Locally homogeneous nearly Kähler manifolds

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October 24, 2014

Abstract

We construct locally homogeneous 6-dimensional nearly Kähler manifolds as quotients of homogeneous nearly Kähler manifolds M by freely acting finite subgroups of $\mathrm{Aut}_0(M)$. We show that non-trivial such groups do only exists if $M = S^3 \times S^3$. In that case we classify all freely acting subgroups of $\mathrm{Aut}_0(M) = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ of the form $A \times B$, where $A \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $B \subset \mathrm{SU}(2)$.

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Introduction

Recall that an almost Hermitian manifold (M, g, J) without nontrivial Kähler local de Rham factor is called (strict) nearly Kähler if $(\nabla_X J)X = 0$ for all $X \in TM$, where ∇ denotes the Levi-Civita connection. It was shown by Nagy [N] that all complete simply connected nearly Kähler manifolds are products of twistor spaces of quaternionic Kähler manifolds of positive scalar curvature, homogeneous spaces and six-dimensional nearly Kähler manifolds.

According to Butruille [B1, B2] there exist only 4 examples of 6-dimensional homogeneous nearly Kähler manifolds $M = G/K$:

1. the sphere $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$,
2. the complex projective space $\mathbb{C}P^3 = \mathrm{Sp}(2)/(\mathrm{U}(1) \times \mathrm{Sp}(1))$,
3. the flag manifold $F_{1,2}(\mathbb{C}^3) = \mathrm{SU}(3)/(\mathrm{U}(1) \times U(1))$,
4. the Lie group $S^3 \times S^3 = \mathrm{SU}(2)^3/\Delta(\mathrm{SU}(2))$, where $\Delta : \mathrm{SU}(2) \hookrightarrow \mathrm{SU}(2)^3$ is the diagonal embedding.

To our knowledge, these exhaust all examples of 6-dimensional nearly Kähler manifolds which have occurred in the literature so far. Incidentally, the second and third examples are precisely the twistor spaces of the 4-dimensional quaternionic Kähler manifolds of positive scalar curvature. Each of these four homogeneous spaces $M = G/K$ is a 3-symmetric space and $G = \mathrm{Aut}_0(M)$ is the maximal connected group of automorphisms of the nearly Kähler structure. The latter statement follows from [GM, Theorem 5.3], which uses [T, Theorem 3.6].

In this paper we are interested in six-dimensional nearly Kähler manifolds M for which the universal covering $\tilde{M}$ is homogeneous. Such manifolds will be called locally homogeneous nearly Kähler manifolds in the following. The classification of these manifolds amounts to the description of the finite subgroups $\Gamma \subset \mathrm{Aut}(\tilde{M})$ which act freely on $\tilde{M}$, for each of the 4 (simply connected) homogeneous nearly Kähler manifolds $\tilde{M}$ from Butruille's list. For simplicity, we will only consider subgroups Γ of $G = \mathrm{Aut}_0(\tilde{M})$. The corresponding locally homogeneous nearly Kähler manifolds M are the quotients $M = \tilde{M}/\Gamma = \Gamma\backslash G/K$, by the natural left-action of $\Gamma \subset G$ on $\tilde{M} = G/K$.

The next proposition shows that it is sufficient to consider the case $\tilde{M} = S^3 \times S^3$, in which case we will classify certain classes of freely acting groups of automorphisms in the main part of the paper.

Proposition 0.1. *Let M be a homogeneous nearly Kähler manifold such that $G = \mathrm{Aut}_0(M)$ admits a nontrivial subgroup acting freely on M. Then the nearly Kähler manifold M is isomorphic to $S^3 \times S^3$.*

Proof. Any element $\gamma \in G$ is contained in some maximal torus T of G. If M is not isomorphic to $S^3 \times S^3$ then the stabilizer $K \subset G$ of a point $o \in M$ is of maximal rank and, hence, contains a maximal torus T_0 of G. Since any two maximal tori are conjugate, there exists an element $a \in G$ such that $aTa^{-1} = T_0$. This implies that $p = a^{-1}o \in M$ is a fixed point of γ. This shows that G does not contain any nontrivial subgroup Γ acting freely on M. □

From now on we consider the case $M = S^3 \times S^3 = G/K$, where $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $K = \Delta(\mathrm{SU}(2)) \subset G$. Notice that the nearly Kähler structure on M can be considered as a left-invariant structure on the Lie group $L = \mathrm{SU}(2) \times \mathrm{SU}(2) = M$. Let L act by left-translations as a subgroup of $G = \mathrm{Aut}_0(M)$, where the inclusion is simply $(a, b) \mapsto (a, b, 1)$. Since the action of $L = \mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \{1\} \subset G$ by left-translations on M is free, any finite subgroup $\Gamma \subset L$ gives rise to a locally homogeneous nearly Kähler manifold M/Γ.

Our main results amount to the classification of all finite subgroups of $G = \mathrm{Aut}_0(M) = L \times \mathrm{SU}(2)$ acting freely on M that (up to a permutation of the three factors of G) are of the form $A \times B$, for some finite subgroups $A \subset L$, $B \subset \mathrm{SU}(2)$. In addition, we classify all finite simple groups $\Gamma \subset G$ that act freely on M, see Theorem 2.2 in Section 2. We refer to Theorem 3.5 in Section 3 for the description of finite subgroups $\Gamma_1 \times \Gamma_2 \times \Gamma_3 \subset G$ acting freely on M that are products of groups $\Gamma_i \subset \mathrm{SU}(2)$ for $i = 1, 2, 3$, and to Theorems 4.3, 4.4, 4.7, 4.8 in Section 4, for the remaining groups. This yields a wealth of new examples of nearly Kähler manifolds.

Acknowledgements: This work was supported by the Collaborative Research Center SFB 676 "Particles, Strings, and the Early Universe" of the Deutsche Forschungsgemeinschaft.

1 Goursat's Theorem

Finite subgroups of the product $G_1 \times G_2$ of two abstract groups are described by Goursat's theorem, see e.g. [FD]. We give the proof of the theorem as it is needed in the sequel.

Theorem 1.1 (Goursat's theorem)**.** *Let G_1, G_2 be groups. There is a one-to-one correspondence between subgroups $C \subset G_1 \times G_2$ and quintuples $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$, where $A_0 \lhd A \subset G_1$, $B_0 \lhd B \subset G_2$ and $\theta : {}^{A}/_{A_0} \longrightarrow {}^{B}/_{B_0}$ is an isomorphism.*

Proof. Let $C \subset G_1 \times G_2$ be a subgroup and denote by $\pi_i : G_1 \times G_2 \longrightarrow G_i$, $i = 1, 2$, the natural projections. Set $A = \pi_1(C) \subset G_1$, $B = \pi_2(C) \subset G_2$, $A_0 = \mathrm{Ker}(\pi_2|_C)$ and $B_0 = \mathrm{Ker}(\pi_1|_C)$. It is readily seen that A_0 and B_0 can be identified with normal subgroups of A and B respectively. We denote these subgroups again by A_0 and B_0. Define a map $\tilde{\theta} : A \longrightarrow {}^{B}/_{B_0}$ as follows. For $a \in A$ pick any $b \in B$ so that $(a, b) \in C$ and set $\tilde{\theta}(a) := bB_0$. One can check that this map is well-defined and factorizes through an isomorphism $\theta : {}^{A}/_{A_0} \longrightarrow {}^{B}/_{B_0}$. This defines a map $C \mapsto \mathcal{Q}(C)$.

Conversely, a quintuple $Q = \{A, A_0, B, B_0, \theta\}$ as described above defines a group $C = \mathcal{G}(Q) \subset G_1 \times G_2$ by setting $C = p^{-1}(\Gamma(\theta))$, where $p : A \times B \longrightarrow {}^{A}\!/\!_{A_0} \times {}^{B}\!/\!_{B_0}$ is the natural homomorphism and $\Gamma(\theta) \subset {}^{A}\!/\!_{A_0} \times {}^{B}\!/\!_{B_0}$ denotes the graph of the homomorphism θ. Observe that $C \subset G_1 \times G_2$ is in fact a fiber product,

$$C = \{(a,b) \in A \times B \ : \ \theta(aA_0) = bB_0\} = \{(a,b) \in A \times B |\, \alpha(a) = \beta(b)\}, \quad (1.1)$$

where

$$\alpha : A \longrightarrow {}^{A}\!/\!_{A_0} \xrightarrow{\ \theta\ } {}^{B}\!/\!_{B_0} \qquad \text{and} \qquad \beta : B \longrightarrow {}^{B}\!/\!_{B_0}$$

are the natural homomorphisms. The maps $\mathcal{Q}$ and $\mathcal{G}$ are inverse to each other. □

Proposition 1.2. *Two subgroups $C, C' \subset G_1 \times G_2$ with corresponding quintuples $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$, $\mathcal{Q}(C') = \{A', A'_0, B', B'_0, \theta'\}$ are conjugate if and only if there exists $(g_1, g_2) \in G_1 \times G_2$ such that $A' = g_1 A g_1^{-1}, B' = g_2 B g_2^{-1}, A'_0 = g_1 A_0 g_1^{-1}, B'_0 = g_2 B_0 g_2^{-1}$ and the diagram*

$$\begin{array}{ccc} A/A_0 & \xrightarrow{\ \theta\ } & B/B_0 \\ {\scriptstyle c(g_1)}\downarrow & & \downarrow{\scriptstyle c(g_2)} \\ A'/A'_0 & \xrightarrow[\ \theta'\ ]{} & B'/B'_0 \end{array}$$

commutes. Where $c(g_i)$ denotes conjugation by $g_i \in G_i$, $i = 1, 2$.

Remark. Sometimes we will consider different subgroups $C = \mathcal{G}(A, A_0, B, B_0, \theta) \subset G_1 \times G_2$ for fixed A, A_0, B, B_0. In that case it is convenient to identify A/A_0 and B/B_0 with the same abstract group F and consider $\theta : A/A_0 \to B/B_0$ as an automorphism of F.

In the remaining parts of this paper, we classify (up to conjugation) finite subgroups $C \subset G$ acting freely on the nearly Kähler manifold M, which are either simple or of the form $D \times E \subset G$, for $D \subset \mathrm{SU}(2)^2$ and $E \subset \mathrm{SU}(2)$ arbitrary. This motivates the following definition.

Definition 1.3. A finite subgroup $C \subset G = \mathrm{SU}(2)^3$ is said to be *splittable* whenever $C = A_1 \times A_2 \times A_3 \subset G$ for some non-trivial subgroups $A_i \subset \mathrm{SU}(2)$ for $i = 1, 2, 3$, and *semi-splittable* if $C = D \times E$ for some non-trivial subgroups $D \subset \mathrm{SU}(2)^2$, $E \subset \mathrm{SU}(2)$. In addition, a semi-splittable group $C \subset G$ is said to be *strict* if it is not splittable.

We are excluding the case of trivial factors in the above definition because the occurrence of a trivial factor implies that C acts freely on M, as mentioned in the introduction.

2 Simple groups

The Lemma below will help us to distinguish subgroups of $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ that act freely on $M = S^3 \times S^3$.

Lemma 2.1. *A subgroup $C \subset G$ acts non freely on M if and only if there is a non-trivial element $(a_1, a_2, a_3) \in C$, so that*

$$\mathrm{Re}(a_1) = \mathrm{Re}(a_2) = \mathrm{Re}(a_3).$$

Proof. Consider the action of $\mathrm{SU}(2) = S^3 \subset \mathbb{H} = \mathbb{R}^4$ on itself given by conjugation. The orbit of a given unit quaternion $1 \neq a = \mathrm{Re}(a) + \mathrm{Im}(a)$ is then of the form

$$\{\mathrm{Re}(a)\} \times S^2(\rho) \subset \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4 \quad , \quad \rho = \sqrt{1 - \mathrm{Re}(a)^2}. \tag{2.1}$$

Let $(a_1, a_2, a_3) \in C$ be a non-trivial element fixing a class $[(a, b, c)] \in M$, i.e. so that $(a, b, c)^{-1}(a_1, a_2, a_3)(a, b, c) = (w, w, w) \in K$. As the real part of a quaternion is invariant under conjugation, we have $\mathrm{Re}(a_1) = \mathrm{Re}(a_2) = \mathrm{Re}(a_3) =$

Re(w). Conversely, suppose there is an element $(1,1,1) \neq (a_1,a_2,a_3) \in C$ with: Re(a_1) = Re(a_2) = Re(a_3). Then, by relation (2.1) the orbits of $a_1,a_2,a_3 \in \mathrm{SU}(2)$ are the same. □

Let us recall now that any finite subgroup of SU(2) is conjugate to one of the so-called ADE groups, see e.g. Theorem 1.2.4 in [To]. These groups are described in terms of generators as follows.

Label	Name	Order	Generators
$\mathbb{A}_{n-1}$	$\mathbb{Z}_n$	n	$e^{\frac{i2\pi}{n}}$
$\mathbb{D}_{n+2}$	$2\mathrm{D}_{2n}$	$4n$	$j, e^{\frac{i\pi}{n}}$
$\mathbb{E}_6$	2T	24	$\frac{1}{2}(1+i)(1+j), \frac{1}{2}(1+j)(1+i)$
$\mathbb{E}_7$	2O	48	$\frac{1}{2}(1+i)(1+j), \frac{1}{\sqrt{2}}(1+i)$
$\mathbb{E}_8$	2I	120	$\frac{1}{2}(1+i)(1+j), \frac{1}{2}(\phi+\phi^{-1}i+j)$

Table 1: Finite subgroups of SU(2).

Here $n \geq 2$ and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. An element-wise description of the ADE groups is provided in Table 2.

Label	Name	Order	Elements
$\mathbb{A}_{n-1}$	$\mathbb{Z}_n$	n	$\{e^{\frac{2\pi i x}{n}} \ : \ x = 0,\ldots,n-1\}$
$\mathbb{D}_{n+2}$	$2\mathrm{D}_{2n}$	$4n$	$\{e^{\frac{i\pi x}{n}} : x = 0,\ldots,2n-1\} \cup j\{e^{\frac{i\pi x}{n}} : x = 0,\ldots,2n-1\}$
$\mathbb{E}_6$	2T	24	$2\mathrm{D}_4 \cup \left\{\frac{\pm1\pm i\pm j\pm k}{2}\right\}$
$\mathbb{E}_7$	2O	48	$2\mathrm{T} \cup e^{\frac{i\pi}{4}}\, 2\mathrm{T}$
$\mathbb{E}_8$	2I	120	$2\mathrm{T} \cup q\, 2\mathrm{T} \cup q^2\, 2\mathrm{T} \cup q^3\, 2\mathrm{T} \cup q^4\, 2\mathrm{T}$

Table 2: Element description for ADE groups.

Where $q = \frac{1}{2}(\phi+\phi^{-1}i+j)$.

Theorem 2.2. *The following are, up to conjugation and permutation of the factors, the only non-trivial simple subgroups $C \subset G$ acting freely on M.*

(a) $\mathbb{Z}_p \times \{1\} \times \{1\}$.

(b) $\Gamma(\varphi(r)) \times \{1\}$, *where* $\varphi(r) \in \mathrm{Aut}(\mathbb{Z}_p)$, *see Section 3.2.*

(c) $C(p,r,s) = \{(x, \varphi(r)x, \varphi(s)x) \ : x \in \mathbb{Z}_p\}$, *where* $\varphi(r), \varphi(s)$ *are automorphisms of* $\mathbb{Z}_p$ *and either* $r \neq \pm 1 \mod p$, *or* $s \neq \pm 1 \mod p$.

Here $p \in \mathbb{Z}$ *is an arbitrary prime number.*

Proof. Let $C \subset G$ be a non trivial simple subgroup and $j \in \{1,2,3\}$ be so that $\pi_j(\Gamma) \neq \{1\}$, where π_j denotes the natural projection. Because C is simple, the taken projection restricts to an isomorphism $(\pi_j)|_C : C \longrightarrow \pi_j(C) \subset G$. However, groups of type DE are non commutative and have non trivial center, whilst groups of type A are commutative and non simple, unless they are of prime order. Consequently, $C \subset G$ has the isomorphism type of $\mathbb{Z}_p$ for some fixed prime number p. We distinguish between the following cases

(a) Let $\pi_1(C) \neq \{1\}$ and $\pi_2(C) = \pi_3(C) = \{1\}$. That is, $C = A \times \{1\} \times \{1\}$ for some finite subgroup $A \subset \mathrm{SU}(2)$. Because $\pi_1|_C$ an isomorphism, the group $C \subset G$ is conjugate to $\mathbb{Z}_p \times \{1\} \times \{1\}$.

(b) Let $\pi_1(C)$, $\pi_2(C) \neq \{1\}$ and $\pi_3(C) = \{1\}$. In particular $C = D \times \{1\} \subset G$ for some finite subgroup $D \subset \mathrm{SU}(2)^2$. Denote by $\mathcal{Q}(D) = \{A, A_0, B, B_0, \theta\}$ the quintuple defining $D \subset \mathrm{SU}(2)^2$, see Theorem 1.1. By construction of $\mathcal{Q}(D)$ and the simplicity of C, we see that $A_0 = B_0 = \{1\}$ and (up to conjugation) $A = B = \mathbb{Z}_p \subset \mathrm{SU}(2)$, and so, the isomorphism θ can be realized as an automorphism of $\mathbb{Z}_p$. It follows that $C \subset G$ is conjugate to $\Gamma(\theta) \times \{1\}$.

(c) Let $\pi_j(C) \neq \{1\}$ for $j = 1,2,3$, $G_1 = \mathrm{SU}(2)^2$ and $G_2 = \mathrm{SU}(2)$. The subgroup $C \subset G_1 \times G_2$ determines the quintuple $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$, where

$$\begin{aligned}
A =& \{(a,b) \in G_1 \;:\; (a,b,c) \in C \text{ for some } c \in \mathrm{SU}(2)\},\\
B =& \{c \in G_2 \;:\; (a,b,c) \in C \text{ for some } c \in \mathrm{SU}(2)\},\\
A_0 =& \{(a,b) \in A \;:\; (a,b,1) \in C\},\\
B_0 =& \{c \in B \;:\; (1,1,c) \in C\},
\end{aligned}$$

and $\theta : {}^{A}\!/\!{}_{A_0} \to {}^{B}\!/\!{}_{B_0}$ is an isomorphism. Using the simplicity of $C \subset G$, we can easily check that A is simple, $A_0 = \{(1,1)\}$ and $B_0 = \{1\}$. Just as in (b), we conclude that (possibly after conjugation) $A = \Gamma(\varphi) \subset G_1$ for some $\varphi \in \mathrm{Aut}(\mathbb{Z}_p)$. The isomorphism θ can be thus realized as an automorphism of $\mathbb{Z}_p$, and so, the group $C \subset G$ is conjugate to a group of the form $\{(x, \varphi(x), \psi(x)) : x \in \mathbb{Z}_p\}$, where φ, ψ are automorphism of $\mathbb{Z}_p$.

We are now left to decide using Lemma 2.1, which of the groups described above act freely on M. To this end, observe that the groups $\mathbb{Z}_p \times \{1\} \times \{1\}$ and $\Gamma(\varphi(r)) \times \{1\}$ act freely on M, so we can suppose $C = C(p,r,s) \subset G$ for integers $r, s \in \mathbb{Z}$ so that $r, s \nmid p$. Lemma 2.1 tells us that the group $C(p,r,s) \subset G$ acts freely on M precisely if the system

$$(1 \pm r)x = 0 \mod p\ ,\ rx = \pm sx \mod p \tag{2.2}$$

admits just the trivial solution. □

3 Splittable groups

In this section we will classify freely acting splittable subgroups $A_1 \times A_2 \times A_3 \subset G = \mathrm{SU}(2)^3$. The correspondence given in Theorem 1.1 together with our knowledge of ADE groups will be used in order to construct all relevant subgroups $C \subset \mathrm{SU}(2)^2$. Thereafter, to verify using Lemma 2.1 when a given subgroup $A_1 \times A_2 \times A_3 \subset G$ acts in the desired fashion amounts to solve certain kinds of equations on integers, such as in (2.2), to which we devote the forthcoming section.

3.1 Integral equations

To begin with, we summarize all real parts of elements in ADE groups.

Label	Name	Real parts
$\mathbb{A}_{n-1}$	$\mathbb{Z}_n$	$\cos\left(\frac{2\pi x}{n}\right) \quad 0 < x \leq n$
$\mathbb{D}_{n+2}$	$2\mathrm{D}_{2n}$	$0, \cos\left(\frac{\pi x}{n}\right) \quad 0 < x \leq 2n$
$\mathbb{E}_6$	2T	$0, \pm 1, \pm\frac{1}{2}$
$\mathbb{E}_7$	2O	$0, \pm 1, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{1}{2\sqrt{2}}$
$\mathbb{E}_8$	2I	$0, \pm 1, \pm\frac{1}{2}, \frac{\pm 1 \pm \sqrt{5}}{4}$

Table 3: Real parts of elements in ADE groups.

We are interested in solving the following type of equations on integers:

Case I $ax + by = c$, for $a, b \in \mathbb{Z} \setminus \{0\}, c \in \mathbb{Z}$. This case corresponds to a linear Diophantine equation which can be completely solved using Bezout's lemma, see e.g. Theorems 2.1.1 and 2.1.2 in [AAC].

Theorem 3.1. *Let $a, b \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{Z}$. The equation*

$$ax + by = c \tag{3.1}$$

is solvable if and only if $\gcd(a,b)|c$, in which case its general solution reads

$$(x, y) = (x_0, y_0) + t(b, -a) \quad t \in \mathbb{Z}. \tag{3.2}$$

Here $(x_0, y_0) \in \mathbb{Z}^2$ is a particular solution of equation (3.1).

Case II $\cos\left(\frac{2\pi x}{n}\right) = b,\ b \in \left\{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{m}} \ :\ m \in \mathbb{N}\right\}$. We determine first all possible values of $m \in \mathbb{N}$ so that $b = \pm\frac{1}{\sqrt{m}}$ can be written as a cosine of some rational multiple of 2π, by making use of the following standard result in algebraic number theory, see e.g. Section 3 of [J].

Theorem 3.2. *Let θ be a rational multiple of 2π. If $\cos(\theta) \in \mathbb{Q}$, then $\cos(\theta) \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$.*

In our situation, it follows that $m \in \{1,2,4\}$, and so, $b \in \mathbb{R}$ belongs to the following list of values: $0, \pm 1, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}$. Table 4 displays the restrictions on $n \in \mathbb{Z}$ in order to get an integral solution $x(n) \in \mathbb{Z}$ for the equation $\cos\left(\frac{2\pi x}{n}\right) = b$.

b	$x(n) \in \mathbb{Q}$	Restriction on n
-1	$\frac{n}{2} + nk$	$2 \mid n$
$-\frac{1}{\sqrt{2}}$	$\frac{3n}{8} + nk, \frac{5}{8}n + nk$	$8 \mid n$
$-\frac{1}{2}$	$\frac{n}{3} + nk$, $\frac{2n}{3} + nk$	$3 \mid n$
0	$\frac{n}{4} + nk$, $\frac{3}{4}n + nk$	$4 \mid n$
$\frac{1}{2}$	$\frac{n}{6} + nk$, $\frac{5n}{6} + nk$	$6 \mid n$
$\frac{1}{\sqrt{2}}$	$\frac{n}{8} + nk$, $\frac{7n}{8} + nk$	$8 \mid n$
1	nk	-

Table 4: Conditions on n.

<u>Case III</u> $\cos\left(\frac{2\pi x}{n}\right) = c,\ c \in \{a + \sqrt{5}b \ : \ a, b \in \mathbb{Q}^\times\}$. Let $c = a + b\sqrt{5}$ with $a, b \in \mathbb{Q}^\times$ and suppose $x(n) \in \mathbb{Z}$ solves the equation in consideration. From this setup, we can easily derive that $x(n) \in \mathbb{Z}$ satisfies the integral equation $\cos(2\theta) + B\cos(\theta) = C$, for $B = -4a$ and $C = 10b^2 - 1 - 2a^2$, and where $\theta = \frac{2\pi x}{n}$. The latter equation can be completely solved by means of Theorem 7 in [CJ], result which is stated in generality sufficient to our needs.

Theorem 3.3 (Conway, Jones)**.** *Suppose we have at most two distinct rational multiples of* π *lying strictly between* 0 *and* $\frac{\pi}{2}$ *for which some rational linear combination of their cosines is rational. Then the appropriate linear combination is proportional to either* $\cos(\pi/5) - \cos(2\pi/5) = \frac{1}{2}$ *or* $\cos(\pi/3) = \frac{1}{2}$.

Table 5 displays the additional values of $c \in \mathbb{Q}[\sqrt{5}]$ we must consider, together with the restrictions to impose on $n \in \mathbb{Z}$ in order to find an integral solution $x(n) \in \mathbb{Z}$ for the equation $\cos\left(\frac{2\pi x}{n}\right) = c$.

$c = a + b\sqrt{5}$	$x(n) \in \mathbb{Q}$	Restriction on n
$\frac{1}{4}(1+\sqrt{5})$	$\frac{n}{10}+kn, \frac{9n}{10}+kn$	$10 \mid n$
$\frac{1}{4}(-1+\sqrt{5})$	$\frac{n}{5}+kn, \frac{4n}{5}+kn$	$5 \mid n$
$\frac{1}{4}(1-\sqrt{5})$	$\frac{3n}{10}+kn, \frac{7n}{10}+kn$	$10 \mid n$
$-\frac{1}{4}(1+\sqrt{5})$	$\frac{2n}{5}+kn, \frac{3n}{5}+kn$	$5 \mid n$

Table 5: Conditions on n.

Remark. In the latter situation $\cos(\theta)$ is an algebraic number of degree 2. It is in fact a zero of the quadratic polynomial $(t-a)^2-5b^2$. The classification of algebraic numbers of low degree and sufficiently small length, see e.g. Section 5 of [J], provides an alternative argument to build Table 5.

3.2 Automorphisms of quotient groups

The present section comprises descriptions of automorphisms groups of quotients of ADE groups that are relevant in the forthcoming sections. The main reference is Section 6.2 in [FD]. This material will be however adapted to our needs.

(1) The group of outer automorphisms of $\mathbb{Z}_n$ is given by

$$\mathrm{Out}(\mathbb{Z}_n) = \{\varphi(r) \ : \ \gcd(r,n) = 1\},$$

where $\varphi(r)$ denotes the map $\mathbb{Z}_n \ni x \mapsto x^r \in \mathbb{Z}_n$ in multiplicative notation.

(2) To describe the outer automorphism group of a dihedral group D_{2n}, consider the following presentation of D_{2n}

$$\mathrm{D}_{2n} = \left\langle x, y \ : \ x^2 = y^n = (xy)^2 = 1 \right\rangle = \{y^p : \ 0 \le p < n\} \cup \{xy^p : \ 0 \le p < n\}.$$

Observe $\mathrm{D}_2 = \mathbb{Z}_2$, so we can assume that $n > 1$. The case $n = 2$ is also special as D_4 is isomorphic to the Klein Vierergruppe. The automorphism group of D_4 isomorphic to $\mathrm{Sym}(3)$ and acts by permutations of the 3 non trivial involutions. The outer automorphism group of D_{2n} for $n > 2$ is

$$\mathrm{Out}(\mathrm{D}_{2n}) = \langle \tau_{a,b} \ : \ (a,b) \in \mathbb{Z}_n^\times \times \mathbb{Z}_n \rangle \cong \mathbb{Z}_n^\times \ltimes \mathbb{Z}_n,$$

where the action of the affine group $\mathbb{Z}_n^\times \ltimes \mathbb{Z}_n$ on D_{2n} is given by

$$\tau_{a,b}(y^p) = y^{ap} \quad , \quad \tau_{a,b}(xy^p) = xy^{ap+b}.$$

(Here $\mathbb{Z}_n$ denotes the additive group and $\mathbb{Z}_n^\times$ the multiplicative group of units in the ring $\mathbb{Z}_n$.)

(3) Since $2\mathrm{D}_2 \cong \mathbb{Z}_4$, we consider $2\mathrm{D}_{2n}$ only for $n > 1$. We have the following presentation:

$$2\mathrm{D}_{2n} = \left\langle s, t : \; s^2 = t^n = (st)^2 \right\rangle = \{t^p : \; 0 \le p < 2n\} \cup \{st^p \; : \; 0 \le p < 2n\}.$$

In fact, we can take $s = je^{i\frac{\pi}{n}}$ and $t = e^{i\frac{\pi}{n}}$ when $2\mathrm{D}_{2n}$ is realized as a subgroup of $\mathrm{SU}(2)$. The outer automorphism group of $2\mathrm{D}_{2n}$ for $n > 2$ is also an affine group:

$$\mathrm{Out}(2\mathrm{D}_{2n}) = \langle \tau_{a,b} \; : \; (a,b) \in \mathbb{Z}_{2n}^\times \times \mathbb{Z}_{2n} \rangle \cong \mathbb{Z}_{2n}^\times \ltimes \mathbb{Z}_{2n},$$

where the action on $2\mathrm{D}_{2n}$ is given by

$$\tau_{a,b}(t^p) = t^{ap} \quad , \quad \tau_{a,b}(st^p) = st^{ap+b}.$$

We need to make a distinction for $n = 2$. Any automorphism of $2\mathrm{D}_4 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathrm{SU}(2)$ is obtained via conjugation with an element in $2\mathrm{O}$ modulo $\mathbb{Z}_2 = \{\pm 1\}$. The point-wise action of ${}^{2\mathrm{O}}/_{\mathbb{Z}_2}$ on $2\mathrm{D}_4$ is described below.

	i	j	k		i	j	k
$[i]$	i	$-j$	$-k$	$[\frac{1}{\sqrt{2}}(1-i)]$	i	$-k$	j
$[j]$	$-i$	j	$-k$	$[\frac{1}{\sqrt{2}}(j+k)]$	$-i$	k	j
$[k]$	$-i$	$-j$	k	$[\frac{1}{\sqrt{2}}(j-k)]$	$-i$	$-k$	$-j$
$[\frac{1}{2}(1+i+j+k)]$	j	k	i	$[\frac{1}{\sqrt{2}}(i+k)]$	k	$-j$	i
$[\frac{1}{2}(1-i-j-k)]$	k	i	j	$[\frac{1}{\sqrt{2}}(1-k)]$	$-j$	i	k
$[\frac{1}{2}(1+i-j-k)]$	$-j$	k	$-i$	$[\frac{1}{\sqrt{2}}(i-k)]$	$-k$	$-j$	$-i$
$[\frac{1}{2}(1+i+j-k)]$	$-k$	i	$-j$	$[\frac{1}{\sqrt{2}}(i+j)]$	j	i	$-k$
$[\frac{1}{2}(1-i+j-k)]$	$-j$	$-k$	i	$[\frac{1}{\sqrt{2}}(1+j)]$	$-k$	j	i
$[\frac{1}{2}(1-i-j+k)]$	j	$-k$	$-i$	$[\frac{1}{\sqrt{2}}(1-j)]$	k	j	$-i$
$[\frac{1}{2}(1-i+j+k)]$	$-k$	$-i$	j	$[\frac{1}{\sqrt{2}}(1+k)]$	j	$-i$	k
$[\frac{1}{2}(1+i-j+k)]$	k	$-i$	$-j$	$[\frac{1}{\sqrt{2}}(i-j)]$	$-j$	$-i$	$-k$
$[\frac{1}{\sqrt{2}}(1+i)]$	i	k	$-j$				

Table 6: Action of $2\mathrm{O}/\mathbb{Z}_2$ on $2\mathrm{D}_4$.

(4) The tetrahedral group T is isomorphic to the alternating group Alt(4), which has automorphism group Sym(4), acting by conjugation on the normal subgroup Alt(4). This corresponds to the action of the octahedral group $\mathrm{O} \cong \mathrm{Sym}(4)$ on its normal subgroup T, which is induced by the action of 2O on the normal subgroup 2T. In fact, it can be derived from Table 6 that the image of $\mathrm{O} = 2\mathrm{O}/\mathbb{Z}_2$ in $\mathrm{Aut}(\mathrm{T}) = \mathrm{Aut}(2\mathrm{T}/\mathbb{Z}_2)$ is isomorphic to Sym(4).

(5) Every automorphism of O is inner.

(6) The icosahedral group I is isomorphic to Alt(5), which is generated by $s = (12)(34)$ and $t = (135)$. Observe these generators satisfy $s^2 = t^3 = (st)^5 = (1)$. The automorphism group of Alt(5) is Sym(5), whilst the outer automorphism group is isomorphic to $\mathbb{Z}_2$. In terms of permutations, the latter is generated by conjugation with an odd permutation, say (35). This sends the generators listed above to (12)(45) and (153) respectively. The action of this automorphism φ on conjugacy classes is described below.

Representative	Size
(1)	1
(123)	20
(12345)	12
(13452)	12
(12)(34)	15

Table 7: Conjugacy classes.

$\mathcal{C}(xyzvw)$	$\varphi(\mathcal{C}(xyzvw))$
$\mathcal{C}(1)$	$\mathcal{C}(1)$
$\mathcal{C}(123)$	$\mathcal{C}(123)$
$\mathcal{C}(12345)$	$\mathcal{C}(13452)$
$\mathcal{C}(13452)$	$\mathcal{C}(12345)$
$\mathcal{C}(12)(34)$	$\mathcal{C}(12)(34)$

Table 8: Action of φ.

Where $\mathcal{C}(xyzvw)$ is the conjugacy class of a permutation $(xyzvw) \in \mathrm{Alt}(5)$.

(7) The outer automorphism group of $2\mathrm{T} \subset \mathrm{SU}(2)$ is generated by an involution that exchanges the generators $s = \frac{1}{2}(1+i)(1+j), t = \frac{1}{2}(1+j)(1+i)$, which satisfy the relations $s^3 = t^3 = (st)^3$. This automorphism is given by conjugation with $\frac{1+j}{\sqrt{2}} \in 2\mathrm{O} \subset \mathrm{SU}(2)$.

(8) The outer automorphism group of $2\mathrm{O} \subset \mathrm{SU}(2)$ is generated by an involution φ fixing s and sending t to $-t$, where $s = \frac{1}{2}(1+i+j+k)$ and $t = e^{\frac{i\pi}{4}}$ generate 2O.

(9) The outer automorphism group of $2\mathrm{I} \subset \mathrm{SU}(2)$ is generated by an involution ψ which fixes s and sends t to $\frac{-\phi^{-1}-\phi i+k}{2}$, where $s = \frac{1}{2}(1+i+j+k)$ and $t = \frac{\phi+\phi^{-1}i+j}{2}$ generate 2I.

The action of the automorphisms $\varphi \in \mathrm{Out}(2\mathrm{O})$ and $\psi \in \mathrm{Out}(2\mathrm{I})$ on conjugacy classes $\mathcal{C}(x)$, for $x \in 2\mathrm{O}$ or $x \in 2\mathrm{I}$ respectively, is described in the following tables.

Representative	Size	Real parts
1	1	1
-1	1	-1
s	8	$\frac{1}{2}$
t	6	$\frac{1}{\sqrt{2}}$
s^2	8	$-\frac{1}{2}$
t^2	8	0
t^3	6	$-\frac{1}{\sqrt{2}}$
st	12	0

Table 9: Conjugacy classes in 2O .

$\mathcal{C}(x)$	$\varphi(\mathcal{C}(x))$	$\mathrm{Re}(\varphi(x))$
$\mathcal{C}(1)$	$\mathcal{C}(1)$	1
$\mathcal{C}(-1)$	$\mathcal{C}(-1)$	-1
$\mathcal{C}(s)$	$\mathcal{C}(s)$	$\frac{1}{2}$
$\mathcal{C}(t)$	$\mathcal{C}(t^3)$	$-\frac{1}{\sqrt{2}}$
$\mathcal{C}(s^2)$	$\mathcal{C}(s^2)$	$-\frac{1}{2}$
$\mathcal{C}(t^2)$	$\mathcal{C}(t^2)$	0
$\mathcal{C}(t^3)$	$\mathcal{C}(t)$	$\frac{1}{\sqrt{2}}$
$\mathcal{C}(st)$	$\mathcal{C}(st)$	0

Table 10: Action of φ.

Representative	Size	Real parts
1	1	1
-1	1	-1
t	12	$\frac{1+\sqrt{5}}{4}$
t^2	12	$-\frac{1-\sqrt{5}}{4}$
t^3	12	$\frac{1-\sqrt{5}}{4}$
t^4	12	$-\frac{1+\sqrt{5}}{4}$
s	20	$\frac{1}{2}$
s^4	20	$-\frac{1}{2}$
st	30	0

Table 11: Conjugacy classes in 2I.

$\mathcal{C}(x)$	$\psi(\mathcal{C}(x))$	$\mathrm{Re}(\psi(x))$
$\mathcal{C}(1)$	$\mathcal{C}(1)$	1
$\mathcal{C}(-1)$	$\mathcal{C}(-1)$	-1
$\mathcal{C}(t)$	$\mathcal{C}(t^3)$	$\frac{1-\sqrt{5}}{4}$
$\mathcal{C}(t^2)$	$\mathcal{C}(t^4)$	$-\frac{1+\sqrt{5}}{4}$
$\mathcal{C}(t^3)$	$\mathcal{C}(t)$	$\frac{1+\sqrt{5}}{4}$
$\mathcal{C}(t^4)$	$\mathcal{C}(t^2)$	$-\frac{1-\sqrt{5}}{4}$
$\mathcal{C}(s)$	$\mathcal{C}(s)$	$\frac{1}{2}$
$\mathcal{C}(s^4)$	$\mathcal{C}(s^4)$	$-\frac{1}{2}$
$\mathcal{C}(st)$	$\mathcal{C}(st)$	0

Table 12: Action of ψ.

3.3 Freely acting splittable groups

We regard splittable groups $A_1 \times A_2 \times A_3 \subset G$ with non-trivial factors $A_i \subset \mathrm{SU}(2)$, for $i = 1, 2, 3$. The following technical result shall be used in the proof of Theorem 3.5.

Lemma 3.4. *Let $n, m \geq 2$ be integers, $k = \gcd(m, n)$ and $m_1 = \frac{m}{k}, n_1 = \frac{n}{k}$. Then, the solution set of*

$$\cos\left(\frac{2\pi x}{n}\right) = \cos\left(\frac{2\pi y}{m}\right) \ , \ (x, y) \in \mathbb{Z}^2,$$

is given by $\{(nq+\varepsilon\ell n_1, \ell m_1) : q,\ell \in \mathbb{Z}, \varepsilon = \pm 1\}$.

Proof. Let $(x,y) \in \mathbb{Z}^2$ be a solution of either of the following equations

$$mx = ny \mod mn \qquad\qquad mx = -ny \mod mn.$$

say the first. In particular, there is an integer $q \in \mathbb{Z}$, so that $m_1 x - n_1 y = mn_1 q$. Theorem 3.1 tells us that $(x,y) \in ((nq,0) + \mathbb{Z}(n_1,m_1))$. The converse of the assertion is straightforward. □

Theorem 3.5. *Any splittable subgroup acting freely on* M*, is up to permutation of the three factors and conjugation in* $G = \mathrm{SU}(2)^3$*, one in the following list.*

Group	Conditions
$\mathbb{Z}_n \times 2\mathrm{I} \times 2\mathrm{I}$	$2,3,5 \nmid n$
$\mathbb{Z}_n \times 2\mathrm{O} \times 2\mathrm{I}$ $\mathbb{Z}_n \times 2\mathrm{O} \times 2\mathrm{O}$ $\mathbb{Z}_n \times 2\mathrm{T} \times 2\mathrm{O}$ $\mathbb{Z}_n \times 2\mathrm{T} \times 2\mathrm{I}$ $\mathbb{Z}_n \times 2\mathrm{T} \times 2\mathrm{T}$	$2,3 \nmid n$
$\mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_l$	$\gcd(n,m,l) = 1$
$\mathbb{Z}_n \times \mathbb{Z}_m \times 2\mathrm{D}_{2l}$	$\gcd(n,m,2l) = 1$
$\mathbb{Z}_n \times 2\mathrm{D}_{2m} \times 2\mathrm{D}_{2l}$	$\gcd(n,2m,2l) = 1$
$\mathbb{Z}_n \times \mathbb{Z}_m \times 2\mathrm{T}$ $\mathbb{Z}_n \times \mathbb{Z}_m \times 2\mathrm{O}$	$2,3 \nmid \gcd(n,m)$
$\mathbb{Z}_n \times 2\mathrm{D}_{2m} \times 2\mathrm{T}$ $\mathbb{Z}_n \times 2\mathrm{D}_{2m} \times 2\mathrm{O}$ $\mathbb{Z}_n \times 2\mathrm{D}_{2m} \times 2\mathrm{I}$	$2,3 \nmid \gcd(n,2m)$
$\mathbb{Z}_n \times \mathbb{Z}_m \times 2\mathrm{I}$	$2,3,5 \nmid \gcd(n,m)$

Table 13: Freely acting splittable groups

Proof. Observe first that a freely acting group $A_1 \times A_2 \times A_3 \subset G$ must have a cyclic factor $\mathbb{Z}_n$ of odd order n, simply because otherwise $(-1-1-1) \in A_1 \times A_2 \times A_3 \subset G$, see Table 3, situation which is not allowed by Lemma 2.1.

A splittable group $\mathbb{Z}_n \times A_2 \times A_3 \subset G$ acts freely on M precisely if $\mathrm{Re}(A_2) \cap \mathrm{Re}(A_3) \cap \mathrm{Re}(\mathbb{Z}_n) = \{1\}$. However, since n is odd, we always have

$$\{-1, 0, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{4}(1 \pm \sqrt{5})\} \cap \mathrm{Re}(\mathbb{Z}_n) = \varnothing,$$

see Table 4 and 5. Moreover, we see that under the additional divisibility assumptions, which can be read off from these tables, also the numbers $-\frac{1}{2}$, $\frac{1}{4}(-1 \pm \sqrt{5})$ are not contained in $\mathrm{Re}(\mathbb{Z}_n)$. This observation gives us already all the freely acting splittable groups $\mathbb{Z}_n \times A_2 \times A_3 \subset G$ with $A_2, A_3 \subset \mathrm{SU}(2)$ of type E satisfying the conditions displayed in Table 13.

As for the cases involving at least one factor A_2 or $A_3 \subset \mathrm{SU}(2)$ of type AD, we observe that $\mathrm{Re}(\mathbb{Z}_n) \cap \mathrm{Re}(2\mathrm{D}_{2m}) = \mathrm{Re}(\mathbb{Z}_{\gcd(n,2m)}), \mathrm{Re}(\mathbb{Z}_n) \cap \mathrm{Re}(\mathbb{Z}_m) = \mathrm{Re}(\mathbb{Z}_{\gcd(n,m)})$, which is a consequence of Lemma 3.4. Similar conclusions as before hold. □

4 Semi-splittable groups

We start the classification of freely acting semi-splittable subgroups $C \times D \subset G$ by considering the classes of such subgroups for which the quintuple $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$ defining $C \subset \mathrm{SU}(2)^2$ is so that $A, B \subset \mathrm{SU}(2)$ are non-trivial ADE groups and one of the following conditions holds.

(a) $A_0 = A$, $B_0 = B$.

(b) $A_0 = B_0 = \{1\}$.

(c) $A_0 \neq \{1\}$ and $B_0 = \{1\}$.

Note that condition (a) is equivalent to $C \times D \subset G$ being splittable. Since splittable groups acting freely on $M = S^3 \times S^3$ have already been classified in the previous sections, we will assume from now on that our group is strictly semi-splittable. Observe also that under the condition (b), the group C is conjugate to the graph $\Gamma(\varphi, A)$ of some automorphism φ of A.

Definition 4.1. A strictly semi-splittable group is said to be of type I or II if it satisfies condition (b) or (c) respectively, and of type III otherwise.

4.1 Type I groups

We regard groups $\Gamma(\varphi, A) \times B \subset G$, where $\Gamma(\varphi, A)$ is the graph of some automorphism $\varphi : A \to A$ of an ADE group $A \subset \mathrm{SU}(2)$ and $B \subset \mathrm{SU}(2)$ is some non trivial ADE group. The following technical result shall be used recurrently in the forthcoming sections.

Lemma 4.2. *Let $n \geq 2, r \in \mathbb{Z}_n^\times$, and set $k = \gcd(1+r, n), n_1 = \frac{n}{k}, r_1 = \frac{1+r}{k}$. Then, $n_1\mathbb{Z}$ is the solution set of the equation: $x = -rx \mod n$.*

Proof. Let $x \in \mathbb{Z}$ be a solution of the equation in question. There is in particular an integer $y \in \mathbb{Z}$, so that $(1+r)x + ny = 0$. If we consider the latter as an equation in $(x, y) \in \mathbb{Z}^2$, then $(x, y) = t(n_1, -r_1)$ for some $t \in \mathbb{Z}$, see Theorem 3.1. This shows the claim as any number $tn_1 \in \mathbb{Z}$ is a solution of: $x = -rx \mod n$. □

Theorem 4.3. *Let $\Gamma(\varphi, A) \times B \subset G$ be a freely acting type I group which is not a subgroup of a freely acting splittable group. Then, it is either $\Gamma(\varphi, 2\mathrm{I}) \times \mathbb{Z}_n$ with $3 \nmid n$, or one of the following groups.*

$\Gamma(\tau_{a,b}, 2\mathrm{D}_{2n}) \times B$, $n > 2$, $a \neq \pm 1 \mod 2n$			
$\tilde{k}_1 > 1$	$\tilde{k}_2 > 1$	B	Conditions
yes	*yes*	$\mathbb{Z}_m$	$4 \nmid m \ \wedge \ \gcd(\tilde{k}_1, m) = \gcd(\tilde{k}_2, m) = 1$
yes	*no*	$\mathbb{Z}_m$	$4 \nmid m \ \wedge \ \gcd(\tilde{k}_1, m) = 1$
no	*yes*	$\mathbb{Z}_m$	$4 \nmid m \ \wedge \ \gcd(\tilde{k}_2, m) = 1$
no	*no*	$\mathbb{Z}_m$	$4 \nmid m$

$\Gamma(\varphi(r), \mathbb{Z}_n) \times B, \quad r \neq \pm 1 \mod n$			
$k_1 > 1$	$k_2 > 1$	B	Conditions
yes	*yes*	$\mathbb{Z}_m$	$\gcd(m, k_1) = \gcd(m, k_2) = 1$
		$2\mathrm{D}_{2m}$	$\gcd(2m, k_2) = \gcd(2m, k_1) = 1$
		$2\mathrm{T}, 2\mathrm{O}$	$2, 3 \nmid k_1, k_2$
		$2\mathrm{I}$	$2, 3, 5 \nmid k_1, k_2$
yes	*no*	$\mathbb{Z}_m$	$\gcd(k_1, m) = 1$
		$2\mathrm{D}_{2m}$	$\gcd(k_1, 2m) = 1$
		$2\mathrm{T}, 2\mathrm{O}$	$2, 3 \nmid k_1$
		$2\mathrm{I}$	$2, 3, 5 \nmid k_1$
no	*yes*	$\mathbb{Z}_m$	$\gcd(k_2, m) = 1$
		$2\mathrm{D}_{2m}$	$\gcd(k_2, 2m) = 1$
		$2\mathrm{T}, 2\mathrm{O}$	$2, 3 \nmid k_2$
		$2\mathrm{I}$	$2, 3, 5 \nmid k_1$
no	*no*	All	-

Table 14: Type I freely acting subgroups.

Where $k_1 = \gcd(1 + r, n)$, $k_2 = \gcd(1 - r, n)$, $\tilde{k}_1 = \gcd(1 + a, 2n)$ *and* $\tilde{k}_2 = \gcd(1 - a, 2n)$.

Proof. Let $A \subset \mathrm{SU}(2)$ be an ADE group and define

$$\mathcal{W}(\varphi, A) = \{\mathrm{Re}(x) : \mathrm{Re}(x) = \mathrm{Re}(\varphi(x))\}.$$

If $\varphi \in \mathrm{Inn}(A)$, then $\mathcal{W}(\varphi, A) = \mathrm{Re}\,(A)$. In consequence, a group $\Gamma(\varphi, A) \times B \subset G$ with $\varphi \in \mathrm{Inn}(A)$ that acts freely on M must be a subgroup of a freely acting splittable group. We consider therefore just outer automorphisms of $A \subset \mathrm{SU}(2)$. Observe that once we have calculated $\mathcal{W}(\varphi, A) \subset \mathbb{R}$, the precise conditions to impose on $\Gamma(\varphi, A) \times B \subset G$ to make it act freely on M are easily obtained from Tables 3, 4 and 5 as in the case of splittable groups. For this reason, we just calculate $\mathcal{W}(\varphi, A)$ for any ADE group $A \subset \mathrm{SU}(2)$ and $\varphi \in \mathrm{Out}(A)$:

(a) let $A \subset \mathrm{SU}(2)$ be a type E group. Since $\mathrm{Out}(A)$ is in this case isomorphic to $\mathbb{Z}_2$, we must consider a single automorphism of the group $A \subset \mathrm{SU}(2)$.

Furthermore, as the only non-trivial class in Out(2T) is represented by an automorphism which can be obtained by conjugation with some element in SU(2), we are left with $A \in \{2\mathrm{O}, 2\mathrm{I}\}$. In these cases, we read from Tables 9-12 that

$$\mathcal{W}(\varphi, 2\mathrm{O}) = \mathcal{W}(\psi, 2\mathrm{I}) = \left\{1, -1, \frac{1}{2}, -\frac{1}{2}, 0\right\}.$$

(b) Let $A = \mathbb{Z}_n$ and $\varphi(r)$ be a non-trivial outer automorphism of $\mathbb{Z}_n$. Elements in $\mathcal{W}(\varphi(r), \mathbb{Z}_n)$ are readily seen to be obtained by solving

$$x = \pm rx \mod n. \tag{4.1}$$

Further, we can suppose $r \neq \pm 1 \mod n$, as otherwise $\mathrm{Re}(\mathbb{Z}_n) = \mathcal{W}(\varphi(r), \mathbb{Z}_n)$. If $\gcd(1 \pm r, n) = 1$, equation (4.1) admits only the trivial solution and, thus, any group $\Gamma(\varphi(r), \mathbb{Z}_n) \times B \subset G$ acts freely on M. Now, suppose that $k_1 = \gcd(1 + r, n) > 1$, $k_2 = \gcd(1 - r, n) = 1$ and write $n = k_1 n_1$, $1 + r = k_1 r_1$. The solution set of equation (4.1) is $n_1 \mathbb{Z}$, see Lemma 4.2, and so

$$\mathcal{W}(\varphi(r), \mathbb{Z}_n) = \left\{\cos\left(\frac{2\pi z}{k_1}\right) \ : \ z \in \mathbb{Z}\right\}.$$

If $k_1, k_2 > 1$, we find out that

$$\mathcal{W}(\mathbb{Z}_n, \varphi(r)) = \left\{\cos\left(\frac{2\pi z}{k_i}\right) \ : \ z \in \mathbb{Z},\ i = 1, 2\right\}.$$

(c) At last, consider $A = 2\mathrm{D}_{2n}$. Since automorphism of $2\mathrm{D}_4$ are obtained by conjugation with an element in $2\mathrm{O} \subset \mathrm{SU}(2)$, we can suppose that $n > 2$. Let $\tau_{a,b}$ be a non-trivial outer automorphism of $2\mathrm{D}_{2n}$. Since

$$\Gamma(\tau_{a,b}, 2\mathrm{D}_{2n}) = \left\{\left(e^{\frac{i\pi x}{n}}, e^{\frac{i\pi a x}{n}}\right), \left(je^{\frac{i\pi x}{n}}, je^{\frac{i\pi(a(x-1)+b+1)}{n}}\right) \ : \ x \in \mathbb{Z}\right\},$$

an element in $\mathcal{W}(2\mathrm{D}_{2n}, \tau_{a,b})$ is either zero or it is obtained by solving the equation: $x = \pm ax \mod 2n$. □

4.2 Type II groups

We regard groups $C \times D \subset G$ so that the quintuple $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$ defining $C \subset \mathrm{SU}(2)^2$ is so that $B_0 = \{1\}$ and $A \subset \mathrm{SU}(2)$ admits a quotient isomorphic (via θ) to some ADE group $B \subset \mathrm{SU}(2)$. As explained in the remark on page 5, we shall identify here A/A_0 with $F = B$ and consider θ as an automorphism of F. Tables 15 and 16 (which are borrowed from [FD]) display all non-trivial normal subgroups $A_0 \subset \mathrm{SU}(2)$ of an ADE group $A \subset \mathrm{SU}(2)$ and the isomorphism type of the corresponding quotient.

$A_0 \lhd A$	A/A_0
$\mathbb{Z}_k \lhd \mathbb{Z}_{kl}$	$\mathbb{Z}_l$
$\mathbb{Z}_{2k} \lhd 2\mathrm{D}_{2kl}$	D_{2l}
$\mathbb{Z}_{2k+1} \lhd 2\mathrm{D}_{2l(2k+1)}$	$2\mathrm{D}_{2l}$
$\mathbb{Z}_{2k+1} \lhd 2\mathrm{D}_{2(2k+1)}$	$\mathbb{Z}_4$
$2\mathrm{D}_{2k} \lhd 2\mathrm{D}_{4k}$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \lhd 2\mathrm{T}$	T

Table 15: Subgroups I.

$A_0 \lhd A$	A/A_0
$2\mathrm{D}_4 \lhd 2\mathrm{T}$	$\mathbb{Z}_3$
$\mathbb{Z}_2 \lhd 2\mathrm{O}$	O
$2\mathrm{D}_4 \lhd 2\mathrm{O}$	D_6
$2\mathrm{T} \lhd 2\mathrm{O}$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \lhd 2\mathrm{I}$	I

Table 16: Subgroups II.

Caveat. It should be stressed here that according to our convention, the group $2\mathrm{D}_{2n}$ is defined for $n > 1$.

Theorem 4.4. *Any type II freely acting group $C \times D \subset G$ that is not a subgroup of a freely acting splittable group belongs to the following list*

Group	Conditions
$\mathcal{G}(2\mathrm{T}, 2\mathrm{D}_4, \mathbb{Z}_3, \{1\}, \varphi(r)) \times \mathbb{Z}_n$	$3 \nmid n$
$\mathcal{G}(2\mathrm{T}, 2\mathrm{D}_4, \mathbb{Z}_3, \{1\}, \varphi(r)) \times 2\mathrm{D}_{2l}$	$3 \nmid 2l$
$\mathcal{G}(2\mathrm{D}_{4k}, 2\mathrm{D}_{2k}, \mathbb{Z}_2, \{1\}, \mathrm{Id}) \times D$ $\mathcal{G}(2\mathrm{O}, 2\mathrm{T}, \mathbb{Z}_2, \{1\}, \mathrm{Id}) \times D$ $\mathcal{G}(2\mathrm{D}_{2k}, \mathbb{Z}_{2k}, \mathbb{Z}_2, \{1\}, \mathrm{Id}) \times D$	–

$\mathcal{G}(\mathbb{Z}_{kl}, \mathbb{Z}_l, \mathbb{Z}_k, \{1\}, \varphi(r)) \times D,\ lr \neq \pm 1 \mod lk$			
$k_1 > 1$	$k_2 > 1$	D	Conditions
yes	*yes*	$\mathbb{Z}_m$ $2\mathrm{D}_{2m}$ $2\mathrm{T}, 2\mathrm{O}$ $2\mathrm{I}$	$\gcd(m, k_1) = \gcd(m, k_2) = 1$ $\gcd(2m, k_2) = \gcd(2m, k_1) = 1$ $2, 3 \nmid k_1, k_2$ $2, 3, 5 \nmid k_1, k_2$
yes	*no*	$\mathbb{Z}_m$ $2\mathrm{D}_{2m}$ $2\mathrm{T}, 2\mathrm{O}$ $2\mathrm{I}$	$\gcd(k_1, m) = 1$ $\gcd(k_1, 2m) = 1$ $2, 3 \nmid k_1$ $2, 3, 5 \nmid k_1$
no	*yes*	$\mathbb{Z}_m$ $2\mathrm{D}_{2m}$ $2\mathrm{T}, 2\mathrm{O}$ $2\mathrm{I}$	$\gcd(k_2, m) = 1$ $\gcd(k_2, 2m) = 1$ $2, 3 \nmid k_2$ $2, 3, 5 \nmid k_1$
no	*no*	All	-

$\mathcal{G}(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l}, \{1\}, c_g \circ \tau_{a,b}) \times D,$ $a(2k+1) \neq \pm 1 \mod 2l(2k+1),\ l > 2$			
$\tilde{k}_1 > 1$	$\tilde{k}_2 > 1$	D	Conditions
yes	*yes*	$\mathbb{Z}_m$	$\gcd(\tilde{k}_1, m) = \gcd(\tilde{k}_2, m) = 1 \ \wedge 4 \nmid m$
yes	*no*	$\mathbb{Z}_m$	$\gcd(\tilde{k}_1, m) = 1 \ \wedge 4 \nmid m$
no	*yes*	$\mathbb{Z}_m$	$\gcd(\tilde{k}_2, m) = 1 \ \wedge 4 \nmid m$
no	*no*	$\mathbb{Z}_m$	$4 \nmid m$

Table 17: Type II freely acting subgroups.

Where $k_1 = \gcd(1+lr, kl)$*,* $k_2 = \gcd(1-lr, kl)$*,* $\tilde{k}_1 = \gcd(1-a(2k+1), 2l(2k+1))$ *and* $\tilde{k}_2 = \gcd(1+a(2k+1), 2l(2k+1))$*.*

Proof. Let $C \times D \subset G$ be a freely acting type III group, $\mathcal{Q}(C) = \{A, A_0, B, \{1\}, \theta\}$ be the quintuple defining $C \subset \mathrm{SU}(2)^2$ and define $\mathcal{W}(C) = \{\mathrm{Re}(x)\ : \mathrm{Re}(x) = \mathrm{Re}(y)\ ,\ (x, y) \in C\}$. The following are the group triples (A, A_0, B) that give rise to a valid choice for $\mathcal{Q}(C)$, that is such that the quotient $B = A/A_0$ is an ADE group, see Tables 15 and 16.

B	(A, A_0)
$\mathbb{Z}_k$	$(\mathbb{Z}_{kl}, \mathbb{Z}_l)$
$\mathbb{Z}_4$	$(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1})$
$\mathbb{Z}_3$	$(2\mathrm{T}, 2\mathrm{D}_4)$
$\mathbb{Z}_2$	$(2\mathrm{D}_{4k}, 2\mathrm{D}_{2k})$
$\mathbb{Z}_2$	$(2\mathrm{O}, 2\mathrm{T})$
$\mathbb{Z}_2$	$(2\mathrm{D}_{2k}, \mathbb{Z}_{2k})$
$2\mathrm{D}_{2l}$	$(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1})$

Table 18: Triples (A, A_0, B).

We calculate $\mathcal{W}(C)$ case by case for triples (A, A_0, B) that are part of $\mathcal{Q}(C)$ in order to read off the conditions for $\mathcal{W}(C) \cap \mathrm{Re}(D) = \{1\}$. For later use we recall, see equation (1.1), that the group C is a fibered product associated with the maps $\alpha : A \to B$ and $\beta = \mathrm{Id}_B : B \to B$.

(a) Let $(A, A_0, B) = (\mathbb{Z}_{kl}, \mathbb{Z}_l, \mathbb{Z}_k)$ and $\theta = \varphi(r)$ some automorphism of $\mathbb{Z}_k$. We have that

$$C = \{([y]_{kl}, [ry]_k) \ : \ y \in \mathbb{Z}\},$$

for some $r \in \mathbb{Z}_k^\times$. Lemma 4.2 helps us determining the solution set of

$$(1 \pm lr)y = 0 \mod kl,$$

whenever $lr \neq \pm 1 \mod lk$. The latter condition can be assumed as otherwise $\mathcal{W}(C) = \mathrm{Re}(\mathbb{Z}_{kl}) = \mathrm{Re}(\mathbb{Z}_k)$ and $C \times D$ is then a subgroup of a freely acting splittable group $A \times B \times D$. We see that under this condition

$$\mathcal{W}(C) = \left\{\cos\left(\frac{2\pi x}{k_i}\right) \ : \ x \in \mathbb{Z}, \ i = 1, 2\right\}.$$

(b) Let $(A, A_0, B) = (2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, \mathbb{Z}_4)$. The map $\alpha : A \to B$ defining $\mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, \mathbb{Z}_4, \{1\}, \varphi(r)) \subset \mathrm{SU}(2)^2$ is given by

$$\alpha : z \mapsto \alpha(z) = \begin{cases} 1 & \text{if} \quad z = e^{\frac{i\pi x}{2k+1}}, \quad x = 0 \mod 2, \\ e^{\frac{i\pi r}{2}} & \text{if} \quad z = je^{\frac{i\pi x}{2k+1}}, \quad x = 0 \mod 2, \\ e^{i\pi r} & \text{if} \quad z = e^{\frac{i\pi x}{2k+1}}, \quad x = 1 \mod 2, \\ e^{\frac{3i\pi r}{2}} & \text{if} \quad z = je^{\frac{i\pi x}{2k+1}}, \quad x = 1 \mod 2. \end{cases}$$

It is not difficult to verify that $(e^{\frac{i\pi x}{2k+1}}, e^{\pi i}) \in C \subset \mathrm{SU}(2)^2$, for x odd and any choice of $r \in \mathbb{Z}_4^\times$, hence $-1 \in \mathcal{W}(C)$. Since $C \times D$ acts freely it follows that $-1 \notin \mathrm{Re}(D)$ and, hence, $D = \mathbb{Z}_n$ for n odd. Using $\mathcal{W}(C) \subset \mathrm{Re}(\mathbb{Z}_4) = \{0, \pm 1\}$, this easily implies that $C \times D \subset G$ is a subgroup of a freely acting splittable group $2\mathrm{D}_{2(2k+1)} \times \mathbb{Z}_4 \times \mathbb{Z}_n$.

(c) Let $(A, A_0, B) = (2\mathrm{T}, 2\mathrm{D}_4, \mathbb{Z}_3)$. The map defining $\mathcal{G}(2\mathrm{T}, 2\mathrm{D}_4, \mathbb{Z}_3, \{1\}, \varphi(r)) \subset \mathrm{SU}(2)^2$ is now given as follows

$$\alpha : z \mapsto \alpha(z) = \begin{cases} 1 & \text{if} \quad z \in 2\mathrm{D}_4, \\ e^{\frac{2\pi i r}{3}} & \text{if} \quad z \in \frac{-1+i+j+k}{2}\, 2\mathrm{D}_4, \\ e^{\frac{4\pi i r}{3}} & \text{if} \quad z \in \frac{1+i+j+k}{2}\, 2\mathrm{D}_4. \end{cases}$$

We can check that $\pm\frac{1}{2} \in \mathcal{W}(C)$ for any choice of $r \in \mathbb{Z}_3^\times$. It follows that $\mathcal{W}(C) = \left\{1, \pm\frac{1}{2}\right\} = \mathrm{Re}(\mathbb{Z}_3)$, and so no new freely acting group is obtained in this way.

(d) Let $(A, A_0, B) = (2\mathrm{D}_{4k}, 2\mathrm{D}_{2k}, \mathbb{Z}_2)$. We have

$$C = \{(e^{\frac{i\pi y}{k}}, 1), (je^{\frac{i\pi y}{k}}, 1), (e^{\frac{i\pi(2y+1)}{2k}}, -1), (je^{\frac{i\pi(2y+1)}{2k}}, -1) \ : \ y \in \mathbb{Z}\},$$

and hence $\mathcal{W}(C) = \{1\}$. We conclude that any group $\mathcal{G}(2\mathrm{D}_{4k}, 2\mathrm{D}_{2k}, \mathbb{Z}_2, \{1\}, \mathrm{Id}) \times D \subset G$ acts freely on M.

(e) Let $(A, A_0, B) = (2\mathrm{O}, 2\mathrm{T}, \mathbb{Z}_2)$. In this case,

$$\alpha : z \mapsto \alpha(z) = \begin{cases} 1 & \text{if} \quad z \in 2\mathrm{T}, \\ -1 & \text{if} \quad z \in e^{\frac{i\pi}{4}}\, 2\mathrm{T}. \end{cases}$$

Because $\mathrm{Re}(e^{\frac{i\pi}{4}}\, 2\mathrm{T}) = \{0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\}$, we have that $\mathcal{W}(C) = \{1\}$. Therefore, every subgroup $\mathcal{G}(2\mathrm{O}, 2\mathrm{T}, \mathbb{Z}_2, \{1\}, \mathrm{Id}) \times D \subset G$ acts freely on M.

(f) Let $(A, A_0, B) = (2\mathrm{D}_{2k}, \mathbb{Z}_{2k}, \mathbb{Z}_2)$. We have that

$$\alpha : z \mapsto \alpha(z) = \begin{cases} 1 & \text{if} \quad z = e^{\frac{i\pi x}{k}}, \\ -1 & \text{if} \quad z = je^{\frac{i\pi x}{k}}. \end{cases}$$

From which, we see that

$$C = \{(e^{\frac{\pi i x}{k}}, 1), (je^{\frac{\pi i x}{k}}, -1) \ : \ x \in \mathbb{Z}\},$$

and so $\mathcal{W}(C) = \{1\}$. Any subgroup $\mathcal{G}(2\mathrm{D}_{2k}, \mathbb{Z}_{2k}, \mathbb{Z}_2, \{1\}, \mathrm{Id}) \times D \subset G$ will then act freely on M.

(g) Let $(A, A_0, B) = (2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l})$ and $l > 2$. Since $2\mathrm{D}_{2l}$ is non-commutative, we must consider also inner automorphisms in the construction of our group $C \subset \mathrm{SU}(2)^2$. Denote by $c_{g(w)}$ the conjugation map by an element $g(w) \in \{e^{\frac{i\pi w}{l}}, je^{\frac{i\pi w}{l}} : w \in \mathbb{Z}\} \subset 2\mathrm{D}_{2l}$. The map defining $C \subset \mathrm{SU}(2)^2$ is given by

$$\alpha : z \mapsto \alpha(z) = \begin{cases} c_{g(w)}(e^{\frac{i\pi xa}{l}}) & \text{if} \quad z = e^{\frac{i\pi x}{l(2k+1)}}, \\ c_{g(w)}(je^{\frac{i\pi(a(x-1)+b+1)}{l}}) & \text{if} \quad z = je^{\frac{i\pi x}{l(2k+1)}}. \end{cases}$$

where $(a, b) \in \mathbb{Z}_{2l}^\times \times \mathbb{Z}_{2l}$. In particular, $C \subset \mathrm{SU}(2)^2$ equals

$$\{(e^{\frac{i\pi x}{l(2k+1)}}, c_{g(w)}(e^{\frac{i\pi xa}{l}})), (je^{\frac{i\pi y}{l(2k+1)}}, c_{g(w)}(je^{\frac{i\pi(a(y-1)+b+1)}{l}})) \ : \ x, y \in \mathbb{Z}\}.$$

An element in $\mathcal{W}(C)$ is thus either zero, or it is obtained from a solution of

$$(1 \pm a(2k+1))x = 0 \mod 2l(2k+1).$$

This situation that can be treated analogously as in case (a). At last, let $l = 2$ and $c_{\tilde{w}}$ be the conjugation map by an element $\tilde{w} \in 2\mathrm{O}$. We find that

$$C = \{(e^{\frac{i\pi x}{2(2k+1)}}, c_{\tilde{w}}(e^{\frac{i\pi x}{2}})), (je^{\frac{i\pi y}{2(2k+1)}}, c_{\tilde{w}}(je^{\frac{i\pi y}{2}})) \ : \ x, y \in \mathbb{Z}\},$$

and so, $\mathcal{W}(C) = \mathrm{Re}(\mathbb{Z}_4)$. No new freely acting subgroups are thus obtained. □

4.3 Type III groups

We seek now to distinguish freely acting type III groups $C \times D \subset G$. To begin with, we assert that such groups must fulfill rather restrictive conditions a priori.

Proposition 4.5. *Let $C \times D \subset G$ be a type III freely acting group and $C = \mathcal{G}(A, A_0, B, B_0, \theta) \subset \mathrm{SU}(2)^2$. Then (up to interchanging the roles of A and B)*

(a) $D = \mathbb{Z}_{2k+1}$ *or*

(b) $(A, A_0, B, B_0) \in \{(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l(2p+1)}, \mathbb{Z}_{2p+1}), (\mathbb{Z}_{3(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{T}, 2\mathrm{D}_4), (\mathbb{Z}_{(2k+1)l}, \mathbb{Z}_{2k+1}, \mathbb{Z}_{pl}, \mathbb{Z}_p), (\mathbb{Z}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{O}, 2\mathrm{T}), (\mathbb{Z}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{4p}, 2\mathrm{D}_{2p}), (\mathbb{Z}_{4(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2(2p+1)}, \mathbb{Z}_{2p+1}), (2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2(2p+1)}, \mathbb{Z}_{2p+1}), (2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, \mathbb{Z}_{4p}, \mathbb{Z}_p)\}$.

Proof. Let $C \times D \subset G$ be a type III freely acting subgroup and $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$ be the quintuple defining $C \subset \mathrm{SU}(2)^2$ via the homomorphisms α and β in equation (1.1). If $D \subset \mathrm{SU}(2)$ is a DE group or a cyclic group of even order, and the groups $A, B \subset \mathrm{SU}(2)$ belong to the following list: groups of type E, $\mathbb{Z}_n$ and $2\mathrm{D}_{2m}$, where n and m are powers of 2. A short glance at Tables 3, 15 and 16 reveals that $-1 \in A_0 \cap B_0 \cap D$. This can not happen if $C \times D \subset G$ acts freely on M, as otherwise $(-1, -1, -1) \in C \times D \subset G$. By listing the remaining possibilities according to Tables 15, 16 and Theorem 1.1, we encounter the necessity of fulfilling (at least) one of the conditions in the theorem if the group $C \times D$ acts freely. ☐

The technical result below will help us distinguishing type III freely acting groups of the forms stated in Proposition 4.5.

Lemma 4.6. *Let $m, n, p \geq 2$. The equation,*

$$\cos\left(\frac{2\pi x}{3n}\right) = c,$$

has a solution

(a) $x \in 1 + 3\mathbb{Z}$ *for* $c = \frac{1}{2}$, *precisely if* $6|(n-2)$ *or* $6|(n+2)$.

(b) $x \in 1 + 3\mathbb{Z}$ *for* $c = -\frac{1}{2}$, *precisely if* $3|(n-1)$ *or* $3|(n+1)$.

(c) $x \in 2 + 3\mathbb{Z}$ *for* $c = \frac{1}{2}$, *precisely if* $6|(n-4)$ *or* $6|(n+4)$.

(d) $x \in 2 + 3\mathbb{Z}$ *for* $c = -\frac{1}{2}$, *precisely if* $3|(n-2)$ *or* $3|(n+2)$.

Theorem 4.7. *For the groups $C = \mathcal{G}(A, A_0, B, B_0, \theta)$ considered below, which include the ones in part (b) of Proposition 4.5, we describe in each case all type III groups $C \times D \subset G$ that acts freely on M.*

(a) Let $C = \mathcal{G}(\mathbb{Z}_{kl}, \mathbb{Z}_k, \mathbb{Z}_{pl}, \mathbb{Z}_p, \varphi(r))$, then $C \times D$ belongs to the following list.

$\mathcal{G}(\mathbb{Z}_{kl}, \mathbb{Z}_k, \mathbb{Z}_{pl}, \mathbb{Z}_p, \varphi(r)) \times D$	
D	Conditions
$\mathbb{Z}_n$	$\gcd(n, ms) = 1$
$2\mathrm{D}_{2n}$	$\gcd(2n, ms) = 1$
$2\mathrm{T}, 2\mathrm{O}$	$2, 3 \nmid ms$
$2\mathrm{I}$	$2, 3, 5 \nmid ms$

Where the above conditions are required for both values of

$$m = m_\varepsilon = \gcd(p - \varepsilon kr, kl, lpk)),$$

for $\varepsilon = \pm 1$, and

$$s = \frac{kl}{\gcd(kl, lpk)}.$$

(b) Let $C = \mathcal{G}(\mathbb{Z}_{3n}, \mathbb{Z}_n, 2\mathrm{T}, 2\mathrm{D}_4, \varphi(r))$ and assume that $C \times D$ is not a subgroup of a splittable freely acting group. Then it is either $\mathcal{G}(\mathbb{Z}_{3n}, \mathbb{Z}_n, 2\mathrm{T}, 2\mathrm{D}_4, \varphi(r)) \times \mathbb{Z}_{2p+1}$ with n even, subject to the conditions:

$$3 \nmid (n-1), (n+1), (n+2), (n-2),$$

or one of the following

$\mathcal{G}(\mathbb{Z}_{3n}, \mathbb{Z}_n, 2\mathrm{T}, 2\mathrm{D}_4, \varphi(r)) \times D, \quad 2 \nmid n$			
$6\|(n \pm 2) \vee 6\|(n \pm 4)$	$3\|(n \pm 1) \vee 3\|(n \pm 2)$	D	Conditions
yes	*yes*	$\mathbb{Z}_m$ $2\mathrm{D}_{2m}$	$3 \nmid m$ $3 \nmid m$
yes	*no*	$\mathbb{Z}_m$ $2\mathrm{D}_{2m}$	$6 \nmid m$ $3 \nmid m$
no	*yes*	$\mathbb{Z}_m$ $2\mathrm{D}_{2m}$	$3 \nmid m$ $3 \nmid m$
no	*no*	All	-

Table 19: Type III freely acting subgroups.

(c) Let $C = \mathcal{G}(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l(2p+1)}, \mathbb{Z}_{2p+1}, c_{g(w)} \circ \tau_{a,b})$ *be such that* $C \times D$ *is not a subgroup of a freely acting splittable group. Then* $l > 2$ *and*

$$C \times D = \mathcal{G}(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l(2p+1)}, \mathbb{Z}_{2p+1}, \tau_{a,b}) \times \mathbb{Z}_{m'}$$

subject to the conditions

$$4 \nmid m' \quad \wedge \quad \gcd(m', ms) = 1.$$

Here the latter condition is required for both values of

$$m = m_\varepsilon = \gcd(2p + 1 - \varepsilon(2k + 1)a, 2l(2k + 1), 2l(2p + 1)(2k + 1)),$$

for $\varepsilon = \pm 1$*, and*

$$s = \frac{2l(2k + 1)}{\gcd(2l(2k + 1), 2l(2p + 1)(2k + 1))}.$$

(d) Let $C = \mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, \mathbb{Z}_{4p}, \mathbb{Z}_p, \varphi(r))$ *be such that* $C \times D$ *is not a subgroup of a freely acting splittable group. Then the latter belongs to the following table.*

$\mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, \mathbb{Z}_{4p}, \mathbb{Z}_p, \varphi(r)) \times D, \quad p$ even	
D	Conditions
$\mathbb{Z}_m, 2\mathrm{D}_{2m}$ $2\mathrm{T}, 2\mathrm{O}$ $2\mathrm{I}$	$\gcd(2k + 1, p, m) = 1$ $2, 3 \nmid \gcd(2k + 1, p)$ $2, 3, 5 \nmid \gcd(2k + 1, p)$

(e) Let $C = \mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2(2p+1)}, \mathbb{Z}_{2p+1}, \varphi(r))$*. Then* $C \times D$ *is a subgroup of a splittable group that acts freely on* M*.*

(f) Let $C = \mathcal{G}(\mathbb{Z}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{O}, 2\mathrm{T}, \mathrm{Id})$*. Then either* $3 \nmid (2k+1)$*, in which case any group* $C \times D$ *acts freely on* M*, or in case* $3 | (2k+1)$*, we have* $D \in \{\mathbb{Z}_m, 2\mathrm{D}_{2m}\}$ *subject to the condition* $3 \nmid m$*.*

(g) Let $C = \mathcal{G}(\mathbb{Z}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{4p}\, 2\mathrm{D}_{2p}, \mathrm{Id})$*. Then* $C \times D$ *is a subgroup of a freely acting splittable group.*

Proof. It suffices to figure out sufficient elements in

$$\mathcal{W}(C) = \{\mathrm{Re}(x) \ : \ \mathrm{Re}(x) = \mathrm{Re}(y), (x, y) \in C\} \subset \mathbb{R}$$

so we can read from them the precise conditions to impose on $C \times D$ to act freely on M.

(a) The maps defining $C = \mathcal{G}(\mathbb{Z}_{kl}, \mathbb{Z}_k, \mathbb{Z}_{pl}, \mathbb{Z}_p, \varphi(r)) \subset \mathrm{SU}(2)^2$ are given by

$$\alpha : \mathbb{Z}_{kl} \ni [x]_{kl} \mapsto [rx]_l \in \mathbb{Z}_l \ , \qquad \beta : \mathbb{Z}_{pl} \ni [y]_{pl} \mapsto [y]_l \in \mathbb{Z}_l.$$

In particular, $C = \{(e^{\frac{2\pi i x}{kl}}, e^{\frac{2\pi i (rx+yl)}{pl}}) \ : \ y \in \mathbb{Z}\}$. To determine $\mathcal{W}(C)$ we must solve the congruence system

$$px = \varepsilon k(rx + yl) \mod lpk$$

separately for $\varepsilon \in \{-1, +1\}$. This leads to the following diophantine equation in three variables

$$\frac{(p - \varepsilon kr)}{m} x - \frac{\varepsilon kl}{m} y + \frac{lpk}{m} z = 0,$$

where $m = \gcd(p - \varepsilon kr, kl, lpk)$. Dividing this equation by $\frac{lk}{m}$ and setting $x = gx'$ for $x' \in \mathbb{Z}$, where $g = \gcd(\frac{kl}{m}, \frac{lpk}{m})$, we obtain the equation

$$\frac{(p - \varepsilon kr)}{kl} gx' - \varepsilon y + pz = 0,$$

which can be easily solved by considering it as an inhomogeneous equation in the variables y and z. The general solution is given by

$$x = gx',$$
$$y = \varepsilon\frac{p - \varepsilon kr}{kl}gx' + \lambda p,$$
$$z = \varepsilon\lambda,$$

where the parameters x' and λ are integers. Therefore

$$\mathcal{W}(C) = \mathrm{Re}(\mathbb{Z}_{ms}), \quad s = \frac{kl}{\gcd(kl, lpk)}.$$

(b) The maps defining the group $C = \mathcal{G}(\mathbb{Z}_{3n}, \mathbb{Z}_n, 2\mathrm{T}, 2\mathrm{D}_4, \varphi(r)) \subset \mathrm{SU}(2)^2$ are given below.

$$\alpha : e^{\frac{2\pi i x}{3n}} \mapsto e^{\frac{2\pi i r x}{3}}, \qquad \beta : z \mapsto \beta(z) = \begin{cases} 1 & \text{if } z \in 2\mathrm{D}_4 \\ e^{\frac{2\pi i}{3}} & \text{if } z \in \frac{-1+i+j+k}{2}\,2\mathrm{D}_4 \\ e^{\frac{4\pi i}{3}} & \text{if } z \in \frac{1+i+j+k}{2}\,2\mathrm{D}_4\,. \end{cases}$$

Thus, for $r = 1$ we get

$$C = \Big\{(e^{\frac{2\pi i x}{n}}, z_0), (e^{\frac{2\pi i(1+3y)}{3n}}, z_1), (e^{\frac{2\pi i(2+3z)}{3n}}, z_2) \;:\; z_0 \in 2\mathrm{D}_4,\ z_1 \in \frac{-1+i+j+k}{2}2\mathrm{D}_4,$$
$$z_2 \in \frac{1+i+j+k}{2}2\mathrm{D}_4,\ x, y, z \in \mathbb{Z}\Big\},$$

whereas for $r = 2$,

$$C = \Big\{(e^{\frac{2\pi i x}{n}}, z_0), (e^{\frac{2\pi i(1+3y)}{3n}}, z_1), (e^{\frac{2\pi i(2+3z)}{3n}}, z_2) \;:\; z_0 \in 2\mathrm{D}_4,\ z_1 \in \frac{1+i+j+k}{2}2\mathrm{D}_4,$$
$$z_2 \in \frac{-1+i+j+k}{2}2\mathrm{D}_4,\ x, y, z \in \mathbb{Z}\Big\},$$

For any value of $r \in \{1, 2\}$, 0 or -1 is an element in $\mathcal{W}(C)$ precisely if $4|n$ or $2|n$ respectively. Moreover, to verify if either $\frac{1}{2} \in \mathcal{W}(C)$ or $-\frac{1}{2} \in \mathcal{W}(C)$ amounts to solve each of the following equations,

$$\cos\left(\frac{2\pi(1+3x)}{3n}\right) = \pm\frac{1}{2}\ ,\ \cos\left(\frac{2\pi(2+3y)}{3n}\right) = \pm\frac{1}{2}.$$

Our analysis must distinguish according to the parity of n. For n even, the element -1 lies in $\mathcal{W}(C)$ for $r = 1, 2$, which implies $D = \mathbb{Z}_{2p+1}$ in order to have a group acting freely. In such case $\frac{1}{2} \notin \mathcal{W}(C) \cap \mathrm{Re}(D)$, see Table 4. Observe also that if $-\frac{1}{2} \in \mathcal{W}(C)$ and the group $C \times D$ acts freely on M, then similarly $3 \nmid (2p+1)$, and so the former group will be a subgroup of the freely acting splittable group $A \times B \times D$. In the opposite case, i.e. $-\frac{1}{2} \notin \mathcal{W}(C)$, we get a new freely acting semi-splittable group, see Lemma 4.6.

Now, in case n is odd, then $\mathcal{W}(C) \subset \{1, \pm\frac{1}{2}\}$. The precise elements in $\mathcal{W}(C)$ can be read off from Lemma 4.6 and a case by case analysis leads to Table 19.

(c) Consider the group $C = \mathcal{G}(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l(2p+1)}, \mathbb{Z}_{2p+1}, c_{g(w)} \circ \tau_{a,b})$ for $l > 2$, and its defining maps

$$\alpha(z) = \begin{cases} c_{g(w)}(e^{\frac{i\pi xa}{l}}) & \text{if } z = e^{\frac{i\pi x}{l(2k+1)}} \\ c_{g(w)}(je^{\frac{i\pi(a(x-1)+b+1)}{l}}) & \text{if } z = je^{\frac{i\pi x}{l(2k+1)}} \end{cases},$$

$$\beta(z) = \begin{cases} e^{\frac{i\pi y}{l}} & \text{if } z = e^{\frac{i\pi y}{l(2p+1)}} \\ je^{\frac{i\pi y}{l}} & \text{if } z = je^{\frac{i\pi y}{l(2p+1)}} \end{cases},$$

where $(a,b) \in \mathbb{Z}_{2l}^{\times} \times \mathbb{Z}_{2l}$ and $g(w) \in \{e^{\frac{i\pi w}{l}}, je^{\frac{i\pi w}{l}} \ : \ w \in \mathbb{Z}\}$. It is easy to see that zero is an element in $\mathcal{W}(C)$ for any $g(w) \in 2\mathrm{D}_{2l}$, and so we see the necessity of having $D = \mathbb{Z}_{m'}$ for an integer $m' \in \mathbb{Z}$ non-divisible by four. Furthermore, since $c_{g(w)}$ sends the element $e^{\frac{i\pi xa}{l}}$ either to itself or its inverse, the rest of the elements in $\mathcal{W}(C)$ different from zero are determined similarly as in (a). In fact, we have to solve

$$Px = \varepsilon(Rx + Ly)(2k+1) \mod LP(2k+1),$$

where $P = 2p+1$, $L = 2l$, $R = \varepsilon' a$ and $\varepsilon' = \varepsilon'(g(w)$ is as follows

$$\varepsilon'(g(w)) = \begin{cases} 1 & \text{if } \ g(w) = e^{\frac{i\pi x}{l}} \\ -1 & \text{if } \ g(w) = je^{\frac{i\pi x}{l}} \end{cases}$$

Summarizing, we obtain

$$\mathcal{W}(C) = \{0\} \cup \mathrm{Re}(\mathbb{Z}_{ms}),$$

where

$$m = \gcd(2p+1-\varepsilon\varepsilon'(2k+1)a, 2l(2k+1), 2l(2p+1)(2k+1)), \quad \varepsilon \in \{-1,1\},$$

and

$$s = \frac{2l(2k+1)}{\gcd(2l(2k+1), 2l(2p+1)(2k+1))}.$$

This leads to the conditions stated in the theorem after absorbing the sign ϵ' into ϵ.

Now, let $l = 2$ and recall that any automorphism of $2\mathrm{D}_4$ is obtained by conjugation φ_w with an element $w \in 2\mathrm{O}$. The maps defining C are:

$$\alpha(z) = \begin{cases} 1 & \text{if } z = e^{\frac{2\pi ix}{(2k+1)}} \\ \varphi_w(j) & \text{if } z = je^{\frac{2\pi ix}{(2k+1)}} \\ \varphi_w(i) & \text{if } z = e^{\frac{i\pi(1+4x)}{2(2k+1)}} \\ -\varphi_w(k) & \text{if } z = je^{\frac{i\pi(1+4x)}{2(2k+1)}} \\ -1 & \text{if } z = e^{\frac{i\pi(1+2x)}{(2k+1)}} \\ -\varphi_w(j) & \text{if } z = je^{\frac{i\pi(1+2x)}{(2k+1)}} \\ -\varphi_w(i) & \text{if } z = e^{\frac{i\pi(3+4x)}{2(2k+1)}} \\ \varphi_w(k) & \text{if } z = je^{\frac{i\pi(3+4x)}{2(2k+1)}} \end{cases} \qquad \beta(z) = \begin{cases} 1 & \text{if } z = e^{\frac{2\pi iy}{(2p+1)}} \\ j & \text{if } z = je^{\frac{2\pi iy}{(2p+1)}} \\ i & \text{if } z = e^{\frac{i\pi(1+4y)}{2(2p+1)}} \\ -k & \text{if } z = je^{\frac{i\pi(1+4y)}{2(2p+1)}} \\ -1 & \text{if } z = e^{\frac{i\pi(1+2y)}{(2p+1)}} \\ -j & \text{if } z = je^{\frac{i\pi(1+2y)}{(2p+1)}} \\ -i & \text{if } z = e^{\frac{i\pi(3+4y)}{2(2p+1)}} \\ k & \text{if } z = je^{\frac{i\pi(3+4y)}{2(2p+1)}} \end{cases}$$

Observe that for any choice of w we have that $\alpha(e^{i\pi}) = \beta(e^{i\pi}) = -1$, and so $D = \mathbb{Z}_m$ with $2 \nmid m$. Moreover, since

$$\alpha(e^{\frac{2\pi ix}{(2k+1)}}) = \beta(e^{\frac{2\pi iy}{(2p+1)}}) = 1,$$

we have $\mathbb{Z}_{2k+1} \times \mathbb{Z}_{2p+1} \subset C$. It follows that $\mathrm{Re}(\mathbb{Z}_{\gcd(2k+1,2p+1)}) \subset \mathcal{W}(C)$, and hence the necessity to have

$$1 = \gcd(2k+1, 2p+1, m) = \gcd(4(2k+1), 4(2p+1), m).$$

(d) Now, consider $C = \mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, \mathbb{Z}_{4p}, \mathbb{Z}_p, \varphi(r))$. The group C is equal to one of the following groups:

$$\left\{(e^{\frac{2\pi ix}{2k+1}}, e^{\frac{2\pi iy}{p}}), (je^{\frac{2\pi ix}{2k+1}}, e^{\frac{\pi i(1+4y)}{2p}}), (e^{\frac{i\pi x}{2k+1}}, e^{\frac{i\pi(1+2y)}{p}}), (je^{\frac{i\pi x}{2k+1}}, e^{\frac{i\pi(3+4y)}{2p}}) \;:\; x,y \in \mathbb{Z}\right\},$$

$$\left\{(e^{\frac{2\pi ix}{2k+1}}, e^{\frac{2\pi iy}{p}}), (je^{\frac{i\pi x}{2k+1}}, e^{\frac{\pi i(1+4y)}{2p}}), (e^{\frac{i\pi x}{2k+1}}, e^{\frac{i\pi(1+2y)}{p}}), (je^{\frac{2\pi ix}{2k+1}}, e^{\frac{i\pi(3+4y)}{2p}}) \;:\; x,y \in \mathbb{Z}\right\}.$$

Let us distinguish according to the parity of p. If p is odd, then for any choice of $\varphi(r)$ we get $-1 \in \mathcal{W}(C)$ and also

$$\mathrm{Re}(\mathbb{Z}_{\gcd(2k+1,p)}) \subset \mathcal{W}(C).$$

In order for $C \times D$ to act freely on M, it is necessary to have $D = \mathbb{Z}_m$ with $2 \nmid m$ and

$$1 = \gcd(2k+1, p, m) = \gcd(2(2k+1), 4p, m).$$

This shows that $C \times D$ is a subgroup of a freely acting splittable group. Let now p be even. In this case $0 \notin \mathcal{W}(C)$ and

$$\mathcal{W}(C) \supset \mathrm{Re}(\mathbb{Z}_{\gcd(2k+1,p)}).$$

The rest of the elements in $\mathcal{W}(C)$ are obtained by solving the following equations

$$px = \varepsilon(2k+1)(1+2y) \mod 2p(2k+1),$$

for any $\varepsilon \in \{1,-1\}$. Because p is even, these equations have no solution. In other words, $\mathcal{W}(C) = \mathrm{Re}(\mathbb{Z}_{\gcd(2k+1,p)})$.

(e) The group $C = \mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2(2p+1)}, \mathbb{Z}_{2p+1}, \varphi(r))$ is either one of the following groups

$$\left\{(e^{\frac{2\pi i x}{2k+1}}, e^{\frac{2\pi i y}{2p+1}}), j(e^{\frac{2\pi i x}{2k+1}}, e^{\frac{2\pi i y}{2p+1}}), (e^{\frac{i\pi(1+2x)}{2k+1}}, e^{\frac{i\pi(1+2y)}{2p+1}}), j(e^{\frac{i\pi(1+2x)}{2k+1}}, e^{\frac{i\pi(1+2y)}{2p+1}}) : \ x, y \in \mathbb{Z}\right\},$$
$$\left\{(e^{\frac{2\pi i x}{2k+1}}, e^{\frac{2\pi i y}{2p+1}}), j(e^{\frac{i\pi(1+2x)}{2k+1}}, e^{\frac{2\pi i y}{2p+1}}), (e^{\frac{i\pi(1+2x)}{2k+1}}, e^{\frac{i\pi(1+2y)}{2p+1}}), j(e^{\frac{2\pi i x}{2k+1}}, e^{\frac{i\pi(1+2y)}{2p+1}}) : \ x, y \in \mathbb{Z}\right\}.$$

As in the first part of (d), we see that $-1 \in \mathcal{W}(C)$ and $\mathcal{W}(C) \supset \mathbb{Z}_{\gcd(2k+1,2p+1)}$. Therefore, we must have $D = \mathbb{Z}_m$ with m odd and

$$1 = \gcd(2k+1, 2p+1, m) = \gcd(2(2k+1), 2(2p+1), m).$$

Again this shows that $C \times D$ is a subgroup of a freely acting splittable group.

(f) The group $C = \mathcal{G}(\mathbb{Z}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{O}, 2\mathrm{T}, \mathrm{Id})$ is given by

$$\left\{\left(e^{\frac{2\pi i x}{2k+1}}, z_1)\right), \left(e^{\frac{\pi i(1+2x)}{2k+1}}, z_2\right) \ : \ z_1 \in 2\mathrm{T}, z_2 \in e^{\frac{i\pi}{4}}\, 2\mathrm{T}\right\}.$$

From this, one can verify that $\mathcal{W}(C) \subset \{1, -\frac{1}{2}\}$. This shows the claim.

(g) Let $C = \mathcal{G}(\mathbb{Z}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{4p}\, 2\mathrm{D}_{2p}, \mathrm{Id})$. It is readily seen that $-1 \in \mathcal{W}(C)$ and that the latter set contains $\mathrm{Re}(\mathbb{Z}_{\gcd(2(2k+1),2p)})$. We conclude the necessity to impose the conditions: $D = \mathbb{Z}_m$ with m odd and

$$1 = \gcd(2(2k+1), 2p, m) = \gcd(2(2k+1), 4p, m).$$

No new freely acting group is obtained in this way. ☐

The following theorem ends up our classification of freely acting finite subgroups $C \times D \subset \mathrm{SU}(2)^3$.

Theorem 4.8. *Let $C \times D$ be a type III freely acting group different from any group occurring in Theorem 4.7. Then, either $C \times D$ is a subgroup of a freely acting splittable group, or it is one of the following groups.*

$\mathcal{G}(A, A_0, B, B_0, \theta) \times \mathbb{Z}_m, \quad 2 \nmid m$				
A	A_0	B	B_0	Conditions
2I	$\mathbb{Z}_2$	2I	$\mathbb{Z}_2$	$3 \nmid m$
$2\mathrm{D}_{6k}$	$\mathbb{Z}_{2k}$	2O	$2\mathrm{D}_4$	$3 \mid k$
$2\mathrm{D}_{2kl}$	$\mathbb{Z}_{2k}$	$2\mathrm{D}_{2pl}$	$\mathbb{Z}_{2p}$	$\gcd(\tilde{m}s, m) = 1$

Where the above conditions for the group $\mathcal{G}(2\mathrm{D}_{2kl}, \mathbb{Z}_{2k}, 2\mathrm{D}_{2pl}, \mathbb{Z}_{2p}, c_w \circ \tau_{a,b}) \times \mathbb{Z}_m$, are required for both values of

$$\tilde{m} = \tilde{m}_\varepsilon = \gcd(p - \varepsilon ka, kl, lpk)),$$

for $\varepsilon = \pm 1$, and

$$s = \frac{kl}{\gcd(kl, lpk)}.$$

Proof. We are left with the case $D = \mathbb{Z}_m$ for $m \in \mathbb{Z}$ odd, see Proposition 4.5. It suffices again to find suitable elements in $\mathcal{W}(C)$. Since we have numerous choices for $C = \mathcal{G}(A, A_0, B, B_0, \theta)$, let us proceed by considering all possible candidates for $F = {}^{B}/_{B_0}$ according to Tables 15 and 16, which were not considered in Theorem 4.7.

(a) Let $F \in \{\mathrm{T},\mathrm{O},\mathrm{I}\}$. In those cases $A = B \in \{2\mathrm{T}, 2\mathrm{O}, 2\mathrm{I}\}$ and $A_0 = B_0 = \mathbb{Z}_2$. Let $\alpha : A \mapsto F$ and $\beta : B \mapsto F$ be the maps defining C. Since the automorphisms θ of ${}^{B}/_{\mathbb{Z}_2}$, for $B = 2\mathrm{T}, 2\mathrm{O}$, is induced by conjugation with a class $[w] \in {}^{2\mathrm{O}}/_{\mathbb{Z}_2}$, we have $\alpha(x) = \beta(c_w(x))$ for any element $x \in B$. It follows that $\mathcal{W}(C) = \mathrm{Re}(B)$. This shows that $C \times D$ is a subgroup of a freely acting splittable group.

We treat now the case $B = 2\mathrm{I}$. Because of the argument above, it suffices to consider the (only) non-trivial outer automorphism of F in the construction of C. Since two elements $[z],[w]$ in ${}^{2\mathrm{I}}/_{\mathbb{Z}_2}$ are conjugate (precisely) when $\mathrm{Re}(z) = \pm\mathrm{Re}(w)$, see Table 11, the group ${}^{2\mathrm{I}}/_{\mathbb{Z}_2}$ has five conjugacy classes $\mathcal{C}([y_i])$, $i = 1,...,5$, with representatives $y_i \in 2\mathrm{I}$ having (up to sign) every possible real part of an element in 2I. On the other hand, if $\varphi : {}^{2\mathrm{I}}/_{\mathbb{Z}_2} \mapsto {}^{2\mathrm{I}}/_{\mathbb{Z}_2}$ is the representative of the generator of $\mathrm{Out}({}^{2\mathrm{I}}/_{\mathbb{Z}_2})$ described in item (6) Section 3.2, then it fixes all conjugacy classes except of the two conjugacy classes of order 12, which are exchanged. It follows that $\mathcal{W}(C) = \{0, \pm 1, \pm\frac{1}{2}\}$. This leads to the additional condition $3 \nmid m$.

(b) Groups $\mathcal{G}(A, A_0, B, B_0, \theta) \subset \mathrm{SU}(2)^2$ giving rise to $F \cong 2\mathrm{D}_{2l}$ were analyzed in Theorem 4.7.

(c) Let $F \cong \mathrm{D}_{2l}$ for some integer $l \geq 2$. Let $C = \mathcal{G}(2\mathrm{O}, 2\mathrm{D}_4, 2\mathrm{O}, 2\mathrm{D}_4, c_w \circ \tau_{a,b})$, where $(a,b) \in \mathbb{Z}_6^\times \times \mathbb{Z}_6$ and $w \in 2\mathrm{O}$. We easily check that

$$\left(-\frac{1}{2}(1+i+j+k), c_w\left(-\frac{1}{2}(1+i+j+k)\right)\right) \in C,$$

whence $a = 1 \mod 6$. As for if $a = 5 \mod 6$, then

$$\left(-\frac{1}{2}(1+i+j+k), c_w\left(\frac{1}{2}(-1+i+j+k)\right)\right) \in C.$$

This implies that $-\frac{1}{2} \in \mathcal{W}(C)$ for any homomorphism $c_w \circ \tau_{a,b}$. This shows that $C \times D$ is a subgroup of a freely acting splittable group.

Now, consider $C = \mathcal{G}(2\mathrm{D}_{6p}, \mathbb{Z}_{2p}, 2\mathrm{O}, 2\mathrm{D}_4, c_{g(w)} \circ \tau_{a,b})$, where $(a,b) \in \mathbb{Z}_6^\times \times \mathbb{Z}_6$ and $g(w) \in 2\mathrm{D}_{6p} = \{e^{\frac{i\pi w}{3p}}, je^{\frac{i\pi w}{3p}} : w \in \mathbb{Z}\}$. For $a = 1 \mod 6$, we have

$$\alpha\left(e^{\frac{2\pi i}{3}}\right) = \begin{cases} \left(\frac{1}{2}(1+i+j+k)\right)^{2p} 2\mathrm{D}_4 & g(w) = e^{\frac{i\pi w}{3p}} \\ \left(\frac{1}{2}(1+i+j+k)\right)^{4p} 2\mathrm{D}_4 & g(w) = je^{\frac{i\pi w}{3p}} \end{cases},$$

whereas for $a = 5 \mod 6$

$$\alpha\left(e^{\frac{2\pi i}{3}}\right) = \begin{cases} \left(\frac{1}{2}(1+i+j+k)\right)^{4p} 2\mathrm{D}_4 & g(w) = e^{\frac{i\pi w}{3p}} \\ \left(\frac{1}{2}(1+i+j+k)\right)^{2p} 2\mathrm{D}_4 & g(w) = je^{\frac{i\pi w}{3p}} \end{cases},$$

where $\beta : 2\mathrm{O} \mapsto {}^{2\mathrm{O}}/{}_{2\mathrm{D}_4}$ is in both cases the natural map. Using the fact that the order of the element

$$\frac{1}{2}(1+i+j+k)\, 2\mathrm{D}_4 \in {}^{2\mathrm{O}}/{}_{2\mathrm{D}_4}$$

is 3, it follows that $-\frac{1}{2} \in \mathcal{W}(C)$ precisely if $3 \nmid p$. In that case $C \times D$ is a subgroup of a freely acting splittable group. Otherwise $\mathcal{W}(C) \cap \mathrm{Re}(D) = \{1\}$, and so the group

$$\mathcal{G}(2\mathrm{D}_{6p}, \mathbb{Z}_{2p}, 2\mathrm{O}, 2\mathrm{D}_4, c_{g(w)} \circ \tau_{a,b}) \times D$$

acts freely on M.

Lastly, consider the case in which $C = \mathcal{G}(2\mathrm{D}_{2kl}, \mathbb{Z}_{2k}, 2\mathrm{D}_{2pl}, \mathbb{Z}_{2p}, c_w \circ \tau_{a,b})$. To determine the elements of $\mathcal{W}(C)$ different from zero, it suffices to consider the subset of $C \subset \mathrm{SU}(2)^2$ given by

$$\{(e^{\frac{i\pi x}{kl}}, e^{\frac{i\pi(ax+ly)}{pl}}) \ : \ x, y \in \mathbb{Z}\},$$

and so solve the equation

$$px = k\varepsilon(ax + ly) \mod 2kpl.$$

In fact, because m is odd, and hence

$$\gcd(2kl, 2pl, m) = \gcd(kl, pl, m),$$

we can consider

$$px = k\varepsilon(ax + ly) \mod kpl$$

instead. This situation was encountered in part (a) of Theorem 4.7, which applied to this situation shows the claim.

(d) Let $F \cong \mathbb{Z}_2$. For the group $C = \mathcal{G}(2\mathrm{O}, 2\mathrm{T}, 2\mathrm{O}, 2\mathrm{T}, \mathrm{Id})$, we have $\mathcal{W}(C) = \mathrm{Re}(2\mathrm{O})$, and so no new freely acting group will be obtained.

Consider the group $C = \mathcal{G}(2\mathrm{O}, 2\mathrm{T}, \mathbb{Z}_{2k}, \mathbb{Z}_k, \mathrm{Id})$ and let

$$\alpha : z \mapsto \alpha(z) = \begin{cases} 1 & \text{if } z \in 2\mathrm{T} \\ -1 & \text{if } z \in e^{\frac{i\pi}{4}}\, 2\mathrm{T} \end{cases}, \quad \beta\left(e^{\frac{i\pi x}{k}}\right) = \begin{cases} 1 & \text{if } x = 0 \mod 2 \\ -1 & \text{if } x = 1 \mod 2 \end{cases}$$

be the maps defining it. We check that $-\frac{1}{2}$ lies in $\mathcal{W}(C)$ precisely if $3|k$. It follows the necessity to impose on $C \times D$ the condition $3 \nmid \gcd(k, m)$, which implies that $3 \nmid \gcd(2k, m)$, and so no new freely acting group will be obtained in this fashion.

The group $C = \mathcal{G}(2\mathrm{D}_{4k}, 2\mathrm{D}_{2k}, 2\mathrm{O}, 2\mathrm{T}, \mathrm{Id})$ is given by

$$\left\{(e^{\frac{i\pi x}{k}}, z), (je^{\frac{i\pi x}{k}}, z), (e^{\frac{i\pi(2x+1)}{2k}}, e^{\frac{i\pi}{4}}z), (je^{\frac{i\pi(2x+1)}{2k}}, e^{\frac{i\pi}{4}}z) \ : \ z \in 2\mathrm{T}, x \in \mathbb{Z}\right\}.$$

A necessary condition to impose on $C \times D$ to act freely on M reads $3 \nmid \gcd(k, m) = \gcd(4k, m)$. Therefore, no new freely acting subgroup will be found in this way.

The group $C = \mathcal{G}(\mathbb{Z}_{2k}, \mathbb{Z}_k, 2\mathrm{D}_{4p}, 2\mathrm{D}_{2p}, \mathrm{Id})$ is given as follows

$$\left\{(e^{\frac{2i\pi x}{k}}, e^{\frac{i\pi y}{p}}), (e^{\frac{2i\pi x}{k}}, je^{\frac{i\pi y}{p}}), (e^{\frac{i\pi(2x+1)}{k}}, e^{\frac{i\pi(2y+1)}{2p}}), (e^{\frac{i\pi(2x+1)}{k}}, je^{\frac{i\pi(2y+1)}{2p}}) \ : \ x, y \in \mathbb{Z}\right\}.$$

In consequence, $\mathcal{W}(C) = \mathrm{Re}(\mathbb{Z}_{\gcd(k,2p)})$. Therefore the necessity to impose

$$1 = \gcd(k, 2p, m) = \gcd(2k, 4p, m),$$

and so there is no new freely acting subgroup.

Analogously, for any group $C = \mathcal{G}(2\mathrm{D}_{4k}, 2\mathrm{D}_{2k}, 2\mathrm{D}_{4p}, 2\mathrm{D}_{2p}, \mathrm{Id})$, we have to have

$$1 = \gcd(4k, 4p, m),$$

and so no new freely acting subgroup is obtained in this fashion.

(e) Let $F \cong \mathbb{Z}_3$. The analysis for $\mathcal{G}(\mathbb{Z}_{3k}, \mathbb{Z}_k, 2\mathrm{T}, 2\mathrm{D}_4, \varphi(r))$ was performed in part (b) of Theorem 4.7, whereas for $C = \mathcal{G}(2\mathrm{T}, 2\mathrm{D}_4, 2\mathrm{T}, 2\mathrm{D}_4, \varphi(r))$, it is not difficult to see that $\mathcal{W}(C) = \mathrm{Re}(2\mathrm{T})$, and so no new freely acting groups are obtained in this case.

(f) The groups $C \subset \mathrm{SU}(2)^2$ leading to $F \cong \mathbb{Z}_l$ for $l \notin \{2,3\}$ were analyzed in Theorem 4.7.

□

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