ZMP-HH / 14-19
Hamburger Beiträge zur Mathematik Nr. 524
September 2014

New R-matrices for small quantum groups

Simon Lentner* and Daniel Nett
Algebra and Number Theory, Hamburg University,
Bundesstraße 55, D-20146 Hamburg

Abstract. It is widely accepted that small quantum groups should possess a quasitriangular structure, even though this is technically not true. In this article we construct explicit R-matrices, sometimes several inequivalent ones, over certain natural extensions of small quantum groups by grouplike elements. The extensions are in correspondence to lattices between root and weight lattice. Our result generalizes a well-known calculation for $u_q(\mathfrak{sl}_2)$ used in logarithmic conformal field theories.

Keywords: Quantum group, R-matrix, braided category
MSC Classification: 16T05

Contents

*Corresponding author:
simon.lentner@uni-hamburg.de, Phone: +49 40 42838 5178, Fax: +49 40 42838-5190

Introduction and Summary

Hopf algebras with R-matrices, so called quasitriangular Hopf algebras, give rise to braided tensor categories, which have many interesting applications: Any braided vector space with a dual can be used to construct knot invariants and, using surgery, a (finite) braided tensor category gives rise to a invariant of 3-manifolds, cf. [Vir06] based on the well-known work [RT90]. In [Ros93, KR02] the case of the representation category of a quantum group is treated. For example, if the R-matrix for the quantum group $U_q(\mathfrak{g})$ in the case $q = i, \mathfrak{g} = \mathfrak{sl}_2$ is evaluated on the standard representation depending on an additional deformation parameter λ, then one obtaines in this way the Alexander-Conway-polynomial. Braided tensor categories with an additional non-degeneracy condition give rise to topological field theories [Tur94, KL01]. Checking which R-matrices below fulfill this additional condition would be an interesting follow-up to the present work.

For quantum groups, Lusztig gives in [Lus93] Sec. 32 essentially an R-matrix, but it is not clear that this gives rise to an R-matrix over the small quantum groups $u_q(\mathfrak{g})$ with q an ℓ-th root of unity. In [Ros93] this has been shown to be true whenever ℓ is odd and prime to the determinant of the Cartan matrix. In other cases Lusztig's small quantum group itself usually does not admit an R-matrix, in many cases even the category is not braided. This has been resolved in two ways in literature:

- Several authors consider slightly smaller quotients (resp. a subcategory), i.e. $K^e = 1$ for e half the exponent in Lusztig's definition, where one can obtain indeed an R-matrix if ℓ is prime to the determinant of the Cartan matrix [Ros93]. For some applications however, it is desirable that the quotient is taken precisely with Lusztig's choice and one wishes to focus on the even case.
- For q an even root of unity, some authors consider R-matrices up to outer automorphism ([Tan92, Res95]), or quadratic extensions of $u_q(\mathfrak{g})$, e.g. explicitly in the case of $u_q(\mathfrak{sl}_2)$ in [RT91, FGST06] and more generally in [GW98] for $u_q(\mathfrak{sl}_n)$. By [Tur94] p. 511 Rosso has already suggested in 1993 that one should consider extensions of $u_q(\mathfrak{g})$ for general $\mathfrak{g}$.

In this article we determine *all* possible R-matrices that can be obtained through Lusztig's ansatz [Lus93] Sec. 32.1, which means to vary the *toral part* R_0 (see below), while at the same time considering extensions of $u_q(\mathfrak{g})$ that are Lie-theoretically motivated and explain the exceptional behaviour with respect to the determinant of the Cartan matrix. In many cases we find several inequivalent choices different from the standard choice of R_0 (most notably $\mathfrak{g} = D_{2n}$), while other cases still do not admit R-matrices. In particular we find indeed that also even ℓ (or divisible by 4 for multiply-laced $\mathfrak{g}$) admit R-matrices for extensions of Lusztig's original quantum group.

More precisely, the extensions $u_q(\mathfrak{g}, \Lambda)$ of $u_q(\mathfrak{g})$ we consider depend on a choice of a lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$ between root and weight lattice, which corresponds to a choice of a complex connected Lie group associated to $\mathfrak{g}$. We first derive a necessary form of the R-matrix, depending only on the fundamental group Λ_W/Λ_R; this amounts to a question in additive combinatorics we have settled in [LN14]. The main calculations concluding the present article is to check sufficiency in terms of certain sublattices of Λ. These sublattices depend heavily on $\mathfrak{g}$ and on the roots of unity in question, in particular in common divisors of ℓ and the determinant of the Cartan matrix, which is the order of Λ_W/Λ_R.

This article is organized as follows.

In Section 1 we fix the Lie theoretic notation and prove some technical preliminaries. In particular, we introduce some sublattices of the weight lattice Λ_W of a simple complex Lie algebra, e.g. the so-called ℓ-centralizer $\mathrm{Cent}^q(\Lambda_R)$ of Λ_R in Λ_W (with respect to the braiding). We then give the definition of the finite dimensional quantum groups $u_q(\mathfrak{g},\Lambda,\Lambda')$ for lattices $\Lambda,\ \Lambda'$, where Λ' is a suitable sublattice of $\mathrm{Cent}^q(\Lambda_R)$. Choices of Λ' correspond to the choice of a quotient, see above. We recall also the definition of an R-matrix.

In Section 2 we review the ansatz $R = R_0\bar{\Theta}$ for R-matrices by Lusztig, with fixed $\Theta \in u_q(\mathfrak{g},\Lambda)^+ \otimes u_q(\mathfrak{g},\Lambda)^-$ and free toral part $R_0 = \sum_{\mu,\nu\in\Lambda/\Lambda'} f(\mu,\nu)K_\mu \otimes K_\nu$. We find equations for the free parameters $f(\mu,\nu)$ that are equivalent to R being an R-matrix and depend on the fundamental group $\pi_1 = \Lambda_W/\Lambda_R$ of $\mathfrak{g}$ and on some sublattices of Λ_W associated to q. This ansatz was also used by Müller [Mül98a, Mül98b] for determining R-matrices for quadratic extensions of $u_q(\mathfrak{sl}_n)$.

In Section 3 we will first consider those equations on $f(\mu,\nu)$, that only depend on π_1 as a group, the so-called *group-equations* for the coefficients of the ansatz in Section 2. We will give all solutions of the group-equations of a group G, where G is cyclic or equal to $\mathbb{Z}_2\times\mathbb{Z}_2$, since these are the relevant cases for $G = \pi_1$ the fundamental group of the Lie algebras in interest. The case $\mathfrak{g} = A_n$ with fundamental group $\mathbb{Z}_{n+1}$ is particularly hard and depends on a question in additive combinatorics, which we settled in [LN14].

We then consider in Section 4 a certain constellation of sublattices of Λ, which we call a diamond. Depending on these sublattices we define *diamond-equations*, derive a necessary condition for the existence of solutions and give again results for the cyclic case.

In Section 5 we give the main result of this article in Theorem A, a list of R-matrices obtained by Lusztig's ansatz. These are obtained by solving the corresponding group- and diamond-equations, depending on the fundamental group π_1 of $\mathfrak{g}$, the lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$, kernel $\Lambda' \subset \mathrm{Cent}^q(\Lambda_W)\cap\Lambda_R$ and the ℓ-th root of unity q. Here, $\mathrm{Cent}^q(\Lambda_W)$ denotes the lattice orthogonal to Λ_W mod ℓ, i.e. the set of $\lambda\in\Lambda$ with $\ell \mid (\lambda,\mu)$ for all weights μ.

We develop general results that allow us to compute all R-matrices fulfilling Lusztig's ansatz depending on $\mathfrak{g},\Lambda,\Lambda'$. Under the additional assumption 1.13 on Λ', which also simplifies some calculations, we find that in fact $\Lambda' = \Lambda_R^{[\ell]}$ is the only choice that allows the existence of an R- matrix.

Theorem A. *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra with root lattice Λ_R, weight lattice Λ_W and fundamental group $\pi_1 = \Lambda_W/\Lambda_R$. Let q be an ℓ-th root of unity, $\ell\in\mathbb{N}$, $\ell > 2$. Then we have the following R-matrix of the form $R = R_0\bar{\Theta}$, with Θ as in Theorem 2.2:*

$$R = \left(\frac{1}{|\Lambda/\Lambda'|}\sum_{(\mu,\nu)\in(\Lambda_1/\Lambda'\times\Lambda_2/\Lambda')} q^{-(\mu,\nu)}\omega(\bar{\mu},\bar{\nu})\ K_\mu\otimes K_\nu\right)\cdot\bar{\Theta},$$

for the quantum group $u_q(\mathfrak{g},\Lambda,\Lambda')$ with Λ_i the preimage of a certain subgroup $H_i\subset\pi_1$ in Λ_W $(i = 1, 2)$, a certain group-pairing $\omega\colon H_1\times H_2\to\mathbb{C}^\times$ and $\Lambda' = \Lambda_R^{[\ell]}$ as in Def. 1.4.

In the following table we list for all root systems the following data, depending on ℓ: Possible choices of H_1, H_2 (in terms of fundamental weights λ_k), the group-pairing ω, and the number of solutions #. If the number has a superscript $*$, we obtain R-matrices for Lusztig's original choice of Λ'. For $\mathfrak{g} = D_n$, $2\mid n$, with $\pi_1 = \mathbb{Z}_2\times\mathbb{Z}_2$ we get the only cases $H_1\neq H_2$ and denote by $\lambda\neq\lambda'\in\{\lambda_{n-1},\lambda_n,\lambda_{n-1}+\lambda_n\}$ arbitrary elements of order 2 in π_1.

$\mathfrak{g}$	ℓ	#	$H_i \cong$	H_i $(i=1,2)$	ω
$A_{n\geq 1}$ $\pi_1=\mathbb{Z}_{n+1}$	ℓ odd		$\mathbb{Z}_d$	$\langle \frac{n+1}{d}\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=\xi_d^k$, if $d \mid (n+1)$, $1\leq k\leq d$ and $\gcd(n+1, d\ell, k\ell-\frac{n+1}{d}n)=1$
	ℓ even	*			
$B_{n\geq 2}$ $\pi_1=\mathbb{Z}_2$	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		1	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=(-1)^{n-1}$
	$\ell\equiv 2 \mod 4$	2	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=\pm 1$
		1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$, if n even
	$\ell\equiv 0 \mod 4$	2*	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=\pm 1$
	$\ell\neq 4$	1*	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$, if n even
$C_{n\geq 3}$ $\pi_1=\mathbb{Z}_2$	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		1	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=-1$
	$\ell\equiv 2 \mod 4$	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		1	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=(-1)^{n-1}$
	$\ell\equiv 0 \mod 4$	2*	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=\pm 1$
	$\ell\neq 4$	1*	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$, if n even
$D_{n\geq 4}$ n even $\pi_1=\mathbb{Z}_2\times\mathbb{Z}_2$	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		3	$\mathbb{Z}_2$	$\langle\lambda\rangle$	$\omega(\lambda,\lambda)=-1$
		6	$\mathbb{Z}_2\neq\mathbb{Z}_2'$	$\langle\lambda\rangle,\langle\lambda'\rangle$	$\omega(\lambda,\lambda')=1$
		1	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\langle\lambda_{n-1},\lambda_n\rangle$	$\omega(\lambda_i,\lambda_j)=1$
		1	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\langle\lambda_{n-1},\lambda_n\rangle$	$\omega(\lambda_i,\lambda_j)=-1$
		2	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\langle\lambda_{n-1},\lambda_n\rangle$	$\omega(\lambda_{n-1},\lambda_{n-1})=\pm 1$ $\omega(\lambda_{n-1},\lambda_n)=1$ $\omega(\lambda_n,\lambda_{n-1})=1$ $\omega(\lambda_n,\lambda_n)=\mp 1$
		2	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\langle\lambda_{n-1},\lambda_n\rangle$	$\omega(\lambda_{n-1},\lambda_{n-1})=-1$ $\omega(\lambda_{n-1},\lambda_n)=\pm 1$ $\omega(\lambda_n,\lambda_{n-1})=\mp 1$ $\omega(\lambda_n,\lambda_n)=-1$
	ℓ even	16*	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\langle\lambda_{n-1},\lambda_n\rangle$	$\omega(\lambda_i,\lambda_j)\in\{\pm 1\}$
$D_{n\geq 5}$ n odd $\pi_1=\mathbb{Z}_4$	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		1	$\mathbb{Z}_2$	$\langle 2\lambda_n\rangle$	$\omega(2\lambda_n,2\lambda_n)=-1$
		2	$\mathbb{Z}_4$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=\pm 1$
	$\ell\equiv 2 \mod 4$	4*	$\mathbb{Z}_4$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=c,\ c^4=1$
	$\ell\equiv 0 \mod 4$	4*	$\mathbb{Z}_4$	$\langle\lambda_n\rangle$	$\omega(\lambda_n,\lambda_n)=c,\ c^4=1$
E_6 $\pi_1=\mathbb{Z}_3$	ℓ odd, $3\nmid\ell$	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		2	$\mathbb{Z}_3$	$\langle\lambda_6\rangle$	$\omega(\lambda_6,\lambda_6)=1,\exp(\frac{2\pi i}{3})$
	ℓ even, $3\nmid\ell$	1*	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		2*	$\mathbb{Z}_3$	$\langle\lambda_6\rangle$	$\omega(\lambda_6,\lambda_6)=1,\exp(2\frac{2\pi i}{3})$
	ℓ odd, $3\mid\ell$	3	$\mathbb{Z}_3$	$\langle\lambda_6\rangle$	$\omega(\lambda_6,\lambda_6)=c,\ c^3=1$
	ℓ even, $3\mid\ell$	3*	$\mathbb{Z}_3$	$\langle\lambda_6\rangle$	$\omega(\lambda_6,\lambda_6)=c,\ c^3=1$
E_7 $\pi_1=\mathbb{Z}_2$	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
		1	$\mathbb{Z}_2$	$\langle\lambda_7\rangle$	$\omega(\lambda_7,\lambda_7)=1$
	ℓ even	2*	$\mathbb{Z}_2$	$\langle\lambda_7\rangle$	$\omega(\lambda_7,\lambda_7)=\pm 1$
E_8	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$
$\pi_1=\mathbb{Z}_1$	ℓ even	1*	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0)=1$

F_4 $\pi_1 = \mathbb{Z}_1$	ℓ odd	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0) = 1$
	$\ell \equiv 2 \mod 4$	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0) = 1$
	$\ell \equiv 0 \mod 4$ $\ell \neq 4$	1^*	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0) = 1$
G_2 $\pi_1 = \mathbb{Z}_1$	ℓ odd $\ell \neq 3$	1	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0) = 1$
	ℓ even $\ell \neq 4, 6$	1^*	$\mathbb{Z}_1$	$\{0\}$	$\omega(0,0) = 1$

Table 1: Solutions for R_0-matrices

The cases B_n, C_n, F_4, $\ell = 4$ and G_2, $\ell = 3, 6$ and $\ell = 4$ respectively, can be obtained in the table for $A_1^{\times n}, D_n, D_4$, and again A_2 and A_3 respectively (cf. [Len14] for details).

Note, that Lusztig's R-matrix for $\Lambda = \Lambda_R$ correspond to the case $H = \mathbb{Z}_1$ and $\omega = 1$. The known quadratic extension for $\mathfrak{sl}_2$ is the case A_1 with $H = \mathbb{Z}_2$ in the example below.

Remark B. We indicate in which sense our results are *not* complete:

- Technically, one could even allow $\Lambda_R \subset \Lambda \subset \Lambda_W^\vee$, but then one would loose the topological interpretation as different choices of a Lie group associated to $\mathfrak{g}$.
- Our additional assumption 1.13 on the considered quotients $\Lambda' \subset \mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R$ was chosen to simplify calculations and prove uniqueness. In general $\Lambda' \in \mathrm{Cent}^q(\Lambda)$ would suffice (and could yield more solutions), but one would have to deal with possible 2-cocycles in $H^2(\Lambda/\Lambda', \pi_1)$ in Lemma 2.5.

Question C. Are *all* R-matrices of $u_q(\mathfrak{g})$ given by Lusztig's ansatz and hence in our list?

Question D. Which R-matrices above give rise to *equivalent* braided tensor categories?

Question E. Which R-matrices in this article are *factorizable* an give hence rise to (non-semisimple) modular tensor categories? What are results for other Nichols algebras?

Example. For $\mathfrak{g} = \mathfrak{sl}_2$ with root system A_1 the fundamental group is $\pi_1 = \mathbb{Z}_2$. Let α be the simple root, generating the root lattice Λ_R, and $\lambda = \frac{1}{2}\alpha$ the fundamental dominant weight, generating the weight lattice Λ_W. We will give the R-matrices for the quantum groups $u = u_q(\mathfrak{g}, \Lambda, \Lambda')$ for ℓ-th root of unity q and lattices $\Lambda_R \subset \Lambda \subset \Lambda_W$ and $\Lambda' = \Lambda_R^{[\ell]}$, which equals in the simply laced case $\ell\Lambda_R$.

The quasi R-matrix Θ (see Theorem 2.2) depends only on the root lattice and exists in $u^+ \otimes u^-$ with Borel parts $u^\pm$, generated by E_α, F_α. With $\ell_\alpha = \ell/\gcd(\ell, 2d_\alpha) = \ell/\gcd(\ell, 2)$ we have

$$\Theta = \sum_{k=0}^{\ell_{\alpha-1}} (-1)^k \frac{(q-q^{-1})^k}{[k]_q!} q^{-k(k-1)/2} E_\alpha^k \otimes F_\alpha^k \quad \text{and} \quad \bar{\Theta} = \sum_{k=0}^{\ell_{\alpha-1}} \frac{(q-q^{-1})^k}{[k]_q!} q^{k(k-1)/2} E_\alpha^k \otimes F_\alpha^k,$$

with q-factorial $[k]_q!$. The toral part R_0-is given by

$$R_0 = \frac{1}{|\Lambda/\Lambda_R^{[\ell]}|} \sum_{\mu,\nu \in \Lambda/\Lambda'} q^{-(\mu,\nu)}\, \omega(\bar{\mu}, \bar{\nu})\, K_\mu \otimes K_\nu,$$

for H and $\omega\colon H\times H\to\mathbb{C}^\times$ as in Table 1. The possible solutions depend on ℓ. We now check the condition $\gcd(2,d\ell,k\ell-2/d)=1$ from the theorem above ($n=1$ and $d=1,2$). For odd ℓ, we get the following solutions by Theorem A:

$$\begin{aligned} H&=\mathbb{Z}_1, &&\omega\colon \mathbb{Z}_1\times\mathbb{Z}_1\to\mathbb{C}^\times,\ \omega(0,0)=1,\\ H&=\mathbb{Z}_2, &&\omega\colon \mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{C}^\times,\ \omega(\lambda,\lambda)=1. \end{aligned}$$

For even ℓ the solution for $H=\mathbb{Z}_1$, i.e. for $\Lambda=\Lambda_R$, does not exist (since $2\mid\ell$ and $2\mid(\ell-2)$), rather we get both possible solutions on the full support $H=\mathbb{Z}_2$:

$$H=\mathbb{Z}_2,\qquad \omega\colon \mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{C}^\times,\ \omega(\lambda,\lambda)=\pm1.$$

In these cases, the R-matrices are explicitly given by

$$R=\frac{1}{2\ell}\sum_{k=0}^{\ell_\alpha-1}\sum_{i,j=0}^{2\ell-1}\frac{(q-q^{-1})^k}{[k]_q!}q^{k(k-1)/2+k(j-i)-\frac{ij}{2}}\,(\pm1)^{ij}\,E_\alpha^kK_\lambda^i\otimes F_\alpha^kK_\lambda^j$$

Acknowledgements. The first author is supported by the DFG Research Training Group 1670. We thank Christoph Schweigert for several helpful discussions.

1. Preliminaries

At first, we fix a convention.

Convention 1.1. *In the following, q is an ℓ-th root of unity. We fix $q=\exp(\frac{2\pi i}{\ell})$ and for $a\in\mathbb{R}$ we set $q^a=\exp(\frac{2\pi i a}{\ell})$, $\ell>2$.*

1.1. **Lie Theory.** Let $\mathfrak{g}$ be a finite-dimensional, semisimple complex Lie algebra with simple roots α_i, indexed by $i\in I$, $|I|=n$, and a set of positive roots Φ^+. Denote the *Killing form* by $(-,-)$, normalized such that $(\alpha,\alpha)=2$ for the short roots α. The *Cartan matrix* is given by

$$a_{ij}=\langle\alpha_i,\alpha_j\rangle=2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}.$$

For a root α we call $d_\alpha:=(\alpha,\alpha)/2$ with $d_\alpha\in\{1,2,3\}$. Especially, $d_i:=d_{\alpha_i}$ and in this notation $(\alpha_i,\alpha_j)=d_ia_{ij}$. The fundamental dominant weights λ_i, $i\in I$, are given by the condition $2(\alpha_i,\lambda_j)/(\alpha_i,\alpha_i)=\delta_{ij}$, hence the Cartan matrix expresses the change of basis from roots to weights.

Definition 1.2. The *root lattice* $\Lambda_R=\Lambda_R(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is the abelian group with rank $\mathrm{rank}(\Lambda_R)=\mathrm{rank}(\mathfrak{g})=|I|$, generated by the simple roots α_i, $i\in I$.

Definition 1.3. The *weight lattice* $\Lambda_W=\Lambda_W(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is the abelian group with rank $\mathrm{rank}(\Lambda_W)=\mathrm{rank}(\mathfrak{g})$, generated by the fundamental dominant weights λ_i, $i\in I$.

The Killing form induces an integral pairing of abelian groups, turning Λ_R into an *integral lattice*. It is standard fact of Lie theory (cf. [Hum72], Section 13.1) that the root lattice is contained in the weight lattice.

Definition 1.4. Let Λ_R, Λ_W the root, resp. weight, lattice of the Lie algebra $\mathfrak{g}$ with generators α_i, resp. λ_i, for $i\in I$.

(i) Following Lusztig, we define $\ell_i := \ell / \gcd(\ell, 2d_i)$, which is the order of q^{2d_i}, where q is a primitive ℓ-th root of unity. More generally, we define for any root $\ell_\alpha := \ell / \gcd(\ell, 2d_\alpha)$. For any positive integer ℓ, the *ℓ-lattice* $\Lambda_R^{(\ell)}$, resp. $\Lambda_W^{(\ell)}$, is defined as

$$\Lambda_R^{(\ell)} = \langle \ell_i \alpha_i,\ i \in I \rangle \quad \text{resp.} \quad \Lambda_W^{(\ell)} = \langle \ell_i \lambda_i,\ i \in I \rangle . \tag{1.1}$$

(ii) For any positive integer ℓ, the lattice $\Lambda_R^{[\ell]}$, resp. $\Lambda_W^{[\ell]}$, is defined as

$$\Lambda_R^{[\ell]} = \left\langle \frac{\ell}{\gcd(\ell, d_i)} \alpha_i,\ i \in I \right\rangle \quad \text{resp.} \quad \Lambda_W^{[\ell]} = \left\langle \frac{\ell}{\gcd(\ell, d_i)} \lambda_i,\ i \in I \right\rangle . \tag{1.2}$$

Definition 1.5. For $\Lambda_1, \Lambda_2 \subset \Lambda_W$ with $\Lambda_2 \subset \Lambda_1$ we define $\mathrm{Cent}^q_{\Lambda_1}(\Lambda_2) = \{\eta \in \Lambda_1 \mid (\eta, \lambda) \in \ell\mathbb{Z}\ \forall \lambda \in \Lambda_2\}$. In the situation $\Lambda_1 = \Lambda_W$ we simply write $\mathrm{Cent}^q_{\Lambda_W}(\Lambda_2) = \mathrm{Cent}^q(\Lambda_2)$.

Especially, the set $\langle K_\eta \mid \eta \in \mathrm{Cent}^q(\Lambda_R) \rangle$ consists of the central group elements of the quantum group $U_q(\mathfrak{g}, \Lambda_W)$, cf. Section 1.2.

Lemma 1.6. *For a Lie algebra $\mathfrak{g}$ we have $\mathrm{Cent}^q(\Lambda_R) = \Lambda_W^{[\ell]}$. We call the elements of* $\mathrm{Cent}^q(\Lambda_R)$ central weights.

Proof. Let $\lambda = \sum_{j \in I} m_j \lambda_j \in \Lambda_W$ with fundamental weights λ_i. For a simple root α_i we have $(\alpha_i, \lambda) = (\alpha_i, \sum_{j \in I} m_j \lambda_j) = d_i m_i$. Thus, λ is central weight if $\ell \mid d_i m_i$ for all i, hence $(\ell / \gcd(\ell, d_i)) \mid m_i$ for all i. ∎

The same calculation gives the following lemma.

Lemma 1.7. *For a Lie algebra $\mathfrak{g}$ we have* $\mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R = \Lambda_R^{[\ell]}$.

1.2. **Quantum groups.** For a finite-dimensional complex simple Lie algebra $\mathfrak{g}$, lattices Λ, Λ' with $\Lambda_R \subset \Lambda \subset \Lambda_W$ and $2\Lambda_R^{(\ell)} \subset \Lambda' \subset \mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R$, and a primitive ℓ-th root of unity q, we aim to define the finite-dimensional quantum group $u_q(\mathfrak{g}, \Lambda, \Lambda')$, also called small quantum group. We construct $u_q(\mathfrak{g}, \Lambda, \Lambda')$ by using rational and integral forms of the deformed universal enveloping algebra $U_q(\mathfrak{g})$ for an indeterminate q. In the following we give the definitions of the quantum groups, following the lines of [Len14]. The different choices of Λ are already in [Lus93], Sec. 2.2. We shall give a dictionary to translate Lusztig's notation to the one used here.

Definition 1.8. For $q \in \mathbb{C}^\times$ or q an indeterminate and $n \leq k \in \mathbb{N}_0$ we define

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]_q! := [1]_1 [2]_q \dots [n]_q \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{[n]_q!}{[k]_q! [n-k]_q!}, & 0 \leq k \leq n, \\ 0, & \text{else.} \end{cases}$$

Definition 1.9. Let q be an indeterminate. For each abelian group Λ with $\Lambda_R \subset \Lambda \subset \Lambda_W$ we define the *rational form* $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ over the ring of rational functions $\Bbbk = \mathbb{Q}(q)$ as follows:

As algebra, let $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ be generated by the group ring $\Bbbk[\Lambda]$, spanned by K_Λ, $\lambda \in \Lambda$, and additional generators E_{α_i}, F_{α_i}, for each simple root α_i, $i \in I$, with relations:

$$K_\lambda E_{\alpha_i} K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_{\alpha_i}, \tag{1.3}$$

$$K_\lambda F_{\alpha_i} K_\lambda^{-1} = q^{-(\lambda, \alpha_i)} F_{\alpha_i}, \tag{1.4}$$

$$E_{\alpha_i} F_{\alpha_j} - F_{\alpha_j} E_{\alpha_i} = \delta_{ij} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_{\alpha_i} - q_{\alpha_i}^{-1}}, \tag{1.5}$$

and Serre relations for any $i \neq j \in I$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_{\alpha_i}^{1-a_{ij}-r} E_{\alpha_j} E_{\alpha_i}^r = 0, \tag{1.6}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{\bar{q}_i} F_{\alpha_i}^{1-a_{ij}-r} F_{\alpha_j} F_{\alpha_i}^r = 0, \tag{1.7}$$

where $\bar{q} := q^{-1}$, the quantum binomial coefficients are defined in Definition 1.8 and by definition $q^{(\alpha_i,\alpha_j)} = (q^{d_i})^{a_{ij}} = q_i^{a_{ij}}$.

As a coalgebra, let the coproduct Δ, the counit ε and the antipode S be defined on the group-Hopf-algebra $\Bbbk[\Lambda]$ as usual

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \qquad \varepsilon(K_\lambda) = 1, \qquad S(K_\lambda) = K_\lambda^{-1} = K_{-\lambda},$$

and on the generator $E_{\alpha_i}, F_{\alpha_i}$, for each simple root α_i, $i \in I$ as follows

$$\begin{aligned} \Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i}, &\quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i}, \\ \varepsilon(E_{\alpha_i}) = 0, &\quad \varepsilon(F_{\alpha_i}) = 0, \\ S(E_{\alpha_i}) = -E_{\alpha_i} K_{\alpha_i}^{-1}, &\quad S(F_{\alpha_i}) = -K_{\alpha_i} F_{\alpha_i}. \end{aligned}$$

This is a Hopf algebra over the field $\Bbbk = \mathbb{Q}(q)$. Moreover, we have a triangular decomposition: Consider the subalgebras $U_q^{\mathbb{Q}(q),+}$, generated by the E_{α_i}, and $U_q^{\mathbb{Q}(q),-}$, generated by the F_{α_i}, and $U_q^{\mathbb{Q}(q),0} = \Bbbk[\Lambda]$, spanned by the K_λ. Then the multiplication in $U_q^{\mathbb{Q}(q)} = U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)$ induces an isomorphism of vector spaces:

$$U_q^{\mathbb{Q}(q),+} \otimes U_q^{\mathbb{Q}(q),0} \otimes U_q^{\mathbb{Q}(q),-} \xrightarrow{\cong} U_q^{\mathbb{Q}(q)}.$$

Definition 1.10. The so-called *restricted integral form* $U_q^{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}, \Lambda)$ is generated as a $\mathbb{Z}[q, q^{-1}]$-algebra by Λ and the following elements in $U_q^{\mathbb{Q}(q),\pm}(\mathfrak{g}, \Lambda)$, called *divided powers*:

$$E_\alpha^{(r)} := \frac{E_\alpha^r}{\prod_{s=1}^r [s]_{q_\alpha}} \quad F_\alpha^{(r)} := \frac{F_\alpha^r}{\prod_{s=1}^r [s]_{\bar{q}_\alpha}} \quad \text{for all } \alpha \in \Phi^+, r > 0,$$

and by the following elements in $U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda)^0$:

$$K_{\alpha_i}^{(r)} = \begin{bmatrix} K_{\alpha_i}; 0 \\ r \end{bmatrix} := \prod_{s=1}^r \frac{K_{\alpha_i} q_{\alpha_i}^{1-s} - K_{\alpha_i}^{-1} q_{\alpha_i}^{s-1}}{q_{\alpha_i}^s - q_{\alpha_i}^{-s}}, \qquad i \in I.$$

These definitions can also be found in Lusztig's book [Lus93]. In order to translate Lusztig's notation to the one used here, one has to match the terms in the following way

Lusztig's notation	notation used here
Index set I	simple roots $\{\alpha_i \mid i \in I\}$
X	root lattice Λ_R
Y	lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$
$i' \in X$	α_i
$i \in Y$	$\frac{\alpha_i}{d_{\alpha_i}} = \alpha_i^\vee$ coroot
$i \cdot j,\ i, j \in \mathbb{Z}[I]$	(α_i, α_j)
$\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i},\ i \in Y, j' \in X$	$\langle \alpha_i, \alpha_j \rangle$
K_i	$K_{\alpha_i^\vee}$
$\tilde{K}_i = K_{\frac{i \cdot i}{2} i}$	K_{α_i}

We now define the *restricted specialization* $U_q(\mathfrak{g}, \Lambda)$. Here, we specialize q to a specific choice $q \in \mathbb{C}^\times$.

Definition 1.11. The infinite-dimensional Hopf algebra $U_q(\mathfrak{g}, \Lambda)$ is defined by

$$U_q(\mathfrak{g}, \Lambda) := U_q^{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}, \Lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}_q,$$

where $\mathbb{C}_q = \mathbb{C}$ with the $\mathbb{Z}[q, q^{-1}]$-module structure defined by the specific value $q \in \mathbb{C}^\times$.

From now on, q will be a primitive ℓ-th root of unity. We choose explicitly $q = \exp(\frac{2\pi i}{\ell})$, see Convention 1.1.

Definition 1.12. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with root system Φ and assume $\mathrm{ord}(q^2) > d_\alpha$ for all $\alpha \in \Phi$. For lattices Λ, Λ' with $\Lambda_R \subset \Lambda \subset \Lambda_W$ and $2\Lambda_R^{(\ell)} \subset \Lambda' \subset \mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R$, we define the *small quantum group* $u_q(\mathfrak{g}, \Lambda, \Lambda')$ as the algebra $U_q(\mathfrak{g}, \Lambda)$ from Definition 1.11, generated by K_λ for $\lambda \in \Lambda$ and E_α, F_α with $\ell_\alpha > 1$, $\alpha \in \Phi^+$ not necessarily simple, together with the relations

$$E_\alpha^{\ell_\alpha} = 0, \quad F_\alpha^{\ell_\alpha} = 0 \quad \text{and} \quad K_\lambda = 1 \text{ for } \lambda \in \Lambda'.$$

The coalgebra structure is again given as in Definition 1.9. This is a finite dimensional Hopf algebra of dimension

$$|\Lambda/\Lambda'| \prod_{\alpha \in \Phi^+,\ \ell_\alpha > 1} \ell_\alpha^2.$$

The fact, that this gives a Hopf algebra for $\Lambda' = 2\Lambda_R^{(\ell)}$ is in Lusztig, [Lus90], Sec. 8.

We fix the assumption on Λ'.

Assumption 1.13. *We assume for the sublattice $\Lambda' \subset \Lambda_W$ in the following that*

$$2\Lambda_R^{(\ell)} \subset \Lambda' \subset \mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R$$

1.3. R-matrices.

Definition 1.14. A Hopf algebra H is called *quasitriangular* if there exists an invertible element $R \in H \otimes H$ such that

$$\Delta^{op}(h) = R\Delta(h)R^{-1}, \tag{1.8}$$
$$(\Delta \otimes Id)(R) = R_{13}R_{23}, \tag{1.9}$$
$$(Id \otimes \Delta)(R) = R_{13}R_{12}, \tag{1.10}$$

with $\Delta^{op}(h) = \tau \circ \Delta(h)$, where $\tau : H \otimes H \longrightarrow H \otimes H,\ a \otimes b \longmapsto b \otimes a$ and $R_{12} = R \otimes 1,\ R_{23} = 1 \otimes R,\ R_{13} = (\tau \otimes Id)(R_{23}) = (Id \otimes \tau)(R_{12}) \in H^{\otimes 3}$. Such an element is called an *R-matrix of H*.

2. Ansatz for R

2.1. **Quasi-R-matrix and Cartan-part.** The goal of this paper is to construct new families of R-matrices for small quantum groups and certain extensions (see Def. 1.12). Our starting point is Lusztig's ansatz in [Lus93], Sec. 32.1, for a universal R-matrix of $U_q(\mathfrak{g}, \Lambda)$. This ansatz has been translated by Müller in his Dissertation [Mül98a], resp. in [Mül98b], for small quantum groups, which we will use in the following. Note, that this ansatz has been successfully generalized to general diagonal Nichols algebras in [AY13].

For a finite-dimensional, semisimple complex Lie algebra $\mathfrak{g}$, an ℓ-th root of unity q and lattices Λ, Λ' as in Section 1.2, we write $u = u_q(\mathfrak{g}, \Lambda, \Lambda')$. Let $\bar{\ }\colon u \to \bar{u}$ be the $\mathbb{Q}$-algebra isomorphism defined by $q \mapsto q^{-1},\ E_{\alpha_i} \mapsto E_{\alpha_i},\ F_{\alpha_i} \mapsto F_{\alpha_i},\ i \in I$, and $K_\lambda \mapsto K_{-\lambda},\ \lambda \in \Lambda$. Then the map $\bar{\ } \otimes \bar{\ }\colon u \otimes u \to \bar{u} \otimes \bar{u}$ is a well-defined $\mathbb{Q}$-algebra isomorphism and we can define a $\mathbb{Q}(q)$-algebra morphism $\bar{\Delta}\colon u \to u \otimes u$ given by $\bar{\Delta}(x) = \overline{\Delta(\bar{x})}$ for all $x \in U$. We have in general $\bar{\Delta} \neq \Delta$.

Assume in the following, that

$$\ell_i > 1 \text{ for all } i \in I, \text{ and } \ell_i > -\langle \alpha_i, \alpha_j \rangle \text{ for all } i, j \text{ with } i \neq j. \tag{2.1}$$

Theorem 2.1 ([Len14])**.** *For a root system Φ of a finite-dimensional simple complex Lie algebra and an ℓ-th root of unity q, the condition* (2.1) *fails only in the following cases (Φ, ℓ). In each case, the small quantum group $u_q(\mathfrak{g})$ is described by a different $\tilde{\Phi}$ fulfilling* (2.1)*, hence the present work also provides results for these cases by consulting the results for $\tilde{\Phi}$.*

Φ	*(all)*	B_n	C_n	F_4	G_2	G_2
ℓ	$1, 2$	4	4	4	$3, 6$	4
$\tilde{\Phi}$	*(empty)*	$\underbrace{A_1 \times \ldots \times A_1}_{n\text{-times}}$	D_n	D_4	A_2	A_3

The following theorem is essentially in [Lus93]. Note that the roles of E, F will be switched in our article to match the usual convention:

Theorem 2.2 (cf. [Mül98b], Thm. 8.2)**.** *(a) There is a unique family of elements $\Theta_\nu \in u_\nu^+ \otimes u_\nu^-$, $\nu \in \Lambda_R$, such that $\Theta_0 = 1 \otimes 1$ and $\Theta = \sum_\nu \Theta_\nu \in u \otimes u$ satisfies $\Delta(x)\Theta = \Theta\bar{\Delta}(x)$ for all $x \in u$.*

(b) Let B be a vector space-basis of u^+, such that $B_\nu = B \cap u_\nu^+$ is a basis of u_ν^+ for all ν. Here, u_ν^+ refers to the natural Λ_R-grading of u^+. Let $\{b^* \mid b \in B_\nu\}$ be the basis of u_ν^- dual to B_ν under the non-degenerate bilinear form $(\,\cdot\,,\cdot\,)\colon u^+ \otimes u^- \to \mathbb{C}$. We have

$$\Theta_\nu = (-1)^{\operatorname{tr}\nu} q_\nu \sum_{b \in B_\nu} b^+ \otimes b^{*-} \in u_\nu^+ \otimes u_\nu^-, \tag{2.2}$$

where $q_\nu = \prod_i q_i^{\nu_i}$, $\operatorname{tr}\nu = \sum_i \nu_i$ for $\nu = \sum_i \nu_i \alpha_i \in \Lambda_R$.

Remark 2.3. (i) The element Θ is called the *Quasi-R-matrix* of $u = u_q(\mathfrak{g}, \Lambda, \Lambda')$.

(ii) Since the element Θ is unique, the expressions $\sum_{b\in B_\nu} b^+ \otimes b^{*-}$ in part (b) of the theorem are independent of the actual choice of the basis B.
(iii) For example, if $\mathfrak{g} = A_1$, i.e. there is only one simple root $\alpha = \alpha_1$, and $E = E_\alpha$, $F = F_\alpha$. Thus we have

$$\Theta = \sum_{n=0}^{\ell_\alpha - 1} (-1)^n \frac{(q-q^{-1})^n}{[n]_q!} q^{-n(n-1)/2} E^n \otimes F^n.$$

(iv) The Quasi-R-matrix Θ is invertible with inverse $\Theta^{-1} = \bar{\Theta}$, i.e. the expression one gets by changing all q to $\bar{q} = q^{-1}$.

Theorem 2.4 (cf. [Mül98b], Theorem 8.11)**.** *Let $\Lambda' \subset \{\mu \in \Lambda \mid K_\mu$ central in $u_q(\mathfrak{g}, \Lambda)\}$ be a subgroup of Λ, and H_1, H_2 be subgroups of Λ/Λ', containing Λ_R/Λ'. In the following, $\mu, \mu_1, \mu_2 \in H_1$ and $\nu, \nu_1, \nu_2 \in H_2$.*

The element $R = R_0\bar{\Theta}$ with $R_0 = \sum_{\mu,\nu} f(\mu,\nu) K_\mu \otimes K_\nu$ is an R-matrix for $u_q(\mathfrak{g}, \Lambda, \Lambda')$, if and only if for all $\alpha \in \Lambda_R$ and μ, ν the following holds:

$$f(\mu+\alpha, \nu) = q^{-(\nu,\alpha)} f(\mu,\nu), \quad f(\mu, \nu+\alpha) = q^{-(\mu,\alpha)} f(\mu,\nu), \tag{2.3}$$

$$\sum_{\substack{\nu_1,\nu_2 \in H_2 \\ \nu_1+\nu_2=\nu}} f(\mu_1,\nu_1) f(\mu_2,\nu_2) = \delta_{\mu_1,\mu_2} f(\mu_1,\nu), \qquad \sum_{\substack{\mu_1,\mu_2 \in H_1 \\ \mu_1+\mu_2=\mu}} f(\mu_1,\nu_1) f(\mu_2,\nu_2) = \delta_{\nu_1,\nu_2} f(\mu,\nu_1), \tag{2.4}$$

$$\sum_\mu f(\mu,\nu) = \delta_{\nu,0}, \quad \sum_\nu f(\mu,\nu) = \delta_{\mu,0}. \tag{2.5}$$

Condition 2.5 follows from 2.3 and 2.4 if there exists $c \in \mathbb{C}$ such that $f(\mu,0) = f(0,\nu) = c$ for all μ, ν. There are conditions on the order of q: For all μ, ν for which there exist $\tilde{\mu}, \tilde{\nu}$ such that $f(\mu,\tilde{\nu}) \neq 0$, $f(\tilde{\mu},\nu) \neq 0$ we have

$$q^{2l_i\langle \mu,\alpha_i\rangle} = q^{2l_i\langle \nu,\alpha_i\rangle} = 1.$$

If this condition is satisfied then f is well-defined on the preimages of $H_1 \times H_2$ under $\Lambda \to \Lambda/\Lambda'$. (In particular, this is the case under our assumption $\Lambda' \subset \mathrm{Cent}^q(\Lambda_W)$.)

2.2. A set of equations.

Lemma 2.5. *Let $\Lambda \subset \Lambda_W$ a sublattice and $\Lambda' \subset \Lambda$. Assume in addition, $\Lambda' \subset \mathrm{Cent}^q(\Lambda_W)$.*

(i) Let $f : \Lambda/\Lambda' \times \Lambda/\Lambda' \to \mathbb{C}$, satisfying condition (2.3) of Theorem 2.4. Then

$$g(\bar{\mu},\bar{\nu}) := |\Lambda_R/\Lambda'| q^{(\mu,\nu)} f(\mu,\nu), \tag{2.6}$$

defines a function $\pi_1 \times \pi_1 \to \mathbb{C}$.

(ii) If, in addition, f satisfies conditions(2.4)-(2.5), the function g in (i) satisfies the following equations:

$$\begin{aligned} \sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \delta_{(\mu_2-\mu_1 \in \mathrm{Cent}^q(\Lambda_R))} q^{(\mu_2-\mu_1,\bar{\nu}_1)} g(\bar{\mu}_1,\bar{\nu}_1) g(\bar{\mu}_2,\bar{\nu}_2) &= \delta_{\mu_1,\mu_2} g(\bar{\mu}_1,\bar{\nu}), \\ \sum_{\bar{\mu}_1+\bar{\mu}_2=\bar{\mu}} \delta_{(\nu_2-\nu_1 \in \mathrm{Cent}^q(\Lambda_R))} q^{(\nu_2-\nu_1,\bar{\mu}_1)} g(\bar{\mu}_1,\bar{\nu}_1) g(\bar{\mu}_2,\bar{\nu}_2) &= \delta_{\nu_1,\nu_2} g(\bar{\mu},\bar{\nu}_1), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \sum_{\bar\nu} \delta_{(\mu\in \mathrm{Cent}^q(\Lambda_R))} q^{-(\mu,\bar\nu)} g(\bar\mu,\bar\nu) &= \delta_{\mu,0}, \\ \sum_{\bar\mu} \delta_{(\nu\in \mathrm{Cent}^q(\Lambda_R))} q^{-(\nu,\bar\mu)} g(\bar\mu,\bar\nu) &= \delta_{\nu,0}. \end{aligned} \tag{2.8}$$

Here, the sums range over π_1 and expressions like $\delta_{(\mu\in \mathrm{Cent}^q(\Lambda_R))}$ equals 1 if μ is a central weight and 0 otherwise.

Before we proceed with the proof we will comment on the relevance of this equations and introduce a definition. For a given Lie algebra $\mathfrak{g}$ with root lattice Λ_R and weight lattice Λ_W the solutions of the $g(\bar\mu,\bar\nu)$-equations give solutions for an R_0 in the ansatz $R = R_0\bar\Theta$. Hence, we get possible R-matrices for the quantum group $u_q(\mathfrak{g},\Lambda_W,\Lambda')$.

We divide the equations in two types.

Definition 2.6. For central weight 0 we call the equations (2.7)-(2.8) *group-equations*:

$$\begin{aligned} g(\bar\mu,\bar\nu) &= \sum_{\bar\nu_1+\bar\nu_2=\bar\nu} g(\bar\mu,\bar\nu_1)g(\bar\mu,\bar\nu_2), \\ g(\bar\mu,\bar\nu) &= \sum_{\bar\mu_1+\bar\mu_2=\bar\mu} g(\bar\mu_1,\bar\nu)g(\bar\mu_2,\bar\nu), \\ 1 &= \sum_{\bar\nu} g(0,\bar\nu), \\ 1 &= \sum_{\bar\mu} g(\bar\mu,0). \end{aligned}$$

For $\pi_1 = \Lambda_W/\Lambda_R$ of order n this gives us $2n^2+2$ group-equations.

For central weight $0 \neq \zeta \in \mathrm{Cent}^q(\Lambda_R)/\Lambda'$, we call the equations (2.7)-(2.8) *diamond-equations* (for reasons that will become transparent later):

$$\begin{aligned} 0 &= \sum_{\bar\nu_1+\bar\nu_2=\bar\nu} q^{(\zeta,\bar\nu_1)} g(\bar\mu,\bar\nu_1)g(\bar\mu+\bar\zeta,\bar\nu_2), \\ 0 &= \sum_{\bar\mu_1+\bar\mu_2=\bar\mu} q^{(\zeta,\bar\mu_1)} g(\bar\mu_1,\bar\nu)g(\bar\mu_2,\bar\nu+\bar\zeta), \\ 0 &= \sum_{\bar\nu} q^{-(\bar\nu,\zeta)} g(\bar\mu+\bar\zeta,\bar\nu), \\ 0 &= \sum_{\bar\mu} q^{-(\bar\mu,\zeta)} g(\bar\mu,\bar\nu+\bar\zeta). \end{aligned}$$

This gives up to $(|\mathrm{Cent}^{[\ell]}(\Lambda_R)/\Lambda'|-1)(2n^2+2)$ diamond-equations.

Proof of Lemma 2.5. (i) Since $\Lambda' \subset \mathrm{Cent}^q(\Lambda_W)$ we have $q^{(\Lambda_W,\Lambda')} = 1$ and terms $q^{(\mu,\nu)}$ for $\mu,\nu \in \Lambda/\Lambda'$ do not depend on the residue class representatives modulo Λ'. We check that the function g is well-defined. Let $\mu,\nu\in\Lambda$ and $\lambda'\in\Lambda'$. Thus,

$$\begin{aligned} g(\mu+\lambda',\nu) &= |\Lambda_R/\Lambda'| q^{(\mu+\lambda',\nu)} f(\mu+\lambda',\nu) \\ &= |\Lambda_R/\Lambda'| q^{(\mu+\lambda',\nu)} q^{-(\lambda',\nu)} f(\mu,\nu) && \text{by eq. (2.3)} \\ &= |\Lambda_R/\Lambda'| q^{(\mu,\nu)} f(\mu,\nu) \\ &= g(\mu,\nu), \end{aligned}$$

and analogously for $g(\mu, \nu + \lambda')$.

(ii) We consider equations (2.4). Let $\nu_i, \nu \in \Lambda/\Lambda'$ and write $\nu_i = \bar{\nu}_i + \alpha_i$ and $\nu = \bar{\nu} + \alpha$ with $\bar{\nu}_i, \bar{\nu} \in \Lambda_W/\Lambda_R$ and $\alpha_i, \alpha \in \Lambda_R$, $i = 1, 2$. For the sum $\nu = \nu_1 + \nu_2$ we get $\bar{\nu} \equiv \bar{\nu}_1 + \bar{\nu}_2$ in Λ_W/Λ_R, i.e. there is a cocycle $\sigma(\nu_1, \nu_2) \in \Lambda_R$ with $\bar{\nu} = \bar{\nu}_1 + \bar{\nu}_2 + \sigma(\nu_1, \nu_2)$ in Λ_W and $\alpha = \alpha_1 + \alpha_2 - \sigma(\nu_1, \nu_2)$. We will write σ for $\sigma(\nu_1, \nu_2)$.

$$\begin{aligned}
&\sum_{\nu_1+\nu_2=\nu} f(\mu_1, \nu_1) f(\mu_2, \nu_2) \\
&\quad = \sum_{\nu_1+\nu_2=\nu} q^{-(\mu_1,\nu_1)+(\bar{\mu}_1,\bar{\nu}_1)-(\mu_2,\nu_2)+(\bar{\mu}_2,\bar{\nu}_2)} f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
&\quad = \sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \sum_{\alpha_1+\alpha_2=\alpha+\sigma} q^{-(\mu_1,\bar{\nu}_1)-(\mu_1,\alpha_1)+(\bar{\mu}_1,\bar{\nu}_1)} q^{-(\mu_2,\bar{\nu}_2)-(\mu_2,\alpha_2)+(\bar{\mu}_2,\bar{\nu}_2)} f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
&\quad = \sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} q^{-(\mu_1,\bar{\nu}_1)+(\bar{\mu}_1,\bar{\nu}_1)-(\mu_2,\bar{\nu}_2)+(\bar{\mu}_2,\bar{\nu}_2)} f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \sum_{\alpha_1+\alpha_2=\alpha+\sigma} q^{-(\mu_1,\alpha_1)-(\mu_2,\alpha_2)}
\end{aligned} \tag{$*$}$$

Firstly, we consider the second sum over the roots (μ_1, μ_2 are fixed).

$$\begin{aligned}
\sum_{\alpha_1+\alpha_2=\alpha+\sigma} q^{-(\mu_1,\alpha_1)-(\mu_2,\alpha_2)} &= \sum_{\alpha_1 \in \Lambda_R/\Lambda'} q^{-(\mu_1,\alpha_1)-(\mu_2,\alpha+\sigma-\alpha_1)} \\
&= q^{-(\mu_2,\alpha+\sigma)} \sum_{\alpha_1 \in \Lambda_R/\Lambda'} q^{(\mu_2-\mu_1,\alpha_1)}
\end{aligned}$$

The last sum equals $|\Lambda_R/\Lambda'|$ iff $\ell \mid (\mu_2 - \mu_1, \alpha_1)$ for all $\alpha_1 \in \Lambda_R/\Lambda'$, i.e. $\mu_2 - \mu_1 \in \mathrm{Cent}^q(\Lambda_R)$, and 0 otherwise. Hence, with $C = |\Lambda_R/\Lambda'| \cdot \delta_{(\mu_2-\mu_1 \in \mathrm{Cent}^{[\ell]}(\Lambda_R))}$, the sum ($*$) simplifies to

$$\begin{aligned}
&C \cdot \sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} q^{-(\mu_1,\bar{\nu}_1)+(\bar{\mu}_1,\bar{\nu}_1)-(\mu_2,\bar{\nu}_2)+(\bar{\mu}_2,\bar{\nu}_2)} q^{-(\mu_2,\alpha+\sigma)} f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
&\quad = C \cdot \sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} q^{-(\mu_1,\bar{\nu}_1)+(\bar{\mu}_1,\bar{\nu}_1)+(\bar{\mu}_2,\bar{\nu}_2)-(\mu_2,\bar{\nu}_1+\bar{\nu}_2+\alpha+\sigma)+(\mu_2,\bar{\nu}_1)} f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
&\quad = C \cdot q^{-(\mu_2,\nu)} \sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} q^{(\mu_2-\mu_1,\bar{\nu}_1)} q^{(\bar{\mu}_1,\bar{\nu}_1)} f(\bar{\mu}_1, \bar{\nu}_1) q^{(\bar{\mu}_2,\bar{\nu}_2)} f(\bar{\mu}_2, \bar{\nu}_2),
\end{aligned}$$

Comparing this with the right hand side of the first equation of (2.4) gives

$$\begin{aligned}
C \cdot q^{-(\mu_2,\nu)} &\sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} q^{(\mu_2-\mu_1,\bar{\nu}_1)} q^{(\bar{\mu}_1,\bar{\nu}_1)} f(\bar{\mu}_1, \bar{\nu}_1) q^{(\bar{\mu}_2,\bar{\nu}_2)} f(\bar{\mu}_2, \bar{\nu}_2) \\
&= \delta_{\mu_1,\mu_2} q^{-(\mu_2,\nu)+(\bar{\mu}_2,\bar{\nu})} f(\bar{\mu}_2, \bar{\nu}),
\end{aligned}$$

and with the definition of $g(\bar{\mu}, \bar{\nu}) = |\Lambda_R/\Lambda'| q^{(\mu,\nu)} f(\mu, \nu)$ we get the following equation

$$\sum_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \delta_{(\mu_2-\mu_1 \in \mathrm{Cent}^q(\Lambda_R))} q^{(\mu_2-\mu_1,\bar{\nu}_1)} g(\bar{\mu}_1, \bar{\nu}_1) g(\bar{\mu}_2, \bar{\nu}_2) = \delta_{\mu_1,\mu_2} g(\bar{\mu}_1, \bar{\nu}).$$

Analogously, we get the equation of the sum $\sum_{\bar{\mu}_1+\bar{\mu}_2=\bar{\mu}}$.

We now consider the equations (2.5). Again, $\nu = \bar{\nu} + \alpha$ as above.

$$\begin{aligned}
\sum_{\nu \in \Lambda/\Lambda'} f(\mu,\nu) &= \sum_{\nu} q^{-(\mu,\nu)+(\bar{\mu},\bar{\nu})} f(\bar{\mu},\bar{\nu}) \\
&= \sum_{\bar{\nu}} q^{(\bar{\mu},\bar{\nu})} f(\bar{\mu},\bar{\nu}) \sum_{\alpha \in \Lambda_R/\Lambda'} q^{-(\mu,\bar{\nu}+\alpha)} \\
&= \sum_{\bar{\nu}} q^{-(\mu-\bar{\mu},\bar{\nu})} f(\bar{\mu},\bar{\nu}) \sum_{\alpha \in \Lambda_R/\Lambda'} q^{-(\mu,\alpha)} \\
&= \delta_{(\mu \in \mathrm{Cent}^{[\ell]}(\Lambda_R))} |\Lambda_R/\Lambda'| \sum_{\bar{\nu}} q^{-(\mu-\bar{\mu},\bar{\nu})} f(\bar{\mu},\bar{\nu}) \\
&= \delta_{(\mu \in \mathrm{Cent}^{[\ell]}(\Lambda_R))} \sum_{\bar{\nu}} q^{-(\mu,\bar{\nu})} g(\bar{\mu},\bar{\nu}) \\
&= \delta_{\mu,0}.
\end{aligned}$$

∎

3. The first type of equations

3.1. Equations of *group-type*.

Definition 3.1. For an abelian group G we define a set of equations for $|G|^2$ variables $g(x,y)$, $x,y \in G$, which we call *group-equations*.

$$g(x,y) = \sum_{y_1+y_2=y} g(x,y_1)g(x,y_2), \tag{3.1}$$

$$g(x,y) = \sum_{x_1+x_2=x} g(x_1,y)g(x_2,y), \tag{3.2}$$

$$1 = \sum_{y \in G} g(0,y), \tag{3.3}$$

$$1 = \sum_{x \in G} g(x,0). \tag{3.4}$$

Thus, there are $2|G|^2 + 2$ group-equations in $|G|^2$ variables with values in $\mathbb{C}$.

These equations are the equations in Lemma 2.5 and the following Definition for central weight $\zeta = 0$.

Theorem 3.2. *Let G be an abelian group of order N, H_1, H_2 subgroups with $|H_1| = |H_2| = d$. Let $\omega\colon H_1 \times H_2 \to \mathbb{C}^\times$ be a pairing of groups. Here, the group G is written additively and $\mathbb{C}^\times$ multiplicatively, thus we have $\omega(x,y)^d = 1$ for all $x \in H_1, y \in H_2$. Then the function*

$$g\colon G \times G \to \mathbb{C},\ (x,y) \mapsto \frac{1}{d}\,\omega(x,y)\delta_{(x\in H_1)}\delta_{(y \in H_2)} \tag{3.5}$$

is a solution of the group-equations (3.1)-(3.4) *of G.*

Proof. Let G, H_1, H_2 and ω be as in the theorem. We insert the function g as in (3.5) in the group-equation (3.1) of G. Let $x, y \in G$.

$$\begin{aligned}\sum_{y_1+y_2=y} g(x,y_1)g(x,y_2) &= \left(\frac{1}{d}\right)^2 \sum_{y_1+y_2=y} \omega(x,y_1)\omega(x,y_2)\delta_{(x\in H_1)}\delta_{(y_1\in H_2)}\delta_{(y_2\in H_2)} \\ &= \left(\frac{1}{d}\right)^2 \sum_{y_1+y_2=y} \omega(x,y_1+y_2)\delta_{(x\in H_1)}\delta_{(y_1\in H_2)}\delta_{(y_2\in H_2)} \\ &= \left(\frac{1}{d}\right)^2 |H_2|\,\omega(x,y)\delta_{(x\in H_1)}\delta_{(y\in H_2)} \\ &= g(x,y).\end{aligned}$$

Analogously for the sum in (3.2). We now insert the function g in (3.3):

$$\sum_{y\in G} g(0,y) = \frac{1}{d}\sum_{y\in G}\omega(0,y)\delta_{(y\in H_2)} = \frac{1}{d}\sum_{y\in H_2} 1 = 1. \qquad \blacksquare$$

Question 3.3. Are these all solutions of the group-equations for a given group G?

3.2. **Results for all fundamental groups of Lie algebras.** We now treat the cases $G = \mathbb{Z}_N$ for $N \geq 1$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, since these are the only examples of fundamental groups π_1 of root systems.

Theorem 3.4. *In the following cases, the functions g of Theorem 3.2 are the only solutions of the group-equations* (3.1)-(3.4) *of G.*

(a) For $G = \mathbb{Z}_N$, the cyclic groups of order N. Here, we get $\sum_{d|N} d$ different solutions.
(b) For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Here we get 35 *different solutions.*

Proof. (a) This is the content of [LN14], Theorem 5.6.
(b) We have checked this explicitly via MAPLE. $\blacksquare$

Example 3.5. Let $G = \mathbb{Z}_N$, $N \geq 1$. For any divisor d of N there is a unique subgroup $H = \frac{N}{d}\mathbb{Z}_N \cong \mathbb{Z}_d$ of G of order d. By Theorem 3.2 we have, that for any pairing $\omega\colon H\times H \to \mathbb{C}^\times$, the function g as in (3.5) is a solution of the group-equations (3.1)-(3.4). We give the solution explicitly. For $H = \langle h\rangle$, $h \in \frac{N}{d}\mathbb{Z}_n$, we get a pairing $\omega\colon H \times H \to \mathbb{C}^\times$ by $\omega(h,h) = \xi$ with ξ a d-th root of unity, not necessarily primitive. Thus, the function (3.5) translates to

$$g\colon G \times G \to \mathbb{C},\ (x,y) \mapsto \frac{1}{d}\,\xi^{\frac{xy}{(N/d)^2}}\,\delta_{(\frac{N}{d}|x)}\delta_{(\frac{N}{d}|y)}. \tag{3.6}$$

Example 3.6. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b\rangle$. For $H_1 = H_2 = G$ there are $2^4 = 16$ possible parings, since a pairing is given by determining the values of $\omega(x,y) = \pm 1$ for $x, y \in \{a, b\}$. In G, there are 3 different subgroups of order 2, hence there are 9 possible pairs (H_1, H_2) of groups H_i of order 2. For each pair, there are two possible choices for $\omega(x,y) = \pm 1$, x, y being the generators of H_1, resp. H_2. Thus, we get 18 pairings for subgroups of order $d = 2$. For $H_1 = H_2 = \{0\}$ there is only one pairing, mapping $(0,0)$ to 1. Thus, we have 35 pairings in total.

#	$H_i \cong$	H_1	H_2	ω
16	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle a,b\rangle$	$\langle a,b\rangle$	$\omega(x,y) = \pm 1$ for $x,y \in \{a,b\}$
9×2	$\mathbb{Z}_2$	$\langle x\rangle$, $x \in \{a,b,a+b\}$	$\langle y\rangle$, $y \in \{a,b,a+b\}$	$\omega(x,y) = \pm 1$
1	$\mathbb{Z}_1$	$\{0\}$	$\{0\}$	$\omega(0,0) = 1$

4. Quotient diamonds and the second type of equations

4.1. Quotient diamonds and equations of *diamond-type*.

Definition 4.1. Let G and A be abelian groups and B, C, D subgroups of A, such that $D = B \cap C$. We call a tuple $(G, A, B, C, D, \varphi_1, \varphi_2)$ with injective group morphisms $\varphi_1 \colon A/B \to G^* = \mathrm{Hom}(G, \mathbb{C}^\times)$ and $\varphi_2 \colon A/C \to G$ a *diamond for* G. We will visualize the situation with the following diagram

$$\begin{tikzcd}[column sep=small] & & A & & \\ G^* \hookleftarrow & B \arrow[ur, dash] & & C \arrow[ul, dash] & \hookrightarrow G \\ & & D \arrow[ul, dash] \arrow[ur, dash] & & \end{tikzcd}$$

Definition 4.2. Let $(G, A, B, C, D, \varphi_1, \varphi_2)$ be a diamond for G. For $a \in A$ and not in $B \cap C$ we define the following equations for the $|G|^2$ variables $g(x, y)$, $x, y \in G$:

$$0 = \sum_{y_1 + y_2 = y, y_i \in G} \varphi_1(a)(y_1) g(x, y_1) g(x + \varphi_2(a), y_2), \tag{4.1}$$

$$0 = \sum_{x_1 + x_2 = x, x_i \in G} \varphi_1(a)(x_1) g(x_1, y) g(x_2, y + \varphi_2(a)), \tag{4.2}$$

$$0 = \sum_{y \in G} (\varphi_1(a)(y))^{-1} g(\varphi_2(a), y), \tag{4.3}$$

$$0 = \sum_{x \in G} (\varphi_1(a)(x))^{-1} g(x, \varphi_2(a)). \tag{4.4}$$

We call this set of equations *diamond-equations* for the diamond of G. Here, $\varphi_i(a)$ denotes the image of $a + B$, resp. $a + C$, for $a \in A$ under φ_1, resp. φ_2.

These are up to $(|A| - 1)(2|G|^2 + 2)$ equations in $|G|^2$ variables with values in $\mathbb{C}$.

We show how these equations arise in the situation of Lemma 2.5.

Lemma 4.3. *Let $G = \pi_1$, the fundamental group of a root system Φ. Assume Λ' is a sublattice of Λ_R, contained in $\mathrm{Cent}^q(\Lambda_W)$. Let $A = \mathrm{Cent}^q(\Lambda_R)/\Lambda'$, $B = \mathrm{Cent}^q(\Lambda_W)/\Lambda'$, $C = \mathrm{Cent}^q(\Lambda_R) \cap \Lambda_R/\Lambda'$ and $D = \mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R/\Lambda'$. Then there exist injections $\varphi_1 \colon A/B \to \pi_1^*$ and $\varphi_2 \colon A/C \to \pi_1$, such that $(G, A, B, C, D, \varphi_1, \varphi_2)$ is a diamond for G.*

$$\begin{tikzcd}[column sep=small] & & \mathrm{Cent}^{[\ell]}(\Lambda_R)/\Lambda' & & \\ \pi_1^* \hookleftarrow & \mathrm{Cent}^{[\ell]}(\Lambda_W)/\Lambda' \arrow[ur, dash] & & \mathrm{Cent}^{[\ell]}(\Lambda_R) \cap \Lambda_R/\Lambda' \arrow[ul, dash] & \hookrightarrow \pi_1 \\ & & \mathrm{Cent}^{[\ell]}(\Lambda_W) \cap \Lambda_R/\Lambda' \arrow[ul, dash] \arrow[ur, dash] & & \end{tikzcd}$$

Proof. Recall from Lemmas 1.6 and 1.7, that we have $\mathrm{Cent}^q(\Lambda_R) = \Lambda_W^{[\ell]}$ and $\mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R = \Lambda_R^{[\ell]}$. We have $A/C \cong \Lambda_W^{[\ell]}/(\Lambda_W^{[\ell]} \cap \Lambda_R)$ and $\Lambda_W^{[\ell]} \subset \Lambda_W$, To show the existence of an injective morphism $\varphi_2 : A/C \to \pi_1$, we define $\tilde{\varphi}_2$ on $\Lambda_W^{[\ell]}$ and calculate the kernel. By Definition 1.4, the generators of $\Lambda_W^{[\ell]}$ are $\ell_{[i]}\lambda_i$ for all $i \in I$, with $\ell_{[i]} := \ell/\gcd(\ell, d_i)$. Thus

$$\tilde{\varphi}_2 \colon \Lambda_W^{[\ell]} \to \pi_1,\ \ell_{[i]}\lambda_i \mapsto \ell_{[i]}\lambda_i + \Lambda_R$$

gives a group morphism. Since $\Lambda' \subset \Lambda_R \cap \Lambda_W^{[\ell]} = \ker \tilde{\varphi}_2$, this induces a well-definend map $\varphi_2 \colon A/\Lambda' \to \pi_1$. Obviously, the kernel of this map is $\Lambda_W^{[\ell]} \cap \Lambda_R$, hence the desired injection $\varphi_2 \colon A/C \to \pi_1$ exists and is given by taking $\lambda + (\Lambda_W^{[\ell]} \cap \Lambda_R)$ modulo Λ_R, $\lambda \in \Lambda_W^{[\ell]}$.

Now, we show the existence of φ_1. The map

$$f \colon \mathrm{Cent}^q(\Lambda_R) \to \mathrm{Hom}(\Lambda_W, \mathbb{C}^\times),\ \lambda \mapsto (\Lambda_W \to \mathbb{C}^\times,\ \eta \mapsto q^{(\lambda,\eta)})$$

is a group morphism. We define $g \colon \mathrm{Hom}(\Lambda_W, \mathbb{C}^\times) \to \mathrm{Hom}(\Lambda_W/\Lambda_R, \mathbb{C}^\times)$ by $g(\psi) := \psi \circ p$, where p is the natural projection $\Lambda_W \to \Lambda_W/\Lambda_R$. Thus, the upper right triangle of the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Cent}^q(\Lambda_R) \arrow[r, "f"] \arrow[d] \arrow[dr, "g \circ f"'] & \mathrm{Hom}(\Lambda_W, \mathbb{C}^\times) \arrow[d, "g"] \\
\mathrm{Cent}^q(\Lambda_R)/\mathrm{Cent}^q(\Lambda_W) \arrow[r, "\varphi_1"'] & \mathrm{Hom}(\pi_1, \mathbb{C}^\times)
\end{tikzcd}$$

There exists $\lambda \in \ker g \circ f$, iff $q^{(\lambda,\bar{\eta})} = 1$ for all $\bar{\eta} \in \pi_1$. Since $\lambda \in \mathrm{Cent}^q(\Lambda_R)$, this is equivalent to $q^{(\lambda,\eta)} = 1$ for all $\eta \in \Lambda_W$, hence $\lambda \in \mathrm{Cent}^q(\Lambda_W)$. Thus, we get φ_1 as desired, which is well defined as map from $\mathrm{Cent}^q(\Lambda_R)/\Lambda'/\mathrm{Cent}^q(\Lambda_W)/\Lambda'$ since $\Lambda' \subset \mathrm{Cent}^q(\Lambda_W) = \ker f$. ∎

Lemma 4.4. *Let $(G, A, B, C, D, \varphi_1, \varphi_2)$ be a diamond as in Lemma 4.3. If $\mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R/\Lambda' \neq 0$, then none of the solutions of the group-equations (3.1)-(3.4) are solutions to the diamond-equations (4.1)-(4.4). Hence under our assumptions 1.13, the existence of an R-matrix requires necessarily the choice $\Lambda' = \mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R$.*

Proof. If $\mathrm{Cent}^q(\Lambda_W) \cap \Lambda_R/\Lambda' \neq 0$, then there exist a root $\zeta \in \mathrm{Cent}^q(\Lambda_W)$, not contained in the kernel Λ'. Thus, there are diamond-equations with $\varphi_1(\zeta) = \mathbf{1}$ and $\varphi_2(\zeta) = 0$, i.e. the set of equations:

$$0 = \sum_{y_1+y_2=y} g(x, y_1)g(x, y_2), \tag{4.5}$$

$$0 = \sum_{x_1+x_2=x} g(x_1, y)g(x_2, y), \tag{4.6}$$

$$0 = \sum_{y \in G} g(0, y), \tag{4.7}$$

$$0 = \sum_{x \in G} g(x, 0). \tag{4.8}$$

Since this are group-equations as in Definition 3.1, but with left-hand side equal to 0, solutions of the group-equations does not solve the diamond-equations in this situation. ∎

Before examining in which case a solution of the group-equations as in Theorem 3.2 is also a solution of the diamond-equations (4.1)-(4.4), we show that it is sufficient to check the diamond-equations (4.3) and (4.4).

Lemma 4.5. *Let G be an abelian group of order N, H_1, H_2 subgroups with $|H_1| = |H_2| = d$ and $\omega\colon H_1 \times H_2 \to \mathbb{C}^\times$ a group-pairing, such that $g\colon G \times G \to \mathbb{C}$, $(x, y) \mapsto 1/d\ \omega(x, y)\delta_{(x\in H_1)}\delta_{(y\in H_2)}$ is a solution of the group-equations (3.1)-(3.4), as in Theorem 3.2. Then the following holds:*
If g is a solution of the diamond-equations (4.1), (4.2), then g solves the diamond-equations (4.3), (4.4) as well.

Proof. Let g be a solution of the group-equations as in Theorem 3.2. Assume that g solves (4.1) and (4.2). Let φ_1, φ_2 as in Definition 4.1 and $0 \neq \zeta \in A$ a non-trivial central weight. Then, for $x, y \in G$ we get by inserting g in (4.1)

$$\begin{aligned}
0 &= \sum_{y_1+y_2=y} \varphi_1(\zeta)(y_1)g(x, y_1)g(x + \varphi_2(\zeta), y_2)\\
&= \sum_{y_1+y_2=y} \varphi_1(\zeta)(y_1)\frac{1}{d^2}\omega(x, y_1)\omega(x + \varphi_2(\zeta), y_2)\delta_{(x\in H_1)}\delta_{(y_1\in H_2)}\delta_{(x+\varphi_2(\zeta)\in H_1)}\delta_{(y_2\in H_2)}\\
&= \delta_{(x\in H_1)}\delta_{(y\in H_2)}\delta_{(\varphi_2(\zeta)\in H_1)}\frac{1}{d^2}\sum_{\substack{y_1+y_2=y\\ y_1,y_2\in H_2}} \varphi_1(\zeta)(y_1)\omega(x, y_1)\omega(x, y_2)\omega(\varphi_2(\zeta), y_2)\\
&= \delta_{(x\in H_1)}\delta_{(y\in H_2)}\delta_{(\varphi_2(\zeta)\in H_1)}\frac{1}{d^2}\omega(x, y)\sum_{\substack{y_1+y_2=y\\ y_1,y_2\in H_2}} \varphi_1(\zeta)(y_1)\omega(\varphi_2(\zeta), y_2)\\
&= \delta_{(x\in H_1)}\delta_{(y\in H_2)}\delta_{(\varphi_2(\zeta)\in H_1)}\frac{1}{d^2}\omega(x, y)\sum_{y_2\in H_2} \varphi_1(\zeta)(y - y_2)\omega(\varphi_2(\zeta), y_2)\\
&= \delta_{(x\in H_1)}\delta_{(y\in H_2)}\delta_{(\varphi_2(\zeta)\in H_1)}\frac{1}{d^2}\omega(x, y)\varphi_1(\zeta)(y)\sum_{y_2\in H_2} \varphi_1(\zeta)(y_2)^{-1}\omega(\varphi_2(\zeta), y_2).
\end{aligned}$$

In particular, this holds for $x = y = 0$, and in this case the expression vanishes iff

$$\delta_{(\varphi_2(\zeta)\in H_1)}\frac{1}{d^2}\sum_{y\in H_2} \varphi_1(\zeta)(y)^{-1}\omega(\varphi_2(\zeta), y) = 0,$$

which is (4.3). Analogously, it follows that if g solves (4.2) it solves (4.4). ∎

4.2. **Cyclic fundamental group $G = \mathbb{Z}_N$.** In the following, G will always be a fundamental group of a simple complex Lie algebra, hence either cyclic or equal to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for the case D_n, n even. In this section, we will derive some results for the cyclic case.

In Example 3.5 we have given solutions of the group-equations for $G = \mathbb{Z}_N$, i.e. for all $d \mid N$ the functions

$$g\colon G \times G \to \mathbb{C},\ (x, y) \mapsto \frac{1}{d}\xi^{\frac{xy}{(N/d)^2}}\,\delta_{(\frac{N}{d}|x)}\delta_{(\frac{N}{d}|y)} \tag{4.9}$$

with ξ a d-th root of unity, not necessarily primitive. In the following, we denote by ξ_d the primitive d-th root of unity $\exp(2\pi i/d)$.

Lemma 4.6. *Let $l \geq 2$, $m \in \mathbb{N}$ and $G = \langle\lambda\rangle \cong \mathbb{Z}_N$. We consider the following diamonds $(G, A, B, C, D, \varphi_1, \varphi_2)$ with $A = \langle a\rangle \cong \mathbb{Z}_N$ and injections φ_1 and φ_2 given by*

$$\tilde{\varphi}_1\colon A \to G^*,\ a \mapsto (\xi_N^m)^{(-)}, \quad \text{with } (\xi_N^m)^{(-)}\colon G \to \mathbb{C}^\times,\ x \mapsto \xi_N^{mx},$$
$$\tilde{\varphi}_2\colon A \to G, \quad a \mapsto l\lambda,$$

with primitive N-th root of unity ξ_N, $B = \ker\tilde{\varphi}_1$ and $C = \ker\tilde{\varphi}_2$ and $D = \{0\}$.

Possible solutions of the group-equations (3.1)-(3.4) *are given for any choice of integers $1 \leq k \leq d$ and $d \mid N$ as in Example 3.5 by*

$$g\colon G \times G \to \mathbb{C},\ (x, y) \mapsto \frac{1}{d}\left(\xi_d^k\right)^{\frac{xy}{(N/d)^2}} \delta_{(\frac{N}{d}|x)}\delta_{(\frac{N}{d}|y)}, \tag{4.10}$$

with primitive d-th root of unity $\xi_d = \exp(2\pi i/d)$. These are solutions also to the diamond-equations (4.1)-(4.4)*, iff $N \mid m, l$ or the following condition hold:*

$$\gcd(N, dl, kl - \frac{N}{d}m) = 1. \tag{4.11}$$

Proof. For $N \mid m, l$ there is no non-trivial diamond-equation, hence all solutions of the group-equations as in Example 3.5 are possible. Assume now, that not both $N \mid m$ and $N \mid l$. We insert the function g from (4.10) in the diamond-equations (4.1)-(4.4) and get requirements for d, k, l and N. By Lemma 4.5 it is sufficient to consider only equations (4.3) and (4.4). Since for cyclic G the function g is symmetric we choose equation (4.3) for the calculation. In the following we omit the $\tilde{}$ on the maps $\tilde{\varphi}_{1/2}\colon A \to G^*$, resp. G. Let $1 \leq z < N$, $a \in A$ and $y \in G$, then

$$\begin{aligned}
\sum_{y=1}^{N} (\varphi_1(za)(y))^{-1}\, g(\varphi_2(za), y) &= \frac{1}{d}\sum_{y=1}^{N} \xi_N^{-zmy}\left(\xi_d^k\right)^{\frac{zly}{(N/d)^2}} \delta_{(\frac{N}{d}|zl)}\delta_{(\frac{N}{d}|y)} \\
&= \frac{1}{d}\sum_{y=1}^{N} \xi_d^{-\frac{zmy}{N/d}}\left(\xi_d^{\frac{zkl}{(N/d)}}\right)^{\frac{y}{N/d}} \delta_{(\frac{N}{d}|zl)}\delta_{(\frac{N}{d}|y)}, \\
&= \frac{1}{d}\sum_{y'=1}^{d}\left(\xi_d^{-zm+\frac{zkl}{(N/d)}}\right)^{y'} \delta_{(\frac{N}{d}|zl)},
\end{aligned}$$

with the substitution $y' = y/(N/d)$. This sum equals 0 iff $N/d \nmid zl$ or $d \nmid z(kl/(N/d) - m)$. This is equivalent to $N \nmid zdl$ or $N \nmid z(kl - (N/d)m)$, hence $N \nmid \gcd(zdl, z(kl - (N/d)m))$. Since this condition has to be fulfilled for all z we get that $N \nmid \gcd(dl, kl - (N/d)m)$, hence $\gcd(N, ld, kl - (N/d)m) = 1$. ∎

We spell out the condition for explicit values m and l.

Example 4.7. Let $G = \mathbb{Z}_N = \langle\lambda\rangle$, $l \geq 2$, $m \in \mathbb{N}$ and diamond $(G, A, B, C, D, \varphi_1, \varphi_2)$ as in Lemma 4.6. Depending on m, l we get the following criteria for solutions of the diamond-equations. Here, we give φ_1 and φ_2 shortly by the generator of its image.

- (I) If $N \mid m$ and $N \mid l$ we have the diamond $(\mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_1, 1, 0)$ and all solutions of the form (4.10) are also solutions to the diamond-equations (4.1)-(4.4). (Since $B, C = A$, there are no non-trivial diamond-equations.)
- (II) If $N \mid m$ and $N \nmid l$ we have the diamond $(\mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_{\gcd(l,N)}, \mathbb{Z}_1, 1, l\lambda)$. In this case the function g as in (4.10) is a solution to the diamond-equations (4.1)-(4.4) if $\gcd(N, dl, kl) = 1$.

(III) If $\gcd(m,N) = 1$ and $N \nmid l$ we have the diamond $(\mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_1, \mathbb{Z}_{\gcd(l,N)}, \mathbb{Z}_1, \xi_N, l\lambda)$. In this case the function g as in (4.10) is a solution to the diamond-equations (4.1)-(4.4) if

$$\gcd(N, dl, kl - \frac{N}{d}m) = 1. \tag{4.12}$$

In most cases, N is prime or equals 1, hence we consider the two special cases
(1) If $d = 1$, (4.12) simplifies to $\gcd(N, l, l - Nm) = 1$, which is equivalent to $\gcd(N, l) = 1$.
(2) If $d = N$, (4.12) simplifies to $\gcd(N, lN, kl - m) = 1$, which is equivalent to $\gcd(N, kl - m) = 1$.

Finally, we consider the Lie algebras with cyclic fundamental group in question and determine the values m and l according to the Lie theoretic data and thereby the corresponding diamonds.

Example 4.8. Let $G = \mathbb{Z}_N$ be the fundamental group of a simple complex Lie algebra $\mathfrak{g}$, generated by the fundamental dominant weight λ_n. Let $\ell \in \mathbb{N}$, $\ell > 2$, $q = \exp(2\pi i/\ell)$, $\ell_{[n]} = \ell/\gcd(\ell, d_n)$, $m_{[n]} := N(\lambda_n, \lambda_n)/\gcd(\ell, d_n)$ and $(G, A, B, C, D, \varphi_1, \varphi_2)$ be a diamond as in Lemma 4.3, such that the corresponding diamond-equations (4.1)-(4.4) have a solution that is also a solution to the group-equations (3.1)-(3.4). Then, the diamond is

$$(G, \mathbb{Z}_N, \mathbb{Z}_{\gcd(m_{[n]},N)}, \mathbb{Z}_{\gcd(\ell_{[n]},N)}, \mathbb{Z}_1, \varphi_1, \varphi_2), \tag{4.13}$$

with injections

$$\varphi_1 \colon A \to G^*,\ \ell_{[n]}\lambda_n \mapsto (\xi_N^{m_{[n]}})^{(-)}, \quad \text{with } (\xi_N^{m_{[n]}})^{(-)} \colon G \to \mathbb{C}^\times,\ x \mapsto \xi_N^{m_{[n]}x},$$
$$\varphi_2 \colon A \to G,\ \ \ell_{[n]}\lambda_n \mapsto \ell_{[n]}\lambda_n,$$

with primitive N-th root of unity $\xi_N = \exp(2\pi i/N)$. The group $A = \mathrm{Cent}^q(\Lambda_R)/\Lambda' = \Lambda_W^{[\ell]}/\Lambda_R^{[\ell]}$ is generated by $\ell_{[n]}\lambda_n$ and $q^{(\ell_{[n]}\lambda_n, \lambda_n)} = (\xi_N^N)^{(\lambda_n,\lambda_n)/\gcd(\ell,d_n)}$. Since the order of $\xi_N^{m_{[n]}}$ in $\mathbb{C}^\times$ is $N/\gcd(m_{[n]}, N)$ and the order of $\ell_{[n]}$ in $\mathbb{Z}_N$ is $N/\gcd(\ell_{[n]}, N)$, the injections φ_1, φ_2 determine the diamond (4.13).

In the following table, we give the values $\ell_{[n]}$ and $m_{[n]}$ for all root systems of simple Lie algebras with cyclic fundamental group.

$\mathfrak{g}$	$A_{n\geq 1}$	$B_{n\geq 2}$	$C_{n\geq 3}$		$D_{n\geq 5}$ n odd	E_6	E_7	E_8	F_4	G_2	
π_1	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	
N	$n+1$	2	2		4	3	2	1	1	1	
d_n	1	1	2		1	1	1	1	1	3	
ℓ	all	all	$2 \nmid \ell$	$2 \mid \ell$	all	all	all	all	all	$3 \nmid \ell$	$3 \mid \ell$
$\gcd(\ell, d_n)$	1	1	1	2	1	1	1	1	1	1	3
(λ_n, λ_n)	$\frac{n}{n+1}$	$\frac{n}{2}$	n		$\frac{n}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	2	1	6	
$\ell_{[n]}$	ℓ	ℓ	ℓ	$\ell/2$	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	$\ell/3$
$m_{[n]}$	n	n	$2n$	n	n	4	3	2	1	6	2
cases	(III)	(I)-(III)	(II)	(I)-(III)	(III)	(III)	(III)	(I)	(I)	(I)	(I)

In the last row we indicate which cases in Example 4.7 apply. This will guide the proof of Theorem A. Note that case (II) only appears for B_n, n even and ℓ odd, and for C_n, even n and $\ell \equiv 2 \mod 4$ or odd ℓ.

4.3. **Example:** B_2**.** For $\mathfrak{g}$ with root system B_2 we have $\pi_1 = \mathbb{Z}_2$. There is one long root, α_1, and one short root, α_2, hence $d_1 = 2$ and $d_2 = 1$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_1, λ_2 are given as in [Hum72], Section 13.2. Here, λ_1 is a root and λ_2 is the generator of the fundamental group $\mathbb{Z}_2$. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \alpha_2\}$.

$$\tilde{C} = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

Thus, $(\lambda_2, \lambda_2) = 1$. The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = \ell\Lambda_W$ and $C = D = \ell\Lambda_R$. Since $(\lambda_2, \lambda_2) = 1$, we have $B = \ell\Lambda_W$. (Since $(\lambda_n, \lambda_n) = n/2$, in the general case B_n, the group $\mathrm{Cent}^q(\Lambda_W)$ depends on n: for even n we have $B = \ell\Lambda_W$, and $B = \ell\Lambda_R$ for odd n.)

(ii) For even ℓ we have $A = C = B = \ell\langle\frac{1}{2}\lambda_1, \lambda_2\rangle$ and $D = \ell\langle\frac{1}{2}\alpha_1, \alpha_2\rangle$. (Again, B depends on n, hence we have $B = \langle\frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_{n-1}, \lambda_n\rangle$ if n is even and $B = \langle\frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_{n-1}, 2\lambda_n\rangle$ if n is odd.)

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ since by the necessary criterion of Lemma 4.4, this is the only case where possible solutions exist. We calculate Lusztig's kernel $2\Lambda_R^{(\ell)}$ as well and compare it with $\Lambda_R^{[\ell]}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n = 2 = N$. Thus, for $\Lambda' = \Lambda_R^{[\ell]}$ the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, 1, \lambda_2)$. By Example 4.7 (II), one has to check for which d, k it is $\gcd(2, d\ell, k\ell) = 1$. This gives the 2 solutions: $(d, k) = (1, 1)$ and $(d, k) = (2, 1)$.

(ii.a) For $\ell \equiv 2 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle\frac{1}{2}\alpha_1, \ldots \frac{1}{2}\alpha_{n-1}, \alpha_n\rangle \neq \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = N$ as above. Here, we have $\gcd(\ell, N) = N = 2$, thus for $\Lambda' = \Lambda_R^{[\ell]}$ we get the quotient diamond $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)$. Thus, all 3 solutions of the group-equations are solutions to the diamond-equations as well by 4.7 (I).

(ii.b) For $\ell \equiv 0 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle\frac{1}{2}\alpha_1, \ldots \frac{1}{2}\alpha_{n-1}, \alpha_n\rangle = 2\Lambda_R^{(\ell)}$. Thus in this case the quotient diamond as in (ii.a) is the same for Lusztig's kernel, namely $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)$ and again all 3 solutions of the group-equations are solutions to the diamond-equations as well.

$$\begin{array}{ccccc} & & \mathbb{Z}_2 & & \\ 1 \leftarrow\!\shortmid & \swarrow & & \searrow & \mapsto \lambda_2 \\ & \mathbb{Z}_2 & & \mathbb{Z}_1 & \\ & \searrow & & \swarrow & \\ & & \mathbb{Z}_1 & & \end{array} \qquad\qquad \begin{array}{ccccc} & & \mathbb{Z}_2 & & \\ 1 \leftarrow\!\shortmid & \swarrow & & \searrow & \mapsto 0 \\ & \mathbb{Z}_2 & & \mathbb{Z}_2 & \\ & \searrow & & \swarrow & \\ & & \mathbb{Z}_1 & & \end{array}$$

quotient diamond in case (i) quotient diamond in cases (ii.a), (ii.b)

5. Proof of Theorem A

We treat the root systems case by case and determine the solutions of diamond equations in Section 4 which are of the form

$$g\colon G\times G\to\mathbb{C},\ (x,y)\mapsto\frac{1}{d}\omega(x,y)\delta_{(x\in H_1)}\delta_{(y\in H_2)}$$

with subgroups H_1,H_2 of $G=\pi_1$ as in Theorem 3.2.
For this, we first determine the lattices $A=\mathrm{Cent}^q(\Lambda_R)=\Lambda_W^{[\ell]},\ B=\mathrm{Cent}^q(\Lambda_W),\ C=\mathrm{Cent}^q(\Lambda_R)\cap\Lambda_R,\ D=\mathrm{Cent}^q(\Lambda_W)\cap\Lambda_R=\Lambda_R^{[\ell]}$, depending on ℓ. For the Lie algebras with cyclic fundamental group (all but for root system D_n with even n), we then determine the values $m_{[n]}$ and $\ell_{[n]}$, depending on ℓ, n and the order of π_1, and thereby the quotient diamonds and which solutions of the group equations are solutions to the corresponding diamond equations. In these cases, the ω-part of the solutions to the group equations are of the form

$$\omega\colon H\times H\to\mathbb{C}^\times,\ (x,y)\mapsto\left(\xi_d^k\right)^{\frac{xy}{(N/d)^2}}$$

for subgroup $H=\frac{N}{d}\mathbb{Z}_N$ of π_1 of order d. We give the solutions by pairs (d,k), which we determine by applying Lemma 4.6 and Example 4.7. An overview of the possible cases gives Example 4.8.
For D_n with even n and fundamental group $\mathbb{Z}_2\times\mathbb{Z}_2$ we also determine all quotient diamonds (depending on ℓ) and check which solutions of the group equations solve the diamond equations in a rather case by case calculation.

(1) For $\mathfrak{g}$ with root system A_n, $n\geq 1$, we have $\pi_1=\mathbb{Z}_{n+1}$ for all n. The simple roots are $\alpha_1,\dots,\alpha_n$ and $d_i=1$ for $1\leq i\leq n$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and λ_n is the generator of the fundamental group $\mathbb{Z}_{n+1}$. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1,\dots,\alpha_n\}$.

$$\tilde{C}=\begin{pmatrix}2&-1&0&.&.&.&.&0\\-1&2&-1&0&.&.&.&0\\0&-1&2&-1&0&.&.&0\\.&.&.&.&.&.&.&.\\0&0&0&0&.&.&-1&2\end{pmatrix}$$

$$id_W^R=a_{ij}\text{ with }a_{ij}=\begin{cases}\frac{1}{n+1}i(n-j+1), & \text{if } i\leq j,\\ \frac{1}{n+1}j(n-i+1), & \text{if } i>j.\end{cases}$$

The lattice diamonds, depending on ℓ, are:
(i) For even ℓ we have $A=\ell\Lambda_W$, $B=D=\ell\Lambda_R$ and $C=\ell/\gcd(n+1,\ell)\Lambda_W$.
(ii) For odd ℓ: the same lattices as in (i).

We calculate the quotient diamonds for kernel $\Lambda'=\Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.
(i) For odd ℓ it is $\Lambda_R^{[\ell]}=\ell\Lambda_R\neq 2\ell\Lambda_R=2\Lambda_R^{(\ell)}$, $\ell_{[n]}=\ell$ and $m_{[n]}=n$. Thus, the quotient diamond is given by $(\mathbb{Z}_{n+1},\mathbb{Z}_{n+1},\mathbb{Z}_1,\mathbb{Z}_{\gcd(\ell,n+1)},\mathbb{Z}_1,\xi_{n+1},\ell\lambda_n)$, hence we are in case (III) of Example 4.7. We get solutions (d,k) iff $\gcd(n+1,d\ell,k\ell-\frac{n+1}{d}n)=1$.

(ii) For odd ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamonds and solutions are as in (i).

(2) For $\mathfrak{g}$ with root system B_n, $n \geq 2$, we have $\pi_1 = \mathbb{Z}_2$ for all n. The long simple roots are $\alpha_1, \dots, \alpha_{n-1}$ and the short simple root α_n, hence $d_i = 2$ for $1 \leq i \leq n-1$ and $d_n = 1$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2. Here, $\lambda_1, \dots, \lambda_{n-1}$ are roots and λ_n is the generator of the fundamental group $\mathbb{Z}_2$. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \dots, \alpha_n\}$.

$$\tilde{C} = \begin{pmatrix} 4 & -2 & 0 & 0 & . & & & 0 \\ -2 & 4 & -2 & 0 & . & & & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & & . & -2 & 4 & -2 \\ 0 & 0 & 0 & & . & 0 & -2 & 2 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} 1 & 1 & . & . & 1 & \frac{1}{2} \\ 1 & 2 & . & . & 2 & 1 \\ . & . & . & . & . & . \\ 1 & 2 & 3 & . & n-1 & \frac{n-1}{2} \\ 1 & 2 & 3 & . & n-1 & \frac{n}{2} \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = \ell\Lambda_W$ and $C = D = \ell\Lambda_R$. Since $(\lambda_n, \lambda_n) = n/2$, the group $\mathrm{Cent}^q(\Lambda_W)$ depends on n. It is $B = \ell\Lambda_W$ for even n and $B = \ell\Lambda_R$ for odd n.

(ii) For even ℓ we have $A = C = \ell\langle \frac{1}{2}\lambda_1, \dots, \frac{1}{2}\lambda_{n-1}, \lambda_n\rangle$ and $D = \ell\langle \frac{1}{2}\alpha_1, \dots, \frac{1}{2}\alpha_{n-1}, \alpha_n\rangle$. Again, B depends on n, and we have $B = \ell\langle \frac{1}{2}\lambda_1, \dots, \frac{1}{2}\lambda_{n-1}, \lambda_n\rangle$ for even n and $B = \ell\langle \frac{1}{2}\lambda_1, \dots, \frac{1}{2}\lambda_{n-1}, 2\lambda_n\rangle$ for odd n.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. In this case, the quotient diamond is given by either $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, 1, \lambda_n)$ for even n, or by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, -1, \lambda_n)$ for odd n. Thus we are either in case (II), or in case (III) of Example 4.7. In the first case (even n) we get solutions by $(d, k) = (1, 1)$ and $(2, 1)$. For odd n we get solutions $(d, k) = (1, 1)$ and $(2, 2)$.

(ii.a) For $\ell \equiv 2 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle \frac{1}{2}\alpha_1, \dots, \frac{1}{2}\alpha_{n-1}, \alpha_n\rangle \neq \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. The quotient diamond is given by either $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)$ for even n, or by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_1, -1, 0)$ for odd n. Thus we are in either in case (I) or in case (III) of Example 4.7. In the first case (even n) we get all possible 3 solutions $(d, k) = (1, 1)$, $(2, 1)$ and $(2, 1)$. For odd n we get solutions $(d, k) = (2, 1)$ and $(2, 2)$.

(ii.b) For $\ell \equiv 0 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle \frac{1}{2}\alpha_1, \dots \frac{1}{2}\alpha_{n-1}, \alpha_n\rangle = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus the quotient diamonds and solutions are as in (ii).

(3) For $\mathfrak{g}$ with root system C_n, $n \geq 3$, we have $\pi_1 = \mathbb{Z}_2$ for all n. The short simple roots are $\alpha_1, \dots, \alpha_{n-1}$ and the long simple root α_n, hence $d_i = 1$ for $1 \leq i \leq n-1$ and $d_n = 2$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and λ_n is the generator of the fundamental group $\mathbb{Z}_2$. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \dots, \alpha_n\}$.

$$\tilde{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & . & & & 0 \\ -1 & 2 & -1 & 0 & . & & & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & & . & -1 & 2 & -2 \\ 0 & 0 & 0 & & . & 0 & -2 & 4 \end{pmatrix} \quad id_W^R = \begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 1 & 2 & . & . & 2 & 2 \\ . & . & . & . & . & . \\ 1 & 2 & . & . & n-1 & n-1 \\ \frac{1}{2} & 1 & . & . & \frac{n-1}{2} & \frac{n}{2} \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = B = \ell\Lambda_W$ and $C = D = \ell\Lambda_R$.

(ii) For $\ell \equiv 2 \mod 4$ we have $A = \ell\langle\lambda_1, \ldots, \lambda_{n-1}, \frac{1}{2}\lambda_n\rangle$ and $C = D = \ell\Lambda_W$. Since $(\lambda_n, \lambda_n) = n$, $B = \mathrm{Cent}^q(\Lambda_W)$ depends on n. For odd n it equals $\ell\Lambda_W$ and for even n it is equal to A.

(iii) For $\ell \equiv 0 \mod 4$ we have $A = C = \ell\langle\lambda_1, \ldots, \lambda_{n-1}, \frac{1}{2}\lambda_n\rangle$ and $D = \ell\Lambda_W$. Here again, $B = \mathrm{Cent}^q(\Lambda_W)$ depends on n. For odd n it equals $\ell\Lambda_W$ and for even n it is equal to A.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 2n$. In this case, the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, 1, \lambda_n)$. Thus we are in case (II) of Example 4.7, hence the 2 solutions are given by $(d, k) = (1, 1)$ and $(2, 1)$.

(ii) For $\ell \equiv 2 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle\alpha_1, \ldots, \alpha_{n-1}, \frac{1}{2}\alpha_n\rangle \neq \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell/2$ and $m_{[n]} = n$. The quotient diamond is given by either $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, 1, \lambda_n)$ for even n, or by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, -1, \lambda_n)$ for odd n. Thus we are in either in case (II) or in case (III) of Example 4.7. In the first case (even n) we get solutions $(d, k) = (1, 1)$ and $(2, 1)$. For odd n we get solutions $(d, k) = (1, 1)$ and $(2, 2)$.

(ii) For $\ell \equiv 0 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle\frac{1}{2}\alpha_1, \ldots \frac{1}{2}\alpha_{n-1}, \alpha_n\rangle = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell/2$ and $m_{[n]} = n$. The quotient diamond is given by either $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)$ for even n, or by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_1, -1, 0)$ for odd n. Thus we are in either in case (I) or in case (III) of Example 4.7. In the first case (even n) we get all 3 possible solutions $(d, k) = (1, 1)$, $(2, 1)$ and $(2, 2)$. For odd n we get solutions $(d, k) = (2, 1)$ and $(2, 2)$.

(4) For $\mathfrak{g}$ with root system D_n, $n \geq 4$ even, we have $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ for all n. The simple roots are $\alpha_1, \ldots, \alpha_n$ and $d_i = 1$ for $1 \leq i \leq n$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and λ_{n-1}, λ_n are the generators of the fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\lambda_{n-1} + \lambda_n$ is the other element of order 2. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_n\}$, and since $d_i = 1$ for all i, also the values (λ_i, λ_j) for $1 \leq i, j \leq n$.

$$\tilde{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & . & & & & 0 \\ -1 & 2 & -1 & 0 & . & & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 2 & -1 & 0 & 0 \\ . & . & . & . & . & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & & . & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & & . & 0 & -1 & 0 & 2 \end{pmatrix} \quad id_W^R = \begin{pmatrix} 1 & 1 & 1 & . & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & . & 2 & 1 & 1 \\ 1 & 2 & 3 & . & 3 & \frac{3}{2} & \frac{3}{2} \\ . & . & . & . & . & . & . \\ 1 & 2 & 3 & . & n-2 & \frac{n-2}{2} & \frac{n-2}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & . & \frac{n-2}{2} & \frac{n}{4} & \frac{n-2}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & . & \frac{n-2}{2} & \frac{n-2}{4} & \frac{n}{4} \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = \ell\Lambda_W$ and $B = C = D = \ell\Lambda_R$.

(ii) For even ℓ we have $A = C = \ell\Lambda_W$ and $B = D = \ell\Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations by a case by case calculation.

(i) For odd ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$. Thus, the quotient diamond is given by $(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \varphi_1, \varphi_2)$ with injections

$$\varphi_1 \colon \ell\langle\lambda_{n-1}, \lambda_n\rangle \to \pi_1^*, \ \ell\lambda_{n-1} \mapsto q^{\ell(\lambda_{n-1}, -)}, \ \ell\lambda_n \mapsto q^{\ell(\lambda_n, -)},$$
$$\varphi_2 \colon \ell\langle\lambda_{n-1}, \lambda_n\rangle \to \pi_1, \ \ell\lambda_{n-1} \mapsto \lambda_{n-1}, \ \ell\lambda_n \mapsto \lambda_n.$$

In the following, we will write $a := \lambda_{n-1}$, $b := \lambda_n$ and $c := \lambda_{n-1} + \lambda_n$ for the 3 elements of order 2 of π_1. Since $(\lambda_j, \lambda_j) = n/4$ for $j \in \{n-1, n\}$, and $(\lambda_i, \lambda_j) = (n-2)/4$ for $i \neq j$, $i, j \in \{n-1, n\}$ we get

φ_1	0	a	b	c	n
0	1	1	1	1	$n \equiv 0 \mod 4$
					$n \equiv 2 \mod 4$
a	1	1	-1	-1	$n \equiv 0 \mod 4$
		-1	1	-1	$n \equiv 2 \mod 4$
b	1	-1	1	-1	$n \equiv 0 \mod 4$
		1	-1	-1	$n \equiv 2 \mod 4$
c	1	-1	-1	1	$n \equiv 0 \mod 4$
					$n \equiv 2 \mod 4$

Since it suffices to consider the diamond equations (4.3) and (4.4) by Lemma 4.5, we check which function

$$g \colon G \times G \to \mathbb{C}, \ (x, y) \mapsto \frac{1}{d}\,\omega(x, y)\delta_{(x \in H_1)}\delta_{(y \in H_2)}$$

with subgroups H_i of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ of order d and a pairing ω as in Example 3.6 is a solution to these equations. We get the following system of equations for g:

$$\begin{aligned}
1 &= g(0,0) + g(a,0) + g(b,0) + g(c,0) \\
1 &= g(0,0) + g(0,a) + g(0,b) + g(0,c) \\
0 &= g(0,a) \pm g(a,a) \mp g(b,a) - g(c,a) \\
0 &= g(a,0) \pm g(a,a) \mp g(a,b) - g(a,c) \\
0 &= g(0,b) \mp g(a,b) \pm g(b,b) - g(c,b) \\
0 &= g(b,0) \mp g(b,a) \pm g(b,b) - g(b,c) \\
0 &= g(0,c) - g(a,c) - g(b,c) + g(c,c) \\
0 &= g(c,0) - g(c,a) - g(c,b) + g(c,c)
\end{aligned} \tag{5.1}$$

where the $\pm, \mp$ possibilities depend on wether $\ell \equiv 0$ or $2 \mod 4$. It is easy to see that the trivial solution on $H_1 = H_2 = \mathbb{Z}_1$ is a solution. For $H_i \cong \mathbb{Z}_2$ the solution has one of the following two structures. For symmetric solutions $H_1 = H_2 = \langle\lambda\rangle$ we get $\omega(\lambda, \lambda) = -1$. If $H_1 = \langle\lambda\rangle \neq \langle\lambda'\rangle = H_2$ we get $\omega(\lambda, \lambda') = 1$. This give all possible 9 solutions with $H_i \cong \mathbb{Z}_2$. Finally, we check which functions on $G =$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ are solutions to the diamond equations. We get 4 symmetric solutions and 2 non-symmetric solutions, which are given by their values $(\omega(x,y))_{x,y\in\{\lambda_{n-1},\lambda_n\}}$ on generator pairs:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

(ii) For even ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$. Thus the quotient diamond is given by $(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_1, \varphi_1, 0)$ and the injection φ_2 is trivial. We get an analogue block of equations as (5.1), but without non-zero "shift" $\varphi_2(x)$, $x \in A$. We can add appropriate equations and get the $1 = 4g(0,0)$, hence only pairings of $H_1 = H_2 = \pi_1$ are solutions. It is now easy to check, that all 16 possible parings on $\pi_1 \times \pi_1$ are solutions to the diamond equations.

(5) For $\mathfrak{g}$ with root system D_n, $n \geq 5$ odd, we have $\pi_1 = \mathbb{Z}_4$ for all n. The root and weight data are as for even n in (4). The weight λ_n is the generator of the fundamental group $\mathbb{Z}_2$.

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = \ell\Lambda_W$ and $B = C = D = \ell\Lambda_R$.

(ii) For $\ell \equiv 2 \mod 4$ we have $A = \ell\Lambda_W$, $C = \ell\langle\lambda_1, \ldots, \lambda_{n-2}, 2\lambda_{n-1}, 2\lambda_n\rangle$ and $B = D = \ell\Lambda_R$.

(iii) For $\ell \equiv 0 \mod 4$ we have $A = C = \ell\Lambda_W$ and $B = D = \ell\Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd ℓ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamond is given by $(\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \xi_4, \lambda_n)$, hence we are in case (III) of Example 4.7. We get solutions $(d,k) = (1,1)$, $(2,1)$, $(4,2)$ and $(4,4)$.

(ii) For $\ell \equiv 2 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamond is given by $(\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_1, \xi_4, 2\lambda_n)$, hence we are in case (III) of Example 4.7. We get all 4 solutions $(d,k) = (4,1)$, $(4,2)$, $(4,3)$ and $(4,4)$ on $H = \mathbb{Z}_4$.

(iii) For $\ell \equiv 0 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamond is given by $(\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_1, \mathbb{Z}_4, \mathbb{Z}_1, \xi_4, 0)$, hence we are in case (III) of Example 4.7. We get the same 4 solutions as in (ii).

(6) For $\mathfrak{g}$ with root system E_6, we have $\pi_1 = \mathbb{Z}_3$. The simple roots are $\alpha_1, \ldots, \alpha_6$ and $d_i = 1$ for $1 \leq i \leq 6$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and λ_6 is the generator of the fundamental group $\mathbb{Z}_3$. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_6\}$, and since $d_i = 1$ for all i, also the values (λ_i, λ_j) for $1 \leq i,j \leq 6$.

$$\tilde{C} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 2 & 3 & 2 & 1 \\ \frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & 3 & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For $3 \nmid \ell$ we have $A = \ell\Lambda_W$ and $B = C = D = \ell\Lambda_R$.

(ii) For $3 \mid \ell$ we have $A = C = \ell\Lambda_W$ and $B = D = \ell\Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i.a) For $3 \nmid \ell$ and ℓ odd it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \xi_3, \lambda_6)$, hence we are in case (III) of Example 4.7. Since $\ell \equiv 2 \mod 3$ we get solutions $(d, k) = (1, 1)$, $(3, 1)$ and $(3, 3)$.

(i.b) For $3 \nmid \ell$ and ℓ even it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \xi_3, \lambda_6)$, and we are again in case (III) of Example 4.7. Since $\ell \equiv 1 \mod 3$ we get solutions $(d, k) = (1, 1)$, $(3, 2)$ and $(3, 3)$.

(ii.a) For $3 \mid \ell$ and ℓ odd it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_1, \xi_3, 0)$, hence we are in case (III) of Example 4.7. We get all 3 solutions $(d, k) = (3, 1)$, $(3, 2)$ and $(3, 3)$ on $\mathbb{Z}_3$.

(ii.b) For $3 \mid \ell$ and ℓ even it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_1, \xi_3, 0)$, and we the same solutions as in (ii.a).

(7) For $\mathfrak{g}$ with root system E_7, we have $\pi_1 = \mathbb{Z}_2$. The simple roots are $\alpha_1, \ldots, \alpha_7$ and $d_i = 1$ for $1 \leq i \leq 7$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and λ_7 is the generator of the fundamental group $\mathbb{Z}_2$. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_7\}$, and since $d_i = 1$ for all i, also the values (λ_i, λ_j) for $1 \leq i, j \leq 7$.

$$\tilde{C} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{7}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = \ell\Lambda_W$ and $B = C = D = \ell\Lambda_R$.

(ii) For even ℓ we have $A = C = \ell\Lambda_W$ and $B = D = \ell\Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For ℓ odd it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 3$. Thus, the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \xi_2, \lambda_7)$ and we are in case (III) of Example 4.7. We get solutions $(d, k) = (1, 1)$ and $(2, 2)$.

(ii) For ℓ even it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 3$. Thus, the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_1, \xi_2, 0)$ and we are again in case (III) of Example 4.7. We get all 2 solutions $(d,k) = (2,1)$ and $(2,2)$ on $\mathbb{Z}_2$.

(8) For $\mathfrak{g}$ with root system E_8, we have $\pi_1 = \mathbb{Z}_1$. The simple roots are $\alpha_1, \ldots, \alpha_8$ and $d_i = 1$ for $1 \leq i \leq 8$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and are roots. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_8\}$, and since $d_i = 1$ for all i, also the values (λ_i, λ_j) for $1 \leq i, j \leq 8$.

$$\tilde{C} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ we have $A = B = C = D = \ell\Lambda_W = \ell\Lambda_R$.

(ii) For even ℓ: same as in (i).

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For ℓ odd it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 2$. Thus, the quotient diamond is given by $(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, 1, 0)$ and we are in case (I) of Example 4.7. We get the only solution $(d,k) = (1,1)$.

(ii) For ℓ even it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 2$. We get the same diamond and solution as in (i).

(9) For $\mathfrak{g}$ with root system F_4, we have $\pi_1 = \mathbb{Z}_1$. The simple roots α_1, α_2 are long, α_3, α_4 are short, hence $d_1 = d_2 = 2$ and $d_3 = d_4 = 1$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and are roots. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_4\}$.

$$\tilde{C} = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} 4 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For odd ℓ, we have $A = B = C = D = \ell\Lambda_W = \ell\Lambda_R$.

(ii) For even ℓ, we have $A = B = C = D = \ell\langle \frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \lambda_3, \lambda_4 \rangle$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For ℓ odd it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 1$. Thus, the quotient diamond is given by $(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, 1, 0)$ and we are in case (I) of Example 4.7. We get the only solution $(d, k) = (1, 1)$.

(ii) For $\ell \equiv 2 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle \frac{1}{2}\alpha_1, \frac{1}{2}\alpha_2, \alpha_3, \alpha_4\rangle \neq \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 1$. We get the same diamond and solution as in (i).

(iii) For $\ell \equiv 0 \mod 4$ it is $\Lambda_R^{[\ell]} = \ell\langle \frac{1}{2}\alpha_1, \frac{1}{2}\alpha_2, \alpha_3, \alpha_4\rangle = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 1$. We get the same diamond and solution as in (i).

(10) For $\mathfrak{g}$ with root system G_2, we have $\pi_1 = \mathbb{Z}_1$. The simple root α_1 is short and α_2 is long, hence $d_1 = 1$ and $d_2 = 3$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights λ_i are given as in [Hum72], Section 13.2, and are roots. The matrix id_W^R gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \dots, \alpha_2\}$.

$$\tilde{C} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix} \qquad id_W^R = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

The lattice diamonds, depending on ℓ, are:

(i) For $3 \nmid \ell$, we have $A = B = C = D = \ell\Lambda_W = \ell\Lambda_R$.

(ii) For $3 \mid \ell$, we have $A = B = C = D = \ell\langle\lambda_1, \frac{1}{3}\lambda_2\rangle$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ and compare it with Lusztig's kernel $2\Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i.a) For $3 \nmid \ell$ and ℓ odd it is $\Lambda_R^{[\ell]} = \ell\Lambda_R \neq 2\ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 6$. Thus, the quotient diamond is given by $(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, 1, 0)$ and we are in case (I) of Example 4.7. We get the only solution $(d, k) = (1, 1)$.

(i.b) For $3 \nmid \ell$ and ℓ even it is $\Lambda_R^{[\ell]} = \ell\Lambda_R = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 6$. We get the same diamond and solution as in (i.a).

(ii.a) For $3 \mid \ell$ and ℓ odd it is $\Lambda_R^{[\ell]} = \ell\langle\alpha_1, \frac{1}{3}\alpha_2\rangle \neq 2\ell\langle\alpha_1, \frac{1}{3}\alpha_2\rangle = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell/3$ and $m_{[n]} = 2$. We get the same diamond and solution as in (i.a).

(ii.b) For $3 \mid \ell$ and ℓ even it is $\Lambda_R^{[\ell]} = \ell\langle\alpha_1, \frac{1}{3}\alpha_2\rangle = 2\Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell/3$ and $m_{[n]} = 2$. We get the same diamond and solution as in (i.a).

References

[AY13] I. Angiono and H. Yamane. *The R-matrix of quantum doubles of Nichols algebras of diagonal type.* Preprint (2013), `arXiv:1304.5752`.

[FGST06] B. Feigin, A. Gainutdinov, A. Semikhatov, I. Tipunin. *Modular Group Representations and Fusion in Logarithmic Conformal Field Theories and in the Quantum Group Center.* Commun. Math. Ohys, 265, 47–93 (2006).

[GW98] S. Gelaki and S. Westreich. *On the Quasitriangularity of $U_q(\mathfrak{sl}_n)'$.* J. Amer. Math. Soc., **57**(1), 105–125 (1998).

[Hum72] J. E. Humphreys. *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics, Vol. 9, Springer, New York, 1972.

[Kas95] C. Kassel. *Quantum groups.* Graduate Texts in Mathematics, Vol. 155, Springer, New York, 1995.

[KL01] T. Kerler and V.V. Lyubashenko. *Non-Semisimple Topological Qunatum Field Theories for 3-Manifolds with Corners.* Lecture Note in Mathematics, Vol. 1765, Springer, Berling, 2001.

[KR02] R. Kashaev and N. Reshetikhin. *Invariants of links with flat connections in their complements. II. Holonomy R-matrices related to quantized universal enveloping algebras at roots of 1.* Preprint (2002), `arXiv:0202212`.

[Len14] S. Lentner. *A Frobenius homomorphism for Lusztig's quantum groups over arbitrary roots of unity.* Preprint (2014), `arXiv:1406.0865`.

[LN14] S. Lentner and D. Nett. *A theorem on roots of unity and a combinatorial principle.* Preprint (2014), `arXiv:1409.5822`.

[Lus90] G. Lusztig. *Quantum groups at roots of 1.* Geometriae Dedicata, **35**(1), 89–113, 1990.

[Lus93] G. Lusztig. *Introduction to Quantum Groups.* Progress in Mathematics, Vol. 110, Birkhäuser, Boston, 1993.

[Mül98a] E. Müller. *Quantengruppen im Einheitswurzelfall.* Dissertation, LMU, München, 1998, `http://www.mathematik.uni-muenchen.de/ ~algebra/Dissertationen/Mueller/doktor.ps`

[Mül98b] E. Müller. *Some topics on Frobenius–Lusztig kernels I, II.* J. Algebra, **206**(2), 624–681 (1998).

[Res95] N. Reshetikhin. *Quasitriangularity of quantum groups at roots of 1.* Commun. Math. Phys., **170**(1), 79–99 (1995).

[Ros93] M. Rosso. *Quantum Groups at a Root of 1 and Tangle Invariants.* Int. J. Mod. Phys. B, **07**(20n21), 3715–3726 (1993).

[RT90] N. Reshetikhin and V.G. Turaev. *Ribbon graphs and their invariants of tangles derived from quantum groups.* Commun. Math. Phys., **127**, 1–26 (1990).

[RT91] N. Reshetikhin and V.G. Turaev. *Invariants of* 3*-manifolds via link polynomials and quantum groups.* Invent. Math., **103**, 547–597 (1991).

[Tan92] T. Tanisaki. *Killing forms, Harish-Chandra homomorphisms and universal R-matrices for quantum algebras.* Int. J. Mod. Phys. A, **07**(supp01b), 941–962 (1992).

[Tur94] V.G. Turaev. *Quantum Invariants of Knots and* 3*-Manifolds.* Studies in Mathematics, Vol. 18, de Gruyter, Berlin, 1994.

[Vir06] *Kirby elements and quantum invariants.* Proc. London Math. Soc. **93**(2), 474–514 (2006).