Completeness of hyperbolic centroaffine hypersurfaces

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Abstract

This paper is concerned with the completeness (with respect to the centroaffine metric) of hyperbolic centroaffine hypersurfaces which are closed in the ambient vector space. We show that completeness holds under generic regularity conditions on the boundary of the convex cone generated by the hypersurface. The main result is that completeness holds for hyperbolic components of level sets of homogeneous cubic polynomials. This implies that every such component defines a complete quaternionic Kähler manifold of negative scalar curvature.

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Contents

Introduction

By a celebrated theorem of Cheng and Yau [CY] a locally strictly convex affine hypersphere which is closed in the ambient vector space is complete with respect to the Blaschke metric. Proper affine hyperspheres are precisely the centroaffine hypersurfaces for which the Blaschke metric coincides with the centroaffine metric (up to a constant factor). In this paper we investigate the completeness of locally strictly convex centroaffine hypersurfaces with respect to the centroaffine metric.

Our main motivation stems from the scalar geometry of 5-dimensional supergravity as described in [GST]. The manifolds carrying this geometry are called *projective special real manifolds*, see Definition 2.1. They form a class of hyperbolic (and thus locally strictly convex) centroaffine hypersurfaces. In Theorem 2.3 and Definition 2.2 we give an intrinsic characterization in terms of the underlying centroaffine geometry. The crucial ingredient is the differential equation (2.2) expressing the covariant derivative of the cubic form in terms of the metric.

Using constructions from supergravity, known as the r-map and the c-map, it was

shown in [CHM], where these constructions are explicitly described, that *every complete* projective special real manifold of dimension n gives rise to a complete quaternionic Kähler manifold of negative scalar curvature of dimension $4n+8$. More specifically, it was shown that given a complete projective special real manifold of dimension n, the r-map associates with it a complete *projective special Kähler domain* (see [CHM, p. 198] for a definition) of real dimension $2n+2$ and that the c-map associates a complete quaternionic Kähler manifold of real dimension $4n+8$ of negative scalar curvature with the latter. This method was used in [CHM, CDL] to construct new explicit examples of complete quaternionic Kähler manifolds of dimension 12 and 16. Moreover, a classification of all complete projective special real manifolds of dimension less than or equal to 2 was given. Based on these results, it was observed [CDL, Cor. 1] that a projective special real manifold of dimension less than or equal to 2 is complete if and only if it is closed and it was asked whether this property extends to higher dimensions. Here we prove that this is indeed the case, see Theorem 2.5. This gives a powerful method for the verification of the completeness of projective special real manifolds and the corresponding quaternionic Kähler manifolds, cf. Theorem 2.6.

Let us now summarize the structure of the paper and mention some further results. In Section 1 we discuss centroaffine structures and centroaffine hypersurfaces from an intrinsic as well as extrinsic point of view. Our main focus is on locally strictly convex centroaffine hypersurfaces and the relation between:

1. closedness (the property of being closed in the ambient space),

2. Euclidean completeness (completeness with respect to the metric induced by a Euclidean scalar product in the ambient space) and

3. completeness (with respect to the centroaffine metric).

Section 1.2 contains some basic results relating these properties. In particular, under natural assumptions, completeness implies closedness and the latter is equivalent to Euclidean completeness, see Proposition 1.8.

After these preliminaries, we concentrate on Euclidean complete hyperbolic centroaffine hypersurfaces $\mathcal{H} \subset \mathbb{R}^{n+1}$ in Section 1.3. We show that $U = \mathbb{R}^{>0} \cdot \mathcal{H}$ is an open convex cone, which is intersected in a relatively compact domain $B = U \cap E \subset E$ by any affine hyperplane E tangent to $\mathcal{H}$. We equip U with a smooth homogeneous function $h : U \to \mathbb{R}$ of degree $k > 1$ such that $\mathcal{H} = \{p \in U \mid h(p) = 1\}$ and with a Lorentzian metric g_L which is a multiple of the Hessian of h. We observe that the completeness of $\mathcal{H}$ is equivalent to the global hyperbolicity of (U, g_L). Then we prove that $\mathcal{H}$ is complete if there exists

$\epsilon \in (0, k)$ such that $f = \left. \sqrt[k-\varepsilon]{h} \right|_B$ is concave, see Lemma 1.15. This allows us to prove the completeness if h is a cubic polynomial and is the key lemma for the proof of Theorem 2.5 about projective special real manifolds. As discussed in the last section of the paper, the result for cubic polynomials can not be extended to real analytic functions but might hold for polynomials of higher degree.

In Section 1.4 we prove that a Euclidean complete hyperbolic centroaffine hypersurface is complete if the boundary of U satisfies certain regularity assumptions, see Theorem 1.18. Furthermore, these conditions are generically satisfied by Theorem 1.20.

In Section 2 we specialize to the case of projective special real manifolds. The main results are the intrinsic characterization developed in Section 2.1 and the equivalence of closedness and completeness proven in Section 2.2, with the application to quaternionic Kähler manifolds in Section 2.3.

Acknowledgements

1 Centroaffine structures

1.1 Centroaffine hypersurfaces and centroaffine structures

In this subsection we review some basic notions from affine differential geometry, see [NS] for a more detailed discussion. Let us consider $\mathbb{R}^{n+1}$ endowed with its canonical flat connection $\tilde{\nabla}$ and the parallel volume form det.

Definition 1.1. *A hypersurface immersion* $\varphi : M \to \mathbb{R}^{n+1}$ *is called* centroaffine *if the position vector field* $\xi : M \to \mathbb{R}^{n+1}$, $p \mapsto \xi_p := \varphi(p)$, *is transversal, that is for all* $p \in M$ *we have* $\varphi(p) \notin d\varphi T_p M$.

A centroaffine hypersurface immersion $\varphi : M \to \mathbb{R}^{n+1}$ induces a torsion-free connection ∇ and a symmetric tensor field g on M such that the Gauß equation

$$\tilde{\nabla}_X d\varphi Y = d\varphi \nabla_X Y + g(X,Y)\xi \tag{1.1}$$

holds for all $X, Y \in \mathfrak{X}(M)$. Furthermore, we have an induced volume form $\nu := \det(\xi, \cdots)$, which is ∇-parallel, as a consequence of the Weingarten equation $\tilde{\nabla}_X \xi = d\varphi X$. In these formulas, $\tilde{\nabla}$ denotes the connection in $\varphi^* T\mathbb{R}^{n+1}$ induced by the connection $\tilde{\nabla}$ in the vector bundle $T\mathbb{R}^{n+1}$. For simplicity of notation, we will usually identify TM with the subbundle $d\varphi TM \subset \varphi^* T\mathbb{R}^{n+1}$ and drop the isomorphism $d\varphi : TM \to d\varphi TM$ in the equations of Gauß and Weingarten.

Definition 1.2. *The above geometric data (∇, g, ν) will be called the* induced (centroaffine) data *of the centroaffine hypersurface immersion φ. The hypersurface (immersion) is called* nondegenerate *if g is nondegenerate and* definite *if g is definite. More specifically, it is called* elliptic *if $g < 0$ and* hyperbolic *if $g > 0$.*

Remark: The above definition is consistent with the usual notions of ellipticity and hyperbolicity in affine differential geometry. In fact, the tensor field $(\operatorname{tr} S)g$ is positive definite in the elliptic case and negative definite in the hyperbolic case, where $S = -\operatorname{Id}$ is the shape tensor associated with our choice of transversal vector field ξ. This ensures for instance that ellipsoids around 0 are elliptic and standard hyperboloids are hyperbolic.

Example: Let $U \subset \mathbb{R}^{n+1}$ be an open subset and $h : U \to \mathbb{R}$ a smooth function which is homogeneous of degree $k \in \mathbb{R}^*$, in the sense that

$$\sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i} h = kh. \tag{1.2}$$

We consider the level set

$$\mathcal{H} := \{x \in U \mid h(x) = 1\},$$

which we assume nonempty. Notice that if h is not the zero function we can always rescale h such that this assumption holds.

Proposition 1.3. *For every homogeneous function h as in the above example the inclusion map*

$$\iota : \mathcal{H} \to \mathbb{R}^{n+1}$$

is a centroaffine hypersurface embedding with

$$g = -\frac{1}{k}\iota^*(\tilde{\nabla}^2 h),$$

where (∇, g, ν) are the induced centroaffine data on $\mathcal{H}$. In particular, $\mathcal{H} \subset \mathbb{R}^{n+1}$ is nondegenerate if and only if the Hessian $\tilde{\nabla}^2 h$ is nondegenerate on $T\mathcal{H}$.

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Proof: By the homogeneity of h, $dh(\xi) = k \neq 0$ on $\mathcal{H}$. Therefore $\mathcal{H}$ is smooth and centroaffine. In order to check the formula for the metric, let X and Y be vector fields defined on some open subset of $\mathbb{R}^{n+1}$, which are tangent to the level sets of h. Then on $\mathcal{H}$ we have

$$
\begin{aligned}
g(X,Y) &= \frac{1}{k}dh(\tilde{\nabla}_X Y) = \frac{1}{k}\big(\tilde{\nabla}_X(dhY) - (\tilde{\nabla}_X dh)Y\big) \\
&= -\frac{1}{k}(\tilde{\nabla}_X dh)Y = -\frac{1}{k}(\tilde{\nabla}^2 h)(X,Y). \qquad \square
\end{aligned}
$$

Locally every centroaffine hypersurface is defined by a homogeneous function:

Proposition 1.4. *Let $\varphi : M \to \mathbb{R}^{n+1}$ be a centroaffine hypersurface immersion, $p \in M$ and $k \in \mathbb{R}^*$. Then there exist open neighbourhoods $U' \subset M$ of p and $U \subset \mathbb{R}^{n+1}$ of $\varphi(p)$ and a smooth homogeneous function of degree k on U such that $\varphi(U') = \{x \in U \mid h(x) = 1\}$.*

Definition 1.5. *A centroaffine structure on a smooth manifold M is a triple (∇, g, ν) consisting of a torsion-free connection, a pseudo-Riemannian metric and a volume form satisfying the following compatibility conditions:*

(i) $\nabla\nu = 0$,

(ii) the curvature tensor R of ∇ is given by

$$
R(X,Y)Z = -\big(g(Y,Z)X - g(X,Z)Y\big), \quad X,Y,Z \in \mathfrak{X}(M),
$$

(iii) ∇g is completely symmetric.

If these conditions are satisfied, (M, ∇, g, ν) is called a centroaffine manifold. The pseudo-Riemannian metric g is called the centroaffine metric and the symmetric tensor field $C := \nabla g$ is called the cubic form of the centroaffine manifold (M, ∇, g, ν).

Theorem 1.6.

(i) Let $\varphi : M \to \mathbb{R}^{n+1}$ be a nondegenerate centroaffine hypersurface immersion of a connected manifold M with induced data (∇, g, ν). Then (M, ∇, g, ν) is a centroaffine manifold.

(ii) Conversely, for a connected and simply connected centroaffine manifold (M, ∇, g, ν), there exists a centroaffine hypersurface immersion $\varphi : M \to \mathbb{R}^{n+1}$ with induced data (∇, g, ν). Furthermore, the immersion φ is unique up to linear unimodular transformations of $\mathbb{R}^{n+1}$.

Proof: To prove (i) it remains to check the equations (ii) and (iii) in Definition 1.5. The first equation is obtained by computing the tangent part of $\tilde{R}(X,Y)Z = 0$ with the help of the equations of Gauß and Weingarten, where $\tilde{R}$ denotes the curvature tensor of $\tilde{\nabla}$. Similarly, the second equation is obtained by computing the part proportional to ξ. (These are in fact special cases of the equations of Gauß and Codazzi for general hypersurface immersions.) The statement (ii) can be proven in a similar way as the fundamental theorem [NS, Thm. 8.1] of affine differential geometry. $\qquad\square$

1.2 Completeness and closedness of centroaffine hypersurfaces

Our overall approach for proving the completeness of a Riemannian manifold is based on the following fact, see [CHM, Lemma 1] for a proof.

Lemma 1.7. *A Riemannian manifold (M, g) is complete if and only if every curve $\gamma : I \to M$ which is not contained in any compact subset of M has infinite length.*

Recall that a submanifold of Euclidean space is called Euclidean complete if it is complete with respect to the Riemannian metric induced by the Euclidean scalar product $\langle \cdot, \cdot \rangle$.

Proposition 1.8. *Let $h : V \to \mathbb{R}$ be a smooth homogeneous function of degree $k \neq 0$ defined on some open set $V \subset \mathbb{R}^{n+1}$ and let $U \subset V$ be an open subset such that $\overline{\mathcal{H}} \subset V$, where $\mathcal{H} := \{x \in U \mid h(x) = 1\}$. Assume that the centroaffine metric of the hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ is definite. Then the following hold for every component $\mathcal{H}_0$ of $\mathcal{H}$.*

(i) If $(\mathcal{H}_0, g)$ is complete then $\mathcal{H}_0 \subset \mathbb{R}^{n+1}$ is a closed subset.

(ii) $\mathcal{H}_0 \subset \mathbb{R}^{n+1}$ is a closed subset if and only if $\mathcal{H}_0 \subset \mathbb{R}^{n+1}$ is Euclidean complete.

Proof: In order to prove (i), let us denote by L the connected component of the level set $\{x \in V \mid h(x) = 1\}$ which contains $\mathcal{H}_0$. Thanks to the assumption $\overline{\mathcal{H}} \subset V$, the closure of $\mathcal{H}_0$ is contained in the level set $\{x \in V \mid h(x) = 1\}$ and thus in L. Therefore, if $\mathcal{H}_0$ is not closed, then there exists a smooth curve $c : [0, 1] \to L$ such that $c(0) \in \mathcal{H}_0$ and $c(1) \notin \mathcal{H}_0$. We can assume without loss of generality that $c([0, 1)) \subset \mathcal{H}_0$. The length of c with respect to the metric g is finite, since it is given by the integral

$$\int_0^1 \sqrt{|g_{c(t)}(c'(t), c'(t))|}\, dt, \quad g = -\frac{1}{k}\iota^* \tilde{\nabla}^2 h,$$

see Proposition 1.3, of a continuous function over a compact interval. The continuity of the integrand follows from the continuity of the Hessian $\tilde{\nabla}^2 h$ on V. This proves (i).

Since the Euclidean length

$$\int_0^1 \sqrt{\langle c'(t), c'(t)\rangle}\,dt$$

is also finite, the same argument shows that Euclidean completeness implies that $\mathcal{H}_0 \subset \mathbb{R}^{n+1}$ is closed. The converse statement follows from the next simple lemma which finishes the proof of (ii). $\qquad\square$

Lemma 1.9. *Let $\varphi : M \to N$ be an embedding into a complete Riemannian manifold (N, g_N). If $\varphi(M) \subset N$ is a closed subset, then $(M, \varphi^* g_N)$ is complete.*

Remark: The lemma does not extend to injective immersions.

1.3 Completeness of hyperbolic centroaffine hypersurfaces

In this section we provide some basic results about Euclidean complete connected hyperbolic centroaffine hypersurfaces $\mathcal{H} \subset \mathbb{R}^{n+1}$, which will be used in Sections 1.4, 2.2 and 2.3. We will first show that the cone

$$U := \mathbb{R}^{>0} \cdot \mathcal{H}$$

is open and convex and that it intersects every affine tangent space $E_p := p + T_p\mathcal{H}$ of $\mathcal{H}$, $p \in \mathcal{H}$, in a relatively compact convex domain $U \cap E_p \subset E_p$. We will then parametrize the hypersurface as a radial graph over such a domain and compute the centroaffine metric in that parametrization as well as its pullback under the radial projection $U \to \mathcal{H}$. The explicit formulas involve a positive homogeneous function h on U that is constant on the hypersurface. Building on these preparations, the upshot of this section is Lemma 1.15, which provides a criterion for the completeness of the centroaffine metric in terms of a concavity property of the function $h|_{U \cap E_p}$. All these results require the Euclidean completeness, with exception of the explicit formulas for the metric, which hold also for local radial parametrizations and projections. Let us emphasize that Lemma 1.15 will play a key role in the proof of Theorem 2.5, which is the main result of this paper. Another important result of this section is Proposition 1.16, which asserts that U carries a natural Lorentzian metric that is globally hyperbolic if and only if the Euclidean complete hypersurface $\mathcal{H}$ is complete with respect to the centroaffine metric.

Proposition 1.10. *The cone U is open and convex. The map $\phi \colon \mathbb{R}^{>0} \times \mathcal{H} \to U$ given by $(\lambda, x) \mapsto \lambda x$ is a diffeomorphism.*

Proof: Since $\mathcal{H}$ is centroaffine, ϕ is a surjective local diffeomorphism. Thus U is open.

By the Sacksteder–van Heijenoort theorem, see for instance [W], any Euclidean complete connected hypersurface with positive sectional curvature of dimension $n \geq 2$ is the boundary of a convex domain $\mathcal{D}$ in $\mathbb{R}^{n+1}$. We want to apply this result to our situation.

For this we have to show that the curvature of $(\mathcal{H}, \iota^*\langle ., .\rangle)$ is positive. Note that by Proposition 1.4 we can assume that locally $\mathcal{H}$ is $\{h = 1\}$ for some homogeneous function $h \colon U' \to \mathbb{R}$ on some subcone $U' \subset U$. It follows that (see [GKM]) the second fundamental form of $(\mathcal{H}, \iota^*\langle ., .\rangle)$ in $(\mathbb{R}^{n+1}, \langle ., .\rangle)$ is $\frac{1}{|\text{grad } h|}\tilde{\nabla}^2 h$ with respect to the normal $-\frac{\text{grad } h}{|\text{grad } h|}$, where grad h denotes the gradient of h with respect to the Euclidean scalar product $\langle ., .\rangle$. By our assumptions the second fundamental form is thus definite. Now the Gauß equation

$$g_{Euc}(R^{Euc}(X,Y)Y, X) = \frac{1}{|\text{grad } h|^2}\left(\tilde{\nabla}^2 h(X, X)\tilde{\nabla}^2 h(Y, Y) - \tilde{\nabla}^2 h(X, Y)^2\right) > 0$$

for the curvature R^{Euc} of the induced metric $g_{Euc} = \iota^*\langle ., .\rangle$ of the immersion $\iota \colon (\mathcal{H}, \iota^*\langle ., .\rangle) \to (\mathbb{R}^{n+1}, \langle ., .\rangle)$ shows that the sectional curvatures are positive. Hence in the case $n \geq 2$, there exists a convex domain $\mathcal{D} \subset \mathbb{R}^{n+1}$ whose boundary is $\mathcal{H}$.

Notice that $\mathcal{H}$ separates U into two domains $U^+ := \mathbb{R}^{>1} \cdot \mathcal{H}$ and $U^- := \mathbb{R}^{<1} \cdot \mathcal{H}$ with common boundary $\mathcal{H}$. The first domain lies on the convex side of $\mathcal{H}$; i.e., every $x \in \mathcal{H}$ has a neighbourhood W_x such that $W_x \cap U^+$ is convex and thus equal to $W_x \cap \mathcal{D}$. This can be seen as follows. The Euclidean second fundamental form of $\mathcal{H}$ with respect to its outer unit normal ν (with respect to the origin) and the centroaffine metric g on $\mathcal{H}$ differ by a positive conformal factor, because they are affine fundamental forms of $\mathcal{H}$ defined by the Gauß equation with respect to vector fields ν and ξ that induce the same orientation of $\mathcal{H}$. Since $\mathcal{H}$ is hyperbolic, its Euclidean second fundamental form with respect to ν is therefore positive definite, which implies the claim that U^+ lies on the convex side of $\mathcal{H}$.

Moreover, U^+ is contained in $\mathcal{D}$, and ϕ is injective. Otherwise some ray $\mathbb{R}^{>1} \cdot x$ with $x \in \mathcal{H}$ would meet the boundary of $\mathcal{D}$, i.e., it would contain some $x' \in \mathcal{H}$. Since $\mathcal{D}$ is convex, the line from x to x' would lie in $\mathcal{D}$, in contradiction to the fact that the convex side of $\mathcal{H}$ at x' is the outer one.

We claim that $\mathcal{D} \subset U^+$. Notice that the origin is not a point of $\mathcal{D}$, because else the convexity of $\mathcal{D}$ and the hyperbolicity of $\mathcal{H}$ would again yield a contradiction. Thus the line segment connecting any point $p \in \mathcal{D}$ to the origin has to intersect $\mathcal{H}$ in some point q. This implies that $p \in \mathbb{R}^{>1} \cdot q \in U^+$, proving that $\mathcal{D} = U^+$. It now follows from the convexity of $\mathcal{D}$ that U^+ and, hence, $U = \mathbb{R}^{>0} \cdot U^+$ is convex if $n \geq 2$.

In the case $n = 1$, $\mathcal{H} \subset (\mathbb{R}^{\geq 1} \cdot x) + T_x\mathcal{H}$ holds for some (in fact, every) $x \in \mathcal{H}$, because otherwise an intermediate value argument (involving x and a hypothetical second element of $\mathcal{H} \cap (x + T_x\mathcal{H})$) would yield a contradiction to the hyperbolicity of $\mathcal{H}$. In particular, $\mathcal{H}$ and, hence, U is contained in the open half-space $\mathbb{R}^{>0} \cdot x + T_x\mathcal{H} \subset \mathbb{R}^2$. This proves the

convexity of the connected cone U if $n = 1$.

Let $\pi \colon \mathbb{R}^2 \setminus \{0\} \to S^1$ be the projection $x \mapsto \frac{x}{|x|}$ and $\mathrm{incl}_{\mathcal{H}} \colon \mathcal{H} \to \mathbb{R}^2 \setminus \{0\}$ the inclusion. Since ϕ has image contained in a half-space, $\pi \circ \mathrm{incl}_{\mathcal{H}} \colon \mathcal{H} \to S^1$, which is by centroaffineness an immersion, is not surjective. This implies that $\pi \circ \mathrm{incl}_{\mathcal{H}}$ and thus ϕ is injective if $n = 1$.

For every n, the injectivity of ϕ shows now that ϕ is a diffeomorphism. $\qquad\square$

Let us recall that $E_p = p + T_p\mathcal{H} \subset \mathbb{R}^{n+1}$ denotes the affine hyperplane tangent to $\mathcal{H}$ at $p \in \mathcal{H}$.

Corollary 1.11. *The intersection $U \cap E_p \subset E_p$ is a convex domain for all $p \in \mathcal{H}$.*

Next we observe that $\mathcal{H} \subset U$ can be described as the level set of a smooth positive function $h \colon U \to \mathbb{R}$ homogeneous of degree $k \in \mathbb{R}^*$. The function $h = h_k$ is defined by

$$h(\lambda x) := \lambda^k \quad \text{for all} \quad \lambda \in \mathbb{R}^{>0}, \ x \in \mathcal{H}.$$

It is well-defined and smooth because ϕ considered in Proposition 1.10 is a diffeomorphism.

We denote by

$$\psi \colon U \to \mathcal{H}, \quad x \mapsto \frac{x}{\sqrt[k]{h(x)}},$$

the radial projection onto $\mathcal{H}$. The restriction

$$\varphi := \psi\big|_{U \cap E_p} \colon U \cap E_p \to \mathcal{H} \tag{1.3}$$

is a parametrization of the hypersurface. In view of Proposition 1.3, the centroaffine metric of $\mathcal{H}$ in this parametrization is given by

$$g = -\frac{1}{k}\varphi^*(\tilde{\nabla}^2 h) = -\frac{1}{k}\psi^*(\tilde{\nabla}^2 h)\big|_{U \cap E_p}.$$

Lemma 1.12.
$$-\psi^*(\tilde{\nabla}^2 h) = -\frac{1}{h}\tilde{\nabla}^2 h + \frac{k-1}{kh^2}dh^2.$$

Proof: For $\lambda > 0$ let us denote by $\mu_\lambda \colon U \to U$ the scalar multiplication by λ. By homogeneity (1.2), we have $\mu_\lambda^* h = \lambda^k h$. Since μ_λ is affine, this implies

$$\mu_\lambda^* \tilde{\nabla}^\ell h = \lambda^k \tilde{\nabla}^\ell h$$

for all $\ell \geq 0$. As a consequence, we also have

$$\tilde{\nabla}^\ell h\big|_{\lambda x} = \lambda^{k-\ell} \tilde{\nabla}^\ell h\big|_x$$

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for all $\ell \geq 0$. In fact,

$$(\mu_\lambda^* \tilde{\nabla}^\ell h)_x(v_1, \ldots, v_\ell) = \tilde{\nabla}^\ell h\big|_{\lambda x}(\lambda v_1, \cdots, \lambda v_\ell) = \lambda^\ell \tilde{\nabla}^\ell h\big|_{\lambda x}(v_1, \cdots, v_\ell)$$

for all $x \in U$, $v_1, \ldots v_\ell \in T_x U = \mathbb{R}^{n+1}$. Next we compute

$$d\psi_x = h(x)^{-\frac{1}{k}} \mathrm{Id} - \frac{1}{k} h(x)^{-\frac{1}{k}-1} dh_x \otimes x.$$

Using these formulas, we calculate

$$
\begin{aligned}
(\psi^* \tilde{\nabla}^2 h)_x &= \tilde{\nabla}^2 h_{\psi(x)}(d\psi_x \cdot, d\psi_x \cdot) = h(x)^{-\frac{k-2}{k}} \tilde{\nabla}^2 h_x(d\psi_x \cdot, d\psi_x \cdot) \\
&= h(x)^{-\frac{k-2}{k}} \left[h(x)^{-\frac{2}{k}} \tilde{\nabla}^2 h_x - \frac{2}{k} h(x)^{-\frac{2}{k}-1} dh_x \otimes \tilde{\nabla}^2 h_x(x, \cdot) \right. \\
&\qquad\qquad \left. + \frac{1}{k^2} h(x)^{-\frac{2}{k}-2} \tilde{\nabla}^2 h_x(x, x) dh_x \otimes dh_x \right] \\
&= h(x)^{-1} \cdot \tilde{\nabla}^2 h_x - \frac{2(k-1)}{k} h(x)^{-2} dh_x \otimes dh_x + \frac{k-1}{k} h(x)^{-3+1} dh_x \otimes dh_x \\
&= h(x)^{-1} \tilde{\nabla}^2 h_x - \frac{k-1}{k} h(x)^{-2} dh_x \otimes dh_x,
\end{aligned}
$$

where at the penultimate step we have used that

$$\tilde{\nabla}^2 h_x(x, \cdot) = (k-1) dh_x \tag{1.4}$$

in combination with $dh_x(x) = k h(x)$. The former equation holds because the partial derivatives of first order of h are homogeneous of degree $k - 1$. $\square$

Corollary 1.13. *The centroaffine metric of the hypersurface $\mathcal{H}$ in the parametrization (1.3) is given by*

$$g = -\frac{1}{k\bar{h}} \tilde{\nabla}^2 \bar{h} + \frac{k-1}{(k\bar{h})^2} d\bar{h}^2 = -\frac{1}{u} \tilde{\nabla}^2 u,$$

where $\bar{h}$ denotes the restriction of h to $U \cap E_p$, $u := \sqrt[k]{\bar{h}}$ and $\tilde{\nabla}$ denotes the flat connection of the affine space E_p.

Remark: The function $\sqrt[k]{h} = \sqrt[k]{h_k}$ coincides with h_1 and is thus independent of k.

Lemma 1.14. *The convex domain $U \cap E_p \subset E_p$ is relatively compact for all $p \in \mathcal{H}$.*

Proof: We consider the positive function $u = \sqrt[k]{h}\big|_{U \cap E_p} = h_1\big|_{U \cap E_p} : U \cap E_p \to \mathbb{R}$, which is concave by Corollary 1.13, since g is positive definite. Let $B_\delta(p)$ be a Euclidean ball in E_p, which is relatively compact in $U \cap E_p$. There exists $\varepsilon > 0$ such that

$$-\varepsilon \langle \cdot, \cdot \rangle \geq \tilde{\nabla}^2 u \tag{1.5}$$

on $B_\delta(p)$. We will compare u to the concave C^1-function $v : E_p \to \mathbb{R}$ defined by

$$v(x) := \begin{cases} 1 - \varepsilon|x - p|^2 & \text{if } x \in B_\delta(p) \\ 1 + \varepsilon\delta^2 - 2\varepsilon\delta|x - p| & \text{otherwise.} \end{cases}$$

We claim that $v \geq u$ on $E \cap E_p$. We have $v(p) = u(p) = 1$ and both functions take their global maximum at p, so $dv_p = du_p = 0$. For any point $x \in U \cap E_p \setminus \{p\}$ we consider the line segment $c : [0,1] \to U \cap E_p$, $t \mapsto (1-t)p + tx$, from p to x. Put

$$f := (v - u) \circ c : [0,1] \to \mathbb{R}$$

and $t_0 := \min\left(1, \frac{\delta}{|x-p|}\right)$. We will prove that $f \geq 0$, which implies $v \geq u$, since x was arbitrary. Notice that f is smooth if $x \in B_\delta(p)$. Otherwise $f|_{[0,t_0]}$ and $f|_{[t_0,1]}$ are smooth. We have $f(0) = f'(0) = 0$ and, in virtue of (1.5), also $(f|_{[0,t_0]})'' \geq 0$. Using the initial conditions, this implies $f' \geq 0$ on $[0, t_0]$ and, hence, $f \geq 0$ on $[0, t_0]$. In particular, $f(t_0) \geq 0$ and $f'(t_0) \geq 0$. This proves that $f \geq 0$ if $t_0 = 1$. Otherwise it suffices to observe that $(f|_{[t_0,1]})'' = -(u \circ c|_{[t_0,1]})'' \geq 0$, which implies $f' \geq 0$ and finally $f \geq 0$ using the inequalities at t_0. So we have proven that $v \geq u$. As a consequence

$$U \cap E_p = u^{-1}((0, \infty)) \subset v^{-1}([0, \infty))$$

and the latter set is compact. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.15. *Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidean complete connected hyperbolic centroaffine hypersurface, $h : U = \mathbb{R}^{>0} \cdot \mathcal{H} \to \mathbb{R}$ the corresponding homogeneous function of degree $k > 0$ and $p \in \mathcal{H}$. Assume that there exists $\varepsilon \in (0, k)$ such that $f = \sqrt[k-\varepsilon]{h}$ is concave, where $\bar{h} = h|_{U \cap E_p}$. Then $\mathcal{H}$ is complete (with respect to the centroaffine metric g).*

Proof: We first compute

$$-\frac{1}{k}\tilde{\nabla}^2 f = \frac{f}{(k - \epsilon)}\left[\left(k - \frac{k}{k - \epsilon}\right)\frac{1}{(k\bar{h})^2}d\bar{h}^2 - \frac{1}{k\bar{h}}\tilde{\nabla}^2\bar{h}\right].$$

Comparing with Corollary 1.13, we see that

$$g = \frac{k - \varepsilon}{f}\left(-\frac{1}{k}\tilde{\nabla}^2 f\right) + \frac{\varepsilon}{(k - \varepsilon)(k\bar{h})^2}d\bar{h}^2.$$

Since the first term is positive semidefinite by assumption, we obtain the estimate

$$g \geq \frac{\varepsilon}{(k - \varepsilon)(k\bar{h})^2}d\bar{h}^2 = \frac{\varepsilon}{k^2(k - \varepsilon)}(d\ln\bar{h})^2, \qquad\qquad (1.6)$$

which implies the completeness as follows. Let $\gamma : I := [0, T) \to \mathcal{H}$, $T \in \mathbb{R}^{>0} \cup \{\infty\}$, be a curve which is not contained in any compact subset of $\mathcal{H}$ and $\gamma_0 : I \to U \cap E_p$ the

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corresponding curve in the parametrization $\varphi : U \cap E_p \xrightarrow{\sim} \mathcal{H}$, see (1.3). Then there exists a sequence $t_i \to T$ such that $\lim_{i \to \infty} h(\gamma_0(t_i)) = 0$. In view of (1.6), putting $f_0 = h \circ \gamma_0$, we can estimate the length of γ as follows:

$$L(\gamma) \geq \underbrace{\frac{1}{k}\sqrt{\frac{\varepsilon}{k - \varepsilon}}}_{C:=} \int_0^{t_i} \left| \frac{d}{dt} \ln f_0 \right| dt$$

$$\geq C \left| \int_0^{t_i} \frac{d}{dt} \ln f_0 \, dt \right| = C \left| \ln f_0(t_i) - \ln f_0(0) \right| \xrightarrow{i \to \infty} \infty. \qquad \square$$

Proposition 1.16.   *Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidean complete connected hyperbolic centroaffine hypersurface, $h : U = \mathbb{R}^{>0} \cdot \mathcal{H} \to \mathbb{R}$ the corresponding homogeneous function of degree $k > 1$. Then*

$$g_L := -\frac{1}{k}\tilde{\nabla}^2 h$$

is a Lorentzian metric on the convex domain U. The Lorentzian manifold (U, g_L) is globally hyperbolic if and only if $\mathcal{H}$ is complete (with respect to the centroaffine metric g).

Proof: By the homogeneity of h, the position vector ξ satisfies

$$\begin{aligned} g_L(\xi, \xi) &= -(k-1)h < 0 \\ g_L(\xi, \cdot) &\overset{(1.4)}{=} -\frac{k-1}{k}dh. \end{aligned} \qquad (1.7)$$

The latter equation shows that ξ is perpendicular to the level sets of h, on which the metric g_L restricts to a positive definite metric. Therefore g_L is Lorentzian.

Also due to the homogeneity of h, the position vector field ξ on U is a homothetic Killing vector field:

$$L_\xi g_L = k g_L.$$

The equation (1.7) shows that it is also a gradient vector field . Thus

$$D^L \xi = \frac{k}{2}\mathrm{Id},$$

where D^L is the Levi-Civita connection of g_L. Rescaling $\zeta := \frac{2}{k}\xi$ we get $D^L\zeta = \mathrm{Id}$. Since the vector field ζ is obviously complete, this implies that (U, g_L) is a metric cone:

$$g_L = -ds^2 + s^2 g.$$

Here U is identified with $\mathbb{R}^{>0} \times \mathcal{H}$ by means of the diffeomorphism

$$\mathbb{R}^{>0} \times \mathcal{H} \ni (s, p) \mapsto \frac{2}{k}sp \in U.$$

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With the substitution $s = e^t$ we obtain

$$g_L = e^{2t}(-dt^2 + g).$$

This shows that the metric g_L is globally hyperbolic if and only if the product metric $-dt^2 + g$ on $\mathbb{R} \times \mathcal{H}$ is. If $(\mathcal{H}, g)$ is complete then the level sets of t are Cauchy hypersurfaces, which implies the global hyperbolicity by [O, Cor. 39]. Otherwise there exists an inextendible geodesic $\gamma : [0, T) \to (\mathcal{H}, g)$ of finite length T. This implies that $J(p, q)$ is noncompact if we put $p = (0, \gamma(0))$, $q = (2T, \gamma(0)) \in \mathbb{R} \times \mathcal{H}$. Recall [O, p. 410] that $J(p, q)$ stands for the smallest set containing all future-pointing causal curves from p to q and that the sets $J(p, q)$ are compact in every globally hyperbolic Lorentzian manifold M, for all $p, q \in M$, see [O, p. 411]. This shows that (U, g_L) is not globally hyperbolic if $(\mathcal{H}, g)$ is incomplete. $\qquad\square$

1.4 Completeness for hyperbolic centroaffine hypersurfaces with regular boundary behaviour

In this subsection, $\mathcal{H} \subset \mathbb{R}^{n+1}$ will be always a Euclidean complete connected hyperbolic centroaffine hypersurface, $U = \mathbb{R}^{>0} \cdot \mathcal{H}$ and $h : U \to \mathbb{R}$ a smooth homogeneous function of degree $k > 1$ such that $\mathcal{H} = \{p \in U \mid h(p) = 1\}$. We assume that h extends to a smooth homogeneous function $h : V \to \mathbb{R}$ defined on some open subset $V \subset \mathbb{R}^{n+1}$ such that $\overline{U} \setminus \{0\} \subset V$. This assumption is satisfied, for instance, if the function $h : U \to \mathbb{R}$ is polynomial. $0 \in \mathbb{R}^{n+1}$ is excluded in order to keep the level of generality. Note that if a homogeneous function is smooth at the origin then the degree of homogeneity k is a nonnegative integer. This follows from the fact that all radial derivatives, especially those of order $n > k$, have to be bounded in 0, which is not possible for negative or non integer degrees of homogeneity.

Definition 1.17. *Under the above assumptions, we say that the hypersurface $\mathcal{H}$ has* regular boundary behaviour *if*

(i) $dh_p \neq 0$ for all $p \in \partial U \setminus \{0\}$. In particular, $\partial U \setminus \{0\}$ is smooth.

(ii) $-\tilde{\nabla}^2 h$ is positive semi-definite on $T(\partial U \setminus \{0\})$ with only one-dimensional kernel.

Example: The curve $\{x(x^2 - y^2) = 1, x > 0\} \subset \mathbb{R}^2$ has regular boundary behaviour (and is therefore complete by the following theorem), whereas $\{x^2 y = 1, x > 0\} \subset \mathbb{R}^2$ does not have regular boundary behaviour (but is still complete). These are the only complete hyperbolic curves defined by a homogeneous cubic polynomial h, up to linear transformations, see [CHM, Cor. 4].

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Theorem 1.18. *Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidean complete hyperbolic centroaffine hypersurface with regular boundary behaviour. Then $\mathcal{H}$ is complete (with respect to the centroaffine metric).*

Before we give the proof we would like to discuss the relation of this result to the literature. Melrose [Me, Ch. 8] considers Riemannian metrics on compact manifolds M with nonempty boundary of the form $x^{2a}dx^2 + x^{2b}H$ where $x \colon M \to [0,\infty)$ is a "boundary defining function", i.e. $\{x = 0\} = \partial M$ and $dx|_{\partial M} \neq 0$ and H is a smooth tensor field inducing a Riemannian metric on the boundary. Up to a constant factor, the centroaffine metrics for functions with regular boundary behaviour correspond to the "marginally complete" case $a = -1$, $b = -1/2$ with $x = \bar{h}$ the boundary defining function and H a constant multiple of $-\tilde{\nabla}^2 \bar{h}$, see Corollary 1.13. In this case the map $x \mapsto y = \sqrt{x}$ transforms the metric to a "conformally compact" metric $\frac{4}{y^2}\left(dy^2 + \frac{1}{4}H\right)$. The completeness follows from the claim [Ma, p. 311] that metrics of this form are complete. For the particular metric considered here this is shown below.

[Ma] further claims that each geodesic ray of the conformally compact metric is asymptotic to a single point in the boundary, the direction approaches the outer normal and the curvature is eventually negative.

[CG13] consider metrics as [Me], with $a = 2b$, but the additional possibility of multiplying the $x^{2a}dx^2$-term by a smooth function C such that the metric still extends smoothly to the boundary. Under these assumptions and with $a \leq -\frac{1}{2}$ they prove completeness of geodesics whose asymptotic tangents are transversal to the boundary. In [CG14] the same authors show that C can be assumed constant under certain conditions by an appropriate choice of the boundary function x.

As a consequence of Proposition 1.16, Theorem 1.18 implies the global hyperbolicity of the Lorentzian metric g_L on $U = \mathbb{R}^{>0} \cdot \mathcal{H}$ for hypersurfaces $\mathcal{H}$ with regular boundary behaviour, see [FHS, Cor. 4.34] for a related result in Lorentzian geometry.

Proof: Let $v \in U$ and E a hyperplane trough v which intersects the convex cone U in a relatively compact domain $B = E \cap U$. Such hyperplanes exist thanks to Lemma 1.14. We denote by $\partial B = E \cap \partial U$ the (smooth) boundary of B in E. For $\epsilon > 0$, we consider the conical hypersurface $F = F_\epsilon$ which is the union of all the rays emanating from ϵv and intersecting ∂B. It is smooth outside the vertex ϵv and so is the homeomorphism $\psi|_{F \cap U} : F \cap U \to \mathcal{H}$. Here we recall that $\psi : U \to \mathcal{H}$ is the map $x \mapsto h(x)^{-1/k}x$.

Lemma 1.19. *There exists $\epsilon_0 > 0$ such that the tensor field $-\tilde{\nabla}^2 h$ restricts to a positive definite metric on a neighbourhood N_0 of ∂B in F_ϵ for all $0 < \epsilon < \epsilon_0$.*

Proof: Let us denote by $\eta \in \mathfrak{X}(V)$ the gradient of $h \in C^\infty(V)$ with respect to the Euclidean scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^{n+1}$. Then, for all $x \in \partial B$,

$$T_x V = T_x \partial U \oplus \mathrm{span}\{\eta_x\}, \quad T_x \partial U = T_x \partial B \oplus \mathrm{span}\{\xi_x\}.$$

The symmetric tensor field $\beta = -\tilde{\nabla}^2 h$ on V is positive definite on $T_x \partial B$, by (ii) in Definition 1.17, and satisfies

$$\beta(\xi_x, \xi_x) = -k(k-1)h(x) = 0,$$
$$\beta(\xi_x, \eta_x) = -(k-1)dh(\eta_x) = -(k-1)\langle \eta_x, \eta_x \rangle =: c < 0,$$

and

$$\beta(\xi_x, y) = -(k-1)dh(y) = 0,$$

for all $y \in T_x \partial B$, as follows from the homogeneity of h, cf. (1.4). Now let $(e_3, \ldots, e_{n+1})$ be a β-orthonormal basis of $T_x \partial B$, which we extend by $e_1 := \eta_x$ and $e_2 = \xi_x$ to a basis $(e_1, \ldots, e_{n+1})$ of $T_x V$. Then

$$\det(\beta(e_i, e_j)_{i,j=1,\ldots,n+1}) = -c^2 \det(\beta(e_a, e_b)_{a,b=3,\ldots,n+1}) < 0.$$

This shows that β_x is a Lorentzian scalar product and implies that the Lorentzian metric g_L is extended by $\frac{1}{k}\beta$ to a Lorentzian metric on a neighbourhood of $\overline{U} \setminus \{0\}$ in V. Now, to prove the lemma, it suffices to check for all $x \in \partial B$ that $\beta(\nu, \nu) > 0$ for a non-zero vector $\nu \in T_x F$ which is orthogonal to the positive definite hyperplane $T_x \partial B \subset T_x F$ with respect to β. As such a vector we can take the orthogonal projection of $\epsilon v - x \in T_x F$ onto the orthogonal complement of $T_x \partial B$ in $T_x F$:

$$\nu = \epsilon v - x - \sum_{a=3}^{n+1} \beta(\epsilon v - x, e_a)e_a = \epsilon v - x - \epsilon \sum_{a=3}^{n+1} \beta(v, e_a)e_a.$$

Then

$$\beta(\nu, \nu) = -2\epsilon\beta(v, x) + \epsilon^2\beta(v, v) - \epsilon^2 \sum_{a=3}^{n+1} \beta(v, e_a)^2.$$

Since $-\beta(v, x) = (k-1)dh(v) > 0$ and ∂B is compact, we see that $\beta(\nu, \nu) > 0$ for all $x \in \partial B$ if ϵ is sufficiently small. $\qquad\square$

Let ϵ, N_0 be as in Lemma 1.19 and put $N = N_0 \cap U$, $F = F_\epsilon$. Then, by Lemma 1.12, we have

$$(\psi^* g)\big|_N = -\frac{1}{kh}\tilde{\nabla}^2 h\big|_N + \frac{k-1}{(kh)^2}dh^2\big|_N > \frac{k-1}{(kh)^2}dh^2\big|_N. \qquad (1.8)$$

This implies the completeness of g as follows. Let $\gamma : [0, b) \to \mathcal{H}$, $0 < b \leq \infty$, be a curve which is not contained in any compact subset of $\mathcal{H}$ and $\gamma_F : [0, b) \to F \cap U$ the

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corresponding curve in $F \cap U$. Then there exists $\delta > 0$ and sequences $t_i \in [0, b)$ and $s_i \in (t_i, b)$ such that $\gamma_F([t_i, s_i]) \subset N$, $h(\gamma_F(t_i)) > \delta$ and $h(\gamma_F(s_i)) \to 0$. In view of (1.8), we have

$$L(\gamma) \geq L\big(\gamma|_{[t_i, s_i]}\big) = L(\gamma_F|_{[t_i, s_i]}) \geq C \int_{t_i}^{s_i} \left| \frac{d}{dt} \ln h \circ \gamma_F \right| dt$$

$$\geq -C \int_{t_i}^{s_i} \frac{d}{dt} \ln h \circ \gamma_F \, dt = C\big(\ln h(\gamma_F(t_i)) - \ln h(\gamma_F(s_i)) \big) \to \infty,$$

where $C = \frac{\sqrt{k-1}}{k}$. This shows that γ has infinite length and proves Theorem 1.18. $\square$

Next we will show that the hypersurfaces with regular boundary behaviour are generic in the class of hypersurfaces considered in this section. In order to make this statement precise, let $V \subset \mathbb{R}^{n+1}$ be an open subset and $k \in (1, \infty)$. We denote by $\mathcal{F} = \mathcal{F}(V, k) \subset C^\infty(V)$ the cone consisting of homogeneous functions h of degree k with the property that there exists an open cone $U \subset V$ such that $\overline{U} \setminus \{0\} \subset V$ and

$$\mathcal{H} = \mathcal{H}(h, U) := \{p \in U \mid h(p) = 1\}$$

is a Euclidean complete connected hyperbolic centroaffine hypersurface. (Notice that for $\mathcal{F}$ to be nonempty V has to contain an open cone U.) We endow $\mathcal{F}$ with the topology induced by the standard Fréchet topology of $C^\infty(V)$. Recall that the latter is the coarsest topology for which the semi-norms $\sup_K \big| \tilde{\nabla}^\ell h \big|$ are continuous for all compact subsets $K \subset V$ and all $\ell = 0, 1, \ldots$, where $| \cdot |$ stands for the Euclidean norm on tensors.

Then we put

$$\mathcal{F}_{reg} = \mathcal{F}_{reg}(V, k)$$
$$:= \big\{ h \in \mathcal{F} \mid \mathcal{H}(h, U) \text{ has regular boundary behaviour for some } U \text{ as above} \big\}.$$

Theorem 1.20.   *$\mathcal{F}_{reg}$ is a dense open subset of $\mathcal{F}$.*

Proof: Let $h \in \mathcal{F}$ and $U \subset V$ an open cone such that $\overline{U} \setminus \{0\} \subset V$ and $\mathcal{H} = \mathcal{H}(h, U)$ is a Euclidean complete connected hyperbolic centroaffine hypersurface. Replacing U by $\{p \in U \mid h(p) > 0\}$, if necessary, we can assume that $h > 0$ on U. Then $U = \mathbb{R}^{>0} \cdot \mathcal{H}$. Further let $p \in \mathcal{H} = \mathcal{H}(h) \subset U$ and $E = E_p$ the affine hyperplane tangent to $\mathcal{H}$ at p. Then we choose linear coordinates $x_1, \ldots, x_{n+1}$ on $\mathbb{R}^{n+1}$ such that $x_1(p) = \cdots = x_n(p) = 0$ and $x_{n+1}\big|_E = 1$. We claim that

$$h_\epsilon := h - \epsilon x_{n+1}^k \in \mathcal{F}_{reg}$$

for all $\epsilon \in (0, 1)$.

Recall first that $\mathcal{H}$ is closed, by Proposition 1.8 (ii). It is mapped to $\mathcal{H}_\epsilon := \mathcal{H}(h_\epsilon, U)$ by the following diffeomorphism of the upper half-space $\{x_{n+1} > 0\} \subset \mathbb{R}^{n+1}$:

$$x = (x_1, \ldots, x_{n+1}) = (\vec{x}, x_{n+1}) \mapsto \left(\frac{\vec{x}}{(1 + \epsilon x_{n+1}^k)^{1/k}}, \ x_{n+1} \right).$$

This shows that $\mathcal{H}_\epsilon$ is a closed connected smooth hypersurface and therefore Euclidean complete, again by Proposition 1.8 (ii). Next we show that the symmetric tensor field $\beta^\epsilon := -\tilde{\nabla}^2 h_\epsilon$ on V is Lorentzian on the cone $U_\epsilon := \mathbb{R}^{>0} \cdot \mathcal{H}_\epsilon$. This implies that $\mathcal{H}_\epsilon$ is hyperbolic and, hence, that $h_\epsilon \in \mathcal{F}$. First we notice that β^ϵ is Lorentzian when evaluated at points of $\partial U_\epsilon \setminus \{0\}$. In fact, since $h_\epsilon - h$ is constant on E, β^ϵ coincides with β on E and is therefore positive definite on $T\partial B_\epsilon$, where ∂B_ϵ is the boundary of the domain $B_\epsilon := U_\epsilon \cap E$ in E. Here we are using that $\partial B_\epsilon = \{p \in E \mid h(p) = \epsilon\}$ is a level set of h. On the other hand, $\xi_x \in T_x \partial U_\epsilon$ is a null vector of β^ϵ for all $x \in \partial B_\epsilon$:

$$\beta^\epsilon(\xi_x, \xi_x) = -k(k-1)h_\epsilon(x) = 0.$$

Moreover, for all $y \in T_x \partial B_\epsilon$ we have

$$\beta^\epsilon(\xi_x, y) = -(k-1)dh_\epsilon(y) - k(k-1)dh(y) = 0.$$

Finally, let η^ϵ be the Euclidean gradient of h_ϵ. Then

$$\beta^\epsilon(\xi_x, \eta_x^\epsilon) = -(k-1)dh_\epsilon(\eta_x^\epsilon) < 0.$$

As in the proof of Lemma 1.19 , this implies that β_x^ϵ has negative determinant and is a Lorentzian scalar product on $T_x V$ for all $x \in \partial B_\epsilon$. By homogeneity, the same is true for all $x \in \partial U_\epsilon \setminus \{0\}$. To prove that β^ϵ is Lorentzian on U_ϵ it suffices now to show that $\det \beta^\epsilon$ is negative on U_ϵ. For all $x \in B_\epsilon$, we have

$$\det \beta_x^\epsilon = \det \beta_x - k(k-1)\epsilon \det A,$$

where the determinant is computed with respect to the basis of $T_x V = \mathbb{R}^{n+1}$ associated with the coordinates $x_1, \ldots, x_{n+1}$ and A is the principal $n \times n$-minor obtained by deleting the last row and column of the Gram matrix of β_x^ϵ. Recall that β_x is Lorentzian and thus $\det \beta_x < 0$. So if $\det A \geq 0$ then it follows that $\det \beta_x^\epsilon < 0$ and we are done. Therefore we can assume that $\det A < 0$ and, since $\epsilon < h(x)$ on B_ϵ,

$$\det \beta_x^\epsilon = \det \beta_x + k(k-1)\epsilon \left| \det A \right| < \det \beta_x + k(k-1)h(x) \left| \det A \right| = \det \beta_x^{h(x)}.$$

Now observe that $x \in \partial B_{h(x)}$ if $h(x) < \max_B h = 1$. It follows from the above discussion that in this case $\beta_x^{h(x)}$ is Lorentzian and $\det \beta_x^{h(x)} < 0$. This shows that $\det \beta_x^{h(x)} \leq 0$ for

all $x \in B_\epsilon$ and implies $\det \beta_x^\epsilon < 0$ for all $x \in B_\epsilon$. By homogeneity, this proves that β_x^ϵ is a Lorentzian metric on U_ϵ.

Finally, we have to show that $\mathcal{H}_\epsilon$ has regular boundary behaviour. Since $h_\epsilon = h - \epsilon$ on E, we have that

$$dh\big|_{T_p E} = dh_\epsilon\big|_{T_p E} \quad \text{and} \quad \beta\big|_{T_p E \times T_p E} = \beta^\epsilon\big|_{T_p E \times T_p E}$$

for all $p \in V \cap E$. As $dh T_p E \neq 0$ for all $p \in B \supset B_\epsilon$ and h_ϵ is homogeneous, condition (i) in Definition 1.17 is clearly satisfied for all $p \in \partial U_\epsilon \setminus \{0\}$. Similarly, since β is positive definite on $T_p \partial B_\epsilon = \ker dh\big|_{T_p E}$ for all $p \in \partial B_\epsilon$ and h_ϵ is homogeneous, we see that also condition (ii) in Definition 1.17 is satisfied on $T(\partial U_\epsilon \setminus \{0\})$. $\qquad\square$

For any integer $k > 1$ let us denote by $\mathcal{P}(k) \subset \mathcal{F}(\mathbb{R}^{n+1}, k)$ the subset consisting of polynomial functions and $\mathcal{P}_{reg}(k) = \mathcal{P}(k) \cap \mathcal{F}_{reg}(\mathbb{R}^{n+1}, k)$.

Corollary 1.21. $\mathcal{P}_{reg}(k) \subset \mathcal{P}(k)$ *is an open dense subset.*

Next we discuss how many functions with Euclidean complete connected hyperbolic centroaffine level sets there are.

Theorem 1.22. *Let $V \subset \mathbb{R}^{n+1}$ be an open cone. Then for every $k \geq 2$, every compact subset $K \subset V$ and every integer $n \geq 1$ there exists $C = C(k, K, n) < \infty$ such that for all smooth functions $h \colon V \to \mathbb{R}$ homogeneous of degree k there exists $h' \in \mathcal{F}_{reg}(V, k)$ such that*

$$\sup_K \big|\tilde{\nabla}^l h - \tilde{\nabla}^l h'\big| \leq C\left(\sup_K \sum_{i=0}^{\max(l,2)} \big|\tilde{\nabla}^i h\big| + 1\right)$$

for every $0 \leq l \leq n$.

Proof: We will distinguish two cases, namely $0 \notin V$ and $0 \in V$.

First assume that $0 \notin V$. Let $p \in V$ with $|p| = 1$. Consider an open subcone V' of V not containing any nonzero vectors orthogonal to p. First we will construct h' on V' and then extend it to V.

Consider an orthonormal basis $\{v_1, \ldots, v_{n+1}\}$ of $\mathbb{R}^{n+1}$ with $v_{n+1} = p$ and dual basis $\{\alpha_1, \ldots, \alpha_{n+1}\}$. For $\eta \in \mathbb{R}$ define the function $h_\eta \colon V' \to \mathbb{R}, x \mapsto h(x) + \eta \alpha_{n+1}(x)^k$. Putting $\eta = |h(p)| + 1$ we have $h_\eta(p) > 0$.

Next consider the smooth and homogeneous function of degree k

$$P \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}, \quad x \mapsto -\alpha_{n+1}(x)^{k-2} \sum_{i=1}^n \alpha_i(x)^2.$$

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Denote with E_p the affine hyperplane in $\mathbb{R}^{n+1}$ intersecting and orthogonal to p, i.e. $E_p = p + \text{span}\{v_1, \ldots, v_n\}$. The second derivative of $P|_{E_p}$ at p is negative definite. In fact we have $\tilde{\nabla}^2(P|_{E_p})_p = -2 \sum_{i=1}^n \alpha_i \otimes \alpha_i$.

For $\lambda \in \mathbb{R}$ define $h_{\eta,\lambda} \colon V' \to \mathbb{R}$, $x \mapsto h_\eta(x) + \lambda P(x)$. Setting $\lambda = \left|\tilde{\nabla}^2(h|_{E_p})_p\right| + 1$ we conclude that $\tilde{\nabla}^2(h_{\eta,\lambda}|_{E_p \cap V})_p < 0$. Hence $\tilde{\nabla}^2(h_{\eta,\lambda}|_{E_p \cap V'})$ is negative definite in a neighbourhood of p in $E_p \cap V'$. Especially p is a local maximum of $h_{\eta,\lambda}|_{E_p}$. For λ sufficiently large the set $E_p \cap (h_{\eta,\lambda})^{-1}[0,\infty)$ is contained in the connected component of $\{q \in V' \cap E_p \mid \tilde{\nabla}^2(h_{\eta,\lambda}|_{E_p \cap V'})_q < 0\}$ around p. Note that since $\tilde{\nabla}^2(h_{\eta,\lambda}|_{E_p \cap V'})_p$ is nondegenerate the map $q \in E_p \cap V' \mapsto d(h_{\eta,\lambda}|_{E_p \cap V'})_q$ is a local diffeomorphism on a neighbourhood U of p in $E_p \cap V'$, i.e. $d(h_{\eta,\lambda}|_{E_p \cap V'})$ vanishes in U only at p. This now implies that the connected component $\mathcal{H}'$ of $h_{\eta,\lambda}^{-1}(1)$ containing p has regular boundary behaviour. The Euclidean completeness of $\mathcal{H}$ follows from Proposition 1.8.

Now consider a smooth function $\mu \colon S^n \to [0,1]$ with support in $V \cap S^n$ and $\mu|_{V' \cap S^n} \equiv 1$ such that $\mu \cdot h_{\eta,\lambda} \leq 0$ on $(V \setminus V') \cap S^n$ and $\mu(q) = 0$ for all $q \in S^n$ orthogonal to p.

Define $\hat{h} :\equiv \mu \cdot h_{\eta,\lambda}|_{V \cap S^n}$ and consider the canonical k-homogeneous extension h' of $\hat{h}$ to $V \setminus \{0\}$. This completes the proof for the case $0 \notin V$.

Now assume $0 \in V$. Then k is a nonnegative integer. Following the above construction we see that $h - h_{\eta,\lambda}$ is polynomial. In this case we can neglect the cutoff function μ and define $h' := h_{\eta,\lambda}$. The claim is now immediate. This completes the proof. $\square$

Corollary 1.23. *For every $k \geq 2$ and every compact set $K \subseteq V$ there exists $C = C(K,k) < \infty$ such that for all polynomials $h \colon \mathbb{R}^{n+1} \to \mathbb{R}$ homogeneous of degree k there exists $h' \in \mathcal{P}_{reg}(k)$ such that*

$$\sup_K \left|\tilde{\nabla}^l h - \tilde{\nabla}^l h'\right| \leq C \sup_K \left(|h| + |\tilde{\nabla}^2 h| + 1\right)$$

for every $l \geq 0$.

2 Projective special real manifolds

2.1 Centroaffine structure and intrinsic characterization of projective special real manifolds

Definition 2.1. *A* projective special real manifold *is a smooth hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ for which there exists a homogeneous cubic polynomial h on $\mathbb{R}^{n+1}$ such that*

(i) $\mathcal{H} \subset \{h = 1\} := \{x \in \mathbb{R}^{n+1} \mid h(x) = 1\}$,

(ii) the Hessian $\tilde{\nabla}^2 h$ is negative definite on $T\mathcal{H}$.

As a consequence of Proposition 1.3, for every projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$, the inclusion $\iota : \mathcal{H} \subset \mathbb{R}^{n+1}$ is a hyperbolic centroaffine immersion and, hence, induces a centroaffine structure (∇, g, ν) on $\mathcal{H}$, such that

$$g = -\frac{1}{3}\iota^*(\tilde{\nabla}^2 h). \tag{2.1}$$

Definition 2.2. *An* intrinsic projective special real manifold *is a centroaffine manifold* (M, ∇, g, ν) *with positive definite metric g such that the covariant derivative of the cubic form $C = \nabla g$ is given by*

$$(\nabla_X C)(Y, Z, W) = g(X, Y)g(Z, W) + g(X, Z)g(W, Y) + g(X, W)g(Y, Z), \tag{2.2}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Remark: The equation (2.2) implies that ∇C is totally symmetric, that is a quartic form.

Theorem 2.3.

(i) *Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a projective special real manifold with induced centroaffine structure (∇, g, ν). Then $(\mathcal{H}, \nabla, g, \nu)$ is an intrinsic projective special real manifold, that is satisfies (2.2).*

(ii) *Conversely, let (M, ∇, g, ν) be a connected and simply connected intrinsic projective special real manifold. Then there exists an embedding $\varphi : M \to \mathbb{R}^{n+1}$ such that $\mathcal{H} := \varphi(M) \subset \mathbb{R}^{n+1}$ is a projective special real manifold. The embedding φ is unique up to linear unimodular transformations of $\mathbb{R}^{n+1}$.*

Remark: A similar characterization in terms of covariant derivatives of C up to order $k-2$ can be given for nondegenerate hypersurfaces which are locally defined by a homogeneous polynomial h of degree $k \geq 2$.

Proof: Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a projective special real manifold with induced centroaffine structure (∇, g, ν). In order to check (2.2), we denote by H the trilinear form on $\mathbb{R}^{n+1}$ such that $H(v, v, v) = h(v)$, for all $v \in \mathbb{R}^{n+1}$. Differentiating the equation $H(\xi, \xi, \xi) = h(\xi) = 1$ yields:

$$0 = H(\xi, \xi, X)$$
$$0 = 2H(\xi, X, Y) + H(\xi, \xi, \nabla_Y X + g(X, Y)\xi) = 2H(\xi, X, Y) + g(X, Y)$$

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for all $X, Y \in \mathfrak{X}(\mathcal{H})$. Thus

$$g = -2H(\xi, \cdot, \cdot)\big|_{T\mathcal{H} \otimes T\mathcal{H}} \tag{2.3}$$

and, hence,

$$C(X, Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = -2H(X, Y, Z),$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{H})$. Thus

$$C = -2H\big|_{T\mathcal{H} \otimes T\mathcal{H} \otimes T\mathcal{H}} \tag{2.4}$$

Next we calculate ∇C using the previous equations:

$$(\nabla_X C)(Y, Z, W)$$
$$= XC(Y, Z, W) - C(\nabla_X Y, Z, W) - C(Y, \nabla_X Z, W) - C(Y, Z, \nabla_X W)$$
$$= -2XH(Y, Z, W) + 2H(\nabla_X Y, Z, W) + 2H(Y, \nabla_X Z, W) + 2H(Y, Z, \nabla_X W)$$
$$\overset{(1.1)}{=} -2g(X, Y)H(\xi, Z, W) - 2g(X, Z)H(Y, \xi, W) - 2g(X, W)H(Y, Z, \xi)$$
$$= g(X, Y)g(Z, W) + g(X, Z)g(Y, W) + g(X, W)g(Y, Z),$$

for all $X, Y, Z, W \in \mathfrak{X}(\mathcal{H})$. This proves (i).

Let (M, ∇, g, ν) be a connected and simply connected intrinsic projective special real manifold. Let us denote by $N = M \times \mathbb{R}$ the trivial line bundle over M, and by ξ_0 its canonical trivializing section. We claim that (2.2) is equivalent to the equation

$$\tilde{\nabla} H = 0,$$

where $\tilde{\nabla} = \tilde{\nabla}^E$ is the flat connection on the vector bundle $E = TM \oplus N$, which is defined by

$$\tilde{\nabla}_X Y := \nabla_X Y + g(X, Y)\xi_0$$
$$\tilde{\nabla}_X \xi_0 := X,$$

for all $X, Y \in \mathfrak{X}(M)$ and $H = H_E \in \Gamma(S^3 E^*)$ is defined by

$$H\big|_{TM^{\otimes 3}} := -\frac{1}{2}C = -\frac{1}{2}\nabla g$$
$$H(\xi_0, \cdot, \cdot)\big|_{TM^{\otimes 2}} := -\frac{1}{2}g$$
$$H(\xi_0, \xi_0, \cdot)\big|_{TM} := 0$$
$$H(\xi_0, \xi_0, \xi_0) := 1.$$

Let us first show that the curvature $\tilde{R}$ of $\tilde{\nabla}$ is zero. The vanishing of the torsion of ∇ implies the equation $\tilde{R}(X, Y)\xi_0 = 0$ and the equations (ii) and (iii) in Definition

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1.5 imply $\tilde{R}(X,Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$. Next we prove that $\tilde{\nabla}H = 0$. For $X, Y, Z, W \in \mathfrak{X}(M)$ we compute:

$$(\tilde{\nabla}_X H)(\xi_0, \xi_0, \xi_0) = X \underbrace{H(\xi_0, \xi_0, \xi_0)}_{=1} - 3 \underbrace{H(\tilde{\nabla}_X \xi_0, \xi_0, \xi_0)}_{=H(X, \xi_0, \xi_0)=0} = 0$$

$$(\tilde{\nabla}_X H)(\xi_0, \xi_0, Y) = XH(\xi_0, \xi_0, Y) - 2H(\xi_0, X, Y) - H(\xi_0, \xi_0, g(X,Y)\xi_0)$$
$$= -2H(\xi_0, X, Y) - g(X, Y) = 0$$

$$(\tilde{\nabla}_X H)(\xi_0, Y, Z) = XH(\xi_0, Y, Z) - H(X, Y, Z) - H(\xi_0, \nabla_X Y, Z) - H(\xi_0, Y, \nabla_X Z)$$
$$= \frac{1}{2}\Big(- Xg(Y, Z) + C(X, Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\Big) = 0$$

$$(\tilde{\nabla}_X H)(Y, Z, W) = XH(Y, Z, W) - H(\tilde{\nabla}_X Y, Z, W) - H(Y, \tilde{\nabla}_X Z, W) - H(Y, Z, \tilde{\nabla}_X W)$$
$$= -\frac{1}{2}(\nabla_X C)(Y, Z, W) - H(\xi_0, Z, W)g(X, Y) - H(\xi_0, Y, W)g(X, Z)$$
$$\quad - H(\xi_0, Y, Z)g(X, W)$$
$$= -\frac{1}{2}\Big((\nabla_X C)(Y, Z, W) - g(Z, W)g(X, Y) - g(Y, W)g(X, Z)$$
$$\quad - g(Y, Z)g(X, W)\Big).$$

This shows that $\tilde{\nabla}H = 0$ if and only if (2.2) holds.

Since $(E, \tilde{\nabla})$ is a flat vector bundle over the simply connected manifold M, there exist an isomorphism $\Phi : (E, \tilde{\nabla}) \to (M \times \mathbb{R}^{n+1}, \tilde{\nabla})$ identifying $(E, \tilde{\nabla})$ with the trivial vector bundle $(M \times \mathbb{R}^{n+1}, \tilde{\nabla})$ endowed with its canonical flat connection $\tilde{\nabla}$. The restriction $\Phi|_{TM}$ to the subbundle $TM \subset E$ is a closed vector valued 1-form $\phi = (\phi_1, \cdots, \phi_{n+1})$ on M. In fact, for all $X, Y \in \mathfrak{X}(M)$ we have

$$X\phi(Y) = X\Phi(Y) = \Phi(\tilde{\nabla}_X Y) = \phi(\nabla_X Y) + g(X, Y)\Phi(\xi_0)$$

and, hence,

$$d\phi(X, Y) = X\phi(Y) - Y\phi(X) - \phi([X, Y]) = \phi\big(\nabla_X Y - \nabla_Y X - [X, Y]\big) = 0.$$

Since M is simply connected, there exists a smooth map $\varphi : M \to \mathbb{R}^{n+1}$ such that $d\varphi = \phi$. It is a hypersurface immersion because Φ being an isomorphism of vector bundles implies that $\phi = \Phi|_{TM}$ is a monomorphism of vector bundles. The vector field $\xi := \Phi(\xi_0) : M \to \mathbb{R}^{n+1}$ is transversal to $\Phi TM = d\varphi TM$ and verifies

$$\tilde{\nabla}_X \xi = \Phi(\tilde{\nabla}_X \xi_0) = \Phi(X) = d\varphi X. \tag{2.5}$$

This implies that there exists $v_0 \in \mathbb{R}^{n+1}$ such that

$$\xi(p) = \varphi(p) + v_0$$

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for all $p \in M$. Composing φ with a translation we can assume that $v_0 = 0$. Then φ is a centroaffine immersion with induced data $(\nabla, g, \det(\xi, \cdots))$. The induced volume form $\det(\xi, \cdots)$ is ∇-parallel (due to (2.5)) and, therefore, coincides with ν up to a constant factor. Rescaling Φ, if necessary, we can assume that $\nu = \det(\xi, \cdots)$. Now the immersion φ is unique up to unimodular linear transformation, by Theorem 1.6. Using the identification Φ of E with the trivial bundle $M \times \mathbb{R}^{n+1}$, the parallel section $H = H_E \in \Gamma(S^3 E^*)$ corresponds to an element $H = H_{\mathbb{R}^{n+1}} \in S^3(\mathbb{R}^{n+1})^*$, which in turn defines a cubic polynomial h on $\mathbb{R}^{n+1}$ such that $h(v) = H(v, v, v)$ for all $v \in \mathbb{R}^{n+1}$. Now it suffices to show that $h \circ \varphi = 1$, which follows from

$$1 = H_E(\xi_0, \xi_0, \xi_0) = H_{\mathbb{R}^{n+1}}(\xi, \xi, \xi) = h(\xi) = h(\varphi).$$

This shows that $\varphi : M \to \mathbb{R}^{n+1}$ is an embedding and that $\mathcal{H} = \varphi(M)$ is a projective special real manifold. $\qquad \square$

2.2 Relation between completeness and closedness of projective special real manifolds

Proposition 2.4. *Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a projective special real manifold with centroaffine metric g, see (2.1). Then the following hold.*

(i) If $(\mathcal{H}, g)$ is complete then $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a closed subset.

(ii) $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a closed subset if and only if $\mathcal{H} \subset \mathbb{R}^{n+1}$ is Euclidean complete.

Proof: Assume that $(\mathcal{H}, g)$ is complete or that $\mathcal{H} \subset \mathbb{R}^{n+1}$ is Euclidean complete. By taking $V = U = \mathbb{R}^{n+1}$ in Proposition 1.8, we see that every component of $\mathcal{H}$ is closed in $\mathbb{R}^{n+1}$ and, hence, coincides with one of the finitely many[1] connected components of $\{h = 1\}$. Then $\mathcal{H}$ is a finite union of closed subsets of $\mathbb{R}^{n+1}$ and, therefore, closed. This proves (i) and one of the implications in (ii). To prove the converse statement in (ii) it is sufficient to remark that the components of the closed subset $\mathcal{H} \subset \mathbb{R}^{n+1}$ are again closed in $\mathbb{R}^{n+1}$ and, therefore, Euclidean complete by Proposition 1.8. $\qquad \square$

Remark: The previous proposition extends [CDL, Prop. 5].

Theorem 2.5. *Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a projective special real manifold endowed with the centroaffine metric g. Then $(\mathcal{H}, g)$ is complete if and only if the subset $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed.*

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[1] The number of connected components of a real algebraic set is finite, see [Mi] and references therein.

Proof: In view of Proposition 2.4 (i), it suffices to show that a closed projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ is complete. We can assume without loss of generality that $\mathcal{H}$ is connected, that is a component of the level set $\{h = 1\}$ of a homogeneous cubic polynomial. By Proposition 2.4 (ii), $\mathcal{H}$ is Euclidean complete and is therefore a Euclidean complete connected hyperbolic centroaffine hypersurface as considered in Lemma 1.15. The unique homogeneous function of degree $k = 3$ on $U = \mathbb{R}^{>0} \cdot \mathcal{H}$ which has the value 1 on $\mathcal{H}$ coincides with the restriction of the polynomial h to U. To prove the completeness we will apply Lemma 1.15 in the case $k = 3$, $\varepsilon = 1$. Thus we have to show that the function $\sqrt{h}|_{U \cap E}$ is concave, where $E := E_p$ is the tangent hyperplane at some point $p \in \mathcal{H}$. Since $U \cap E \subset E$ is relatively compact (see Lemma 1.14), for every $x \in U \cap E$ and $v \in T_p\mathcal{H}$ there exists $-\infty < a < b < \infty$ such that the line $x + \mathbb{R}v \subset E$ intersects the domain $U \cap E$ in the bounded segment

$$\{x + tv \mid a < t < b\}.$$

We consider the polynomial function $h_0 : \mathbb{R} \to \mathbb{R}$ defined by $h_0(t) := h(x + tv)$. It suffices to check that $\sqrt{h_0}'' \le 0$ on (a, b). We compute

$$4h_0^{3/2}\sqrt{h_0}'' = 2h_0 h_0'' - (h_0')^2$$

and

$$(2h_0 h_0'' - (h_0')^2)' = 2h_0 h_0'''.$$

Since h_0''' is constant, this shows that the function $f_0 := 2h_0 h_0'' - (h_0')^2$ is monotone on (a, b). Observing that $f_0(a) = -(h_0'(a))^2 \le 0$ and $f_0(b) = -(h_0'(b))^2 \le 0$, we see that $f_0 \le 0$ on (a, b). This proves that $\sqrt{h_0}'' \le 0$ on (a, b). $\square$

2.3   Applications

Theorem 2.6.   *Let h be a cubic homogeneous polynomial on $\mathbb{R}^{n+1}$ and $\mathcal{H}$ a locally strictly convex (i.e. definite) component of the level set $\{h = 1\}$. Then $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a complete projective special real manifold, which defines a complete quaternionic Kähler manifold of negative scalar curvature diffeomorphic to $\mathbb{R}^{4n+8}$ by applying first the r-map and then the c-map.*

Remark: Notice that the components of the hypersurface

$$\{x \in \mathbb{R}^{n+1} | h(x) = 1 \text{ and } g_x \text{ is definite}\}$$

are locally strictly convex but are not necessarily components of the level set $\{h = 1\}$. In fact, they are in general not closed in the ambient space and therefore incomplete.

Proof: We claim that the centroaffine hypersurface $\mathcal{H}$ is hyperbolic. Assume it is elliptic.

If $n \geq 2$ and $\mathcal{H}$ is compact, then Hopf's characterization of ovaloids [H, p. 122] implies $\mathbb{R}^{>0} \cdot \mathcal{H} = \mathbb{R}^{n+1} \setminus \{0\}$. Thus h is positive on $\mathbb{R}^{n+1} \setminus \{0\}$ and has odd degree, a contradiction.

If $n \geq 2$ and $\mathcal{H}$ is noncompact, then the Stoker–Wu theorem [W] yields an element $A \in \mathrm{GL}(n+1)$ such that $\mathcal{H}' := A(\mathcal{H}) \subset \mathbb{R}^n \times \mathbb{R}$ is the graph of a strictly convex function $f \colon \Omega \to \mathbb{R}$, where Ω is a convex open subset of $\mathbb{R}^n$ and f achieves its minimum at some $x_0 \in \Omega$. (To apply the theorem, we used that $\mathcal{H}$ is closed in $\mathbb{R}^{n+1}$ and thus Euclidean complete by Proposition 1.8.) Ellipticity implies that $0 \in \mathbb{R}^{n+1}$ lies in the strict epigraph $C := \{(x, y) \mid x \in \Omega,\, y > f(x)\}$, which is convex and has boundary $\mathcal{H}'$. The convex set $C \cap (\mathbb{R}^n \times \{0\})$ is relatively compact in $\mathbb{R}^{n+1}$, as one sees easily by considering lines from $(x_0, f(x_0))$ to other points on $\mathcal{H}'$, taking the strict convexity of f near x_0 and the convexity of C into account. Thus every ray from 0 in $P := \mathbb{R}^n \times \{0\}$ meets $\mathcal{H}'$. We infer that the homogeneous polynomial $h \circ A^{-1}|_P$ is positive on $P \setminus \{0\}$ and has odd degree. This is again a contradiction.

If $n = 1$, then still $\mathcal{H}$, being a closed embedded centroaffine curve of elliptic type, is the boundary of a convex domain containing the origin in its interior. Therefore, $\mathcal{H} \subset \{h = 1\}$ intersects every line through the origin. The set $\{h = 0\}$ contains at least one such line, because h has odd degree. Once more, that is a contradiction.

Hence $\mathcal{H}$ is hyperbolic, as claimed. Now the completeness of $\mathcal{H}$ is a consequence of Theorem 2.5. According to [CHM, Thm. 4], the complete projective special real manifold $\mathcal{H}$ defines a complete projective special Kähler domain M of dimension $2n + 2$ by the r-map. The domain is diffeomorphic to TU, where $U = \mathbb{R}^{>0} \cdot \mathcal{H}$. By Proposition 1.10, U is diffeomorphic to a convex domain. Therefore M is diffeomorphic to $\mathbb{R}^{2n+2}$. Next, the complete projective special Kähler domain M defines a complete quaternionic Kähler manifold N of negative scalar curvature by the c-map, see [CHM, Thm. 5]. As a differentiable manifold, N is a product $M \times G$, where G is the solvable Iwasawa subgroup of $\mathrm{SU}(1, n + 3)$. The latter Lie group is diffeomorphic to $\mathbb{R}^{2n+6}$. $\qquad\square$

Theorem 2.7. *Let h be a cubic homogeneous polynomial on $\mathbb{R}^{n+1}$ and $\mathcal{H}$ a locally strictly convex component of the level set $\{h = 1\}$. Then*

$$g_L := -\frac{1}{3} \tilde{\nabla}^2 h$$

is a globally hyperbolic Lorentzian metric on the convex domain $U = \mathbb{R}^{>0} \cdot \mathcal{H}$.

Proof: As in the previous theorem, the assumptions imply that the centroaffine hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ is hyperbolic. Thus the result is an immediate consequence of Theorem 2.5 and Proposition 1.16. $\qquad\square$

2.4 An open problem

For each two natural numbers $n \geq 1$ and $k \geq 2$, one can consider the following statement:

$\mathcal{S}(n,k)$: *For every homogeneous polynomial h of degree k on $\mathbb{R}^{n+1}$, every locally strictly convex component $\mathcal{H}$ of the level set $\{h = 1\}$ is complete with respect to the centroaffine metric.*

Corollary 1.21 and Theorem 1.18 show that for all $n \geq 1$ and $k \geq 2$, the property in $\mathcal{S}(n,k)$ is true at least for generic polynomials.

As an immediate consequence of Theorem 2.5, $\mathcal{S}(n,k)$ is true for all $n \geq 1$ in the case $k \in \{2,3\}$:

Corollary 2.8. *Let h be a homogeneous polynomial of degree $k \in \{2,3\}$ on $\mathbb{R}^{n+1}$ and $\mathcal{H}$ a locally strictly convex component of the level set $\{h = 1\}$. Then $\mathcal{H} \subset \mathbb{R}^{n+1}$ is complete with respect to the centroaffine metric.*

Proof: The case $k = 2$ is trivial since in that case the tensor field $-\frac{1}{2}\tilde{\nabla}^2 h$ on $\mathbb{R}^{n+1}$ inducing the centroaffine metric is constant. The case $k = 3$ is part of Theorem 2.5. $\quad\square$

Moreover, $\mathcal{S}(1,k)$ is true for every $k \geq 2$:

Theorem 2.9. *Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $k \geq 2$ and $\mathcal{H}$ a locally strictly convex connected component of $\{h = 1\}$. Then $\mathcal{H}$ is complete with respect to the centroaffine metric.*

Proof: Since everything is invariant under linear unimodular transformations we can assume that $U = \{x, y > 0\}$, i.e. U is the first quadrant in the plane.

Choose a smooth curve S in the first quadrant transversal to the position vector field such that its closure connects the points $(0,1)$ and $(1,0)$ and is parallel to the x-axis near $(0,1)$ and parallel to the y-axis near $(1,0)$. Recall from Lemma 1.12 that

$$-\psi^*(\tilde{\nabla}^2 h) = -\frac{1}{h}\tilde{\nabla}^2 h + \frac{k-1}{kh^2} dh \otimes dh$$

on $S \cap U$ for the map $\psi(x,y) = \frac{1}{\sqrt[k]{h(x,y)}}(x,y)$.

For the question of completeness we are only interested in the behaviour of $-\psi^*(\tilde{\nabla}^2 h)\big|_S$ near $(1,0)$ and $(0,1)$. Again by the invariance under linear unimodular transformations we only need to consider the problem near $(0,1)$. We want to apply the method of Lemma

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1.15 for $\varepsilon = 1$. Therefore we have to show that $\left(\sqrt[k-1]{h}\big|_S \right)'' \leq 0$ near $(0,1)$. Since S is parallel to the x-axis in this area the claim follows from

$$0 \geq (k-1)h^{\frac{2k-3}{k-1}}\frac{\partial^2}{\partial x^2}\left(\sqrt[k-1]{h} \right) = \frac{2-k}{k-1}\left(\frac{\partial h}{\partial x} \right)^2 + h\frac{\partial^2 h}{\partial x^2}.$$

Note that the right-hand side is polynomial so we only need to consider the monomial $x^l y^{k-l}$ with l minimal appearing in h. We know that $1 \leq l \leq k-1$ since h vanishes on both the x- and the y-axis. Further note that the respective coefficient of $x^l y^{k-l}$ is positive since $h|_U > 0$. Then we have

$$\frac{2-k}{k-1}\left(lx^{l-1} \right)^2 + x^l l(l-1)x^{l-2} = \frac{x^{2l-2}}{k-1}\left(l^2 - l(k-1) \right) \leq 0$$

for $x \geq 0$. Now we can use the method of Lemma 1.15 on S near $(0,1)$ and the theorem follows. $\qquad\square$

If we consider instead of polynomials the larger class of analytic functions, Theorem 2.9 becomes false, as the following counterexample shows:

Example: Let $k > 1$. The homogeneous function

$$h(x,y) = \left(\frac{xy}{x+y} \right)^k$$

is real analytic (and even rational if k is an integer) on the quadrant $U := \{x > 0, y > 0\}$. The hypersurface $\mathcal{H} = \{p \in U \mid h(p) = 1\}$ is obviously closed in $\mathbb{R}^2$, Euclidean complete and can be parametrized by

$$\varphi : B \to \mathcal{H}, \quad p \mapsto h(p)^{-1/k}p,$$

where B is the intersection of U with the line $\{x + y = 1\}$. According to Corollary 1.13, in this parametrization, the centroaffine metric is computed from $u = \sqrt[k]{h}\big|_B = xy\big|_B = x(1-x)$ by

$$g = -\frac{1}{u}\tilde{\nabla}^2 u = \frac{2dx^2}{x(1-x)}.$$

The centroaffine length of the curve $\mathcal{H}$ is thus

$$\sqrt{2}\int_0^1 \frac{dx}{\sqrt{x(1-x)}} < \infty.$$

Since $\mathcal{H}$ is symmetric with respect to the axis $y = x$, this implies that $\mathcal{H}$ is incomplete.

Open problem 2.10. *Given $n \geq 2$ and $k \geq 4$, decide whether the statement $\mathcal{S}(n,k)$ is true.*

Note that it is not possible to generalize the proof of Corollary 2.8, which is based on Lemma 1.15, to any $k \geq 4$. In order to do that, one would have to prove that there exists a constant $c < \frac{k-1}{k}$ such that

$$c\eta'(x)^2 - \eta(x)\eta''(x) \geq 0$$

holds for all $x \in [0, 1]$ and all polynomials $\eta \colon \mathbb{R} \to \mathbb{R}$ of degree $\leq k$ which

(A) satisfy $\eta(0) = \eta(1) = 0$ and are positive on the interval $(0, 1)$,

(B) satisfy $\frac{k-1}{k}(\eta')^2 - \eta\eta'' > 0$ on $(0, 1)$.

Then for $\epsilon = \frac{k}{1-c}\left(\frac{k-1}{k} - c\right)$, the computations in the proof of Lemma 1.15 would show that for each affine line L in E_p which meets U, the function $\eta_L := h|_L$ (which is a polynomial of degree $\leq k$ because h is a polynomial of degree k, and which has an affine reparametrization $\eta \colon \mathbb{R} \to \mathbb{R}$ with $\eta(0) = \eta(1) = 0$ such that $\eta > 0$ and $\frac{k-1}{k}(\eta')^2 - \eta\eta'' \geq 0$ hold on $(0, 1)$) makes $\sqrt[k]{\eta_L}$ concave on $L \cap U$. This would imply the completeness: $\mathcal{S}(n, k)$ would be true for the considered k and all $n \geq 1$.

For $k = 3$, the proof of Theorem 2.5 shows that $c = \frac{1}{2}$ works. For $k = 2$, it is easy to see that $c = 0$ works. Unfortunately, a constant c with the desired property does not exist for $k \geq 4$. The following example demonstrates this in the case $k = 4$:

Example: For each $a \in \mathbb{R}^{\geq 0}$, the fourth-order polynomial

$$\eta_a(x) := x(1 - x)\left(\left(x - \tfrac{3}{20}\right)^2 + \tfrac{51}{20^2} + a\right)$$

is obviously positive on $(0, 1)$ and vanishes at 0 and 1. We consider

$$P_a(x) := \tfrac{3}{4}\eta_a'(x)^2 - \eta_a(x)\eta_a''(x)$$
$$= \frac{3\left(14x^2 + 6x - 3\right)^2}{40^2} + \frac{-80x^4 + 188x^3 - 42x^2 - 24x + 9}{40}a + \frac{4x^2 - 4x + 3}{4}a^2.$$

As $14x^2 + 6x - 3$ has precisely one zero in the interval $[0, 1]$, namely $x_0 := -\frac{3}{14} + \frac{1}{14}\sqrt{51} \approx 0.2958$, $P_0|_{[0,1]}$ is nonnegative and vanishes precisely at x_0.

Since the polynomial $Q := -80x^4 + 188x^3 - 42x^2 - 24x + 9$ is positive at x_0, namely $Q(x_0) \approx 2.479$, there exists an $a_0 \in \mathbb{R}^{>0}$ such that for every $a \in (0, a_0]$, P_a is positive on $[0, 1]$. (One can even check that Q is positive on $[0, 1]$. Since also $4x^2 - 4x + 3 = 2(x - 1)^2 + 2x^2 + 1$ is positive, P_a is therefore positive on $[0, 1]$ for every $a > 0$.)

Thus η_a satisfies (A) and (B) with $k = 4$ for all $a \in (0, a_0]$. Assume that there exists a constant $c < \frac{3}{4}$ such that $c\eta_a'(x)^2 \geq \eta_a(x)\eta_a''(x)$ holds for all $a \in (0, a_0]$ and $x \in [0, 1]$.

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By continuity, $c\eta_0'(x)^2 \geq \eta_0(x)\eta_0''(x)$ would hold for all $x \in [0,1]$, in particular for x_0. We would obtain $c\eta_0'(x_0)^2 \geq \eta_0(x_0)\eta_0''(x_0) = \frac{3}{4}\eta_0'(x_0)^2$, hence $\eta_0'(x_0) = 0$. But that is false:

$$\eta_0'(x_0) = -4x_0^3 + \tfrac{39}{10}x_0^2 - \tfrac{9}{10}x_0 + \tfrac{3}{20} \approx 0.1215.$$

Thus for $k = 4$, there is no constant $c < \frac{k-1}{k}$ with the desired property described above.

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