On higher topological Hochschild homology of rings of integers

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Abstract

We determine higher topological Hochschild homology of rings of integers in number fields with coefficients in suitable residue fields. We use the iterative description of higher *THH* for this and Postnikov arguments that allow us reduce the necessary computations to calculations in homological algebra, starting from the results of Bökstedt and Lindenstrauss-Madsen on (ordinary) topological Hochschild homology.

1 Introduction

Topological Hochschild homology, *THH*, of rings of integers in number fields is well-understood: Bökstedt [Bö∞] calculated *THH* of the integers and in [LM00] the general case is covered. The aim of this paper is the calculation of higher order topological Hochschild homology of rings of integers in number fields with coefficients in a suitable residue field. Topological Hochschild homology of a (simplicial) ring or ring spectrum R with coefficients in a (simplicial) module (spectrum) N is obtained as a simplicial object where one uses a simplicial model of the 1-sphere, S^1, and glues N to the basepoint of S^1 and R to all other simplices in S^1. Topological Hochschild homology of order n of R with coefficients in N, $THH^{[n]}(R,N)$ is the analogue of this where we use $S^n = (S^1)^{\wedge n}$ as a simplicial model of the n-sphere and glue again N to the basepoint and R to all other simplices of S^n. For a definition using the topological space $\mathbb{S}^1$ see [MSV97]; for an approach preserving the epicyclic structure, see e.g., [BCD10] or [V∞].

There are natural stabilization maps $\pi_*(THH^{[n]}(R,N)) \to \pi_{*+1}(THH^{[n+1]}(R,N))$ whose colimit gives the topological André-Quillen homology of R with coefficients in N as defined in [Ba99].

If $N = R$, then we will abbreviate $THH^{[n]}(R,R)$ with $THH^{[n]}(R)$. As usual we abuse notation and write $THH^{[n]}(A)$ or $THH^{[n]}(A,M)$ for $THH^{[n]}(HA)$ or $THH^{[n]}(HA,HM)$, respectively, if A is a commutative discrete or simplicial ring, M a discrete or simplicial A-module and H denots the Eilenberg-Mac Lane spectrum functor. We will phrase our results in terms of iterated Tor groups:

Definition 1.1 Let k be a field. Let $B^1_k(x)$ be the polynomial algebra $P_k(x)$ on a generator x in degree $2m$. Inductively, we define the k-algebra $B^n_k(x) = \mathrm{Tor}_*^{B^{n-1}_k(x)}(k,k)$.

By Cartan [C54], $B^2_k(x)$ is the exterior algebra $\Lambda_k(\sigma x)$ on a generator σx in degree $2m+1$; after that, we get a divided power algebra on a generator of degree $2m+2$, and after that, the formulas become more complicated (but see [BLPRZ∞] for an illustration of what the $B^n_k(x)$ look like when k is a finite field).

In the following, all tensors are over $\mathbb{F}_p$ unless otherwise explicitly marked. Our first result is:

Theorem 3.1 Let $n \geq 1$. Then $THH^{[n]}_*(\mathbb{Z},\mathbb{F}_p) \cong B^n_{\mathbb{F}_p}(x) \otimes B^{n+1}_{\mathbb{F}_p}(y)$ where $|x| = 2p$ and $|y| = 2p-2$.

The calculation of $THH^{[n]}_*(\mathbb{Z},\mathbb{F}_p)$ along with a calculation of the Bockstein spectral sequence on it (which we do not do in this paper) would give us $THH^{[n]}_*(\mathbb{Z})$. If one wanted to do a similar calculation for more general number rings A, observe first that for any n and any commutative ring A, $THH^{[n]}_0(A) \cong A$, since in the definition of $THH^{[n]}(A)$, both d_0 and d_1 have to multiply all copies of HA indexed on all the 1-simplices into the copy that sits over the basepoint, and so by the commutativity of HA, d_0 and d_1 induce the same maps on homology.

The Bökstedt spectral sequence for higher topological Hochschild homology (Proposition 7.2 in [BLPRZ∞]) with rational coefficients is a spectral sequence with

$$E^2_{*,*} = HH^{[n]}(H_*(HA;\mathbb{Q})) \Rightarrow H_*(THH^{[n]}(A);\mathbb{Q}).$$

Since $H_*(HA;\mathbb{Q})$ consists just of $A\otimes\mathbb{Q}$ in dimension zero and since for a number ring A, $A\otimes\mathbb{Q}$ is étale over $\mathbb{Q}$, by Theorem 9.1(a) of [BLPRZ∞], for $*>0$, $THH^{[n]}_*(A)$ consists entirely of torsion.

However, since A is a number ring, and hence a Dedekind domain, any finitely generated torsion module over it is a finite direct sum of modules $A/P_i^{k_i}$, with P_i nonzero prime ideals and $k_i \geq 1$. For each such prime ideal P_i, there is a unique prime $p\in\mathbb{Z}$ for which $pA\subseteq P_i$, and if we consider $THH^{[n]}(A)^\wedge_p$, its homotopy groups in positive dimensions will be all the modules $A/P_i^{a_i}$ where $pA\subseteq P_i$, and none of the others. The methods of Addendum 6.2 in [HM97], which show that for any number ring A, $THH(A)^\wedge_p \simeq THH(A^\wedge_p)^\wedge_p$, also show that for general $n\geq 1$,

$$THH^{[n]}(A)^\wedge_p \simeq THH^{[n]}(A^\wedge_p)^\wedge_p.$$

So in order to understand the P_i torsion in $THH^{[n]}(A)$, we could see it instead in

$$THH^{[n]}(A^\wedge_p) \cong THH^{[n]}(\prod_{P_i \text{ prime},\ pA\subseteq P_i} A^\wedge_{P_i}) \simeq \prod_{P_i \text{ prime},\ pA\subseteq P_i} THH^{[n]}(A^\wedge_{P_i}).$$

Then, like calculating $THH^{[n]}_*(\mathbb{Z},\mathbb{F}_p)$ was an intermediate goal in the calculation of $THH^{[n]}_*(\mathbb{Z})$, calculating $THH^{[n]}_*(A^\wedge_{P_i},(A^\wedge_{P_i})/P_i)$ is an intermediate goal in the calculation of $THH^{[n]}_*(A^\wedge_{P_i})$.

We calculate the groups $THH^{[n]}_*(A^\wedge_{P_i},(A^\wedge_{P_i})/P_i)$ below, obtaining

Theorem 4.3 Let A be the ring of integers in a number field, and let P be a nonzero prime ideal in A. Denote the residue field A/P by $\mathbb{F}_q$. Then

$$THH^{[n]}_*(A^\wedge_P, A/P) \cong B^n_{\mathbb{F}_q}(x_P)\otimes_{\mathbb{F}_q} B^{n+1}_{\mathbb{F}_q}(y_P)$$

where

(i) $|x_P|=2$ and $|y_P|=0$ if A is ramified over $\mathbb{Z}$ at P, and

(ii) $|x_P|=2p$ and $|y_P|=2p-2$, if A is unramified over $\mathbb{Z}$ at P.

This gives the homotopy groups of

$$THH^{[n]}(A,A/P_i) \simeq THH^{[n]}(A^\wedge_{P_i}) \wedge_{H(A^\wedge_{P_i})} H(A/P_i).$$

As in the $n=1$ case, multiplying $H\mathbb{Z}$ into the copy of $H(A^\wedge_{P_i})$ over the basepoint shows that $THH^{[n]}(A^\wedge_{P_i})$ is a retract of $H\mathbb{Z}\wedge THH^{[n]}(A^\wedge_{P_i})$, and so additively, it is a product of Eilenberg-Mac Lane spectra. Any shifted copy $H(A/P_i^{a_i})$, $a_i\geq 1$, that we have in $THH^{[n]}(A^\wedge_{P_i})$ will yield two correspondingly shifted copies of $H(A/P_i)$ (one with the same shift, one with that shift plus one) in $THH^{[n]}(A^\wedge_{P_i})\wedge_{H(A^\wedge_{P_i})} H(A/P_i)$. Again one can then read off the rank of the P_i-torsion from $THH^{[n]}(A^\wedge_{P_i},A/P_i)$, and to understand what the torsion actually is, one would need to look at Bockstein-type operators associated with multiplication by a uniformizer of $A^\wedge_{P_i}$. We plan to do the necessary Bockstein spectral sequence calculations in a future project.

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2 Identifying square zero extensions

Let k be a commutative ring, and let Hk be the associated Eilenberg-Mac Lane commutative ring spectrum. We will show that there is exactly one homotopy type of augmented commutative Hk-algebras C with homotopy $\pi_*C \cong \Lambda_k(x)$ where x is a generator in a given positive degree. That is, there is a chain of stable equivalences of augmented commutative Hk-algebras $C \simeq Hk \vee \Sigma^m Hk$ where the Hk-module $\Sigma^m Hk$ is the m-fold suspension of Hk and $Hk \vee \Sigma^m Hk$ is the square-zero extension of Hk by $\Sigma^m Hk$.

We learned of such a fact when $k = \mathbb{F}_p$ from Michael Mandell who proves it by means of topological André-Quillen homology and uses it in a program joint with Maria Basterra.

Proposition 2.1 Let C be a commutative augmented Hk-algebra and assume that there is an isomorphism of graded commutative k-algebras $\pi_*C \cong \Lambda_k(x)$ where $|x| = m > 0$. Then there is a chain of stable equivalences of commutative augmented Hk-algebras between C and $Hk \vee \Sigma^m Hk$.

Proof: For concreteness, we formulate the proof in symmetric spectra. Let S be the sphere spectrum, and P_S and P_{Hk} the free commutative algebra functors (adjoint to the forgetful functor with values in S or Hk-modules). We may assume that C is fibrant in the positive Hk-model structure of Shipley [S04]. Let M be a positively Hk-cofibrant resolution of $\Sigma^m Hk$; for concreteness $M = Hk \wedge F_1(\mathbb{S}^{m+1})$, where F_1 is the adjoint to the evaluation on the first level. Represent x by an Hk-module morphism $M \to C$. Now,

$$P_{Hk}(M) = \bigvee_{n \geq 0} (M^{\wedge_{Hk} n})_{\Sigma_n} \cong Hk \wedge P_S(F_1(\mathbb{S}^{m+1}))$$

is the free commutative Hk-algebra on M and we consider the induced map $f\colon P_{Hk}(M) \to C$. Since

$$(F_1((\mathbb{S}^{m+1})^{\wedge n})_{\Sigma_n} \simeq (F_1((\mathbb{S}^{m+1})^{\wedge n})_{h\Sigma_n}$$

is $(mn-1)$-connected and $m > 0$, f is $2m \geq (m+1)$-connected.

Taking the mth Postnikov section P_m of $P_{Hk}(M)$ in the setting of commutative Hk-algebra spectra (as done in the [EKMM97] setting in [Ba99, §8]) gives a map of commutative Hk-algebra spectra $P_m \to C$, which is an isomorphism on the non-zero homotopy groups in degree 0 and m. As both spectra are semistable, this map is a stable equivalence of symmetric spectra. Hence, C and P_m are of the same stable homotopy type, and repeating the argument with a fibrant model for $Hk \vee \Sigma^m Hk$ we get the promised chain of stable equivalences connecting C and $Hk \vee \Sigma^m Hk$. ∎

3 The calculation of $THH_*^{[n]}(\mathbb{Z}, \mathbb{F}_p)$

Our goal in this section is to prove

Theorem 3.1 Let $n \geq 1$ and p be any prime. Then $THH_*^{[n]}(\mathbb{Z}, \mathbb{F}_p) \cong B^n_{\mathbb{F}_p}(x) \otimes B^{n+1}_{\mathbb{F}_p}(y)$ where $|x| = 2p$ and $|y| = 2p - 2$.

To this end we use the iterative description of $THH^{[n]}$: the n-sphere $\mathbb{S}^n$ can be decomposed into two hemispheres whose intersection is the equator, $\mathbb{S}^n = \mathbb{D}^n \cup_{\mathbb{S}^{n-1}} \mathbb{D}^n$. This decomposition yields [V∞] that

$$THH^{[n]}(\mathbb{Z}, \mathbb{F}_p) \simeq H\mathbb{F}_p \wedge^L_{THH^{[n-1]}(\mathbb{Z}, \mathbb{F}_p)} H\mathbb{F}_p \tag{3.1.0}$$

where $\wedge^L$ denotes the derived smash product.

Note that for any commutative ring spectrum R, $THH(R)$ is a commutative R-algebra spectrum [EKMM97, IX.2.2], in particular, $THH(\mathbb{Z})$ is a commutative $H\mathbb{Z}$-algebra spectrum. As $THH(\mathbb{Z}, \mathbb{F}_p) = THH(\mathbb{Z}) \wedge_{H\mathbb{Z}} H\mathbb{F}_p$, we have a commutative $H\mathbb{F}_p$-algebra structure on $THH(\mathbb{Z}, \mathbb{F}_p)$ and the multiplication on $H\mathbb{Z}$ and the $H\mathbb{Z}$-module structure of $H\mathbb{F}_p$ give rise to a canonical augmentation from $THH(\mathbb{Z}, \mathbb{F}_p)$ to $H\mathbb{F}_p$. Thus $THH(\mathbb{Z}, \mathbb{F}_p)$ is a commutative augmented $H\mathbb{F}_p$-algebra spectrum.

Bökstedt [Bö∞] calculated $THH_*(\mathbb{Z})$ and his result gives the $n = 1$ case of the theorem,

$$THH_*(\mathbb{Z}, \mathbb{F}_p) \cong \mathbb{F}_p[x_{2p}] \otimes \Lambda(z_{2p-1})$$

since $B_{\mathbb{F}_p}(y_{2p-2})$ is isomorphic to $\Lambda(z_{2p-1})$. We use the $(2p-1)$st Postnikov section of commutative augmented $H\mathbb{F}_p$-algebras to map $THH_*(\mathbb{Z}, \mathbb{F}_p)$ to something which by Proposition 2.1 has to be weakly equivalent to $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$. Then we consider the homotopy pushout diagram in the category of commutative augmented $H\mathbb{F}_p$-algebras

$$\begin{tikzcd}
THH(\mathbb{Z}, \mathbb{F}_p) \arrow[r] \arrow[d] & H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \arrow[d, "f"] \\
H\mathbb{F}_p \arrow[r] & (H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p) \wedge^L_{THH(\mathbb{Z}, \mathbb{F}_p)} H\mathbb{F}_p.
\end{tikzcd}$$

A bar spectral sequence argument tells us that the homotopy groups of $(H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p) \wedge^L_{THH(\mathbb{Z}, \mathbb{F}_p)} H\mathbb{F}_p$ are isomorphic to $\Lambda(y_{2p+1})$ and using Proposition 2.1 again we see that the homotopy pushout is a commutative augmented $H\mathbb{F}_p$-algebra which is weakly equivalent to $H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p$.

Lemma 3.2 Let $f\colon H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \to (H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p) \wedge^L_{THH(\mathbb{Z}, \mathbb{F}_p)} H\mathbb{F}_p. \cong H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p$ be the map in the homotopy category of augmented commutative $H\mathbb{F}_p$-algebras induced by the pushout above. Then f factors through the augmentation $\varepsilon\colon H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \to H\mathbb{F}_p$.

Proof: Consider the diagram

$$\begin{tikzcd}
H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \arrow[r, "="] \arrow[d, "\epsilon"] & H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \arrow[d, "f"] \\
H\mathbb{F}_p \arrow[r] & H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p,
\end{tikzcd}$$

which we obtain from the leftmost square of the previous diagram because the augmentation $THH(\mathbb{Z}, \mathbb{F}_p) \to H\mathbb{F}_p$ has to factor through the Postnikov map to $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$. This diagram commutes, so the maps to the bottom right factor through the pushout $H\mathbb{F}_p \wedge_{H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p} (H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p) \simeq H\mathbb{F}_p$ of the top left corner, so f factors through ϵ. ∎

Hence, $THH^{[2]}(\mathbb{Z}, \mathbb{F}_p)$ can be described by the following iterated homotopy pushout diagram.

$$\begin{tikzcd}
THH(\mathbb{Z}, \mathbb{F}_p) \arrow[r] \arrow[dd] & H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \arrow[rr] \arrow[dd, "f"] \arrow[dr, "\varepsilon"] & & H\mathbb{F}_p \arrow[dr] \arrow[dd] & \\
& & H\mathbb{F}_p \arrow[dl, "\eta"] \arrow[rr, crossing over] & & \Gamma \arrow[dl] \\
H\mathbb{F}_p \arrow[r] & H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p \arrow[rr] & & THH^{[2]}(\mathbb{Z}, \mathbb{F}_p) &
\end{tikzcd}$$

Here, Γ denotes the homotopy pushout of the upper right subdiagram in the category of commutative $H\mathbb{F}_p$-algebras, and as above we get

$$\Gamma = H\mathbb{F}_p \wedge^L_{H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p} H\mathbb{F}_p.$$

We have again a Tor-spectral sequence converging to the homotopy groups of the spectrum Γ with

$$E^2_{*,*} = \mathrm{Tor}^{\Lambda(z_{2p-1})}_{*,*}(\mathbb{F}_p, \mathbb{F}_p)$$

and hence $\pi_*(\Gamma)$ is isomorphic to a divided power algebra over $\mathbb{F}_p$ on a generator in degree $2p$, $\Gamma(a_{2p})$. In the iterated homotopy pushout diagram all maps involved are maps of commutative S-algebras and thus we can identify $THH^{[2]}(\mathbb{Z}, \mathbb{F}_p)$ as a commutative $H\mathbb{F}_p$-algebra as

$$THH^{[2]}(\mathbb{Z}, \mathbb{F}_p) \simeq (H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p) \wedge^L_{H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p} H\mathbb{F}_p \simeq (H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \Gamma \cong \Gamma \vee \Sigma^{2p+1}\Gamma.$$

Thus, $THH^{[2]}(\mathbb{Z}, \mathbb{F}_p)$ is equivalent to the bar construction $B_{H\mathbb{F}_p}(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p)$. Its homotopy groups are

$$THH^{[2]}_*(\mathbb{Z}, \mathbb{F}_p) \cong \Gamma(a_{2p}) \otimes \Lambda(y_{2p+1}).$$

We use this to determine higher THH via iterated bar constructions. We know that $THH^{[n+1]}(H\mathbb{Z}, H\mathbb{F}_p)$ is equivalent to the derived smash product

$$H\mathbb{F}_p \wedge^L_{THH^{[n]}(H\mathbb{Z}, H\mathbb{F}_p)} H\mathbb{F}_p$$

whose homotopy groups are the ones of the bar construction $B_{H\mathbb{F}_p}(H\mathbb{F}_p, THH^{[n]}(H\mathbb{Z}, H\mathbb{F}_p), H\mathbb{F}_p)$ and iteratively, we can express $THH^{[n]}(H\mathbb{Z}, H\mathbb{F}_p)$ again in terms of such a bar construction as long as n is greater than two. For $n = 2$ we know the answer by the above argument. For larger n we can determine the homotopy groups of $THH^{[n+1]}(H\mathbb{Z}, H\mathbb{F}_p)$ iteratively. Abbreviating $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$ to $E(z)$ and $H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p$ to $E(y)$ we define

$$B^{(n)} := B_{H\mathbb{F}_p}(H\mathbb{F}_p, \ldots, B(H\mathbb{F}_p, B(H\mathbb{F}_p, E(z), E(y)), H\mathbb{F}_p), \ldots, H\mathbb{F}_p)$$

with $n-1$ pairs of outer terms of $H\mathbb{F}_p$. We denote by $\underline{E(y)}$ the constant simplicial $H\mathbb{F}_p$-algebra spectrum on $E(y)$.

Lemma 3.3 As n-simplicial commutative $H\mathbb{F}_p$-algebras

$$B^{(n)} \simeq B^{(n)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B^{(n-1)}_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p)$$

for all $n \geq 2$.

Proof: We show the claim directly for $n = 2$: $B^{(2)}$ is

$$B_{H\mathbb{F}_p}(H\mathbb{F}_p, B_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), E(y)), H\mathbb{F}_p).$$

As we know from Lemma 3.2 that the $E(z)$-module structure of $E(y)$ factors via the augmentation map through the $H\mathbb{F}_p$-module structure of $E(y)$, we get that $B_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), E(y))$ can be split as an augmented simplicial commutative $H\mathbb{F}_p$-algebra as $B_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \underline{E(y)}$ and thus we get a weak equivalence of bisimplicial commutative $H\mathbb{F}_p$-algebra spectra:

$$\begin{aligned}
&B_{H\mathbb{F}_p}(H\mathbb{F}_p, B_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), E(y)), H\mathbb{F}_p) \\
\simeq &B_{H\mathbb{F}_p}(H\mathbb{F}_p, B_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \underline{E(y)}, H\mathbb{F}_p) \\
\simeq &B_{H\mathbb{F}_p}(H\mathbb{F}_p, B_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p), H\mathbb{F}_p) \wedge B_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p) \\
= &B^{(2)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \wedge B_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p).
\end{aligned}$$

For the second weak equivalence we use that the bar construction $B_R(R, X \wedge_R Y, R)$ of the smash product of two commutative simplicial R-algebra spectra X and Y is equivalent as a bisimplicial commutative R-algebra to $B_R(R, X, R) \wedge_R B(R, Y, R)$.

By induction we assume that n is bigger than 2 and that we know the result for all $k < n$. Then

$$\begin{aligned}
B^{(n)} &= B(H\mathbb{F}_p, B^{(n-1)}, H\mathbb{F}_p) \\
&\simeq B(H\mathbb{F}_p, B^{(n-1)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B^{(n-2)}_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p), H\mathbb{F}_p) \\
&\simeq B_{H\mathbb{F}_p}(H\mathbb{F}_p, B^{(n-1)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, B^{(n-2)}_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p), H\mathbb{F}_p) \\
&= B^{(n)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B^{(n-1)}_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p).
\end{aligned}$$

■

We view $THH^{(n)}(H\mathbb{Z}, H\mathbb{F}_p)$ as a simplicial commutative $H\mathbb{F}_p$-algebra for all $n \geq 1$ and therefore describe $THH^{(n+1)}(H\mathbb{Z}, H\mathbb{F}_p)$ as the diagonal of the bar construction $B_{H\mathbb{F}_p}(H\mathbb{F}_p, THH^{[n]}(H\mathbb{Z}, H\mathbb{F}_p), H\mathbb{F}_p)$.

Corollary 3.4 We obtain, that

$$THH_*^{(n+1)}(H\mathbb{Z}, H\mathbb{F}_p) \cong \pi_*\mathrm{diag}B^{(n)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \otimes_{\mathbb{F}_p} \pi_*\mathrm{diag}B^{(n-1)}_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p).$$

For sake of definiteness in the following we will work in the category of symmetric spectra in simplicial sets, Sp^Σ [HSS00]. The Eilenberg-Mac Lane spectrum gives rise to a functor

$$H\colon \mathsf{sAb} \to \mathsf{Sp}^\Sigma$$

such that $HA(n) = \mathrm{diag}(A \otimes \tilde{\mathbb{Z}}(S^n))$ where $S^n = (S^1)^{\wedge n}$ and $\tilde{\mathbb{Z}}(-)$ denotes the free abelian group generated by all non-basepoint elements. This functor is lax symmetric monoidal [S∞, 2.7,3.11]. A square-zero extension $H\mathbb{F}_p \vee \Sigma^n H\mathbb{F}_p$ (for $n \geq 1$) can be modelled by $\mathbb{F}_p(\Delta_n/\partial\Delta_n)$:

Lemma 3.5 There is a stable equivalence of commutative symmetric ring spectra $\psi\colon H\mathbb{F}_p(\Delta_n/\partial\Delta_n) \to H\mathbb{F}_p \vee \Sigma^n H\mathbb{F}_p$.

Proof: There are two non-degenerate simplices in $\Delta_n/\partial\Delta_n$: a zero-simplex $*$ corresponding to the unique basepoint and an n-simplex corresponding to the identity map $\mathrm{id}_{[n]}$ on the set $[n] = \{0, \ldots, n\}$. We can represent any simplex in $\Delta_n/\partial\Delta_n$ as $s_{i_\ell} \circ \ldots \circ s_{i_0}(*)$ or $s_{i_\ell} \circ \ldots \circ s_{i_0}(\mathrm{id}_{[n]})$. We define the map $\psi(m)_\ell$ from $H\mathbb{F}_p(\Delta_n/\partial\Delta_n)(m)_\ell = \mathbb{F}_p(\Delta_n/\partial\Delta_n)_\ell \otimes \tilde{\mathbb{Z}}(S^m_\ell)$ to $(H\mathbb{F}_p \vee \Sigma^n H\mathbb{F}_p)(m)_\ell = \mathbb{F}_p \otimes \tilde{\mathbb{Z}}(S^m_\ell) \vee \Delta_n/\partial\Delta_n \wedge \mathbb{F}_p \otimes \tilde{\mathbb{Z}}(S^m_\ell)$ on generators by setting

$$\psi(s_{i_\ell} \circ \ldots \circ s_{i_0}(*) \otimes x) = 1 \otimes x \in \mathbb{F}_p \otimes \tilde{\mathbb{Z}}(S^m_\ell)$$
$$\psi(s_{i_\ell} \circ \ldots \circ s_{i_0}(\mathrm{id}_{[n]}) \otimes x) = [s_{i_\ell} \circ \ldots \circ s_{i_0}(\mathrm{id}_{[n]}), x] \in \Delta_n/\partial\Delta_n \wedge \mathbb{F}_p \otimes \tilde{\mathbb{Z}}(S^m_\ell)$$

for $x \in S^m_\ell$ and by extending it in a bilinear manner. This map is well-defined and multiplicative. Both spectra have finite stable homotopy groups and are therefore semistable. It thus suffices to show that ψ is a stable homotopy equivalence. The stable homotopy groups on both sides are exterior algebras on a generator in degree n and ψ induces the map on stable homotopy groups that sends 1 to 1 and maps the generator in degree n to a degree n generator. ■

We have a weak equivalence

$$B_{H\mathbb{F}_p}(H\mathbb{F}_p, HA, HC) \to H(B_{\mathbb{F}_p}(\mathbb{F}_p, A, C))$$

for all simplicial $\mathbb{F}_p$-algebras A and C. Let N denote the normalization functor from simplicial $\mathbb{F}_p$-vector spaces to non-negatively graded chain complexes over $\mathbb{F}_p$. This is a lax symmetric monoidal functor, so it sends simplicial commutative $\mathbb{F}_p$-algebras to commutative differential graded $\mathbb{F}_p$-algebras. Note that we obtain isomorphisms of differential graded $\mathbb{F}_p$-algebras

$$N(\mathbb{F}_p(\Delta_{2p-1}/\partial\Delta_{2p-1})) \cong \Lambda_{\mathbb{F}_p}(z), \qquad N(\mathbb{F}_p(\Delta_{2p+1}/\partial\Delta_{2p+1})) \cong \Lambda_{\mathbb{F}_p}(y)$$

because for positive n, $\Delta_n/\partial\Delta_n$ has only a non-degenerate zero cell and a non-degenerate n-cell. Note that $B^{(n)}_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p(\Delta_{2p-1}/\partial\Delta_{2p-1}), \mathbb{F}_p)$ is an $(n+1)$-fold simplicial commutative $\mathbb{F}_p$-algebra. We can calculate the homotopy groups of its diagonal as the homology of the total complex associated to the bicomplex

$$C_{r,s} = N_r\mathrm{diag}^n B^{(n)}_{\mathbb{F}_p}(\mathbb{F}_p, N_s\mathbb{F}_p(\Delta_{2p-1}/\partial\Delta_{2p-1}), \mathbb{F}_p) \cong N_r\mathrm{diag}^n B^{(n)}_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p(\Delta_{2p-1}/\partial\Delta_{2p-1})_s, \mathbb{F}_p).$$

As the differential in s-direction is trivial the spectral sequence collapses at the E^2-term with total homology isomorphic to the given by the homology of the differential graded n-fold bar construction. These homology groups were calculated in [BLPRZ∞] and we obtain that $\pi_*\mathrm{diag}B^{(n)}_{H\mathbb{F}_p}(H\mathbb{F}_p, E(z), H\mathbb{F}_p) \cong B^{n+2}_{\mathbb{F}_p}(y)$. Similarly, $\pi_*\mathrm{diag}B^{(n-1)}_{H\mathbb{F}_p}(H\mathbb{F}_p, \underline{E(y)}, H\mathbb{F}_p) \cong B^{n+1}_{\mathbb{F}_p}(x)$. This proves Theorem 3.1.

4 The number ring case

The aim of this section is to prove Theorem 4.3, calculating the higher topological Hochschild homology of number rings with coefficients in the residue field. The calculation starts with the following observation.

Lemma 4.1 Let B be a characteristic zero complete discrete valuation ring with residue field $\mathbb{F}_q$ of characteristic $p > 0$ and let P denote the ideal of all elements with positive valuation in B. Then

$$HH_1^{\mathbb{Z}_p}(B, B/P) \cong \begin{cases} B/P, & \text{if } B \text{ is ramified over } \mathbb{Z}_p \text{ at } P, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: By Proposition 12 in Chapter 3 of [S79], B is generated over $\mathbb{Z}_p$ by a single element, $B = \mathbb{Z}_p[x]/(f(x))$ for some monic polynomial f. By a well-known calculation which can be traced back to [T57], for any ring R and monic polynomial $f(x)$ over it, $HH_1^R(R[x]/(f(x))) \cong R[x]/(f(x), f'(x))$. By Corollary 2 of Chapter 3 of [S79], the ideal $(f'(x))$ in $\mathbb{Z}_p[x]/(f(x))$ is equal to the different ideal $\mathcal{D}_{B/\mathbb{Z}_p}$, so the result for B over $\mathbb{Z}_p$ becomes

$$HH_1^{\mathbb{Z}_p}(B) \cong B/\mathcal{D}_{B/\mathbb{Z}_p}.$$

Since B is commutative, we also know that $HH_0^{\mathbb{Z}_p}(B) \cong B$ is free over B, hence $\mathrm{Tor}_s^B(HH_0^{\mathbb{Z}_p}(B), B/P) = 0$ for $s > 0$. If we tensor the Hochschild complex of B over B with B/P, then we get by the universal coefficient theorem, that

$$HH_1^{\mathbb{Z}_p}(B, B/P) \cong HH_1^{\mathbb{Z}_p}(B) \otimes_B B/P \cong B/\mathcal{D}_{B/\mathbb{Z}_p} \otimes_B B/P \cong B/(\mathcal{D}_{B/\mathbb{Z}_p}, P).$$

If $\mathcal{D}_{B/\mathbb{Z}_p} \subseteq P$, this is just B/P, but if not, by the maximality of P in B, $B/(\mathcal{D}_{B/\mathbb{Z}_p}, P) \cong 0$. Theorem 1 in Chapter 3 of [S79] says that an extension B of $\mathbb{Z}_p$ is ramified at an ideal P of B if and only if P divides the different ideal $\mathcal{D}_{B/\mathbb{Z}_p}$. ∎

From this we establish the one-dimensional (ordinary topological Hochschild homology) case of Theorem 4.3, which is closely related to Theorem 4.4 in [LM00], and in fact in the unramified case is exactly Theorem 4.4 (i) there (since in the unramified case $P = pA$ for a rational prime p). The symbol x_i in the statement indicates a generator of degree i.

Proposition 4.2 Let A be the ring of integers in a number field, and let P be a nonzero prime ideal in A. Denote the residue field A/P by $\mathbb{F}_q$.

(i) If A is ramified over $\mathbb{Z}$ at P, $THH_*^{[1]}(A_P^\wedge, \mathbb{F}_q) \cong \mathbb{F}_q[x_2] \otimes_{\mathbb{F}_q} \Lambda_{\mathbb{F}_q}[x_1]$.

(ii) If A is unramified over $\mathbb{Z}$ at P, $THH_*^{[1]}(A_P^\wedge, \mathbb{F}_q) \cong \mathbb{F}_q[x_{2p}] \otimes_{\mathbb{F}_q} \Lambda_{\mathbb{F}_q}[x_{2p-1}]$.

Proof: We set $B = A_P^\wedge$, to get a ring that satisfies the conditions of Lemma 4.1. We now use P to denote the ideal in B obtained as PB for the ideal P of A. We use Morten Brun's spectral sequence from Theorem 3.3 in [LM00] for the map $B \to B/P$. This gives a multiplicative spectral sequence

$$E_{r,s}^2 = THH_r(B/P, \mathrm{Tor}_s^B(B/P, B/P)) \Rightarrow THH_{r+s}(B, B/P).$$

Since P is a principal ideal in B, generated by any uniformizer π, the resolution

$$0 \longrightarrow B \xrightarrow{\cdot\pi} B \longrightarrow B/P \longrightarrow 0$$

shows that $\mathrm{Tor}_*^B(B/P, B/P) \cong \Lambda_{\mathbb{F}_q}(\tau_1)$ for a 1-dimensional generator τ_1, where $B/P = (A_P^\wedge)/P \cong A/P = \mathbb{F}_q$.

Bökstedt showed in [Bö∞] that $THH_*(\mathbb{F}_p) \cong \mathbb{F}_p[u_2]$, and since $HH_*(\mathbb{F}_q)$ consists only of $\mathbb{F}_q$ in dimension zero, the spectral sequence of Theorem 2.2 in [Li00]

$$E_{r,s}^2 = HH_r^{\mathbb{F}_p}(\mathbb{F}_q, THH_s(\mathbb{F}_p; \mathbb{F}_q)) \Rightarrow THH_{r+s}(\mathbb{F}_q)$$

consists only of $\mathbb{F}_q \otimes \mathbb{F}_p[u_2]$ in the zeroth row, we get that

$$THH_*(\mathbb{F}_q) \cong \mathbb{F}_q[u_2].$$

Thus Brun's spectral sequence takes the form

$$E^2_{*,0} \cong \mathbb{F}_q[u_2], \quad E^2_{*,1} \cong \tau_1 \cdot \mathbb{F}_q[u_2].$$

From Lemma 4.1 and the fact that Hochschild and topological Hochschild homology agree in degree 1, we know that we end up with nothing in total degree 1 if B is unramified over $\mathbb{Z}_p$, and with a copy of $\mathbb{F}_q$ if B is ramified. So we get

$$d^2(u_2) = \begin{cases} 0, & \text{if } B \text{ is ramified over } \mathbb{Z}_p \text{ at } P, \\ (\text{unit}) \cdot \tau_1, & \text{otherwise.} \end{cases}$$

In the ramified case we already know that d^2 vanishes on 1 and τ_1, since there is nothing these elements could hit. Therefore, we get that $d^2 = 0$. As d^2 is the last differential that could be nontrivial, $E^\infty_{*,*} \cong E^2_{*,*} \cong \Lambda_{\mathbb{F}_q}(\tau_1) \otimes_{\mathbb{F}_q} \mathbb{F}_q[u_2]$, and since this is the multiplication with the fewest relations possible that could be defined on a graded-commutative algebra with this linear structure, extensions cannot give any other multiplicative structure and we get

$$THH_*(B, B/P) \cong \Lambda_{\mathbb{F}_q}(\tau_1) \otimes_{\mathbb{F}_q} \mathbb{F}_q[u_2].$$

In the unramified case, knowing what d^2 does on the generators shows us that $d^2(u_2^a) = (\text{unit}) \cdot \tau_1 \cdot u_2^{a-1}$ when p does not divide a, but $d^2(u_2^{pk}) = 0$ and nothing hits the elements $\tau_1 \cdot u_2^{pk-1}$. Again d^2 is the last differential that could be nonzero so $E^\infty_{*,*} \cong E^3_{*,*} \cong \Lambda_{\mathbb{F}_q}(\tau_1 \cdot u_2^{p-1}) \otimes_{\mathbb{F}_q} \mathbb{F}_q[u_2^p]$, and since this is again a multiplication with the fewest relations possible that could be defined on a graded-commutative algebra with this linear structure, extensions cannot give any other multiplicative structure and we get

$$THH_*(B, B/P) \cong \Lambda_{\mathbb{F}_q}(\tau_1 \cdot u_2^{p-1}) \otimes_{\mathbb{F}_q} \mathbb{F}_q[u_2^p].$$

∎

Theorem 4.3 Let A be the ring of integers in a number field, and let P be a nonzero prime ideal in A. Denote the residue field A/P by $\mathbb{F}_q$.

Then

$$THH^{[n]}_*(A^\wedge_P, \mathbb{F}_q) \cong B^n_{\mathbb{F}_q}(x_P) \otimes_{\mathbb{F}_q} B^{n+1}_{\mathbb{F}_q}(y_P)$$

where

(i) $|x_P| = 2$ and $|y_P| = 0$ if A is ramified over $\mathbb{Z}$ at P, and

(ii) $|x_P| = 2p$ and $|y_P| = 2p - 2$, if A is unramified over $\mathbb{Z}$ at P.

Proof: The $n = 1$ case is true by Proposition 4.2, with $x = x_2$ and y zero-dimensional (so that $B_{\mathbb{F}_q}(y) \cong \Lambda_{\mathbb{F}_q}(x_1)$) in the ramified case, and with $x = x_{2p}$ and y of dimension $2p - 2$ (so that $B_{\mathbb{F}_q}(y) \cong \Lambda_{\mathbb{F}_q}(x_{2p-1})$) in the unramified case.

The rest of the proof proceeds by exact analogy to the calculation of $THH^{[n]}_*(\mathbb{Z}, \mathbb{F}_p)$. ∎

Note 4.4 The ramified case Theorem 4.3 (i) can actually be proven quite algebraically by noting that for an arbitrary flat ring A and A-bimodule M, the linearization map $THH(A, M) \to H(HH^{\mathbb{Z}}(A, M))$ is 3-connected so that the first Postnikov sections of $THH(A, M)$ and $H(HH^{\mathbb{Z}}(A, M))$ agree. As a matter of fact, when A is a $\mathbb{Z}_{(p)}$-algebra, Bökstedt's calculation of the topological Hochschild homology of the integers gives that Theorem 2.3 of [Li00] implies that this can be improved to saying that $THH(A, M) \to H(HH^{\mathbb{Z}}(A, M))$ is $(2p-1)$-connected. This means that the Postnikov section involved in the crucial step moving from THH to to the algebraic $THH^{[2]}$ coincides with that of Hochschild homology. This was how we originally established the calculation in the ramified case.

Note 4.5 The unramified case, Theorem 4.3(ii), could have also been deduced from Theorem 3.1 by showing that $THH_*^{[n]}(A_P^\wedge, A/P) \cong THH_*^{[n]}(\mathbb{Z}_p, \mathbb{F}_p) \otimes \mathbb{F}_q$ (where $\mathbb{F}_q = A/P$) as an augmented $\mathbb{F}_q$-algebra, where the augmentation on the right-hand side comes from the augmentation of $THH_*^{[n]}(\mathbb{Z}, \mathbb{F}_p)$ over $\mathbb{F}_p$, tensored with the identity of $\mathbb{F}_q$. This is true for $n = 1$ by Theorem 4.4 (i) of [LM00], and then we proceed inductively, using the decomposition from (3.1.0). This yields in this case a decomposition

$$THH^{[n+1]}(A_P^\wedge, \mathbb{F}_q) \simeq H\mathbb{F}_q \wedge^L_{THH^{[n]}(A_P^\wedge, \mathbb{F}_q)} H\mathbb{F}_q.$$

and a multiplicative spectral sequence $\mathrm{Tor}_{*,*}^{THH_*^{[n]}(A_P^\wedge, \mathbb{F}_q)}(\mathbb{F}_q, \mathbb{F}_q) \Rightarrow THH_*^{[n+1]}(A_P^\wedge, \mathbb{F}_q)$, which can be rewritten by the inductive hypothesis as

$$\mathrm{Tor}_{*,*}^{THH_*^{[n]}(\mathbb{Z}_p,\mathbb{F}_p)\otimes\mathbb{F}_q}(\mathbb{F}_p\otimes\mathbb{F}_q, \mathbb{F}_p\otimes\mathbb{F}_q) \cong \mathrm{Tor}_{*,*}^{THH_*^{[n]}(\mathbb{Z}_p,\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)\otimes \mathrm{Tor}_*^{\mathbb{F}_q}(\mathbb{F}_q, \mathbb{F}_q) \cong \mathrm{Tor}_{*,*}^{THH_*^{[n]}(\mathbb{Z},\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)\otimes\mathbb{F}_q,$$

where the first factor is the image of the E^2-term of the spectral sequence calculating $THH_*^{[n+1]}(\mathbb{Z}, \mathbb{F}_p)$ and the second term is in $E^2_{0,0}$ and therefore can cause no nontrivial differentials or multiplicative extensions. This splitting is a splitting of algebras and the augmentation is that of the first factor tensored with the identity of the second.

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