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Abstract

For given integers k and r, the Folkman number $f(k;r)$ is the smallest number of vertices in a graph G which contains no clique on $k+1$ vertices, yet for every partition of its edges into r parts, some part contains a clique of order k. The existence (finiteness) of Folkman numbers was established by Folkman (1970) for $r=2$ and by Nešetřil and Rödl (1976) for arbitrary r, but these proofs led to very weak upper bounds on $f(k;r)$.

Recently, Conlon and Gowers and independently the authors obtained a doubly exponential bound on $f(k;2)$. Here, we establish a further improvement by showing an upper bound on $f(k;r)$ which is exponential in a polynomial of k and r. This is comparable to the known lower bound $2^{\Omega(rk)}$. Our proof relies on a recent result of Saxton and Thomason (or, alternatively, on a recent result of Balogh, Morris, and Samotij) from which we deduce a quantitative version of Ramsey's theorem in random graphs.

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1 Introduction

For two graphs, G and F, and an integer $r \geq 2$ we write $G \to (F)_r$ if every r-coloring of the edges of G results in a monochromatic copy of F. By a copy we mean here a subgraph of G isomorphic to F. Let K_k stand for the complete graph on k vertices and let $R(k;r)$ be the r-color Ramsey number, that is, the smallest integer n such that $K_n \to (K_k)_r$. As it is customary, we suppress $r = 2$ and write $R(k) := R(k;2)$ as well as $G \to F$ for $G \to (F)_2$.

In 1967 Erdős and Hajnal [8] asked if for some l, $k+1 \leq l < R(k)$, there exists a graph G such that $G \to K_k$ and $G \not\supset K_l$. Graham [12] answered this question in positive for $k = 3$ and $l = 6$ (with a graph on eight vertices), and Pósa (unpublished) for $k = 3$ and $l = 5$. Folkman [10] proved, by an explicit construction, that such a graph exists for every $k \geq 3$ and $l = k+1$. He also raised the question to extend his result for more than two colors, since his construction was bound to two colors.

For integers k and r, a graph G is called *$(k;r)$-Folkman* if $G \to (K_k)_r$ and $G \not\supset K_{k+1}$. We define the *r-color Folkman number* for K_k by

$$f(k;r) = \min\{n \in \mathbb{N} \colon \exists \ G \text{ such that } |V(G)| = n \text{ and } G \text{ is } (k;r)\text{-Folkman}\}.$$

For $r = 2$ we set $f(k) := f(k;2)$. It follows from [10] that $f(k)$ is well defined for every integer k, i.e., $f(k) < \infty$. This was extended by Nešetřil and Rödl [17], who showed that $f(k;r) < \infty$ for an arbitrary number of colors r.

Already the determination of $f(3)$ is a difficult, open problem. In 1975, Erdős [7] offered max(100 dollars, 300 Swiss francs) for a proof or disproof of $f(3) < 10^{10}$. For the history of improvements of this bound see [5], where a computer assisted construction is given yielding $f(3) < 1000$. For general k, the only previously known upper bounds on $f(k)$ come from the constructive proofs in [10] and [17]. However, these bounds are tower functions of height polynomial in k. On the other hand, since $f(k) \geq R(k)$, it follows by the well known lower bound on the Ramsey number that $f(k) \geq 2^{k/2}$, which for $k = 3$ was improved to $f(3) \geq 19$ [19].

We prove an upper bound on $f(k;r)$ which is exponential in a polynomial of k and r. Set $R := R(k;r)$ for the r-color Ramsey number for K_k. It is known that there exists some $c > 0$ such that for every $r \geq 2$ and $k \geq 3$ we have

$$2^{crk} < R < r^{rk}.$$

The upper bound already appeared in the work of Skolem [25]. The lower bound obtained from a random r-coloring of the complete graphs is of the form $r^{k/2}$. However, Lefmann [14] noted that the simple inequality $R(k; s+t) \geq (R(k;s)-1)(R(k;t)-1)+1$ yields a lower bound of the form $2^{kr/4}$. Using iteratively random 3-colorings in this "product-type" construction yields a slightly better lower bound of the form $3^{rk/6}$. Our main result establishes an upper bound on the Folkman number $f(k;r)$ of similar order of magnitude.

Theorem 1. *For all integers $r \geq 2$ and $k \geq 3$,*

$$f(k;r) \leq k^{400k^4} R^{40k^2} \leq 2^{c(k^4 \log k + k^3 r \log r)}.$$

for some $c > 0$ independent of r and k.

To prove Theorem 1, we consider a random graph $G(n,p)$, $p = Cn^{-\frac{2}{k+1}}$, where $n = n(k,r)$ and $C = C(n,k,r)$ and carefully estimate from below the probabilities $\mathbb{P}(G(n,p) \to (K_k)_r)$ and $\mathbb{P}(G(n,p) \not\supset K_{k+1})$, so that their sum is strictly greater than 1. The latter probability is easily bounded by the FKG inequality. However, to set a bound on $\mathbb{P}(G(n,p) \to (K_k)_r)$ we rely on a recent general result of Saxton and Thomason [24], elaborating on ideas of Nenadov and Steger [15] (see Remark 3).

Remark 1. Instead of the Saxton-Thomason theorem, we could have used a concurrent result of Balogh, Morris, and Samotij [1], which, by using our method, yields only a slightly worse upper bound on the Folkman numbers $f(k;r)$ than Theorem 1 (the k^4 in the exponent has to be replaced k^6).

Remark 2. In a related paper [23], we combined ideas from [9, 20, 22] and, for $r = 2$, obtained another proof of the Ramsey threshold theorem that yields a self-contained derivation of a double-exponential bound for the two-color Folkman numbers $f(k)$. Independently, a similar double-exponential bound for $f(k;r)$, for $r \geq 2$, was obtained by Conlon and Gowers [2] by a different method.

Motivated by the original question of Erdős and Hajnal, one can also define, for $r = 2$, $k \geq 3$, and $k+1 \leq l \leq R(k)$, *a relaxed Folkman number* as

$$f(k,l) = \min\{n : \exists \; G \text{ such that } |V(G)| = n, \quad G \to K_k \text{ and } G \not\supset K_l\}.$$

Note that $f(k,k+1) = f(k)$. As mentioned above, Graham [12] found out that $f(3,6) = 8$, while Nenov [16] and Piwakowski, Radziszowski and Urbański [18] determined that $f(3,5) = 15$ (see also [26]). Of course, the problem is easier when the difference $l - k$ is bigger. Our final result provides an exponential bound of the form $f(k,l) \leq \exp\{-ck\}$, when l is close to but bigger than $4k$ (the constant c is proportional to the reciprocal of the difference between l/k and 4).

Theorem 2. *For every $0 < \alpha < \frac{1}{4}$ there exists k_0 such that for k and l satisfying $k \geq k_0$ and $k \leq \alpha l$,*

$$f(k;l) \leq 2^{4k/(1-4\alpha)}.$$

It would be interesting to decide if the true order of the logarithm of $f(k,k+1) = f(k)$ is also linear in k.

The paper is organized as follows. In the next section we prove our main result, Theorem 1, while Theorem 2 is proved in Section 3. Finally, a short Section 4 offers a brief discussion of the analogous problem for hypergraphs.

Most logarithms in this paper are binary and are denoted by log. Only occasionally, when citing a result from [24] (Theorem 5 in Section 2 below), we will use the natural logarithms, denoted by ln.

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2 Proof of Theorem 1

We will prove Theorem 1 by the probabilistic method. Let $G(n,p)$ be the binomial random graph, where each of the $\binom{n}{2}$ possible edges is present, independently, with probability p. We are going to show that for every $n \geq k^{40k^4}R^{10k^2}$ and a suitable function $p = p(n)$, with positive probability, $G(n,p)$ has simultaneously two properties: $G(n,p) \to (K_k)_r$ and $G(n,p) \not\supset K_{k+1}$. Of course, this will imply that there exists an $(k;r)$-Folkman graph on n vertices. We begin with a simple lower bound on $\mathbb{P}(G(n,p) \not\supset K_{k+1})$.

Lemma 3. *For all $k,n \geq 3$, and $C > 0$, if $p = Cn^{-2/(k+1)} \leq \frac{1}{2}$ then*

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp(-C^{\binom{k+1}{2}}n)\,.$$

Proof. By applying the FKG inequality (see, e.g., [13, Theorem 2.12 and Corollary 2.13]), we obtain the bound

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) \geq \left(1 - p^{\binom{k+1}{2}}\right)^{\binom{n}{k+1}} \geq \exp\left(-2C^{\binom{k+1}{2}}n^{-k}\binom{n}{k+1}\right) > \exp\left(-C^{\binom{k+1}{2}}n\right)\,,$$

where we also used the inequalities $\binom{n}{k+1} < n^{k+1}/2$ and $1 - x \geq e^{-2x}$ for $0 < x < \frac{1}{2}$. $\square$

The main ingredient of the proof of Theorem 1 traces back to a theorem from [20] establishing edge probability thresholds for Ramsey properties of $G(n,p)$. A special case of that result states that for all integers $k \geq 3$ and $r \geq 2$ there exists a constant C such that if $p = p(n) \geq Cn^{-\frac{2}{k+1}}$ then $\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (K_k)_r) = 1$.

Adapting an idea of Nenadov and Steger [15] (see Remark 3 for more on that), and based on a result of Saxton and Thomason [24], we obtain the following quantitative version of the above random graph theorem. Recall our notation $R = R(k;r)$ for the r-color Ramsey number and notice an easy lower bound

$$R(k;r) > 2r \tag{1}$$

valid for all $r \geq 2$ and $k \geq 3$ (just consider a factorization of K_{2r}).

Lemma 4. *For all integers $r \geq 2$, $k \geq 3$, and*

$$n \geq k^{400k^4}R^{40k^2}, \tag{2}$$

the following holds. Set

$$b = \frac{1}{2R^2}, \quad C = 2^{5\sqrt{\log n \log k}} R^{16}, \quad \textit{and} \quad p = Cn^{-\frac{2}{k+1}}. \tag{3}$$

Then

$$\mathbb{P}(G(n,p) \to (K_k)_r) \geq 1 - \exp\left(-bp\tbinom{n}{2}\right).$$

We devote the next two subsections to the proof of Lemma 4. Now, we deduce Theorem 1 from Lemmas 3 and 4.

Proof of Theorem 1. For given r and k, let n be as in (2), and let b, C, and p be as in (3). Below we will show that these parameters satisfy not only the assumptions of Lemma 4, but also the assumption $p \leq \frac{1}{2}$ of Lemma 3, as well as an additional inequality

$$n \geq (3/b)^{\frac{k+1}{k-1}} C^{\binom{k+2}{2}}. \tag{4}$$

With these two inequalities at hand, we may quickly finish the proof of Theorem 1. Indeed, (4) implies that

$$bp\binom{n}{2} \geq \frac{1}{3}bpn^2 = (b/3)Cn^{1+\frac{k-1}{k+1}} \overset{(4)}{\geq} C^{\binom{k+1}{2}}n \tag{5}$$

which, by Lemma 3, implies in turn that

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left(-bp\tbinom{n}{2}\right).$$

Since, by Lemma 4,

$$\mathbb{P}(G(n,p) \to (K_k)_r) \geq 1 - \exp\left(-bp\tbinom{n}{2}\right),$$

we conclude that

$$\mathbb{P}(G(n,p) \to (K_k)_r \text{ and } G(n,p) \not\supset K_{k+1}) > 0.$$

Thus, there exists a $(k;r)$-Folkman graph on n vertices, and thus, $f(k) \leq k^{400k^4} R^{40k^2}$.

It remains to show that $p \leq \frac{1}{2}$ and that (4) holds. The first inequality is equivalent to

$$n \geq (2C)^{\frac{k+1}{2}}. \tag{6}$$

We will now show that this inequality is a consequence of (4) and then establish (4) itself. Since $C > 2$ and $3/b \overset{(3)}{=} 6R^2 \geq 1$, we infer that

$$(3/b)^{\frac{k+1}{k-1}} C^{\binom{k+2}{2}} \geq C^{\binom{k+2}{2}} \geq (2C)^{\frac{k+1}{2}},$$

and hence, (6) indeed follows from (4).

Finally, we establish (4). In doing so we will use again the identity $3/b \overset{(3)}{=} 6R^2$, as well as the inequalities $36 \le C$, which follows from (2) and (3), $\binom{k+2}{2} \le k^2+1 \le 2k^2-1$, and $\frac{k+1}{k-1} \le 2$, valid for all $k \ge 3$. The R-H-S of (4) can be bounded from above by

$$(6R^2)^{\frac{k+1}{k-1}} C^{\binom{k+2}{2}} \le 36R^4 C^{\binom{k+2}{2}} \le R^4 C^{k^2+2} \le 2^{10k^2\sqrt{\log n \log k}} R^{20k^2}.$$

Hence, it suffices to show that

$$n \ge 2^{10k^2\sqrt{\log n \log k}} R^{20k^2}. \tag{7}$$

Observe that, by (2), $\frac{1}{2}\log n \ge 20k^2 \log R$, and thus, it remains to check that

$$\frac{1}{2}\log n \ge 10k^2\sqrt{\log n \log k},$$

or equivalently that

$$\log n \ge 400k^4 \log k.$$

This, however, follows trivially from (2). □

2.1 The proof of Lemma 4 – preparations

In this and the next subsection we present a proof of Lemma 4, which is inspired by the work of Nenadov and Steger [15] and is based on a recent general result of Saxton and Thomason [24] on the distribution of independent sets in hypergraphs. For a hypergraph H, a subset $I \subseteq V(H)$ is *independent* if the subhypergraph $H[I]$ induced by I in H has no edges.

For an h-graph H, the degree $d(J)$ of a set $J \subset V(H)$ is the number of edges of H containing J. (Since in our paper letter r is reserved for the number of colors, we will use h for hypergraph uniformity.) We will write $d(v)$ for $d(\{v\})$, the ordinary vertex degree. We further define, for a vertex $v \in V(H)$ and $j = 2, \dots, h$, the maximum j-degree of v as

$$d_j(v) = \max\left\{ d(J) : v \in J \subset \binom{V(H)}{j} \right\}.$$

Finally, the co-degree function of H with a formal variable τ is defined in [24] as

$$\delta(H,\tau) = \frac{2^{\binom{h}{2}-1}}{nd} \sum_{j=2}^{h} \frac{\sum_v d_j(v)}{2^{\binom{j-1}{2}} \tau^{j-1}}, \tag{8}$$

where the inner sum is taken over all vertices $v \in V(H)$ and d is the average vertex degree in H, that is, $d = \frac{1}{n}\sum_v d(v)$.

Theorem 5 below is an abridged version of [24, Corollary 3.6, page 14], where we suppress part of conclusion (a) (about the sets T_i), as well as the "Moreover" part therein, since we do not use this additional information here. Most importantly, we provide an explicit value of the constant $c = c(h)$ as $864(h!)^3h$. Indeed, it follows from the calculations in [24, page 32] that

$$c \le \frac{288(h!)^2h}{\ln 1/\varepsilon}\left(1 + \frac{\ln \varepsilon}{\ln(1 - 1/(2(h!)))}\right) \le \frac{288(h!)^2h}{\ln 1/\varepsilon}\,(1 + 2(h!)\ln 1/\varepsilon) \le 864(h!)^3h,$$

since $\ln(1-x) \le -x$ for $x < 1$. In part (c) of the theorem below, for convenience, we switch from ln to log, but only on the R-H-S of the upper bound on $\ln|\mathcal{C}|$.

Theorem 5 (Saxton & Thomason, [24])**.** *Let H be an h-graph on vertex set $[n]$ and let ε and τ be two real numbers such that $0 < \varepsilon < 1/2$,*

$$\tau \le 1/(144(h!)^2h) \qquad \textit{and} \qquad \delta(H,\tau) \le \varepsilon/(12(h!)).$$

Then there exists a collection $\mathcal{C}$ of subsets of $[n]$ such that the following three properties hold.

(a) For every independent set I in H there exists a set $C \in \mathcal{C}$ such that $I \subset C$.

(b) For all $C \in \mathcal{C}$, we have $e(H[C]) \le \varepsilon e(H)$.

(c) $\ln|\mathcal{C}| \le c\log(1/\varepsilon)\tau\log(1/\tau)n$, where $c = 864(h!)^3h$.

We will now tailor the above result to our application. The hypergraphs we consider have a very symmetric structure. Given k and n, let $H(n,k)$ be the hypergraph with vertex set $\binom{[n]}{2}$, the edges of which correspond to all copies of K_k in the K_n with vertex set $[n]$. Thus, $H(n,k)$ has $\binom{n}{2}$ vertices, $\binom{n}{k}$ edges, and is $\binom{k}{2}$-uniform and $\binom{n-2}{k-2}$-regular.

For $J \subseteq \binom{[n]}{2}$, the degree of J in $H(n,k)$ is $d(J) = \binom{n-v_J}{k-v_J}$, where v_J is the number of vertices in J treated as a graph on $[n]$ rather than a subset of vertices of $H(n,k)$. Thus, over all J with $|J| = j$, $d(J)$ is maximized by the smallest possible value of v_J, that is, when $v_J = l_j$, the smallest integer l such that $j \le \binom{l}{2}$. Consequently, for every vertex v of $H(n,k)$ (that is, an edge of K_n on $[n]$) and for each $j = 2, \ldots, \binom{k}{2}$, we have

$$d_j(v) = \binom{n - l_j}{k - l_j}.$$

Clearly, $l_j \ge 3$ for $j \ge 2$, which will be used later.

Let

$$\delta(n,k,\tau) := \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4}k^{k-2}}{\tau^{j-1}n^{l_j-2}}.$$

The co-degree function of $H(n,k)$ can be bounded by $\delta(n,k,\tau)$.

Claim 6.

$$\delta(H(n,k),\tau) \le \delta(n,k,\tau).$$

Proof. By the definition of $\delta(H,\tau)$ in (8) with h replaced by $\binom{k}{2}$, n by $\binom{n}{2}$, d by $\binom{n-2}{k-2}$, $d_j(v)$ by $\binom{n-l_j}{k-l_j}$, and with $2^{\binom{j-1}{2}}$ dropped out from the denominator, we have

$$\delta(H(n,k),\tau) \le 2^{k^4} \sum_{j=2}^{\binom{k}{2}} \frac{\binom{n-l_j}{k-l_j}}{\tau^{j-1}\binom{n-2}{k-2}}.$$

Now, observe that $\frac{\binom{n-l_j}{k-l_j}}{\binom{n-2}{k-2}} \le (k/n)^{l_j-2}$ and $l_j \le k$. □

The most important property of hypergraph $H(n,k)$ is that a subset S of the vertices of H corresponds to a graph G with vertex set $[n]$ and edge set S, and S is an independent set in $H(n,k)$ if and only if the corresponding graph G is K_k-free. We now apply Theorem 5 to $H(n,k)$.

Corollary 7. *Let $k \ge 3$, $n \ge 3$, and let ϵ and τ be two real numbers such that $0 < \varepsilon < 1/2$,*

$$\tau \le \left(k^2!\right)^{-2} \qquad \text{and} \qquad \delta(n,k,\tau) \le \frac{\varepsilon}{k^2!}\,. \tag{9}$$

Then there exists a collection $\mathcal{C}$ of subgraphs of K_n such that the following three properties hold.

- (*a*) *For every K_k-free graph $G \subseteq K_n$ there exists a graph $C \in \mathcal{C}$ such that $G \subset C$.*
- (*b*) *For all $C \in \mathcal{C}$, C contains at most $\varepsilon\binom{n}{k}$ copies of K_k.*
- (*c*) $\ln|\mathcal{C}| \le (2k^2)!\log(1/\varepsilon)\tau\log(1/\tau)\binom{n}{2}$.

Proof. Note that for $k \ge 3$,

$$k^2! > 12\tbinom{k}{2}! \quad \text{and, consequently,} \quad (k^2!)^2 > 144\tbinom{k}{2}!\tbinom{k}{2},$$

and that, by Claim 6, $\delta(H(n,k),\tau) \le \delta(n,k,\tau)$. Thus, the assumptions of Theorem 5 hold for $H := H(n,k)$ with $h = \binom{k}{2}$, and its conclusions (a-c) translate into the corresponding properties (a-c) of Corollary 7. Finally, notice that

$$(2k^2)! > c = 864\left(\tbinom{k}{2}!\right)^3\tbinom{k}{2}.$$

□

In the next subsection we deduce Lemma 4 from Corollary 7. First, however, we make a simple observation about the number of monochromatic copies of K_k in every coloring of K_n. Recall that $R = R(k;r)$ is the r-color Ramsey number for K_k and set

$$\alpha = \binom{R}{k}^{-1}. \tag{10}$$

Proposition 8. *Let $n \geq R$. For every $(r+1)$-coloring of the edges of K_n either there are more than $\frac{\alpha}{2}\binom{n}{k}$ monochromatic copies of K_k colored by the first r colors, or more than $\frac{1}{R^2}\binom{n}{2}$ edges receive color $r+1$.*

Proof. Consider an $(r+1)$-coloring of the edges of K_n. Let $x\binom{n}{R}$ be the number of the R-element subsets of the vertices of K_n with no edge colored by color $r+1$. By the definition of R, each of these subsets induces in K_n a monochromatic copy of K_k. Thus, counting repetitions, there are at least

$$x\frac{\binom{n}{R}}{\binom{n-k}{R-k}} = x\frac{\binom{n}{k}}{\binom{R}{k}} = x\alpha\binom{n}{k}$$

monochromatic copies of K_k colored by one of the first r colors. Suppose that their number is at most

$$\frac{\alpha}{2}\binom{n}{k}.$$

Then $x \leq \frac{1}{2}$, that is, at least a half of the R-element subsets of $V(K_n)$ contain at least one edge colored by $r+1$. Hence, color $r+1$ appears on at least

$$\frac{\frac{1}{2}\binom{n}{R}}{\binom{n-2}{R-2}} = \frac{\frac{1}{2}\binom{n}{2}}{\binom{R}{2}} > \frac{1}{R^2}\binom{n}{2}$$

edges of K_n. This completes the proof. □

2.2 Proof of Lemma 4 – details

Let $r \geq 2$, $k \geq 3$, and let n, b, C, and p be as in Lemma 4, see (3) and (2). We have to show that

$$\mathbb{P}(G(n,p) \to (K_k)_r) \geq 1 - \exp(-bp\tbinom{n}{2}).$$

First we set up a few auxiliary constants required for the application of Corollary 7. Recalling that α is defined in (10), let

$$\varepsilon = \frac{\alpha}{2r}, \tag{11}$$

$$C_0 = 2^{4\sqrt{\log n}}R^{10/k}, \quad \text{and} \quad \tau = C_0 n^{-\frac{2}{k+1}}. \tag{12}$$

We will now prove that the above defined constants ε and τ satisfy the assumptions of Corollary 7.

Claim 9. *Inequalities (9) hold true for every $k \geq 3$.*

Proof. In order to verify the first inequality in (9), note that by the definitions of τ and C_0 in (12) and the obvious bound $x! < x^x$,

$$(k^2!)^2\tau \leq k^{4k^2}2^{4\sqrt{\log n}}R^{10/k}n^{-\frac{2}{k+1}}. \tag{13}$$

It remains to show that the R-H-S of (13) is smaller than one, or, by taking logarithms, that

$$4k^2 \log k + 4\sqrt{\log n} + \frac{10}{k} \log R < \frac{2}{k+1} \log n.$$

This, however, follows from

$$4\sqrt{\log n} < \frac{1}{k+1} \log n,$$

or equivalently,

$$16(k+1)^2 < \log n,$$

and from

$$4k^2(k+1) \log k + \frac{10}{k}(k+1) \log R < \log n,$$

both of which are true by the lower bound on n in (2).

To prove the second inequality in (9), note that since $\tau \le 1$ and $j \le \binom{l_j}{2}$, the quantity $\tau^{j-1} n^{l_j-2}$ is minimized when $j = \binom{l_j}{2}$. Thus, we have

$$\tau^{j-1} \cdot n^{l_j-2} \ge \tau^{\binom{l_j}{2}-1} \cdot n^{l_j-2} = C_0^{\binom{l_j}{2}-1} n^{-\frac{(l_j-2)(l_j+1)}{k+1}+l_j-2} = C_0^{\binom{l_j}{2}-1} n^{\frac{(l_j-2)(k-l_j)}{k+1}}. \tag{14}$$

Recall that for $j \ge 2$ we have $l_j \ge 3$. In what follows we obtain a lower bound on the R-H-S of (14) by distinguishing two cases: $l_j < k$ and $l_j = k$. If $l_j < k$, then $(l_j - 2)(k - l_j)$ is minimized for $l_j = 3$ and $l_j = k-1$ and owing to $C_0 > 1$ we infer

$$\tau^{j-1} \cdot n^{l_j-2} \overset{(14)}{\ge} C_0^{\binom{l_j}{2}-1} n^{\frac{(l_j-2)(k-l_j)}{k+1}} > n^{\frac{k-3}{k+1}} \overset{(2)}{\ge} k^{80k^4} R^{8k^2},$$

where we also used the bound $\frac{k+1}{k-3} \le 5$ for all $k \ge 4$, which holds due to $3 \le l_j < k$. If, on the other hand, $l_j = k$, then, by the definition of C_0 in (12) and the bound on n in (2),

$$C_0 \ge 2^{80k^2} R^{10/k}. \tag{15}$$

Hence, in view of (15), and the fact that $\binom{k}{2} - 1 \ge \frac{1}{5}k^2$ for $k \ge 3$, we have that

$$\tau^{j-1} \cdot n^{l_j-2} \overset{(14)}{\ge} C_0^{\binom{k}{2}-1} \ge \left(2^{80k^2} R^{10/k}\right)^{k^2/5} = 2^{16k^4} R^{2k}.$$

Consequently, using the trivial bounds $k^k \cdot k^2! < 2^{15k^4}$, $\binom{R}{k} < R^k$, and $R^k \overset{(1)}{>} r$, we conclude that

$$\sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{\tau^{j-1} n^{l_j-2}} \le \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{2^{16k^4} R^{2k}} \le \frac{k^k}{2^{15k^4} R^{2k}} \le \frac{1}{2r\binom{R}{k} \cdot k^2!} \overset{(10),(11)}{=} \frac{\varepsilon}{k^2!},$$

which concludes this proof. □

In view of Claim 9, the conclusions of Corollary 7 hold true with ε and τ defined in, resp., (11) and (12). That is, there exists a collection $\mathcal{C}$ of subgraphs of K_n such that Properties (a), (b), and (c) of Corollary 7 are satisfied for these specific values of ε and τ.

To continue with the proof of Lemma 4 consider a random graph $G(n,p)$ and let $\mathcal{E}$ be the event that $G(n,p) \not\to (K_k)_r$. For each $G \in \mathcal{E}$, there exists an r-coloring $\varphi\colon E(G) \to [r]$ yielding no monochromatic copy of K_k. (Further on we will call such a coloring *proper*.) In other words, there are K_k-free graphs $G_1, \ldots, G_r$, defined by $G_i = \varphi^{-1}(i)$, such that $G_1 \uplus \ldots \uplus G_r = G$. According to Property (a) of Corollary 7, for every $i \in [r]$ there exists a graph $C_i \in \mathcal{C}$ such that $G_i \subseteq C_i$. Consequently,

$$G \cap \left(K_n \setminus \bigcup_{i=1}^{r} C_i \right) = \emptyset.$$

Notice that there are only at most $|\mathcal{C}|^r$ distinct graphs $K_n \setminus \bigcup_{i=1}^r C_i$. Moreover, we next show that all these graphs are dense (see Claim 10). Hence, as it is extremely unlikely for a random graph $G(n,p)$ to be completely disjoint from one of the few given dense graphs, it will ultimately follow that $\mathbb{P}(\mathcal{E}) = o(1)$.

Claim 10. *For all $C_1, \ldots, C_r \in \mathcal{C}$,*

$$|K_n \setminus \bigcup_{i=1}^{r} C_i| \geq \binom{n}{2} / R^2.$$

Proof. The graphs C_i, $i \in [r]$, together with $K_n \setminus \bigcup_{i=1}^r C_i$, form an $(r+1)$-coloring of K_n, more precisely, an $(r+1)$-coloring where, for each $i = 1, \ldots, r$, the edges of color i are contained in C_i, while all edges of $K_n \setminus \bigcup_{i=1}^r C_i$ are colored with color $r+1$. (Note that this coloring may not be unique, as the graphs C_i are not necessarily mutually disjoint.) By Proposition 8, this $(r+1)$-coloring yields either more than $(\alpha/2)\binom{n}{k}$ monochromatic copies of K_k in the first r colors or more than $\binom{n}{2}/R^2$ edges in the last color. Since for each $i \in [r]$, the i-th color class is contained in C_i, it follows from Property (b) that there at most

$$r \cdot \varepsilon \binom{n}{k} \overset{(11)}{=} \frac{\alpha}{2} \binom{n}{k}$$

monochromatic copies of K_k in the first r colors. Consequently, we must have

$$|K_n \setminus \bigcup_{i=1}^{r} C_i| > \frac{1}{R^2} \binom{n}{2}, \tag{16}$$

which concludes the proof. □

Based on Claim 10 we can now bound $\mathbb{P}(\mathcal{E}) = \mathbb{P}(G(n,p) \not\to (K_s)_r)$ from above.

Claim 11.

$$\mathbb{P}(G(n,p) \not\to (K_s)_r) \leq |\mathcal{C}|^r \exp\left\{ -\frac{p\binom{n}{2}}{R^2} \right\}$$

Proof. Let $\mathcal{F}$ be the event that $G(n,p) \cap (K_n \setminus \bigcup_{i=1}^r C_i) = \emptyset$ for at least one r-tuple of graphs $C_i \in \mathcal{C}$, $i = 1, \ldots, r$. We have $\mathcal{E} \subseteq \mathcal{F}$. Indeed, if $G \in \mathcal{E}$ then there is a proper coloring φ of G and graphs $C_1, \ldots, C_r \in \mathcal{C}$ such that $G \subseteq \bigcup_{i=1}^r C_i$ and, by Claim 10, $K_n \setminus \bigcup_{i=1}^r C_i$ has at least $\frac{1}{R^2}\binom{n}{2}$ edges and is disjoint from G. Thus, $G \in \mathcal{F}$. Consequently,

$$\mathbb{P}(G(n,p) \not\to (K_k)_r) \leq \mathbb{P}(\mathcal{F}).$$

To estimate $\mathbb{P}(\mathcal{F})$ we write $\mathcal{F} = \bigcup \mathcal{F}(C_1, \ldots, C_r)$, where the summation runs over all collections $(C_1, \ldots, C_r)$ with $C_i \in \mathcal{C}$, $i = 1, \ldots, r$, and the event $\mathcal{F}(C_1, \ldots, C_r)$ means that $G(n,p) \cap (K_n \setminus \bigcup_{i=1}^r C_i) = \emptyset$. Clearly,

$$\mathbb{P}(\mathcal{F}(C_1, \ldots, C_r)) = (1-p)^{|K_n \setminus \bigcup_{i=1}^r C_i|} \leq (1-p)^{\binom{n}{2}/R^2},$$

where the last inequality follows by Claim 10. Finally, applying the union bound, we have

$$\mathbb{P}(G(n,p) \not\to (K_s)_r) \leq \mathbb{P}(\mathcal{F}) \leq |\mathcal{C}|^r (1-p)^{\binom{n}{2}/R^2} \leq |\mathcal{C}|^r \exp\left\{-\frac{p\binom{n}{2}}{R^2}\right\}. \quad \square$$

Observe that by Property (c) of Corollary 7,

$$|\mathcal{C}|^r \leq \exp\left\{r(2k^2)! \log(1/\varepsilon)\tau \log(1/\tau)\binom{n}{2}\right\}. \tag{17}$$

In view of Claim 11 and inequality (17), to complete the proof of Lemma 4, it suffices to show that

$$r(2k^2)! \log(1/\varepsilon)\tau \log(1/\tau)\binom{n}{2} \leq \frac{p\binom{n}{2}}{2R^2},$$

or, equivalently, after applying the definitions of p and τ ((3) and (12), resp.) and dividing sidewise by $n^{-\frac{2}{k+1}}\binom{n}{2}$, that

$$r(2k^2)! \log(1/\varepsilon) C_0 \log(1/\tau) \leq C/(2R^2). \tag{18}$$

To this end, observe that, since $C_0 \geq 1$ and, by (1), $R > 2r$, we have

$$\log(1/\tau) \overset{(12)}{\leq} \tfrac{2}{k+1} \log n$$

and

$$\log(1/\varepsilon) \overset{(11)}{=} \log(2r\tbinom{R}{k}) \leq (k+1)\log R\,.$$

Hence, the L-H-S of (18) can be upper bounded by $2r(2k^2)! C_0 \log R \log n$. Consequently, using also the bounds $(2k^2)! < (2k)^{4k^2}$ and, again, $R > 2r$, we realize that (18) will follow from

$$2R^3 \log R \cdot (2k)^{4k^2} \log n \leq C/C_0. \tag{19}$$

On the other hand,

$$C/C_0 \overset{(3),(12)}{=} 2^{5\sqrt{\log n \log k}-4\sqrt{\log n}} R^{16-10/k} \geq 2^{\sqrt{\log n \log k}+4\sqrt{\log n}(\sqrt{\log k}-1)} R^{12}.$$

Thus, (19) is an immediate consequence of the following two inequalities, which are themselves easy consequences of (2):

$$2^{\sqrt{\log n \log k}} \overset{(2)}{\geq} 2^{20k^2 \log k} \geq (2k)^{4k^2}$$

and

$$2^{4\sqrt{\log n}(\sqrt{\log k}-1)} > 2^{\sqrt{\log n}} \geq \log n.$$

For the latter inequality we first used $k \geq 3$ and $\sqrt{\log 3} > \frac{5}{4}$, and then the fact that $2^{\sqrt{x}} \geq x$ for all $x \geq 16$, which can be easily verified by checking the first derivative (note that by (2), $\log n \geq 16$). This completes the proof of Lemma 4.

Remark 3. The idea of utilizing hypergraph containers for Ramsey properties of random graphs comes from a recent paper by Nenadov and Steger [15] (see also [11], Ch. 7) where the authors give a short proof of the main theorem from [20] establishing an edge probability threshold for the property $G(n,p) \to (F)_r$. Let us point to some similarities and differences between their and our approach. For clarity of the comparison, let us restrict ourselves to the case $F = K_k$ considered in our paper (the generalization to an arbitrary graph F is quite straightforward).

In [15] the goal is to prove an asymptotic result with $n \to \infty$ and all other parameters fixed. Consequently, they do not optimize, or even specify constants. Our task is to provide as good as possible upper bound on n in terms of k and r, so there is no asymptotics.

The observation that a K_k-free coloring of the edges of $G(n,p)$ yields r independent sets in the hypergraph $H(n,k)$, and therefore, by the Saxton-Thomason Theorem there are r graphs C_i, $i = 1, \ldots, r$, each with only a few copies of K_k, whose union contains all the edges of $G(n,p)$, was made in [15]. Also there one can find a statement similar to our Proposition 8 (Corollary 3 in [15].) These two facts lead to similar estimates of the probability that $G(n,p)$ is not Ramsey. However, Nenadov and Steger, assuming that C is a constant, are forced to use Theorem 2.3 from [24] which involves the sequences of sets T_i. In our setting, we choose $C = C(n)$ in a balanced way, allowing us to go through with the estimates of $\mathbb{P}(G(n,p) \not\to (K_k)_r)$ without introducing the T_i's, while, on the other hand, keeping the upper bound on n exponential in k. In fact, as observed by Conlon and Gowers [2], the approach via random graphs cannot yield a better than double-exponential upper bound on n if one assumes that p is at the Ramsey threshold, i.e., if C is a constant.

3 Relaxed Folkman numbers

In this section we prove Theorem 2. We will need an elementary fact about Ramsey properties of quasi-random graphs. For constants ϱ and d with $0 < d, \varrho \leq 1$, we say that an n-vertex graph Γ is (ϱ, d)-*dense* if every induced subgraph on $m \geq \varrho n$ vertices contains at least $d(m^2/2)$ edges. It follows by an easy averaging argument that it suffices to check the above inequality only for $m = \lceil \varrho n \rceil$. Note also that every induced subgraph of a (ϱ, d)-dense n-vertex graph on at least cn vertices is $(\frac{\varrho}{c}, d)$-dense.

It turns out that for a suitable choice of the parameters, (ϱ, d)-dense graphs are Ramsey.

Proposition 12. *For every $k \geq 2$ and every $d \in (0,1)$, if $n \geq (2/d)^{2k-4}$ and $0 < \varrho \leq (d/2)^{2k-4}$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains a monochromatic copy of K_k.*

Proof. For a two-coloring of the edges of a graph Γ we call a sequence of vertices $(v_1, \dots, v_l)$ *canonical* if for each $i = 1, \dots, l-1$ all the edges $\{v_i, v_j\}$, for $j > i$, are of the same color.

We will first show by induction on l that for every $l \geq 2$ and $d \in (0,1)$, if $n \geq (2/d)^{l-2}$ and $0 < \varrho \leq (d/2)^{l-2}$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains a canonical sequence of length l.

For $l = 2$, every ordered pair of adjacent vertices is a canonical sequence. Assume that the statement is true for some $l \geq 2$ and consider an n-vertex (ϱ, d)-dense graph Γ, where $\varrho \leq (d/2)^{l-1}$ and $n \geq (2/d)^{l-1}$. As observed above, there is a vertex u with degree at least dn. Let M_u be a set of at least $dn/2$ neighbors of u connected to u by edges of the same color. Let $\Gamma_u = \Gamma[M_u]$ be the subgraph of Γ induced by the set M_u. Note that Γ_u has $n_u \geq dn/2 \geq (2/d)^{l-2}$ vertices and is (ϱ_u, d)-dense with $\varrho_u \leq (d/2)^{l-2}$. Hence, by the induction assumption, there is a canonical sequence of length l in Γ_u. This sequences preceded by the vertex u makes a canonical sequence of length $l+1$ in Γ.

To complete the proof of Proposition 12, set $l = 2k-2$ above and observe that every canonical sequence $(v_1, \dots, v_{2k-2})$ contains a monochromatic copy of K_k. Indeed, among the vertices $v_1, \dots, v_{2k-3}$, some $k-1$ have the same color on all the "forward" edges. These vertices together with vertex v_{2k-2} form a monochromatic copy of K_k. $\square$

Proof of Theorem 2. Let $n = 2^{4k/(1-4\alpha)}$. Consider a random graph $G(n,p)$ where

$$p = 2n^{-\frac{7+4\alpha}{16k}} = 2^{-\frac{20\alpha+3}{4(1-4\alpha)}}.$$

By elementary estimates one can bound the expected number of l-cliques in $G(n,p)$ by

$$\left(\frac{en}{l} p^{\frac{l-1}{2}}\right)^l.$$

Thus, if

$$\frac{l-1}{2} \geq \frac{\log n}{\log(1/p)} = \frac{16k}{20\alpha+3}$$

then, as $k \to \infty$, a.a.s. there are no l-cliques in $G(n,p)$. By assumption,

$$\frac{l-1}{2} \geq \frac{k-\alpha}{2\alpha} \geq \frac{16k}{20\alpha+3},$$

where the last inequality, equivalent to $(3-12\alpha)k \geq 20\alpha^2 + 3\alpha$, holds if $k \geq \frac{2}{3(1-4\alpha)}$ (we used here the assumption that $\alpha < \frac{1}{4}$).

Further, by a straightforward application of Chernoff's bound (see, e.g., [13, ineq. (2.6)]), a.a.s. $G(n,p)$ is $(\varrho, p-o(p))$-dense, where $\varrho = \frac{\log^2 n}{n}$, say. Indeed, setting $t = \varrho n = \log^2 n$, $\epsilon = \epsilon(n) = (\log n)^{-1/3}$, and $d = (1-\epsilon)p$, the probability that a fixed set T of t vertices spans in $G(n,p)$ fewer than $dt^2/2$ edges is at most

$$\mathbb{P}(e(T) \leq (1-\epsilon)pt^2/2) \leq \mathbb{P}\left(e(T) \leq (1-\epsilon/2)p\binom{t}{2}\right) \leq \exp\left\{-\frac{\epsilon^2}{8}p\binom{t}{2}\right\} \leq \exp\left\{-\frac{\epsilon^2}{24}pt^2\right\}.$$

Finally, note that the above bound, even multiplied by $\binom{n}{t}$, the number of all t-element subsets of vertices in $G(n,p)$, still converges to zero (recall that p is a constant).

Using that $\epsilon k = O(\log^{2/3} n)$ one can easily verify that both assumptions of Proposition 12,that is, $n \geq (2/d)^{2k-4}$ and $\varrho \leq (d/2)^{2k-4}$, hold true. Indeed, dropping the subtrahend 4 for simplicity,

$$(d/2)^{2k} = (1-\epsilon)^{2k} n^{-1+\delta} \geq \varrho \geq \frac{1}{n},$$

for n large enough, that is, for k large enough.

In conclusion, a.a.s. $G(n,p)$ is such that

- it contains no K_l, and
- for every two-coloring of its edges, there is a monochromatic copy of K_k.

Hence, there exists an n-vertex graph with the above two properties and, consequently, $f(k,l) \leq n = 2^{4k/(1-4\alpha)}$. □

4 Hypergraph Folkman numbers

Hypergraph Folkman numbers are defined in an analogous way to their graph counterparts. Given three integers h, k, and r, the h-uniform Folkman number $f_h(k;r)$ is the minimum number of vertices in an h-uniform hypergraph H such that $H \to (K_k^{(h)})_r$ but $H \not\supset K_{k+1}^{(h)}$. Here $K_k^{(h)}$ stands for the complete h-uniform hypergraph on k vertices, that is, one with $\binom{k}{h}$ edges. The finiteness of hypergraph Folkman numbers was proved by Nešetřil and Rödl in [17, Colloary 6, page 206] and besides the gigantic upper bound stemming from their construction, no reasonable bounds have been proven so far. Much better understood are the vertex-Folkman numbers (where instead of edges, the vertices are colored), which for both, graphs and hypergraphs, are bounded from above by an almost quadratic function of k, while from below the bound is only linear in k (see [6, 4].

The study of Ramsey properties of random hypergraphs began in [21] where a threshold was found for $K_4^{(3)}$, the 3-uniform clique on 4 vertices. Also there a general conjecture was stated that a theorem analogous to that in [20] holds for hypergraphs too. This was confirmed for h-partite h-uniform hypergraphs in [22], and, finally, for all h-uniform hypergraphs in [9] and, independently, in [3].

As remarked by Nenadov and Steger in [15], the Saxton-Thomason (or the Balogh-Morris-Samotij) theorem should also yield a much simpler proof of the hypergraph Ramsey threshold theorem from [9, 3]. We believe that, similarly, our quantitative approach should also provide an upper bound on the hypergraph Folkman numbers $f_h(k; r)$, exponential in a polynomial of k and r.

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