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A decomposition of the Brauer-Picard group of the representation category of a finite group

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Abstract. We present an approach of calculating the group of braided autoequivalences of the category of representations of the Drinfeld double of a finite group G and thus the Brauer-Picard group of $\mathrm{Rep}(G)$. We obtain two natural subgroups and a subset as candidates for generators. As our main result we prove that any element of the Brauer-Picard group, fulfilling an additional cohomological condition, decomposes into an ordered product of our candidates.
For elementary abelian groups G our decomposition reduces to the Bruhat decomposition of the Brauer-Picard group, which is in this case a Lie group over a finite field. Our results are motivated by and have applications to symmetries and defects in $3d$-TQFT and group extensions of fusion categories.

Contents

1. Introduction

For a finite tensor category $\mathcal{C}$ the *Brauer-Picard group* $\mathrm{BrPic}(\mathcal{C})$ is defined as the group of equivalence classes of invertible $\mathcal{C}$-bimodule categories. This group is an important invariant of the tensor category $\mathcal{C}$ and appears at several essential places in representation theory, for example in the classification problem of G-extensions of fusion categories, see [ENO09]. In mathematical physics (bi-)module categories appear as boundary conditions and defects in $3d$-TQFT, in particular the Brauer-Picard group is a symmetry group of such theories, see [FSV13], [FPSV14].
By a result of [ENO09] for a finite tensor category $\mathcal{C}$ (see [DN12] for $\mathcal{C}$ not semisimple) there exists a group isomorphism to the group of equivalence classes of *braided autoequivalences* of the *Drinfeld center* $Z(\mathcal{C})$:

$$\mathrm{BrPic}(\mathcal{C}) \cong \mathrm{Aut}_{br}(Z(\mathcal{C}))$$

In the case $\mathcal{C} = \mathrm{Rep}(G)$ of finite dimensional complex representations of a finite group G (respectively $\mathcal{C} = \mathrm{Vect}_G$ which has the same Drinfeld center) computing the Brauer-Picard group is already an interesting and non-trivial task. This group appears as the symmetry group of (extended) Dijkgraaf Witten theories based on the structure group G. See [DW90] for the original work on Chern-Simons with finite gauge group G and see [FQ93], [Mo13] for the extended case. In [O03] the authors have obtained a parametrization of Vect_G-bimodule categories in terms of certain subgroups $L \subset G \times G^{op}$ and 2-cocycles μ on L and [Dav10] has determined a condition when such pairs correspond to invertible bimodule categories. However, the necessary calculations to determine $\mathrm{BrPic}(\mathcal{C})$ seem to be notoriously hard and the above approach gives little information about the group structure. In [NR14] the authors use the isomorphism to $\mathrm{Aut}_{br}(Z(\mathcal{C}))$ in order to compute the Brauer-Picard group for several groups G using the following strategy: They enumerate all subcategories $\mathcal{L} \subset Z(\mathcal{C})$ that are braided equivalent to $\mathcal{C} = \mathrm{Rep}(G)$, then they prove that $\mathrm{Aut}_{br}(Z(\mathcal{C}))$ acts transitively on this set. Finally they determine the stabilizer of the standard subcategory $\mathcal{C} \subset Z(\mathcal{C})$ with trivial braiding.

For G abelian, the second author's joint paper [FPSV14] determines a set of generators of the Brauer-Picard group and provides a field theoretic interpretation of the isomorphism $\mathrm{BrPic}(\mathcal{C}) \cong \mathrm{Aut}_{br}(Z(\mathcal{C}))$ in terms of 3d-Dijkgraaf-Witten theory with defects. Results for Brauer-Picard groups of other categories $\mathcal{C}$ include representations of the Taft algebra in [FMM14] and of supergroups in [Mom12],[BN14].
An alternative characterization of elements in $\mathrm{Aut}_{br}(Z(H\text{-mod}))$ in terms of quantum commutative Bigalois objects was given in [ZZ13].

In this article we propose an approach to calculate $\mathrm{BrPic}(\mathcal{C})$ for $\mathcal{C} = H$-mod, the category of finite-dimensional representations of a finite-dimensional Hopf algebra H. Let $\mathcal{C}$ be any tensor category. Then there exists a well-known group homomorphism:

$$\mathrm{Ind}_{\mathcal{C}}:\ \mathrm{Aut}_{mon}(\mathcal{C}) \to \mathrm{BrPic}(\mathcal{C}) \cong \mathrm{Aut}_{br}(Z(\mathcal{C}))$$

The image of this map gives us a natural subgroup of the Brauer-Picard group. Choosing different categories $\mathcal{C}'$ with equivalent Drinfeld center $F : Z(\mathcal{C}') \xrightarrow{\sim} Z(\mathcal{C})$ produces different subgroups:

$$\mathrm{Ind}_{\mathcal{C}'}:\ \mathrm{Aut}_{mon}(\mathcal{C}') \to \mathrm{BrPic}(\mathcal{C}') \cong \mathrm{Aut}_{br}(Z(\mathcal{C}')) \overset{F}{\cong} \mathrm{Aut}_{br}(Z(\mathcal{C}))$$

We consider the case $\mathcal{C} = H$-mod where $Z(H\text{-mod}) = DH\text{-mod} = H^* \bowtie H\text{-mod}$. Then we have a second canonical choice, namely $\mathcal{C}' = H^*$-mod. Note that both subgroups

$\mathrm{im}(\mathrm{Ind}_{\mathcal{C}}), \mathrm{im}(\mathrm{Ind}_{\mathcal{C}'})$ contain a common subgroup $\mathcal{V}$ which is the image of $\mathrm{Out}_{Hopf}(H)$.

The two subgroups defined above use the isomorphism in [ENO09]. Now we introduce an additional set $\mathcal{R} \subset \mathrm{Aut}_{br}(DH\text{-mod})$ that does not use this isomorphism: In the first author's paper [BLS15] the notion of a *partial dualization* $r(H)$ of a Hopf algebra H was defined. These operation generalizes Lusztig's reflection operators on quantum groups, strongly relying on [HS13]. It was proven in [BLS15] Thm. 4.5 that $Z(H\text{-mod}) \cong Z(r(H)\text{-mod})$, which implies that H-mod and $r(H)$-mod have equivalent bimodule categories, one can say that H and $r(H)$ are 2-Morita equivalent. In particular, the Brauer-Picard group is invariant under partial dualizations. One can easily give conditions that ensure the existence of an Hopf isomorphism $f : r(H) \cong H$, hence each pair (r, f) induces an equivalence $Z(H\text{-mod}) \cong Z(r(H)\text{-mod}) \cong_f Z(H\text{-mod})$ and thus provides additional elements in $\mathrm{Aut}_{br}(DH\text{-mod})$. A natural question is:

Question 1.1. *Do the previously defined subgroups and the subset of partial dualizations generate the group* $\mathrm{BrPic}(H\text{-mod})$*? Does* $\mathrm{BrPic}(H\text{-mod})$ *decompose as an ordered product of these subsets?*

Question 1.2. *The elements of* $\mathrm{im}(\mathrm{Ind}_{\mathcal{C}}), \mathrm{im}(\mathrm{Ind}_{\mathcal{C}'})$ *are by definition realized as different bimodule category structures of the abelian categories* $\mathcal{C}$ *and* $\mathcal{C}'$ *respectively. What are the bimodule categories associated to the partial dualizations?*

Question 1.3. *What are the three types of group extensions of the fusion category* $\mathcal{C}$ *associated by the isomorphism in* [ENO09] *to the two subgroups groups and groups generated by partial dualizations?*

A decomposition as described in Question 1.1 would give us effective control over the Brauer-Picard group $\mathrm{BrPic}(\mathcal{C})$ trough explicit and natural generators. Additionally, these generators have an interesting field theoretic interpretation (see next page).

In the present article we solely consider the case $H = kG$ with G a finite group, hence $\mathcal{C} = \mathrm{Vect}_G$, $\mathcal{C}' = \mathrm{Rep}(G)$; in this case the subgroups $\mathrm{Aut}_{mon}(\mathrm{Vect}_G)$ and to a lesser extend $\mathrm{Aut}_{mon}(\mathrm{Rep}(G))$ are well-known. As a main result, we prove that the decomposition described in Question 1.1 holds true for the subgroup of elements in $\mathrm{BrPic}(\mathrm{Vect}_G)$, which fulfill the additional cohomological property of *laziness* in the sense of Schauenburg [Schau02] (see Definition 2.16 below). This condition is automatically fulfilled in the case that G is abelian. Further, for some known examples, we check that the decomposition holds also for the full Brauer-Picard group (see Section 6). One important example is the following

Example (Sec. 6.3)**.** *Let* $G \cong \mathbb{Z}_p^n$ *with p prime number. Our decomposition reduces to the Bruhat decomposition of* $\mathrm{BrPic}(\mathrm{Vect}_G)$*, which is the Lie group* $\mathrm{Sp}_{2n}(\mathbb{F}_2)$ *resp.* $\mathrm{O}_{2n}(\mathbb{F}_p)$ *over the finite field* $\mathbb{F}_p$ *for* $p \geq 3$*. In this case, the images of* $\mathrm{Ind}_{\mathcal{C}}$ *resp.* $\mathrm{Ind}_{\mathcal{C}'}$ *in these Lie groups are lower resp. upper triangular matrices, intersecting in the subgroup* $\mathrm{Out}(G) = \mathrm{GL}_n(\mathbb{F}_p)$*. The partial dualizations are Weyl group elements.*

More precisely, our result reduces to the Bruhat decomposition of the Lie groups C_n *resp.* D_n *relative to the parabolic subsystem* A_{n-1}*. In particular there are* $n+1$ *double cosets of the parabolic Weyl group* $\mathbb{S}_n$*, accounting for the* $n+1$ *non-isomorphic partial dualizations on subgroups* $\mathbb{Z}_p^k$ *for* $k = 0, ..., n$*.*

Our general decomposition is modeled after this example and retains roughly what remains of the Bruhat decomposition for a Lie group over a ring (say in the case $G = \mathbb{Z}_k^n$ with k not prime), but it is not a Bruhat decomposition in general. Moreover, for G non-abelian the subgroups $\mathrm{Ind}_{\mathcal{C}}, \mathrm{Ind}_{\mathcal{C}'}$ in $\mathrm{Aut}_{br}(DG\text{-mod})$ are not isomorphic. Additionally, we exhibit a rare class of braided autoequivalences acting as the identity functor on objects and morphisms but having a non-trivial monoidal structure.

From a mathematical physics perspective these subgroups arise as follows: A Dijkgraaf-Witten theory has as input data a finite group G and a 3-cocycle ω on G. It is a topological gauge theory with principal G-bundles on a manifold M as classical fields. Since for a finite group G all G-bundles are flat, they already form the configuration space. The ω corresponds to a Lagrangian functional (in our article ω is trivial). We are now interested to calculate the symmetry group of the quantized field theory, which is per definition the group of invertible defects and hence $\mathrm{BrPic}(\mathrm{Vect}_G)$.

A first obvious subgroup of the automorphism group of such a theory is $\mathcal{V} = \mathrm{Out}(G)$. Since it already exists at the classical level, we call this a classical symmetry. More symmetries can be obtained by the following idea: equivalence classes of fields are principle G-bundles and thus in bijection with homotopy classes of maps from M to BG, the classifying space of G. One may view this gauge theory as a σ-model with target space BG. Then the 3-cocycle ω can be viewed as a background field on the target space and the choice of ω corresponds to the choice of a 2-gerbe. We obtain a subgroup of automorphisms of the theory by taking the symmetry group of this 2-gerbe. This gives us other classical symmetries, the so-called *background field symmetries* $\mathrm{H}^2(G, k^\times)$ of this 2-gerbe. Our subgroup $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) = \mathcal{B} \rtimes \mathcal{V}$ where $\mathcal{B} \cong \mathrm{H}^2(G, k^\times)$ is therefore the semidirect product of the two classical symmetry groups from above. An interesting implication of our result is that in order to obtain the full automorphism group one seems to require a second σ-model associated to $\mathcal{C}' = \mathrm{Rep}(G)$ (leading to the same quantum field theory) and this alternative σ-model induces another subgroup of background field symmetries $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)})$. This subgroup contains again $\mathcal{V} = \mathrm{Out}(G)$, but is in the general case *not* a semidirect product $\mathcal{E} \rtimes \mathcal{V}$.

The elements in $\mathcal{R}$ correspond to so-called *partial EM-dualities*. These kind of dualities in gauge theories have been discussed e.g. in [KaW07]. In the non-abelian case it turns out $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G})$ and $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)})$ are not conjugate subgroups via $\mathcal{R}$; they are usually not even isomorphic.

We now outline the structure of this article and give details on our methods and results:

In Section 2 we give some preliminaries: We recall the definition the Drinfeld double $H = DG$ and list the irreducible modules $\mathcal{O}_g^\chi$ to be able to express our result also in this explicit basis. Further, we give an introduction to Hopf Bigalois objects: These are certain H^*-H^*-bicomodule algebras A such that the functor $A \otimes_H \bullet$ gives an element in $\mathrm{Aut}_{mon}(H\text{-mod})$ - all monoidal equivalences of H-mod arise in this way for some Bigalois object. At the end of this section we present a special class of Bigalois objects called *lazy*: These are described by pairs (ϕ, σ) where $\phi \in \mathrm{Aut}_{Hopf}(H\text{-mod})$ describes the action of $A \otimes_H \bullet$ on the set of objects in H-mod and where $\sigma \in \mathrm{Z}^2_L(H^*)$ is a so-called lazy 2-cocycle that describes the monoidal structure on the functor $A \otimes_H \bullet$. An essential tool will be the Kac-Schauenburg sequence expressing $\mathrm{Z}^2_L(DH)$ as direct product of three subgroups $\mathrm{Z}^2_L(H), \mathrm{Z}^2_L(H^*), \mathrm{P}(H, H^*)$. This result also holds in the non-lazy cases, but then 2-cocycles do not form a group and the decomposition into pairs (ϕ, σ) is no semidirect product any more.

Section 3 is devoted to the Hopf algebra automorphisms $\mathrm{Aut}_{Hopf}(DG)$. We give a double coset decomposition of $\mathrm{Aut}_{Hopf}(DG)$ with respect to certain natural subgroups. For this we use the classification of bicrossed products of Hopf algebras and their homomorphisms [ABM12] in the special case $DG = k^G \rtimes kG$ [Keil13]. In this approach, Hopf automorphisms of DG are described in terms of 2×2-matrices with entries determined by certain group homomorphisms (see Proposition 3.1). In the case that G has no abelian direct factors, Keilberg determined completely $\mathrm{Aut}_{Hopf}(DG)$ in terms of an exact factorization of $\mathrm{Aut}_{Hopf}(DG)$ into natural subgroups, see Proposition 3.4 - 3.7 and Theorem 3.8. These subgroups are the upper triangular matrices E, lower triangular matrices B, certain diagonal matrices $V \cong \mathrm{Aut}(G)$ and other diagonal matrices V_c (those will not give rise to braided autoequivalences). Schauenburg and Keilberg have determined $\mathrm{Aut}_{Hopf}(DG)$ in [KS14] for general G with a different approach, which we did not find suitable to analyze the subgroup of braided monoidal autoequivalences.

Our main observation for the structure of $\mathrm{Aut}_{Hopf}(DG\text{-mod})$ is that when G contains a direct abelian factor C we need a new class of Hopf automorphisms R of DG called *reflections* of C. These automorphisms exchange a direct factor C with its dual $\hat{C}$ and are a special case of the partial dualization discussed above. This leads us to the main result of Section 3:

Theorem (3.10)**.**
With the subgroups B, E, V, V_c and subsets R, R_t of $\mathrm{Aut}_{Hopf}(DG)$ defined above the following holds:

(ii) *For every $\phi \in \mathrm{Aut}_{Hopf}(DG)$ there is a twisted reflection $r \in R_t$ such that ϕ is an element in the double coset*

$$[(V_c \rtimes V) \ltimes B] \cdot r \cdot [(V_c \rtimes V) \ltimes E]$$

(v) *For every $\phi \in \mathrm{Aut}_{Hopf}(DG)$ there is a reflection $r = r_{(C,H,\delta)} \in R$ such that ϕ is an element in*

$$((V_c \rtimes V) \ltimes B)E \cdot r$$

In the case of an additive group $G = \mathbb{F}_p^n$ (Example 3.12) all twisted reflections are proper reflections and we obtain the Bruhat decomposition of $\mathrm{Aut}_{Hopf}(DG) = \mathrm{GL}_{2n}(\mathbb{F}_p)$, which is a Lie group of type A_{2n-1}, relative to the parabolic subsystem $A_{n-1} \times A_{n-1}$. More precisely, the Levi subgroup is $V_c \rtimes V = \mathrm{GL}_n(\mathbb{F}_p) \times \mathrm{GL}_n(\mathbb{F}_p)$ and $E \cong B$ are the solvable part of the parabolic subgroup. Representatives of the reflections are representatives of the $n+1$ double cosets of the Weyl group $\mathbb{S}_{2n}$ with respect to the parabolic Weyl group $\mathbb{S}_n \times \mathbb{S}_n$. In more general cases usually $E \ncong B$.

In Section 4 we construct braided lazy autoequivalences of DG-mod. Such functors can be parametrized (via Bigalois objects) by pairs

$$(\phi, \sigma) \in \mathrm{Aut}_{Hopf}(DG) \ltimes \mathrm{Z}^2_L(DG^*) =: \underline{\mathrm{Aut}}_{mon,L}(DG\text{-mod})$$

The pair (ϕ, σ) corresponds to the functor (F_ϕ, J^σ) acting on objects via ϕ together with a monoidal structure J determined by the 2-cocycle σ. Two different pairs may give naturally equivalent functors yielding a map $\underline{\mathrm{Aut}}_{mon}(DG\text{-mod}) \to \mathrm{Aut}_{mon}(DG\text{-mod})$. It restricts on $\underline{\mathrm{Aut}}_{mon,L}(DG\text{-mod})$ to a group homomorphism to $\mathrm{Aut}_{mon,L}(DG\text{-mod})$.

Justified by the fact that two monoidal autoequivalences in $\mathrm{Aut}_{Hopf}(DG) \ltimes \mathrm{Z}^2_L(DG^*)$ that differ by an inner Hopf automorphism and a closed 2-cocycle are monoidally equivalent, we often consider the obvious intermediate quotient group $\widetilde{\mathrm{Aut}}_{mon,L}(DG\text{-mod})$ of

$\underline{\mathrm{Aut}}_{mon,L}(DG\text{-mod})$ by replacing automorphisms by outer Hopf automorphism and 2-cocycles by cohomology classes. We denote subgroups of this group by a tilde. However the homomorphism from this quotient to $\mathrm{Aut}_{mon,L}(DG\text{-mod})$ is still not injective.

An element $(\phi,\sigma) \in \underline{\mathrm{Aut}}_{mon,L}(DG\text{-mod})$ is called *braided* iff the associated functor is a braided autoequivalence. We call this subgroup $\underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$ and give criteria (see Lemma 4.4 and equation 23) when (ϕ,σ) is braided. Note that monoidal natural equivalences may hold between braided and non-braided autoequivalences, so the image $\mathrm{Aut}_{br,L}(DG\text{-mod})$ consists of equivalence classes of functors with at least one braided representative.

The ultimate goal is to determine these subgroups, but it seems to be too hard to tackle directly. Rather, we proceed as follows: Consider the subgroups V, B, E as well as the subset R in $\mathrm{Aut}_{Hopf}(DG)$ from Section 6.3. The key observation for the present article is that for each of the subsets any suitable element ϕ can be combined rather uniquely with a specific element σ in one of the three terms of the Kac-Schauenburg sequence for $Z^2_L(DG^*)$, such that the pair (ϕ,σ) becomes braided. We thus define subgroups $\underline{\mathcal{V}}_L$, $\underline{\mathcal{B}}_L$, $\underline{\mathcal{E}}_L$ and a subset $\underline{\mathcal{R}}_L$ of $\underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$ in Propositions 4.5, 4.9, 4.14 and 4.22.

- The subgroup $\underline{\mathcal{V}}_L \cong \mathrm{Aut}(G)$ consists of pairs $(\phi,1)$ induced by group automorphisms ϕ of G endowed with trivial monoidal structure. In the quotient we have $\mathcal{V}_L = \widetilde{\mathcal{V}}_L \cong \mathrm{Out}(G)$.
- The subgroup $\underline{\mathcal{E}}_L$ consists of suitable elements $\phi \in E$ combined with a specific cocycle $\sigma \in \mathrm{Z}^2(k^G)$. More precisely, the group homomorphism $\underline{\mathcal{E}}_L \to \mathrm{Aut}_{Hopf}(DG)$ mapping $(\phi,\sigma) \mapsto \phi$ induces an isomorphism
$$\widetilde{\mathcal{E}}_L \xrightarrow{\sim} Z(G) \wedge Z(G) \subset E$$
The image $\mathcal{E}$ corresponds to the lazy elements in the image of
$$\mathrm{Ind}_{\mathrm{Rep}(G)}:\ \mathrm{Aut}_{mon}(\mathrm{Rep}(G)) \to \mathrm{BrPic}(\mathrm{Rep}(G))$$
(up to $\mathcal{V}$). Lazy here implies they arise from $\mathrm{Aut}_{mon}(\mathrm{Rep}(Z(G)))$.
- The subgroup $\underline{\mathcal{B}}_L$ is similar to $\underline{\mathcal{E}}_L$, combining an element $\phi \in B$ with a cocycle $\sigma \in \mathrm{Z}^2(kG)$. The image $\mathcal{B}$ corresponds to the lazy elements in the image of
$$\mathrm{Ind}_{\mathrm{Vect}_G}:\ \mathrm{Aut}_{mon}(\mathrm{Vect}_G) \to \mathrm{BrPic}(\mathrm{Rep}(G))$$
(up to $\underline{\mathcal{V}}_L$). Lazy here implies they arise from $\mathrm{Aut}_{mon}(\mathrm{Vect}_{G_{ab}})$. In this case $(\phi,\sigma) \mapsto \phi$ gives again a the surjective group homomorphism
$$\widetilde{\mathcal{B}}_L \longrightarrow \hat{G}_{ab} \wedge \hat{G}_{ab} \subset B$$
which is *not* always injective. Rather $\widetilde{\mathcal{B}}_L$ is a central extension of $\hat{G}_{ab} \wedge \hat{G}_{ab}$ by the group of *distinguished cohomology classes* (c.f. [Higgs87]).

  In these non-abelian, rather rare cases we have hence additional nontrivial braided autoequivalences in $\mathrm{Aut}_{br}(DG\text{-mod})$, which are trivial on objects ($\phi = 1$) with nontrivial monoidal structure J^σ. The first nontrivial example arises for G a certain non-abelian group of order p^9, see Example 4.13.
- The reflections $\underline{\mathcal{R}}_L = \widetilde{\mathcal{R}}_L$ arise as follows: For every decomposition $G = H \times C$ we consider the reflection $r_{(H,C,\delta)} \in R$ and we find the unique 2-cocycle induced by a pairing $\lambda \in \mathrm{P}(H,H^*)$ such that the element (r,λ) is the braided autoequivalence obtained in [BLS15] and especially we have $\underline{\mathcal{R}}_L \cong R$. Note however that contrary to the abelian case, $\underline{\mathcal{R}}_L$ does not conjugate $\underline{\mathcal{E}}_L$ and $\underline{\mathcal{B}}_L$. Also, there are other partial dualizations for *semidirect* products appearing in $\mathrm{BrPic}(\mathrm{Rep}(G))$,

but these do not give lazy elements, see the example $G = \mathbb{S}_3$ in Section 6.

In Section 5 we finally prove the main result of this article:

Theorem (5.1)**.**

- *Let $G = H \times C$ where H purely non-abelian and C is* elementary *abelian. Then $\underline{\mathrm{Aut}}_{br,L}(DG\text{-}mod)$ has a double coset decomposition*
$$\underline{\mathrm{Aut}}_{br,L}(DG\text{-}mod) = \bigsqcup_{(r,\lambda)\in\mathcal{R}_L/\sim} \underline{\mathcal{V}}_L\underline{\mathcal{B}}_L \cdot (r,\lambda) \cdot \underline{\mathcal{V}}_L\underline{\mathcal{E}}_L$$
and similarly in the quotient $\mathrm{Aut}_{br,L}(DG\text{-mod}) \subset \mathrm{Aut}_{br}(DG\text{-mod})$.
- *Let G be a finite group with not necessary elementary abelian direct factors. For every element $(\phi,\sigma) \in \underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$ there exists a $(r,\lambda) \in \underline{\mathcal{R}}_L$ such that (ϕ,σ) is in*
$$[\underline{\mathcal{V}}_L \ltimes \underline{\mathcal{B}}_L)\underline{\mathcal{E}}_L] \cdot (r,\lambda)$$
and similarly in the quotient $\mathrm{Aut}_{br,L}(DG\text{-mod}) \subset \mathrm{Aut}_{br}(DG\text{-mod})$.

The proof proceeds roughly as follows: Take an arbitrary element (ϕ,σ), write $\phi \in \mathrm{Aut}_{Hopf}(DG)$ according to the decomposition obtained in Section 3 and then step-by-step prove from the braiding condition that the respective factor in V, E, B, R lies in the image of the maps $(\phi,\sigma) \to \phi$ of the respective subgroups $\underline{\mathcal{V}}_L, \underline{\mathcal{E}}_L, \underline{\mathcal{B}}_L, \underline{\mathcal{R}}_L$. At this point we can use in each step that by construction we can construct an element in $\mathrm{Aut}_{br}(DG\text{-mod})$, which can be subtracted from (ϕ,σ), thereby simplifying ϕ.

We close in Section 6 by several examples, compare our findings to the full Brauer-Picard group obtained in [NR14] and show that the answer to Question 1.1 is positive in these cases.

2. Preliminaries

We will work with a field k that is algebraically closed and has characteristic zero. We denote by $\hat{G}$ the group of 1-dimensional characters of G.

2.1. Modules over the Drinfeld double.

We assume the reader is familiar with Hopf algebras and the representation theory of Hopf algebras to the extend as given in the standard literature e.g. [Kass94]. Denote by kG the group algebra and by $k^G = kG^*$ the dual of the group algebra. Both have well known Hopf algebra structures. The Hopf algebra kG acts on itself and on k^G by conjugation: $h \triangleright g = hgh^{-1}$ and $h \triangleright e_g = e_{hgh^{-1}}$. Starting from a finite dimensional Hopf algebra H one can construct the Drinfeld double DH that is $H^{*\mathrm{cop}} \otimes H$ as a coalgebra (see e.g. [Kass94]). Here we will be mainly interested in the Drinfeld double of the Hopf algebra kG for a finite group G.

Definition 2.1. *Let G be a finite group. We denote the vector space basis of DG by $\{e_x \times y\}_{x,y\in G}$ where a function $e_x \in k^G$ is defined by $e_x(y) = \delta_{x,y}$.*
Then DG has the following Hopf algebra structure:

Unital algebra:
$$(e_x \times y)(e_{x'} \times y') = e_x(yx'y^{-1})(e_x \times yy') \qquad 1_{DG} = \sum_{x\in G}(e_x \times 1_G)$$

Counital coalgebra:

$$\Delta(e_x \times y) = \sum_{x_1 x_2 = x} (e_{x_1} \times y) \otimes (e_{x_2} \times y) \qquad \epsilon(e_x \times y) = \delta_{x,1_G}$$

Antipode:

$$S(e_x \times y) = e_{y^{-1}x^{-1}y} \times y^{-1}$$

Later we will also use the Hopf algebra DG^* the dual Hopf algebra of DG. Recall that the multiplication in DG^* is given by

$$(x \times e_y)(x' \times e_{y'}) = (xx' \times e_y * e_{y'})$$

and comultiplication by

$$\Delta(x \times e_y) = \sum_{y_1 y_2 = y} (x \times e_{y_1}) \otimes (y_1^{-1} x y_1 \times e_{y_2})$$

In the case the group $G = A$ is abelian we have that $DA \simeq k(\hat{A} \times A)$ and $DA^* \simeq k(A \times \hat{A})$ are isomorphic to each other as Hopf algebras. In general there is no Hopf automorphism from DG to DG^*.

Lemma 2.2. *Let us denote the category of left DG-modules by DG-mod. We recall that this is a semisimple braided tensor category as follows:*

- *The simple objects of DG-mod are induced modules $\mathcal{O}_g^\rho := kG \otimes_{k\mathrm{Cent}(g)} V$, where $[g] \subset G$ is a conjugacy class and $\rho : \mathrm{Cent}(g) \to \mathrm{GL}(V)$ an isomorphism class of an irreducible representation of the centralizer of a representative $g \in [g]$. We have the following left DG-action on $\mathcal{O}_g^\rho$:*

  $$(e_h \times t).(y \otimes v) := e_h((ty)g(ty)^{-1})(ty \otimes v)$$

  More explicitly: $\mathcal{O}_g^\rho$ is a G-graded vector space consisting of $|[g]|$ copies of V:

  $$\mathcal{O}_g^\rho := \bigoplus_{g' \in [g]} V_{g'}, \quad V_{g'} := V$$

  Then the action of an element $(e_h \times 1) \in DG$ is given by projecting to the homogeneous component V_h. Choose a set of coset representatives such that $G = \bigcup_i s_i \mathrm{Cent}(g)$. Then the action of an element $(1 \times h) \in DG$ is given by

  $$\begin{aligned} V_{g'} &\to V_{hg'h^{-1}} \\ v &\mapsto (1 \times h).v := \rho(s_j h s_i^{-1})v \end{aligned}$$

  where the s_i, s_j are determined by $s_i g s_i^{-1} = g'$ and $s_j g s_j = hg'h^{-1}$.
- *The monoidal structure on DG-mod is given by the tensor product of DG-modules, i.e. with the diagonal action on the tensor product.*
- *The braiding $\{c_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M \mid M, N \in DG\text{-mod}\}$ on DG-mod is defined by the universal R-matrix*

  $$R = \sum_{g \in G} (e_g \times 1) \otimes (1 \times g) = R_1 \otimes R_2 \in DG \otimes DG$$

  $$c_{M,N}(m \otimes n) = \tau(R.(m \otimes n)) = R_2.n \otimes R_1.m$$

Note that DG-mod is equivalent as braided monoidal category to the category of G-Yetter-Drinfeld-modules and the Drinfeld center of the category of G-graded vector spaces.

Definition 2.3.

- *Let* $\underline{\mathrm{Aut}}_{mon}(DG\text{-mod})$ *be the functor category of monoidal autoequivalences of* DG-mod *and natural monoidal isomorphisms and* $\mathrm{Aut}_{mon}(DG\text{-mod})$ *be the group of isomorphism classes of monoidal autoequivalences of* DG-mod.
- *Let* $\underline{\mathrm{Aut}}_{br}(DG\text{-mod})$ *be the functor category of braided autoequivalences of* DG-mod *and natural monoidal isomorphisms and* $\mathrm{Aut}_{br}(DG\text{-mod})$ *be the group of isomorphism classes of monoidal autoequivalences of* DG-mod.

Note that a natural monoidal transformation between two braided functors is automatically a natural braided transformation. On the other hand there do exist natural monoidal transformations between a braided and a non-braided functor. Hence $\mathrm{Aut}_{br}(DG\text{-mod})$ consists of classes where there *exists* a representative that is braided.

2.2. **Hopf-Galois-Extensions.**

In order to study braided automorphisms of DG-mod we will make use of the theory of Hopf-Galois extensions. For this our main source is [Schau96] and [Schau91]. The motivation for this approach lies mainly in the relationship between Galois extensions and monoidal functors as formulated in e.g. in [Schau96] and also stated in Proposition 2. Namely, monoidal functors between the category of L-comodules and the category of H-comodules are in one-to-one correspondence with L-H-Bigalois objects. For this reason we are lead to the study of DG^*-Bigalois extensions. Since DG is finite dimensional we can use the fact that Bigalois objects over a finite dimensional Hopf algebra can essentially be described by an automorphism of H and a 2-cocycle on H. We will see in a later section that it is possible to manage the automorphism group of DG, but not so much the quite large set of cocycles. There is however a special class of 2-cocycles that are called lazy (or sometimes invariant) which can be controlled quite well and give us a large class of Bigalois objects. Before that we would like to introduce some basic notions and properties of Hopf-Galois extensions.

Definition 2.4. *For a bialgebra H let* comod-H *be the category of right H-comodules. A right H-comodule algebra A is an algebra object in* comod-H *such that $\delta(xy) = x_0y_0 \otimes x_1y_1$. The subset of H-coinvariants $A^{coH} := \{a \in A \mid \delta(a) = a \otimes 1\}$ is a subalgebra of A. A right H-comodule algebra A is called a* right H-Galois extension of $B := A^{coH}$ *if A is faithfully flat over B and the Galois map*

$$\begin{array}{ccccc} A \otimes_B A & \xrightarrow{id_A \otimes \delta} & A \otimes A \otimes H & \xrightarrow{\mu_A \otimes id_H} & A \otimes H \\ x \otimes y & \longmapsto & x \otimes y_0 \otimes y_1 & \longmapsto & xy_0 \otimes y_1 \end{array}$$

is a bijection. A morphism of right H-Galois objects is an H-colinear algebra morphism. Left H-Galois extensions are defined similarly. Denote by $\mathrm{Gal}_B(H)$ the set of equivalence classes of right H-Galois extensions of B and by H-Gal_B the equivalence classes of left H-Galois extensions of B. A right (left) H-Galois object *is a Galois extension of the base field $B = k$.*

Example 2.5. *A Hopf algebra H has a natural structure of an H-Galois object where the coaction is given by the comultiplication in H.*
Note that a bialgebra H is a H-Galois object if and only if H is a Hopf algebra.

Definition 2.6. *Let L, H be two Hopf algebras. An L-H-*Bigalois object *A is an L-H-bicomodule algebra which is a left H-Galois object and a right L-Galois object. Denote by* $\mathrm{Bigal}(L, H)$ *the set of isomorphism classes of L-H-Bigalois objects and by* $\mathrm{Bigal}(H)$ *the set of isomorphism classes of H-H-Bigalois objects.*

Recall that the *cotensor product* of a right L-comodule (A, δ_R) and a left L-comodule (B, δ_L) is defined by

$$A \square_L B := \{a \otimes b \in A \otimes B \mid \delta_R(a) \otimes b = a \otimes \delta_L(b)\}$$

Moreover, if A is an E-L-Bigalois object and B an L-H-Bigalois object then the cotensor product $A \square_L B$ is an E-H-Bigalois object.

Proposition 2.7. *The cotensor product gives* $\mathrm{Bigal}(H)$ *a group structure. The Hopf algebra H with the natural H-H-Bigalois object structure is the unit in the group* $\mathrm{Bigal}(H)$*. Further, we can define* Bigal *to be a groupoid where the objects are given by Hopf algebras and the morphisms between two Hopf algebras L, H are given by elements in* $\mathrm{Bigal}(L, H)$*. The composition of morphisms is the cotensor product.*

Proposition 2.8. [Schau96] *Given an right H-Galois object A there exists a Hopf algebra $L(H, A)$ such that A is an $L(H, A)$-H-Bigalois object. If L' is another Hopf algebra such that A is an L'-H-Bigalois object then there is an isomorphism $L(H, A) \cong L'$ that is compatible with the respective coactions on A. In particular, for a given right H-Galois object A the set of all $L(H, A)$-H-Bigalois object structures on A is parametrized by* $\mathrm{Aut}_{Hopf}(L(H, A))$ *up to coinner Hopf automorphisms.*

Example 2.9. *For a Hopf algebra automorphism $\phi \in \mathrm{Aut}_{Hopf}(H)$ we obtain an H-H-Bigalois object ${}_{\phi}H$ where the left H-coaction is the coproduct post-composed with ϕ. This yields a group homomorphism $\mathrm{Aut}_{Hopf}(H) \to \mathrm{Bigal}(H)$ which in general is neither surjective nor injective.*

Let us now discuss the relation between Galois extensions and monoidal functors, which we will later use to study the group $\mathrm{Aut}_{mon}(DG\text{-mod})$. For this recall that given a Hopf algebra H a *fiber functor* $H\text{-comod} \to \mathrm{Vect}_k$ is a k-linear, monoidal, exact and faithful functor that preserves colimits.

Proposition 2.10. *(*[Schau96] *Def. 5.1, Theorem 5.2)*
Let H be a Hopf algebra, $\mathrm{Gal}(H)$ *the set of isomorphism classes of H-Galois objects and* $\mathrm{Fun}_{fib}(H\text{-comod}, \mathrm{Vect}_k)$ *the set of monoidal isomorphism classes of fiber functors. Then there is a bijection of sets*

$$\begin{aligned} \mathrm{Gal}(H) &\xrightarrow{\sim} \mathrm{Fun}_{fib}(H\text{-comod}, \mathrm{Vect}_k) \\ A &\mapsto (A \square_H \bullet, J^A) \end{aligned} \tag{1}$$

where the monoidal structure J^A of the functor $A \square_H \bullet$ is given by

$$\begin{aligned} J^A_{V,W} : (A \square_H V) \otimes_k (A \square_H W) &\xrightarrow{\sim} A \square_H (V \otimes_k W) \\ \left(\sum x_i \otimes v_i\right) \otimes \left(\sum y_i \otimes w_i\right) &\mapsto \sum x_i y_i \otimes v_i \otimes w_i \end{aligned} \tag{2}$$

Proposition 2.11. *(*[Schau96] *Theorem 5.5, Corollary 5.7)*
Let H, L *be two Hopf algebras and* $\mathrm{Aut}_{mon}(L\text{-comod}, H\text{-comod})$ *be the group of isomorphism classes of monoidal equivalences. Then there is a group isomorphism:*

$$\mathrm{Bigal}(L, H) \to \mathrm{Aut}_{mon}(L\text{-comod}, H\text{-comod})$$
$$A \mapsto (A\square_H \bullet, J^A)$$

In particular, these give us isomorphisms

$$\mathrm{Bigal}(H) \simeq \mathrm{Aut}_{mon}(H\text{-comod}) \qquad \mathrm{Bigal}(H^*) \simeq \mathrm{Aut}_{mon}(H\text{-mod})$$

Example 2.12. *In Example 2.9 we obtained for each* $\phi \in \mathrm{Aut}_{Hopf}(H) \cong \mathrm{Aut}_{Hopf}(H^*)$ *a* H^**-Bigalois object* ${}_\phi H^*$ *isomorphic to* H^* *as an algebra but with right comodule structure post-composed by* ϕ*. Under the isomorphism above, this corresponds to the monoidal autoequivalence* (F_ϕ, J^{triv}) *mapping an* H*-module* M *to the* H*-module* ${}_\phi M$ *given by pre-composing the module structure with* ϕ *(and a trivial monoidal structure).*

There is a large class of H-Galois extensions which come from twisting the algebra structure. In the case when H is finite-dimensional or pointed all H-Galois extensions arise in this way.

Definition 2.13.

(i) *Denote by* $\mathrm{Reg}^1(H)$ *the group of convolution invertible,* k*-linear maps* $\eta : H \to k$ *such that* $\eta(1) = 1$*.*

(ii) *Let* $\mathrm{Reg}^2(H)$ *be the group of convolution invertible,* k*-linear maps* $\sigma : H \otimes H \to k$ *such that* $\sigma(1, h) = \epsilon(h) = \sigma(h, 1)$

(iii) *A left* 2-cocycle *on* H *is a map* $\sigma \in \mathrm{Reg}^2(H)$ *such that for all* $a, b \in H$

$$\sigma(a_1, b_1)\sigma(a_2b_2, c) = \sigma(b_1, c_1)\sigma(a, b_2c_2) \tag{3}$$

We denote the set of 2*-cocycles on* H *by* $\mathrm{Z}^2(H)$*.*

(iv) *We define a map* $d : \mathrm{Reg}^1(H) \to \mathrm{Reg}^2(H)$ *by*

$$\mathrm{d}\eta(a, b) = \eta(a_1)\eta(b_1)\eta^{-1}(a_2b_2) \tag{4}$$

for all $a, b \in H$*. We have* $\mathrm{d}\eta \in \mathrm{Z}^2(H)$ *and denote by* $\mathrm{H}^2(H)$ *the set of* 2*-cocycles modulo the image of* d*.*

For other properties of d *see* [BC04] *Lemma 1.6.*

Note that we are using Sweedler notation: $\Delta(h) = h_1 \otimes h_2$ for $h \in H$.

Proposition 2.14. [Schau91]
An H*-Galois object* A *is called* cleft *if one of the following equivalent conditions hold:*

- *There exists an* H*-comodule isomorphism* $H \simeq A$*.*
- *There exists an* H*-colinear convolution invertible map* $H \to A$
- *There exists a* 2*-cocycle* σ *such that* $A \simeq {}_\sigma H$ *as* H*-comodule algebras, where* ${}_\sigma H$ *is has the* H*-comodule structure of* H *and the following twisted algebra structure*

$$a \cdot_\sigma b = \sigma(a_1, b_1)a_2b_2$$

Two sufficient conditions such that any H*-Galois object* A *is cleft are:*

- H *is finite dimensional.*
- H *is pointed (all simple comodules are* 1*-dimensional).*

In this case the map $\sigma \mapsto {}_\sigma H$ induces an bijection $\mathrm{H}^2(H) \cong \mathrm{Gal}(H)$.
Also, if we are in the cleft case the unique Hopf algebra $L(H,A)$ from Proposition 2.8 is given by the Doi twist $L(H,A) := {}_\sigma H_{\sigma^{-1}}$ which is H as a coalgebra and has the following twisted algebra structure:

$$a \cdot b := \sigma(a_1, b_1) a_2 b_2 \sigma^{-1}(a_3, b_3)$$

It is important to note that $\mathrm{Z}^2(H)$ (as well as $\mathrm{H}^2(H)$) is *not* a group, because the convolution of two 2-cocycles is not a 2-cocycle in general. The convolution product $\sigma * \tau$ for $\sigma \in \mathrm{Z}^2(H)$ and $\tau \in \mathrm{Z}^2(L)$ is a 2-cocycle in $Z^2(H)$ if $L \simeq {}_\sigma H_{\sigma^{-1}}$. One should rather, analogous to Proposition 2.7, consider the groupoid where the objects are Hopf algebras and the morphisms between two Hopf algebras H, L are given by those 2-cocycles $\sigma \in \mathrm{Z}^2(H)$ that have the property $L \simeq {}_\sigma H_{\sigma^{-1}}$.

Moreover, if $F : H\text{-comod} \to \mathrm{Vect}_k$ is a fiber functor then F corresponds under (1) to a cleft H-Galois object if and only if F is isomorphic to the forgetful functor if and only if F preserves the dimensions of the k-vector spaces underlying the H-comodules.

Corollary 2.15. *For a finite-dimensional Hopf algebra H we have a surjection of sets*

$$\mathrm{Z}^2(H^*) \to \mathrm{Fun}_{fib}(H\text{-mod}, \mathrm{Vect}_k) \qquad J^\sigma_{V,W} : V \otimes_k W \xrightarrow{\simeq} V \otimes_k W$$
$$\sigma \mapsto (\mathrm{Forget}, J^\sigma) \qquad v \otimes w \mapsto \sigma_1.v \otimes \sigma_2.w$$

Here we have identified $\sigma : H^* \otimes H^* \to k$ with an element $\sigma = \sigma_1 \otimes \sigma_2 \in H \otimes H$. Further, this map induces a bijection of sets $\mathrm{H}^2(H^*) \simeq \mathrm{Fun}_{fib}(H\text{-mod}, \mathrm{Vect}_k)$.

Proof. Use Proposition 2.10 and the fact that since H is finite dimensional all H-Galois objects are cleft. Hence the corresponding functor is the forgetful functor on objects and morphisms. Further, it is a straightforward calculation that if use the canonical equivalence comod-$H^* \cong H$-mod and view the 2-cocycle as an element $\sigma \in H \otimes H$ that the monoidal structure as defined in (2) is equivalent to acting with σ. Bijectivity follows from Proposition 2.14 ☐

In general it is very hard to control the sets $\mathrm{Z}^2(H)$, $\mathrm{H}^2(H)$ and even more the subset of Galois objects with $L(A,H) \cong H$. For this reason we consider below H-Galois objects that have an additional property. These H-Galois objects can be characterized as having a canonical choice of a Bigalois object structure and lead ultimately to Lemma 2.26. An implication of that property is that they can be described by certain cohomology groups.

2.3. Lazy Bigalois Objects and Lazy Cohomology.

Definition 2.16.

- *An H-H-Bigalois object A is called* bicleft *if and only if $A \cong H$ as H-bicomodules. The group* $\mathrm{Bigal}_{bicleft}(H)$ *of bicleft H-H-Bigalois objects is a normal subgroup of* $\mathrm{Bigal}(H)$.
- *A right H-Galois object A is called* lazy *if there exists a unique left H-Galois structure such that A is bicleft. Hence it is a Galois object where there is a canonical $L(H,A) \cong H$.*
- *An H-H-Bigalois object A is called* lazy *if it is lazy as a right H-Galois object. Denote the group of lazy H-H-Bigalois objects by* $\mathrm{Bigal}_{lazy}(H)$.

We have an isomorphism of groups

$$\mathrm{Bigal}_{lazy}(H) \cong \mathrm{Aut}_{Hopf}(H) \ltimes \mathrm{Bigal}_{bicleft}(H)$$

Further, there is a bijection of sets $\mathrm{Bigal}_{bicleft}(H) \cong \mathrm{Gal}_{lazy}(H)$.

Example 2.17. *If H is cocommutative, then all H-Galois objects and H-H-Bigalois objects are lazy. We will see later, that for $H = k^G$ with e.g. $|G|$ odd all H-H-Bigalois objects are lazy, even though there are L-H-Bigalois objects with $L(H,A) \ncong H$.*

Since we are here mainly interested in the cleft case, hence the case where all H-Galois objects are of the form ${}_\sigma H$ for a 2-cocycle σ, we discuss what additional property on σ corresponds to the lazy property of the Galois object ${}_\sigma H$.

Definition 2.18. [BC04]

- *An $\eta \in \mathrm{Reg}^1(H)$ is called* lazy *if it has the additional property $\eta * \mathrm{id} = \mathrm{id} * \eta$. Denote by $\mathrm{Reg}^1_L(H)$ the subgroup of such lazy regular maps.*
- *A $\sigma \in \mathrm{Reg}^2(H)$ is called* lazy *if it commutes with the multiplication on H:*

  $$\sigma * \mu_H = \mu_H * \sigma$$

  and denote the subgroup of such lazy regular maps by $\mathrm{Reg}^2_L(H)$.
- *A 2-cocycle $\sigma \in \mathrm{Z}^2(H)$ is called* lazy *if $\sigma \in \mathrm{Reg}^2_L(H)$. Denote by $\mathrm{Z}^2_L(H)$ the subgroup of lazy 2-cocycles.*
- *The homomorphism* d *in Definition 2.13 maps $\mathrm{Reg}^1_L(H)$ to the center of $\mathrm{Z}^2_L(H)$.*
- *An $\eta \in \mathrm{Reg}^1(H)$ is called* almost lazy *if $\mathrm{d}\eta$ is lazy. Denote by $\mathrm{Reg}^1_{aL}(H)$ the group of such almost lazy regular maps.*
- *The lazy cohomology groups are then defined by $\mathrm{H}^1_L(H) := \mathrm{Z}^1_L(H) := \ker(\mathrm{d})$ and*

  $$\mathrm{H}^2_L(H) := \mathrm{Z}^2_L(H)/B^2_L(H) \quad B^2_L(H) := \mathrm{d}(\mathrm{Reg}^1_L(H))$$

Proposition 2.19. *(*[BC04] *Proposition 3.6, Proposition 3.7)*
An H-Galois object A is bicleft if and only if there exists a lazy $\sigma \in Z^2_L(H)$ such that ${}_\sigma H_{\sigma^{-1}} \simeq A$ as H-bicomodule algebras. Further, the group of bicleft H-Bigalois objects is a normal subgroup of $\mathrm{Bigal}(H)$ *and* $\mathrm{H}^2_L(H) \simeq \mathrm{Gal}_{bicleft}(H)$.

Example 2.20.

(1) *For $H = kG$ with G a finite group the 2-cocycles are k-linear maps $\sigma : kG \otimes kG \to k$ fulfilling $\sigma(ab,c)\sigma(a,b) = \sigma(b,c)\sigma(a,bc)$. It is easy to see that the lazy condition is automatically fulfilled and that* d *is the differential corresponding to the bar complex. We conclude that 2-cocycles and lazy 2-cocycles on kG are just ordinary group 2-cocycles and that the (lazy) 2-cohomology is just group cohomology $\mathrm{H}^2(kG) = \mathrm{H}^2_L(kG) = \mathrm{H}^2(G)$*

(2) *For $H = k^G$ with G a finite group, a left 2-cocycle α is lazy if and only if for all $x, y, g \in G$ holds: $\alpha(e_x, e_y) = \alpha(e_{gxg^{-1}}, e_{gyg^{-1}})$. An $\eta \in Reg^1(k^G)$ is lazy if and only if $\eta(e_x) = \eta(e_{gxg^{-1}})$ for all $g, x \in G$.*
The non cocommutativity of k^G makes a more explicit characterization quite hard. In fact, will see in Section 4.4 that one needs much machinery to determine $H^2_L(k^G)$.

Corollary 2.21. *Let H be a finite (or pointed) Hopf algebra, then there is a natural group homomorphism $Z^2_L(H) \to \mathrm{Gal}_{lazy}(H); \sigma \mapsto {}_\sigma H$ which induces a group isomorphism $Z^2_L(H)/\mathrm{d}(\mathrm{Reg}^1_{aL}(H)) \simeq \mathrm{Gal}_{lazy}(H)$.*

Definition 2.22.

- *For a Hopf algebra H denote by* $\mathrm{Int}(H)$ *the subgroup of* internal *Hopf automorphisms. These are such $\phi \in \mathrm{Aut}_{Hopf}(\mathrm{H})$ that are of the form $\phi(h) = xhx^{-1}$ for some invertible $x \in \mathrm{H}$ and x has the property that for all $h \in \mathrm{H}$*
$$(x \otimes x)\Delta(x^{-1})\Delta(h) = \Delta(h)(x \otimes x)\Delta(x^{-1}) \tag{5}$$
Note: For an invertible element $x \in H$ the conjugation $\phi(h) = xhx^{-1}$ is an algebra automorphism. It is a coalgebra automorphism if and only if (5) *holds.*
- *Denote by* $\mathrm{Inn}(H) \subset \mathrm{Int}(H)$ *the subgroup of* inner *Hopf automorphisms, hence $\phi \in \mathrm{Aut}_{Hopf}(H)$ of the form $\phi(h) = xhx^{-1}$ for some group-like $x \in H$.*
- *Let* $\mathrm{Out}(H) := \mathrm{Aut}_{Hopf}(H)/\mathrm{Inn}(H)$ *the subgroup of outer Hopf automorphisms.*

Example 2.23. *For $H = DG$ the group-like elements are $G(DG) = \hat{G} \times G$, hence* $\mathrm{Inn}(DG) \cong \mathrm{Inn}(G)$. *More precisely, each inner automorphism $\phi \in \mathrm{Inn}(DG)$ is of the form $\phi(e_g \times h) = e_{tgt^{-1}} \times tgt^{-1}$ for some $t \in G$.*

We now discuss how the previously defined subgroups interact:

Lemma 2.24. *(*[BC04] *Lemma 1.15)*
$\mathrm{Aut}_{Hopf}(H)$ *acts on $Z^2_L(H^*)$ by $\phi.\sigma = (\phi \otimes \phi)(\sigma)$ where we identify a 2-cocycle in H^* with $\sigma = \sigma_1 \otimes \sigma_2 \in H \otimes H$. Then for all $\phi \in \mathrm{Aut}_{Hopf}(H)$ we have:*

- *If $\omega, \sigma \in \mathrm{Z}^2(H^*)$ then $\phi.(\sigma * \omega) = (\phi.\sigma) * (\phi.\omega)$.*
- *If $\gamma \in \mathrm{Reg}^1(H^*)$ then $\phi.d\gamma = d(\gamma \circ \phi)$.*
- $\mathrm{Inn}(H)$ *acts trivially on* $\mathrm{Reg}^2_L(H^*)$.
- *This action induces an action of* $\mathrm{Out}(H)$ *on lazy cohomology* $\mathrm{H}^2_L(H^*)$

Lemma 2.25. *Let $\phi \in \mathrm{Aut}_{Hopf}(H)$ be a Hopf automorphism and let $\sigma \in Z^2_L(H^*)$ a lazy 2-cocycle then the following are equivalent*

- *The functor (F_ϕ, J^σ) is monoidally equivalent to* (id, J^{triv})
- *$\phi = x \cdot \mathrm{id}_H \cdot x^{-1} \in \mathrm{Int}(H)$ for some invertible element $x \in H$ and*
$$\sigma = \Delta(x)(x^{-1} \otimes x^{-1}) \tag{6}$$

Proof. Let η be the monoidal equivalence $(\mathrm{id}, J^{triv}) \sim (F_\phi, J^\sigma)$. Then in particular there is an H-module isomorphism for the regular H-module $\eta_H : H \xrightarrow{\sim} F_\phi(H) =: {}_\phi H$ such that $\eta_H \circ f = f \circ \eta_H$ for all H-module homomorphisms $f : H \to H$. Note that η_H is determined by what it does on 1_H hence by an invertible element $x := \eta(1) \in H$. Further, every H-module morphism $f : H \to H$ is determined by an $h := f(1) \in H$ and then the naturality property of η implies $\phi(h) = xhx^{-1}$. Since ϕ is a Hopf automorphism there are additional conditions on x. It is obvious that ϕ is an algebra automorphism for all invertible $x \in H$ but it is a coalgebra automorphism if and only if
$$\Delta(x^{-1})\Delta(h)\Delta(x) = (x^{-1} \otimes x^{-1})\Delta(h)(x \otimes x)$$
which is equivalent to (5). Further, by definition there has to be an H-module isomorphism $\eta_{H\otimes H} : H \otimes H \xrightarrow{\sim} {}_\phi(H \otimes H)$ such that for all H-module morphisms $r : H \to H \otimes H$ the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\eta_H} & {}_\phi H \\ r\downarrow & & r\downarrow \\ H \otimes H & \xrightarrow{\eta_{H\otimes H}} & {}_\phi(H \otimes H) \end{array}$$

Again, r is determined by $y := r(1) \in H \otimes H$ then the diagram above implies $\eta_{H\otimes H}(y) = x.y$ for all $y \in H \otimes H$. Now we use that η is monoidal, hence in particular the diagram

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\eta_H \otimes \eta_H} & {}_{\phi}H \otimes {}_{\phi}H \\ \downarrow J^{triv}_{H\otimes H} & & \downarrow J^{\sigma}_{H\otimes H} \\ H \otimes H & \xrightarrow{\eta_{H\otimes H}} & {}_{\phi}(H \otimes H) \end{array}$$

commutes which implies that $J^{\sigma}_{H\otimes H}(xh \otimes xh') = \Delta(x)(h \otimes h')$ for all $h, h' \in H$ which is equivalent to (6).
On the other hand let $(\phi, \sigma) \in \mathrm{Aut}_{Hopf}(H) \times Z^2_L(H^*)$ such that $\phi(h) = xhx^{-1}$ for some invertible $x \in H$ and such that equations (5) and (6) hold. It can be checked by direct calculation that ϕ is a well-defined algebra automorphism because x is invertible and coalgebra morphism because of (5). We claim that the following family of morphisms $\eta = \{\eta_M : M \to {}_{\phi}M; \eta_M(m) = x.m\}$ is a monoidal equivalence between (id, J^{triv}) and (F_ϕ, J^σ). The fact that η is a natural equivalence follows from construction. The fact that η is monoidal follows from (6). □

We collect the results of this section in the following Lemma.

Lemma 2.26. *There is an assignment*

$$\begin{array}{lllll} \Psi: & \mathrm{Aut}_{Hopf}(H) \times \mathrm{Z}^2_L(H^*) & \to \mathrm{Bigal}(H^*) & \to & \underline{\mathrm{Aut}}_{mon}(H\text{-mod}) \\ & (\phi, \sigma) & \mapsto {}_{\phi}({}_{\sigma}H^*) & \mapsto & {}_{\phi}({}_{\sigma}H^*) \otimes_H \bullet \ = (F_\phi, J^\sigma) \end{array}$$

where ${}_{\phi}({}_{\sigma}H^*)$ is the cleft lazy H^*-H^*-Bigalois object defined by H^* with multiplication deformed by σ, right H^*-coaction given by comultiplication and left H^*-coaction given by comultiplication post-composed with ϕ. We obtain a group homomorphism

$$\mathrm{Aut}_{Hopf}(H) \ltimes \mathrm{Z}^2_L(H^*) \to \mathrm{Aut}_{mon}(H\text{-mod})$$

where $\mathrm{Aut}_{Hopf}(H)$ acts on $\mathrm{Z}^2_L(H^*)$ as defined in Proposition 2.24. This map factors into a group homomorphism

$$\mathrm{Out}_{Hopf}(H) \ltimes \mathrm{H}^2_L(H^*) \to \mathrm{Aut}_{mon}(H\text{-mod}) \qquad (7)$$

that in general is neither injective nor surjective. In particular, the subgroup $\mathrm{Aut}_{Hopf}(H)$ is mapped to monoidal autoequivalences of the form (F_ϕ, J^{triv}) given by a pullback of the H-module action along an $\phi \in \mathrm{Aut}_{Hopf}(DG)$ and trivial monoidal structure (see Example 2.12). The subgroup $\mathrm{Z}^2_L(H^*)$ is mapped to monoidal autoequivalences of the form (id, J^σ), which act trivial on objects and morphisms but have a non-trivial monoidal structure given by J^σ (see Corollary 2.15).

Proof. We first check that (7) is indeed a homomorphism. The composition in the semi-direct product $(\phi, \sigma)(\phi', \sigma') = (\phi' \circ \phi, (\phi.\sigma')\sigma)$ is mapped to the functor $(F_{\phi'\circ\phi}, J^{(\phi.\sigma')\sigma})$. On the other hand the composition (F_ϕ, J^σ) with $(F_{\phi'}, J^{\sigma'})$ gives $F_{\phi'} \circ F_\phi = F_{\phi'\circ\phi}$ with the monoidal structure $F_{\phi'}(J^\sigma_{M,N}) \circ J^{\sigma'}_{M_\phi, N_\phi}(m \otimes n) = \sigma.(\phi.\sigma').(m \otimes n) = J^{\sigma*(\phi.\sigma')}_{M,N}(m \otimes n)$
Let us now show that the map factorizes as indicated: The kernel of $\mathrm{Aut}_{Hopf}(H) \ltimes \mathrm{Z}^2_L(H^*) \to \mathrm{Out}_{Hopf}(H) \ltimes H^2_L(H^*)$ is given by the set of all (ϕ, σ) with $\phi(h) = tht^{-1}$ for some grouplike element $t \in G(H)$ and $\sigma = \mathrm{d}\eta$ for $\eta \in \mathrm{Reg}^1_L(H^*)$. To see that the functor $\Psi(\phi, \sigma)$ is in this case trivial up to monoidal natural transformations we apply Lemma 2.3 for the element $x = \eta^{-1} \cdot t$ in H: We only have to check that indeed

$$\phi(h) = tht^{-1} = xhx^{-1}$$

since $\eta \in \mathrm{Reg}^1_L(H^*)$ is by definition (convolution-) central in $(H^*)^* = H$, and that

$$\sigma = \mathrm{d}(\eta^{-1}) = (\mathrm{d}\eta)^{-1} = \Delta(\eta)(\eta^{-1} \otimes \eta^{-1}) = \Delta(x)(x^{-1} \otimes x^{-1})$$

since t is grouplike. □

Note that according to Lemma 2.3 there are in general invertible $x \in H$ that are not group-likes but still give functors $\Psi(\phi, \sigma)$ that are trivial up to monoidal natural transformations and which are not zero in $\mathrm{Out}_{Hopf}(H) \ltimes H^2_L(H^*)$. In fact, we have an exact sequence

$$0 \to \mathrm{Int}(H)/\mathrm{Inn}(H) \to \mathrm{Out}(H) \ltimes \mathrm{H}^2_L(H^*) \to \mathrm{Aut}_{mon}(H\text{-mod})$$

3. Decomposition of $\mathrm{Aut}_{Hopf}(DG)$

Let us recall that our approach to determine $\mathrm{Aut}_{br}(H\text{-mod})$ for $H = DG$ can be summarized by the following roadmap

$$\mathrm{Aut}_{br}(H\text{-mod}) \subset \mathrm{Aut}_{mon}(H\text{-mod}) \cong \mathrm{Bigal}(H^*) \leftarrow \mathrm{Aut}_{Hopf}(H) \ltimes Z^2_L(H^*)$$

In this section we give a description of $\mathrm{Aut}_{Hopf}(DG)$. More precisely, we determine in Theorem 3.10 a decomposition of $\mathrm{Aut}_{Hopf}(DG)$ into double cosets similar to the Bruhat-decomposition of a Lie group. We will use this decomposition in order to construct appropriate subgroups of $\mathrm{Aut}_{br}(DG\text{-mod})$ and its decomposition.

Our results in this section rely on the approach [ABM12] Corollary 3.3 and on the works of Keilberg [Keil13]. He has determined a product decomposition (exact factorization) of $\mathrm{Aut}_{Hopf}(DG)$ whenever G does not contain abelian direct factors. In [KS14] Keilberg and Schauenburg determined $\mathrm{Aut}_{Hopf}(DG)$ in the general case, hence when G is allowed to have abelian direct factors using an approach that differs from ours and which we did not find helpful for our purposes.

Proposition 3.1. *(*[Keil13] *1.1, 1.2)*
The underlying set of $\mathrm{Aut}_{Hopf}(DG)$ *is in bijection to the set of matrices* $\begin{pmatrix} u & b \\ a & v \end{pmatrix}$ *where*

$$\begin{aligned} &u : k^G \to k^G && \text{is a Hopf algebra morphism} \\ &b : G \to \hat{G} && \text{is a group homomorphism} \\ &a : k^G \to kG && \text{is a Hopf algebra morphism} \\ &v : G \to G && \text{is a group homomorphism} \end{aligned}$$

fulfilling the following three additional conditions for all $f \in k^G$ *and* $g \in G$*:*

$$u(f_1) \times a(f_2) = u(f_2) \times a(f_1) \qquad v(g) \triangleright u(f) = u(g \triangleright f) \qquad v(g)a(f) = a(g \triangleright f)v(g) \tag{8}$$

This bijection maps such a matrix to an automorphism $\phi \in \mathrm{Aut}_{Hopf}(DG)$ *defined by:*

$$\phi(f \times g) = u(f_1)b(g) \times a(f_2)v(g) \qquad \forall f \in k^G \quad \forall g \in G$$

We equip this set of matrices with matrix multiplication where we use convolution to add and composition to multiply then the above bijection is a group homomorphism.

Example 3.2. *For* $G = A$ *a finite abelian group we get an isomorphism*

$$\mathrm{Aut}_{Hopf}(DA) \simeq \mathrm{Aut}(\hat{A} \times A)$$

where all maps u, b, a, v *are group homomorphisms.*

In the following we will often express the maps u, a, b in terms of a canonical basis in the following way

$$u(e_g) = \sum_{h\in G} u(e_g)(h)e_h \qquad b(g) = \sum_{h\in G} b(g)(h)e_h$$
$$a(e_g) = \sum_{h\in G} e_h(a(e_g))h = \sum_{h\in G} a_g^h h$$

We denote by e.g. $u^* : kG \to kG$ the dual map of $u : k^G \to k^G$, hence $e_h(u^*(g)) = u(e_h)(g)$ and similarly for a, b, v. An automorphism $\phi \in \mathrm{Aut}_{Hopf}(DG)$ can then be given in the following way:

$$\phi(e_g \times h) = \sum_{x,y\in G} b(h)(x)\, a_{gu^*(x)^{-1}}^{v(h)^{-1}y}\, (e_x \times y)$$

In the matrix notation $\left(\begin{smallmatrix} u & b \\ a & v \end{smallmatrix}\right)$ we use the following conventions:

- $u \equiv 0$ denotes the map $k^G \to k^G; e_g \mapsto (h \mapsto \delta_{g,1_G})$ and $u \equiv 1$ the identity map on k^G.
- $b \equiv 0$ denotes the map $kG \to k^G; g \mapsto 1_{k^G}$
- $a \equiv 0$ denotes the map $k^G \to kG; e_g \mapsto 1_G\delta_{g,1_G}$
- $v \equiv 0$ denotes the map $G \to G; g \mapsto 1_G$ and $v \equiv 1$ the identity map on G.

Lemma 3.3. *(*[Keil13] *2.1, 2.4, 2.10, 2.12)*
Let $\left(\begin{smallmatrix} u & b \\ a & v \end{smallmatrix}\right) \in \mathrm{Aut}_{Hopf}(DG\text{-mod})$. *Then the following holds:*

- *The Hopf morphism a is uniquely determined by a group isomorphism $\hat{A} \cong B$ where A, B are abelian subgroup of G such that* $\mathrm{im}(a^*) = kA$ *and* $\mathrm{im}(a) = kB$. *The map a is then given by composing $k^G \hookrightarrow k^A \cong k\hat{A} \cong kB \hookrightarrow kG$.*
- *v maps $C_G(A)$ to $C_G(B)$. In particular $C_G(v(A)) \subset C_G(B)$ with equality if v is an isomorphism.*
- $A, B \leq Z(G)$, $G = Z(G)\mathrm{im}(v)$, $\mathrm{im}(a)\mathrm{im}(v) = \mathrm{im}(v)\mathrm{im}(a)$.
- *$u^* \circ v$ is a normal group homomorphism.*
- *The kernels of v and u^* are contained in an abelian direct factor of G.*

We now introduce several important subgroups of $\mathrm{Aut}_{Hopf}(DG)$. The first subgroup illustrates how a group automorphism of G induces an Hopf automorphism of DG.

Proposition 3.4. *(*[Keil13] *4.3)*
There is a natural subgroup of $\mathrm{Aut}_{Hopf}(DG)$ *given by:*

$$V := \left\{ \begin{pmatrix} (v^{-1})^* & 0 \\ 0 & v \end{pmatrix} \mid v \in \mathrm{Aut}(G) \right\}$$

An element in V corresponds to the following automorphism of DG:

$$e_g \times h \mapsto e_{v(g)} \times v(h)$$

We obviously have an isomorphism of groups: $V \simeq \mathrm{Aut}(G)$.

Then we have two group of 'strict upper triangular matrices' and 'strict lower triangular matrices' which come from the abelianization $G_{ab} = G/[G, G]$ and the center $Z(G)$ respectively.

Proposition 3.5. *(*[Keil13] *4.4)*
There is a natural abelian subgroup of $\mathrm{Aut}_{Hopf}(DG)$ *given by:*

$$B := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathrm{Hom}(G_{ab}, \widehat{G}_{ab}) \right\}$$

An element in B *corresponds to the following isomorphism of* DG*:*

$$e_g \times h \mapsto b(h)(g)\; e_g \times h$$

We have an isomorphism of groups $B \cong \mathrm{Hom}(G_{ab}, \widehat{G}_{ab}) \cong \widehat{G}_{ab} \otimes \widehat{G}_{ab}$ *where the* Hom*-space is equipped with the convolution product and the tensor product is over* $\mathbb{Z}$*.*

Proposition 3.6. *(*[Keil13] *4.1, 4.2)*
There is natural abelian subgroup of $\mathrm{Aut}_{Hopf}(DG)$ *given by:*

$$E := \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in \mathrm{Hom}(\widehat{Z(G)}, Z(G)) \right\}$$

An element in E *corresponds to the following isomorphism of* DG*:*

$$e_g \times h \mapsto \sum_{g_1 g_2 = g} e_{g_1} \times a(e_{g_2})h$$

We have an isomorphism $E \cong \mathrm{Hom}(\widehat{Z(G)}, Z(G)) \cong Z(G) \otimes Z(G)$ *where the* Hom*-space is equipped with the convolution product and the tensor product is over* $\mathbb{Z}$*.*

Proposition 3.7. *(*[Keil13] *4.5)*
There is a subgroup of $\mathrm{Aut}_{Hopf}(DG)$ *given by:*

$$V_c := \left\{ \begin{pmatrix} (v^{-1})^* & 0 \\ 0 & 1 \end{pmatrix} \mid v \in \mathrm{Aut}_c(G) \right\}$$

where $\mathrm{Aut}_c(G) := \{v \in \mathrm{Aut}(G) | v(g)g^{-1} \in Z(G)\}$*. An element in* V_c *corresponds to the following map*

$$e_g \times h \mapsto e_{v(g)} \times h$$

We can now state the main result of [Keil13] which determines $\mathrm{Aut}_{Hopf}(DG)$ in the case that G is purely non-abelian (i.e. has no direct abelian factors):

Theorem 3.8. *(*[Keil13] *Theorem 5.7)*
Let G *be a purely non-abelian finite group. There is an exact factorization into subgroups*

$$\begin{aligned} \mathrm{Aut}_{Hopf}(DG) &\simeq E((V_c \rtimes V) \ltimes B) \\ &\simeq ((V_c \rtimes V) \ltimes B)E \end{aligned}$$

The main step in the proof is using the fact that $\ker(u^*)$ and $\ker(v)$ are contained in direct factors of G according to Lemma 3.3. Clearly, if G is purely non-abelian then u, v have to be invertible. This leads directly to the above decomposition. The exact factorization fails to be true in the presence of direct abelian factors. In this case neither u nor v have to be invertible, but their kernels are still contained in an direct abelian factor. From this point of view it seems natural to introduce an additional class of automorphisms of DG that act on direct abelian factors of G. In particular, this will be maps that exchange an abelian factor $kC \subset k^G \rtimes kG$ with its dual $k^C \subset k^G \rtimes kG$.

Proposition 3.9. *Let $R_t(G)$ be the set of all tuples (H, C, δ, ν), where C is an abelian subgroup of G and H is a subgroup of G, such that $G = H \times C$, $\delta : kC \xrightarrow{\sim} k^C$ a Hopf isomorphism and $\nu : C \to C$ a nilpotent homomorphism.*

(i) *For (H, C, δ, ν) we define a* twisted reflection $r_{(H,C,\delta,\nu)} : DG \to DG$ *of C by:*

$$(f_H, f_C) \times (h, c) \mapsto (f_H, \delta(c)) \times (h, \delta^{-1}(f_C)\nu(c))$$

We write p_H, p_C for the two projections to H, C and ι_H, ι_C for the two embeddings. Then the matrix corresponding to a twisted reflection is given by

$$\begin{pmatrix} (\iota_H \circ p_H)^* & p_C^* \circ \delta \circ p_C \\ \iota_C \circ \delta^{-1} \circ \iota_C^* & p_H + \nu \end{pmatrix}$$

All twisted reflections are Hopf automorphisms.

(ii) *Denote by R the subset of $R_t(G)$ elements with $\nu = 1_C$. We call the corresponding Hopf automorphisms* reflections *of C. For two triples (H, C, δ) and (H', C', δ') in R with $C \cong C'$ the elements $r_{(H,C,\delta)}, r_{(H',C',\delta')}$ are conjugate in $\mathrm{Aut}_{Hopf}(DG)$ (by an element in $V \cong \mathrm{Aut}(G)$).*

In the following we will sometimes restrict ourselves to a set of representatives $r_{[C]}$ for each isomorphism type $[C]$ of C. In order to simplify the notation we abbreviate the matrix corresponding to a (twisted) reflection (H, C, δ, ν) by $\begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H+\nu \end{pmatrix}$.

Proof. For convenience we denote $g_H := p_H(g)$, $g_C := p_C(g)$ etc.
(i) In order to prove that a reflection is indeed an automorphism of DG we have to check the three equations (8). The first equation holds because for all $x, g \in G$ we have:

$$(\delta^{-1} \circ \iota_C^*)(e_{p_H(x)^{-1}g}) = \delta^{-1}(e_{x_H^{-1}g} \circ \iota_C) = \delta_{x_H,1}\delta_{g_H,1}\delta^{-1}(e_{g_C}) = (\delta^{-1} \circ \iota_C^*)(e_{gp_H(x)^{-1}})$$

For the second equation of (8) we have to check that with $u = \hat{p}_H$ and $v = p_H + \nu$ we get $u(e_y)(v(g)^{-1}xv(g)) = u(e_{gyg^{-1}})(x)$ for all $x, y, g \in G$. This is indeed true:

$$e_y(u^*(v(g)xv(g))) = e_y(p_H(g^{-1}xg)) = \delta_{g_C y_C g_C^{-1},1} e_{g_H y_H g_H^{-1}}(x_H) = e_{gyg^{-1}}(p_H(x))$$

The last equation also holds since $g_H\nu(g_C)\delta^{-1}(e_x \circ \iota_C) = \delta^{-1}(e_{gxg^{-1}} \circ \iota_C)g_H\nu(g_C)$
(ii) Let H, C, H', C' be subgroups of G such that $H \times C = G = H' \times C'$ and $v_C : C \cong C'$ (such that $v_C^*\delta = \delta' v_C$). Then we claim that there is an automorphism $v \in \mathrm{Aut}(G)$ such that $v(H) = H'$ and $v(C) = C'$ (such that $v|_C = v_C$), which immediately implies that v conjugates $r_{H,C,\delta}$ to $r_{H',C',\delta'}$.

Hence it suffices to show that given an $\varphi : H \times C \cong H' \times C'$ and an isomorphism $v_C : C \to C'$ then we also have an isomorphism $H \cong H'$ (note that for infinite groups this statement is *wrong*, as for example we can have $C = C' = H \times H' \times H \times H' \times \cdots$). Let us write φ as a matrix:

$$\varphi = \begin{pmatrix} \varphi_{H,H'} & \varphi_{C,H'} \\ \varphi_{H,C'} & \varphi_{C,C'} \end{pmatrix}$$

- If $\varphi_{C,C'}$ is invertible we can find an isomorphism $\varphi' : H \times C \cong H' \times C'$ that is diagonal, hence with $\varphi'_{H,C'} = 0 = \varphi'_{C,H'}$ by

$$\varphi' := \begin{pmatrix} 1 & -\varphi_{C,H'}\varphi_{C,C'}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_{H,H'} & \varphi_{C,H'} \\ \varphi_{H,C'} & \varphi_{C,C'} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varphi_{C,C'}^{-1}\varphi_{H,C'} & 1 \end{pmatrix}$$

  which implies that $\varphi'_{H,H'} : H \cong H'$ is an isomorphism.

- If $\varphi_{C,C'}$ is not invertible we assume w.l.o.g (induction otherwise) that $C = \mathbb{Z}_{p^n}$. In this case, $\varphi_{C,H'}$ has to be injective (and $\varphi_{H,C}$ has to be surjective) in order for φ to be bijective. Again by column-transformation we thus obtain an isomorphism φ' such that $\varphi'_{C,C'} = 0$. We thus obtain a direct factor $\varphi'_{C,H'}(C)$ in H' and a direct factor $\varphi'^{-1}_{H,C'}(C)$ of H. We will now write φ' as a 3×3-matrix

$$\varphi' = \begin{pmatrix} \phi'_{\tilde{H}',\tilde{H}'} & * & 0 \\ * & * & \varphi_{C,H'} \\ 0 & \varphi_{H,C'} & 0 \end{pmatrix} : \tilde{H} \times \varphi'^{-1}_{H,C'}(C) \times C \to \tilde{H}' \times \varphi'_{C,H'}(C) \times C$$

Again, we can eliminate the $*$ hence also get an isomorphism $H \cong H'$ as above.

□

Theorem 3.10.

(i) *Let G be a finite group, then $\mathrm{Aut}_{Hopf}(DG)$ is generated by the subgroups V, V_c, B, E and the set of reflections R.*

(ii) *For every $\phi \in \mathrm{Aut}_{Hopf}(DG)$ there is a twisted reflection $r = r_{(H,C,\delta,\nu)} \in R_t(G)$ such that ϕ is an element in the double coset*

$$[(V_c \rtimes V) \ltimes B] \cdot r \cdot [(V_c \rtimes V) \ltimes E]$$

(iii) *Two double cosets corresponding to reflections $(C, H, \delta), (C', H', \delta') \in R$ are equal if and only if $C \simeq C'$.*

(iv) *For every $\phi \in \mathrm{Aut}_{Hopf}(DG)$ there is a reflection $r = r_{(C,H,\delta)} \in R$ such that ϕ is an element in*

$$r \cdot [B((V_c \rtimes V) \ltimes E)]$$

(v) *For every $\phi \in \mathrm{Aut}_{Hopf}(DG)$ there is a reflection $r = r_{(C,H,\delta)} \in R$ such that ϕ is an element in*

$$[((V_c \rtimes V) \ltimes B)E] \cdot r$$

Before we turn to the proof we illustrate the statement of Theorem 3.10 on some examples:

Example 3.11. *In the case G is purely non-abelian, there are no (non-trivial) reflections. We get the result of Theorem 3.8.*

Example 3.12. *For $G = (\mathbb{F}_p^n, +)$ a finite vector space, we have directly*

$$\mathrm{Aut}_{Hopf}(DG) = \mathrm{Aut}(\mathbb{F}_p^2 \times \mathbb{F}_p^2) \cong \mathrm{GL}_{2n}(\mathbb{F}_p)$$

On the other hand the previously defined subgroups are in this case:

- $V \cong \mathrm{Aut}(G) = \mathrm{GL}_n(\mathbb{F}_p)$ *and* $V_c \ltimes V \cong \mathrm{GL}_n(\mathbb{F}_p) \times \mathrm{GL}_n(\mathbb{F}_p)$
- $B \cong \widehat{G}_{ab} \otimes \widehat{G}_{ab} = \mathbb{F}_p^{n\times n}$ *as additive group.*
- $E \cong Z(G) \otimes Z(G) = \mathbb{F}_p^{n\times n}$ *as additive group.*

The set R is very large: For each dimension $d \in \{0, \ldots, n\}$ there is a unique isomorphism type $C \cong \mathbb{F}_p^d$. The possible subgroups of this type $C \subset G$ are the Grassmannian $Gr(n, d, G)$, the possible $\delta : C \to \hat{C}$ are parametrized by $\mathrm{GL}_d(\mathbb{F}_p)$ and in this fashion R can be enumerated.

On the other hand, we have only $n+1$ representatives $r_{[C]}$ for each dimension d, given for example by permutation matrices

$$\left(\begin{array}{cc|cc} 0 & 0 & \mathbb{1}_d & 0 \\ 0 & \mathbb{1}_{n-d} & 0 & 0 \\ \hline \mathbb{1}_d & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{n-d} \end{array}\right)$$

One checks this indeed gives a decomposition of $\mathrm{GL}_{2n}(\mathbb{F}_p)$ into V_cVB-V_cVE-cosets, e.g.

$$\begin{array}{rcccccc} \mathrm{GL}_4(\mathbb{F}_p) & = & (V_cVB\cdot r_{[1]}\cdot V_cVE) & \cup & (V_cVB\cdot r_{[\mathbb{F}_p]}\cdot V_cVE) & \cup & (V_cVB\cdot r_{[\mathbb{F}_p^2]}\cdot V_cVE) \\ |\mathrm{GL}_4(\mathbb{F}_p)| & = & p^8|\mathrm{GL}_2(\mathbb{F}_p)|^2 & + & \frac{p^3|\mathrm{GL}_2(\mathbb{F}_p)|^4}{(p-1)^4} & + & p^4|\mathrm{GL}_2(\mathbb{F}_p)|^2 \\ & = & p^8(p^2-1)^2(p^2-p)^2 & + & \frac{p^3(p^2-1)^4(p^2-p)^4}{(p-1)^4} & + & p^4(p^2-1)^2(p^2-p)^2 \\ & = & (p^4-1)(p^4-p)(p^4-p^2)(p^4-p^3) & & & & \end{array}$$

It corresponds to a decomposition of the Lie algebra A_{2n-1} according to the $A_{n-1}\times A_{n-1}$ parabolic subsystem. Especially on the level of Weyl groups we have a decomposition as double cosets of the parabolic Weyl group

$$\begin{aligned} \mathbb{S}_{2n} &= (\mathbb{S}_n\times\mathbb{S}_n)1(\mathbb{S}_n\times\mathbb{S}_n)\ \cup\ (\mathbb{S}_n\times\mathbb{S}_n)(1,1+n)(\mathbb{S}_n\times\mathbb{S}_n)\ \cup \\ &\quad \cdots\ \cup\ (\mathbb{S}_n\times\mathbb{S}_n)(1,1+n)(2,2+n)\cdots(n,2n)(\mathbb{S}_n\times\mathbb{S}_n) \\ \textit{e.g.}\quad |\mathbb{S}_4| &= 4+16+4 \end{aligned}$$

In this case, the full Weyl group $\mathbb{S}_{2n}$ of $\mathrm{GL}_{2n}(\mathbb{F}_p)$ is the set of all reflections (as defined above) that preserve a given decomposition $G=\mathbb{F}_p\times\cdots\times\mathbb{F}_p$.

Proof of Theorem 3.10.
(i) follows immediately from (iv).
(ii) From above we know that $\ker(v)$ is contained in an abelian direct factor G. The other factor can be abelian or not, but we can decompose it into a purely non-abelian factor times an abelian factor. Hence we arrive at the decomposition $G=H\times C$ where H is purely non-abelian and where $\ker(v)$ is contained in an direct abelian factor of C. Since C is a finite abelian group there is an $n\in\mathbb{N}$ and an isomorphism

$$C\cong C_1\times ...\times C_n$$

where C_i are cyclic groups of order $p_i^{k_i}$ for some prime numbers p_i and $k_i\in\mathbb{N}$ with $p_i^{k_i}\leq p_j^{k_j}$ for $i\leq j$.
A general Hopf automorphism $\phi\in\mathrm{Aut}_{Hopf}(DG)$ can then be written in matrix form with respect to the decomposition $G=H\times C_1\times C_2\times\cdots\times C_n$ as

$$\phi=\left(\begin{array}{ccc|ccc} u_{H,H} & \cdots & u_{C_n,H} & b_{H,H} & \cdots & b_{C_n,H} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{H,C_n} & \cdots & u_{C_n,C_n} & b_{H,C_n} & \cdots & b_{C_n,C_n} \\ \hline a_{H,H} & \cdots & a_{C_n,H} & v_{H,H} & \cdots & v_{C_n,H} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{H,C_n} & \cdots & a_{C_n,C_n} & v_{H,C_n} & \cdots & v_{C_n,C_n} \end{array}\right) \tag{9}$$

Let c_n be the generator of C_n. Since ϕ is an automorphism we know that the order of $\phi(c_n)$ is also $p_n^{k_n}$. This implies that one of the $b_{C_n,H}(c_n),\cdots,b_{C_n,C_n}(c_n),v_{C_n,H}(c_n),\cdots,v_{C_n,C_n}(c_n)$ has order $p_n^{k_n}$. Then we can have three possible cases:

Case (1): One of the v_{C_n,C_n} or b_{C_n,C_n} is injective.
Case (2): One of the v_{C_n,C_m} or b_{C_n,C_m} is injective and $m<n$.

Case (3): One of the $v_{C_n,H}$ or $b_{C_n,H}$ is injective.

Case (1): If v_{C_n,C_n} is injective, then it has to be bijective (since C_n is finite). We can construct an element in B and an element in V_c :

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & v_{C_n,H}v^{-1}_{C_n,C_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & v^{-1}_{C_n,C_n} \end{array}\right) \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & b_{C_n,H}v^{-1}_{C_n,C_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & b_{C_n,C_n}v^{-1}_{C_n,C_n} \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right) \tag{10}$$

Multiplying (9) from the *left* by (10) we eliminate the $2n$-column. Similarly, multiplying with elements of E and V_c from the *right* we can eliminate the $2n$-row.

$$\overset{B,V_c}{\rightsquigarrow} \left(\begin{array}{ccc|cccc} * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ \hline * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ * & \cdots & * & * & \cdots & * & 1 \end{array}\right) \overset{E,V_c}{\rightsquigarrow} \left(\begin{array}{ccc|cccc} * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ \hline * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{array}\right) \tag{11}$$

In the case that b_{C_n,C_n} is injective, hence bijective, we construct an element in V_c:

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & b_{C_n,H}b^{-1}_{C_n,C_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & b_{C_n,C_n}b^{-1}_{C_n,C_n} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right) \tag{12}$$

Since the upper left quadrant of (12) is the dual of an automorphism that takes values in abelian factors of G it is indeed an element of V_c.

Multiplying (9) from the *left* by (12) we eliminate the upper half of the $2n$-column and by multiplying with elements of E and V_c from the *right* we eliminate the n-row:

$$\overset{B,V_c}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & b_{C_n,C_n} \\ \hline * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \end{array}\right) \overset{E,V_c}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & b_{C_n,C_n} \\ \hline * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \end{array}\right) \tag{13}$$

Case (2): If $v_{C_n,C_m}(c_n)$ is injective it is also bijective, since $|C_m| \leq |C_n|$ by construction. Then we must have $p_n = p_m$ and $k_n = k_m$. Then let $w \in \mathrm{Aut}_c(G)$ be an automorphism such that $w(C_n) = C_m$, $w(C_m) = C_n$ and identity elsewhere. Multiplying with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & w \end{smallmatrix}\right) \in V_c$ from the left returns us to Case (1) when v_{C_n,C_n} is invertible. Similarly, if b_{C_n,C_m} is injective, it has to be bijective because of the same order argument as above.

Exchanging the C_n with C_m by an $\left(\begin{smallmatrix} w^* & 0 \\ 0 & 1 \end{smallmatrix}\right) \in V_c$ we return to the Case (1) when b_{C_n,C_n} is invertible.

Case (3): If $v_{C_n,H}$ is injective, we can also assume that $v_{C_n,H}$ is the only injective map in the last column of ϕ, else we choose Case (1) or Case (2). Since the whole matrix ϕ is invertible, there exists an inverse matrix $\phi^{-1} = \left(\begin{smallmatrix} u' & b' \\ a' & v' \end{smallmatrix}\right)$ and then the multiplication of the last right column of ϕ with the last upper row of ϕ^{-1} has to be 1. Therefore there has to be a homomorphism $v'_{H,C_n} : H \to C_n$ such that $w := v'_{H,C_n} \circ v_{C_n,H} : C_n \to C_n$ is injective, and therefore bijective. We have an exact sequence

$$0 \to \ker(v'_{H,C_n}) \to H \overset{v'_{H,C_n}}{\to} C_n \to 0$$

which splits on the right via $v_{C_n,H} \circ w^{-1}$. Restricting to group-like objects $G(DG) = \hat{G} \times G$ the row $a_{H,H} a_{C_1,H} ... v_{H,H} ... v_{C_n,H}$ gives a surjection from $\hat{G} \times G$ to H. It maps central elements to central elements because it is surjective and hence the restriction to C_n, namely $v_{C_n,H}$, has central image. This implies that C_n is a direct abelian factor of H which is not possible, because H is purely non-abelian per construction.

Hence we end up with either the form (13) or the form (14). Now we inductively move on to C_{n-1}, C_{n-2}, ..., C_1 where we permute parts with the non-invertible $v's$ to the right lower corner by multiplying with elements of V_c. Since $\ker(v)$ has trivial intersection with H per construction, the map $v_{H,H}$ is invertible. As in the Case (1) we can use row and column manipulation to get zeros below and above $v_{H,H}$ as well as left and right. Note that the elements we are constructing are always in V_c because either have abelian image per definition or are restrictions on abelian direct factors of surjections. Only $v_{H,H}, u_{H,H}$ do in general not induce V_c elements like that. But corresponding to the automorphism $v_{H,H}^{-1}$ there is a matrix in V. Multiplying with this matrix changes the remaining $u_{H,H}$ to $v^*_{H,H} \circ u_{H,H}$ and the $v_{H,H}$ to id_H.

Now we now consider a generator χ_n of $\hat{C}_n$ and conclude from the fact that ϕ is an automorphism the analogous case differentiation from above but now for entries in the remaining u and a. With the same arguments as above we move through the columns corresponding to $\hat{C}_{n-1}, ..., \hat{C}_1, \hat{H}$ and end up with a matrix of the following form:

$$\left(\begin{array}{ccc|ccc} u_{H,H} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_m \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & A_m & 0 & 0 & V_m \end{array}\right) \tag{14}$$

where $k + m + 1 = n$, B_m, A_m are diagonal $m \times m$-matrices with isomorphisms on the diagonal, I_k an $k \times k$-identity matrix and V_m a $m \times m$-matrix with non-invertible homomorphisms as entries. Further, since H is purely non-abelian and by Lemma 3.3 $\ker(u)$ is contained in an abelian direct factor we deduce that $u_{H,H}$ is an isomorphism. Also by Lemma 3.3 we know that the composition

$$\begin{pmatrix} u^*_{H,H} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & V_m \end{pmatrix} = \begin{pmatrix} u^*_{H,H} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has to be a normal homomorphism, hence $u_{H,H}$ has to be a central automorphism. Therefore we get (14) with $u_{H,H} = 1$ by multiplying with the inverse in V_c. Our final step is normalizing the A_m by multiplying with an element in V_c corresponding to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & B_m^{-1}A_m^{-1} \end{pmatrix}$$

Hence we end up with a twisted reflection:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_m \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & B_m^{-1} & 0 & 0 & V_m \end{array}\right) \tag{15}$$

(iii) If $C \cong C'$ it follows from Proposition 3.9 (ii) that $r_{H,C,\delta}$ and $r_{H',C',\delta'}$ are conjugate to each other. This implies that their double cosets have non trivial intersection and hence are equal.
Assume that the two double cosets corresponding to $r'_{(H',C',\delta')}$ and $r_{(H,C,\delta)}$ are equal. Then there are $w, w', v, v' \in \mathrm{Aut}(G)$, $a \in \mathrm{Hom}(\widehat{Z(G)}, Z(G))$, $b' \in \mathrm{Hom}(G, \widehat{G})$ such that

$$\begin{pmatrix} w^* & a \\ 0 & v \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta_C^{-1} & p_H \end{pmatrix} = \begin{pmatrix} \hat{p}_{H'} & \delta' \\ \delta'^{-1} & p_{H'} \end{pmatrix} \begin{pmatrix} w'^* & 0 \\ b' & v' \end{pmatrix}$$

This implies in particular $v \circ p_H = p_{H'} \circ v'$. Then $C = \ker(p_H) = \ker(v^{-1} \circ p_{H'} \circ v') = v'^{-1}(\ker(p_{H'})) = v'^{-1}(C')$. Hence v' restricted to C gives the isomorphism $C \simeq C'$.

(iv) Let ϕ be a general element in $\mathrm{Aut}_{Hopf}(DG\text{-mod})$ as in (11). We use the same arguments as in (ii) to get to the case differentiation for every column. We can produce zeros in each row by multiplying ϕ with E and V_c form the right, except when we have invertible entries in a and u. In this case multiplying with E, V_c from the right can only produce zeros in u and a respectively. Hence we end up with:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & I_k & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & B_m \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & B_m^{-1} & * & * & * \end{array}\right) \tag{16}$$

where B_m is a diagonal $m \times m$-matrix with isomorphisms on the diagonal, I_k an $k \times k$-identity matrix and $k + m = n + 1$. Multiplying again from the right but now with elements of B (which was not allowed in (ii)) and elements of V_c we eliminate the $*$ which results in a (non-twisted) reflection.

(v) This is similar to above. We start with (11) an move from column to column as in (ii) and (iv). We identify the invertible entries and can clean up each *column* by multiplying with elements of B and V_c from the *left*. Finally as in *(ii)* we get a non-invertible twist V_m below B_m but contrary to *(ii)* we can multiply with E from the *left* and eliminate the V_m. Therefore we again end up with a (non-twisted) reflection. This concludes the proof of Theorem 3.10. □

4. Subgroups of $\mathrm{Aut}_{br}(DG\text{-mod})$

In the previous section we have calculated $\mathrm{Aut}_{Hopf}(DG)$ in terms of a V_cVB-V_cVE-double cosets decomposition with representatives R, where V, V_c, E, B are natural subgroups of $\mathrm{Aut}_{Hopf}(DG)$. In the present section, we will construct certain subgroups of $\mathrm{Aut}_{Hopf}(DG) \ltimes Z^2_L(DG^*)$ that will be denoted by $\underline{\mathcal{V}}_L, \underline{\mathcal{E}}_L, \underline{\mathcal{B}}_L$ together with a set $\underline{\mathcal{R}}_L$. These will play an essential role in the double coset decomposition of $\mathrm{Aut}_{br}(DG\text{-mod})$. The following observations and properties are essential: The natural projection $\mathrm{Aut}_{Hopf}(DG) \ltimes Z^2_L(DG^*) \to \mathrm{Aut}_{Hopf}(DG)$ maps them to V, E, B and R; The map

$$\Psi : \mathrm{Aut}_{Hopf}(DG) \ltimes \mathrm{Z}^2_L(DG^*) \to \underline{\mathrm{Aut}}_{mon}(DG\text{-mod})$$

introduced in Lemma 2.26 maps them to *braided* monoidal autoequivalences; and the main observation is that each factor in the decomposition:

$$\mathrm{Z}^2_L(k^G) \times \mathrm{Z}^2(G) \times \mathrm{P}(kG, k^G) \xrightarrow{\sim} \mathrm{Z}^2_L(DG^*)$$

from Lemma 4.2 should correspond to one of the following: E, B, R. This leads us to an ansatz of how to construct the $\underline{\mathcal{V}}_L, \underline{\mathcal{E}}_L, \underline{\mathcal{B}}_L$. In order to control the coset decomposition we further mod out the obvious monoidally trivial elements, hence those that come from inner Hopf automorphisms and exact 2-cocycles and then explicitly calculate the images of the above projections.

4.1. **General considerations.**

Let H be a Hopf algebra and $(\phi, \sigma) \in \mathrm{Aut}_{Hopf}(H) \ltimes \mathrm{Z}^2_L(H^*)$, then we obtained in Lemma 2.26 an associated monoidal autoequivalence (F_ϕ, J^σ) of H-mod. We now apply this explicitly to the case $H = DG$. For convenience we use the following convention: Let M be a DG-module with action ρ and $\phi = \left(\begin{smallmatrix} u & b \\ a & v \end{smallmatrix}\right) \in \mathrm{Aut}_{Hopf}(DG)$ then F_ϕ assigns M to the DG-module ${}_\phi M$ with the action $\rho \circ \phi^* \otimes \mathrm{id}_M$ where the Hopf automorphism

$$\phi^* = \begin{pmatrix} v^* & b^* \\ a^* & u^* \end{pmatrix} \tag{17}$$

has as entries the duals of the Hopf morphisms u, v, b, a.

Lemma 4.1. *Write any $\phi \in \mathrm{Aut}_{Hopf}(DG)$ in the form $\begin{pmatrix} u & b \\ a & v \end{pmatrix}$ as in Proposition 3.1, then F_ϕ has the following explicit form on simple DG-modules:*

$$F_\phi(\mathcal{O}^\rho_g) = \mathcal{O}^{(\rho\circ u^*)b(g)}_{a(\rho')v(g)}$$

where we denote by $\rho' : Z(G) \to k^*$ the one-dimensional representation such that the any central element $z \in Z(G)$ act in ρ by multiplication with the scalar $\rho'(z)$. In particular $\rho|_{Z(G)} = \dim(\rho) \cdot \rho'$.

Proof. We first check that the G-coaction resp. k^G-action on $F_\phi(\mathcal{O}_g^\rho)$ is as asserted:

$$\begin{aligned}
\phi^*(e_x \times 1).(1 \otimes w) &= \sum_{kl=x} (v^*(e_k) \times a^*(e_l)).(1 \otimes w) \\
&= \sum_{kl=x,k',l'} v^*(e_k)(k')e_{l'}(a^*(e_l))(e_{k'} \times l').(1 \otimes w) \\
&= \sum_{kl=x,k',l'} e_k(v(k'))e_l(a(e_{l'}))(e_{k'} \times l').(1 \otimes w) \\
&= \sum_{kl=x,l'} e_k(v(l'gl'^{-1}))e_l(a(e_{l'}))l' \otimes w \\
&= \sum_{kl=x,l' \in Z(G)} e_k(v(g))e_l(a(e_{l'}))\hat{\rho}(l')1 \otimes w \\
&= \sum_{v(g)l=x} \delta_{l,a(\hat{\rho})} 1 \otimes w \\
&= \delta_{x,v(g)a(\hat{\rho})} 1 \otimes w
\end{aligned}$$

For $y \in \mathrm{Cent}(a(\rho')v(g))$ we check that the action is as asserted. We first collect the following facts:

- By definition $a(\rho') \in Z(G)$, so $y \in \mathrm{Cent}(v(g))$.
- By Lemma 3.3 we have that $u^* \circ v$ is a normal group homomorphism, hence

$$(u^* \circ v)(u^*(y)gu^*(y)^{-1}) = u^*(y) \cdot u^*(v(g)) \cdot u^*(y)^{-1} = u^*(yv(g)y^{-1}) = u^*(v(g))$$

  Therefore $[u^*(y), g] = u^*(y)gu^*(y)^{-1}g^{-1} \in \ker(u^* \circ v)$.
- Moreover by Lemma 3.3 we have that $\ker(u^*)$ is in a direct abelian factor C of G, hence the commutator $v([u^*(y), g]) = [v(u^*(y)), v(g)] \in \ker(u^*)$ is already equal to 1 (for we may consider the projection p_C of the commutator being equal to 1 but the projection is the identity on C). Thus $[u^*(y), g] \in \ker(v)$. By the same Lemma, $\ker(v)$ is in a direct abelian factor of G hence again the commutator $[u^*(y), g]$ is already equal 1. This finally shows $u^*(y) \in \mathrm{Cent}(g)$.

Now we calculate using $u^*(y) \in \mathrm{Cent}(g)$ and $b(hgh^{-1}) = b(g)$:

$$\begin{aligned}
\phi^*(1 \times y).(1 \otimes v) &= (b^*(y) \times u^*(y)).(1 \otimes v) \\
&= \sum_k b^*(y)(k)(e_k \times u^*(y)).(1 \otimes v) \\
&= \sum_k b(k)(y)e_k(u^*(y)gu^*(y)^{-1})(u^*(y) \otimes v) \\
&= b(u^*(y)gu^*(y)^{-1})(y)(u^*(y) \otimes v) \\
&= b(g)(y)(1 \otimes \rho(u^*(y))(v)) \\
&= 1 \otimes [(\rho \circ u^*)b(g)](y)v
\end{aligned}$$

Since these two actions characterize the simple DG-module we have verified the claim.
□

Definition 4.2. *(*[Bich04]*)*
Let us define the group of central bialgebra pairings $\mathrm{P}(kG, k^G)$ *to be convolution invertible* k*-linear maps* $\lambda : kG \otimes k^G \to k$ *such that for all* $x, t \in G,\ f \in k^G$*:*

$$\lambda(gh, f) = \lambda(g, f_1)\lambda(h, f_2) \qquad \lambda(g, f * f') = \lambda(g, f)\lambda(g, f') \qquad \lambda(g, e_x) = \lambda(g, e_{txt^{-1}})$$
$$\lambda(1, f) = \epsilon_{k^G}(f) \qquad \lambda(g, 1) = \epsilon_{kG}(g)$$

with convolution as its group structure. Then there is a natural group homomorphism

$$\mathrm{P}(kG, k^G) \to \mathrm{Z}^2_L(DG^*); \lambda \mapsto \sigma_\lambda$$

where

$$\sigma_\lambda(x \times e_y, z \times e_w) = \lambda(z, e_y)\epsilon_{kG}(x)\epsilon_{k^G}(e_w)$$

This homorphism induces a well-defined homomorphism $\mathrm{P}(kG, k^G) \to \mathrm{H}^2_L(DG^*)$.

Theorem 4.3. *(*[Bich04] *Corollary 4.11)*
There is an isomorphism of groups:

$$\mathrm{H}^2_L(k^G) \times \mathrm{H}^2(G) \times \mathrm{P}(kG, k^G) \xrightarrow{\sim} \mathrm{H}^2_L(DG^*)$$
$$(\alpha, \beta, \lambda) \mapsto (\alpha \otimes \beta) * \sigma_\lambda$$

In particular, every $\sigma \in Z^2_L(DG^*)$ has the following form:

$$\sigma(x \times e_y, z \times e_w) = \sum_{y_1 y_2 = y} \alpha(e_{y_1}, e_w)\beta(x, z)\lambda(w^{-1}zw, e_{y_2})$$

We now want to know which elements in $\mathrm{Aut}_{Hopf}(DG) \ltimes Z^2_L(DG^*)$ correspond to *braided* monoidal autoequivalences. Note that we have to be a bit careful whether we are asking for a specific functor to be braided or for an equivalence class in $\mathrm{Aut}_{mon}(DG\text{-mod})$ to be braided. It is in general possible that there are non-braided functors in an equivalence class $[(F, J)] \in \mathrm{Aut}_{mon}(DG\text{-mod})$ for a braided representative (F, J).

The functor $\Psi(\phi, \sigma)$ is braided if and only if the following diagram commutes:

$$\begin{array}{ccc}
{}_\phi M \otimes {}_\phi N & \xrightarrow{F_\phi(J^\sigma_{M,N})} & {}_\phi(M \otimes N) \\
\Big\downarrow c_{{}_\phi M, {}_\phi N} & & \Big\downarrow F_\phi(c_{M,N}) \\
{}_\phi N \otimes {}_\phi M & \xrightarrow[F_\phi(J^\sigma_{N,M})]{} & {}_\phi(N \otimes M)
\end{array}$$

for all $M, N \in DG$-mod. This is equivalent to the fact that for all DG-modules M, N

$$R_2.\sigma_2.n \otimes R_1.\sigma_1.m = \sigma_1.\phi^*(R_2).n \otimes \sigma_2.\phi^*(R_1).m \tag{18}$$

holds for all $m \in M$ and $n \in N$. Recall from Lemma 2.2 the R-matrix $R = R_1 \otimes R_2 = \sum_{x \in G}(e_x \times 1) \otimes (1 \times x)$ and the braiding c. As above, we identify $\sigma \in Z^2_L(DG^*)$ with an element $\sigma = \sigma_1 \otimes \sigma_2 \in DG \otimes DG$ and take the action of ${}_\phi M$ to be the pre-composition with ϕ^* as defined in (17). We collect some of the resulting equations in the following Lemma.

Lemma 4.4. *Let* $\phi = \begin{pmatrix} u & b \\ a & v \end{pmatrix} \in \mathrm{Aut}_{Hopf}(DG)$ *and* $\sigma = (\alpha \otimes \beta) * \sigma_\lambda \in Z^2_L(DG^*)$ *such that* $\Psi(\phi, \sigma)$ *is braided then the following equations have to hold:*

$$\beta(g, g^{-1}hg) = \beta(h, g)b(h)(v(g)) \tag{19}$$

$$\alpha(\rho, \chi) = \alpha(\chi, \rho)u(\chi)(a(\rho)) \tag{20}$$

$$\lambda(g, f) = b(g)(a(f)) \tag{21}$$

$$\rho(g) = u(\rho)[v(g)]b(g)[a(\rho)] \tag{22}$$

for all $\rho, \chi \in \hat{G}$, $g, h \in G$ *and* $f \in k^G$.

Proof. Evaluating equation (18) we get

$$\sum_{g,t,h,d,k \in G} \sigma(g \times e_t, h \times e_d)(1 \times k).(e_h \times d).n \otimes (e_k \times 1).(e_g \times t).m$$
$$= \sum_{g,t,h,d,k \in G} \sigma(g \times e_t, h \times e_d)(e_g \times t).\phi^*(1 \times k).n \otimes (e_h \times d).\phi^*(e_k \times 1).m$$

The left hand side is equal to

$$\sum_{g,t,h,d \in G} \sigma(g \times e_t, h \times e_d)(e_{ghg^{-1}} \times gd).n \otimes (e_g \times t).m$$
$$= \sum_{g,t,h,d \in G} \sigma(g \times e_t, g^{-1}hg \times e_{g^{-1}d})(e_h \times d).n \otimes (e_g \times t).m$$

and the right hand side is equal to

$$\sum_{g,k,t,h,d,k_1k_2=k} \sigma(g \times e_t, h \times e_d)(e_g \times t).(b^*(k) \times u^*(k)).n \otimes (e_h \times d).(v^*(e_{k_1}) \times a^*(e_{k_2})).m$$
$$= \sum_{g,t,h,d,w,y,z,x} \sigma(g \times e_t, h \times e_d)a^w_x b(y)(v(z)w)(e_g \times t).(e_y \times u^*(v(z)w)).n \otimes (e_h \times d).(e_z \times x).m$$
$$= \sum_{g,t,h,d,w,y,z,x} \sigma(g \times e_t, h \times e_d)a^w_x b(y)(v(z)w)(\delta_{g,tyt^{-1}} e_g \times tu^*(v(z)w)).n \otimes (\delta_{h,dzd^{-1}} e_h \times \mathrm{d}x).m$$
$$= \sum_{t,d,y,z,x,w} \sigma(y \times e_t, z \times e_d)b(y)(v(d^{-1}zd)w)a^w_x(e_y \times tu^*(v(d^{-1}zd)w)).n \otimes (e_z \times \mathrm{d}x).m$$
$$= \sum_{t,d,y,h,x,w} \sigma(h \times e_d, g \times e_t)b(h)(v(t^{-1}gt)w)a^w_x(e_h \times du^*(v(t^{-1}gt)w)).n \otimes (e_g \times tx).m$$
$$= \sum_{\substack{t,h,g,d,x,w,d' \\ d=d'u^* \circ v(xt^{-1}gtx^{-1})u^*(w)}} \sigma(h \times e_{d'}, g \times e_{tx^{-1}})b(h)(v(g)w)a^w_x(e_h \times d).n \otimes (e_g \times t).m$$

Here we have used several times that the homomorphism a is supported on $Z(G)$ and that b maps G to the character group $\hat{G}$ which is abelian. We now that the above equality of the right and left hand side have to hold in particular for the regular DG-module and the elements $m = n = 1$. This implies:

$$\sigma(g \times e_t, g^{-1}hg \times e_{g^{-1}d}) = \sum_{\substack{x,w,d' \\ d=d'u^*(w)(u^* \circ v)(t^{-1}gt)}} \sigma(h \times e_{d'}, g \times e_{tx^{-1}})b(h)(v(g)w)a^w_x$$

On the other hand, if this equation holds, then also the right and left hand side are equal. Using Theorem 4.3 we can now write

$$\sum_{\substack{t_1,t_2\\t_1t_2=t}} \alpha(e_{t_1},e_{g^{-1}d})\beta(g,g^{-1}hg)\lambda(d^{-1}hd,e_{t_2}) = \sum_{\substack{x,w,d_1',d_2'\\d=d_1'd_2'u^*(w)t^{-1}(u^*\circ v)(g)t}} \alpha(e_{d_1'},e_{tx^{-1}})\beta(h,g)\lambda(t^{-1}gt,e_{d_2'})b(h)(v(g)w)a_x^w \tag{23}$$

We sum over all $t,d\in G$ and get equation (19). Using (19) we can simplify (23) to:

$$\sum_{\substack{t_1,t_2\\t_1t_2=t}} \alpha(e_{t_1},e_{g^{-1}d})\lambda(h,e_{t_2}) = \sum_{\substack{x,w,d'\\d=d'u^*(w)t^{-1}(u^*\circ v)(g)t}} \alpha(e_{d_1'},e_{tx^{-1}})\lambda(t^{-1}gt,e_{d_2'})b(h)(w)a_x^w$$

If we then sum in this equation over all $d\in G$ we get equation (21). Hence the first equation (23) simplifies again:

$$\sum_{\substack{t_1,t_2\\t_1t_2=t}} \alpha(e_{t_1},e_{g^{-1}d})\lambda(h,e_{t_2}) = \sum_{\substack{x,w,d'\\d=d'u^*(w)t^{-1}(u^*\circ v)(g)t}} \alpha(e_{d_1'},e_{tx^{-1}})\lambda(g,e_{d_2'})b(h)(w)e_w(a(e_x))$$

If we consider the case $h=g=1$ we get:

$$\alpha(e_t,e_d) = \sum_{x,w\in Z(G)} \alpha(e_{du^*(w)^{-1}},e_{tx^{-1}})a_x^w$$

For arbitrary $\chi,\rho\in\hat{G}$ we multiplying the last equation with $\rho(t),\chi(d)$ and sum over all $t,d\in G$. This leads to the equation (20). In order to get the last equation we multiply both sides of equation (23) with $\rho(t),\chi(d)$ for some $\chi,\rho\in\hat{G}$ and sum over all $t,d\in G$. Then the left hand side of equation (23) equals to: $\alpha(\chi,\rho)\beta(g,g^{-1}hg)\lambda(h,\chi)\rho(g)$ and the right hand side equals to:

$$\begin{aligned}&\sum_{x,w\in Z(G)} \chi(x)\rho(u^*(w)u^*(v(g)))\alpha(\rho,\chi)\beta(h,g)\lambda(g,\rho)b(h)(v(g)w)e_w(a(e_x))\\ =&\rho(u^*(a(\chi)))\rho((u^*\circ v)(g))\alpha(\rho,\chi)\beta(h,g)\lambda(g,\rho)b(h)(v(g)a(\chi))\end{aligned}$$

Using equality on both sides and equations (19),(20),(21) we get the last equation (22). □

4.2. **Automorphism Symmetries.**

We have seen in Definition 3.4 that a group automorphism $v\in\mathrm{Aut}(G)$ induces a Hopf automorphism $\begin{pmatrix}(v^{-1})^* & 0\\ 0 & v\end{pmatrix}\in V\subset\mathrm{Aut}_{Hopf}(DG)$. We now show that automorphisms of G also naturally induce braided autoequivalences of DG.

Proposition 4.5.

(i) *Consider the subgroup $\underline{\mathcal{V}}_L := V\times 1$ of $\mathrm{Aut}_{Hopf}(DG)\ltimes Z_L^2(DG^*)$. For an element $(v,1)\in\underline{\mathcal{V}}_L$ the corresponding monoidal functor $\Psi(v,1)=(F_v,J^{triv})$ with trivial monoidal structure is given on simple objects by*

$$F_v(\mathcal{O}_g^\rho)=\mathcal{O}_{v(g)}^{v^{-1*}(\rho)}$$

(ii) *Every $\Psi(v,1)$ is braided.*

(iii) *Let $\widetilde{\mathcal{V}}_L$ be the image of $\underline{\mathcal{V}}_L$ in $\mathrm{Out}_{Hopf}(DG)\ltimes H_L^2(DG^*)$, then we have $\widetilde{\mathcal{V}}_L\cong\mathrm{Out}(G)$.*

Proof. (i),(iii) Obvious from the above and Lemma 4.1.
(ii) An element in $\mathrm{Aut}_{Hopf}(DG) \times \mathrm{Z}^2_L(DG^*)$ is braided if and only if equation (23) is satisfied. For an element $(v, 1)$ this simplifies:

$$\delta_{a,d}\delta_{b,1} = \sum_{\substack{x,w,d'_1,d'_2 \in G \\ d = d'_1 d'_2 w b^{-1} ab}} \delta_{d'_1,1}\delta_{b,x}\delta_{d'_2,1}\delta_{w,1}\delta_{x,1} = \delta_{b,1}\delta_{d,b^{-1}ab}$$

which is clearly true for all $a, b, d \in G$.
(iv) The intersection of $\underline{\mathcal{V}}_L$ with the kernel $\mathrm{Inn}(G) \ltimes B^2_L(DG^*)$ is clearly $\mathrm{Inn}(G)$. □

Example 4.6. *For $G = \mathbb{F}_p^n$ we have $V = \mathrm{GL}_n(\mathbb{F}_p)$.*

Example 4.7. *The* extraspecial p-group p_+^{2n+1} *is a group of order p^{2n+1} generated by elements x_i, y_i for $i \in \{1, 2, \ldots n\}$ and the following relations (especially $2_+^{2+1} = \mathbb{D}_4$)*

$$x_i^p = y_i^p = 1 \qquad [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1, \; for \; i \neq j \qquad [x_i, y_i] = z \in Z(p_+^{2n+1})$$

Then the inner automorphism group is $\mathrm{Inn}(G) \cong \mathbb{Z}_p^{2n}$ and the automorphism group is $\mathrm{Out}(G) = \mathbb{Z}_{p-1} \ltimes \mathrm{Sp}_{2n}(\mathbb{F}_p)$ for $p \neq 2$ resp. $\mathrm{Out}(G) = \mathrm{SO}_{2n}(\mathbb{F}_2)$ for $p = 2$, see [Win72].

4.3. B-Symmetries.

Definition 4.8. *Let G be a finite group, one calls a cohomology class $[\beta] \in \mathrm{H}^2(G, k^\times)$* distinguished *if one of the following equivalent conditions is fulfilled* [Higgs87]*:*

- *The twisted group ring $k_\beta G$ has the same number of irreducible representations as kG. Note that $k_\beta G$ for $[\beta] \neq 1$ has no 1-dimensional representations.*
- *The centers are of equal dimension $\dim Z(k_\beta G) = \dim Z(kG)$.*
- *All conjugacy classes $[x] \subset G$ are β-*regular*, i.e. for all $g \in \mathrm{Cent}(x)$ we have $\beta(g, x) = \beta(x, g)$.*

The conditions are clearly independent of the representing 2-cocycle β and the set of distinguished cohomology classes forms a subgroup $\mathrm{H}^2_{dist}(G)$.

In fact, nontrivial distinguished classes are quite rare and we give in Example 4.13 a non-abelian group with p^9 elements which admits such a class.

Proposition 4.9.

(i) *The group $B \times Z^2(G)$ is a subgroup of $\mathrm{Aut}_{Hopf}(DG) \ltimes Z^2_L(DG^*)$. An element (b, β) corresponds to the monoidal functor (F_b, J^β) given by $F_b(\mathcal{O}_g^\rho) = \mathcal{O}_g^{\rho * b(g)}$ with monoidal structure*

$$\mathcal{O}_g^{\rho * b(g)} \otimes \mathcal{O}_h^{\chi * b(h)} \to F_b(\mathcal{O}_g^\rho \otimes \mathcal{O}_h^\chi)$$
$$(s_m \otimes v) \otimes (r_n \otimes w) \mapsto \beta(g_m, h_n)(s_m \otimes v) \otimes (r_n \otimes w)$$

where $\{s_m\}, \{r_n\} \subset G$ are choices of representatives of $G/\mathrm{Cent}(g)$ and $G/\mathrm{Cent}(h)$ respectively and where $g_m = s_m g s_m^{-1}, h_n = r_n h r_n^{-1}$.

(ii) *The subgroup $\underline{\mathcal{B}}_L$ of $\mathrm{Z}^2(G) \times B$ defined by*

$$\underline{\mathcal{B}}_L := \{(b, \beta) \in B \times Z^2(G) \mid b(g)(h) = \frac{\beta(h, g)}{\beta(g, h)} \quad \forall g, h \in G\}$$

consists of all elements $(b, \beta) \in B \times Z^2(G)$ such that $\Psi(\beta, b)$ is a braided autoequivalence.

(iii) *Let* $B_{alt} \cong \hat{G}_{ab} \wedge \hat{G}_{ab}$ *be the subgroup of alternating homomorphisms of* B, *i.e.* $b \in \mathrm{Hom}(G_{ab}, \hat{G}_{ab})$ *with* $b(g)(h) = b(h)(g)^{-1}$. *Then the following group homomorphism is well-defined and surjective:*

$$\underline{\mathcal{B}}_L \to B_{alt}, \quad (b, \beta) \mapsto b$$

(iv) *Let* $\widetilde{\mathcal{B}}_L$ *be the image of* $\underline{\mathcal{B}}_L$ *in* $\mathrm{Out}_{Hopf}(DG) \ltimes \mathrm{H}^2_L(DG^*)$, *then we have a central extension*

$$1 \to \mathrm{H}^2_{dist}(G) \to \widetilde{\mathcal{B}}_L \to B_{alt} \to 1$$

Note that not *all representing* 2*-cocycles* β *of a distinguished class lead to braided autoequivalences* $\Psi(\beta, 1)$.

Before we proceed with the proof, we give some examples:

Example 4.10. *For* $G = \mathbb{F}_p^n$ *we have* $B = \hat{G} \otimes \hat{G} = \mathbb{F}_p^{n \times n}$ *respectively* $B_{alt} = \hat{G} \wedge \hat{G} = \mathbb{F}_p^{\binom{n}{2}}$ *the additive group of* $n \times n$*-matrices resp. skew-symmetric* $n \times n$*-matrices (for* $p = 2$ *we additionally demand all diagonal entries are zero).*
For an abelian group there are no distinguished 2*-cohomology-classes, hence* $\widetilde{\mathcal{B}}_L \cong B_{alt}$ *where* $b \in B_{alt}$ *corresponds to* $(b, \beta) \in \underline{\mathcal{B}}_L$ *where* β *is any* 2*-cocycle with* $\beta(g, h)\beta(h, g) = b(g)(h)$, *which precisely determines a cohomology class* $[\beta]$ *in this case.*

Example 4.11. *For* $G = \mathbb{D}_4 = \langle x, y \rangle, x^2 = y^2 = 1$ *we have* $G_{ab} = \langle \bar{x}, \bar{y} \rangle \cong \mathbb{Z}_2^2$ *and* $B = \mathrm{Hom}(G_{ab}, \hat{G}_{ab}) = \mathbb{Z}_2^{2\times 2}$ *and* $B_{alt} = \mathbb{Z}_2 = \{1, b\}$ *with* $b(\bar{x})(\bar{y}) = b(\bar{y})(\bar{x}) = -1$. *It is known that* $\mathrm{H}^2(G, k^\times) = \mathbb{Z}_2 = \{[1], [\alpha]\}$, *where the nontrivial* 2*-cocycles* α *have a nontrivial restriction to the abelian subgroups* $\langle x, z \rangle, \langle y, z \rangle \cong \mathbb{Z}_2^2$ *of* G. *Especially* $[\alpha]$ *is no distinguished* 2*-cohomology-classes and by definition of* $\underline{\mathcal{B}}_L$*:*

$$\underline{\mathcal{B}}_L = \{(1, sym),\ (b, \beta \cdot sym)\}$$

where β *is the pullback of any nontrivial* 2*-cocycle in* G_{ab} *with* $\beta(x, y)\beta(y, x)^{-1} = -1$ *and* sym *denotes any symmetric* 2*-cocycles. Especially* $[\beta] = [1]$ *as one checks on the abelian subgroups and thus by definition*

$$\widetilde{\mathcal{B}}_L = \{(1, [1]),\ (b, [1])\} \cong \mathbb{Z}_2$$

However, these $(1, 1)$ *and* (b, β), *which are pull-backs of two different braided autoequivalences on* G_{ab}, *give rise to the* same *braided equivalence up to monoidal isomorphisms on* G. *Especially in this case we have a* non-injective *homomorphism.*

$$\widetilde{\mathcal{B}}_L \to \mathrm{Aut}_{mon}(DG\text{-mod})$$

This is proven explicitly by applying Lemma 2.3 to the element $\eta \times 1 \in DG$ *where for* $\eta \in \mathrm{Reg}^1_{aL}(G)$ *we take any* η *fulfilling* $\eta(tgt^{-1}) = \eta(g) \cdot b(t)(g)$, *e.g. by choosing* $\eta(g_i)$=1 *for representatives* g_i *of each conjugacy class and* $\eta(tg_it^{-1}) = b(t)(g_i)$ *else.*

More generally for the examples $G = p_+^{2n+1}$ we have B, B_{alt} as for the abelian group $\mathbb{F}_p^{2n}$, but (presumably) all braided autoequivalences in $\underline{\mathcal{B}}_L(\mathbb{F}_p^{2n})$ pull back to a single trivial braided autoequivalence on G.

Question 4.12. *We conjecture that in general the kernel of* $\widetilde{\mathcal{B}}_L \to \mathrm{Aut}_{mon}(DG\text{-mod})$ *consist of those* (b, β) *for which* $[\beta] = [1]$ *i.e. the remaining non-injectivity is controlled by the non-injectivity of the pullback* $H^2(G_{ab}) \to H^2(G)$.

We give now an example where $\widetilde{\mathcal{B}}_L \to B_{alt}$ is not injective and we get new braided autoequivalences $(1, \beta)$ compared to the abelian case:

Example 4.13. *In* [Higgs87] *p. 277 a group G of order p^9 with $H^2_{dist}(G) = \mathbb{Z}_p$ is constructed as follow: We start with the group $\tilde{G}$ of order p^{10} generated by x_1, x_2, x_3, x_4 of order p, all commutators $[x_i, x_j], i \neq j$ nontrivial of order p and central. Then $\tilde{G}$ is a central extension of $G := \tilde{G}/\langle s\rangle$ where $s := [x_1, x_2][x_3, x_4]$, and this central extension corresponds to a class of distinguished 2-cocycles $\langle\sigma\rangle = \mathbb{Z}_p = \mathrm{H}^2_{dist}(G) = \mathrm{H}^2(G)$ (this is a consequence of the fact that s cannot be written as a single commutator).*

The distinguished 2-cocycle corresponds to a braided equivalence (id, J^σ) trivial on objects. From $G_{ab} \cong \mathbb{Z}_p^4$, hence $B_{alt} = \mathbb{Z}_p^4 \wedge \mathbb{Z}_p^4 = \mathbb{Z}_p^6$ we have a central extension

$$1 \to \mathbb{Z}_p \to \widetilde{\mathcal{B}}_L \to \mathbb{Z}_p^6 \to 1$$

In fact we assume that the sequence splits and the braided autoequivalence (id, J^σ) is the only nontrivial generator of the image $\Psi(\widetilde{\mathcal{B}}_L) \subset \mathrm{Aut}_{br}(DG\text{-mod})$, since the pullback $\mathrm{H}^2(G_{ab}) \to \mathrm{H}^2(G)$ is trivial.

Proof of Lemma 4.9. (i): Let us show that B acts trivially on $\mathrm{Z}^2(G)$:

$$b.\beta = \sum_{x,y,g,h} ((\epsilon_{k^G\otimes k^G} \otimes \beta) * \epsilon_{kG\otimes k^G})(x \times e_y, g \times e_h) \begin{pmatrix} 1 & b^* \\ 0 & 1 \end{pmatrix} (e_x \times y) \otimes \begin{pmatrix} 1 & b^* \\ 0 & 1 \end{pmatrix} (e_g \times h)$$

$$= \sum_{x,g} \beta(x, g)(e_x \times 1) \otimes (e_g \times 1) = \beta$$

For the action on simple DG-modules use Lemma 4.1. The rest of the statements are easy calculations.

(ii): Assume $\Psi(b, \beta)$ is braided then according to Lemma 4.4 we get for $v = \mathrm{id}$:

$$b(g)(h) = \beta(h, g)\beta(hgh^{-1}, h)^{-1} \quad \forall g, h \in G \tag{24}$$

Because β is closed we have: $1 = d\beta(h, gh^{-1}, h) = \frac{\beta(gh^{-1},h)\beta(h,g)}{\beta(hgh^{-1},h)\beta(h,gh^{-1})}$ and therefore:

$$b(g)(h) = \beta(h, g)\beta(hgh^{-1}, h)^{-1} = \beta^{-1}(gh^{-1}, h)\beta(h, gh^{-1})$$

$$\Leftrightarrow b(g)(h) = b(g)(h)b(h)(h) = b(gh)(h) = \beta^{-1}(g, h)\beta(h, g) \tag{25}$$

In the proof of Lemma 4.4 we also have shown that $\Psi(b, \beta)$ is braided if and only if (23) holds. In this case (23) reduces to (24), hence $\Psi(b, \beta)$ is braided. Since the product of braided autoequivalences is braided this also shows that $\underline{\mathcal{B}}_L$ is in fact a subgroup of $B \times Z^2(G)$.

(iii) By definition of $\underline{\mathcal{B}}_L$ we have $b \in B_{alt}$. We now show surjectivity: Let $G_{ab} = G/[G, G]$ be the abelianization of G and $\hat{\beta}_b \in Z^2(G_{ab})$ an abelian 2-cocycle defined uniquely up to cohomology by $b(g)(h) = \hat{\beta}_b(h, g)\hat{\beta}_b(hgh^{-1}, h)^{-1} = \hat{\beta}_b(h, g)\hat{\beta}_b(g, h)^{-1}$ for $g, h \in G_{ab}$. Further, we have a canonical surjective homomorphism $\iota : G \to G_{ab}$ which induces a pullback $\iota^* : Z^2(G_{ab}) \to Z^2(G)$, hence we define $\beta_b := \iota^*\hat{\beta}_b$.

(iv) By (iii) the map $(b, \beta) \mapsto b$ is a group homomorphism $\underline{\mathcal{B}}_L \to B_{alt}$ and this factorizes to a group homomorphism $\widetilde{\mathcal{B}}_L \to B_{alt}$, since $(\mathrm{Inn}(G)\times \mathrm{B}^2(G))\cap(B\times \mathrm{Z}^2(G)) = 1$. The kernel of this homomorphism consists of all $(1, [\beta]) \in \widetilde{\mathcal{B}}_L$, hence all $(1, [\beta])$ where $[\beta]$ has at least one representative β with $\beta(g, x) = \beta(x, g)$ for all $g, x \in G$. We denote this kernel by K and note that it is central in $\underline{\mathcal{B}}_L$.

It remains to show $K = \mathrm{H}^2_{dist}(G)$: Whenever $[\beta] \in K$ then there exists a representative β with $\beta(g, x) = \beta(x, g)$ for all $g, x \in G$, in particular for any elements $g \in \mathrm{Cent}(x)$, which implies any conjugacy class $[x]$ is β-regular and thus $[\beta]\in \mathrm{H}^2_{dist}(G)$. For the other direction we need a specific choice of representative: Suppose $[\beta]\in \mathrm{H}^2_{dist}(G)$ and thus all x are β-regular; by [Higgs87] Lm. 2.1(i) there exists a representative β with

$$\frac{\beta(g,x)\beta(gx,g^{-1})}{\beta(g,g^{-1})}=1$$

for all β-regular x (i.e. here all x) and all g. An easy cohomology calculation shows indeed

$$\frac{\beta(g,x)}{\beta(gxg^{-1},g)}=\frac{\beta(g,x)}{\beta(gxg^{-1},g)}\cdot\frac{\beta(gx,g^{-1})\beta(gxg^{-1},g)}{\beta(gx,1)\beta(g,g^{-1})}=1$$

hence $(1,\beta)\in \underline{\mathcal{B}}_L$ by equation (24). $\square$

4.4. **E-Symmetries.**

It is now natural to construct a subgroup of $E\times \mathrm{Z}^2_L(k^G)$ in a similar fashion. For this we use the classification of Galois algebras and Bigalois algebras in [Dav01].

Proposition 4.14.

(i) *The group $E\times \mathrm{Z}^2_L(k^G)$ is a subgroup of $\mathrm{Aut}_{Hopf}(DG)\ltimes \mathrm{Z}^2_L(DG^*)$. An element (a,α) corresponds to the monoidal functor $\Psi(a,\alpha)=(F_a,J^\alpha)$ given on simple objects by $F_a(\mathcal{O}_g^\rho)=\mathcal{O}^\rho_{a(\rho')g}$, with the monoidal structure (in (v) we give easy representatives).*

$$\mathcal{O}^\rho_{a(\hat\rho)g}\otimes\mathcal{O}^\rho_{a(\hat\chi)h}\to F_a(\mathcal{O}^\rho_g\otimes\mathcal{O}^\chi_h)$$

$$(s_m\otimes v)\otimes(r_n\otimes w)\mapsto\sum_{\substack{i,j;x\in\mathrm{Cent}(g)\\ y\in\mathrm{Cent}(h)}}\alpha(e_{s_ixs_m^{-1}},e_{r_jyr_n^{-1}})[s_i\otimes\rho(x)(v)]\otimes[r_j\otimes\chi(y)(w)]$$

where we denote by $\rho':Z(G)\to k^*$ the one-dimensional representation such that the any central element $z\in Z(G)$ act in ρ by multiplication with the scalar $\rho'(z)$ and $\{s_i\},\{r_j\}\subset G$ are choices of representatives of $G/\mathrm{Cent}(g)$ and $G/\mathrm{Cent}(h)$ respectively.

(ii) *The subgroup $\underline{\mathcal{E}}_L\subset E\times Z^2_L(k^G)$ defined by*

$$\underline{\mathcal{E}}_L:=\{(a,\alpha)\in E\times Z^2_L(k^G)\mid \forall g,t,h\in G:\alpha(e_t,e_{ght})=\alpha(e_t,e_{hg^{-1}t})$$
$$\alpha(e_t,e_h)=\sum_{x,y\in Z(G)}\alpha(e_{hy^{-1}},e_{tx^{-1}})e_y(a(e_x))\}$$

consists of all elements $(a,\alpha)\in E\times \mathrm{Z}^2_L(k^G)$ such that the monoidal autoequivalence $\Psi(a,\alpha)$ is braided.

(iii) *Let $E_{alt}\cong Z(G)\wedge Z(G)$ be the subgroup of alternating homomorphisms in $E=\mathrm{Hom}(\widehat{Z(G)},Z(G))=Z(G)\otimes Z(G)$, i.e. the set of homomorphisms $a:\widehat{Z(G)}\to Z(G)$ with $\rho(a(\chi))=\chi(a(\rho))^{-1}$ and $\chi(a(\chi))$ for all $\chi,\rho\in\widehat{Z(G)}$. Then the following group homomorphism is well-defined and surjective:*

$$\underline{\mathcal{E}}_L\to E_{alt},\quad (a,\alpha)\mapsto a$$

(iv) *Let $\widetilde{\mathcal{E}}_L$ be the image of $\underline{\mathcal{E}}_L$ in $\mathrm{Out}_{Hopf}(DG)\ltimes \mathrm{H}^2_L(DG^*)$, then the previous group homomorphism factorizes to an isomorphism*

$$\widetilde{\mathcal{E}}_L\cong E_{alt}$$

For each $a\in E_{alt}$ we have a representative functor $\Psi(a,\alpha)=(F_a,J^\alpha)$ for a certain α obtained by pull-back from the center of G. More precisely, the *functor is given by* $F_a(\mathcal{O}_g^\rho) = \mathcal{O}_{a(\rho')g}^\rho$ *and the monoidal structure given by a scalar*

$$\begin{aligned}\mathcal{O}_{a(\rho')g}^\rho \otimes \mathcal{O}_{a(\chi')h}^\rho &\to F_a(\mathcal{O}_g^\rho \otimes \mathcal{O}_h^\chi)\\ m \otimes n &\mapsto \alpha'(\rho', \chi') \cdot (m \otimes n)\end{aligned}$$

where $\alpha' \in \mathrm{Z}^2(\widehat{Z(G)})$ *is any 2-cocycle in the cohomology class associated to the alternating form* $a \in E_{alt}$ *on the abelian group* $\widehat{Z(G)}$.

Before we proceed to the proof we give some examples:

Example 4.15. *For* $G = \mathbb{D}_4 = \langle x, y\rangle, x^2 = y^2 = 1$ *we have* $Z(G) = \langle [x,y]\rangle \cong \mathbb{Z}_2$ *and hence* $E = \mathrm{Hom}(\widehat{Z(G)}, Z(G)) = \mathbb{Z}_2$ *and* $E_{alt} = 1$. *More generally for the examples* $G = p_+^{2n+1}$ *we have* $E = \mathbb{Z}_p \otimes \mathbb{Z}_p = \mathbb{Z}_p$ *and* $E_{alt} = \mathbb{Z}_p \wedge \mathbb{Z}_p = 1$ *and hence* $\widetilde{\mathcal{E}}_L = 1$.

Example 4.16. *For the group of order* p^9 *in Example 4.13 we have* $Z(G) = \mathbb{Z}_p^5$ *generated by all commutators* $[x_i, x_j], i \neq j$ *modulo the relation* $[x_1, x_2][x_3, x_4]$. *Hence* $E_{alt} = \mathbb{Z}_p^5 \wedge \mathbb{Z}_p^5 \cong \mathbb{Z}_p^{\binom{5}{2}} = \mathbb{Z}_p^{10}$ *and respectively* $\widetilde{\mathcal{E}}_L = \mathbb{Z}_p^{10}$.

Proof of Proposition 4.14.

(i) Let us show that E acts trivially on $Z_L^2(k^G)$. For this we calculate:

$$\begin{aligned}a.\alpha &= \sum_{x,y,z,w} ((\alpha \otimes \epsilon_{kG\otimes kG}) * \epsilon_{kG\otimes k^G})(x \times e_y, z \times e_w) \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (e_x \times y) \otimes \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (e_z \times w)\\ &= \sum_{y,w} \alpha(e_y, e_w) \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (1 \times y) \otimes \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (1 \times w) = \sum_{y,w} \alpha(e_y, e_w)(1 \times y) \otimes (1 \times w) = \alpha\end{aligned}$$

For the action on simple DG-modules use Lemma 4.1. The rest are easy calculations.

(ii) Let $(a, \alpha) \in E \times Z_L^2(k^G)$. Then we have shown above that $\Psi(a, \alpha)$ is braided if and only if equation (23) holds. Hence, iff for all $g, t, d \in G$:

$$\alpha(e_t, e_{gd}) = \sum_{h,x\in Z(G)} \alpha(e_{dh^{-1}(t^{-1}g^{-1}t)}, e_{tx^{-1}}) e_h(a(e_x)) \tag{26}$$

Setting $g = 1$ gives us the second defining equation of $\mathcal{E}_L$. Further, (26) is equivalent to

$$\alpha(e_t, e_{gdt^{-1}gt}) = \sum_{h,x\in Z(G)} \alpha(e_{dh^{-1}}, e_{tx^{-1}}) e_h(a(e_x)) \tag{27}$$

and therefore: $\alpha(e_t, e_{gdt^{-1}gt}) = \alpha(e_t, e_d)$ which is equivalent to the first defining equation of $\mathcal{E}_L$. Since the product of braided autoequivalences is braided this also shows that $\mathcal{E}_L$ is in fact a subgroup of $E \times Z_L^2(k^G)$.

(iii) We first note that by equation (20) for $u = \mathrm{id}$ we have $a \in E_{alt}$. We now show surjectivity: Since $Z(G)$ is an abelian group there exists a unique (up to cohomology) 2-cocycle $\alpha \in \mathrm{H}^2(\widehat{Z(G)})$ with can be pulled back to a 2-cocycle in $Z_L^2(k^G)$. Then (a, α) is in $\mathcal{E}_L$ which proves surjectivity.

(iv) Before we show the isomorphism we obtain the description of the explicit representatives: In (iii) we constructed preimages (a, α) of each $a \in E_{alt}$ by pulling back a 2-cocycle $\alpha' \in \mathrm{Z}^2(\widehat{Z(G)})$ in the cohomology class associated to a. We now apply the explicit formula in (i) and use that α is only nonzero on e_g, e_h with $g, h \in Z(G)$: Hence we have only nonzero summands for $s_m^{-1}s_i \in Z(G)$, hence $i = m$ and similarly $j = n$. Moreover, ρ, χ reduce on $Z(G)$ to one dimensional representations ρ', χ'. Evaluating the resulting sum we get the asserted form.

Next we note that the group homomorphism $\mathcal{E}_L \to E_{alt}$ in (iii) factorizes to a group homomorphism $\widetilde{\mathcal{E}}_L \to E_{alt}$, since $(\mathrm{Inn}(G) \times \mathrm{B}^2_L(k^G)) \cap (E \times \mathrm{Z}^2_L(k^G)) = 1$. The kernel of this homomorphism consists of all $(1, [\alpha]) \in \widetilde{\mathcal{E}}_L$, i.e. all classes $[\alpha]$ such that there exists a lazy representative $\alpha \in Z^2_L(k^G)$. Then, by definition of $\mathcal{E}_L$, the following is fulfilled for a pair $(1, \alpha) \in \mathcal{E}_L$:

$$\alpha(e_t, e_{ght}) = \alpha(e_t, e_{hg^{-1}t}) \quad \alpha(e_g, e_h) = \alpha(e_h, e_g)$$

We now need to show that such a symmetric α is already cohomologically trivial. The proof of this part of the proposition is more tedious and will require the rest of this section. It uses the classification of *all* lazy 2-cocycles on k^G. We now obtain this by using Movshev's classification of k^G-Galois objects and the additional result in the form presented in Davydov [Dav01] and apply these to the lazy case.

Let us recall that for a finite group G a left G-algebra (equivalently a right k^G-comodule algebra) is an associative algebra R together with a left G-action given by a homomorphism $G \to \mathrm{Aut}(R)$. A G-algebra is Galois if the algebra homomorphism

$$\theta: \ R \otimes kG \to \mathrm{End}(R) \qquad \theta(r \otimes g)(r') = r(g.r')$$

is an isomorphism. Therefore, a left G-Galois algebra is a right k^G-Galois object. For any group S and any 2-cocycle $\eta \in Z^2(S)$ we view the twisted group algebra $k_\eta S$ as a S-algebra with the associative multiplication given by $g \cdot h = \eta(g, h)gh$ and the left S-action $g.h = g \cdot h \cdot g^{-1}$. Given a subgroup S of G and a S-algebra M there is a natural way to construct a G-algebra by induction:

$$ind_G^S(M) := \{r : G \to B \mid r(sg) = s.r(g)\}$$

which is an associative algebra with the pointwise multiplication of maps and has a left G-action given by $(g'.r)(g) = r(gg')$. We are now ready to present the first classification result due to Movshev. Note that this is extended in [Dav01] for arbitrary characteristic and in [GM10] for algebraically not closed:

Lemma 4.17. *(*[Dav01] *Theorem 3.8)*
Let k be an algebraically closed field of characteristic zero. Then there is a bijection between isomorphism classes of right k^G-Galois objects and conjugacy classes $(S, [\eta])$ where S is a subgroup of G and $[\eta] \in \mathrm{H}^2(S, k^\times)$ for η a non-degenerate 2-cocycle. The isomorphism assigns to a conjugacy class $(S, [\eta])$ the isomorphism class of

$$R(S, \eta) := \{r : G \to k_\eta S \mid r(sg) = s.r(g) \ \forall s \in S, g \in G\}$$

with multiplication given by pointwise multiplication of functions and with a G-action given by $(g.r)(h) = r(hg)$ or equivalently a k^G-coaction given by $\delta(r) = \sum_g e_g \otimes g.r$.

Lemma 4.18. *(*[Dav01]*: Proposition 6.1)*
Let $R(S, \eta)$ be a right k^G-Galois object as above and let $G' := \mathrm{Aut}_G(R)$. The following are equivalent:

- *$R(S,\eta)$ is an $k^{G'}$-k^G-Bigalois object with the natural left $k^{G'}$-coaction resp. right G'-action $r.f = f(r)$.*
- *$|G'| = |G|$.*
- *S is an abelian normal subgroup of G and the class $[\eta]$ is G-invariant.*

Relevant to this article is to find all Bigalois objects with $G \cong G'$ from this description. We now determine for the previously previously introduced Bigalois objects the explicit 2-cocycles:

Lemma 4.19. *Let S be a normal abelian subgroup of G and $\eta \in Z^2(S)$ a non-degenerate 2-cocycle i.e. $\langle s,t\rangle := \eta(s,t)\eta(t,s)^{-1}$ is a non-degenerate bimultiplicative symplectic form. Further, assume for simplicity that $\eta(s,s^{-1}) = 1$ for all $s \in S$ (Note that this is always possible up to cohomology). Then there is an isomorphism of k^G-comodule algebras: $(k^G)_\alpha \simeq R(S,\eta)$ where $\alpha \in \mathrm{Z}^2(k^G)$ is defined by*

$$\alpha(\varphi,\varphi') = \sum_{r,t,r',t' \in S} \eta(t,t')\langle t,r\rangle\langle t',r'\rangle\varphi(r)\varphi'(r')$$

Proof. We show that an isomorphism $\Phi : (k^G)_\alpha \mapsto R(S,\eta)$ is given by

$$\Phi : \varphi \mapsto \frac{1}{|S|}\left(h \mapsto \sum_{t,r\in S} t\varphi(rh)\langle t,r\rangle\right)$$

We check that $\Phi(\varphi)$ is a kS-linear map:

$$\begin{aligned}
\Phi(\varphi)(sg) &= \frac{1}{|S|}\sum_{t,r\in S} t\varphi(rsg)\langle t,r\rangle = \frac{1}{|S|}\sum_{t,r'} t\varphi(r'g)\langle t,r'\rangle\langle t,s\rangle^{-1} \\
&= \frac{1}{|S|}\sum_{t,r'} \left(\eta(s,t)t\eta(t,s)^{-1}\right)\varphi(r'g)\langle t,r'\rangle \\
&= \frac{1}{|S|}\sum_{t,r'} \left(\eta(s,t)t\frac{\eta(st,s^{-1})}{\eta(t,1)\eta(s,s^{-1})}\right)\varphi(r'g)\langle t,r'\rangle \\
&= \frac{1}{|S|}\sum_{t,r'} (s.t)\,\varphi(r'g)\langle t,r'\rangle = s.\Phi(\varphi)(g)
\end{aligned}$$

Next we check this is a kG-module morphism:

$$\Phi(h.\varphi) = \Phi(\varphi^{(2)}(h)\varphi^{(1)}) = \Phi(g' \mapsto \varphi(g'h)) = \frac{1}{|S|}(g \mapsto \sum_{t,r\in S} t\varphi(rgh)\langle t,r\rangle) = h.\Phi(\varphi)$$

Further, it is an algebra morphism: $\Phi(1)(g) = \frac{1}{|S|^2}\sum_{t,r\in S} t\langle t,r\rangle = \sum_{t\in S} t\delta_{t,1} = 1$

$$
\begin{aligned}
\big(\Phi(\varphi)\Phi(\varphi')\big)(g) &= \frac{1}{|S|^2}\left(\sum_{t,r\in S} t\varphi(rg)\langle t,r\rangle\right)\cdot_\eta\left(\sum_{t',r'\in S} t'\varphi(r'g)\langle t',r'\rangle\right)\\
&= \frac{1}{|S|^2}\sum_{t,r,t',r'\in S} tt'\eta(t,t')\cdot\varphi(rg)\langle t,r\rangle\varphi'(r'g)\langle t',r'\rangle\\
&= \frac{1}{|S|}\sum_{r,t,r',t'\in S}\left(\frac{1}{|S|}\sum_{\tilde t,\tilde r\in S}\tilde t\langle\tilde t,\tilde r\rangle\langle t,\tilde r^{-1}\rangle\langle t',\tilde r^{-1}\rangle\right)\\
&\cdot\varphi_{(1)}(r)\varphi'_{(1)}(r')\langle t,r\rangle\langle t',r'\rangle\eta(t,t')\cdot\varphi_{(2)}(g)\varphi'_{(2)}(g)\\
&= \frac{1}{|S|}\sum_{\tilde t,\tilde r,r,t,r',t'\in S}\tilde t\langle\tilde t,\tilde r\rangle\\
&\cdot\varphi_{(1)}(r)\varphi'_{(1)}(r')\langle t,r\tilde r^{-1}\rangle\langle t',r'\tilde r^{-1}\rangle\eta(t,t')\cdot\varphi_{(2)}(g)\varphi'_{(2)}(g)\\
\overset{\substack{r''=r\tilde r^{-1}\\ r'''=r'\tilde r^{-1}}}{}&= \frac{1}{|S|}\sum_{\tilde t,\tilde r,r'',t,r''',t'\in S}\tilde t\langle\tilde t,\tilde r\rangle\\
&\cdot\varphi_{(1)}(r'')\varphi'_{(1)}(r''')\langle t,r''\rangle\langle t',r'''\rangle\eta(t,t')\cdot\varphi_{(2)}(\tilde r)\varphi'_{(2)}(\tilde r)\cdot\varphi_{(3)}(g)\varphi'_{(3)}(g)\\
&= \frac{1}{|S|}\sum_{\tilde t,\tilde r\in S}\tilde t\langle\tilde t,\tilde r\rangle\cdot\alpha(\varphi_{(1)},\varphi'_{(1)})\varphi_{(2)}(\tilde r g)\varphi'_{(2)}(\tilde r g) = \Phi(\varphi\cdot_\alpha\varphi')(g)
\end{aligned}
$$

We finally check bijectivity of Φ. We first note that for any coset $Sg \subset G$ the functions φ which are nonzero only on Sg are sent to functions $\Phi(\varphi)$ which are nonzero only on Sg. With the fixed representative g of a coset Sg we consider the basis e_{sg} for k^{Sg}. By construction this element is mapped to the following element in $\mathrm{Hom}_{kS}(k[Sg], kS)$:

$$\Phi(e_{sg}) = \frac{1}{|S|}\left(s'g \mapsto \sum_{t,r\in S} te_{sg}(rs'g)\langle t,r\rangle\right) = \frac{1}{|S|}\left(s'g\mapsto\sum_{t\in S} t\langle t,s\rangle\langle t,s'\rangle^{-1}\right)$$

This is by construction a Fourier transform with a non-degenerate form $\langle,\rangle$, which implies bijectivity. More explicitly, we show injectivity by considering $s'=1$, then we get elements $\sum_{t\in S} t\langle t,s\rangle$ in S, which are linearly independent by the assumed non-degeneracy. Now bijectivity follows because source and target have dimension $|S|$. □

In particular, the assumption in Lemma 4.4 is that the class $[\eta]\in H^2(S)$ is G-invariant and hence the symplectic form $\langle,\rangle$ on S is G-invariant. Then the criterion in Example 2.20 2 shows easily α is lazy iff it is G-invariant and an easy calculation shows:

Corollary 4.20. *A lazy cocycle for k^G is up to cohomology the restriction of a G-invariant 2-cocycle η (not just an G-invariant class $[\eta]$) on a normal abelian subgroup S in the sense of Lemma 4.19.*

We now conclude the proof of Proposition 4.14 (iv) as follows: Let $\alpha\in Z^2_L(k^G)$ be a 2-cocycle with the additional property $\alpha(e_g,e_h)=\alpha(e_h,e_g)$, then the Galois objects ${}_\alpha(k^G)$ is actually a commutative algebra. Since any lazy 2-cocycle is up to cohomology the restriction of a 2-cocycle η on a normal abelian subgroup $S\subset G$ the symmetry condition implies also η symmetric, but then η is cohomologically trivial and so is the restriction. □

4.5. **Partial E-M Dualizations.**

In [BLS15] a family of braided functors has been obtained, which are generalizations of Lusztig's reflection operators T_i on quantum groups, reflecting on the simple root α_i. In the present article these no reappear as the set of representatives of the double coset decomposition; for the example $G = \mathbb{F}_p^n$ they are indeed again connected to the Weyl group.

Lemma 4.21. *Let $G = H \rtimes C$, then applying* [BLS15] *Cor. 4.5 to $H := kG$ with respect to the so-called* partial dualization *datum $\mathcal{A} = (kG \xrightarrow{\pi} kC, k^C, \omega)$ returns a braided functor with a nontrivial monoidal structure*

$$D(kG)\text{-mod} \to D(r_{\mathcal{A}}(kG))\text{-mod}$$

between the Drinfeld centers of kG-mod and $r_{\mathcal{A}}(kG)$-mod, where $r_{\mathcal{A}}(H)$ is the partial dualization *of H defined in the cited article.*

If C is abelian*, we may fix an isomorphism $\delta : kC \cong k^C$ and if in addition C is a* direct factor *then $r_{\mathcal{A}}(kG) \cong kG$, so we obtain a braided autoequivalence of DG-mod.*

Recall that R was the set of triples (H, C, δ) such that $G = H \times C$ and $\delta : kC \to k^C$ a Hopf isomorphism. Corresponding to that triple there is unique Hopf automorphism of DG that we called $r_{(H,C,\delta)}$ that exchanges the C and $\hat{C}$. We will identify the triple (H, C, δ) with the corresponding automorphism $r = r_{(H,C,\delta)}$ and the other way around.

Proposition 4.22.

(i) *Consider the subset $R \times \mathrm{P}(kG, k^G)$ in $\mathrm{Aut}_{Hopf}(DG) \ltimes \mathrm{Z}^2_L(DG^*)$. An element (r, λ) corresponds to the monoidal functor $\Psi(r, \lambda) = (F_r, J^\lambda)$ given on simple objects by $F_r(\mathcal{O}_{hc}^{\rho_H\rho_C}) = \mathcal{O}_{\delta^{-1}(\rho_C)h}^{\rho_H\delta(c)}$, where we decompose any group element and representation according to the choice $G = H \times C$ into $h \in H, c \in C$ resp. $\rho_H \in \mathrm{Cent}_H(h)$-mod, $\rho_C \in \mathrm{Cent}_C(c)$-mod. The monoidal structure is given by*

$$\mathcal{O}_{\delta^{-1}(\rho_C)h}^{\rho_H\delta(c)} \otimes \mathcal{O}_{\delta^{-1}(\chi_C)h'}^{\chi_H\delta(c')} \to F_r(\mathcal{O}_{hc}^{\rho_H\rho_C} \otimes \mathcal{O}_{h'c'}^{\chi_H\chi_C})$$
$$(s_m \otimes v) \otimes (r_n \otimes w) \mapsto \sum_{\substack{i \\ z\in \mathrm{Cent}(hc)}} \lambda((h'c')_n, e_{s_i z s_m^{-1}})[s_i \otimes \rho(z)(v)] \otimes (r_n \otimes w)$$

where $\{s_m\}, \{r_n\} \subset G$ are choices of representatives of $G/\mathrm{Cent}(g)$ and $G/\mathrm{Cent}(h)$ respectively and where $(h'c')_n = r_n h'c' r_n^{-1}$.

(ii) *Define the following set uniquely parametrized by decompositions $G = H \times C$:*

$$\mathcal{R}_L := \{(r_{(H,C,\delta)}, \lambda) \in R \times P(kG, k^G) \mid \forall (h, c), (h', c') \in H \times C :$$
$$\lambda(hc, e_{h'c'}) = \delta_{c,c'}\epsilon(h)\epsilon(e_{h'}), \quad \delta(c)(\delta^{-1}(e_{c'})) = \delta_{c,c'}\}$$

Then $\Psi(r_{(H,C,\delta)}, \lambda)$ is a braided autoequivalence iff $(r_{(H,C,\delta)}, \lambda) \in \mathcal{R}_L$.

(iii) *For $(r_{(H,C,\delta)}, \lambda) \in \mathcal{R}_L$ the monoidal structure of $\Psi(r_{(H,C,\delta)}, \lambda)$ simplifies to just a multiplication with a scalar:*

$$\mathcal{O}_{\delta^{-1}(\rho_C)h}^{\rho_H\delta(c)} \otimes \mathcal{O}_{\delta^{-1}(\chi_C)h'}^{\chi_H\delta(c')} \to F_r(\mathcal{O}_{hc}^{\rho_H\rho_C} \otimes \mathcal{O}_{h'c'}^{\chi_H\chi_C})$$
$$m \otimes n \mapsto \rho_C(c') \cdot (m \otimes n)$$

Proof.
(i) For the action on simple DG-modules use Lemma 4.1.
(ii) For $(r_{(H,C,\delta)},\lambda) \in R\times P(kG,k^G)$ the functor $\Psi(r,\lambda)$ is braided iff the equation (23) holds. Let us denote an element in the group $G = H \times C$ by $g = g_H g_C$ and recall that we write p_C, p_H for the obvious projections. Then we calculate:

$$\sum_{x,y,z\in G} \lambda(y^{-1}xy, e_z)(e_x \times y) \otimes (e_y \times z) = \sum_{x,y,z,w\in G} \delta_{y,w}\lambda(y^{-1}xy, e_z)(e_x \times y) \otimes (e_w \times z)$$

has to be equal to

$$\sum_{w,y,g_1,g_2} \lambda(w, e_y)(1 \times y)(\delta^*((g_1g_2)_C) \times (g_1g_2)_H) \otimes (e_w \times 1)(e_{g_1} \circ p_H \times \delta^{-1*}(e_{g_2} \circ p_C))$$

$$= \sum_{x,y,g_1,g_2,w,z} \lambda(w, e_y)(1 \times y)\delta^*((g_1g_2)_C)(r)e_z(\delta^{-1*}(e_{g_2} \circ p_C))(e_x \times (g_1g_2)_H) \otimes (e_w \times 1)(e_{g_1} \circ p_H \times z)$$

$$= \sum_{\substack{x,y,g_1,g_2,w,z\\(g_2)_H=1}} \delta_{z_H,1}\lambda(w, e_y)\delta(x_C)((g_1g_2)_C)e_{(g_2)_C}(\delta^{-1}(e_{z_C}))(1 \times y)(e_x \times (g_1g_2)_H) \otimes (e_w \times 1)(e_{g_1} \circ p_H \times z)$$

$$= \sum_{\substack{w,y,g_1,g_2\\(g_2)_H=1,w_H=(g_1)_H}} \delta_{z_H,1}\lambda(w, e_y)\delta(x_C)((g_2)_C)e_{(g_2)\circ p_C}(\delta^{-1}(e_{z_C}))(e_{yxy^{-1}} \times y(g_1)_H) \otimes (e_w \times z)$$

$$= \sum_{x,y,w,z} \delta_{z_H,1}\lambda(w, e_{yw_H^{-1}})\delta(x_C)((\delta^{-1}(e_{z_C})))(e_x \times y) \otimes (e_w \times z)$$

This is equivalent to saying that for all $x, y, w, z \in G$ the following holds:

$$\delta_{y,w}\lambda(y^{-1}xy, e_z) = \delta_{z_H,1}\lambda(w, e_{yw_H^{-1}})\delta(x_C)((\delta^{-1}(e_{z_C})))$$

Further, we see that (r,λ) fulfills this equation if and only if $\lambda(hc, e_{h'c'}) = \delta(c)(\delta^{-1}(e_{c'}))\epsilon(h)\epsilon(e_{h'}) = \delta_{c,c'}\epsilon(h)\epsilon(e_{h'})$ for all $hc, h'c' \in H \times C$.

(iii) This is a simple calculation using that C is abelian and then that $\lambda(hc, e_{h'c'}) = \delta_{c,c'}\epsilon(h)\epsilon(e_{h'})$ implies $i = m$ and only leaves the term $\delta_{c',z}$. □

5. Main Result

Recall that we have defined certain characteristic elements of $\mathrm{Aut}_{br}(DG\text{-mod})$ in the Propositions 4.5, 4.9, 4.14, 4.22 and showed how they can explicitly calculated. In our main result we show that these elements generate $\mathrm{Aut}_{br}(DG\text{-mod})$.

Theorem 5.1.

(i) *Let $G = H \times C$ where H purely non-abelian and C is* elementary *abelian. Then the subgroup of $\mathrm{Aut}_{Hopf}(DG) \ltimes Z^2_L(DG^*)$ defined by*

$$\underline{\mathrm{Aut}}_{br,L}(DG\text{-mod}) := \{(\phi,\sigma) \in \mathrm{Aut}_{Hopf}(DG) \ltimes \mathrm{Z}^2_L(DG^*) \mid \Psi(\phi,\sigma) \text{ braided } \}$$

has the following decomposition into disjoint double cosets

$$\underline{\mathrm{Aut}}_{br,L}(DG\text{-}mod) = \bigsqcup_{(r,\lambda)\in\mathcal{R}_L/\sim} \underline{\mathcal{V}}_L\underline{\mathcal{B}}_L \cdot (r,\lambda) \cdot \underline{\mathcal{V}}_L\underline{\mathcal{E}}_L$$

Similarly its quotient $\mathrm{Aut}_{br,L}(DG\text{-mod})$ has a decomposition into double cosets

$$\mathrm{Aut}_{br,L}(DG\text{-}mod) = \bigsqcup_{(r,\lambda)\in\mathcal{R}_L/\sim} \mathcal{V}_L\mathcal{B}_L \cdot (r,\lambda) \cdot \mathcal{V}_L\mathcal{E}_L$$

(ii) *Let G be a finite group with not necessary elementary abelian direct factors. For every element $(\phi,\sigma)\in \underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$ there exists a $(r,\lambda)\in \underline{\mathcal{R}}_L$ such that (ϕ,σ) is in*

$$(r,\lambda)\cdot[\underline{\mathcal{B}}_L(\underline{\mathcal{V}}_L\ltimes\underline{\mathcal{E}}_L)]$$

and similarly in the quotient $\mathrm{Aut}_{br,L}(DG\text{-mod})$.

(iii) *Let G be a finite group with not necessary elementary abelian direct factors. For every element $(\phi,\sigma)\in \underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$ there exists a $(r,\lambda)\in \underline{\mathcal{R}}_L$ such that (ϕ,σ) is in*

$$[\underline{\mathcal{V}}_L\ltimes\underline{\mathcal{B}}_L)\underline{\mathcal{E}}_L]\cdot(r,\lambda)$$

and similarly in the quotient $\mathrm{Aut}_{br,L}(DG\text{-mod})$.

Before we turn to the proof, we add some useful facts proven above. In the subsequent section we give examples and discuss how the full Brauer-Picard group is described in this way.

- Ψ from Lemma 2.26 induces a group homomorphism that factorizes to
$$\underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})\to\widetilde{\mathrm{Aut}}_{br}(DG\text{-mod})$$
- Ψ is not necessarily injective, as Example 4.11 shows.
- The group structure of $\widetilde{\mathrm{Aut}}_{br,L}(DG\text{-mod})$ can be almost completely read off using he maps from $\widetilde{\mathcal{V}}_L,\widetilde{\mathcal{B}}_L,\widetilde{\mathcal{E}}_L,\widetilde{\mathcal{R}}_L$ to the known groups (resp. set) $\mathrm{Out}(G),B_{alt},E_{alt},R$ in terms of matrices. Only $\widetilde{\mathcal{B}}_L\to B_{alt}$ is not necessarily a bijection in rare cases (in these cases additional cohomology calculations are necessary to determine the group structure).

Proof of Theorem 5.1.
(i) We start with a general element $(\phi,\sigma)\in\underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$. As in Theorem 3.10 (ii) we write ϕ as a product of elements in V,V_c,B,E,R. Since we only have elementary abelian direct factors the twist ν is zero. The general procedure is to multiply the element (ϕ,σ) with specific elements of $\underline{\mathcal{V}}_L,\underline{\mathcal{B}}_L,\underline{\mathcal{E}}_L$ from both sides in order to simplify the general form. We will use the symbol $\rightsquigarrow$ after an multiplication and warn that the u,v,b,a before and after the multiplication are in general different. Also, as in the proof of Theorem 3.10 we will use the matrix notation with respect to the product $DG=k^G\rtimes kG$ and also with respect to the product $G=H\times C$. For example we write an $v\in\mathrm{Aut}(H\times C)$ as $\left(\begin{smallmatrix}v_{H,H}&v_{C,H}\\v_{H,C}&v_{C,C}\end{smallmatrix}\right)$ and similarly for the u,b,a.
First, it is easy to see that we can find elements $\underline{\mathcal{V}}_L$ such that (ϕ,σ) becomes a pair where the automorphism ϕ has the form

$$\rightsquigarrow\begin{pmatrix}v^*&0\\0&1\end{pmatrix}\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}\hat{p}_H&\delta\\\delta^{-1}&p_H\end{pmatrix}\begin{pmatrix}1&0\\0&w\end{pmatrix}\begin{pmatrix}1&0\\a&1\end{pmatrix}\tag{28}$$

and where the 2-cocycle σ stays the same, since the cocycles in $\underline{\mathcal{V}}_L$ are trivial. Here we used that V normalizes V_c and E. Hence with this step we have eliminated the $\underline{\mathcal{V}}_L\cong\mathrm{Aut}(G)$ parts in ϕ. Further, we use the fact that the subgroup $\mathrm{Aut}_c(G)$ normalizes the subgroup B and arrive at

$$\rightsquigarrow\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}v^*&0\\0&1\end{pmatrix}\begin{pmatrix}\hat{p}_H&\delta\\\delta^{-1}&p_H\end{pmatrix}\begin{pmatrix}1&0\\0&w\end{pmatrix}\begin{pmatrix}1&0\\a&1\end{pmatrix}\tag{29}$$

$$=\begin{pmatrix}v^*\hat{p}_H+b\delta^{-1}+v^*\delta wa+bp_Hwa&v^*\delta w+bp_Hw\\\delta^{-1}+p_Hwa&p_Hw\end{pmatrix}\tag{30}$$

Since (30) together with the 2-cocycle σ is braided we deduce from Lemma 4.4 equation (19) that

$$1 = [\delta \circ p_C \circ w(g)(v \circ p_H \circ w(g))] \cdot [b \circ p_H \circ w(g)(p_H \circ w(g))] \tag{31}$$

for all $g \in G$. In particular for $g = w^{-1}(h)$ with arbitrary $h \in H$:

$$1 = b(h)(h) = b_{H,H}(h)(h) \tag{32}$$

which implies that $b_{H,H}$ is alternating. Further, taking $g = w^{-1}(h, c)$ in (31) we get $\delta(c)(v(h)) = 1$ for all $c \in C, h \in H$, hence $v_{H,C} = 0$. Taking the inverse of (29) and arguing analogously on the inverse matrix we deduce that $a_{H,H}$ is alternating and that $(w^{-1})_{C,H} = 0$ and therefore $w_{C,H} = 0$. Both such alternating $b_{H,H}$ can be trivially extended to alternating $b = \begin{pmatrix} b_{H,H} & 0 \\ 0 & 0 \end{pmatrix}$ on G and similarly for $a_{H,H}$. Now we use Propositions 4.9 (iii) and 4.14 (iii): For these alternating a, b exist 2-cocycles $\beta_b \in Z^2(G)$ and $\alpha_a \in Z^2_L(k^G)$ such that $(b, \beta_b) \in \underline{\mathcal{B}}_L$ and $(a, \alpha_a) \in \underline{\mathcal{E}}_L$. Multiplying equation (29) with the inverses of (b, β_b) and (a, α_a) we simplify equation (29) to

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{33}$$

with $a = \begin{pmatrix} 0 & a_{C,H} \\ a_{H,C} & a_{C,C} \end{pmatrix}$, $b = \begin{pmatrix} 0 & b_{C,H} \\ b_{H,C} & b_{C,C} \end{pmatrix}$, $v = \begin{pmatrix} v_{H,H} & v_{C,H} \\ 0 & v_{C,C} \end{pmatrix}$ and $w = \begin{pmatrix} w_{H,H} & 0 \\ w_{H,C} & w_{C,C} \end{pmatrix}$ where the 2-cocycle σ changes to some 2-cocycle σ'. The b and a can be simplified even further by using the fact that we can construct alternating $\tilde{b} = \begin{pmatrix} 0 & \tilde{b}_{C,H} \\ -b_{H,C} & 0 \end{pmatrix}$ with $\tilde{b}_{C,H}(c)(h) = -1/b_{H,C}(h)(c)$ and similarly an alternating $\tilde{a}$. For these maps there exists again 2-cocycles that lift them to elements in $\underline{\mathcal{B}}_L$ and $\underline{\mathcal{E}}_L$ respectively. As before, we multiplying equation (33) with the inverses of the lifts and get:

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{34}$$

with $a = \begin{pmatrix} 0 & 0 \\ a_{H,C} & a_{C,C} \end{pmatrix}$, $b = \begin{pmatrix} 0 & b_{C,H} \\ 0 & b_{C,C} \end{pmatrix}$, $v = \begin{pmatrix} v_{H,H} & v_{C,H} \\ 0 & v_{C,C} \end{pmatrix}$ and $w = \begin{pmatrix} w_{H,H} & 0 \\ w_{H,C} & w_{C,C} \end{pmatrix}$ Now we commute the matrix corresponding to b to the right as follows:

$$\begin{pmatrix} 1 & \begin{pmatrix} 0 & b_{C,H} \\ 0 & b_{C,C} \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H + \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \begin{pmatrix} 0 & \tilde{b}_{C,H} \\ 0 & \tilde{b}_{C,C} \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} \underbrace{\begin{pmatrix} \begin{pmatrix} 1 & \tilde{b}_{C,H}\delta^{-1} \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 1 \end{pmatrix}}_{\in V_c} \underbrace{\begin{pmatrix} 1 & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & \delta^{-1}\tilde{b}_{C,C}\delta^{-1} \end{pmatrix} & 1 \end{pmatrix}}_{\in E} \tag{37}$$

By commuting the V_c elements in the decomposition to the right, multiplying with V as in the first step and then commuting back we thus arrived at the following form:

$$\rightsquigarrow \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{38}$$

with $a = \begin{pmatrix} 0 & 0 \\ 0 & a_{C,C} \end{pmatrix}$, $v = \begin{pmatrix} v_{H,H} & v_{C,H} \\ 0 & v_{C,C} \end{pmatrix}$ and $w = \begin{pmatrix} w_{H,H} & 0 \\ w_{H,C} & w_{C,C} \end{pmatrix}$. Here we eliminated the $a_{H,C}$ part, similarly as the $b_{C,H}$ part, by commuting the corresponding matrix to the left, past trough the reflection. This gives us again an element in V_c which we can

absorb.
Now consider the inverse of (38):

$$\begin{pmatrix} \hat{p}_H(v^*)^{-1} & \delta \\ -a\hat{p}_H(v^*)^{-1} + w^{-1}\delta^{-1}v^{*-1} & -a\delta + w^{-1}p_H \end{pmatrix}$$

is again braided, hence we use as before Lemma 4.4 equation (19) to get:

$$1 = \delta(p_C(g))(a(\delta(p_C(g))w_{H,C}^{-1}(p_H(g)))) = \delta(g_C)(a_{C,C}(\delta(g_C)))\delta(g_C)(w_{H,C}^{-1}(g_H)) \quad (39)$$

Since this has to hold for all $g = g_H g_C \in H \times C$ we argue as before and get that $a_{C,C}$ is alternating and that $w_{H,C}^{-1} = 0$ and therefore $w_{H,C} = 0$. So we can eliminate the $a_{C,C}$ part by the same arguments as before. Using Lemma 4.4 equation (20) on (38) we deduce: $v_{C,H} = 0$. Since v is diagonal we can commute the matrix to the right through the reflection. We then get a product of a reflection $\delta' = v_{C,C}^* \circ \delta$, $H = H'$ and v. In other words, diagonal elements w.r.t a decomposition $G = H \times C$ of V_c normalize reflections of the form (H, C, δ). We can lift any reflection to an element in $\underline{\mathcal{R}}_L$ according to Proposition 4.22 (iii). Thus we arrive at:

$$\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} w_{H,H} & 0 \\ 0 & w_{C,C} \end{pmatrix} \end{pmatrix} \quad (40)$$

Applying Lemma 4.4 equation (22) on (40) we get that $\chi(g) = \chi(w(g))$ for all $g \in G$, hence $w = \mathrm{id}$. During all of the above multiplications the 2-cocycle σ changed to some cocycle σ' so that now we are left with $(1, \sigma') = (1, \beta)(1, \lambda)(1, \alpha) \in \underline{\mathrm{Aut}}_{br,L}(DG\text{-mod})$. From equation (23) follows that β, α are symmetric and λ is trivial. Therefore we know $(1, \beta) \in \underline{\mathcal{B}}_L$, $(1, \alpha) \in \underline{\mathcal{E}}_L$ and $(1, \lambda) \in \underline{\mathcal{R}}_L$. Hence we can get rid of $(1, \sigma')$ by multiplying with the inverses of the lifts. This proves the claimed decomposition.

(ii) By Theorem 3.10 (iv) we write

$$\phi = \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (41)$$

where we have already eliminated the V element since it normalizes E and every V has a lift to $\underline{\mathcal{V}}_L$. Similarly, we know from Proposition 4.22 that (up to an V that ensures $\delta(c)(\delta(e_{c'})) = \delta_{c,c'}$) every reflection r has a lift $(r, \lambda) \in \underline{\mathcal{R}}_L$. Hence we multiply (ϕ, σ) with the inverse $(r, \lambda)^{-1}$ from the left so that ϕ changes to:

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} v^* + ba & b \\ a & 1 \end{pmatrix} \quad (42)$$

Since this element has to be braided, using Lemma 4.4 equation 19 together with 25 it follows that b is alternating on G. From Lemma 4.4 equation 22 follows that $v = \mathrm{id}_G$ and then that a is alternating. Hence we can construct lifts to $\underline{\mathcal{B}}_L$ and $\underline{\mathcal{E}}_L$ and multiplying with the corresponding inverses just leaves us with $(1, \beta)(1, \lambda)(1, \alpha)$. As in (i) these can be lifted which finally proves the claim.
The proof of (iii) is completely analogous to (ii). □

6. Examples and the full Brauer-Picard group

We now discuss the results of this paper for several classes of groups G. In particular, we compare our results to the examples obtained in [NR14]. In all these cases we verify that the decomposition we proposed in Question 1.1 is also true for the full Brauer-Picard group (i.e. for non-lazy elements).

The approach in [NR14] is to study $\mathrm{Aut}_{br}(DG\text{-mod})$ via its action on a set $\mathbb{L}$ of so-called Lagrangian subcategories $\mathcal{L} \subset DG$-mod, which are parametrized by pairs $(N, [\mu])$ where N is a normal abelian subgroup of G and $[\mu]$ is a G-invariant 2-cohomology class on N. The associated Lagrangian subcategory is explicitly generated as abelian category by the following simple objects:

$$\mathcal{L}_{N,\mu} := \left\langle \mathcal{O}_g^\chi \mid g \in N,\ \chi(h) = \mu(g,h)\mu(h,g)^{-1}\ \forall h \in N \right\rangle$$

Now it is proven in [NR14] Prop. 7.6 that the *orbit* $\mathbb{L}_0 \subset \mathbb{L}$ of the standard Lagrangian subcategory $\mathcal{L}_{1,1} = \langle \mathcal{O}_1^\chi \rangle = \mathrm{Rep}(G)$ under the action of $\mathrm{Aut}_{br}(DG\text{-mod})$ is characterized by being the set of all Lagrangian subcategories equivalent to $\mathrm{Rep}(G)$, i.e. $\mathrm{Aut}_{br}(DG\text{-mod})$ acts transitively on all $\mathcal{L}_{N,\mu} \cong \mathrm{Rep}(G)$. Moreover it is shown in Cor. 6.9 and Lm. 6.10 that the stabilizer of the standard Lagrangian subcategory $\mathcal{L}_{1,1}$ is the image of the injective group homomorphism

$$\mathrm{Ind}_{\mathrm{Vect}_G} :\ \mathrm{Aut}_{mon}(\mathrm{Vect}_G) \to \mathrm{Aut}_{br}(DG\text{-mod})$$

$$\mathrm{Aut}_{mon}(\mathrm{Vect}_G) \cong \mathrm{Out}(G) \ltimes \mathrm{H}^2(G, k^\times)$$

In each example class of groups G below we determine our lazy subgroups $\mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L, \mathcal{V}_L$ and present them acting on $\mathbb{L}$ in the context of [NR14]. We then give the full Brauer-Picard group in the known examples and show that in each case the decomposition we proposed in Question 1.1 does indeed hold also in the non-lazy case.

For the first examples we will explicitly calculate the action of the groups $\mathrm{Ind}_{\mathrm{Vect}_G}, \mathrm{Ind}_{\mathrm{Rep}(G)}$ and the partial dualizations r_N on DG-mod in terms of the simple objects $\mathcal{O}_g^\chi$.

To discuss the later examples involving some larger groups we sketch at this point some formula to calculate the action of (non-lazy) partial dualizations N and $\mathrm{Ind}_{\mathrm{Rep}(G)}$ if applied to $\mathcal{O}_1^\chi$ and hence we obtain the image of $\mathcal{L}_{1,1} = \mathrm{Rep}(G)$. These considerations are not necessary for the first examples and are just an outlook on the general theory to be more thoroughly treated in a future article:

6.1. **General considerations on non-lazy reflections.** In [BLS15] we assume to be given a finite-dimensional Hopf algebra H and a split Hopf algebra map $H \to A$ with Hopf kernel K (i.e. $H = K \rtimes A$ is a Radford biproduct). From this data we construct a different Hopf algebra $r(H)$ with a split Hopf algebra map $r(H) \to A^*$ and discuss its properties and relations to H, most significant we have a category equivalence DH -mod $\cong Dr(H)$ -mod.

For the present article, suppose now $r(H) \cong H$ or $\cong H^*$, then we obtain thus braided autoequivalences of DH-mod. We now spell this property out for $H = kG$ a group ring: By construction, $r(H) \cong \Omega(K) \rtimes A^*$ where $\Omega_A : DA\text{-mod} \to DA^*\text{-mod}$ is a functor essentially interchanging H-action and coaction. Thus whenever $G = N \rtimes Q$ is a semidirect product (with nontrivial action) we would obtain a nontrivial coaction on $\Omega_A(K)$ and hence a non-cocommutative Hopf algebra $kN \rtimes k^Q$; this example was discussed in [BLS15]. *However*, iff N is abelian, then $r(H)$ is isomorphic to the dual group ring k^G. We thus obtain a braided autoequivalence

$$r_N :\ DG\text{-mod} \overset{r_{G\to Q}}{\longrightarrow} D\left(kN \rtimes k^Q\right)\text{-mod} \overset{\Omega_H}{\longrightarrow} D\left(k^N \rtimes kQ\right)\text{-mod} \cong DG\text{-mod}$$

We now wish to describe this functor explicitly on $\mathcal{O}_g^\chi$: Let $G = N \rtimes Q$ be a given decomposition into a semidirect product with N abelian, assume further we can fix an isomorphism $\delta : k^N \to kN$ invariant under G-conjugation, then we wish calculate the action of a (non-lazy) partial dualization $r_N(\mathcal{O}_1^\chi) = \mathcal{O}_g^\rho$ from [BLS15], where χ is an irreducible representation. Since r_N is a monoidal autoequivalence, we have necessarily ρ irreducible as well as for reasons of dimensionality $\dim(\chi) = |[g]| \cdot \dim(\rho)$; note that the forgetful functor is preserved.

Clifford theory states that for an arbitrary normal subgroup $N \subset G$ the restriction of an irreducible character $\chi|_N$ decomposes into a direct sum of irreducible N-characters

$$\chi = e \sum_{i=1}^{t} \chi_i$$

where the multiplicity e is a natural number and where the χ_i form a G-orbit under conjugation action on N and hence on $\mathrm{Rep}(N)$. The subgroups $I_i \subset G/N$ fixing one χ_i are called inertia subgroups, they are conjugate to each other and $[G/N : I_i] = t$.

Now since N is assumed abelian we obtain 1-dimensional representations $\chi_i \in \hat{N}$ forming a G-conjugacy class. Then $n_i := \delta(\chi_i)$ are group elements in N and the form a single conjugacy class in G. We now indeed prove that this is the coaction on $r_{(N,\delta)}(\mathcal{O}_1^\chi)$. Let $M = \bigoplus_{j=0}^{t} M_j \otimes v_j k$ be the decomposition of the representation χ. Then for any $m_j \otimes v_j \in V$ we have:

$$\begin{aligned} e_{n_i}.r_{(N,\delta)}(m_j \otimes v_j) &= \delta(e_{n_i}).(m_j \otimes v_j) \\ &= m_j \otimes \chi_j(\delta(e_{n_i}))v_j \\ &= m_j \otimes e_{n_i}(\delta(\chi_j))v_j \\ &= m_j \otimes e_{n_i}(n_j)v_j \\ &= \delta_{i,j} \cdot m_j \otimes v_j \end{aligned}$$

Now we turn to the action of the centralizer of n_i, which decomposes $\mathrm{Cent}(n_i) = N \rtimes I_i$. The representation M has the isotypical component M_i of dimension e, and since we have a semidirect product decomposition, we may restrict this representation to I_i and again extend trivially to $\mathrm{Cent}(n_i)$. Overall we get

$$r_{C,\delta} : \ \mathcal{O}_1^\chi \longmapsto \mathcal{O}_{[n_1]}^{M_i|_{I_i}}$$

6.2. **General considerations on non-lazy induction.** We now turn to the subgroups of $\mathrm{Aut}_{br}(DG\text{-mod})$ defined to be the images of the functors

$$\mathrm{Ind}_{\mathrm{Vect}_G} : \ \mathrm{Aut}_{mon}(\mathrm{Vect}_G) \to \mathrm{Aut}_{br}(DG\text{-mod})$$

$$\mathrm{Ind}_{\mathrm{Rep}(G)} : \ \mathrm{Aut}_{mon}(\mathrm{Rep}(G)) \to \mathrm{Aut}_{br}(DG\text{-mod})$$

We already know from [NR14]

$$\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) = \mathrm{Aut}_{mon}(\mathrm{Vect}_G) = \mathrm{Out}(G) \ltimes \mathrm{H}^2(G, k^\times)$$

The subgroup $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)})$ is much harder to compute. The group $\mathrm{Aut}_{mon}(\mathrm{Rep}(G))$ is studied in [Dav01]: He considered pairs (N, α) of an abelian subgroup with a G-invariant cohomology class $[\alpha]$. Then a subset of these pairs is in bijective correspondence with all $F_{N,\alpha} \in \mathrm{Aut}_{mon}(\mathrm{Rep}(G))$. The subset contains all pairs where α is G-invariant even as an 2-cocycle (these are the lazy cases).

Remark 6.1. *An interesting counterexample is* $G = 2^n \rtimes \mathrm{Sp}_{2n}$ *which has a pair* (N, α) *such that the associated functor is a monoidal equivalence*

$$F_{N,\alpha}:\ 2^n \rtimes \mathrm{Sp}_{2n}\text{-mod} \xrightarrow{\sim} 2^n.\mathrm{Sp}_{2n}\text{-mod}$$

These groups are only for $n = 1$ *isomorphic, namely both to* $\mathbb{S}_4$*, which leads in particular to a nontrivial (and non-lazy) monoidal autoequivalence, see below.*

We need to determine for any $F \in \mathrm{Aut}_{mon}(\mathcal{C})$, $\mathcal{C} = \mathrm{Rep}(G)$ the action of the image

$$E_F := \mathrm{Ind}_{\mathrm{Rep}(G)}(F) \in \mathrm{Aut}_{br}(DG\text{-mod})$$

The functor $\mathrm{Ind}_{\mathcal{C}}$ relies on the isomorphism $\mathrm{BrPic}(\mathcal{C}) \to \mathrm{Aut}_{br}(Z(\mathcal{C}))$ in [ENO09]. This is unfortunately not a very explicit isomorphism. In [NR14] formulae (16),(17) it is worked out on the level of objects and then applied for Vect_G, but for $\mathrm{Rep}(G)$ it seems hard to explicitly compute the image object from this. We can easily derive at least an "equation" in the sense that we derive a necessary condition on $\mathcal{O}, \mathcal{O}' \in Z(\mathcal{C})$ which is necessarily true whenever for

$$E_F(\mathcal{O}) = \mathcal{O}'$$

By [ENO09] we consider $\mathcal{M} = \mathcal{C}$ as a right module category over $\mathcal{C}$ via $\bullet \otimes V$ and a left module category over $\mathcal{C}$ via $F(V) \otimes \bullet$. The map E_F is then constructed by the fact that on both sides the pairs $(V, c) \in Z(\mathcal{C})$ of objects and half-braidings (i.e. comodule structure) act as bimodule category morphism on the bimodule category $\mathcal{M}$. The half-braidings determine the coherence transformations of the bimodule category morphism. On level of objects we simply have the necessary condition:

$$E_F(V, c) = (V', c') \quad \Rightarrow \quad F(V) \otimes X \cong X \otimes V' \quad \forall X \in \mathcal{M}$$

Thus $F(V) \cong V'$ and the functor E_F factorizes to F under the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$. In our particular case $\mathcal{C} = \mathrm{Rep}(G)$ this implies:

$$E_F(\mathcal{O}_g^\chi) = \mathcal{O}_{g'}^{\chi'} \quad \Rightarrow \quad F(\mathrm{Ind}_{\mathrm{Cent}(g)}^G(\chi)) \cong \mathrm{Ind}_{\mathrm{Cent}(g')}^G(\chi')$$

Thus, possible images of E_F are determined by the character table of G and induction-restriction table with $\mathrm{Cent}(g), \mathrm{Cent}(g')$. We continue for the special case $g = 1$ to determine the possible images $E_F(\mathcal{O}_1^\chi)$ and hence $E_F(\mathcal{L}_{1,1})$. Our formula above implies:

$$F(\chi) = \mathrm{Ind}_{\mathrm{Cent}(g')}^G(\chi')$$

In particular $\mathrm{Ind}_{\mathrm{Cent}(g')}^G(\chi')$ has to be irreducible.

Question 6.2. *Derive a closed formula at least for* $E_F(\mathcal{O}_1^\chi)$*. It seems there is a very similar formula to the previous section involving Clifford theory (simply omit the restriction to* I_i*).*

6.3. **Elementary abelian groups.** For $G = \mathbb{F}_p^n$ a finite vector space we already known directly

$$\mathrm{Aut}_{br}(DG\text{-mod}) = \begin{cases} O_{2n}(\mathbb{F}_p), & p \neq 2 \\ Sp_{2n}(\mathbb{F}_p), & p = 2 \end{cases}$$

For abelian groups, all 2-cocycles over DG are lazy and the results of this article gives a product decomposition of $\mathrm{BrPic}(\mathrm{Rep}(G))$. The lazy subgroups we defined are in this example:

- $\widetilde{\mathcal{V}}_L \cong \mathrm{Out}(G) = \mathrm{GL}_n(\mathbb{F}_p)$.
- $\widetilde{\mathcal{B}}_L \cong B_{alt} \cong \mathbb{F}_p^{\binom{n}{2}}$ as additive group.
- $\widetilde{\mathcal{E}}_L \cong E_{alt} \cong \mathbb{F}_p^{\binom{n}{2}}$ as additive group.

The set $\mathcal{R}_L/\sim$ consists of $n+1$ representatives $r_{[C]}$, one for each possible dimension d of a direct factor $\mathbb{F}_p^d \cong C \subset G$, and $r_{[C]}$ is an actual reflection on the subspace C with a suitable monoidal structure determined by the pairing λ. Especially the generator $r_{[G]}$ conjugates $\widetilde{\mathcal{B}}_L$ and $\widetilde{\mathcal{E}}_L$.

In the case the double coset decomposition is a variant of the Bruhat decomposition of $O_{2n}(\mathbb{F}_p)$ of type D_n for $2 \nmid p$ resp. $Sp_{2n}(\mathbb{F}_2)$ of type C_n. More precisely, one takes the Bruhat decomposition with respect to the parabolic subsystem of type A_{n-1} with Levi subgroup $\widetilde{\mathcal{V}}_L = GL_n(\mathbb{F}_p)$ and parabolic subgroup $\widetilde{\mathcal{V}}_L\,\widetilde{\mathcal{B}}_L$.

We now discuss how this example acts on Lagrangian subspaces in the sense of [NR14]: $\mathbb{L}_0 = \mathbb{L}$ is parametrized by pairs $(N, [\mu])$ where N is a subvector space of $\mathbb{F}_p^n$ and $[\mu]$ is uniquely defined by an alternating bilinear form $\langle , \rangle_\mu$ on N. Choosing a complement $\mathbb{F}_p^n = N \oplus N'$ we have

$$\mathcal{L}_{N,[\mu]} = \left\{ \mathcal{O}_g^{\chi_{N'} \langle g, - \rangle} \right\}$$

where $g \in N$ and $\chi_{N'} \in \hat{N}'$ are free. The action of our subgroups are as follows:

- Elements in $\widetilde{\mathcal{V}}_L = \mathrm{Out}(G) = \mathrm{GL}_n(\mathbb{F}_p)$ act in the obvious way.
- Partial dualization $r_N \in \mathcal{R}_L$ on N maps $\mathcal{L}_{1,1}$ to $\mathcal{L}_{N,1}$.
- Any $[\beta] \in \mathrm{H}^2(G, k^\times) \cong G \wedge G \subset \widetilde{\mathcal{B}}_L$ acts by

$$\mathcal{O}_g^\chi \mapsto \mathcal{O}_g^{\chi \cdot \langle g, - \rangle_\beta}$$

  Especially it stabilizes $\mathcal{L}_{1,1}$ and sends $\mathcal{L}_{N,1} \mapsto \mathcal{L}_{N,\beta}$.
- $\widetilde{\mathcal{E}}_L \cong \widetilde{\mathcal{B}}_L$ acts similarly by a 2-cocycle $\alpha \in H^2(k^G) \cong \mathrm{H}^2(G, k^\times)$ which we write accordingly as $\alpha_1 \wedge \alpha_2 \in kG \otimes kG$:

$$\mathcal{O}_g^\chi \mapsto \mathcal{O}_{g \cdot \alpha_1 \chi(\alpha_2)}^\chi$$

  In particular in our case it sends $\mathcal{L}_{1,1}$ to $\mathcal{L}_{N,1}$ with N the subspace of $\mathbb{F}_p^n$ generated by α. Note that in our case $\widetilde{\mathcal{E}}_L$ is not necessary to generate $\mathrm{Aut}_{br}(DG\text{-mod})$, since it is conjugate to $\mathcal{B}_L$ via the full dualization r_G.

6.4. **Simple groups.** Let G be a simple group, then our result returns

- $\widetilde{\mathcal{V}}_L = \mathrm{Out}(G)$
- $\widetilde{\mathcal{B}}_L = \hat{G}_{ab} \wedge \hat{G}_{ab} = 1$, where it is a known result, that simple groups have no distinguished 2-cocycles.
- $\widetilde{\mathcal{E}}_L = Z(G) \wedge Z(G) = 1$
- $\mathcal{R}_L = 1$

hence the only *lazy* autoequivalences are induced by outer automorphisms.

On the other hand by [NR14] we have no normal abelian subgroups except $\{1\}$ and hence the only Lagrangian subcategory is $\mathcal{L}_{1,1}$ and the stabilizer is $\mathrm{Out}(G) \ltimes \mathrm{H}^2(G, k^\times)$ is equal to $\mathrm{Aut}_{br}(DG\text{-mod})$.

Observe that in this example we obtain also a decomposition of the full Brauer-Picard group and our Question 1.1 is answered positively: Namely, $\mathrm{Aut}_{br}(DG\text{-mod})$ is equal the image of the induction $\mathrm{Ind}_{\mathrm{Vect}_G}$, while the other subgroups are trivial.

6.5. **Lie groups and quasisimple groups.** Lie groups over finite fields $G(\mathbb{F}_q), q = p^k$ have (with small exceptions) the property $G_{ab} = 1$ and there are no semidirect factors. On the other hand depending on the choice they may contain a nontrivial center $Z(G)$. This is comparable to their complex counterpart, where the center of the simple-connected form $Z(G_{sc}(\mathbb{C}))$ is equal to the fundamental group $\pi_1(G_{ad}(\mathbb{C}))$ of the adjoint form with no center $Z(G_{ad}(\mathbb{C})) = 1$. In exceptional cases for q, the maximal central extension may be larger than π_1. Similarly central extensions of the sporadic groups G may be considered; all these groups appear in any insolvable group as part of the Fitting group.

Definition 6.3. *A group G is called* quasisimple *if it is a perfect central extension of a simple group:*

$$Z \to G \to H \qquad Z = Z(G), [G,G] = G$$

As long as $\mathrm{H}^2(Z, \mathbb{C}^\times) = 1$, e.g. because Z is cyclic, there is no difference to the simple case (in [NR14] however there are more abelian normal subgroups to consider). *Nontrivial* $\widetilde{\mathcal{E}}_L$-terms appear as soon as $\mathrm{H}^2(Z, \mathbb{C}^\times) \neq 1$. This is only the case for $D_n(\mathbb{F}_q) = \mathrm{SO}_{2n}(\mathbb{F}_q)$ with $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and in some exceptional cases as follows. For example the last exceptional cover G is a subgroup of the monster.

Z	$\to G \to$	H	$\widetilde{\mathcal{E}}_L$
$\mathbb{Z}_2 \times \mathbb{Z}_2$		$D_n(\mathbb{F}_q)$	$\mathbb{Z}_2$
$\mathbb{Z}_4 \times \mathbb{Z}_4$		$A_2(\mathbb{F}_{2^2})$	$\mathbb{Z}_4$
$\mathbb{Z}_3 \times \mathbb{Z}_3$		${}^2A_3(\mathbb{F}_{3^2})$	$\mathbb{Z}_3$
$\mathbb{Z}_2 \times \mathbb{Z}_2$		${}^2A_5(\mathbb{F}_{2^2})$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$		${}^2B_2(\mathbb{F}_{2^3})$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$		${}^2E_6(\mathbb{F}_{2^2})$	$\mathbb{Z}_2$

$\mathrm{Out}(H)$ typically consists of scalar- and Galois-automorphisms of the base field $\mathbb{F}_q$, extended by the group of diagram automorphisms; in particular for D_4 it involves the triviality automorphisms $\mathbb{S}_3$. Note further that any automorphism on G preserves the center Z, hence it factors to an automorphism in H. The kernel of this group homomorphism $\mathrm{Out}(G) \to \mathrm{Out}(H)$ is trivial, since all elements in Z are products of commutators of G elements, yielding $\mathrm{Out}(G) \cong \mathrm{Out}(H)$.

Claim 6.4. *For the groups G above our result returns*

- $\widetilde{\mathcal{V}}_L = \mathrm{Out}(H)$
- $\widetilde{\mathcal{B}}_L = \hat{H} \wedge \hat{H} = 1$
- $\widetilde{\mathcal{E}}_L = \mathbb{Z}_n \wedge \mathbb{Z}_n = \mathbb{Z}_n$ *with $n \in \{2, 3, 4\}$ as indicated in the table.*
- $\mathcal{R}_L = 1$, *as there are no direct factors.*

Hence the only lazy *braided autoequivalences are the elements in $\widetilde{\mathcal{E}}_L$.*

We now turn our attention to the full Brauer-Picard group using the methods in [NR14]: The only normal divisors are $N = 1, \mathbb{Z}_n \subset Z, Z$ and all 2-cocycles are trivial except for the nondegenerate cocycle $\alpha^k, 1 \leq k \leq n$ on $Z = \mathbb{Z}_n \times \mathbb{Z}_n$.

We now observe that $\mathcal{L}_{\mathbb{Z}_n,1}, \mathcal{L}_{Z,1} \ncong \mathrm{Rep}(G)$, because the 1-dimensional (invertible) objects in $\mathrm{Rep}(G)$ are $\mathcal{O}_g^\chi$ for all $\chi|_N = 1$, but both $\mathcal{L}_{\mathbb{Z}_n,1}, \mathcal{L}_{Z,1}$ contain in addition all such $\mathcal{O}_{z,\chi}$ for a central element. The only possible remaining elements in $\mathbb{L}_0$ are the Lagrangian subcategories $\mathcal{L}_{1,1}, \mathcal{L}_{Z,\alpha}$.

But α defined on the central normal subgroup is a nondegenerate G-invariant 2-cocycle on Z in the sense of [Dav01], so we obtain the lazy 2-cocycles $\alpha^k \in Z^2_L(k^G)$ and thus our nontrival $\widetilde{\mathcal{E}}_L$:

$$\mathcal{L}_{1,1} \overset{(a,\alpha)^k\in\mathcal{E}_L}{\longleftrightarrow} \mathcal{L}_{Z,\alpha^k}$$

Hence the Brauer-Picard group factorizes into the stabilizer of $\mathcal{L}_{1,1}$, which is $\mathrm{Out}(G) \ltimes \mathrm{H}^2(G,k^\times)$, and $\widetilde{\mathcal{E}}_L = \langle (a,\alpha)\rangle$.

Claim 6.5. *The decomposition we proposed in Question 1.1 is also true for the full Brauer-Picard group of the G above. More precisely*

$$\mathrm{BrPic}(\mathrm{Rep}(G)) = \mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) \cdot \mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) \cdot \mathcal{R}$$
$$\mathbb{S}_3 = \mathrm{Out}(G) \ltimes \mathrm{H}^2(G,k^\times) \cdot \mathbb{Z}_n \cdot 1$$

- $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) = \mathrm{H}^2(G,k^\times)$, *in addition to the lazy case.*
- $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) = \widetilde{\mathcal{E}}_L \cong \mathbb{Z}_n$
- *No reflections, as there is no semidirect decomposition of G.*

6.6. **Symmetric group $\mathbb{S}_3$.**

Claim 6.6. *For $G = \mathbb{S}_3$ our result returns*

- $\widetilde{\mathcal{V}}_L = \mathrm{Out}(\mathbb{S}_3) = 1$
- $\widetilde{\mathcal{B}}_L = \hat{\mathbb{S}}_3 \wedge \hat{\mathbb{S}}_3 = \mathbb{Z}_2 \wedge \mathbb{Z}_2 = 1$
- $\widetilde{\mathcal{E}}_L = Z(\mathbb{S}_3) \wedge Z(\mathbb{S}_3) = 1$
- $\mathcal{R}_L = 1$, *as there are no direct factors.*

Hence there are no lazy *braided autoequivalences.*

We now turn our attention to the full Brauer-Picard group of $\mathbb{S}_3$ which was computed in [NR14] Sec. 8.1: We have the Lagrangian subcategories $\mathcal{L}_{1,1}, \mathcal{L}_{\langle(123)\rangle,1}$ and stabilizer $\mathrm{Out}(\mathbb{S}_3) \ltimes \mathrm{H}^2(\mathbb{S}_3,k^\times) = 1$. Hence $\mathrm{Aut}_{br}(D\mathbb{S}_3\text{-mod}) = \mathbb{Z}_2$.

Claim 6.7. *The decomposition we proposed in Question 1.1 is also true for the full Brauer-Picard group of $\mathbb{S}_3$. More precisely*

$$\mathrm{BrPic}(\mathrm{Rep}(\mathbb{S}_3)) = \mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) \cdot \mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) \cdot \mathcal{R}$$
$$\mathbb{Z}_2 = 1 \cdot 1 \cdot \mathbb{Z}_2$$

- $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) = 1$
- $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) = 1$
- *Reflections $\mathbb{Z}_2$, generated by the partial dualizations r_N on the semidirect decomposition $\mathbb{S}_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ with abelian normal subgroup $N = \mathbb{Z}_3$. More precisely r interchanges $\mathcal{L}_{1,1}, \mathcal{L}_{\langle(123)\rangle,1}$, the action on O_g^χ is made explicit in the proof.*

Proof. First, $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G})$ is the stabilizer $\mathrm{Out}(\mathbb{S}_3) \ltimes \mathrm{H}^2(\mathbb{S}_3,k^\times) = 1$. Second, [Dav01] states that $\mathrm{Aut}_{mon}(\mathrm{Rep}(G))$ is a subset of the set of pairs consisting of a abelian normal subgroup and a *nondegenerate* G-invariant cohomology class on this subgroup. The only nontrivial normal abelian subgroup for $\mathbb{S}_3$ is cyclic and hence there is no such pair, thus $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) = 1$.

As an initial example, we want to calculate the effect of the non-lazy partial dualization r on the decomposition $\mathbb{S}_3 = \langle(123)\rangle \rtimes \langle(12)\rangle$. Choose an isomorphism $k^{\mathbb{Z}_3} \cong k\mathbb{Z}_3$ by fixing ζ a primitive 3rd root unity:

$$e_{(123)} \mapsto \frac{1}{3}\left(1 + \zeta(123) + \zeta^2(132)\right)$$

Let $V \in \mathrm{Rep}(\mathbb{S}_3)$ and denote the irreducible representations by $triv, sgn, ref$. We determine the comodule structure on $r(\mathcal{O}_1^V)$ by calculating the action of the projectors:

$$\begin{aligned}
e_{(12)}|_{r(\mathcal{O}_1^V)} &= e_{(12)}|_{\mathcal{O}_1^V} = 0 \\
e_{(123)}|_{r(\mathcal{O}_1^V)} &= \frac{1}{3}\left(1+\zeta(123)+\zeta^2(132)\right)|_{\mathcal{O}_1^V} \\
e_{(132)}|_{r(\mathcal{O}_1^V)} &= \frac{1}{3}\left(1+\zeta^2(123)+\zeta(132)\right)|_{\mathcal{O}_1^V} \\
e_{(123)}, e_{(132)}|_{r(\mathcal{O}_1^{triv})} &= \frac{1}{3}\left(1+\zeta+\zeta^2\right) = 0 \\
e_{(123)}, e_{(132)}|_{r(\mathcal{O}_1^{sgn})} &= \frac{1}{3}\left(1+\zeta+\zeta^2\right) = 0 \\
e_{(123)}|_{r(\mathcal{O}_1^{ref})} &= \frac{1}{3}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\zeta\begin{pmatrix}0&-1\\1&-1\end{pmatrix}+\zeta^2\begin{pmatrix}-1&1\\-1&0\end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix}1-\zeta & \zeta-\zeta^2\\ -\zeta+\zeta^2 & 1-\zeta^2\end{pmatrix} \\
e_{(132)}|_{r(\mathcal{O}_1^{ref})} &= \frac{1}{3}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\zeta^2\begin{pmatrix}0&-1\\1&-1\end{pmatrix}+\zeta\begin{pmatrix}-1&1\\-1&0\end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix}1-\zeta^2 & \zeta^2-\zeta\\ -\zeta^2+\zeta & 1-\zeta\end{pmatrix}
\end{aligned}$$

On $r(\mathcal{O}_1^{ref})$ we see that $e_{(132)}, e_{(132)}$ are projectors to the subspaces

$$r(\mathcal{O}_1^{ref}) = \begin{pmatrix}-\zeta^2\\1\end{pmatrix}k \oplus \begin{pmatrix}-\zeta\\1\end{pmatrix}k$$

and hence as comodule $r(\mathcal{O}_1^{ref}) = \mathcal{O}_{(123)}^{\chi}$. To determine χ we check the action of the generator $(123) \in \mathrm{Cent}((123))$

$$(123)|_{r(\mathcal{O}_1^{ref})} = \left(e_1 + \zeta e_{(123)} + \zeta^2 e_{(132)}\right)|_{\mathcal{O}_1^{ref}} = 1$$

Hence the partial dualization r maps

$$\mathcal{L}_{1,1} = \left\{\mathcal{O}_1^{triv},\ \mathcal{O}_1^{sgn},\ \mathcal{O}_1^{ref}\right\} \longmapsto \mathcal{L}_{(123),1} = \left\{\mathcal{O}_1^{triv},\ \mathcal{O}_1^{sgn},\ \mathcal{O}_{(123)}^1\right\}$$

Even though the proof is finished, we interpret the result in terms of the general considerations in Section 6.1: The Clifford decomposition of the restrictions $triv|_N, sgn|_N, ref|_N$ to $N = \mathbb{Z}_3$ is $1, 1, \zeta \oplus \zeta^2$ respectively. In the last case $\mathbb{Z}_2$ is acting by interchanging the summands (resp. by Galois action), the inertia group being trivial. We get hence also in this way

$$r(\mathcal{O}_1^{ref}) = \mathcal{O}_{(123)}^1$$

□

6.7. **Symmetric group $\mathbb{S}_4$.** Let $V \in \mathrm{Rep}(\mathbb{S}_4)$ and denote the irreducible representations by $triv, sgn, ref2, ref3, ref3 \otimes sgn$ and denote the unique abelian normal subgroup by $N = \{1, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Claim 6.8. *For $G = \mathbb{S}_4$ our result returns*

- $\widetilde{\mathcal{V}}_L = \mathrm{Out}(\mathbb{S}_4) = 1$
- $\widetilde{\mathcal{B}}_L = \hat{\mathbb{S}}_4 \wedge \hat{\mathbb{S}}_4 = \mathbb{Z}_2 \wedge \mathbb{Z}_2 = 1$
- $\widetilde{\mathcal{E}}_L = Z(\mathbb{S}_4) \wedge Z(\mathbb{S}_4) = 1$
- $\mathcal{R}_L = 1$, *as there are no direct factors.*

Hence there are no lazy *braided autoequivalences.*

We now turn our attention to the full Brauer-Picard group of $\mathbb{S}_4$ which was computed in [NR14] Sec. 8.2: We have three Lagrangian subcategories $\mathcal{L}_{1,1}, \mathcal{L}_{N,1}, \mathcal{L}_{N,\mu}$ for $N = \{1, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and stabilizer $\mathrm{Out}(\mathbb{S}_4) \ltimes \mathrm{H}^2(\mathbb{S}_4, k^\times) = \mathbb{Z}_2$. In particular $\mathrm{Aut}_{br}(D\mathbb{S}_4\text{-mod})$ has order 6. One checks, that the nontrivial $[\beta] \in \mathrm{H}^2(\mathbb{S}_4, k^\times)$ restricts to the nontrivial $[\mu]$ on N, hence

$$[\beta]: \; \mathcal{L}_{1,1}, \mathcal{L}_{N,1}, \mathcal{L}_{N,\mu} \longmapsto \mathcal{L}_{1,1}, \mathcal{L}_{N,\mu}, \mathcal{L}_{N,1}$$

and by order and injectivity $\mathrm{Aut}_{br}(D\mathbb{S}_4\text{-mod}) \cong \mathbb{S}_3$.

Claim 6.9. *The decomposition we proposed in Question 1.1 is also true for the full Brauer-Picard group of* $\mathbb{S}_4$*. More precisely*

$$\mathrm{BrPic}(\mathrm{Rep}(\mathbb{S}_4)) = \mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) \cdot \mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) \cdot \mathcal{R}$$
$$\mathbb{S}_3 = \mathbb{Z}_2 \cdot \mathbb{Z}_2 \cdot \mathbb{Z}_2$$

- $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G}) = \mathbb{Z}_2$ *generated by the nontrivial cohomology class* $[\beta]$ *of* $\mathbb{S}_4$ *with action on* $\mathbb{L}_0$ *described above. Note that* $[\beta]$ *restricts to the unique nontrivial cohomology class on* N*.*
- $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(G)}) = \mathbb{Z}_2$ *generated by the non-lazy monoidal autoequivalence* F *of* $\mathrm{Rep}(\mathbb{S}_4)$*, described in detail in the last section of* [Dav01]*. More precisely, we show that the image* $E_F \in \mathrm{Aut}_{br}(DG\text{-mod})$ *interchanges* $\mathcal{L}_{1,1}, \mathcal{L}_{\langle(123)\rangle,\mu}$*.*
- *Reflections* $\mathcal{R} \cong \mathbb{Z}_2$*, generated by the reflection* r *on the semidirect decomposition* $\mathbb{S}_4 = N \rtimes \mathbb{S}_3$ *with abelian kernel* N*. More precisely* r *interchanges* $\mathcal{L}_{1,1}, \mathcal{L}_{\langle(123)\rangle,1}$*.*

Proof. The stabilizer $\mathrm{im}(\mathrm{Ind}_{\mathrm{Vect}_G})$ and its action on $\mathbb{L}_0$ has already been calculated. To compute $\mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(\mathbb{S}_4)})$ note that $\mathrm{Aut}_{mon}(\mathrm{Rep}(\mathbb{S}_4))$ has been explicitly computed in the last section of [Dav01]: Since there is a single nontrivial normal subgroup $N = \mathbb{Z}_2 \times \mathbb{Z}_2$ and a single nondegenerate 2-cocycle α on N, which is G-invariant *only* as a cohomology class $[\alpha]$. He shows that it gives in fact rise to a (non-lazy) monoidal autoequivalence F of $\mathrm{Rep}(\mathbb{S}_4)$ interchanging

$$ref3 \leftrightarrow ref3 \otimes sgn \qquad [(12)] \leftrightarrow [(1234)]$$

visible as a symmetry of the character table.

We now compute the effect of $E_F \in \mathrm{im}(\mathrm{Ind}_{\mathrm{Rep}(\mathbb{S}_4)})$ in particular for all $\mathcal{O}_1^\chi$. First, $\chi = triv, sgn, ref2$ restrict to a trivial representation on N and are hence fixed. Secondly, the possible images

$$E_F(\mathcal{O}_1^{ref3}) = \mathcal{O}_g^\chi, \quad E_F(\mathcal{O}_1^{ref3 \otimes sgn}) = \mathcal{O}_{g'}^{\chi'}$$

belong to the G-conjugacy classes in N, i.e. $g, g' = 1$ or $g, g' = (12)(34)$. Also they have to fulfill the characterization outlined in general considerations on non-lazy induction (section 6.2):

$$F(ref) = ref \otimes sgn \stackrel{!}{=} \mathrm{Ind}_{\mathrm{Cent}(g)}^G(\chi)$$
$$F(ref \otimes sgn) = ref \stackrel{!}{=} \mathrm{Ind}_{\mathrm{Cent}(g')}^G(\chi')$$

Now $g = g = 1$ would imply $E_F(\mathcal{L}_0) = \mathcal{L}_0$ and thus E_F in the stabilizer, which is $\mathrm{Out}(\mathbb{S}_4) \ltimes \mathrm{H}^2(\mathbb{S}_4, k^\times)$, but this is not possible since E_F acts nontrivial on objects, not induced by an automorphism. Hence we have to solve

$$F(ref) = ref \otimes sgn \stackrel{!}{=} \mathrm{Ind}_{\mathrm{Cent}(12)(34)}^G(\chi)$$

$$F(ref \otimes sgn) = ref \overset{!}{=} \mathrm{Ind}_{\mathrm{Cent}(12)(34)}^{G}(\chi')$$

where $\mathrm{Cent}(12)(34) = \langle (12), (13)(24) \rangle \cong \mathbb{D}_4$ and the character table quickly returns the only possible χ, χ' by restriction

$$E_F(ref) = \mathcal{O}_{(12)(34)}^{--} \qquad E_F(ref \otimes sgn) = \mathcal{O}_{(12)(34)}^{+-}$$

We see that $\chi|_N$ and $\chi'|_N$ are nontrivial, hence in $\mathcal{L}_{N,\mu}$ for μ nontrivial and

$$E_F:\ \mathcal{L}_{1,1} = \left\{ \mathcal{O}_1^{triv}, \mathcal{O}_1^{sgn}, \mathcal{O}_1^{ref2}, \mathcal{O}_1^{ref3}, \mathcal{O}_1^{ref3\otimes sgn}, \right\}$$

$$\longmapsto \mathcal{L}_{N,\mu} = \left\{ \mathcal{O}_1^{triv}, \mathcal{O}_1^{sgn}, \mathcal{O}_1^{ref3}, \mathcal{O}_{[(12)(34)]}^{--}, \mathcal{O}_{[(12)(34)]}^{+-}, \right\}$$

We finally calculate the action of the partial dualization r on the decomposition $\mathbb{S}_4 = N \rtimes \mathbb{S}_3$. The general considerations in Section 6.1 imply the following for the images $r(\mathcal{O}_1^{\chi})$: Since $\chi = triv, sgn, ref2$ restrict to the trivial representation on N, these are fixed. For $\chi = ref3, \chi' = ref3 \otimes sgn$ the restrictions are easily determined by the character table to both be

$$\chi|_N = \chi'|_N = (-+) \oplus (+-) \oplus (--)$$

which returns via $\delta : k^N \to kN$ precisely the conjugacy class $[(12)(34)]$ and the inertia subgroup is $I = N \rtimes \langle (12) \rangle$. To see the action on the centralizer, we restrict the representations χ, χ' to I and extend it trivially to $I = \mathrm{Cent}(12)(34) = \langle (12), (13)(24) \rangle \cong \mathbb{D}_4$ yielding finally:

$$r(\mathcal{O}_1^{ref}) = \mathcal{O}_{(12)(34)}^{++} \qquad r(\mathcal{O}_1^{ref\otimes sgn}) = \mathcal{O}_{(12)(34)}^{-+}$$

$$r:\ \mathcal{L}_{1,1} = \left\{ \mathcal{O}_1^{triv}, \mathcal{O}_1^{sgn}, \mathcal{O}_1^{ref2}, \mathcal{O}_1^{ref3}, \mathcal{O}_1^{ref3\otimes sgn}, \right\}$$

$$\longmapsto \mathcal{L}_{N,1} = \left\{ \mathcal{O}_1^{triv}, \mathcal{O}_1^{sgn}, \mathcal{O}_1^{ref3}, \mathcal{O}_{[(12)(34)]}^{++}, \mathcal{O}_{[(12)(34)]}^{-+}, \right\}$$

□

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References

[ABM12] A. L. Agore, C. G. Bontea, G. Militaru, *Classifying Bicrossed Products of Hopf Algebras*, arXiv:1205.6110v4 [math.QA]

[BC04] J. Bichon, G. Carnovale, *Lazy cohomology: an analogue of the Schur multiplier for arbitrary Hopf algebras*, arXiv:math/0410263v2 [math.QA]

[Bich04] J. Bichon, *Galois and Bigalois objects over monomial non semisimple Hopf algebras*, arXiv:math/0404059 [math.QA]

[BLS15] A. Barvels, S. Lentner, C. Schweigert, *Partially dualized Hopf algebras have equivalent Yetter-Drinfel'd modules*, J. Algebra 430 (2015), 303-342, arXiv:1402.2214 [math.QA]

[BN14] C.-G. Bontea, D. Nikshych: On the Brauer-Picard group of a finite symmetric tensor category, Preprint (2014), arXiv:1408.6445v1 [math.QA]

[CP04] J. Cuadra, F. Panaite, *Extending lazy 2-cocycles on Hopf algebras and lifting projective representations afforded by them*, arXiv:math/0412425v1 [math.QA]

[Dav01] A. A. Davydov, *Galois Algebras and Monoidal Functors between Categories of Representations of Finite Groups*, J. Algebra 244 (2001), 273-301.

[Dav10] A. A. Davydov, *Modular invariants for group-theoretical modular data. I*, J. Algebra 323 (2010) ,13211348, arXiv:0908.1044 [math.QA]

[DN12] A. A. Davydov, D. Nikshych, *The Picard crossed module of a braided tensor category*, Preprint (2012), arXiv:1202.0061.

[DW90] R. Dijkgraaf, E. Witten, *Topological Gauge Theories and Group Cohomology*, Commun. Math. Phys. 129 (1990), 393

[ENO09] P. Etingof, D. Nikshych, V. Ostrik: Fusion categories and homotopy theory, Preprint (2009), arXiv:0909.3140v2 [math.QA]

[FMM14] B. Femic, A. Mejia Castaño, M. Mombelli, *Invertible bimodule categories over the representation category of a Hopf algebra*, J. Pure Appl. Algebra 218 (2014) 2096-2118, arXiv:1402.2955 [math.QA]

[FS15] J. Fuchs, C. Schweigert, *Symmetries and defects in three-dimensional topological field theory*, Preprint (2015), arXiv:1501.01885v1 [hep-th]

[FPSV14] J. Fuchs, J. Priel, C. Schweigert, A. Valentino, *On the Brauer groups of symmetries of abelian Dijkgraaf-Witten theories*, arXiv:1404.6646v1 [hep-th]

[FSV13] J. Fuchs, C. Schweigert, A. Valentino, *Bicategories for boundary conditions and for surface defects in 3-d TFT*, Commun. Math. Phys. 321 (2013) 543-575, arXiv:1203.4568v3 [hep-th]

[FSV14] J. Fuchs, C. Schweigert, A. Valentino, *A geometric approach to boundaries and surface defects in Dijkgraaf-Witten theories*, Commun. Math. Phys. 332 (2014) 981-1015, arXiv:1307.3632v3 [hep-th]

[FQ93] D. Freed, F. Quinn, *Chern-Simons theory with finite gauge group* Commun.Math.Phys. 156:435-472, 1993, arXiv:hep-th/9111004

[GM10] C. Galindo, M. Medina, *On the classification of Galois objects for finite groups* Comm. Algebra 40, No. 7 (2012), 2577-2595, arXiv:1006.3872v1 [math.QA]

[Higgs87] R. J. Higgs,*Projective characters of degree one and the inflation-restriction sequence*, J. Austral. Math. Soc. (Series A) 46 (1989), 272-280.

[HS13] I. Heckenberger, H.-J. Schneider: *YetterDrinfeld modules over bosonizations of dually paired Hopf algebras*, Advances in Mathematics 244 (2013), 354-394.

[Kass94] C. Kassel, *Quantum groups*, Springer-Verlag GTM 155, Heidelberg 1994

[KaW07] A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. 1 (2007) 1-236, arXiv:hep-th/0604151

[Keil13] M. Keilberg, *Automorphisms of the doubles of purely non-abelian finite groups*, arXiv:1311.0575v3 [math.QA]

[KS14] M. Keilberg, P. Schauenburg, *On tensor factorizations of Hopf algebras*, Preprint (2014), arXiv:1410.6290 [math.RA]

[Mom12] M. Mombelli, *The Brauer-Picard group of the representation category of finite supergroup algebras, The Brauer-Picard group of the representation category of finite supergroup algebras*, Rev. Unin Mat. Argent. Vol. 55 No. 1 (2014) 83-117, arXiv:1202.6238v3 [math.QA]

[Mo13] J. C. Morton, *Cohomological twisting of 2-linearization and extended TQFT* J. Homotopy and Related Structures 8 1-61 arXiv:1003.5603 [math.QA]

[NR14] D. Nikshych, B. Riepel, *Categorical Lagrangian Grassmannians and Brauer-Picard groups of pointed fusion categories*, J. Algebra 411 (2014), 191-214.

[O03] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups 8 (2003), no. 2, 177206.

[Schau96] P. Schauenburg, *Hopf bigalois extensions*, Comm. Algebra 24, No. 12 (1996), 3797-3825.

[Schau91] P. Schauenburg, *Hopf-Galois and Bi-Galois Extensions*, Unpublished, http://schauenburg.perso.math.cnrs.fr/papers/hgbgt.pdf

[Schau02] P. Schauenburg, *Hopf bimodules, coquasibialgebras, and an exact sequence of Kac*, Adv. Math. 165 (2002), 194-263.

[Win72] D. L. Winter, *The automorphism group of an extraspecial p-group*, Rocky Mountain J. Math. 2 (1972), no. 2, 159-168.

[ZZ13] H. Zhu, Y. Zhang, *Braided autoequivalences and quantum commutative bi-Galois objects*, Journal of Pure and Applied Algebra 219 Issue 9 (2015), 4144-4167.