname{Diff}(M)$). Since C^k-regularity is an extension property [NS12, Appendix B] it follows that also $\operatorname{Bis}\left(\frac{P\times P}{H}\right)$ is C^k-regular, what we were after to show. ■

4.2. Let $\mathcal{G}$ be a Lie groupoid. Define the map

$$f\colon [0,1] \times G \times C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}} \to T^\alpha G, \quad (t, g, \eta) \mapsto TR_g \eta^\wedge(t, \beta(g)) := TR_g \eta(t)(\beta(g)).$$

This map makes sense, since (6) shows for fixed $(t, \eta) \in [0,1] \times C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))$ that $f(t, \cdot, \eta)$ is the right-invariant vector field associated to $\eta(t)$ and thus takes its values in $T^\alpha G$. We now consider the parameter dependent initial value problem:

$$\begin{cases} x'(t) &= f(t, x(t), \eta) = TR_{x(t)} \eta^\wedge(t, \beta(x(t))), \\ x(t_0) &= g_0, \qquad (t_0, g_0) \in [0,1] \times G \end{cases} \tag{20}$$

To prove the regularity of $\operatorname{Bis}(\mathcal{G})$ we will study the flow of the differential equation. ■

Recall the following definition of $C^{r,s}$-mappings from [AS12].

4.3. Let E_1, E_2 and F be locally convex spaces, U and V open subsets of E_1 and E_2, respectively, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. A mapping $f\colon U \times V \to F$ is called a $C^{r,s}$-map if for all $i, j \in \mathbb{N}_0$ such that $i \leq r, j \leq s$, the iterated directional derivative

$$d^{(i,j)} f(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) := (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y)$$

exists for all $x \in U, y \in V, w_1, \ldots, w_i \in E_1, v_1, \ldots, v_j \in E_2$ and yields continuous maps

$$\begin{aligned} d^{(i,j)} f\colon U \times V \times E_1^i \times E_2^j &\to F, \\ (x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) &\mapsto (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y). \end{aligned}$$

One can extend the definition of $C^{r,s}$-maps to mappings on locally convex domains with dense interior (cf. Definition A.2). In addition, there are chain rules for $C^{r,s}$-mappings allowing us to naturally extend the notion of $C^{r,s}$-maps to maps defined on products of locally convex manifolds with values in a locally convex manifold. ■

For further results and details on the calculus of $C^{r,s}$-maps we refer to [AS12].

Proposition 4.4. *a) The map f from 4.2 is of class $C^{0,\infty}$ with respect to the splitting $[0,1] \times (G \times C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))_{c.o.})$ and satisfies*

$$f(t,g,\eta) \in T^\alpha_g G \text{ for } (t,g,\eta) \in [0,1] \times G \times C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G}))).$$

Assume in addition that G is a Banach-manifold. Then the following holds:

*b) There is a zero-neighbourhood $\Omega \Subset \Gamma(\mathbf{L}(\mathcal{G}))$ such that for every $(t_0, g_0, \eta) \in [0,1] \times G \times C^0([0,1], \Omega)$ the initial value problem (20) admits a unique maximal solution $\varphi_{t_0,g_0,\eta} \colon [0,1] \to G$. Here we defined $C^0([0,1], \Omega) := \left\{X \in C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G}))) \middle| X([0,1]) \subseteq \Omega\right\} \Subset C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))_{c.o.}$.
Hence, we obtain the flow of (20) as*

$$\mathrm{Fl}^f \colon [0,1] \times [0,1] \times G \times C^0([0,1], \Omega) \to G, (t_0, t, g_0, \eta) \mapsto \varphi_{t_0,g_0,\eta}(t).$$

c) The map $\mathrm{Fl}^f_0 := \mathrm{Fl}^f(0, \cdot) \colon [0,1] \times (G \times C^0([0,1], \Omega)) \to G, (t,g,\eta) \mapsto \mathrm{Fl}^f(0,t,g,\eta)$ is of class $C^{1,\infty}$.

d) Fix $(s,t,\eta) \in [0,1] \times [0,1] \times C^0([0,1], \Omega)$ then the map $\beta \circ \mathrm{Fl}^f(s,t,\cdot,\eta) \circ 1 \colon M \to M$ is a diffeomorphism.

e) Fix $\eta \in C^0([0,1], \Omega)$, then $H_\eta \colon [0,1] \times M \to G, (t,x) \mapsto \mathrm{Fl}^f_0(t, 1_x, \eta)$ is a $C^{1,\infty}$-mapping which induces a C^1-map

$$c_\eta \colon [0,1] \to \mathrm{Bis}(\mathcal{G}), t \mapsto \mathrm{Fl}^f_0(t, \cdot, \eta).$$

We postpone the rather technical proof of Proposition 4.4 to Section 5.

Theorem 4.5. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and assume that G is a Banach manifold and that M is compact. Then the Lie group $\mathrm{Bis}(\mathcal{G})$ is C^k-regular for each $k \in \mathbb{N}_0 \cup \{\infty\}$. In particular, the Lie group $\mathrm{Bis}(\mathcal{G})$ is regular in the sense of Milnor.*

Proof. Let $\Omega \Subset \Gamma(\mathbf{L}(\mathcal{G})) = T_1 \mathrm{Bis}(\mathcal{G}) = \mathbf{L}(\mathrm{Bis}(\mathcal{G}))$ be the zero-neighbourhood constructed in Proposition 4.4 b). Combine Proposition 4.4 c) and d) with Theorem A.8 d) to obtain a smooth map

$$\mathrm{evol} \colon C^0([0,1], \Omega) \to \mathrm{Bis}(\mathcal{G}), \quad \eta \mapsto \mathrm{Fl}^f_0(1, \cdot, \eta) \circ 1 = c_\eta(1).$$

We claim that $c_\eta \colon [0,1] \to \mathrm{Bis}(\mathcal{G})$ is the product integral of $\eta \colon [0,1] \to \Omega \subseteq \mathbf{L}(\mathrm{Bis}(\mathcal{G}))$, i.e. it solves the initial value problem (cf. (1))

$$\begin{cases} \gamma'(t) &= T_1 \rho_{\gamma(t)}(\eta(t)) = \eta(t).\gamma(t) \\ \gamma(0) &= 1 \end{cases}.$$

If this is true, then the proof can be completed as follows: For each $\eta \in C^0([0,1], \Omega) \Subset C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))$ there is a product integral and the evolution evol is smooth. Then $\mathrm{Bis}(\mathcal{G})$ is C^0-regular by [Dah12, Proposition 1.3.10]. Since C^0-regularity implies C^k-regularity for all $k \geq 0$ the assertion follows.

Proof of the claim: Fix $\eta \in C^0([0,1], \Omega)$ and observe that $c_\eta \colon [0,1] \to \mathrm{Bis}(\mathcal{G})$ is a C^1-curve by Proposition 4.4 e). Furthermore, $c_\eta(0) = H_\eta(0, \cdot) = \mathrm{Fl}^f_0(0, \cdot, \eta) \circ 1 = 1 \in \mathrm{Bis}(\mathcal{G})$. Let us now compute the derivative $\frac{\partial}{\partial t} c_\eta(t)$ for fixed $t \in [0,1]$. To this end choose a smooth curve $k \colon]-\varepsilon, \varepsilon[\to \mathrm{Bis}(\mathcal{G})$ (for some $\varepsilon > 0$) with $k(0) = 1$ and $k'(0) = \eta(t) \in T_1 \mathrm{Bis}(\mathcal{G})$. Recall that $\Phi_{M,G} \colon TC^\infty(M,G) \to C^\infty(M, TG), [t \mapsto h(t)] \mapsto (m \mapsto [t \mapsto h^\wedge(t,m)])$ is an isomorphism of vector bundles by Theorem A.9. Therefore, we can compute the derivative as follows:

$$\begin{aligned}
\frac{\partial}{\partial t} c_\eta(t) &= [s \mapsto c_\eta(t+s)] = \Phi^{-1}_{M,G}\left(m \mapsto [s \mapsto \mathrm{Fl}^f_0(t+s, 1_m, \eta)]\right) \\
&\overset{(20)}{=} \Phi^{-1}_{M,G}\left(m \mapsto TR_{\mathrm{Fl}^f_0(t,1_m,\eta)} \eta^\wedge(t, \beta(\mathrm{Fl}^f_0(t, 1_m, \eta)))\right) \\
&= \Phi^{-1}_{M,G}\left(m \mapsto TR_{c^\wedge_\eta(t,m)} \circ \eta^\wedge(t, \cdot) \circ (\beta \circ c_\eta)^\wedge(t,m))\right) \\
&= \Phi^{-1}_{M,G}\left(m \mapsto [s \mapsto R_{c^\wedge_\eta(t,m)} \circ k^\wedge(s, \cdot) \circ \beta \circ c^\wedge_\eta(t,m)]\right) \\
&\overset{(2)}{=} \Phi^{-1}_{M,G}\left(m \mapsto [s \mapsto ((\rho_{c_\eta(t)} \circ k)^\wedge(s,m)]\right) = [s \mapsto \rho_{c_\eta(t)} \circ k(s)] = T_1 \rho_{c_\eta(t)}([s \mapsto k(s)]) \\
&= T_1 \rho_{c_\eta(t)} \eta(t)
\end{aligned}$$

As $t \in [0,1]$ was arbitrary, c_η is the product integral for $\eta \colon [0,1] \to \mathbf{L}(\mathrm{Bis}(\mathcal{G}))$. ■

5 Proof of Proposition 4.4

In this section we exhibit the technical proof of Proposition 4.4. Let us first recall its content:

5.1. a) The map f from 4.2 is of class $C^{0,\infty}$ with respect to the splitting $[0,1]\times(G\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}})$ and satisfies

$$f(t,g,\eta)\in T^\alpha_g G \text{ for } (t,g,\eta)\in[0,1]\times G\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G}))).$$

Assume in addition that G is a Banach-manifold. Then the following holds:

b) There is a zero-neighbourhood $\Omega \Subset \Gamma(\mathbf{L}(\mathcal{G}))$ such that for every $(t_0,g_0,\eta)\in[0,1]\times G\times C^0([0,1],\Omega)$ the initial value problem (20) admits a unique maximal solution $\varphi_{t_0,g_0,\eta}\colon[0,1]\to G$. Here we defined $C^0([0,1],\Omega):=\left\{X\in C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))\middle| X([0,1])\subseteq\Omega\right\}\Subset C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}}$.
Hence, we obtain the *flow* of (20) as

$$\mathrm{Fl}^f\colon[0,1]\times[0,1]\times G\times C^0([0,1],\Omega)\to G,(t_0,t,g_0,\eta)\mapsto\varphi_{t_0,g_0,\eta}(t).$$

c) The map $\mathrm{Fl}^f_0:=\mathrm{Fl}^f(0,\cdot)\colon[0,1]\times(G\times C^0([0,1],\Omega))\to G,(t,g,\eta)\mapsto\mathrm{Fl}^f(0,t,g,\eta)$ is of class $C^{1,\infty}$.

d) Fix $(s,t,\eta)\in[0,1]\times[0,1]\times C^0([0,1],\Omega)$ then the map $\beta\circ\mathrm{Fl}^f(s,t,\cdot,\eta)\circ 1\colon M\to M$ is a diffeomorphism.

e) Fix $\eta\in C^0([0,1],\Omega)$, then $H_\eta\colon[0,1]\times M\to G,(t,x)\mapsto\mathrm{Fl}^f_0(t,1_x,\eta)$ is a $C^{1,\infty}$-mapping which induces a C^1-map

$$c_\eta\colon[0,1]\to\mathrm{Bis}(\mathcal{G}),t\mapsto\mathrm{Fl}^f_0(t,\cdot,\eta).$$

■

Proof (of Proposition 4.4). a) To prove that f is a $C^{0,\infty}$-map, define first an auxiliary map

$$f_0\colon[0,1]\times M\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}}\to T^\alpha G,(t,x,\eta)\mapsto\eta^\wedge(t,x)=\eta(t)(x).$$

We will first prove that f_0 is of class $C^{0,\infty}$ with respect to the splitting $[0,1]\times(M\times C^r([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}})$. To this end consider the evaluation maps $\mathrm{ev}_0\colon C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}}\times[0,1]\to\Gamma(\mathbf{L}(\mathcal{G})),\mathrm{ev}_0(\eta,t)=\eta(t)$ and $\widetilde{\mathrm{ev}}\colon\Gamma(\mathbf{L}(\mathcal{G}))\times M\to T^\alpha G,\widetilde{\mathrm{ev}}(X,y)=X(y)$. Clearly $f_0(t,x,\eta)=\widetilde{\mathrm{ev}}\circ(\mathrm{ev}_0(\eta,t),g)$. Since ev_0 is a $C^{\infty,0}$-map by [AS12, Proposition 3.20], a combination of the chain rules [AS12, Lemma 3.17 and Lemma 3.19] for $C^{r,s}$-mappings shows that f_0 will be $C^{0,\infty}$ if $\widetilde{\mathrm{ev}}$ is smooth. To see that $\widetilde{\mathrm{ev}}$ is smooth we compute in bundle charts. Consider the vector bundle $\pi_\alpha\colon T^\alpha G\to G$ and denote its typical fibre by E, We choose a local trivialisation $\kappa\colon\pi_\alpha^{-1}(U_\kappa)\to U_\kappa\times E$ such that $U_\kappa\cap 1(M)\neq\emptyset$. By construction 1.4, the vector bundle $\mathbf{L}(G)\to M$ is the pullback bundle of $T^\alpha G$ over the embedding 1. Hence κ induces the trivialisation $1^*\kappa\colon(1*\pi_\alpha)^{-1}(1^{-1}(U_\kappa))\to 1^{-1}(U_\kappa)\times E,Y\mapsto(1^*\pi_\alpha(Y),\kappa(\pi_\alpha^*1(Y)))$ of $\mathbf{L}(G)\to M$. Shrinking U_κ we may assume that $W:=1^{-1}(U_\kappa)$ is the domain of a manifold chart (ψ,W) of M. Recall from [Woc13, Proposition 7.3 and Lemma 5.5] that the map $\theta_{\kappa,\psi}\colon\Gamma(\mathbf{L}(\mathcal{G}))\to C^\infty(\psi(W),E),X\mapsto\mathrm{pr}_2\circ 1^*\kappa\circ X\circ\psi^{-1}$ is continuous linear, whence smooth. We obtain a commutative diagram with smooth columns

$$\begin{array}{ccc}
\Gamma(\mathbf{L}(\mathcal{G}))\times W & \xrightarrow{\widetilde{\mathrm{ev}}|^{\pi_\alpha^{-1}(U_\kappa)}_{\Gamma(\mathbf{L}(\mathcal{G}))\times W}} & \pi_\alpha^{-1}(U_\kappa)\\
{\scriptstyle(\theta_{\kappa,\psi}\times\psi)}\downarrow & & \uparrow{\scriptstyle\kappa^{-1}}\\
C^\infty(\psi(W),E)\times\psi(W) & \xrightarrow{(1\circ\mathrm{pr}_2,\mathrm{ev})} & U_\kappa\times E
\end{array}$$

where $\mathrm{ev}\colon C^\infty(\psi(W),E)\times\psi(W)\to E,(\lambda,x)\mapsto\lambda(x)$ is the evaluation map. By [AS12, Proposition 3.20] (which is applicable by [Woc13, Lemma 5.3]) the map ev is smooth. Furthermore, the local trivialisation κ was chosen arbitrarily, the map $\widetilde{\mathrm{ev}}$ is smooth and in conclusion f_0 is of class $C^{0,\infty}$.

We claim that the following diagram makes sense and commutes:

$$\begin{array}{ccc}
[0,1]\times G\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}} & \xrightarrow{f} & TG\\
{\scriptstyle(\mathrm{id}_{[0,1]}\times(\beta,\mathrm{id}_G)\times\mathrm{id}_{C^r([0,1],\Gamma(\mathbf{L}(\mathcal{G})))})}\downarrow & & \uparrow{\scriptstyle Tm}\\
[0,1]\times M\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}}\times G & \xrightarrow{f_0\times 0} & TG\times_{T\alpha,T\beta}TG
\end{array}\qquad(21)$$

Assume for a moment that the claim is true and (21) commutes. Then f is a $C^{0,\infty}$-map by the chain rules for $C^{0,\infty}$ mappings and smooth maps [AS12, Lemma 3.17 and Lemma 3.18]. By construction (see 4.2) the map f factors through the split submanifold $T^\alpha G$, whence assertion a) follows.

Proof of the claim: Fix $(t, g, \eta) \in [0,1] \times G \times C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))$. Notice first that the composition makes sense: By definition we have $f_0(t, \beta(g), \eta) = T^\alpha_{1(\beta(g))}G$. Hence $T\alpha(f_0(t, \beta(g), \eta)) = 0_{\beta(g)} = T\beta(0(g))$. In conclusion $f_0 \times 0$ factors through $TG \times_{T\alpha, T\beta} TG$. We are left to prove that (21) commutes. To see this we will use the explicit formula for the multiplication in the tangent prolongation. However, since f_0 takes its image only in tangent spaces over units in G, the formula simplifies (cf. [Mac05, Theorem 1.4.14, eq. (4)]) to

$$\begin{aligned} Tm(f_0(t, \beta(g), \eta), 0(g)) &= T(R_g)(f_0(t, \beta(g), \eta)) - T(1)T\alpha(f_0(t, \beta(g), \eta)) + 0(g) \\ &= T(R_g)(\eta^\wedge(t, \beta(g))) - \underbrace{T(1)T\alpha(f_0(t, \beta(g), \eta)))}_{=0(1(\beta(g))} = f(t, g, \eta). \end{aligned}$$

b) In a) we have seen that f is a mapping of class $C^{0,\infty}$ such that for all $g \in G$ we have $f(\cdot, g, \cdot) \in T_g G$. Now G is a smooth Banach-manifold by assumption and the map f satisfies the assumptions of [AS12, 5.12]. Hence [AS12, 5.12] yields for all choices $(t_0, g_0, \eta) \in [0,1] \times G \times C^0([0,1], \Gamma(\mathbf{L}(\mathcal{G})))$ a unique maximal solution $\varphi_{t_0, g_0, \eta} \colon J_{t_0, g_0, \eta} \to G$ of (20) defined on some (relatively) open interval $t_0 \in J_{t_0, g_0, \eta} \subseteq [0,1]$. We claim that it is possible to construct an zero-neighbourhood $\Omega \Subset \Gamma(\mathbf{L}(\mathcal{G}))$ such that for all $(t_0, g_0, \eta) \in [0,1] \times G \times C^0([0,1], \Omega)$ the maximal solution $\varphi_{t_0, g_0, \eta}$ is defined on $[0,1]$. If this is true, then the flow map Fl^f is defined on $[0,1] \times [0,1] \times (G \times C^0([0,1], \Omega))$.

Construction of Ω: We construct the neighbourhood Ω via a local argument in charts. Let us thus fix the following symbols for the rest of this proof:

- F is the Banach-space on which G is modelled,
- E denotes the complemented subspace of F on which $T^\alpha G$ is modelled (cf. 1.3).

Step 1: Reduction to initial values in M. First recall that for $t \in J_{t_0, g_0, \eta}$ the following holds

$$\frac{\partial}{\partial t} \varphi_{t_0, g_0, \eta}(t) = f(t, \varphi_{t_0, g_0, \eta}(t), \eta) = TR_{\varphi_{t_0, g_0, \eta}(t)} \eta^\wedge(t, \beta(\varphi_{t_0, g_0, \eta}(t))) \overset{(6)}{=} \overrightarrow{\eta(t)}(\varphi_{t_0, g_0, \eta}(t)). \tag{22}$$

We conclude that $\varphi_{t_0, g_0, \eta}$ is an integral curve for the time-dependent right-invariant vector field $\overrightarrow{\eta} \colon [0,1] \to \Gamma^\rho(T^\alpha G), t \mapsto \overrightarrow{\eta(t)}$. Arguing as in [Mac05, p. 132 3.6] we derive the following information on the integral curves: Recall that $\overrightarrow{\eta(t)}$ is α-vertical for each $t \in [0,1]$. Thus (22) yields for $g_0 \in G$ the equation $\alpha \circ \varphi_{t_0, g_0, \eta} = \alpha(g_0)$. In particular, the integral curve through $1_{\beta(g_0)}$ restricts to a mapping $J_{t_0, 1_{\beta(g_0)}, \eta} \to \alpha^{-1}(\beta(g_0))$. Thus $c \colon J_{t_0, 1_{\beta(g_0)}, \eta} \to G, t \mapsto R_{g_0} \circ \varphi_{t_0, 1_{\beta(g_0)}, \eta}(t)$ is defined. A quick computation shows that $c(t_0) = g_0$ and by (22) we derive $\frac{\partial}{\partial t} c(t) = \overrightarrow{\eta(t)}(c(t))$. This proves that $c(t)$ is the solution of (20) passing through g_0 at time 0. We conclude that it suffices to construct a zero-neighbourhood Ω such that the solutions $\varphi_{t_0, 1_x, \eta}$ are defined on $[0,1]$ for all $(x, \eta) \in M \times C^0([0,1], \Omega)$.

Step 2: Integral curves on $[0,1]$ for all initial values in a neighbourhood of $x_0 \in M$. Fix $x_0 \in M$. We choose a submersion chart $\kappa_{x_0} \colon U_{x_0} \to V_{x_0} \subseteq F$ for α whose domain contains 1_{x_0}. Then the tangent chart $T\kappa_{x_0}$ is a submersion chart for $T\alpha$, whence it restricts to the bundle $\pi_\alpha \colon T^\alpha G \to G$ and yields a bundle trivialisation $T^\alpha \kappa_{x_0} \colon \pi_\alpha^{-1}(U_{x_0}) \to V_{x_0} \times E$ defined via $T^\alpha \kappa_{x_0} = T\kappa_{x_0}|_{\pi_\alpha^{-1}(U_{x_0})}^{V_{x_0} \times E}$. We remark that $T^\alpha \kappa_{x_0}$ induces a trivialisation of the pullback bundle $\mathbf{L}(G) \to M$. In the following we will identify $\mathbf{L}(G) \to M$ with the restriction of $T^\alpha G$ to $1(M)$ and the fibres $\mathbf{L}(G)_x$ with $T^\alpha_{1_x} G$. Under this identification the pullback trivialisation is just the restriction of $T^\alpha \kappa_{x_0}$ to $\pi_\alpha^{-1}(1(M))$.

Note that the map $1\circ\beta$ fixes units. Hence, by replacing U_{x_0} with the open set $U_{x_0}\cap(1\circ\beta)^{-1}(U_{x_0})\ni 1_{x_0}$ for each $g\in U_{x_0}$ the unit $1_{\beta(g)}$ is also contained in U_{x_0}. Denote the (smooth) inclusion of E into F by I_E^F. We define the map

$$h_{x_0}\colon [0,1]\times(V_{x_0}\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}})\to F,(t,x,\eta)\mapsto I_E^F\circ\text{pr}_2\circ T^\alpha\kappa_{x_0}\circ f(t,\kappa_{x_0}^{-1}(x),\eta) \tag{23}$$

which is of class $C^{0,\infty}$ by the chain rules [AS12, Lemma 3.17 and Lemma 3.18] and part (a). For later use we record that for all $X\in\pi_\alpha^{-1}(U_{x_0})\subseteq TG$ we have $I_E^F\circ\text{pr}_2\circ T^\alpha\kappa_{x_0}(X)=\text{pr}_2\circ T\kappa_{x_0}(X)$. Note that since $1\circ\beta(U_{x_0})\subseteq U_{x_0}$, we can rewrite $h_{x_0}(t,y,\eta)$ for fixed $(t,y)\in[0,1]\times V_{x_0}$ as

$$\begin{aligned}h_{x_0}(t,y,\eta)&=\text{pr}_2\circ T\kappa_{x_0}\circ f(t,\kappa_{x_0}^{-1}(y),\eta)=\text{pr}_2\circ T\kappa_{x_0}\circ TR_{\kappa_{x_0}^{-1}(y)}(\underbrace{\eta^\wedge(t,\beta(\kappa_{x_0}^{-1}(y)))}_{\in\pi_\alpha^{-1}(U_{x_0})})\\&=\text{pr}_2\circ T\kappa_{x_0}\circ TR_{\kappa_{x_0}^{-1}(y)}\circ T^\alpha\kappa_{x_0}^{-1}\circ T^\alpha\kappa_{x_0}(\eta^\wedge(t,\beta(\kappa_{x_0}^{-1}(y))))\\&=\text{pr}_2\circ T\kappa_{x_0}\circ TR_{\kappa_{x_0}^{-1}(y)}\circ T^\alpha\kappa_{x_0}^{-1}(\kappa_{x_0}(1_{\beta(\kappa_{x_0}^{-1}(y))}),\text{pr}_2\,T^\alpha\kappa_{x_0}(\eta^\wedge(t,\beta(\kappa_{x_0}^{-1}(y))))).\end{aligned}$$

Hence we obtain for each $y\in V_{x_0}$ a continuous linear map

$$l_{x_0,y}\colon E\to F, l_{x_0,y}(\omega)=\text{pr}_2\circ T\kappa_{x_0}\circ TR_{\kappa_{x_0}^{-1}(y)}\circ T^\alpha\kappa_{x_0}^{-1}(\kappa_{x_0}(1_{\beta(\kappa_{x_0}^{-1}(y))}),\omega).$$

Set $z_0:=\kappa_{x_0}(1_{x_0})$. By Lemma 5.2 the assignment $l_{x_0}\colon V_{x_0}\to\mathcal{L}(E,F),y\mapsto l_{x_0,y}$ is continuous. Hence we obtain an open z_0-neighbourhood $W_{z_0}\subseteq V_{x_0}$ such that $\sup_{w\in W_{z_0}}\|l_{x_0,w}\|_{\text{op}}\leq B_{x_0}:=\|l_{x_0,z_0}\|_{\text{op}}+1$. Now β is a submersion, whence an open map. We conclude from $\beta\kappa_{x_0}^{-1}(z_0)=x_0$ that $\beta\circ\kappa_{x_0}^{-1}(V_{x_0})$ is an open neighbourhood of x_0. Choose a compact x_0-neighbourhood $A_{x_0}\subseteq\beta(U_{x_0})$ and note that $z_0\in(\beta\circ\kappa_{x_0}^{-1})^{-1}(A_{x_0}^\circ)\Subset V_{x_0}$. Hence there is $R_{x_0}>0$ with $\overline{B_{2R_{x_0}}(z_0)}\subseteq W_{z_0}\cap(\beta\circ\kappa_{x_0}^{-1})^{-1}(A_{x_0}^\circ))$.

By Lemma 5.3 we can shrink R_{x_0} and choose a zero-neighbourhood $N_0\Subset\Gamma(\mathbf{L}(\mathcal{G}))$ such that the map h_{x_0} is uniformly Lipschitz continuous in the Banach-space component on $[0,1]\times C^0([0,1],N_0)\times\overline{B_{2R_{x_0}}(z_0)}$. Fix $v\in B_{R_{x_0}}(z_0)$ and estimate the supremum of the norm of h_{x_0} on $[0,1]\times C^0([0,1],N_0)\times\overline{B_{R_{x_0}}(v)}$. By choice of v the ball $\overline{B_{R_{x_0}}(v)}$ is contained in $B_{2R_{x_0}}(z_0)\subseteq W_{x_0}$. Thus for all $y\in\overline{B_{R_{x_0}}(v)}$ we have the upper bound B_{x_0} for $\|l_{x_0,y}\|_{\text{op}}$. We obtain an estimate the supremum of $\|h_{x_0}(t,y,\eta)\|_F$ over $(t,y,\eta)\in[0,1]\times C^0([0,1],N_0)\times\overline{B_{R_{x_0}}(v)}$ as

$$\begin{aligned}\sup_{(t,y,\eta)}\|h_{x_0}(t,y,\eta)\|_F&\leq\sup_{(t,y,\eta)}\|l_{x_0,y}\|_{\text{op}}\left\|\text{pr}_2\,T^\alpha\kappa_{x_0}\circ\eta^\wedge(t,\beta(\kappa_{x_0}^{-1}(y)))\right\|_E\\&\leq B_{x_0}\sup_{(t,y,\eta)}\left\|\text{pr}_2\,T^\alpha\kappa_{x_0}\circ\eta^\wedge(t,\beta(\kappa_{x_0}^{-1}(y)))\right\|_E.\end{aligned}\tag{24}$$

By construction of $B_{2R_{x_0}}(z_0)$ we have $\beta\circ\kappa_{x_0}^{-1}(B_{2R_{x_0}}(z_0))\subseteq A_{x_0}$ and A_{x_0} is a compact subset of $1^{-1}(U_{x_0})\subseteq M$. Now $T^\alpha\kappa_{x_0}$ restricts to a trivialisation of the pullback bundle $\mathbf{L}(G)\to M$ (identified with a subset of $T^\alpha G$). Recall that A_{x_0} is a compact set. Hence for each open set $O\subseteq E$ by definition of the compact open topology and Definition A.6 the set $\lfloor A_{x_0},O\rfloor:=\{f\in C^\infty(\beta(U_{x_0}),E)\mid f(A_{x_0})\subseteq O\}$ is open in $C^\infty(\beta(U_{x_0}),E)$. Let $\text{res}\colon\Gamma(\mathbf{L}(\mathcal{G}))\to\Gamma(\mathbf{L}(\mathcal{G})|_{\beta(U_{x_0})}),f\mapsto f|_{\beta(U_{x_0})}$. Then a combination of [Woc13, Lemma 5.5] with Theorem A.7 implies that

$$D:=\left\{X\in\Gamma(\mathbf{L}(\mathcal{G}))\middle|\sup_{y\in A_{x_0}}\|\text{pr}_2\circ T^\alpha\kappa_{x_0}\circ X(y)\|_E<\frac{R_{x_0}}{B_{x_0}}\right\}=((\text{pr}_2\circ T^\alpha\kappa_{x_0})_*\circ\text{res})^{-1}\left(B_{\frac{R_{x_0}}{B_{x_0}}}(0)\right)$$

is an open neighbourhood of the zero-section in $\Gamma(\mathbf{L}(\mathcal{G}))$.

Define $\Omega_0:=D\cap N_0$. Clearly $\Omega_0\Subset\Gamma(\mathbf{L}(\mathcal{G}))$ is a zero-neighbourhood and $C^0([0,1],\Omega_0)$ is open in $C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}}$. From (24) we derive that for all $v\in B_{R_{x_0}}(z_0)$ and $\eta\in C^0([0,1],\Omega_{x_0})$ the estimate:

$$\sup_{(t,y,\eta)\in[0,1]\times B_{R_{x_0}}(v)\times C^0([0,1],\Omega_{x_0})}\|h_{x_0}(t,y,\eta)\|_F<R_{x_0}\tag{25}$$

We will now solve the initial value problem (20) for fixed $\eta \in C^0([0,1],\Omega_{x_0})$ and $(t_0,v) \in [0,1]\times B_{R_{x_0}}(z_0)$:

$$\begin{cases} \frac{\partial}{\partial t}c(t) = h_{x_0}(t,c(t),\eta), \\ c(t_0) = v. \end{cases} \tag{26}$$

From the argument above, we know that $h_{x_0}(\cdot,\eta)\colon [0,1]\times B_{2R_{x_0}}(z_0) \to F$ satisfies a uniform Lipschitz condition in the Banach space component (i.e. in $B_{2R_{x_0}}(z_0)$). In addition, (25) shows that for all $\eta \in C^0([0,1],\Omega_{x_0})$ we obtain $M := \sup_{(t,x)\in[0,1]\times\overline{B_{R_{x_0}}(v)}} \|h_{x_0}(t,x,\eta)\|_E < R_{x_0}$. Observe that $\frac{R_{x_0}}{M} \geq 1$. Now a combination of [Ama90, 7.4 Local Existence and Uniqueness Theorem] with [Ama90, Remark 7.10 (a)] shows that the initial value problem (26) admits a unique solution $c_{t_0,v,\eta}\colon [0,1] \to B_{R_{x_0}}(v)$. [4] It is easy to see that $\kappa_{x_0}^{-1}\circ c_{t_0,v,\eta}$ is just the integral curve $\varphi_{t_0,\kappa_{x_0}^{-1}(v),\eta}$, whence this curve must exist on $[0,1]$. Hence for all $(t_0,y,\eta)\in[0,1]\times\kappa_{x_0}(B_{R_{x_0}}(z_0))\times C^0([0,1],\Omega_{x_0})$ the initial value problem (20) admits a unique solution $\varphi_{t_0,y,\eta}$ on $[0,1]$. Finally we remark that $\kappa_{x_0}(B_{R_{x_0}}(z_0))$ is an open neighbourhood of 1_{x_0}.

Step 3: Define Ω. We construct for each $x\in M$ as in Step 2 an open neighbourhood $\mathcal{N}_x \subseteq G$ of x and a zero-neighbourhood $\Omega_x \Subset \Gamma(\mathbf{L}(\mathcal{G}))$. By construction the solution $\varphi_{t_0,y,\eta}$ of (20) exists on $[0,1]$ for all $(t_0,y,\eta)\in[0,1]\times\mathcal{N}_x\times C^0([0,1],\Omega_x)$. Since M is compact, there is a finite set $x_1,\ldots,x_n\in M, n\in\mathbb{N}$ such that $G_{x_0}\subseteq\bigcup_{1\leq i\leq n}\mathcal{N}_{x_i}$. Then $\Omega := \bigcap_{1\leq i\leq n}\Omega_{x_i}$ is an open zero-neighbourhood in $\Gamma(\mathbf{L}(\mathcal{G})))$. By construction for all $(t_0,x,\eta)\in[0,1]\times M\times C^0([0,1],\Omega)$ the solution $\varphi_{t_0,1_x,\eta}$ of (20) exists on $[0,1]$.

c) Let $r\in\mathbb{N}_0\cup\{\infty\}$. For $\eta\in C^0([0,1],\Omega)$ we derive from (b) that the integral curves for (20) exist on $[0,1]$. From (a) we know that f is of class $C^{0,\infty}$. Hence [AS12, Proposition 5.13] implies that for fixed $t_0\in[0,1]$ the map $\mathrm{Fl}^f(t_0,\cdot)\colon [0,1]\times(G\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G}))_{\text{c.o.}})\to G$ is a mapping of class $C^{1,\infty}$. Specialising to $t_0=0$ we see that Fl^f_0 is of class $C^{1,\infty}$.

d) Fix $\eta\in C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))$ and $s,t\in[0,1]$. We have to prove that $\Psi_{s,t}:=\beta\circ\mathrm{Fl}^f(s,t,\cdot,\eta)\circ 1\colon M\to M$ is a diffeomorphism. The map $\Psi_{s,t}$ is smooth as a composition of smooth mappings. We claim that the inverse of $\Psi_{s,t}$ is given by $\Psi_{t,s}:=\beta\circ\mathrm{Fl}^f(t,s,\cdot,\eta)\circ 1$. To see this recall the following properties of the flow for $t_0,t_1,t_2\in[0,1]$

$$\mathrm{Fl}^f(t_1,t_2,\cdot,\eta)\circ\mathrm{Fl}^f(t_0,t_1,\cdot,\eta)=\mathrm{Fl}^f(t_0,t_2,\cdot,\eta) \quad\text{and}\quad \mathrm{Fl}^f(t_0,t_0,\cdot,\eta)=\mathrm{id}_G(\cdot) \tag{27}$$

$$\forall (h,g)\in G\times_{\alpha,\beta}G,\ \text{by (b) and (c) Step 1: } R_g\circ\mathrm{Fl}^f(t_0,t_1,h,\eta)=\mathrm{Fl}^f(t_0,t_1,\cdot,\eta)\circ R_g(h) \tag{28}$$

Furthermore, we observe for $g\in G$ that $R_g^{-1}=R_{g^{-1}}$. Hence for $x\in M$ a combination of (27) and (28) yields $R^{-1}_{\mathrm{Fl}^f(t,s,1_x,\eta)}(1_x)=\mathrm{Fl}^f(s,t,\cdot,\eta)\circ R^{-1}_{\mathrm{Fl}^f(t,s,1_x,\eta)}\circ\mathrm{Fl}^f(t,s,\cdot,\eta)(1_x)=\mathrm{Fl}^f(s,t,\cdot,\eta)(1_{\beta(\mathrm{Fl}^f(t,s,1_x,\eta))})$. Together with $\beta(R_g(h))=\beta(h)$ for all $(h,g)\in G\times_{\alpha,\beta}G$, the last observation enables the following computation:

$$\begin{aligned} x=\beta(1_x)&=\beta\circ R^{-1}_{\mathrm{Fl}^f(t,s,1_x,\eta)}(1_x)=\beta\circ\mathrm{Fl}^f(s,t,\cdot,\eta)\circ R^{-1}_{\mathrm{Fl}^f(t,s,1_x,\eta)}\circ\mathrm{Fl}^f(t,s,\cdot,\eta)(1_x)\\ &=\beta\circ\mathrm{Fl}^f(s,t,\cdot,\eta)\circ 1\circ\beta\circ\mathrm{Fl}^f(t,s,\cdot,\eta)(1_x)=\Psi_{s,t}\circ\Psi_{t,s}(x)\end{aligned}$$

Interchanging the roles of $\Psi_{s,t}$ and $\Psi_{t,s}$ we see that indeed $\Psi_{s,t}^{-1}=\Psi_{t,s}$ and $\Psi_{s,t}$ is a diffeomorphism.

e) The map $1\colon M\to G$ is smooth. Thus the chain rule [AS12, Lemma 3.17] and (c) show that $H_\eta=\mathrm{Fl}^f_0(\cdot,\eta)\circ\mathrm{id}_{[0,1]}\times 1$ is a $C^{1,\infty}$-map. From (22) and step 1 in (a) we infer $\alpha\circ c_\eta(t)=\mathrm{id}_M$. Then a combination of [Glö12, 1.7] and (d) shows that $c_\eta(t)\in\mathrm{Bis}(\mathcal{G})$ and thus $c_\eta\colon[0,1]\to\mathrm{Bis}(\mathcal{G})$ makes sense.

To see that c_η is continuous, recall from Definition A.6 that the topology on $C^\infty(M,G)$ is initial with respect to the family $(T^n\colon C^\infty(M,G)\to C^0(T^nM,T^nG)_{\text{c.o.}}, f\mapsto T^nf)_{n\in\mathbb{N}_0}$. Now $\mathrm{Bis}(\mathcal{G})$ is an embedded

[4] Note that the proof of [Ama90, 7.4] for infinite-dimensional Banach spaces requires a uniform Lipschitz condition on all of $\overline{B_{2R_{x_0}}(z_0)}$ (cf. [Ama90, Remark 7.5 (b)]).

submanifold whence the topology on $\mathrm{Bis}(\mathcal{G})$ is the subspace topology induced by $C^\infty(M,G)$. Clearly c_η will be continuous if for all $n \in \mathbb{N}_0$ the map $T^n \circ c_\eta$ is continuous. Observe that $c_\eta^\wedge = H_\eta$ and the mapping $H_\eta\colon [0,1]\times M \to G$ is of class $C^{1,\infty}$. For $\xi \in T^nM$ and $t\in[0,1]$ fixed, we obtain the formula $T^n c_\eta(t)(\xi) = T^n H_\eta(t,\cdot)(\xi)$ (where we compute the tangent only with respect to the argument in M). We compute locally to exploit the $C^{1,\infty}$-property of H_η: Choose an open t-neighbourhood $U_t \Subset [0,1]$ with inclusion $i_t\colon U_t \hookrightarrow [0,1]$ and charts κ of M and λ of G such that $\lambda\circ H_\eta \circ (i_t\times\kappa)$ is defined. Then we see (cf. [Woc13, Lemma 5.3]) that

$$T^n\lambda \circ T^n H_\eta(t,\cdot)\circ T^n\kappa = T^n(\lambda\circ H_\eta(t,\cdot)\circ\kappa) = T^{n-1}(\lambda\circ H_\eta(t,\cdot)\circ\kappa)\times(dT^{n-1}(\lambda\circ H_\eta(t,\cdot)\circ\kappa)). \quad (29)$$

Denote by $\mathbb{R}^k$ the model space of M and let $f\colon \mathbb{R}^k \supseteq U \to F$ be a smooth mapping from an open subset into a locally convex space. Recall from [Glö02, p. 49] the following variant of the usual differential for f: Set $d^0 f = f$, $d^1 f = df$ and $d^n f = d(d^{n-1}f)\colon U\times(\mathbb{R}^k)^{2^n-1}\to F$. Note that by [Glö02, Lemma 1.14] these differentials exist for any C^∞-map f. Thus (29) shows that we can recursively split the tangent into a product of derivatives $d^r(\lambda\circ H_\eta(t,\cdot)\circ\kappa)$ composed with projections. Furthermore, the formula in the proof of [Glö02, Lemma 1.14] and the $C^{1,\infty}$-property of H_η show that $d^r(\lambda\circ H_\eta(t,\cdot)\circ\kappa_2)$ depends continuously on t. In conclusion, $T^{0,n}H_\eta\colon [0,1]\times T^nM \to T^nG, (t,\xi)\mapsto T^nH_\eta(t,\cdot)(\xi)$ is continuous. The manifold M is finite dimensional and so is T^nM for all $n\in\mathbb{N}_0$. In particular, T^nM is locally compact for $n\in\mathbb{N}_0$ and we derive from [Eng89, Theorem 3.4.1] that $T^nc_\eta = (T^{0,n}H_\eta)^\vee$ is continuous.

We will prove now that c_η is of class C^1. It suffices to prove that c_η is locally of class C^1. To do so fix $s\in[0,1]$ and recall some facts from the construction of Ω in (b). The set Ω was constructed with respect to a finite family $\mathcal{N}_i \Subset G, 1\leq i\leq n$ which satisfies:

(i) $1(M) \subseteq \bigcup_{1\leq i\leq n}\mathcal{N}_i$,

(ii) For each $1\leq i\leq n$ there is a manifold chart $\kappa_i\colon U_i\to V_i\subseteq F$, such that $\mathrm{Fl}^f(0,\cdot)|_{[0,1]\times\mathcal{N}_i\times C^0([0,1],\Omega)}$ takes its values in U_i (cf. (b) Step 2). Moreover, κ_i is a submersion chart for α, whence $T^\alpha\kappa_i := T\kappa_i|^E_{T^\alpha U_i}\colon T^\alpha U_i \to U_i\times E$ is a trivialisation of $T^\alpha G$.

Set $g := c_\eta(s)\in\mathrm{Bis}(\mathcal{G})$. Recall from Proposition 2.6 the form of a manifold chart around g of $\mathrm{Bis}(\mathcal{G})$: $\varphi_g\colon O_g\to\varphi_g(O_g)\Subset\Gamma(g^*T^\alpha G)$, where $O_g := \{s\in\mathrm{Bis}(\mathcal{G}) \mid \forall x\in M,\ (s(x),g(x))\in Q\}$ for a a fixed neighbourhood Q of the diagonal in $N\times N$.

Consider the $C^{1,\infty}$-mapping $c_\eta^\wedge := H_\eta\colon [0,1]\times M\to G$. Then $(g\circ\mathrm{pr}_2, c_\gamma^\wedge)\colon [0,1]\times M\to G\times G$ is continuous with $\{s\}\times M \subseteq (g\circ\mathrm{pr}_2,c_\gamma^\wedge)^{-1}(V)\Subset[0,1]\times M$. Hence there is a relatively open interval $s\in J_s\subseteq[0,1]$ such that $c_\eta(\overline{J_s})\subseteq O_g$. We will prove that $\varphi_g\circ c_\eta|_{J_s}\colon J_s\to\Gamma(g^*T^\alpha G)$ is C^1. Note first that $c_\eta(t)$ maps $1^{-1}(\mathcal{N}_i)\Subset M$ into the chart domain U_i for all $t\in J_s$. In particular, for $z\in 1^{-1}(\mathcal{N}_i)$ the compact set $c_\eta^\wedge(\overline{J_s}\times\{z\})\times\{g(z)\}$ is contained in $V\cap(U_i\times U_i)$. We apply Wallace Theorem [Eng89, 3.2.10] to obtain open neighbourhoods $U_{z,s},V_z\Subset G$ with $c_\eta^\wedge(\overline{J_s}\times\{z\})\times\{g(z)\}\subseteq U_{z,s}\times V_z\Subset V\cap(U_i\times U_i)$. The set $(c_\eta^\wedge)^{-1}(U_{z,s})$ is an open neighbourhood of $\overline{J_s}\times\{z\}\in[0,1]\times 1^{-1}(\mathcal{N}_i)$. Apply Wallace Theorem again to find an open z-neighbourhood $W_z\Subset 1^{-1}(\mathcal{N}_i)$ such that $g(W_z)\subseteq V_z$ and $J_s\times W_z\subseteq(c_\eta^\wedge)^{-1}(U_{z,s})$. By (i) the open sets $(1^{-1}(\mathcal{N}_i))_{1\leq i\leq n}$ cover M. Thus we repeat the construction of W_z for all $z\in M$. By compactness of M, there are finitely many $z_j, 1\leq j\leq m$ such that $M=\bigcup_{1\leq j\leq m}W_{z_j}$. For each $1\leq j\leq m$ we choose κ_{i_j} such that $c_\eta(J_s\times W_{z_j})\times g(W_{z_j})\subseteq U_{z_j,s}\times V_{z_j}\subseteq U_{i_j}\times U_{i_j}$. Note that by (ii) the trivialisations $T^\alpha\kappa_{i_j}$ of $T^\alpha G$ induce an atlas of trivialisations for the bundle $g^*T^\alpha G$. From [Woc13, Proposition 7.3] we recall that the topology of $\Gamma(g^*T^\alpha G)$ is initial with respect to

$$\Phi\colon\Gamma(g^*T^\alpha G)\to\prod_{1\leq j\leq n}C^\infty(W_{z_j},E), \sigma\mapsto\mathrm{pr}_2\circ T^\alpha\kappa_{i_j}\circ\sigma|_{W_{z_j}}$$

and that Φ is a linear topological embedding with closed image. Thus $\varphi_g\circ c_\eta|_{J_s}$ will be of class C^1 if and only if for each $1\leq j\leq m$ the map $p_j\circ\Phi\circ\varphi_g\circ c_\eta|_{J_s}\colon J_s\to C^\infty(W_{z_j},E)$ is C^1. Here p_j is the projection onto the j-th component of the product. The manifold $W_{z_j}\Subset M$ is finite dimensional and $J_s\Subset[0,1]$ is a locally convex subset with dense interior of $\mathbb{R}$. We can thus apply the exponential law for $C^{1,\infty}$-maps [AS12, Theorem 4.6 (d)]: The map $\varphi_g\circ c_\eta|_{J_s}$ will be of class C^1 if and only if for each

$1 \leq j \leq m$ the map $(\mathrm{pr}_j \circ \Phi \circ \varphi_g \circ c_\eta|_{J_s})^\wedge \colon J_s \times W_{z_j} \to E$ is of class $C^{1,\infty}$. Let $\pi_\alpha \colon T^\alpha G \to G$ be the bundle projection and denote by Σ the local addition on G adapted to α. Using the description of the chart φ_g we compute an explicit formula for $(\mathrm{pr}_j \circ \Phi \circ \varphi_g \circ c_\eta|_{J_s})^\wedge$:

$$\begin{aligned}(\mathrm{pr}_j \circ \Phi \circ \varphi_g \circ c_\eta|_{J_s})^\wedge(t,x) &= (\mathrm{pr}_2 T^\alpha \kappa_{i_j} \circ (\pi_\alpha, \Sigma)^{-1} \circ (g, c_\eta|_{J_s}))^\wedge(t,x) \\ &= (\mathrm{pr}_2 T^\alpha \kappa_{i_j} \circ (\pi_\alpha, \Sigma)^{-1} \circ (g, c_\eta(t))(x)) \\ &= \mathrm{pr}_2 T^\alpha \kappa_{i_j} \circ (\pi_\alpha, \Sigma)^{-1} \circ (g \circ \mathrm{pr}_{W_{z_j}}, c_\eta^\wedge|_{J_s \times W_{z_j}})(t,x)\end{aligned} \tag{30}$$

Here $\mathrm{pr}_{W_{z_j}} \colon J_s \times W_{z_j} \to W_{z_j}$ is the canonical (smooth) projection. By construction of the open sets W_{z_j}, the smooth mapping $\mathrm{pr}_2 T^\alpha \kappa_i \circ (\pi_\alpha, \Sigma)^{-1} \circ (\mathrm{id}_{U_{z,s}}, g)$ is defined on the product $U_{z_j,s} \times W_{z_j}$. Furthermore $c_\eta^\wedge|_{J_s \times W_{z_j}} = H_\eta|_{J_s \times W_{z_j}}$ holds and thus $c_\eta^\wedge|_{J_s \times W_{z_j}}$ is of class $C^{1,\infty}$. We have $c_\eta^\wedge(J_s \times W_{z_j}) \subseteq U_{z_j,s}$. Computing in local charts, the chain rule [AS12, Lemma 3.19] together with (30) implies that $(\mathrm{pr}_j \circ \Phi \circ \varphi_g \circ c_\eta|_{J_s})^\wedge$ is a mapping of class $C^{1,\infty}$. We conclude that $c_\eta|_{J_s} \colon J_s \to \mathrm{Bis}(\mathcal{G})$ is of class C^1, whence the assertion follows. ■

Lemma 5.2. *In the situation of Proposition 4.4 b) denote by F the model space of G and by E the typical fibre of $T^\alpha G$. Let $\kappa \colon U \to V \subseteq F$ be a submersion chart for α such that for all $g \in U$ we have $1_{\beta(g)} \in U$. Furthermore, we denote by $T^\alpha \kappa$ the trivialisation of the bundle $T^\alpha G$ obtained by restriction of $T\kappa$ to $T^\alpha G$. Then the map*

$$l \colon V \to \mathcal{L}(E,F), y \mapsto l_y \qquad \text{with } l_y(\omega) = \mathrm{pr}_2 \circ T\kappa \circ TR_{\kappa^{-1}(y)} \circ T^\alpha \kappa^{-1}(\kappa(1_{\beta(\kappa^{-1}(y))}), \omega)$$

is continuous with respect to the operator-norm topology on $\mathcal{L}(E,F)$.

Proof. Before we tackle the continuity, we begin with some preliminaries: For each $y \in V$, the element $(1_{\beta(\kappa(y))}, \kappa^{-1}(y))$ is contained in the domain $G \times_{\alpha,\beta} G$ of the groupoid multiplication m. Define the map $u \colon V \to G \times_{\alpha,\beta} G, v \mapsto (1_{\beta(\kappa^{-1}(v))}, \kappa^{-1}(v))$. As $G \times_{\alpha,\beta} G$ is a split submanifold of $G \times G$ (modelled on a complemented subspace H of $F \times F$), the map u is smooth as a mapping into the submanifold. Note that for all $g \in U$ the unit $1_{\beta(g)}$ is contained in U, whence $u(V) \subseteq (U \times U) \cap (G \times_{\alpha,\beta} G)$. It suffices to check continuity of l locally. To this end fix $v \in V$. Let $\tau \colon U_\tau \to V_\tau \subseteq F \times F$ be a submanifold chart for $G \times_{\alpha,\beta} G$ around $u(v)$, i.e. $\tau(U_\tau \cap (G \times_{\alpha,\beta} G)) = V_\tau \cap H$ and $u(v) \in U_\tau$. Shrinking U_τ we can assume that $m(U_\tau \cap (G \times_{\alpha,\beta} G)) \subseteq U$ holds. We obtain an open v-neighbourhood $W_v := u^{-1}(U_\tau) \subseteq V \subseteq F$ such that $u(W_v) \subseteq U_\tau \cap ((U \times U) \times (G \times_{\alpha,\beta} G))$. Now back to l: As explained in Proposition 4.4 a), we can rewrite the formula for l for all $y \in W_v$ as follows

$$\begin{aligned} l(y) &= \mathrm{pr}_2 \circ T\kappa \circ Tm(T^\alpha \kappa^{-1}(\kappa(1_{\beta(\kappa^{-1}(y))}), \omega), 0(\kappa^{-1}(y))) \\ &= \mathrm{pr}_2 \circ T\kappa \circ Tm(T^\alpha \kappa^{-1} \times T^\alpha \kappa^{-1}((\kappa(1_{\beta(\kappa^{-1}(y))}), \omega)(y,0))) \\ &= \mathrm{pr}_2 \circ T(\kappa \circ m \circ \tau^{-1}|_{V_\tau \cap H})(T\tau \circ (T^\alpha \kappa^{-1} \times T^\alpha \kappa^{-1})((\kappa(1_{\beta(\kappa^{-1}(y))}), \omega)(y,0))) \end{aligned} \tag{31}$$

The above formula shows that the map l splits into several components. We exploit this splitting to prove continuity of $l|_{W_v}$. First consider $\mathrm{pr}_2 \circ T(\kappa \circ m \circ \tau^{-1}|_{V_\tau \cap H})$. The mapping $\kappa \circ m \circ \tau^{-1}|_{V_\tau \cap H} \colon V_\tau \cap H \to V$ is well-defined and smooth. We see that $\mathrm{pr}_2 \circ T(\kappa \circ m \circ \tau^{-1}|_{V_\tau \cap H}) = d(\kappa \circ m \circ \tau^{-1}|_{V_\tau \cap H}) \colon (V_\tau \cap H) \times H \to F$. Since H and F are Banach spaces, [Mil82, Lemma 2.10] implies that $p \colon V_\tau \cap H \to \mathcal{L}(H,F), y \mapsto d(\kappa \circ m \circ \tau^{-1}|_{V_\tau \cap H})(y, \cdot)$ is continuous.

Now we deal with $T\tau \circ (T^\alpha \kappa^{-1} \times T^\alpha \kappa^{-1})$: Recall that $\kappa \colon U \to V \subseteq F$ is a chart for G and for the bundle trivialisation we have $T^\alpha \kappa = T\kappa|_{\pi_\alpha^{-1}(U)}^{V \times E}$. Hence $T\tau \circ (T^\alpha \kappa^{-1} \times T^\alpha \kappa^{-1})$ is the restriction of $T(\tau \circ (\kappa^{-1} \times \kappa^{-1}))$ to the subset $((\kappa \times \kappa((U \times U) \cap U_\tau)) \times E \times E)$. Again from [Mil82, Lemma 2.10] it follows that

$$q \colon \kappa \times \kappa((U \times U) \cap U_\tau) \to \mathcal{L}(F \times F, F \times F), x \mapsto d(\tau \circ \kappa^{-1} \times \kappa^{-1}|_{\kappa \times \kappa((U \times U) \cap U_\tau)})(x, \cdot)$$

is a continuous map. Let I_E^F be the canonical (smooth) inclusion of E into F. Then for all x in the domain of q we derive

$$q(x) \circ (I_E^F \times I_E^F)(\cdot) = \mathrm{pr}_2 T\tau \circ (T^\alpha \kappa^{-1} \times T^\alpha \kappa^{-1})(x, \cdot). \tag{32}$$

We define the canonical inclusion $I_E^{E\times E}\colon E \to E\times\{0\} \subseteq E\times E$ and note that $(I_E^F \times I_E^F)\circ I_E^{E\times E} \in \mathcal{L}(E, F\times F)$. The Banach space H is a split subspace of $F \times F$. Hence the projection $\pi_H\colon F\times F \to H$ is continuous. The composition of continuous linear maps between Banach spaces is jointly continuous. Thus we obtain a continuous map

$$z\colon \kappa\times\kappa((U\times U)\cap U_\tau) \to \mathcal{L}(E,H), x \mapsto \pi_H \circ q(x)\circ (I_E^F\times I_E^F)\circ I_E^{E\times E}$$

Now for $x \in (U\times U)\cap U_\tau \cap G \times_{\alpha,\beta} G$ we set $(x_1,x_2) := \kappa\times\kappa(x)$ and let $Y\in E$. Then the identity (32) together with τ being a submanifold chart for $G\times_{\alpha,\beta} G$ shows

$$q(x)\circ(I_E^F\times I_E^F)\circ I_E^{E\times E}(Y) = \mathrm{pr}_2\, T_x\tau(\underbrace{T_{x_1}^{\alpha}\kappa^{-1}(Y)\times 0_{\kappa^{-1}(x_2)}}_{\in TG\times_{T\alpha,T\beta}TG\overset{(5)}{=}T(G\times_{\alpha,\beta}G)}) \in H$$

In particular, $z(x)(Y) = q(x)\circ(I_E^F\times I_E^F)(Y,0)$. With the help of (32) insert the mappings p and z into (31) to derive for $y\in W_v$ the identity

$$l(y) = p(\tau\circ u(y))\circ z((\kappa\times\kappa)\circ u(y)) \in \mathcal{L}(E,F).$$

We conclude that $l|_{W_v}$ is continuous and the assertion of the Lemma follows. ■

Lemma 5.3. *Let $h_{x_0}\colon [0,1]\times V_{x_0}\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G}))) \to F$ be the map defined in (23). Then there is an open zero-neighbourhood $N_0 \Subset \Gamma(\mathbf{L}(\mathcal{G}))$ and $\varepsilon>0$ such that $h_{x_0}|_{[0,1]\times C^0([0,1],N_0)\times B_\varepsilon(z_0)}$ is uniformly Lipschitz continuous with respect to the Banach space component.*
Here we define $C^0([0,1],N_0) = \{\eta\in C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G}))) \mid \eta([0,1])\subseteq N_0\} \Subset C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{c.o.}$.

Proof. Reordering the product on which h_{x_0} is defined, we identify h_{x_0} with a $C^{0,\infty}$-mapping with respect to the decomposition $([0,1]\times C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))_{\text{c.o.}})\times V_{x_0}$. By [Ama90, Proposition 6.3] the map h_{x_0} is locally Lipschitz continuous with respect to the Banach space component. Consider the constant map $\mathbf{0}\colon [0,1]\to \Gamma(\mathbf{L}(\mathcal{G}))$ whose image is the zero section in $\Gamma(\mathbf{L}(\mathcal{G}))$. Since $[0,1]\times\{\mathbf{0}\}\times\{z_0\}$ is compact, there are finitely many indices $1\le i\le n, n\in\mathbb{N}$ such that $[0,1]=\bigcup_{1\le i\le n} J_i$ for $J_i\Subset[0,1]$ and the following holds:

For each $1\le i\le n$ there is $U_i \Subset C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))$ and $\varepsilon_i>0$ such that on $J_i\times U_i\times B_{\varepsilon_i}(z_0)$ the mapping h_{x_0} is Lipschitz continuous with respect to $z\in B_{\varepsilon_i}(z_0)$.

We let λ_i be the minimal Lipschitz constant for h_{x_0} on $J_i\times U_i\times B_{\varepsilon_i}(z_0)$. Let $\mathcal{P}$ be a subbasis for the topology of $\Gamma(\mathbf{L}(\mathcal{G}))$. Since $[0,1]$ is compact, the sets $C^0([0,1],W) := \{f\in C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))|f([0,1])\subseteq W\}$ with $W\in\mathcal{P}$ form a subbasis of the compact-open topology on $C^0([0,1],\Gamma(\mathbf{L}(\mathcal{G})))$. Hence there are $W_1,\ldots W_m\in\mathcal{P}, m\in\mathbb{N}$ such that $C^0([0,1],\bigcap_{1\le j\le m}W_j) = \bigcap_{1\le j\le m} C^0([0,1],W_j)\subseteq\bigcap_{1\le i\le n}U_i$ is a $\mathbf{0}$-neighbourhood. Define $N_0 := \bigcap_{1\le j\le n} W_j$. Since $\mathbf{0}\in C^0([0,1],N_0)$ and thus $\mathbf{0}(1)=0\in N_0$ holds, $N_0\Subset\Gamma(\mathbf{L}(\mathcal{G}))$ is a zero-neighbourhood. Now define $L:=\max_{1\le i\le n}\{\lambda_i\}$ and $\varepsilon = \frac{\min_{1\le i\le n}\varepsilon_i}{2}$. We consider $(t,\eta,x)(t,\eta,y)\in[0,1]\times C^0([0,1],N_0)\times B_\varepsilon(z_0)$ such that $t\in J_i$. From $\|x-y\|_E<\varepsilon\le\frac{\varepsilon_i}{2}$ we derive $\|y-z_0\|_E\le\|y-x\|_E+\|x-z_0\|_E<\varepsilon_i$. Thus $\|h_{x_0}(t,\eta,x)-h_{x_0}(t,\eta,y)\|_F\le\lambda_i\|x-y\|_E\le L\|x-y\|_E$ and L is a Lipschitz constant for h_{x_0} on $[0,1]\times C^0([0,1],N_0)\times B_\varepsilon(0)$. In conclusion, h_{x_0} is uniformly Lipschitz continuous with respect to the Banach space component. ■

A Locally convex manifolds and spaces of smooth maps

In this appendix we collect the necessary background on the theory of manifolds that are modelled on locally convex spaces and how spaces of smooth maps can be equipped with such a structure. Let us first recall some basic facts concerning differential calculus in locally convex spaces. We follow [Glö02, BGN04].

Definition A.1. Let E,F be locally convex spaces, $U\subseteq E$ be an open subset, $f\colon U\to F$ a map and $r\in\mathbb{N}_0\cup\{\infty\}$. If it exists, we define for $(x,h)\in U\times E$ the directional derivative

$$df(x,h) := D_hf(x) := \lim_{t\to 0} t^{-1}(f(x+th)-f(x)).$$

We say that f is C^r if the iterated directional derivatives

$$d^{(k)}f(x, y_1, \ldots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$ and $y_1, \ldots, y_k \in E$ and define continuous maps $d^{(k)}f \colon U \times E^k \to F$. If f is C^∞ it is also called smooth. We abbreviate $df := d^{(1)}f$.

From this definition of smooth map there is an associated concept of locally convex manifold, i.e., a Hausdorff space that is locally homeomorphic to open subsets of locally convex spaces with smooth chart changes. See [Woc13, Nee06, Glö02] for more details. ■

Definition A.2 (Differentials on non-open sets). a) A subset U of a locally convex space E is called *locally convex* if every $x \in U$ has a convex neighbourhood V in U.

b) Let $U \subseteq E$ be a locally convex subset with dense interior and F a locally convex space. A continuous mapping $f \colon U \to F$ is called C^r if $f|_{U^\circ} \colon U^\circ \to F$ is C^r and each of the maps $d^{(k)}(f|_{U^\circ}) \colon U^\circ \times E^k \to F$ admits a continuous extension $d^{(k)}f \colon U \times E^k \to F$ (which is then necessarily unique). Analogously, we say that a continuous map $g \colon U \to M$ to a smooth manifold M is of class C^r if the tangent maps $T^k(f|_{U^\circ}) \colon U^\circ \times E^{2^k-1} \to T^kM$ exist and admit a continuous extension $T^kf \colon U \times E^{2^k-1} \to T^kM$. Note that we defined C^k-mappings on locally convex sets with dense interior in two ways for topological vector spaces (when viewed as manifolds). However, by [Glö02, Lemma 1.14] both conditions yield the same class of mappings. If $U \subseteq \mathbb{R}$ and g is C^1, we obtain a continuous map $g' \colon U \to TM, g'(x) := T_xg(1)$. We shall write $\frac{\partial}{\partial x} g(x) := g'(x)$. ■

Definition A.3. Let M be a smooth manifold. Then M is called *Banach* (or *Fréchet*) manifold if all its modelling spaces are Banach (or Fréchet) spaces. The manifold M is called *locally metrisable* if the underlying topological space is locally metrisable (equivalently if all modelling spaces of M are metrizable). It is called *metrizable* if it is metrisable as a topological space (equivalently locally metrisable and paracompact). ■

Definition A.4. Suppose M is a smooth manifold. Then a *local addition* on M is a smooth map $\Sigma \colon U \Subset TM \to M$, defined on an open neighbourhood U of the submanifold $M \subseteq TM$ such that

a) $\pi \times \Sigma \colon U \to M \times M$, $v \mapsto (\pi(v), \Sigma(v))$ is a diffeomorphism onto an open neighbourhood of the diagonal $\Delta M \subseteq M \times M$ and

b) $\Sigma(0_m) = m$ for all $m \in M$.

We say that M *admits a local addition* if there exist a local addition on M. ■

Lemma A.5. *(cf. [Mic80, 10.11]) Suppose that $\Sigma \colon U \Subset TM \to M$ is a local addition on M and that $\tau \colon T(TM) \to T(TM)$ is the canonical flip on $T(TM)$. Then $T\Sigma \circ \tau \colon \tau(TU) \Subset T(TM) \to TM$ is a local addition on TM. In particular, TM admits a local addition if M does so.*

Proof. Let $0_M \colon M \to TM$ denote the zero section of $\pi_M \colon TM \to M$.

The diffeomorphism $\tau \colon T(TM) \to T(TM)$ is locally given by $(m, x, y, z) \mapsto (m, y, x, z)$ and makes the diagrams

$$\begin{array}{ccc} T(TM) & \xrightarrow{\tau} & T(TM) \\ {\scriptstyle T\pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{TM}} \\ TM & = & TM \end{array} \quad \text{and} \quad \begin{array}{ccc} T(TM) & \xrightarrow{\tau} & T(TM) \\ {\scriptstyle T0_M}\big\uparrow & & \big\uparrow{\scriptstyle 0_{TM}} \\ TM & = & TM \end{array}$$

commute [Mic80, 1.19]. Then $\Sigma \circ 0_M = \mathrm{id}_M$ implies that $T\Sigma$ is defined on the open neighbourhood TU of $T0_M(TM)$ in $T(TM)$ and satisfies $T\Sigma \circ T0_M = \mathrm{id}_{TM}$. This implies that $T\Sigma \circ \tau$ is defined on the open neighbourhood $\tau(TU)$ of $0_{TM}(TM)$. It satisfies $T\Sigma \circ \tau \circ 0_{TM} = \mathrm{id}_{TM}$ and thus A.4 b) by construction. Moreover, if $\pi_M \times \Sigma$ is a diffeomorphism from U onto $V \Subset M \times M$, then $T(\pi_M \times \Sigma) = (T\pi_M \times T\Sigma)$ is a diffeomorphism from TU onto $TV \Subset T(M \times M) = TM \times TM$. Thus $(\pi_{TM} \times T\Sigma \circ \tau)$ is a diffeomorphism from $\tau(TU)$ onto TV. This establishes A.4 a). ■

Definition A.6. Let M, N be smooth manifolds. Then we endow the smooth maps $C^\infty(M,N)$ with the initial topology with respect to

$$C^\infty(M,N) \hookrightarrow \prod_{k\in\mathbb{N}_0} C^0(T^kM, T^kN)_{c.o.}, \quad f \mapsto (T^kf)_{k\in\mathbb{N}_0},$$

where $C^0(T^kM, T^kN)_{c.o.}$ denotes the space of continuous functions endowed with the compact-open topology.■

From [Woc13, Proposition 7.3 and Theorem 5.14] we recall the following result.

Theorem A.7. *Let $E \to M$ be a vector bundle over the compact manifold M such that the fibres are locally convex spaces. Then the space of sections $\Gamma(M \leftarrow E)$ is a closed subspace of $C^\infty(M,E)$ and a locally convex space with respect to point-wise addition and scalar multiplication. If the fibres of $E \to M$ are metrisable, then so is $\Gamma(M \leftarrow E)$ and if the fibres are Fréchet spaces, then so is $\Gamma(M \leftarrow E)$.*

Our main tool will be the following excerpt from [Woc13, Theorem 7.6].

Theorem A.8. *Let M be a compact manifold and N be a locally convex and locally metrisable manifold that admits a local addition $\Sigma\colon U \Subset TN \to N$. Set $V := (\pi \times \Sigma)(U)$, which is an open neighbourhood of the diagonal ΔN in $N \times N$. For each $f \in C^\infty(M,N)$ we set*

$$O_f := \{g \in C^\infty(M,N) \mid (f(x), g(x)) \in V\}.$$

Then the following assertions hold.

a) The set O_f contains f, is open in $C^\infty(M,N)$ and the formula $(f(x), g(x)) = (f(x), \Sigma(\varphi_f(g)(m)))$ determines a homeomorphism

$$\varphi_f\colon O_f \to \{h \in C^\infty(M,TN) \mid \pi(h(x)) = f(x)\} \cong \Gamma(f^*(TN))$$

from O_f onto the open subset $\{h \in C^\infty(M,TN) \mid \pi(h(x)) = f(x)\} \cap C^\infty(M,U)$ of $\Gamma(f^*(TN))$.

b) The family $(\varphi_f\colon O_f \to \varphi_f(O_f))_{f\in C^\infty(M,N)}$ is an atlas, turning $C^\infty(M,N)$ into a smooth locally convex and locally metrisable manifold.

c) The manifold structure on $C^\infty(M,N)$ from b) is independent of the choice of the local addition Σ.

d) If L is another locally convex and locally metrisable manifold, then a map $f\colon L \times M \to N$ is smooth if and only if $\widehat{f}\colon L \to C^\infty(M,N)$ is smooth. In other words,

$$C^\infty(L \times M, N) \to C^\infty(L, C^\infty(M,N)), \quad f \mapsto \widehat{f}$$

is a bijection (which is even natural).

e) Let M' be compact and N' be locally metrisable such that N' admits a local addition. If $\mu\colon M' \to M$, $\nu\colon N \to N'$ are smooth, then

$$\nu_*\mu^*\colon C^\infty(M,N) \to C^\infty(M',N'), \quad \gamma \mapsto \nu \circ \gamma \circ \mu$$

is smooth.

f) If M' is another compact manifold, then the composition map

$$\circ\colon C^\infty(M',N) \times C^\infty(M,M') \to C^\infty(M,N), \quad (\gamma,\eta) \mapsto \gamma \circ \eta$$

is smooth.

Theorem A.9. *Let M be a compact manifold and N be a locally convex and locally metrisable manifold that admits a local addition. There is an isomorphism of vector bundles*

$$\begin{array}{ccc} TC^\infty(M,N) & \xrightarrow{\Phi_{M,N}} & C^\infty(M,TN) \\ & {\scriptstyle \pi_{TC^\infty(M,N)}}\searrow \quad \swarrow{\scriptstyle (\pi_{TN})_*} & \\ & C^\infty(M,N) & \end{array}$$

given by

$$\Phi_{M,N}\colon TC^\infty(M,N) \to C^\infty(M,TN), \quad [t \mapsto \eta(t)] \mapsto (m \mapsto [t \mapsto \eta^\wedge(t,m)])\,.$$

Here we have identified tangent vectors in $C^\infty(M,N)$ with equivalence classes $[\eta]$ of smooth curves $\eta\colon]{-\varepsilon},\varepsilon[\to C^\infty(M,N)$ for some $\varepsilon > 0$. The isomorphism $\varphi_{M,N}$ is natural with respect to the morphisms from e), i.e., the diagrams

$$\begin{array}{ccc} TC^\infty(M,N) & \xrightarrow{\Phi_{M,N}} & C^\infty(M,TN) \\ {\scriptstyle T(\mu^*)}\downarrow & & \downarrow{\scriptstyle \mu_*} \\ TC^\infty(M',N) & \xrightarrow{\Phi_{M',N}} & C^\infty(M',TN) \\ {\scriptstyle T(\nu_*)}\downarrow & & \downarrow{\scriptstyle (T\nu)_*} \\ TC^\infty(M',N') & \xrightarrow{\Phi_{M',N'}} & C^\infty(M',TN') \end{array} \quad \text{and} \quad \begin{array}{ccc} TC^\infty(M,N) & \xrightarrow{\Phi_{M,N}} & C^\infty(M,TN) \\ {\scriptstyle T(\nu_*)}\downarrow & & \downarrow{\scriptstyle (T\nu)_*} \\ TC^\infty(M,N') & \xrightarrow{\Phi_{M,N'}} & C^\infty(M,TN') \\ {\scriptstyle T(\mu^*)}\downarrow & & \downarrow{\scriptstyle \mu_*} \\ TC^\infty(M',N') & \xrightarrow{\Phi_{M',N'}} & C^\infty(M',TN') \end{array}$$

commute. In particular, $T_fC^\infty(M,N)$ is naturally isomorphic (as a topological vector space) to $\Gamma(f^*TN)$ and with respect to this isomorphism we have

$$\begin{aligned} T_f(\mu^*)&\colon \Gamma(f^*TN) \to \Gamma((f\circ\mu)^*TN), \quad &\sigma \mapsto \sigma\circ\mu \\ T_f(\nu_*)&\colon \Gamma(f^*TN) \to \Gamma((\nu\circ f)^*TN'), \quad &\sigma \mapsto T\nu\circ\sigma. \end{aligned}$$

Proof. First note that TN is also locally convex and locally metrisable and from Lemma A.5 we infer that it also admits a local addition. Let $\Sigma\colon TN \supseteq \Omega \to N$ be the local addition on N and $\tau\colon T^2N \to T^2N$ be the canonical flip (cf. Lemma A.5). Then $T\Sigma\circ\tau$ is a local addition on TN. Furthermore, M is compact and thus Theorem A.8 implies that $C^\infty(M,N)$, $TC^\infty(M,N)$ and $C^\infty(M,TN)$ are locally convex manifolds. We can now argue as in [Mic80, 10.12] to see that the charts $(\varphi_{0\circ f})_{f\in C^\infty(M,N)}$ cover $C^\infty(M,TN)$. In fact, the charts $(\varphi_{0\circ f})_{f\in C^\infty(M,N)}$ are bundle trivialisations for $(\pi_{TN})_*\colon C^\infty(M,TN)\to C^\infty(M,N)$ (see [Mic80, 10.12 2. Claim]). The map $\Phi_{M,N}$ will be an isomorphism of vector bundles if we can show that it coincides fibre-wise with the isomorphism of vector bundles constructed in the proof of [Mic80, Theorem 10.13]. Note that the proof of [Mic80, Theorem 10.13] deals only with the case of a finite-dimensional target N. However, the local addition constructed in Lemma A.5 allows us to copy the proof of [Mic80, Theorem 10.13] almost verbatim[5]. To prove that $\Phi_{M,N}$ is indeed of the claimed form, fix $f \in C^\infty(M,N)$. We will evaluate $\varphi_{0\circ f}\circ\Phi_{M,N}$ on the equivalence class $[t\mapsto c(t)]$ of a smooth curve $c\colon]-\varepsilon,\varepsilon[\to C^\infty(M,N)$ with $c(0)=f$:

$$\begin{aligned} \varphi_{0\circ f}\circ\Phi_{M,N}([t\mapsto c(t)]) &= \varphi_{0\circ f}(m\mapsto [t\mapsto c^\wedge(t,m)]) \\ &= \big(m\mapsto (\pi_{T^2N}, T\Sigma\circ\tau)^{-1}(0\circ f(m),[t\mapsto c^\wedge(t,m)])\big) \end{aligned} \tag{33}$$

By construction we obtain an element in $\Gamma((0\circ f)^*T^2N) = \Gamma((0\circ f)^*T^2N|N)$ where $T^2N|N$ is the restriction of the bundle T^2N to the zero-section of TN. Consider the vertical lift $V_{TN}\colon TN\oplus TN \to V(TN)$ given locally by $V((x,a),(x,b)) := (x,a,0,b)$. Recall that τ and V_{TN} are vector bundle isomorphisms. Now we argue as in [Mic80, 10.12] to obtain a canonical isomorphism

$$I_f := (f^*(V_{TN})^{-1}\circ f^*\tau)_*\colon \Gamma((0\circ f)^*T^2N|N) \to \Gamma(f^*TN)\oplus\Gamma(f^*TN).$$

(Notice that there is some abuse in notation for $f^*\tau$, explained in detail in [Mic80, 10.12]). We will now prove that I_f is the inverse of $\varphi_{0\circ f}\circ\Phi_{M,N}\circ T\varphi_f^{-1}$. A computation in canonical coordinates for T^2N yields

$$\begin{aligned} T\varphi_f([t\mapsto c(t)])] &= (m\mapsto [t\mapsto (\pi_{TN},\Sigma)^{-1}(f(m),c^\wedge(t,m))])) \\ &= (m\mapsto V_{TN}^{-1}\circ T(\pi_{TN},\Sigma)^{-1}(0\circ f,[t\mapsto c^\wedge(t,\cdot)])) \in \Gamma(f^*TN)\oplus\Gamma(f^*TN). \end{aligned} \tag{34}$$

[5] The changes needed are restrictions of some mappings to open subsets since contrary to [Mic80, Theorem 10.13] our local additions are not defined on the whole tangent bundle.

Here we have used the identifications $C^\infty_f(M, TN \oplus TN) \cong \Gamma(f^*(TN \oplus TN)) \cong \Gamma(f^*TN) \oplus \Gamma(f^*TN)$. Since τ is an involution on T^2N we can compute as follows

$$\begin{aligned} I_f \circ \varphi_{0\circ f} \circ \Phi_{M,N}([t \mapsto c(t)]) &\overset{(33)}{=} \left(m \mapsto V_{TN}^{-1} \circ \tau \circ (\pi_{T^2N}, T\Sigma \circ \tau)^{-1}(0 \circ f(m), [t \mapsto c^\wedge(t,m)])\right) \\ &= \left(m \mapsto V_{TN}^{-1} \circ (\pi_{T^2N} \circ \tau, T\Sigma \circ \tau \circ \tau)^{-1}(0 \circ f(m), [t \mapsto c^\wedge(t,m)])\right) \\ &= \left(m \mapsto V_{TN}^{-1} \circ T(\pi_{TN}, \Sigma)^{-1}(0 \circ f, [t \mapsto c^\wedge(t,\cdot)])\right). \end{aligned} \tag{35}$$

Hence the right hand side of (35) coincides with the right hand side of (34). Summing up the map I_f is the inverse of $\Phi_{M,N}|^{C^\infty_{0\circ f}(M,TN)}_{T_f C^\infty(M,N)}$. We conclude that $\Phi^{-1}_{M,N}$ is the isomorphism of vector bundles described in [Mic80, Theorem 10.13]. The statements concerning the tangent maps of the smooth maps discussed in Theorem A.8 e) then follow from [Mic80, Corollary 10.14]. ■

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