A SIMPLICIAL COMPLEX OF NICHOLS ALGEBRAS

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ABSTRACT. We translate the concept of restriction of an arrangement in terms of Hopf algebras. In consequence, every Nichols algebra gives rise to a simplicial complex decorated by Nichols algebras with restricted root systems. As applications, some of these Nichols algebras provide Weyl groupoids which do not arise for Nichols algebras over finite groups and in fact we realize all root systems of finite Weyl groupoids of rank greater than three. Further, our result explains the root systems of the folded Nichols algebras over nonabelian groups and of generalized Satake diagrams.

CONTENTS

1. Introduction

Nichols algebras are Hopf algebras in a braided category, that enjoy certain universal properties and appear naturally in every pointed Hopf algebra. The most prominent examples are the Borel parts $u_q(\mathfrak{g})^{\pm}$ of the small quantum groups. In fact the Lie-theoretical flavour is retained in general, and we know from [AHS10] that any Nichols algebra is controlled by a Weyl groupoid. This is a natural generalization of a Weyl group, where different Dynkin diagrams are attached to the different groupoid objects. Since [Cun11] we have a good perception of a Weyl groupoid as an arrangement of hyperplanes, and in [CH13] all finite Weyl groupoids were classified: There are infinitely many finite Weyl groupoids of rank 2, an additional infinite series between D_n and C_n, as well as 74 sporadic cases up to rank 8.

However, compared to the theory of semisimple Lie algebras where root systems provide a complete classification (at least over the complex numbers), for Nichols algebras the situation is more complicated: There are non-isomorphic Nichols algebras whose corresponding Weyl groupoids are equivalent, and it is not known yet whether all Weyl groupoids arise as symmetry structures of Nichols algebras (not even in the case of finite Weyl groupoids). The latter problem has been brought up on several occasions, notably by the first author at the Oberwolfach Mini-Workshop on "Nichols Algebras and Weyl Groupoids" in 2012:

Question 1.1. Is there a Weyl groupoid which does not occur as symmetry structure of a Nichols algebra?[1] Is there a characterization of those root systems that cannot appear as root systems of Nichols algebras?

One way to answer part of these questions is to introduce constructions producing new Nichols algebras (and hopefully new Weyl groupoids as well). For example, *folding of root systems* has been investigated by the second author to construct new large rank Nichols algebras over nonabelian groups [Len14a]. As proven recently by Heckenberger and Vendramin [HV14], these Nichols algebras are the only examples in large rank and characteristic zero. There are four exceptions of rank

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[1]For Nichols algebras over groups, the answer is positive.

two and three which are not obtained by this folding construction, and for rank one the classification is still largely open. The second author hence asked at the Oberwolfach Mini-Workshop on "Infinite dimensional Hopf algebras" in 2014:

Question 1.2. Systematically explain the impact of the folding contruction on the root system[2]. Are there more general folding constructions on the root system of a Nichols algebra? Is there a closed construction of all Nichols algebras over nonabelian groups in this way?

In this article we discuss such a construction, which partly answers the previous questions, namely *restriction* on the arrangement of hyperplanes given by the root system. More precisely we proceed as follows:

The set $\mathcal{A}$ of hyperplanes $\alpha^{\perp} \subset V$ orthogonal to the roots is a *crystallographic arrangement* and this induces an equivalence between crystallographic arrangements and finite root systems. For details see the preliminaries in Section 2.

Let X be a subspace of V. Then the restriction $\mathcal{A}^X$ is the set of hyperplanes in $\mathcal{A}$ of the form $X \cap H$ for $H \in \mathcal{A}$. This construction is discussed in Section 3. Typically, not all hyperplanes $H \in \mathcal{A}$ give rise to hyperplanes $X \cap H$ in $\mathcal{A}^X$, and several hyperplanes in $\mathcal{A}$ may give rise to the same hyperplane in $\mathcal{A}^X$.

There are two important special cases:
• The case when X is an intersection of some hyperplanes of $\mathcal{A}$, we call this *parabolic restriction*. Then $\mathcal{A}^X$ is automatically again crystallographic. In fact, most finite Weyl groupoids appear in this way, including examples which are not attained from Nichols algebras over groups.
• The case when X is the fixpoint set of an automorphism g of $\mathcal{A}$. In general, $\mathcal{A}^X$ must not be crystallographic, but in several cases it is. In the special case where g permutes a set of simple roots (called *permutation restriction*), these restrictions describe the root system of the folded Nichols algebra over the centrally extended group.

In Section 4 we turn our attention to Weyl groupoids of Nichols algebras. Let $J \subset I$ be a subset of simple roots for some Nichols algebra $\mathcal{B}(M)$, denote by $M_J \subset M$ the corresponding sub-Yetter-Drinfel'd module and consider the associated Nichols

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[2]This has previously been calculated by hand.

algebra of coinvariants

$$\mathcal{B}(\bar{M}) := \mathcal{B}(M)^{coin(\mathcal{B}(M_J))}.$$

This is a Nichols algebra in the category of $\mathcal{B}(M_J)$-Yetter-Drinfel's modules. The key result of this article in Theorem 4.13 is that the root system of $\mathcal{B}(\bar{M})$ is the parabolic restriction $\mathcal{A}^X$, $X = J^\perp$. Moreover, $\mathcal{B}(M_J)$ (determining the category) corresponds precisely to the sub-arrangement of hyperplanes in $\mathcal{A}$ that do not give rise to hyperplanes in $\mathcal{A}^X$, while the number of different hyperplanes in $\mathcal{A}$ that give rise to the same hyperplane in $\mathcal{A}^X$, determines the dimensions of the Yetter-Drinfel'd modules in the PBW-basis of $\mathcal{B}(\bar{M})$.

In Section 5 we conclude by viewing the set of all parabolic restrictions of a Nichols algebra as a simplicial complex. In this picture, the reflection operation for Nichols algebras has a particularly nice interpretation in terms of restrictions and dualization.
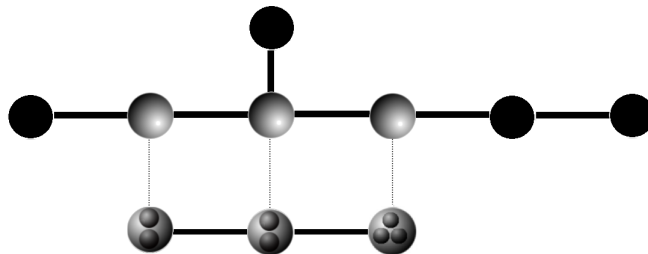
We now discuss *applications*:

As first result, we obtain Nichols algebras with Weyl groupoids, which were not attained from a previously known Nichols algebra over a group:

Theorem (4.14)**.**

(1) *There exist Nichols algebras whose corresponding crystallographic arrangements are the sporadic arrangements of rank three labeled $7, 13, 14, 15, 20, 23$, although these Nichols algebras do not exist over any finite group.*

(2) *Since every crystallographic arrangement of rank greater than three is a restriction of a Weyl arrangement, every crystallographic arrangement of rank greater than three is symmetry structure of some Nichols algebra.*

Example 1.3. Let $\mathcal{B}(M) = u_q(E_7)^+$, which is a Nichols algebra of dimension $\mathrm{ord}(q^2)^{63}$. We consider the parabolic restriction indicated in the diagram:

Then the restriction of the Weyl group arrangement E_7 with 63 roots has a Weyl groupoid of sporadic type 7 and 13 roots (one of which is nonreduced), namely

$\bar{R}_+^a$	$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$	$(1,1,0)$	$(0,1,1)$	$(1,1,1)$	$+2(1,1,1)$
multiplicity	2	2	3	4	6	12	$+3$
$\bar{R}_+^a$	$(1,2,1)$	$(2,2,1)$	$(1,2,2)$	$(2,3,2)$	$(2,3,3)$	$(2,4,3)$	$(3,4,3)$
multiplicity	6	3	6	6	2	1	2

This yields a Nichols algebra $\mathcal{B}(\bar{M})$ of dimension $\mathrm{ord}(q^2)^{58}$ over an object M of rank 3 and dimension $2+2+3$ in the braided category $^{\mathcal{B}(M_J)}_{\mathcal{B}(M_J)}\mathcal{YD}$ with $J = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7\}$, where $\mathcal{B}(M_J)$ is an ordinary Borel part $u_q(A_1 \times A_1 \times A_2)^+$.

The graphical representation using black dots is suggested by *Satake diagrams* of Lie algebras[3]. Indeed, we define *folding* of a crystallographic arrangement (resp. the root system of a Weyl groupoid) as the restriction to the fixed-point set X of some automorphism group of the root system, and we prove a respective Lemma 3.9 that enables us to classify such scenarios again by Satake diagrams. Note that we even get new Satake diagrams for ordinary Lie algebras, where the restricted root system corresponds to a Weyl groupoid. This gives rise to the following potential applications:

First, it points to the more general folding construction for Nichols algebras: In this article we do *not* define the respective (non-parabolic) restriction of Nichols algebras, that should come with an additional group action on the root spaces M_α, but in the cases in [Len14a] the restriced root system $\mathcal{A}^X$ yields precisely the correct root system. In Example 4.12 we identify an inclusion chain of three Nichols algebra, for which restriction would reproduce the correct root system and root space dimensions for the exceptional Nichols algebras of rank $3, 2, 1$ over nonabelian groups involving $\mathbb{S}_3$ in [HV14]:
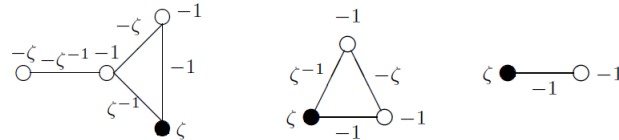
Secondly, Satake diagrams for Nichols algebras of Lie type appear prominently in the theory of quantum symmetric pairs, introduced by Noumi, Sugitani, and Dijkhuizen for $\mathfrak{g}$ of classical type in connection with the reflection equation and

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[3]We thank S. Kolb for having brought up this topic to our attention at the Oberwolfach miniworkshop mentioned above.

in theoretical physics (see e.g. [NS95]) and independently by G. Letzter from the Iwasawa decomposition for the quantum group (see e.g. [Let97]), generalized to Kac Moody algebras by [Kolb14]. Our results show that one can consider more general Satake diagrams for Lie type root systems, which do not yield Lie type root systems anymore, as in the example above. On the other hand they allow Satake diagrams to be considered for arbitrary Nichols algebras, in particular for super Lie algebras and color Lie algebras. Conversely, these authors have explicit methods to construct coideal subalgebras corresponding to the fixed-point set of the root system automorphism. This might give the right inspiration for constructing the respective restricted Nichols algebra in the general (non-parabolic, non-permutation) case.

2. Weyl groupoids and crystallographic arrangements

2.1. **Simplicial arrangements.** Let $r \in \mathbb{N}$, $V := \mathbb{R}^r$. For $\alpha \in V^*$ we write $\alpha^\perp = \ker(\alpha)$. We first recall the definition of a simplicial arrangement (compare [OT92, 1.2, 5.1]).

Definition 2.1. An *arrangement of hyperplanes* $\mathcal{A}$ is a finite set of hyperplanes in V. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (*chambers*) of $V \setminus \bigcup_{H \in \mathcal{A}} H$. If every chamber K is an *open simplicial cone*, i.e. there exist $\alpha_1^\vee, \ldots, \alpha_r^\vee \in V$ such that

$$K = \left\{ \sum_{i=1}^r a_i \alpha_i^\vee \mid a_i > 0 \quad \text{for all} \quad i = 1, \ldots, r \right\} =: \langle \alpha_1^\vee, \ldots, \alpha_r^\vee \rangle_{>0},$$

then $\mathcal{A}$ is called a *simplicial arrangement*.

Example 2.2.

(1) The picture on the left is a simplicial arrangement in $\mathbb{R}^2$. The picture on the right is a representation of a three dimensional simplicial arrangement in the projective plane. The simplicial cones become triangles in this representation.
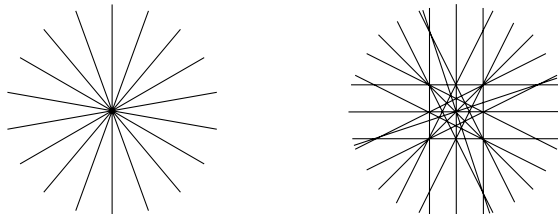
(2) Let W be a real reflection group, R the set of roots of W. Then $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ is a simplicial arrangement.

2.2. **Crystallographic arrangements.** Let $\mathcal{A} = \{H_1, \ldots, H_n\}$, $|\mathcal{A}| = n$ be simplicial. For each H_i, $i = 1, \ldots, n$ we choose an element $x_i \in V^*$ such that $H_i = x_i^\perp$ and let $R := \{\pm x_1, \ldots, \pm x_n\} \subseteq V^*$.

For each chamber $K \in \mathcal{K}(\mathcal{A})$ set

$$\Delta^K = \{ \text{ normal vectors in } R \text{ of the walls of } K \text{ pointing to the inside } \}.$$

If $\alpha_1^\vee, \ldots, \alpha_r^\vee$ is the dual basis to $\Delta^K = \{\alpha_1, \ldots, \alpha_r\}$, then $K = \langle \alpha_1^\vee, \ldots, \alpha_r^\vee \rangle_{>0}$ since $\mathcal{A}$ is simplicial.

We are now ready for the main definition.

Definition 2.3. Let $\mathcal{A}$ be a simplicial arrangement and $R \subseteq V^*$ a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$:

$$R \subseteq \sum_{\alpha \in \Delta^K} \mathbb{Z}\alpha.$$

Two crystallographic arrangements $(\mathcal{A}, R)$, $(\mathcal{A}', \bar{R})$ in V are called *equivalent* if there exists $\psi \in \mathrm{Aut}(V^*)$ with $\psi(R) = \bar{R}$. We then write $(\mathcal{A}, R) \cong (\mathcal{A}', \bar{R})$.

Example 2.4.   (1) Let R be the set of roots of the root system of a crystallographic Coxeter group. Then $(\{\alpha^\perp \mid \alpha \in R\}, R)$ is a crystallographic arrangement.

(2) If $R_+ := \{(1,0), (3,1), (2,1), (5,3), (3,2), (1,1), (0,1)\}$, then $(\{\alpha^\perp \mid \alpha \in R_+\}, R_+ \dot\cup -R_+)$ is a crystallographic arrangement.

As most crystallographic arrangements are not defined by reflection groups, they have less symmetries in general. As a substitute for this large symmetry group, it turns out that a corresponding *Weyl groupoid* is the right symmetry structure.

2.3. **Cartan graphs and Weyl groupoids.** We now define the notion of a Weyl groupoid which was introduced by Heckenberger and Yamane [HY08] and reformulated in [CH09]. But before we present the axioms, let us look at an example.

Example 2.5. Let $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$, and

$$R_+^a := \{\alpha_1, \alpha_2, \alpha_3, (0,1,1), (0,1,2), (1,0,1), (1,1,1), (1,1,2)\}.$$

For $1 \le i, j \le 3$, define entries $c_{i,j}$ of a matrix C by

$$c_{i,j} := -\max\{k \mid k\alpha_i + \alpha_j \in R_+^a\}, \qquad c_{i,i} := 2 \qquad \text{for} \quad i \ne j,$$

thus

$$C^a = (c_{i,j})_{i,j} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

This is a *generalized Cartan matrix*. It defines *reflections* via

$$\sigma_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i \qquad \text{for } j = 1, 2, 3.$$

For instance,

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\sigma_1(R_+^a) = \{-\alpha_1, \alpha_2, (1,0,1), (1,1,1), (2,1,2), \alpha_3, (0,1,1), (1,1,2)\}.$$

The elements of $\sigma_1(R_+^a)$ are *positive* or *negative*. Let $R^a = R_+^a \cup -R_+^a$ and $R^b = \sigma_1(R^a) =: R_+^b \cup -R_+^b$. Again, one can construct a Cartan matrix from R_+^b and it gives new reflections. In this example, we obtain a diagram:

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \overset{\sigma_3}{\rule{1cm}{0.4pt}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \overset{\sigma_2}{\rule{1cm}{0.4pt}} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\Big| \sigma_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big| \sigma_1$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix} \qquad\qquad\qquad\qquad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

For the general definition, we first recall:

Definition 2.6. Let $I := \{1, \dots, r\}$ and $\{\alpha_i \mid i \in I\}$ the standard basis of $\mathbb{Z}^I$. A *generalized Cartan matrix* $C = (c_{ij})_{i,j \in I}$ is a matrix in $\mathbb{Z}^{I \times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,

(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

The above diagram of matrices is a *Cartan graph*[4]:

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[4]In earlier papers, Cartan graphs were called Cartan schemes. The new term was chosen by Andruskiewitsch, Heckenberger and Schneider to avoid confusion with geometric schemes.

Definition 2.7. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k \in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I \times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$$

is called a *Cartan graph* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

Definition 2.8. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan graph. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

(1) $$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I.$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ^a_i with $i \in I$ and $a \in A$, where σ^a_i is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The cardinality of I is the *rank of* $\mathcal{W}(\mathcal{C})$.

Definition 2.9. A Cartan graph is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan graph is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}^a\}$ for all $a \in A$. There is a straight forward notion of *equivalence* of Cartan graphs which we skip here.

Let $\mathcal{C}$ be a Cartan graph. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}^a \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \mid k \in \mathbb{N}_0, \, i_1, \ldots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a \in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called positive (resp. negative) if $\alpha \in \mathbb{N}_0^I$ (resp. $\alpha \in -\mathbb{N}_0^I$).

Definition 2.10. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subseteq \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i \rho_j)^{m^a_{i,j}}(a) = a$.

The root system $\mathcal{R}$ is called *finite* if for all $a \in A$ the set R^a is finite. By [CH09, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R} = \mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case.

Remark 2.11. If $\mathcal{C}$ is a Cartan graph and there exists a root system of type $\mathcal{C}$, then $\mathcal{C}$ satisfies

(C3) If $a, b \in A$ and $\mathrm{id} \in \mathrm{Hom}(a, b)$, then $a = b$.

Example 2.12 (Lie type). Let $\mathfrak{g}$ be a semisimple finite-dimensional complex Lie algebra. Then this is uniquely determined (up to isomorphisms) by its corresponding root system, which is the root system of a finite Weyl group W. The corresponding Cartan graph has exactly one object a where C^a is the Cartan matrix of W. The set R^a is the root system of W.

2.4. **Classification of finite Weyl groupoids.** Connected simply connected Cartan graphs for which the real roots are a finite root system (these are also called *universal finite Weyl groupoids*) are in one-to-one correspondence with crystallographic arrangements. Under the correnspondence, every chamber of the arrangements corresponds to an object; the sets R^a are the coordinates of R with respect to the basis Δ^K where K corresponds to a.

Theorem 2.13. *[see* [Cun11]*] Let $\mathfrak{A}$ be the set of all crystallographic arrangements and $\mathfrak{C}$ be the set of all connected simply connected Cartan graphs for which the real roots are a finite root system. Then the map*

$$\mathfrak{C}/_{\cong} \to \mathfrak{A}/_{\cong}, \quad \overline{\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})} \mapsto \overline{(\{\alpha^\perp \mid \alpha \in R^a\}, R^a)},$$

where a is any object of $\mathcal{C}$, is a bijection.

A series of five papers by the first author and Heckenberger culminates in the following complete classification of finite Weyl groupoids and thus equivalently of crystallographic arrangements.

Theorem 2.14 (see [CH13]). *There are exactly three families of irreducible crystallographic arrangements:*

(1) *The family of rank two parametrized by triangulations of convex n-gons by non-intersecting diagonals.*

(2) *For each rank $r > 2$, arrangements of type A_r, B_r, C_r and D_r, and a further series of $r - 1$ arrangements denoted $\mathcal{A}_r^k(2)$ in [OT92, 6.4].*

(3) Further 74 "sporadic" arrangements of rank r, $3 \leq r \leq 8$.

Remark 2.15. Theorem 2.14 classifies simply-connected Cartan graphs. Every Cartan graph has a simply-connected cover. For example the Cartan graph of Lie type has a single Cartan matrix and the Weyl groupoid has a single object with automorphism group the Weyl group W. The simply-connected cover has $|W|$ Cartan matrices, all of the same type, and its Weyl groupoid has $|W|$ objects and no nontrivial automorphisms.

Automorphism groups and minimal quotients of finite Weyl groupoids have all been determined in [CH13]. Whether one can consider such non-simply-connected quotients depends on additional data in the application. For example, the Lie algebra $\mathfrak{sl}_3$ resp. the Lie superalgebra $\mathfrak{sl}(2|1)$ have the same associated arrangement A_2, but in the first case one usually considers the Weyl group (one object), while in the latter case some roots are labeled differently, so there are two types of chambers.

3. Restrictions of arrangements and root systems

Definition 3.1 ([OT92, 1.12-1.14]). Let $\mathcal{A}$ be an arrangement in V. We denote $L(\mathcal{A})$ the set of all nonempty intersections of elements of $\mathcal{A}$.

For a subspace $X \leq V$, define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ called the *localization at X* by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Define an arrangement $\mathcal{A}^X$ in X called the *restriction to X* by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \backslash \mathcal{A}_X \text{ and } X \cap H \neq \varnothing\}.$$

If $\mathcal{A}$ is a reflection arrangement of a Coxeter group W, then localizations of $\mathcal{A}$ are the reflection arrangements of parabolic subgroups of W. Thus localizations are easy to understand from the algebraic point of view. Restrictions $\mathcal{A}^X$ however are not reflection arrangements in general, even in the case when X is in the intersection lattice $L(\mathcal{A})$. For crystallographic restrictions on the other hand, we get our first class of examples:

3.1. **Parabolic restriction.** In this subsection we discuss the following case of restriction:

Definition 3.2. A *parabolic restriction* of an arrangement $\mathcal{A}$ is a restriction $\mathcal{A}^X$ to an intersection of existing hyperplanes $X \in L(\mathcal{A})$.

In contrast to parabolic localization $\mathcal{A}_X$ this corresponds to *quotienting out* a parabolic subgroup and the result is in general not a reflection arrangement. Note that any parabolic restriction can be obtained by repeatedly restricting to a single hyperplane $H \in \mathcal{A}$.

It is an easy fact that (see [CRT12], [BC12]):

Lemma 3.3. *Let $(\mathcal{A}, R)$ by a crystallographic arrangement, then any parabolic restriction $\mathcal{A}^X$ to some $X \in L(\mathcal{A})$ is again a crystallographic arrangement.*
More precisely, a root system $\bar{R}$ for $\mathcal{A}^X$ is given as follows: Suppose without loss of generality $X \in \mathcal{A}$ a single hyperplane and a chamber K chosen adjacient to X, say by suitable numbering $X = \alpha_1^\perp$. Then the restriction of $\mathcal{A}$ to $X = \alpha_1^\perp$ is

$$\mathcal{A}^{\alpha_1^\perp} = \{\beta'^\perp \mid \beta \in R, \ \beta \neq \alpha_1\},$$

where if $\beta = \sum_{\alpha \in \Delta^K} b_\alpha \alpha$, then

$$\beta' = \frac{1}{\gcd(b_\alpha \mid \alpha_1 \neq \alpha \in \Delta^K)} \sum_{\alpha_1 \neq \alpha \in \Delta^K} b_\alpha \alpha.$$

Thus we obtain the restriction by deleting the coordinate to α_1 and reducing to the shortest vector in the lattice.

Example 3.4. Let $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$.

a) Let $R_+^a := \{\alpha_1, \alpha_2, \alpha_3, (0,1,1), (1,1,0), (1,1,1)\}$, and let $\mathcal{A} := \{\alpha^\perp \mid \alpha \in R_+^a\}$ be the crystallographic arrangement of Lie type A_3. Then the restriction $\mathcal{A}^H$ of $\mathcal{A}$ to any hyperplane $H = \alpha_i^\perp$ is the crystallographic arrangement of type A_2 defined by $\bar{R}_+^a = \{(1,0), (0,1), (1,1)\}$.

b) Let $R_+^a := \{\alpha_1, \alpha_2, \alpha_3, (0,1,1), (0,1,2), (1,0,1), (1,1,1), (1,1,2)\}$, and let $\mathcal{A} := \{\alpha^\perp \mid \alpha \in R_+^a\}$ (which is not of Lie type). Then the restriction $\mathcal{A}^H$ of $\mathcal{A}$ to the hyperplane $H = \alpha_1^\perp$ is the crystallographic arrangement of Lie type B_2 (or C_2 depending on the used definition) defined by $\bar{R}_+^a = \{(1,0), (0,1), (1,1), (1,2)\}$.

c) Now let

$$R_+^a = \{(0,0,1), (0,1,0), (0,1,1), (0,1,2), (1,0,0), (1,1,0), (1,1,1), (1,1,2), (1,2,2)\},$$

$\mathcal{A} := \{\alpha^\perp \mid \alpha \in R_+^a\}$, thus R^a is a root system of type B_3. Choose $H := (1,0,0)^\perp$. Then $\mathcal{A}^H$ is the set of kernels of

$$\bar{R}_+^a = \{(0,1), (1,0), (1,1), (1,2)\}.$$

Looking more closely, we notice that there are several hyperplanes in $\mathcal{A}$ which restrict to the same element of $\mathcal{A}^H$. For example, three different roots map to the root $(1,1)$, namely $(0,1,1),(1,1,1),(1,2,2)$.

We can see two effects at this last example:

(1) It may be useful to keep track of the number of hyperplanes falling together under restriction. Thus instead of considering arrangements of hyperplanes, one should consider arrangements of hyperplanes with multiplicities (these are called *multiarrangements*).

(2) A-priori, the vectors with the deleted coordinate do not form a reduced root system, i.e. they may have the form $k\alpha$ for $k \in \mathbb{Z}\backslash\{\pm 1\}$ and α in the lattice. The reduction requires to rescale a root, for example $(1,2,2) \mapsto \frac{1}{2}(2,2)$.

It would make sense to include the information of these two situations into the setting of Cartan graphs, but this would possibly make things more complicated than necessary. Notice that in the connected simply-connected case, crystallographic arrangements, Weyl groupoids, or Cartan graphs are all uniquely determined by the roots at a single object, hence by a single set $R_+^a \subseteq \mathbb{Z}^r$.

Definition 3.5 (compare [CH13]). We will say that a finite set $\Phi \subseteq \mathbb{Z}^r$ is a *reduced root set of rank r* if there exists a Cartan graph $\mathcal{C}$ of rank r and an injective linear map $w : \mathbb{Z}^r \to \mathbb{Z}^r$ such that $w((R^{\mathrm{re}})^a) = \Phi$ for some object a.

Definition 3.6. A *(nonreduced) root set of rank r* is a set $\Phi \subseteq \mathbb{Z}^r$ such that

$$\left\{ \frac{1}{\gcd(a_1,\ldots,a_r)}\alpha \mid \alpha = (a_1,\ldots,a_r) \in \Phi \right\}$$

is a reduced root set. The *simple roots* of Φ are the simple roots of the root system at the object of the Cartan graph given by the definition of 'root set'.

A *root multiset* is a root set Φ together with a map $\Phi \to \mathbb{N}$, $\alpha \mapsto m_\alpha$.

3.2. **All parabolic restrictions of finite root systems.** By Theorem 2.13 the root systems are in $1:1$ correspondence with crystallographic arrangements. Thus Theorem 2.14 is a complete classification of simply-connected Cartan graphs.

In the following we compute all restrictions to existing hyperplanes (parabolic restrictions, see Definition 3.2) of all root systems in terms of their arrangement:

Theorem 3.7. *In rank three, the only crystallographic arrangements which are (parabolic) restrictions from higher dimensional crystallographic arrangements are*

those labeled

$$1, 2, 3, 6, 7, 8, 9, 13, 14, 15, 20, 23$$

in [CH13]. *From rank four to rank eight, only the reflection arrangements of types* E_6, E_7, *and* E_8 *are* not *restrictions of higher dimensional Weyl arrangements of the infinite series. Every crystallographic arrangement of rank greater than 8 is restriction of a Weyl arrangement.*

Proof. The first two assertion are straightforward calculation performed by the computer using the classification, see Theorem 2.14. The last assertion is proven as follows: Every crystallographic arrangement in dimension greater than 8 is either a Weyl arrangement or an arrangement denoted $\mathcal{A}_\ell^k(2)$ in [OT92, 6]. But according to [OT92,Table 6.2], $\mathcal{A}_\ell^k(2)$ is the restriction of an arrangement $\mathcal{A}_{\ell'}^0$ for some ℓ' large enough, and $\mathcal{A}_{\ell'}^0$ is the arrangement of type $D_{\ell'}$. $\qquad\square$

3.3. **Folding restriction.** A second important source of examples is as follows: We say that a finite group G acts on an arrangement $\mathcal{A}$ if G acts on V such that $g.\mathcal{A} = \mathcal{A}$ for any $g \in G$. Moreover we say G acts on a crystallographic arrangement $(\mathcal{A}, R)$ if $g.R = R$ for all $g \in G$, where g acts on V^* by $(g.\alpha)(v) = \alpha(g^{-1}(v))$ for $\alpha \in V^*$.

If there exists a chamber K where G permutes the simple roots, we call the action a *permutation action* and it is equivalent to a permutation action of G on the Dynkin diagram resp. Cartan matrix of R in K.

Definition 3.8. A *folding restriction* of an arrangement $\mathcal{A}$ with an action of G is the restriction $\mathcal{A}^X$ to the subspace $X = V^G$ of fixed points. We similarly define a folding restriction and a permutation restriction of a crystallographic arrangement $(\mathcal{A}, R)$.

In the sequel we will use the following notation. If $(\mathcal{A}, R)$ is crystallographic and K is a chamber of $\mathcal{A}$, then

$$R_+^K := \left\{ \alpha \in R \mid \alpha \in \sum_{\gamma \in \Delta^K} \mathbb{N}_0 \gamma \right\}.$$

Thus $R = R_+^K \dot\cup - R_+^K$ for every K. The sets R_+^K should not be confused with the positive roots R_+^a at an object of a Weyl groupoid. If a is the object corresponding to the chamber K, then $R^a \subseteq \mathbb{Z}^r$ is the set of coordinate vectors with respect to Δ^K.

We show the following easy characterization which generalizes the approach in [Ar62] for Lie type arrangements to classify real Lie algebras:

Lemma 3.9. *Let g be an involutive automorphism of the crystallographic arrangement $(\mathcal{A}, R)$. Then the folding restriction can be decomposed into two steps: First a parabolic restriction of $\mathcal{A}$ to $X_1 \in L(\mathcal{A})$ where X_1 is a suitable subspace invariant under g, then by a permutation restriction of $\mathcal{A}^{X_1}$ with respect to a suitable chamber K_2.*

Proof. The proof yields X_1, a_2 and the explicit permutation action, and provides an efficient diagrammatic description of the possible actions of cyclic groups on crystallographic arrangements (comparable to Satake diagrams in [Ar62]):

a) Let K be a chamber, such that $|g.R_+^K \cap -R_+^K|$ is minimal, which surely exists. Denote $\Delta = \Delta^K$ the positive simple roots at K.

b) If $\alpha \in \Delta$, then $g.\alpha \in R_+^K$ or $g.\alpha = -\alpha$: Otherwise, consider the chamber K' adjacent to K with $R_+^{K'} = (R_+^K \backslash \{\alpha\}) \cup \{-\alpha\}$. If $g.\alpha = -\beta \in -R_+^K$ and $\beta \neq \alpha$, then

$$|g.R_+^{K'} \cap -R_+^{K'}| = |g.R_+^K \cap -R_+^K| - 1,$$

which contradicts the assumed minimality. Note that we use here that g is involutive, such that $g.\alpha = -\beta$ also imples $g^{-1}.\alpha = -\beta$, so no root in R_+^K maps to α.

c) Let $\Delta_1 := (-g^{-1}.\Delta) \cap \Delta$, thus the set of $\alpha \in \Delta$ with $g.\alpha = -\alpha$. Define $X_1 := \Delta_1^\perp \in L(\mathcal{A})$. Since $g.\alpha = -\alpha$ for $\alpha \in \Delta_1$, X_1 is invariant under g. Further, if $v \in X = V^G$, then $g.v = v$. Hence $\alpha(v) = (-\alpha)(v)$ for all $\alpha \in \Delta_1$, i.e. $X \subseteq X_1$.

d) Consider the partition $\Delta := \Delta_1 \cup \Delta_2$, in particular $g.\Delta_2 \subset R_+^K$. We claim that there is a permutation $g(\_)$ of the set Δ_2, such that $g.\alpha_i = \alpha_{g(i)} + \mathbb{N}\Delta_1$. Indeed, modulo $\mathbb{R}\Delta_1$ the actions of g, g^{-1} are inverse positive integer matrices, hence permutation matrices.

e) Consider the parabolic restriction $\mathcal{A}^{X_1}$ and let $K_2 := K \cap X_1$ be the respective chamber, then by the above, g acts as a permutation on the crystallographic arrangement $\mathcal{A}^{X_1}$ and the full restriction to $X = V^G \subseteq X_1$ is hence a permutation restriction. □

Remark 3.10. Even if $(\mathcal{A}^X, \bar{R})$ is not crystallographic, it could be crystallographic with respect to a different choice of roots.

It is *not* true that every parabolic restriction can be obtained from a folding restriction for some suitable automorphism g:

Lemma 3.11. *For $J \subset I$, there exists an automorphism g, such that parabolic restriction to $X = J^\perp$ coincides with folding restriction by g, if and only if the permutation automorphism (i.e. diagram automorphism) $f_J := -w_J$ (on the parabolic subsystem generated by J with w_J the longest element), together with the identity permutation on all simple roots $I \backslash J$ is a permuatation automorphism for the entire arrangement.*

Take as counterexample $A_2 \subset A_3$ or also $A_2 \subset A_4$. The condition is however always fulfilled for parabolic restrictions by one simple root!

Proof. Let f be such an extension of $f_J = -w_J$ and consider the automorphism $g := w_J f$. On the subsystem R_J^K it acts as $-$id, while on the remaining positive roots it acts by adding terms in $R_{J,+}^K$ i.e. as the identity modulo R_J^K. We now apply Lemma 3.9: We see that for a simple root $\alpha_i \in \Delta^K$, by construction we have $g.\alpha \in R_+^K$ for $i \in I \backslash J$ or $g.\alpha_i = -\alpha_i$ for $i \in J$. Hence $\Delta_1 = J$ and thus the first parabolic restriction $X_1 = J^\perp$ is the parabolic restriction in question. Furthermore, since g acts on the remaining simple roots by identity modulo R_J^K, the second permutation restriction is trivial.

Vice-versa, for any automorphism g by Lemma 3.9 we have a chamber K and a set of simple roots $\Delta_1 \subset \Delta$ with the property $g.\alpha_i = -\alpha_i$, and a permutation π on the other roots $\Delta \backslash \Delta_1$ such that $g.\alpha \in \pi(\alpha) + R_{+,J}^K$. The folding restriction by g is a parabolic restriction to Δ_1 and then a permutation restriction.

Since it needs to coincide by assumption with parabolic restriction to J, we have $J \subset \Delta$, moreover $J = \Delta$ and $\sigma = $ id. We can consider $w_J^{-1} g$, which is a permutation automorphism f. On the subsystem J, f is equal to $-w_J =: f_J$ and on the remaining roots the identity, as claimed. $\square$

Remark 3.12. Compare the condition of this Lemma to the condition fulfilled by the quotient root systems used for parabolic induction of cuspidal representations in Lusztig's character theory of finite Lie groups.

Note that the permutation induces a permutation automorphism of root systems already on the localization to Δ_2, in particular a diagram automorphism of the sub-Dynkin diagram. One may hence enumerate all possible automorphism by listing root systems, where for some object a, R^a has a symmetric sub-diagram

(and say color all nodes in Δ_1 black) as done for Satake diagrams. Alternatively (and maybe more feasible for us) one may compute all parabolic restrictions as done below and then enumerate all full permutation automorphisms. A different way to classify root system automorphisms (even if g is *not involutive*) is

Lemma 3.13. *For any automorphism g of a crystallographic arrangement and any chamber K we can write $g = wf$, where w is the Weyl groupoid element defined by $w(K) = g(K)$ and f is a permutation automorphism of the object K (so in particular a diagram automorphism of the respective Dynkin diagram, possibly trivial).*

Proof. This is easily proven as for Lie algebras: The autmorphism g has to send a chamber K to some chamber K'. By transitivity of the Weyl groupoid on the set of chambers we find a (unique) w with $w(K) = K'$. Then both $w^{-1}g$ and $g^{-1}w$ leave K invariant and act hence as mutually inverse, integral positive matrix on R_+^K. They are hence again permutation matrices. □

It is not as easy as for parabolic restriction to determine the resulting root system of a folding restriction. In the special case of permutation restriction, the new set of roots $\bar{R}$ consists of the orbits of R under G:

Example 3.14. In all of the following examples, the diagram automorphism inducing g is pictured, as well as the resulting root system including multiplicities. Note that foldings for Lie type such as a)-c) are of course well-known.

a) Let $R_+^a = \{(1,0),(0,1)\}$ be of Lie type $A_1 \times A_1$ and g the permutation transposing $\alpha_1 \leftrightarrow \alpha_2$. A basis for V is $\alpha_1^\vee, \alpha_2^\vee$, hence the invariant subspace is $X = v\mathbb{R} := (\alpha_1^\vee + \alpha_2^\vee)\mathbb{R}$. The restriction of the arrangement is of course of type A_1 (two of the four chambers are intersected). The restrictions of the roots $\alpha_1, \alpha_2 \in R$ to $(\mathcal{A}^X, \bar{R})$ are both $\bar{\alpha}_1 := (v \mapsto 1)$ which can be expressed in terms of the orbit $G\alpha_1 = \{\alpha_1, \alpha_2\}$ as follows

$$\bar{\alpha}_1 = \frac{\alpha_1 + \alpha_2}{2} = \alpha_{\{1,2\}} \qquad \text{where } \alpha_{G\alpha} := \frac{1}{|G|}\sum_{g \in G} g.\alpha$$

(this also justifies our initial choice of scaling $v = \bar{\alpha}_1^\vee$). Hence $\bar{R}_+^a$ is reduced of Lie type A_1 with root $\bar{\alpha}_1 = \alpha_{\{1,2\}}$ of multiplicity 2.
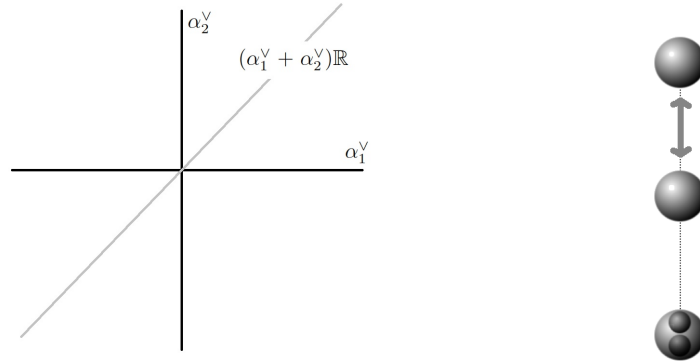
b) Let $R^a_+ = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,1,0), (0,1,0,1)\}$ be of Lie type $A_2 \times A_2$ and g the permutation transposing $\alpha_1 \leftrightarrow \alpha_2, \alpha_3 \leftrightarrow \alpha_4$. Then the restriction is $\bar{R}^a_+ = \{(1,0), (0,1), (1,1)\}$ with two simple roots and one non-simple root each corresponding to an orbit of length 2:

$$\bar{\alpha}_1 = \alpha_{\{1,2\}}$$

$$\bar{\alpha}_2 = \alpha_{\{3,4\}}$$

$$\bar{\alpha}_1 + \bar{\alpha}_2 = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{2} = \alpha_{\{13,24\}}$$

The restriction is hence a reduced root system of Lie type A_2 and all roots have multiplicity 2.
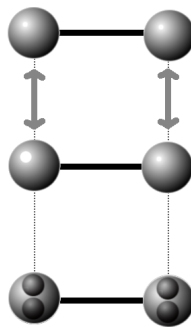
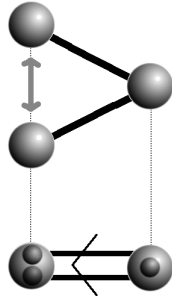c) Let $R^a_+ = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,1,1)\}$ be of Lie type A_3 and g the permutation transposing $\alpha_1 \leftrightarrow \alpha_3$. Then the restriction is $\bar{R}^a_+ =$

$\{(1,0),(0,1),(1,1),(2,1)\}$ with the following roots:

$$\bar{\alpha}_1 = \alpha_{\{1,3\}}$$

$$\bar{\alpha}_2 = \alpha_2$$

$$\bar{\alpha}_1 + \bar{\alpha}_2 = \frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} = \alpha_{\{12,23\}}$$

$$2\bar{\alpha}_1 + \bar{\alpha}_2 = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_{123}$$

The restriction is hence reduced of Lie type B_2 with short roots of multiplicity 2 and long roots of multiplicity 1.

d) Let $R_+^a = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$, which is *not* of Lie type, let g be the permutation transposing $\alpha_1 \leftrightarrow \alpha_3$. Then the restriction is $\bar{R}_+^a = \{(1,0),(0,1),(1,1),(2,0),(2,1)\}$ and hence of Lie type B_2 and multiplicities $1,2$ as in the previous case, but this time non-reduced, i.e.

$$[M_{\bar{\alpha}_1}, M_{\bar{\alpha}_1}] = M_{2\bar{\alpha}_1} \neq 0$$
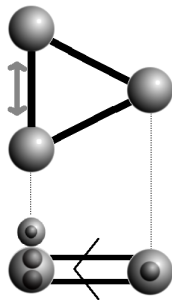
A similar effect appears already when folding a so-called loop A_2, see [Len14a].

We finally give examples for general folding restrictions.

Example 3.15. Similarly to Satake diagram (first example), according to the proof of Lemma 3.9, we draw the permutation of the simple roots in Δ_2 and denote the simple roots in Δ_1 by blackend dots.

a) Let $R_+^K = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,1,1)\}$ be of Lie type A_3 and g the automorphism

$$g := \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have $g.R_+^K \cap -R_+^K = \{-\alpha_2\}$ which is minimal (so K is already chosen suitably) and thus $\Delta_1 = \{\alpha_2\}$ and $\Delta_2 = \{\alpha_1, \alpha_3\}$. Moreover, modulo Δ_1 we have a permutation $g : \alpha_1 \leftrightarrow \alpha_2$. The restriction to $X_1 = \alpha_2^\perp$ is of type A_2 with multiplicities 2 for the two simple roots and 1 for the non-simple root. Note that while the root system is of Lie type, the multiplicities are not invariant under all reflections and we thus get a Weyl groupoid covering the Weyl group A_3 (similar to a Lie superalgebra). As a second step, the permutation restriction by the transposition g yields a non-reduced root system of Lie type A_1 where $\bar{\alpha}_1$ has multiplicity 4 and $2\bar{\alpha}_1$ has multiplicity 1.
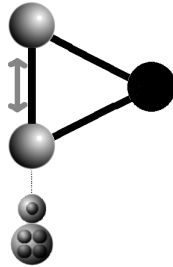
4. ARRANGEMENTS AND NICHOLS ALGEBRAS

4.1. **Nichols algebras.** Let $\mathcal{C}$ be a braided category, then there is a straightforward notion of a Hopf algebra in $\mathcal{C}$. For this and especially the notion of Yetter-Drinfel'd modules in braided categories see e.g. [BLS14]. To simplify the discussion we assume in this article $\mathcal{C} = {}_h^h\mathcal{YD}$ where h is a complex Hopf algebra.

Definition 4.1. Let $M = M_1 \oplus \cdots \oplus M_n$ be a semisimple object in $\mathcal{C}$ with simple summands M_i. We call n the *rank* of M. The *Nichols algebra* $\mathcal{B}(M)$ is an $\mathbb{N}$-graded Hopf algebra in the category $\mathcal{C}$ with $\mathcal{B}(M)_0 = \mathbb{1}\mathbb{C}$ the identity object in $\mathcal{C}$ and $\mathcal{B}(M)_1 = M$ primitive elements $\Delta(m) = 1 \otimes m + m \otimes 1$ which generate $\mathcal{B}(M)$

as an algebra.

The Nichols algebra $\mathcal{B}(M)$ can be defined by the following equivalent properties:

- Every $\mathbb{N}$-graded Hopf algebra B' with the above properties admits a unique graded Hopf algebra surjection $B' \to \mathcal{B}(M)$ which is the identity on M. That is, $\mathcal{B}(M)$ is the quotient of the tensor algebra $T(M)$ by the largest Hopf ideal $\mathfrak{J}$ in degree ≥ 2.
- The pairing $M \otimes M^* \to \mathbb{C}$ induces a Hopf pairing $\mathcal{B}(M) \otimes \mathcal{B}(M^*) \to \mathbb{C}$ which is nondegenerate. That is, $\mathcal{B}(M)$ is the quotient of the tensor algebra $T(M)$ by the radical of the Hopf pairing $T(M) \otimes T(M^*) \to \mathbb{C}$.
- More explicitly, $\mathcal{B}(M)$ is the quotient of $T(M)$ in each degree k by the kernel of the *quantum symmetrizer*, which is defined in terms of the braid group action $M^{\otimes k} \to M^{\otimes k}$.

While the last characterization enables one in principle to compute $\mathcal{B}(M)$ in each degree, it is extremely difficult to find generators and relations for $\mathcal{B}(M)$ or even determine for a given M if it is finite-dimensional.

Since $\mathcal{B}(M)$ is an $\mathbb{N}$-graded algebra with finite-dimensional degree layers, we may consider the *Hilbert series*, a formal power series

$$\mathcal{H}(t) := \sum_{k \geq 0} \dim \left(\mathcal{B}(M)_k \right) t^k.$$

In particular, if the dimension is finite then $\dim(\mathcal{B}(M)) = \mathcal{H}(1)$. We frequently use the symbol $(n)_t := 1 + t + t^2 + \cdots + t^{n-1}$.

The first examples serves to fix notation:

Example 4.2. Let $h = \mathbb{C}[\Gamma]$ and Γ be an abelian group. Then the braided category $\mathcal{C} = {}^h_h \mathcal{YD}$ is semisimple. The finite-dimensional simple objects are 1-dimensional vector spaces $M_i = x_i \mathbb{C}$ together with a group element $g_i \in \Gamma$ and a linear character $\chi_i : G \to \mathbb{C}^\times$. The braiding is given by

$$M_i \otimes M_j \to M_j \otimes M_i,$$

$$x_i \otimes x_j \mapsto q_{ij}\, x_j \otimes x_i, \quad q_{ij} := \chi_j(g_i).$$

We call such a braiding *diagonal* and q_{ij} the *braiding matrix*. From the third characterization we see that the Nichols algebra $\mathcal{B}(M)$ depends only on the braiding matrix $(q_{ij})_{i,j}$ of M. Conversely, every diagonal braiding can be realized as Yetter-Drinfel'd module, say over $\Gamma = \mathbb{Z}^n$.

Example 4.3. Let $q \in \mathbb{C}^\times$. Let $M = M_1$ be the one-dimensional complex braided vector space with basis x_1 and braiding matrix $(q_{11}) = (q)$. The Nichols algebra is

$$\mathcal{B}(M) \cong \mathbb{C}[x_1]/(x_1^\ell)$$

if q is a primitive ℓ-th root of unity, and $\mathcal{B}(M) \cong \mathbb{C}[x_1]$ if $q = 1$ or not a root of unity. In the first case, the Hilbert series is $(\ell)_t$, in the second case $\frac{1}{1-t}$.

The next example exhibits the role of Nichols algebras as quantum Borel parts.

Example 4.4. Let q be a primitive ℓ-th root of unity and assume for simplification $2, 3 \nmid \ell$ and $\ell > 3$. Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra of rank n, let $\alpha_1, \ldots \alpha_n$ be a set of simple roots, R_+ the set of positive roots, and let $(,)$ be the Killing form, normalized to $(\alpha_i, \alpha_i) = 2$ for short simple roots α_i.
Consider the Yetter-Drinfel'd modules $M = M_1 \oplus \cdots \oplus M_n$ over the abelian group $\Gamma = \mathbb{Z}_\ell^n$ generated by g_i and $M_i = E_i\mathbb{C}$ with group element g_i and character $\chi_i(g_j) := q^{(\alpha_i, \alpha_j)}$.
Then $\mathcal{B}(M)$ is finite-dimensional with Hilbert series $\prod_{\alpha \in R_+}(\ell)_{t^{deg(\alpha)}}$, thus of dimension $\ell^{|R_+|}$. In fact $\mathcal{B}(M) \cong u_q(\mathfrak{g})^+$ is as an algebra the positive part of the small quantum group.

4.2. The Weyl groupoid of a Nichols algebra.

The structure theory of Nichols algebras is dominated by the structure of the Weyl groupoid, which generalizes the role of reflection operators in quantum groups as introduced by Lusztig [Lusz90], and allows to define a root system for the Nichols algebra. For M, q_{ij} diagonally braided (see above) Heckenberger has introduced in a series of papers reflections and an arithmetic root system in terms of the bicharacter induced by q_{ij} and finally classified all finite-dimensional Nichols algebras in terms of their root systems in [Heck09]. More generally in [AHS10][HS10] Andruskiewitsch, Heckenberger, and Schneider have introduced a Weyl groupoid and root system for arbitrary semisimple $M \in {}^h_h\mathcal{YD}$. We shall sketch their approach in the form discussed in [HS13] or [BLS14]:

Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite-dimensional semisimple object in $\mathcal{C} = {}^h_h\mathcal{YD}$ and assume for simplicity that $\mathcal{B}(M)$ is already finite-dimensional. For any i the projection $M \to M_i$ induces a projection $\pi_{\mathcal{B}(M)} \to \mathcal{B}(M_i)$. By the *Radford projection theorem* we may write

$$\mathcal{B}(M) \cong K \rtimes \mathcal{B}(M_i), \qquad K = \mathcal{B}(M)^{coin\,\mathcal{B}(M_i)} := \{h \mid (\mathrm{id} \otimes \pi_i)\Delta(h) = h \otimes 1\}$$

where the space of coinvariants K with respect to π_i is a Hopf algebra in the braided category $\substack{\mathcal{B}(M_i) \\ \mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})$ and $\rtimes$ is the *Radford biproduct*.

By the second characterization of Nichols algebras we have a non-degenerate Hopf pairing $\mathcal{B}(M) \otimes \mathcal{B}(M^*) \to \mathbb{C}$. One can show that this induces a category equivalence $\Omega : \substack{\mathcal{B}(M) \\ \mathcal{B}(M)}\mathcal{YD} \to \substack{\mathcal{B}(M^*) \\ \mathcal{B}(M^*)}\mathcal{YD}$. Now we may turn $\Omega(K) \in \substack{\mathcal{B}(M^*) \\ \mathcal{B}(M^*)}\mathcal{YD}$ again into a Hopf algebra in $\mathcal{C}$, the reflection:

$$r_i(\mathcal{B}(M)) := \Omega(K) \rtimes \mathcal{B}(M^*).$$

The previous operation is neither restricted to Nichols algebras nor to the category $\mathcal{C} = \substack{h \\ h}\mathcal{YD}$ and was dubbed *partial dualization* in [BLS14]. In general $H, r(H)$ can be quite different although one can prove $\substack{H \\ H}\mathcal{YD} \cong \substack{r(h) \\ r(h)}\mathcal{YD}$.

For a Nichols algebra $\mathcal{B}(M)$, Andruskiewitsch, Heckenberger and Schneider describe this operation in much more detail: In particular, $r_i(\mathcal{B}(M)) \cong \mathcal{B}(R_i(M))$ is again a Nichols algebra for some explicit $R_i(M) \in \mathcal{C}$. We summarize some results of [AHS10], [HS10], and [HS13]:

Theorem 4.5. *Let h be a complex Hopf algebra. Let M_i be a finite collection of simple h-Yetter-Drinfel'd modules. Consider $M := \bigoplus_{i=1}^n M_i \in \substack{h \\ h}\mathcal{YD}$ and assume that the associated Nichols algebra $H := \mathcal{B}(M)$ is finite-dimensional. Then the following assertions hold:*

- *By construction, the Nichols algebras $\mathcal{B}(M), r_i(\mathcal{B}(M))$ have the same dimension as complex vector spaces.*
- *For $i \in I$, denote by $\hat{M}_i$ the braided subspace*

$$\hat{M}_i = M_1 \oplus \ldots \oplus M_{i-1} \oplus M_{i+1} \oplus \ldots \oplus M_n$$

of M. Denote by $\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$ the braided vector space obtained as the image of $\hat{M}_i \subset \mathcal{B}(M)$ under the adjoint action of the Hopf subalgebra $\mathcal{B}(M_i) \subset \mathcal{B}(M)$. Then, there is a unique isomorphism [HS13, Prop. 8.6]

$$K_i \cong \mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i))$$

of Hopf algebras in the braided category $\substack{\mathcal{B}(M_i) \\ \mathcal{B}(M_i)}\mathcal{YD}\left(\substack{h \\ h}\mathcal{YD}\right)$ which is the identity on $\mathrm{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$.

- *Define, with the usual convention for the sign,*

$$-c_{ij} := \max\{m \mid \mathrm{ad}_{M_i}^m(M_j) \neq 0\} \qquad c_{ii} = 2.$$

Fix $i \in I$ and denote for $j \neq i$

$$V_j := \mathrm{ad}_{M_i}^{-c_{ij}}(M_j) \subset \mathcal{B}(M) .$$

The braided vector space

$$R_i(M) = V_1 \oplus \cdots M_i^* \cdots \oplus V_n \in {}_h^h \mathcal{YD}$$

is called the the i-th reflection of the braided vector space M. Then there is a unique isomorphism [HS13, Thm. 8.9] of Hopf algebras in ${}_h^h \mathcal{YD}$

$$r_i(\mathcal{B}(M_1 \oplus \cdots \oplus M_n)) \cong \mathcal{B}(V_1 \oplus \cdots M_i^* \cdots \oplus V_n)$$

which is the identity on M.

- *With the same definition for c_{ij} for $i \neq j$ and $c_{ii} := 2$, the matrix $(c_{ij})_{i,j=1,\ldots n}$ is a generalized Cartan matrix [AHS10, Thm. 3.12]. Moreover, one has $r_i^2(\mathcal{B}(M)) \cong \mathcal{B}(M)$, and the Cartan matrices coincide, $c_{ij}^M = c_{ij}^{r_i(M)}$. One obtains a Cartan graph where each object $a \in A$ corresponds to some reflection $\mathcal{B}(M^a)$.*

- *The maps r_i give rise to a Weyl groupoid: The objects are the different isomorphism classes of tuples $(M_1, \ldots, M_n)$ and the formal morphisms $M \to r_i(M)$ are generated by reflections $\mathbb{Z}^n \to \mathbb{Z}^n$ with respect to c_{ij}^M.*

- *Let R_+^a the set of positive roots, and choose a reduced expression for the longest element in the Weyl groupoid, then there is an isomorphism of $\mathbb{N}$-graded objects in $\mathcal{C}$ (not algebras), or PBW-basis:*

$$\mathcal{B}(M^a) = \bigotimes_{\alpha \in R_+^a} M_\alpha.$$

Here M_α are certain simple object in $\mathcal{C}$, namely $= M_i^a$ if H_α is adjacent to the object a. For details, we refer to [AHS10, Sect. 3.5] and [HS10, Sect. 5].

Note that a crucial point is that the algebras $\mathcal{B}(M), r_i(\mathcal{B}(M))$ can be quite different and in particular their Cartan matrix $(c_{ij})_{ij}$ may be different – in contrast to quantum groups, where all reflections are isomorphic as algebras. This is why we obtain a Weyl groupoid instead of a Weyl group, a Cartan graph instead of a single Cartan matrix, and more root systems than for semisimple Lie algebras.

For $(M, (q_{ij})_{i,j})$ diagonal, Heckenberger had already described $r_i(M), q'_{ij}$ as the explicit base transformation of the bicharacter induced by the reflection $\alpha_j \mapsto \alpha_j - c_{ij}\alpha_i$, $\alpha_i \mapsto -\alpha_i$, where the Cartan matrix $(c_{ij})_{ij}$ associated to $(M, (q_{ij})_{i,j})$ can

be calculated by

$$c_{ii} = 2, \qquad c_{ij} = -min\{m \mid (m+1)_{q_{ii}} = 0 \text{ or } q_{ii}q_{ij}q_{ji} = 1\}, \quad i \neq j.$$

Theorem 4.6 (Heckenberger, [Heck09]). *Any finite-dimensional complex Nichols algebra in the categoy of Yetter-Drinfel'd modules over an abelian group $^\Gamma_\Gamma \mathcal{YD}$ appears in Heckenberger's list* [Heck09].

The respective root systems are many, but not all possible root systems and Weyl groupoids as classified by Heckenberger and the first author, see Section 2.4. For instance, there are infinitely many Weyl groupoids of rank 2 and many more exceptional Weyl groupoids in rank 3.

The following class of examples of type B_2 contains as special case the Borel part $u_q(B_2)^+$ when the braiding matrix $(q_{ij})_{ij}$ is chosen symmetric.

Example 4.7. Let $q, q^4 \neq 1$ be an ℓ-th root of unity and $M_{\alpha_1}, M_{\alpha_2}$ be 1-dimensional Yetter-Drinfel'd modules (say over $\Gamma = \mathbb{Z}^2$) such that the braiding matrix fulfills

$$q_{\alpha_1\alpha_1} = q^2, \qquad q_{\alpha_i\alpha_j}q_{\alpha_j\alpha_i} = q^{-4}, \qquad q_{\alpha_2\alpha_2} = q^4.$$

The previous formula yields the Cartan matrix and Dynkin diagram of $M^a :=$ $M_{\alpha_1} \oplus M_{\alpha_2}$:

$$(c^a_{ij})_{i,j} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad \text{}$$

The reflection on the short root α_1 given by this Cartan matrix gives for $r_1(M)$ $\alpha'_1 = -\alpha_1$ and $\alpha'_2 = \alpha_2 + 2\alpha_1$. For the braiding matrix of $r_1(M)$ we thus get by base transformation of the bicharacter:

$$q_{\alpha'_1\alpha'_1} = q_{\alpha_1\alpha_1} = q^2$$
$$q_{\alpha'_2\alpha'_2} = q_{\alpha_2\alpha_2}q^2_{\alpha_1\alpha_2}q^2_{\alpha_2\alpha_1}q^4_{\alpha_1\alpha_1} = q^{4-8+8} = q^4$$
$$q_{\alpha'_1\alpha'_2} = q^{-1}_{\alpha_1\alpha_2}q^{-2}_{\alpha_1\alpha_1}$$
$$q_{\alpha'_2\alpha'_1} = q^{-1}_{\alpha_1\alpha_2}q^{-2}_{\alpha_1\alpha_1}$$

We observe that in general the Yetter-Drinfel'd module $r_1(M)$ given by $q_{\alpha'_i\alpha'_j}$ is *not* isomorphic to M (for the special case $q_{ij} = q_{ji}$ appearing in $u_q(B_2)^+$ it is!). However we easily check

$$q'_{\alpha_1\alpha_2}q'_{\alpha_2\alpha_1} = q^{-1}_{\alpha_1\alpha_2}q^{-1}_{\alpha_2\alpha_1}q^{-4}_{\alpha_1\alpha_1} = q^{4-8} = q_{\alpha_i\alpha_j}q_{\alpha_j\alpha_i},$$

so $r_1(M) = M_{\alpha'_1} \oplus M_{\alpha'_2}$ has the same Cartan matrix as M (this can not be deduced merely from the Dynkin diagram). The same can be checked for $r_2(M)$. We thus obtain a Cartan graph with a single Cartan matrix for all reflections M^a. The arrangement and set of positive roots is of Lie type B_2:

$$R_+^a = \{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\} = \{(1,0),(0,1),(1,1),(2,1)\}$$

with notation $\alpha_{12} := \alpha_1 + \alpha_2$, $\alpha_{112} := 2\alpha_1 + \alpha_1$. The self-braidings are

$$q_{\alpha_1\alpha_1} = q_{\alpha_{12}\alpha_{12}} = q^2, \qquad q_{\alpha_2\alpha_2} = q_{\alpha_{112}\alpha_{112}} = q^4.$$

Hence the PBW-basis Theorem gives an isomorphism of $\mathbb{N}$-graded Γ-Yetter-Drinfel'd modules:

$$\mathcal{B}(M^a) \cong \bigotimes_{\alpha \in R_+^a} M_\alpha$$
$$= \mathbb{C}[t_1]/(t_1^{\mathrm{ord}(q^2)}) \otimes \mathbb{C}[t_2]/(t_2^{\mathrm{ord}(q^4)}) \otimes \mathbb{C}[t_{12}]/(t_{12}^{\mathrm{ord}(q^2)}) \otimes \mathbb{C}[t_{112}]/(t_{112}^{\mathrm{ord}(q^4)}).$$

For ℓ odd, q^2, q^4 both have order ℓ, hence the Hilbert series is $(\ell)_t^2(\ell)_{t^2}(\ell)_{t^3}$ and in particular the dimension is ℓ^4.

We also give an example which is not of Lie type and which we will use in the following. It is a finite-dimensional Nichols algebra $\mathcal{B}(M)$ of diagonal type of rank 3 appearing in [Heck09, row 11]. The rows $9, 10$ define the same arrangement, but different roots of unity involved and hence more types of objects with the same Cartan matrix. The arrangement associated to this Nichols algebra as well as all restrictions are calculated in Section 5.3.

Example 4.8. Let $M_{\alpha_1}, M_{\alpha_2}, M_{\alpha_3}$ be 1-dimensional Yetter-Drinfel'd modules (say over $\mathbb{Z}^3$) such that the braiding matrix fulfills

$$q_{\alpha_i\alpha_i} = -1, \qquad q_{\alpha_i\alpha_j}q_{\alpha_j\alpha_i} = \zeta,$$

with $i \neq j$ and ζ a primitive third root of unity. The associated crystallographic arrangement has 7 roots. It is called $\mathcal{A}_3^1(2)$ and the first member of a series.

$$R_+^a = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}.$$

From this we easily see the Dynkin diagram/Cartan matrix is a simply-laced triangle and not of Lie type:
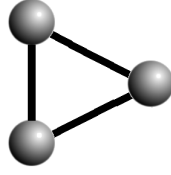
The self-braidings of these roots are

$$q_{\alpha_i \alpha_i} = -1, \qquad q_{\alpha_{ij}, \alpha_{ij}} = \zeta, \qquad q_{\alpha_{123}} = -1.$$

Hence the Nichols algebra $\mathcal{B}(M^a)$, $M^a = M_{\alpha_1} \oplus M_{\alpha_2} \oplus M_{\alpha_3}$ has Hilbert series $(2)_t^3 (3)_{t^2}^3 (2)_{t^3}$ and dimension 432.

We calculate the reflection to an object a' with respect to α_2: The simple roots at a' are then $\alpha_1' := \alpha_{12}$, $\alpha_2' := -\alpha_2$, $\alpha_3' := \alpha_{23}$ and the braiding matrix fulfills

$$q_{\alpha_1' \alpha_1'} = q_{\alpha_3' \alpha_3'} = \zeta, \qquad q_{\alpha_2' \alpha_2'} = -1,$$

$$q_{\alpha_1' \alpha_2'} q_{\alpha_2' \alpha_1'} = q_{\alpha_2' \alpha_3'} q_{\alpha_3' \alpha_2'} = \zeta^{-1}, \qquad q_{\alpha_1' \alpha_3'} q_{\alpha_3' \alpha_1'} = 1.$$

The respective roots in the basis $\alpha_1', \alpha_2', \alpha_3'$ can easily be either again calculated or directly read off from transforming R_+^a:

$$R_+^{a'} = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,2,1)\}$$

which has Dynkin diagram/Cartan matrix A_2, but is not of Lie type, since it has a different root system.

The Hilbert series is now $(2)_t (3)_t^2 (2)_{t^2}^2 (2)_{t^3} (3)_{t^4}$ but of course the dimension is still 432.

We finally give an example over the nonabelian dihedral group $\mathbb{D}_4$. It has been constructed first in [MS00] in the context of Coxeter groups. Later, in [Len14a] it has been constructed by the second author as smallest example of a new family of large-rank finite-dimensional indecomposable Nichols algebras, introducing the folding construction of non-diagonal Nichols algebras from diagonal ones (in this case $A_2 \times A_2$ with $q = -1$). The reader may compare the root system below with the root system in 3.14 b). A similar construction for c) returns a decomposable Nichols algebra of type B_2.
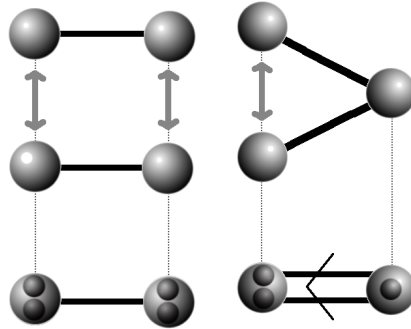
Example 4.9. Let $\Gamma = \mathbb{D}_4 = \langle a, b \rangle / (a^4 = b^2 = 1, ba = a^{-1}b)$ then one can define 2-dimensional simple Yetter-Drinfel'd modules $M_{\alpha_1}, M_{\alpha_2}$ associated to the conjugacy classes $[ab], [b]$. In fact, the Nichols algebra $\mathcal{B}(M)$ for $M = M_{\alpha_1} \oplus M_{\alpha_2}$ has a root system of type A_2 and $M_{\alpha_{12}}$ is also 2-dimensional and associated to the conjugacy class $[a]$. Altogether this Nichols algebra has Hilbert series $(2)_t^4 (2)_{t^2}^2$ and thus dimension 2^6, indeed just like $u_q(A_2 \times A_2)^+$ with $q = i$.

Note that in [Heck09] all finite-dimensional complex Nichols algebras over abelian groups (equivalently with diagonal braiding) have been classified, and very recently in [HV14] all finite-dimensional complex indecomposable Nichols algebras over nonabelian groups. For later use we note that only a subset of all finite Weyl groupoids (Theorem 2.14) appear in these cases:

Corollary 4.10. *Using the labeling introduced in* [CH13]*, the following crystallographic arrangements of rank three come from Weyl groupoids of finite-dimensional Nichols algebras of diagonal type,*

types $A_3, B_3, C_3, \mathcal{A}_3^1(2), \mathcal{A}_3^2(2)$, *and the sporadic ones labeled* $1, 2, 3, 6, 8, 9$,

and the following come from complex finite-dimensional indecomposable Nichols algebras over nonabelian groups,

types A_3, C_3, *and the sporadic one labeled* 9.

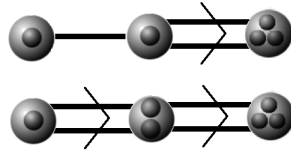*Note that by Theorem 3.7, all of these are (parabolic) restrictions of Weyl arrangements.*

Question 4.11. As already mentioned in the previous example, all such Nichols algebras of rank < 4 over nonabelian groups had been uniformly constructed in [Len14a] by *folding*; they have root systems of type $A_n, C_n, E_6, E_7, E_8, F_4$. The

resulting root system was calculated by hand and can now be understood as a restriction. It would be interesting to give a larger class of constructions:

- By considering foldings of Nichols algebras of non-Lie type, such as Example 3.14 d).
- By considering Satake-type foldings with nontrivial Δ_1, which tends to drastically decrease the rank. This leads a-priori to Nichols algebras in other braided categories.
- In the affine setting, compare with the quantum affine algebra foldings in [Len14b].

In particular one would like to construct via some generalized folding the remaining exceptional cases of low-rank $2, 3$ classified in [HV14] and known (and possibly new) cases of rank 1.

Example 4.12. In rank 3, apart from A_3, C_3, there is only one finite dimensional exceptional Nichols algebra over a nonabelian group (defined for any field with a third root of unity or any field characteristic 3). It is *not* of Lie type, but has the root system rank 3 no. 9 with 13 roots and root space dimensions (in the first object a with B_3 Cartan matrix, see [HV14, Lm. 8.8]):


$\alpha \in R_+^a$	$\dim(M_\alpha)$
$1, 2, 12, 12^2 3^4, 12^3 3^4, 1^2 2^3 3^4$	1
$2 3^2, 12 3^2, 12^2 3^2$	2
$3, 23, 123, 12^2 3^3$	3

The rank 2 parabolic generated by α_2, α_3 is an exceptional Nichols algebra over a nonabelian group and has a standard root system of type B_2. The three-dimensional rank 1 parabolics are the Nichols algebras of dimension 12 over the nonabelian group $\mathbb{S}_3$, which was the first known Nichols algebra over a nonabelian group [MS00].

Ideally one would want to construct also these Nichols algebras via restriction, and we start our search with purely parabolic restrictions (no diagram automorphism). *Indeed* we find, that a restriction of the sporadic root system of rank 4

no. 7 with 25 roots (and the parabolics rank 3 no. 2 and B_2) have precisely the right multiplicities. Moreover, they can be realized (in characteristic 0 with ζ a primitive third root of unity) by an inclusion chain of diagonal Nichols algebras, namely row 17, row 16 and row 6 for $q = -1$:[5]
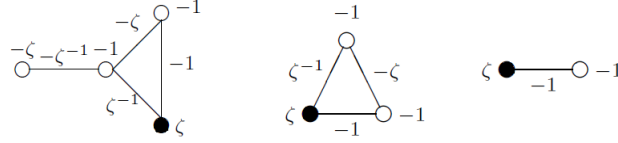
Hence we *can* obtain Nichols algebras with the right root system and root space dimensions in the category of $\mathbb{C}[x]/(x^3)$-Yetter-Drinfel'd modules. However the Hilbert series' do *not* coincide completely, e.g. in the smallest example we obtain a Nichols algebra with Hilbert series $(2)_t(3)_t(6)_t$ instead of $(2)_t(3)_t(2)_t$ over the group $\mathbb{S}_3$, which suggests to somehow consider a subalgebra of graded index $(3)_{t^2}$. If and how our Nichols algebras above can indeed be turned into Nichols algebras over the respective nonabelian groups is not clear at this point (for the folding in [Len14a] this was a central extension, but this cannot be true for $\mathbb{S}_3$).

4.3. **Restrictions of Nichols algebras.** Let $\mathcal{B}(M^a)$ be the Nichols algebra of a semisimple object $M^a = M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_n}$ in a braided category $\mathcal{C}$, and assume for now $\mathcal{C} = {}^h_h\mathcal{YD}$. Let $(\mathcal{A}, R)$ be the associated n-dimensional crystallographic arrangement where the chamber a has positive simple roots $\Delta^a = \{\alpha_1, \ldots, \alpha_n\}$.

Fix some α_i. Recall that in the construction of the reflection r_{α_i} in Section 4.2 we have considered the algebra of coinvariants $\mathcal{B}(\bar{M})$ of the Hopf algebra map $\pi : \mathcal{B}(M^a) \to \mathcal{B}(M_{\alpha_i})$, which is a Hopf algebra in a different braided category

$$\mathcal{B}(\bar{M}) := \mathcal{B}(M^a)^{coin\,\mathcal{B}(M_i)} \in {}^{\mathcal{B}(M_{\alpha_i})}_{\mathcal{B}(M_{\alpha_i})}\mathcal{YD}(\mathcal{C}),$$

and by Theorem 4.5 it is a Nichols algebra $\mathcal{B}(\bar{M})$ in this category and thus has an associated root system. The aim of the next theorem is to show that the root system of $\mathcal{B}(\bar{M})$ is precisely the restricted root system $(\mathcal{A}^X, \bar{R})$ with $X = \alpha_i^\perp$. It moreover gives interpretations for the restriction multiplicities and possible non-reducedness of $(\mathcal{A}^X, \bar{R})$ in terms of $\mathcal{B}(\bar{M})$. Repeating this argument for a subset J of simple roots, we get a similar statement for an arbitrary intersection of hyperplanes $X = J^\perp$.

---

[5]Images from Heckenberger's list modified by adding a black dot.

Let us give in advance a *heuristic argument*, that underlies the proof below: The restriction to X induces a map of sets (additive where appropriate) between the set of roots $R \to \bar{R} \cup \{0\}$ and we denote by $\bar{\alpha} \in \bar{R}_+^a \cup \{0\}$ the image of a root $\alpha \in R_+^a$.

- The preimage of 0 are all roots α where $X \subset \alpha^\perp$, i.e. in terms of arrangements the localization $\mathcal{A}_X$, i.e. in terms of roots the parabolic subsystem generated by J.

  In the PBW-basis of $\mathcal{B}(M) = \bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha)$ the respective factors $\mathcal{B}(M_\beta)$ for all β with $\bar{\beta} = 0$ hence form precisely the PBW-basis of the subalgebra $\mathcal{B}(M_J)$ with $M_J = \bigoplus_{i \in J} M_{\alpha_i}$.

- The preimage of any $\bar{\alpha} \neq 0$ is a root α up to addition of roots in J.

  Hence *morally*, adding all preimages of $\bar{\alpha}$ should yield an irreducible $\mathcal{B}(M_J)$-Yetter-Drinfel'd submodule, with rank, as h-Yetter-Drinfel'd module, given by the number of preimages and hence the restriction multiplicity.

$$\bar{M}_{\bar{\alpha}} := \bigoplus_{\beta, \bar{\beta} = \bar{\alpha}} M_\beta.$$

  However, since in general $M_{\alpha_i + \beta} \neq \mathrm{ad}_{M_{\alpha_i}}(M_\beta)$ this is *not* true, but we can argue similarly with other $\bar{M}_{\bar{\alpha}}$ defined via the right hand side expression.

- Finally, in the PBW-decomposition of $\mathcal{B}(\bar{M})$ only the roots $\bar{\alpha}$ with $\frac{1}{k}\bar{\alpha} \notin \bar{R}_+$ appear. The other factors $\mathcal{B}(M_\beta), \bar{\beta} \in \mathbb{N}\bar{\alpha}$ appear implicitly as higher commutators of $\mathcal{B}(\bar{M}_{\bar{\alpha}})$. This is why we work with the non-reduced system.

Finally this yields a PBW-decomposition of $\mathcal{B}(\bar{M})$ according to $(\mathcal{A}^X, \bar{R}_+^a)$:

$$\mathcal{B}(\bar{M}_{\bar{\alpha}}) = \bigotimes_{\beta, \bar{\beta} \in \mathbb{N}\bar{\alpha}} \mathcal{B}(M_\beta), \qquad \mathcal{B}(\bar{M}) = \bigotimes_{\bar{\alpha} \in \bar{R}_+^a} \mathcal{B}(\bar{M}_{\bar{\alpha}}).$$

We now turn to the exact proof:

Theorem 4.13.

a) *Assume $M^a = M_i \oplus M_j$ is of rank 2 and $\mathcal{B}(M)$ is finite-dimensional. Then there is an isomorphism of $\mathbb{N}$-graded ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})$-objects*

$$\bigotimes_{\alpha_i \neq \alpha \in R_+^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(\bar{M}_{\bar{\alpha}_j}), \qquad \bar{M}_{\bar{\alpha}_j} := \mathrm{ad}_{\mathcal{B}(M_i)}(M_j).$$

   (Note that the tensor factors are not $\mathcal{B}(M_i)$-Yetter-Drinfel'd modules.)

b) *In the previous case the restricted root systems of $\mathcal{A}^{\alpha_i^\perp}$ and of $\mathcal{B}(\bar{M})$ are trivially coinciding (type A_1), but moreover the restriction multiplicity of $\bar{\alpha}_j \in \bar{R}_+^a$ equals the rank of the new simple summand $\bar{M}_{\bar{\alpha}_j}$ as semisimple object in $\mathcal{C}$. Non-reducedness, i.e. $2\bar{\alpha}_j \in \bar{R}_+^a$, implies $[\bar{M}_{\bar{\alpha}_j}, \bar{M}_{\bar{\alpha}_j}] \neq 0$.*

c) Let $M^a = \bigoplus_{i \in I} M_i$ and assume $\mathcal{B}(M)$ is finite-dimensional with root system R_+^a. For $J \subset I$ let $M_J = \bigoplus_{i \in J} M_i$ and let $(\mathcal{A}^X, \bar{R}), X = J^\perp$ be the restricted root system. Then there exists an isomorphism

$$\bigotimes_{\bar{\alpha} \in \bar{R}_+^a} \mathcal{B}(\bar{M}_{\bar{\alpha}}) \cong \mathcal{B}(M^a)^{coin\, \mathcal{B}(M_J)}$$

of $\mathbb{N}^{I \setminus J}$-graded objects in $\,^{\mathcal{B}(M_J)}_{\mathcal{B}(M_J)}\mathcal{YD}(\mathcal{C})$ for suitable $\bar{M}_{\bar{\alpha}} \in \,^{\mathcal{B}(M_J)}_{\mathcal{B}(M_J)}\mathcal{YD}(\mathcal{C})$ where in particular $\bar{M}_{\bar{\alpha}_j} := \mathrm{ad}_{\mathcal{B}(M_J)}(M_j)$ for $j \notin J$. Moreover, the restriction multiplicity of $\bar{\alpha}$ determines the rank of the new simple summand $M_{\bar{\alpha}}$ as semisimple object in $\mathcal{C}$ and non-reducedness, i.e. $2\bar{\alpha} \in \bar{R}_+^a$ implies $[\bar{M}_{\bar{\alpha}}, \bar{M}_{\bar{\alpha}}] \neq 0$.

d) In the previous situation, for the Nichols algebra $\mathcal{B}(\bar{M}) := \mathcal{B}(M^a)^{coin\, \mathcal{B}(M_J)}$ in the category $\,^{\mathcal{B}(M_J)}_{\mathcal{B}(M_J)}\mathcal{YD}(\mathcal{C})$, the associated root system (crystallographic arrangement) in the sense of the Theorem 4.5 coincides with the reduced crystallographic arrangement of the restricted root system $(\mathcal{A}^X, \bar{R})$ with $X = J^\perp$ in the sense of Lemma 3.3.

Proof.

a) By Theorem 4.5 (PBW basis) we have an isomorphism of $\mathbb{N}^2$-graded $\mathcal{C}$-objects

$$\bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(M),$$

and since the map is multiplication in $\mathcal{B}(M)$ it is an isomorphism in $\,^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathbb{C})$ (of course the tensor factors are no submodules in this matter).

By the same theorem we have an isomorphism of $\mathbb{N}$-graded $\,^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})$-objects

$$\mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_{\alpha_i})}(M_{\alpha_j})) \cong \mathcal{B}(M)^{coin\, \mathcal{B}(M_i)}.$$

By the Radford projection theorem we have an isomorphism of $\mathcal{C}$-algebras

$$\mathcal{B}(M) \cong \mathcal{B}(M_i) \otimes \mathcal{B}(M)^{coin\, \mathcal{B}(M_i)},$$
$$x \mapsto \pi(x^{(1)}) \otimes \pi(S(x^{(2)}))x^{(3)},$$
$$ab \leftarrow\!\shortmid a \otimes b.$$

and since the $\mathbb{N}$-grading in the rank 1 Nichols algebra $\mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_{\alpha_i})}(M_{\alpha_j}))$ is given by the radical filtration and the $\mathcal{B}(M_i)$-action by the adjoint action, this is even an isomorphism of $\mathbb{N}$-graded $\,^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})$-objects.

Altogether we get a surjective map of $\mathbb{N}$-graded $\,^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})$-objects

$$\bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha) \xrightarrow{\text{mult.}} \mathcal{B}(M) \xrightarrow{x \mapsto \pi(S(x^{(1)}))x^{(2)}} \mathcal{B}(M)^{coin\, \mathcal{B}(M_i)} \cong \mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_i)}(M_j)).$$

We immediately see that $\cdots \otimes \mathcal{B}(M_i)^+$ is in the kernel. Hence this factorizes to the following surjective map, which is even an isomorphism since the dimensions are equal:

$$\bigotimes_{\alpha_i \neq \alpha \in R_+^a} \mathcal{B}(M_\alpha) \xrightarrow{\cong} \mathcal{B}(\mathrm{ad}_{\mathcal{B}(M_i)}(M_j)).$$

b) By definition

$$\bar{M} = \mathrm{ad}_{\mathcal{B}(M_i)}(M_j) = \bigoplus_{k=0}^{\infty} \mathrm{ad}_{M_i}^k(M_j) = \bigoplus_{k=0}^{-c_{ij}} \mathrm{ad}_{M_i}^k(M_j)$$

and by Theorem 4.5 all $\mathrm{ad}_{M_i}^k(M_j)$ are simple $\mathcal{C}$-objects. Hence the rank of $\bar{M}$ as $\mathcal{C}$-object is $1 - c_{ij}$. On the other hand the roots restricting to $\bar{\alpha}_j$ in $\mathcal{A}^{\alpha_i^\perp}$ are all elements in the root string $\alpha_j, \alpha_j + \alpha_i, \ldots, \alpha_j - c_{ij}\alpha_i$, so the two numbers coincide as claimed. The claim about non-reducedness comes from considering the dimensions, since $\bar{M}$ generates $\mathcal{B}(\bar{M})$.

c) Since both parabolic restriction and taking coinvariants can be performed root-by-root it suffices to treat the case $J = \{\alpha_1\}$. As in a) we have by the PBW basis theorem an isomorphism of $\mathbb{N}^{I \setminus J}$-graded objects in ${}_{\mathcal{B}(M_J)}^{\mathcal{B}(M_J)}\mathcal{YD}(\mathcal{C})$

$$\bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(M^a)$$

and by the Radford projection theorem an isomorphism of $\mathbb{N}^{I \setminus J}$-graded objects in ${}_{\mathcal{B}(M_J)}^{\mathcal{B}(M_J)}\mathcal{YD}(\mathcal{C})$

$$\bigotimes_{\alpha \in R_+^a \setminus R_{J,+}^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(M^a)^{coin\, \mathcal{B}(M_J)}$$

where $R_{J,+}^a := R_+^a \cap J\mathbb{N}$ is the parabolic subsystem (localization). Now restriction to $(\mathcal{A}^{J^\perp}, \bar{R})$ defines a surjective map (even compatible with addition where appropriate)

$$R_+^a \setminus R_{J,+}^a \to \bar{R}_+^a$$

where the preimages of $\bar{\alpha}_j$ is the root string $(\alpha_j + \mathbb{N}\alpha_i) \cap R_+^a$. Similarly all other preimages of $\bar{\alpha}$ are root strings after reflection, i.e. in some other object a' where α is in a parabolic subsystem $\langle \alpha_i, \alpha_j \rangle$. Reordering (which is clearly isomorphism) and using for each $\bar{\alpha}$ the rank 2 result of a) yields the desired isomorphism. The claims on multiplicities and nonreducedness follow from c).

d) Since we obtained in c) a decomposition

$$\bigotimes_{\bar{\alpha} \in \bar{R}_+^a} \mathcal{B}(M_{\bar{\alpha}}) \cong \mathcal{B}(\bar{M})$$

of the Nichols algebra in $\mathbb{N}^{I\setminus J}$-graded $\mathcal{B}(M_J)$-Yetter-Drinfel'd modules, and the degrees of this decomposition is by construction given by $\mathcal{A}^X$, we can apply the uniqueness result in [HS10] Theorem 4.5 (2).[6] $\square$

4.4. Nichols algebras with new Weyl groupoids.

Recall from Corollary 4.10 that only certain root systems appear as root systems of Nichols algebras in $^{\mathbb{C}[G]}_{\mathbb{C}[G]}\mathcal{YD}$. Parabolic restriction produces Nichols algebras in other braided categories

$$\mathcal{B}(\bar{M}) \coloneqq \mathcal{B}(M)^{coin\,\mathcal{B}(M_J)} \in {}^{\mathcal{B}(M_J)}_{\mathcal{B}(M_J)}\mathcal{YD}(\mathcal{C})$$

and by Theorem 4.13 the root system and Weyl groupoid corresponds to the restricted root system $(\mathcal{A}^X, \bar{R})$, $X = J^\perp$. We hence find an answer to Question 1.1:

Theorem 4.14.

(1) *There exist Nichols algebras whose corresponding crystallographic arrangement are the sporadic arrangements of rank three labeled $7, 13, 14, 15, 20, 23$, although such a Nichols algebra does not exist over any finite group. These have $13, 16, 17, 17, 19, 19$ positive roots respectively, and are restrictions of the reflection arrangements of type $E_7, E_8, E_8, E_8, E_8, E_8$ respectively.*

(2) *Since every crystallographic arrangement of rank greater than three is a (parabolic) restriction of a Weyl arrangement, every crystallographic arrangement of rank greater than three is symmetry structure of some Nichols algebra.*

Question 4.15. Do all root systems appear as root systems of some Nichols algebra in some braided category $\mathcal{C}$?

We now give first an example with a new sporadic Weyl groupoid of rank three labeled 7:

Example 4.16. Let $\mathcal{B}(M) = u_q(E_7)^+$, which is a Nichols algebra of dimension $\operatorname{ord}(q^2)^{63}$. We consider the parabolic restriction indicated in the diagram:

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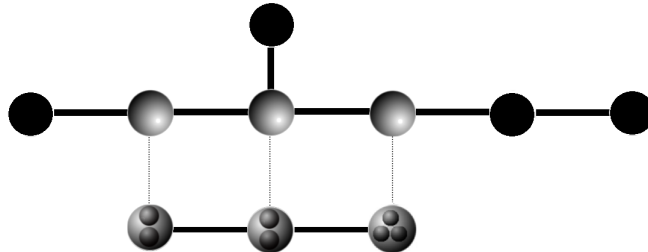
[6]We thank Jing Wang for pointing out this uniqueness result for the root system.

Then the restriction of the Weyl group arrangement E_7 with 63 roots has a Weyl groupoid of sporadic type 7 and 13 roots (one of which is nonreduced), namely

R_+^a	$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$	$(1,1,0)$	$(0,1,1)$	$(1,1,1)$	$+2(1,1,1)$
multiplicity	2	2	3	4	6	12	$+3$
R_+^a	$(1,2,1)$	$(2,2,1)$	$(1,2,2)$	$(2,3,2)$	$(2,3,3)$	$(2,4,3)$	$(3,4,3)$
multiplicity	6	3	6	6	2	1	2

This yields a Nichols algebra $\mathcal{B}(\bar{M})$ of dimension $\mathrm{ord}(q^2)^{58}$ over an object M of rank 3 and dimension $2 + 2 + 3$ in the braided category $^{\mathcal{B}(M_J)}_{\mathcal{B}(M_J)}\mathcal{YD}$ with $J = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7\}$, where $\mathcal{B}(M_J)$ is an ordinary Borel part $u_q(A_1 \times A_1 \times A_2)^+$.

We continue with some smaller, more explicit examples with common Weyl groupoids that illustrate the impact of parabolic restriction:

Example 4.17. Let $M_1 = x_1\mathbb{C}$ and $M_2 = x_2\mathbb{C}$ be Yetter-Drinfel'd modules over $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g_1, g_2 \rangle$ with

$$\chi_1(g_1) = -1, \qquad \chi_2(g_1) = -1,$$
$$\chi_1(g_2) = 1, \qquad \chi_2(g_2) = -1.$$

Then the Nichols algebra $\mathcal{B}(M_1 \oplus M_2)$ is of Lie type A_2,

$$R_+^a = \{(1,0), (0,1), (1,1)\},$$

and multiplication in the Nichols algebra induces an isomorphism of graded vector spaces

$$\mathcal{B}(x_1) \otimes \mathcal{B}(x_1x_2 + x_2x_1) \otimes \mathcal{B}(x_2) \cong \mathcal{B}(M \oplus N).$$

The Hilbert series is $(2)_t^2(2)_{t^2}$ and the dimension is 2^3. Consider the coinvariants $K = \mathcal{B}(M)^{coin\,\mathcal{B}(M_1)}$ with respect to the projection $\mathcal{B}(M \oplus N) \to \mathcal{B}(M)$, it is generated by $x_2, x_1x_2 + x_2x_1$. By Theorem 4.5, K is the Nichols algebra of the $\mathcal{B}(M) \rtimes k[\Gamma]$-Yetter-Drinfel'd module

$$K = \mathrm{ad}_{\mathcal{B}(M)}(N) = \mathrm{ad}_1(x_2) + \mathrm{ad}_{x_1}(x_2) = x_2\mathbb{C} \oplus (x_1x_2 + x_2x_1)\mathbb{C}$$

which is simple as such; hence the root system of $\mathcal{B}(K)$ is of type A_1 with node multiplicity 2. This is in agreement with the root system $(\mathcal{A}^{H_{\alpha_1}}, \bar{R})$ of the restriction, where $\alpha_1 = (1,0)$ is removed and both α_2, α_{12} restricted to H_{α_1} yield the simple root in $\bar{R}$.

We next consider an example which is not of Lie type:

Example 4.18. Consider the diagonal Nichols algebra $\mathcal{B}(M)$ of rank 3, thoroughly treated in Example 4.8, which has braiding matrix

$$q_{\alpha_i\alpha_i} = -1, \qquad q_{\alpha_i\alpha_j}q_{\alpha_j\alpha_i} = \zeta,$$

and the following root system and vector space PBW-basis:

$$R_+^a = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\},$$

$$\mathcal{B}(M) \cong \mathcal{B}(x_{\alpha_1}) \otimes \mathcal{B}(x_{\alpha_2}) \otimes \mathcal{B}(x_{\alpha_3}) \otimes \mathcal{B}(x_{\alpha_{12}}) \otimes \mathcal{B}(x_{\alpha_{13}}) \otimes \mathcal{B}(x_{\alpha_{23}}) \otimes \mathcal{B}(x_{\alpha_{123}})$$

$$\cong \mathbb{C}[x_{\alpha_1}]/(x_{\alpha_1}^2) \otimes \mathbb{C}[x_{\alpha_2}]/(x_{\alpha_2}^2) \otimes \mathbb{C}[x_{\alpha_3}]/(x_{\alpha_3}^2)$$

$$\otimes \mathbb{C}[x_{\alpha_{12}}]/(x_{\alpha_{12}}^3) \otimes \mathbb{C}[x_{\alpha_{13}}]/(x_{\alpha_{13}}^3) \otimes \mathbb{C}[x_{\alpha_{23}}]/(x_{\alpha_{23}}^3)$$

$$\otimes \mathbb{C}[x_{\alpha_{123}}]/(x_{\alpha_{123}}^2).$$

Consider the projection to the Nichols algebra

$$\mathcal{B}(M_{\alpha_2}) \cong \mathbb{C}[x_{\alpha_2}]/(x_{\alpha_2}^2).$$

The action and coaction of x_{α_2} span the following simple 2-dimensional Yetter-Drinfel'd submodules over $\mathcal{B}(M_{\alpha_2}) \rtimes \mathbb{C}[\Gamma]$:

$$M_{\bar{\alpha}_1} := M_{\alpha_1} \oplus M_{\alpha_{12}},$$

$$M_{\bar{\alpha}_2} := M_{\alpha_3} \oplus M_{\alpha_{23}},$$

$$M_{\bar{\alpha}_{12}} := M_{\alpha_{13}} \oplus M_{\alpha_{123}}.$$

We obtain the new generating Yetter Drinfel'd according to Theorem 4.5:

$$\mathcal{B}(M)^{coin\,\mathcal{B}(M_{\alpha_2})} = \mathcal{B}(\bar{(}M)), \qquad \bar{M} = \mathrm{ad}_{\mathcal{B}(M_{\alpha_2})}(M_{\alpha_1} \oplus \mathbb{M}_{\alpha_3}) = M_{\bar{\alpha}_1} \oplus M_{\bar{\alpha}_2}.$$

Altogether, this agrees with the crystallographic arrangement $(\mathcal{A}^{H_{\alpha_2}}, \bar{R})$ which is of type A_2 and all multiplicities 2. We get a new PBW-basis and Hilbert series

$$\mathcal{B}(\bar{M}) \cong \mathcal{B}(M_{\bar{\alpha}_1}) \otimes \mathcal{B}(M_{\bar{\alpha}_2}) \otimes \mathcal{B}(M_{\bar{\alpha}_{12}}),$$

$$\mathcal{H}_{\mathcal{B}(\bar{M})}(t) = (2)_t^2 (3)_t^2 (3)_{t^2} (2)_{t^2}.$$

5. A SIMPLICIAL COMPLEX OF NICHOLS ALGEBRAS

5.1. **The simplicial complex.** As in [CMW15], to a crystallographic arrangement $(\mathcal{A}, R)$ of rank n we associate the *simplicial complex* S on V whose $(n-1)$-cells are the chambers of $\mathcal{A}$ intersected with the $n-1$-sphere. This complex $S \subseteq V$ is topologically isomorphic to a sphere. Sometimes it will be convenient to add a single (-1)-cell corresponding to the origin $0 \in V$. We need the following facts (see [CMW15] for proofs and more details):

Lemma 5.1.

a) *Let K be a chamber and Δ^K be the associated set of simple roots. Then the adjacent $(k-1)$ cells $x \subseteq K$ are in bijection with the subsets $J_{x,K} \subseteq \Delta^K$ of order $n-k$. Moreover $x \subseteq X_x := J_{x,K}^{\perp}$ is of dimension k.*

b) *Let $K_1 \neq K_2$ be chambers and let w be the Weyl groupoid element with $w.K_1 = K_2$. Then the intersection $K_1 \cap K_2$ is a common adjacent $(k-1)$-cell $x \subseteq K_1, K_2$, w is contained in the Weyl groupoid of the localization $\mathcal{A}_{X_x}$, and $w.J_{x,K_1} = J_{x,K_2}$.*

5.2. **The decoration by Nichols algebras in braided categories.** Suppose we have a Nichols algebra $\mathcal{B}(M)$, for now in the braided category $\mathcal{C} = {}^h_h\mathcal{YD}$. We consider the crystallographic arrangement $(\mathcal{A}, R)$ with roots $\alpha \in R$ labeled by objects $M_\alpha \in \mathcal{C}$ from Theorem 4.5. More precisely, the initial Nichols algebra is $\mathcal{B}(M^a)$ associated to some chamber a of $\mathcal{A}$ and one set of positive simple roots Δ_a^+ such that $M^a = \bigoplus_{\alpha \in \Delta_a^+} M_\alpha$, while the other chambers correspond to Weyl equivalent Nichols algebras $\mathcal{B}(M^{a'})$. Thus it would be better to think of $(\mathcal{A}, R)$ being associated to the entire Weyl equivalence class and a specific representing Nichols algebra $\mathcal{B}(M^a) \in \mathcal{C}$ being associated to each $(n-1)$-cell a. We now wish to associate Nichols algebra data to all $(k-1)$-cells x.

Definition 5.2. As in Lemma 5.1, let x be a $(k-1)$-cell, let $X_x \in L(\mathcal{A})$ the associated subspace of V, let a be any adjacent chamber with simple roots Δ^a and $J_{x,a} \subseteq \Delta^a$. Then we associate to the pair (x, a) the braided category of the localized Nichols algebra

$$\mathcal{C}_x := {}^{\mathcal{B}(M_{x,a})}_{\mathcal{B}(M_{x,a})}\mathcal{YD}, \qquad M_x := \bigoplus_{\alpha \in J_{x,a}} M_\alpha$$

and the restricted Nichols algebra in this category as in Theorem 4.13

$$\mathcal{B}(M)^x := \mathcal{B}(M^a)^{coin\,\mathcal{B}(M_{x,a})} \cong \mathcal{B}(M^{x,a}), \qquad M^{x,a} := \mathrm{ad}_{\mathcal{B}(M_{x,a})}\left(\bigoplus_{\alpha \in \Delta^a \setminus J_{x,a}} M_\alpha\right).$$

Remark 5.3. The previous definition is *independent* of the chosen adjacent chamber a in the following sense: Let a_1, a_2 be chambers adjacent to x, then by Lemma 5.1 b) we have an element w in the Weyl groupoid of $\mathcal{A}$, which is contained in the localization to X_x, such that $w.a_1 = a_2$ and $w.J_{x,a_1} = J_{x,a_2}$. Thus $R_w(M_{x,a_1}) = M_{x,a_2}$ and the Nichols algebras $\mathcal{B}(M_{x,a_1}), \mathcal{B}(M_{x,a_2})$ are Weyl equivalent. By [BLS14] Thm. 4.4 this implies there is a category isomorphism

$$r_w : {}^{\mathcal{B}(M_{x,a_1})}_{\mathcal{B}(M_{x,a_1})}\mathcal{YD} \cong {}^{\mathcal{B}(M_{x,a_2})}_{\mathcal{B}(M_{x,a_2})}\mathcal{YD},$$

$$r_w : \ \mathcal{B}(M^{x,a_1}) \mapsto \mathcal{B}(M^{x,a_2}).$$

In particular, the vector space, the root system and Dynkin diagram, and the vector space dimension of the nodes $\bigoplus_i M_{\bar{\alpha}_i} = M^{x,a}$ (multiplicities) are independent of a (in contrast to $M_{x,a}$).

Corollary 5.4. *In this sense, the simplicial complex is decorated by sets of equivalent Nichols algebras in an equivalence class of braided categories and by Dynkin diagrams and root multiplicities.*

We first give the two extremal examples:

Example 5.5. The chambers $x = a \in A$ are the $(n-1)$-cells. Each corresponds to the empty subsets $J_{a,a} = \{\}$ and is contained only in the chamber a itself. It is hence decorated with the unique category

$$\mathcal{C}_a = \ {}^{\mathcal{B}(M_{x,a})}_{\mathcal{B}(M_{x,a})}\mathcal{YD} = \ {}^{1_{\mathcal{C}}}_{1_{\mathcal{C}}}\mathcal{YD} = 1_{\mathcal{C}}$$

and the following Nichols algebra in this category:

$$\mathcal{B}(M)^a = \mathcal{B}(M^a)^{coin\, 1_{\mathcal{B}(M^a)}} = \mathcal{B}(M^a) \in \mathcal{C}.$$

Hence as expected we associate to each $(n-1)$-cell a the Nichols algebra $\mathcal{B}(M^a)$ in the base category $\mathcal{C}$.

Example 5.6. The center point 0 is a (-1)-cell adjacent to any chamber a. It corresponds to the full set of simple roots depending on the chamber $J_{0,a} = \Delta^a$ and is hence decorated with the equivalence class of categories

$$\mathcal{C}_0 = \ {}^{\mathcal{B}(M_{0,a})}_{\mathcal{B}(M_{0,a})}\mathcal{YD} = \ {}^{\mathcal{B}(M^a)}_{\mathcal{B}(M^a)}\mathcal{YD}$$

and the following Nichols algebras in each category:

$$\mathcal{B}(M)^0 = \mathcal{B}(M^a)^{coin\, \mathcal{B}(M^a)} = 1_{{}^{\mathcal{B}(M^a)}_{\mathcal{B}(M^a)}\mathcal{YD}}.$$

Hence we associate to the (-1)-cell in the center the trivial Nichols algebras, i.e. on unit objects in all braided categories ${}^{\mathcal{B}(M^a)}_{\mathcal{B}(M^a)}\mathcal{YD}$ over some Weyl equivalence representative of the full Nichols algebra $\mathcal{B}(M^a)$.

We also note that *simple reflection* can be visualized nicely from this picture: Let a_1, a_2 be adjacent chambers, meaning $a_1 \cap a_2$ is an $(n-2)$-cell, i.e. in a unique hyperplane $H_\alpha, \alpha \in \Delta^{a_1}$. Then the reflection $r_\alpha : \mathcal{B}(M^{a_1}) \to \mathcal{B}(M^{a_2})$ constitutes of the following steps:

- Restriction to the hyperplane H_α yields the restricted Nichols algebra $K :=$ $\mathcal{B}(M^{a_1})^{coin\,\mathcal{B}(M_\alpha)} \in {}^{\mathcal{B}(M_\alpha)}_{\mathcal{B}(M_\alpha)}\mathcal{YD} = C_x$.
- The orientation chance $H_\alpha = H_{-\alpha}$ i.e. $J_{x,a_1} = \{\alpha\} \mapsto \{-\alpha\} = J_{x,a_2}$ is simply a dualization $\mathcal{B}(M_\alpha) \mapsto \mathcal{B}(M_\alpha^*)$, yielding equivalent categories $C_x = {}^{\mathcal{B}(M_\alpha)}_{\mathcal{B}(M_\alpha)}\mathcal{YD} \cong {}^{\mathcal{B}(M_\alpha)^*}_{\mathcal{B}(M_\alpha)^*}\mathcal{YD}$ and especially as image of K a Nichols algebra $L \in {}^{\mathcal{B}(M_\alpha^*)}_{\mathcal{B}(M_\alpha^*)}\mathcal{YD}$.
- Now K is the restriction of the reflected Nichols algebra $\mathcal{B}(M^{a_2})$ to the hyperplane $x = H_{-\alpha}$ and we may conversely obtain the reflected Nichols algebra from K by a Radford biproduct.

5.3. **Example: A Nichols algebra of rank 3.** Consider the Nichols algebra thoroughly treated in Example 4.8. It was of rank 3, and had two types of chambers a, a' with the following sets of 7 positive roots

$$R_+^a = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$$

$$R_+^{a'} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,2,1)\}$$

$$\text{where } \alpha_1' = \alpha_{12}, \alpha_2 = -\alpha_2, \alpha_3' = \alpha_{23}$$

The following picture now shows the arrangement in perspective: It is basically a cuboctahedron (an archimedian solid) with 8 equilateral triangles and 6 squares, which has each square subdivided into 4 right triangles (by the three orthogonal planes $H_{\alpha_{ij}}$). There are hence 7 hyperplanes and $8 + 24 = 32$ chambers.
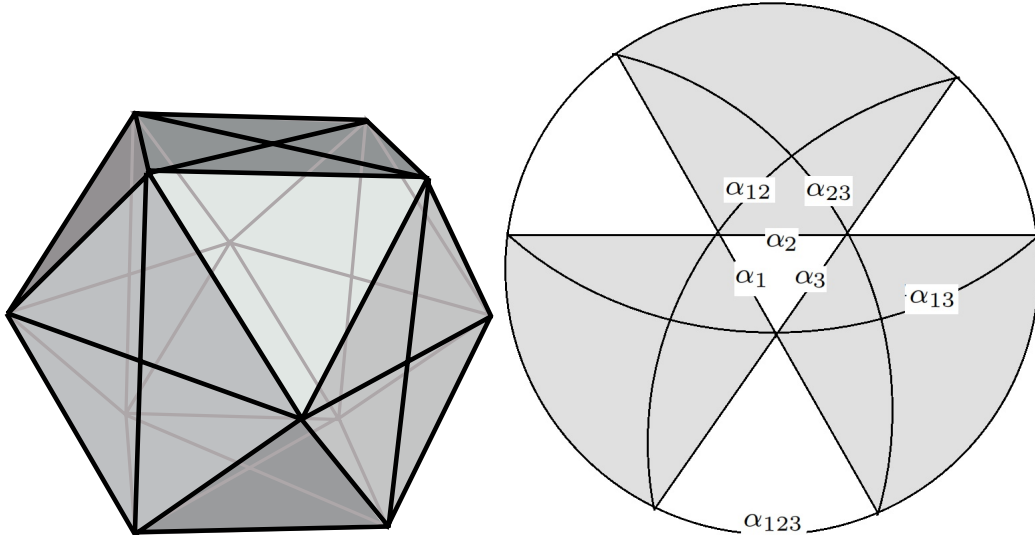
So we have a spherical simplicial complex of dimension $3-1$. We now calculate the categories and Nichols algebras associated to each $(k-1)$-cell:

- 2-Cells: There are two types of chambers:
  - There are 8 chambers associated to equilateral triangles as the example $x = a$ bounded by $H_{\alpha_1}, H_{\alpha_2}, H_{\alpha_3}$. It is associated to the Nichols algebra $\mathcal{B}(M^a)$ in the base category $\mathcal{C}_x = \mathcal{C}$ with Dynkin diagram a cycle and Hilbert series $(2)_t^3(3)_{t^2}^3(2)_{t^3}$
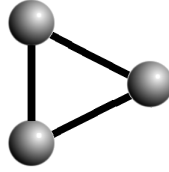
  - There are 24 chambers associated to right triangles as in the example $x = a'$ bounded by $H_{\alpha_{12}}, H_{-\alpha_2}, H_{\alpha_{23}}$ (note $H_{-\alpha_2} = H_{\alpha_2}$). It is associated to the Nichols algebra $\mathcal{B}(M^{a'})$ in the base category $\mathcal{C}_x = \mathcal{C}$ with Dynkin diagram A_2 (but one more root) and Hilbert series $(2)_t(3)_t^2(2)_{t^2}^2(2)_{t^3}(3)_{t^4}$.
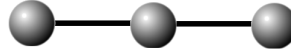
- 1-Cells: There are two types of edges:
  - There are 24 edges between an equilateral and right triangle (the edges of the cuboctahedron) as in the examples a, a' the common hyperplane $x = H_{\alpha_2}$. It is associated to the following restricted Nichols algebra $\mathcal{B}(M^{x,a})$ in the category

  $$\mathcal{C}_x = {}_{\mathcal{B}(M_{\alpha_2})}^{\mathcal{B}(M_{\alpha_2})}\mathcal{YD}, \text{ where } \mathcal{B}(M_{\alpha_2}) \cong \mathbb{C}[t_{\alpha_2}]/(t_{\alpha_2}^2)$$
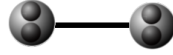
  Note that we calculate the restriction from the chamber a, but the calculation from a' would return the same result except $\alpha_2 \to -\alpha_2$. The generating Yetter-Drinfel'd modules is $M^{x,a} = M_1^{x,a} \oplus M_2^{x,a}$, the Nichols algebra is of Lie type A_3. More precisely, the simple summands are

  $$M_1^{x,a} = M_{\alpha_1} \oplus M_{\alpha_{12}} \qquad M_2^{x,a} = M_{\alpha_3} \oplus M_{\alpha_{23}},$$

  $$[M_1^{x,a}, M_2^{x,a}] = M_{\alpha_{13}} \oplus M_{\alpha_{123}}.$$

  In particular, the Hilbert series of $\mathcal{B}(M^{x,a})$ is $(2)_t^2(3)_t^2(3)_{t^2}(2)_{t^2}$.

— There are 24 edges between two right triangles (half subdividing diagonals in the cuboctahedron squares) as in the examples a' the hyperplane $x = H_{\alpha_{23}}$. It is associated to the following restricted Nichols algebra $\mathcal{B}(M^{x,a})$ in the category

$$\mathcal{C}_x = {}^{\mathcal{B}(M_{\alpha_{23}})}_{\mathcal{B}(M_{\alpha_{23}})}\mathcal{YD}, \text{ where } \mathcal{B}(M_{\alpha_{23}}) \cong \mathbb{C}[t_{\alpha_{23}}]/(t^3_{\alpha_{23}})$$

The generating Yetter-Drinfel'd modules is $M^{x,a} = M^{x,a}_1 \oplus M^{x,a}_2$, the Nichols algebra is of type B_2. More precisely, the simple summands are

$$M^{x,a}_1 = M_{-\alpha_2} \oplus M_{\alpha_1} \qquad M^{x,a}_2 = M_{\alpha_{23}}$$

$$[M^{x,a}_1, M^{x,a}_2] = M_{\alpha_3} \oplus M_{\alpha_{123}} \qquad [M^{x,a}_1, [M^{x,a}_1, M^{x,a}_2]] = M_{\alpha_{13}}$$

In particular, the Hilbert series of $\mathcal{B}(M^{x,a})$ is $(2)^2_t(3)_t(2)^2_{t^2}(3)_{t^3}$.

- 0-Cells: There are two types of 0-cells:
  — There are 12 vertices with altogether three intersecting hyperplanes (the cuboctahedron vertices) as in the examples a the bounding hyperplanes $x = H_{\alpha_1} \cap H_{\alpha_3}$. It is associated to the following restricted Nichols algebra $\mathcal{B}(M^{x,a})$ in the category

$$\mathcal{C}_x = {}^{\mathcal{B}(M_{\alpha_1} \oplus M_{\alpha_3})}_{\mathcal{B}(M_{\alpha_1} \oplus M_{\alpha_3})}\mathcal{YD}$$

where $\mathcal{B}(M_{\alpha_1} \oplus M_{\alpha_3})$ is a Nichols algebra of dimension $2^23 = 12$ with Dynkin diagram A_2 (the Borel part of $\mathfrak{sl}(2|1)$ at $q = \zeta$): The generating Yetter-Drinfel'd module $M^{x,a}$ is simple, the Nichols algebra of type A_1:

$$M^{x,a} = M_{\alpha_2} \oplus M_{\alpha_{12}} \oplus M_{\alpha_{23}} \oplus M_{\alpha_{123}}$$

In particular, the Hilbert series of $\mathcal{B}(M^{x,a})$ is $(2)^2_t(3)^2_t$.

  — There are 6 vertices with two orthogonally intersecting hyperplanes (the subdivision centers of the cuboctahedron squares) as in the examples a' the bounding hyperplanes $x = H_{\alpha_{12}} \cap H_{\alpha_{23}}$. It is associated

to the following restricted Nichols algebra $\mathcal{B}(M^{x,a})$ in the category

$$\mathcal{C}_x = {}^{\mathcal{B}(M_{\alpha_{12}} \oplus M_{\alpha_{23}})}_{\mathcal{B}(M_{\alpha_{12}} \oplus M_{\alpha_{23}})}\mathcal{YD}, \text{ where}$$

$$\mathcal{B}(M_{\alpha_{12}} \oplus M_{\alpha_{23}}) \cong \mathbb{C}[t_{\alpha_{12}}]/(t^3_{\alpha_{12}}) \otimes \mathbb{C}[t_{\alpha_{23}}]/(t^3_{\alpha_{23}})$$

is a Nichols algebra of type $A_1 \times A_1$. The generating Yetter-Drinfel'd modules $M^{x,a}$ is simple, the Nichols algebra of non-reduced type A_1:

$$M^{x,a} = M_{-\alpha_2} \oplus M_{\alpha_{12}} \oplus M_{\alpha_{23}} \oplus M_{\alpha_{123}} \qquad [M^{x,a}, M^{x,a}] = M_{\alpha_{13}}$$

In particular, the Hilbert series of $\mathcal{B}(M^{x,a})$ is $(2)_t^4(3)_{t^2}$.

- Again the (-1)-cell $x = 0$ in the center is the trivial Nichols algebra $1\mathbb{C}$ over any Weyl representative of the full Nichols algebra $\mathcal{C}_x = {}^{\mathcal{B}(M^a)}_{\mathcal{B}(M^a)}\mathcal{YD}$.

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