Free Actions of Compact Groups on C*-Algebras, Part I

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Abstract

We study free and compact group actions on unital C*-algebras. In particular, we provide a complete classification theory of these actions for compact abelian groups and explain its relation to the classical classification theory of principal bundles.

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1 Introduction

In this article we study free actions of compact groups on unital C*-algebras. This class of actions was first introduced under the name *saturated actions* by Rieffel [44] and equivalent characterizations where given by Ellwood [19] and by Gottman, Lazar, and Peligrad [20] (see also [6]). Other related notions of freeness were studied by Phillips [41] in connection with K-theory (see also [42]).

Free actions do not admit degeneracies that may be present in general actions. For this reasons they are easier to understand and to classify. In fact, free ergodic actions, i. e., free actions with trivial fixed point algebra, were completely classified by the remarkable work of Wassermann [56, 57, 58]. According to [56], for a compact group G there is a 1-to-1 correspondence between free ergodic actions of G and 2-cocycles of the dual group. Extending this result beyond the ergodic case is however not straightforward because, even for a commutative fix point algebra, the action cannot necessarily be decomposed into a bundle of ergodic actions. For compact abelian groups our results about free, but not necessary ergodic actions may be regarded as a generalization of the classification given by Wasserman.

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The study of non-ergodic free actions is also motivated by the established theory of principal bundles. By a classical result, having a free action of a compact group G on a compact Hausdorff space P is equivalent saying that P carries the structure of a principal bundle over the quotient $X := P/G$ with structure group G. Very well-understood is the case of locally trivial principal bundles, that is, if P is glued together from spaces of the form $U \times G$ with an open subset $U \subseteq X$. This gluing immediatly leads to G-valued cocycles. The corresponding homology, called Čech cohomology, then gives a complete classification of locally trivial principal bundles with base space X and structure group G (see [50]). For principal bundles that are not locally trivial, however, there is no obvious classification available. Our results provide such a classification in the case of an abelian structure group.

Passing to noncommutative geometry poses the question how to extend the concept of principal bundles to noncommutative geometry. In the case of vector bundles the Theorem of Serre and Swan (cf. [48]) gives the essential clue: The category of vector bundles over a compact space X is equivalent to the category of finitely generated and projective $C(X)$-modules. This observation leads to a notion of noncommutative vector bundles and is the connection between the topological K-theory based on vector bundles and the K-theory for C*-algebras. For principal bundles, free and proper actions offer a good candidate for a notion of noncommutative principal bundles (see e. g. [5, 6, 19, 42]). A similar geometric approach based on transformation groups was developed by one of the authors [53, 54]. In a purely algebraic setting, the well-established theory of Hopf–Galois extensions provides a wider framework comprising coactions of Hopf algebras (e. g. [23, 29, 46]). We also would like to mention the related notion of noncommutative principal torus bundles proposed by Echterhoff, Nest, and Oyono-Oyono [18] (see also [24]), which relies on a noncommutative version of Green's Theorem. Considering free actions as noncommutative principal bundles, the present article characterizes principal bundles in terms of associated vector bundles. This leads to a complete classification of all principal bundles over a compact noncommutative base space with a (classical) compact abelian structure group.

Extending the classical theory of principal bundles to noncommutative geometry is not of purely mathematical interest. In fact, noncommutative principal bundles become more and more prevalent in geometry and physics. For instance, Ammann and Bär [2, 3] study the properties of the Riemannian spin geometry of a smooth principal $U(1)$-bundle. Under suitable hypotheses, they relate the spin structure and the Dirac operator on the total space to the spin structure and the Dirac operator on the base space. A noncommutative generalization of these results was developed by Dabrowski, Sitarz, and Zucca [13, 14] using ideas from the theory of Hopf–Galois extensions. Noncommutative principal bundles also appear in the study of 3-dimensional topological quantum field theories that are based on the modular tensor category of representations of the Drinfeld double (cf. [30]). In this context, special types of Hopf–Galois extensions correspond to symmetries of the theory or, equivalently, to invertible defects. As such, they are connected to module categories and in particular to the Brauer–Picard group of pointed fusion categories. Furthermore, T-duality is considered to be an important symmetry of string theories ([1, 10]). It is known that a circle bundle with H-flux given by a Neveu–Schwarz 3-form admits a T-dual circle bundle with dual H-flux. However, it is also known that in general torus bundles with H-flux do not necessarily have a T-dual that is itself a classical torus bundle. Mathai and Rosenberg [32, 33] showed that this problem is resolved by passing to noncommutative spaces. For example, it turns out that every principal $\mathbb{T}^2$-bundle with H-flux does indeed admit a T-dual but its T-dual is non-classical. It is a bundle of noncommutative 2-tori, which can (locally) be realized as a noncommutative principal $\mathbb{T}^2$-bundle in the sense of [18]. Our classification result may lead to a better understanding of T-duals and the question of their existence.

The present article investigates the structure of free actions of compact groups from a geometrical point of view. One of our main objectives in the first part of the article is to provide methods to construct or deconstruct examples of free actions. It turns out that the isotypic components of the action admit a Morita equivalence bimodule structure. For compact abelian groups this allows us to completely classify free actions on unital C*-algebras. The resulting classification is the main objective for the later part of the article. More detailedly, the paper is organized as follows.

After briefly introducing the necessary tools from geometry, representation theory, and operator algebras in Section 2, the first part of Section 3 discusses the different equivalent characterizations of freeness (Theorem 3.10). In the second part of Section 3 we provide some methods to construct new free actions from given ones. As an example we present a one-parameter family of free $\mathrm{SU}_2(\mathbb{C})$-actions related to the Connes–Landi spheres (cf. [29]).

Section 4 is devoted to the study of Hilbert structures on modules which are naturally associated to a C*-dynamical system. We would like to point out that these modules have a natural interpretation as noncommutative vector bundles. Of particular importance in this context are Theorem 4.10 and its Corollary 4.15, which characterize freeness in terms of Hilbert bimodule maps. This will be the foundation of our later classification results.

The aim of Section 5 is to give a deeper insight into the theory for compact abelian groups. To get more comfortable we first present some examples, among them the noncommutative tori, the group C*-algebra of the discrete two-dimensional Heisenberg group and Woronowicz's twisted SU(2). Furthermore, we show how Morita self-equivalences over the corresponding fixed point algebra enter the game in form of the Picard group. This leads to an invariant of the action given by a group homomorphism $\varphi : \hat{G} \to \mathrm{Pic}(\mathcal{B})$ between the corresponding dual group $\hat{G}$ and the Picard group of the fixed point algebra $\mathcal{B}$ (Proposition 5.11).

The map φ alone does not distinguish all free actions. However, using some additional data, Section 6 provides a universal construction of, firstly, a unital C*-algebra and, secondly, a free G-action on this C*-algebra with fixed point algebra $\mathcal{B}$ (Theorem 6.17 and Proposition 6.20). We then classify all free actions by the map φ and the additional data used for the construction. Our approach is to some extend similar to the classical theory of group extensions (cf. [31]). For this reason we call the underlying classification data a *factor system*.

Section 7 is finally dedicated to a classification of free action of compact abelian group on unital C*-algebras. For this purpose we additionally fix a group homomorphism $\varphi : \hat{G} \to \mathrm{Pic}(\mathcal{B})$ and restrict our attention to C*-dynamical system with the given φ as invariant, as discussed in Section 5. The main result is that, if such dynamical systems exist, all free actions associated to the triple $(\mathcal{B}, G, \varphi)$ are parametrized, up to 2-coboundaries, by 2-cocycles on the dual group $\widehat{G}$ with values in the group $UZ(\mathcal{B})$ of central unitary elements in $\mathcal{B}$ (Theorem 7.8 and Corollary 7.9*)*. In other words, the set in question is a principal homogenous space with respect to a classical cohomology group $H^2(\widehat{G}, UZ(\mathcal{B}))$. In the remaining part of this section we provide a group theoretic criterion for the existence of free action with invariant φ, i. e., factor system associated to the triple $(\mathcal{B}, G, \varphi)$.

As already mentioned, for a Hausdorff space X and a compact group G, locally trivial principal G-bundles over X are classified by the Čech cohomology of the pair (X, G). Since each locally trivial principal bundle gives rise to a free action in our sense, it is natural to ask how the Čech cohomology for the pair (X, G) is related to the classification theory presented in Section 7. This question is the main drive for Section 8. Finally, in Section 9 we discuss a few examples.

We would like to point out that with little effort the arguments and the results which are presented in this article for actions of compact abelian groups extend to coactions of group C*-algebras of finite groups. We also would like to mention that this article is the first part of a larger program aiming at classifying more general free actions on C*-algebras. This classification could be used to develope a fundamental group for C*-algebras or an approach to noncommutative gerbes (cf. [35, 36]).

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2 Preliminaries and Notations

Our study is concerned with free and compact group actions on C*-algebras and their classification. As a consequence, we use and blend tools from geometry, representation theory and operator algebras. In this preliminary section we provide the most important definitions and notations which are repeatedly used in this article.

Principal Bundles and Free Group Actions

Let P and X be compact spaces. Furthermore, let G be a compact group. A locally trivial principal bundle is a quintuple (P, X, G, q, σ), where $q : P \to X$ is a continuous map and $\sigma : P \times G \to P$ a continuous action, with the property of local triviality: Each point $x \in X$ has an open neighbourhood U for which there exists a homeomorphism $\varphi_U : U \times G \to q^{-1}(U)$ satisfying $q \circ \varphi_U = \mathrm{pr}_U$ and additionally the equivariance property

$$\varphi_U(x, gh) = \varphi_U(x, g).h$$

for $x \in U$ and $g, h \in G$. It follows that the map q is surjective, that the action σ is free and proper, and that the natural map $P/G \mapsto X$, $p.G \mapsto q(p)$ is a homeomorphism. In particular, we recall that the action σ is called free if and only if all stabilizer groups $G_p := \{g \in G \mid \sigma(p, g) = p\}$, $p \in P$, are trivial. For a solid background on free group actions and principal bundles we refer to [26, 28, 39].

Representations

All tensor products of vector spaces are taken with respect to the algebraic tensor product denoted by $\otimes$ if not mentioned otherwise. For a finite-dimensional Hilbert space V with inner product ${}_V\langle\cdot,\cdot\rangle$, we write tr_V for the canonical trace on $\mathcal{L}(V)$ and $\theta_{v,w} \in \mathcal{L}(V)$ for the rank one operator defined for two elements $v, w \in V$ by $\theta_{v,w}(\xi) := {}_V\langle\xi, w\rangle \cdot v$. For the corresponding dual vector space we write $\bar{V}$. Moreover, we use of the linear isomorphisms $\varphi_{V,\bar{V}} : V \otimes \bar{V} \to \mathcal{L}(V)$, $v \otimes \bar{w} \mapsto \theta_{v,w}$ and $\varphi_{V,\bar{V}} : V \otimes \bar{V} \to \mathcal{L}(V)$, $v \otimes \bar{w} \mapsto \theta_{v,w}$. Given a finite-dimensional unitary representation (π, V) of a group G, we denote the dual representation by $(\bar{\pi}, \bar{V})$ and the adjoint representation by $(\mathrm{Ad}[\pi], \mathcal{L}(V))$. Besides those, the induced representation

$$R[\pi] : G \times \mathcal{L}(V) \to \mathcal{L}(V), \quad (g, S) \mapsto S\pi_g^*$$

of G on $\mathcal{L}(V)$ is of particular interest to us.

C*-Dynamical Systems

Let $\mathcal{A}$ be a unital C*-algebra and G a compact group that acts on $\mathcal{A}$ by *-automorphism $\alpha_g : \mathcal{A} \to \mathcal{A}$ $(g \in G)$ such that the map $G \times \mathcal{A} \to \mathcal{A}$, $(g, a) \mapsto \alpha_g(a)$ is continuous. Throughout this paper we call such a triple $(\mathcal{A}, G, \alpha)$ a C*-dynamical system. We usually denote by $\mathcal{B} := \mathcal{A}^G$ the corresponding fixed point C*-algebra of the action α and we write $P : \mathcal{A} \to \mathcal{A}$ for the conditional expectation given by

$$P(x) := \int_G \alpha_g(x)\, dg.$$

At this point it is worth mentioning that all integrals over compact groups are understood to be with respect to probability Haar measure. More generally, for a finite-dimensional unitary representation (π, V) of G we write $P_\pi : \mathcal{A} \to \mathcal{A}$ for the continuous G-equivariant projection onto the isotypic component $A(\pi) := P_\pi(A)$ given by

$$P_\pi(x) := \dim V \cdot \int_G \mathrm{tr}_V(\pi_g^*) \cdot \alpha_g(x)\, dg.$$

It is a consequence of the Peter–Weyl Theorem that the algebraic direct sum $\bigoplus_{\pi \in \widehat{G}} A(\pi)$ is a dense *-subalgebra of $\mathcal{A}$. Moreover, we point out that each continuous group action $\sigma : P \times G \to P$ of a compact group G on a compact space P gives rise to a C*-dynamical system $(C(P), G, \alpha_\sigma)$ defined by

$$\alpha_\sigma : G \times C(P) \to C(P), \quad (g, f) \mapsto f \circ \sigma_g.$$

Hilbert Module Structures

A huge part of this paper is concerned with Hilbert module structures. For the readers' convenience we recall some of the central definitions. Let $\mathcal{B}$ be a unital C*-algebra. A right pre-Hilbert $\mathcal{B}$-module is a vector space M which is a right $\mathcal{B}$-module equipped with a positive definite $\mathcal{B}$-valued sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ satisfying

$$\langle x, y.b \rangle_{\mathcal{B}} = \langle x, y \rangle_{\mathcal{B}} b \quad \text{and} \quad \langle x, y \rangle_{\mathcal{B}}^* = \langle y, x \rangle_{\mathcal{B}}$$

for all $x, y \in M$ and $b \in \mathcal{B}$. A right Hilbert $\mathcal{B}$-module is a right pre-Hilbert $\mathcal{B}$-module M which is complete with respect to the norm given by $\|x\|^2 = \|\langle x, x \rangle_{\mathcal{B}}\|$ for $x \in M$. It is called a full right Hilbert $\mathcal{B}$-module if the ideal $\mathrm{span}\{\langle x, y \rangle_{\mathcal{B}} \mid x, y \in M\}$ is dense in $\mathcal{B}$. Left (pre-) Hilbert $\mathcal{B}$-modules are defined in a similar way. Next, let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras. A right (pre-) Hilbert $\mathcal{A}-\mathcal{B}$-bimodule is a right (pre-) Hilbert $\mathcal{B}$-module M that is also a left $\mathcal{A}$-module satisfying

$$a.(x.b) = (a.m).b \quad \text{and} \quad \langle a.x, y \rangle_{\mathcal{B}} = {}_{\mathcal{B}}\langle x, a^*.y \rangle_{\mathcal{B}}$$

for all $x, y \in M$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In other words, a right (pre-) Hilbert $\mathcal{A}-\mathcal{B}$-bimodule is a right (pre-) Hilbert $\mathcal{B}$-module M together with a *-representation of $\mathcal{A}$ as adjointable operators on M. For later purposes we recall that in this situation the right $\mathcal{B}$-valued inner product satisfies the inequality

$$\langle a.x, a.x\rangle_{\mathcal{B}} \leq \|a\|^2 \langle x, x\rangle_{\mathcal{B}}$$

for all $x \in M$ and $a \in \mathcal{A}$. Moreover, we point out that right Hilbert $\mathcal{A}-\mathcal{B}$-bimodules are sometimes called $\mathcal{A}-\mathcal{B}$ correspondences in the literature. Eventually we also need the notion of (internal) tensor products. In fact, given a right (pre-) Hilbert $\mathcal{A}-\mathcal{B}$-bimodule M and a right (pre-) Hilbert $\mathcal{B}-\mathcal{B}$-bimodule M, their algebraic $\mathcal{B}$-tensor product $M \otimes_{\mathcal{B}} N$ carries a natural right pre-Hilbert $\mathcal{A}-\mathcal{B}$-bimodule structure with right $\mathcal{B}$-valued inner product given by

$$\langle x_1 \otimes_{\mathcal{B}} y_1, x_2 \otimes_{\mathcal{B}} y_2\rangle_{\mathcal{B}} := \langle x_1, \langle y_1, y_2\rangle_{\mathcal{B}}.x_2\rangle_{\mathcal{B}}$$

for $x_1, x_2 \in M$ and $y_1, y_2 \in N$. In particular, its completion $M \widehat{\otimes}_{\mathcal{B}} N$ (with respect to the induced norm) is a right Hilbert $\mathcal{A}-\mathcal{B}$-bimodule. Left (pre-) Hilbert $\mathcal{A}-\mathcal{B}$-bimodules are defined in a similar way. A (pre-)Hilbert $\mathcal{A}-\mathcal{B}$-bimodule is an $\mathcal{A}-\mathcal{B}$-bimodule M which is a left (pre-) Hilbert $\mathcal{A}$-module and a right (pre-) Hilbert $\mathcal{B}$-module satisfying

$$\langle a.x, y\rangle_{\mathcal{B}} = \langle x, a^*.y\rangle_{\mathcal{B}}, \quad {}_{\mathcal{A}}\langle x.b, y\rangle = {}_{\mathcal{A}}\langle x, y.b^*\rangle \quad \text{and} \quad {}_{\mathcal{A}}\langle x, y\rangle.z = x.\langle y, z\rangle_{\mathcal{B}}$$

for all $x, y, z \in M$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. As a consequence of the structure, the induced norms are equal. A Morita equivalence $\mathcal{A}-\mathcal{B}$-bimodule is a Hilbert $\mathcal{A}-\mathcal{B}$-bimodule with full inner products. In this context it is also important to mention that the algebraic $\mathcal{B}$-tensor product $M \otimes_{\mathcal{B}} N$ of a (pre-) Hilbert $\mathcal{A}-\mathcal{B}$-bimodule M and a (pre-) Hilbert $\mathcal{B}-\mathcal{B}$-bimodule N carries a natural pre-Hilbert $\mathcal{A}-\mathcal{B}$-bimodule structure with inner products given by

$${}_{\mathcal{A}}\langle x_1 \otimes_{\mathcal{B}} y_1, x_2 \otimes_{\mathcal{B}} y_2\rangle := {}_{\mathcal{A}}\langle x_1.{}_{\mathcal{B}}\langle y_1, y_2\rangle, x_2\rangle$$

and

$$\langle x_1 \otimes_{\mathcal{B}} y_1, x_2 \otimes_{\mathcal{B}} y_2\rangle_{\mathcal{B}} := \langle y_1, \langle x_1, x_2\rangle_{\mathcal{B}}.y_2\rangle_{\mathcal{B}}$$

for $x_1, x_2 \in M$ and $y_1, y_2 \in N$. Unsurprisingly, its completion $M \widehat{\otimes}_{\mathcal{B}} N$ is a Hilbert $\mathcal{A}-\mathcal{B}$-bimodule. Last but not least, if M is a Morita equivalence $\mathcal{A}-\mathcal{B}$-bimodule and N is a Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule, then it is easily checked that the completion $M \widehat{\otimes}_{\mathcal{B}} N$ is a Morita equivalence $\mathcal{A}-\mathcal{B}$-bimodule. For a detailed background on Hilbert module structures we refer to [4, 7, 17, 43].

3 On Free Group Actions: Some General Theory

The aim of this section is to discuss some of the forms of free and compact group actions on C*-algebras that have been used. In particular, we give some indications of their strengths and relationships to each other. Furthermore, we provide some methods to construct new free actions from given ones. As an example we present a one-parameter family of free $\mathrm{SU}_2(\mathbb{C})$-actions which are related to the Connes–Landi spheres. We write $\otimes_{\min}$ for the spatial tensor product of C*-algebras.

Proposition 3.1. ([41, *Proposition 7.1.3*]). *Let* $(\mathcal{A}, G, \alpha)$ *be a* C^**-dynamical system. Then the following definitions make a suitable completion of* $\mathcal{A}$ *into a Hilbert* $\mathcal{A} \rtimes_\alpha G - \mathcal{B}$*-bimodule:*

(i) $f.x := \int_G f(g)\alpha_g(x)\, dg$ *for* $f \in L^1(G, \mathcal{A}, \alpha)$ *and* $x \in \mathcal{A}$.

(ii) $x.b := xb$ *for* $x \in \mathcal{A}$ *and* $b \in \mathcal{B}$.

(iii) ${}_{\mathcal{A} \rtimes_\alpha G}\langle x, y\rangle$ *is the function* $g \mapsto x\alpha_g(y^*)$ *for* $x, y \in \mathcal{A}$.

(iv) $\langle x, y\rangle_{\mathcal{B}} := \int_G \alpha_g(x^* y)\, dg$ *for* $x, y \in \mathcal{A}$.

It is easily seen that the module under consideration in the previous statement is almost a Morita equivalence$\mathcal{A} \rtimes_\alpha G - \mathcal{B}$-bimodule. In fact, the only missing condition is that the range of $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes_\alpha G}$ need not be dense. The imminent definition was originally introduced by M. Rieffel and has a number of good properties that resemble the classical theory of free and compact group actions as we will soon see below.

Definition 3.2. ([41, *Definition 7.1.4*]). We call a C*-dynamical system $(\mathcal{A}, G, \alpha)$ *free* if the bimodule from Proposition 3.1 is a Morita equivalence bimodule, that is, the range of $\langle \cdot, \cdot \rangle_{\mathcal{A} \rtimes_\alpha G}$ is dense in the crossed product $\mathcal{A} \rtimes_\alpha G$.

Remark 3.3. We point out that M. Rieffel used the notion "*saturated*" instead of free, i. a., because of its relation to Fell bundles in the case of compact abelian group actions. Moreover, we recall that [44, Definition 1.6] provides a notion for free actions of locally compact groups which is consistent with Definition 3.2 for compact groups.

The next result shows that Definition 3.2 extends the classical notion of free and compact group actions.

Theorem 3.4. ([41, *Proposition 7.1.12 and Theorem 7.2.6*]). *Let* P *be a compact space and* G *a compact group. A continuous group action* $\sigma : P \times G \to P$ *is free if and only if the corresponding* C^**-dynamical system* $(C(P), G, \alpha_\sigma)$ *is free in the sense of Definition 3.2.*

Another hint for the strength of Definition 3.2 comes from the following observation: Let (P, X, G, q, σ) be a locally trivial principal bundle and (π, V) a finite-dimensional unitary representation of G. Then it is a well-known fact that the isotypic component $C(P)(\pi)$ is finitely generated and projective as a right $C(X)$-module (cf. [51, Proposition 8.5.2]). In the C*-algebraic setting a similar statement is valid.

Theorem 3.5. ([15, *Theorem 1.2*]). *Let* $(\mathcal{A}, G, \alpha)$ *be a free* C^**-dynamical system and* (π, V) *a finite-dimensional unitary representation of* G*. Then the corresponding isotypic component* $A(\pi)$ *is finitely generated and projective as a right* $\mathcal{B}$*-module.*

We proceed with introducing two more notions which will turn out to be equivalent characterizations of noncommutative freeness. The first notion is a C*-algebraic version of the purely algebraic Hopf–Galois condition (cf. [23, 46]) and is due to D. A. Ellwood.

Definition 3.6. ([19, *Definition 2.4*]). We say that a C*-dynamical system $(\mathcal{A}, G, \alpha)$ satisfies the *Ellwood condition* if the map

$$\Phi : \mathcal{A} \otimes \mathcal{A} \to C(G, \mathcal{A}), \quad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range (with respect to the canonical C*-norm on $C(G, \mathcal{A})$).

The second notion is of representation-theoretic nature and makes use of the so-called generalized isotypic components of a C*-dynamical system.

Definition 3.7. (*Generalized isotypic components*). Let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system and (π, V) a finite-dimensional unitary representation of G.

(a) We consider the C*-algebra $\mathcal{A} \otimes \mathcal{L}(V)$ equipped with the linear action $\alpha \otimes R[\pi]$ of G and denote by $A_2(\pi)$ the corresponding fixed point space, i. e.,

$$A_2(\pi) := \{s \in \mathcal{A} \otimes \mathcal{L}(V) \mid (\alpha_g \otimes R[\pi_g])(s) = s \ \text{ for all } \ g \in G\}.$$

(b) We consider the C*-algebra $\mathcal{A} \otimes \mathcal{L}(V)$ equipped with the action $\alpha \otimes \mathrm{Ad}[\pi]$ of G by *-automorphisms and denote by $\mathcal{C}(\pi)$ the corresponding fixed point algebra, i. e.,

$$\mathcal{C}(\pi) := \{s \in \mathcal{A} \otimes \mathcal{L}(V) \mid (\alpha_g \otimes \mathrm{Ad}[\pi_g])(s) = s \ \text{ for all } \ g \in G\}.$$

Both spaces carry a natural $\mathcal{B} - \mathcal{B}$-bimodule structure induced by the algebra structure of $\mathcal{A}$. Furthermore, a simple calculation shows that $A(\pi) = (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{tr}_V)(A_2(\pi))$ and that $\mathcal{B} = (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{tr}_V)(\mathcal{C}(\pi))$.

Remark 3.8. Suppose that (π, V) is an irreducible unitary representation of G. Then the map $\mathrm{id}_{\mathcal{A}} \otimes \mathrm{tr}_V : \mathcal{A} \otimes \mathcal{L}(V) \to \mathcal{A}$ restricts to a topological G-equivariant isomorphism between the spaces $A(\pi)$ and $A_2(\pi)$. Indeed, the map $q : \mathcal{A} \to A_2(\pi)$ given by

$$q(x) := \dim V \cdot \int_G \alpha_g(x) \otimes \pi_g^* \, dg$$

satisfies $(\mathrm{id}_{\mathcal{A}} \otimes \mathrm{tr}_V) \circ q = P_\pi$ and $q \circ (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{tr}_V) = \mathrm{id}_{A_2(\pi)}$, so that the restriction of q to A_π inverts the corestriction of $\mathrm{id}_{\mathcal{A}} \otimes \mathrm{tr}_V$ onto its image.

We are now ready to present the second notion which is of major relevance in our attempt to classify free C*-dynamical systems.

Definition 3.9. ([20, Definition 1.1 (b)]). Let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system. We call the set

$$\mathrm{Sp}(\alpha) := \left\{[(\pi, V)] \in \widehat{G} \mid \overline{\mathrm{span}\{x^*y \mid x, y \in A_2(\pi)\}} = \mathcal{C}(\pi)\right\}$$

the *Averson spectrum* of $(\mathcal{A}, G, \alpha)$.

As the following result finally shows, all the previous notions agree.

Theorem 3.10. *Let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system. Then the following statements are equivalent:*

(a) *The C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free.*

(b) *The C*-dynamical system $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition.*

(c) *The C*-dynamical system $(\mathcal{A}, G, \alpha)$ satisfies $Sp(\alpha) = \widehat{G}$.*

The equivalence between (a) and (b) was proved quite recently in [15], although it has been known that these two conditions are closely related to each other (cf. [55] for the case of compact Lie group actions). A proof of the equivalence between (a) and (c) can be found in [20]. We point out that condition (c) means that the generalized isotypic components $A_2(\pi)$ carry the structure of a Morita equivalence $B \otimes \mathcal{L}(V) - \mathcal{C}(\pi)$-bimodule (see Corollary 4.16 below).

In the remaining part of this section we provide some methods to construct new free actions from given ones. As an application, we present a one-parameter family of free $\mathrm{SU}_2(\mathbb{C})$-actions which are related to the Connes–Landi spheres.

Proposition 3.11. *Let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system and H a closed subgroup of G. Then also the restricted C*-dynamical system $(\mathcal{A}, H, \alpha_{|H})$ is free.*

Proof. Since $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition, the surjectivity of the restriction map $C(G, \mathcal{A}) \to C(H, \mathcal{A})$, $f \mapsto f_{|H}$ implies that the map

$$\Phi : \mathcal{A} \otimes \mathcal{A} \to C(H, \mathcal{A}), \quad \Phi(x \otimes y)(h) := x\alpha_h(y)$$

has dense range. That is, $(\mathcal{A}, H, \alpha_{|H})$ also satisfies the Ellwood condition. □

Proposition 3.12. *Let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system and N a normal subgroup of G. Then also the induced C*-dynamical system $(\mathcal{A}^N, G/N, \alpha_{|G/N})$ is free.*

Proof. Since $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition, the map

$$\Phi : \mathcal{A} \otimes \mathcal{A} \to C(G, \mathcal{A}), \quad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range. Moreover, the C*-algebra $C(G/N, \mathcal{A}^N)$ is naturally identified with functions in $C(G, \mathcal{A})$ satisfying $f(g) = \alpha_{n1}(f(gn_2))$ for all $g \in G$ and $n_1, n_2 \in N$. In other words, $C(G/N, \mathcal{A}^N)$ is the fixed point algebra of the action $\alpha \otimes \mathrm{rt}$ of $N \times N$ on $C(G) \otimes_{\min} \mathcal{A} = C(G, \mathcal{A})$, where $\mathrm{rt} : N \times C(G) \to C(G)$, $\mathrm{rt}(n, f)(g) := f(gn)$ denotes the right-translation action by N. Let $P_N : \mathcal{A} \to \mathcal{A}$ and $P_{N \times N} : C(G, \mathcal{A}) \to C(G, \mathcal{A})$ be the conditional expectations for the actions $\alpha_{|N}$ and $\alpha \otimes \mathrm{rt}$, respectively. Then we obtain for $x, y \in \mathcal{A}$

$$\begin{aligned}\Phi(P_N(x) \otimes P_N(y)) &= \int_{N\times N} \alpha_{n_1}(x)\alpha_{gn_2}(y)\, dn_1 dn_2 = \int_{N\times N} \alpha_{n_1}(x)\alpha_{n_2 g}(y)\, dn_1 dn_2 \\ &= \int_{N\times N} \alpha_{n_1}(x\alpha_{n_2 g}(y))\, dn_1 dn_2 = \int_{N\times N} \alpha_{n_1}(x\alpha_{gn_2}(y))\, dn_1 dn_2 \\ &= P_{N\times N}(\Phi(x \otimes y)).\end{aligned}$$

It follows that the restricted map

$$\Phi_{|\mathcal{A}^N \otimes \mathcal{A}^N} : \mathcal{A}^N \otimes \mathcal{A}^N \to C(G, A), \quad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range in the C*-subalgebra $C(G/N, \mathcal{A}^N)$. That is, $(\mathcal{A}^N, G/N, \alpha_{|G/N})$ also satisfies the Ellwood condition. □

Proposition 3.13. *Let $(\mathcal{A}, G, \alpha)$ and $(\mathcal{C}, H, \gamma)$ be free C*-dynamical systems. Then also their tensor product $(\mathcal{A} \otimes_{\min} \mathcal{C}, G \times H, \alpha \otimes \gamma)$ is free.*

Proof. We first note that the map

$$\begin{gathered}\Phi : \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{A} \otimes \mathcal{C} \to C(G \times H, \mathcal{A} \otimes_{\min} \mathcal{C}), \\ \Phi(x \otimes y \otimes u \otimes v)(h) := x\alpha_h(y) \otimes u\gamma_h(v)\end{gathered}$$

is, up to a permutation of the tensor factors, an amplification of the corresponding maps induced by $(\mathcal{A}, G, \alpha)$ and $(\mathcal{C}, H, \gamma)$. Therefore, $(\mathcal{A} \otimes_{\min} \mathcal{C}, G \times H, \alpha \otimes \gamma)$ inherits the Ellwood condition from $(\mathcal{A}, G, \alpha)$ and $(\mathcal{C}, H, \gamma)$. □

Remark 3.14. Let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system. Furthermore, let $\mathcal{C}$ be any unital C*-algebra. Then Proposition 3.13 applied to the trivial group H implies that the C*-dynamical system $(\mathcal{A} \otimes_{\min} \mathcal{C}, G, \alpha \otimes \mathrm{id}_{\mathcal{C}})$ is free. More generally, if $(\mathcal{C}, G, \gamma)$ is any C*-dynamical system, then is is not hard to check that the C*-dynamical system $(\mathcal{A} \otimes_{\min} \mathcal{C}, G, \alpha \otimes \gamma)$ satisfies the Ellwood condition, i. e., $(\mathcal{A} \otimes_{\min} \mathcal{C}, G, \alpha \otimes \gamma)$ is free. This observation corresponds in the classical setting to the situation of endowing the cartesian product of a free and compact G-space X and any compact G-space Y with the free diagonal action of G.

We proceed with a construction which is slightly more involved. For this purpose, let $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}, H, \beta)$ be C*-dynamical systems with commuting actions. Then it is easily seen that the map

$$\alpha \circ \beta : (G \times H) \times \mathcal{A} \to \mathcal{A}, \quad ((g, h), x) \mapsto (\alpha_g \circ \beta_h)(x)$$

defines a continuous action of $G \times H$ on $\mathcal{A}$ by algebra automorphisms and therefore leads to the C*-dynamical system $(\mathcal{A}, G \times H, \alpha \circ \beta)$. Furthermore, let $(\mathcal{C}, H, \gamma)$ be any another C*-dynamical system. Then the fixed point algebra $(\mathcal{A} \otimes_{\min} \mathcal{C})^H$ with respect to the tensor product action $\beta \otimes \gamma$ of H on $\mathcal{A} \otimes_{\min} \mathcal{C}$ gives rise to the C*-dynamical system $((\mathcal{A} \otimes_{\min} \mathcal{C})^H, G, \alpha \otimes \mathrm{id}_{\mathcal{C}})$. Indeed, since the actions α and β commute by assumption, the action of G on $(\mathcal{A} \otimes_{\min} \mathcal{C})^H$ given by $\alpha \otimes \mathrm{id}_{\mathcal{C}}$ is well-defined.

Theorem 3.15. *Let $(\mathcal{A}, G, \alpha)$ and $(\mathcal{C}, H, \gamma)$ be free C*-dynamical systems. Furthermore, let $(\mathcal{A}, H, \beta)$ be any another C*-dynamical system such that the actions α and β commute. Then the following assertions hold:*

(*a*) *The C*-dynamical system $((\mathcal{A} \otimes_{\min} \mathcal{C}), G \times H, (\alpha \circ \beta) \otimes \gamma)$ is free.*

(*b*) *The C*-dynamical system $((\mathcal{A} \otimes_{\min} \mathcal{C})^H, G, \alpha \otimes \mathrm{id}_{\mathcal{C}})$ is free.*

Proof. (a) We first note that $(\mathcal{A}, G, \alpha)$ and $(\mathcal{C}, H, \gamma)$ both satisfy the Ellwood condition from which we conclude that the maps

$$\Phi_1 : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{C} \to C(G, \mathcal{A} \otimes_{\min} \mathcal{C}), \quad \Phi_1(x_1 \otimes x_2 \otimes y)(g) := x_1 \alpha_g(x_2) \otimes y$$

and $\Phi_2 : (\mathcal{A} \otimes \mathcal{C}) \otimes (\mathcal{A} \otimes \mathcal{C}) \to C(H, \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{C})$ given by

$$\Phi_2\big((x_1 \otimes y_1) \otimes (x_2 \otimes y_2)\big)(h) := x_1 \otimes \beta_h(x_2) \otimes y_1 \gamma_h(y_2)$$

have dense range. It follows, identifying $C(H, C(G, \otimes \mathcal{A} \otimes_{\min} \mathcal{C}))$ with $C(G \times H, \mathcal{A} \otimes_{\min} \mathcal{C})$, that also their amplified composition

$$\Phi := (\mathrm{id}_{C(H)} \otimes \Phi_1) \circ \Phi_2 : (\mathcal{A} \otimes \mathcal{C}) \otimes (\mathcal{A} \otimes \mathcal{C}) \to C(G \times H, \mathcal{A} \otimes_{\min} \mathcal{C})$$

given by

$$\Phi\big((x_1 \otimes y_1) \otimes (x_2 \otimes y_2)\big)(g, h) = x_1 (\alpha_g \beta_h)(x_2) \otimes y_1 \gamma_h(y_2)$$

has dense range. That is, $((\mathcal{A} \otimes_{\min} \mathcal{C}), G \times H, (\alpha \circ \beta) \otimes \gamma)$ satisfies the Ellwood condition.

(b) To verify the second assertion we simply apply Proposition 3.12 to the C*-dynamical system in part (a) and the normal subgroup $\{\mathbb{1}_G\} \times H$ of $G \times H$. $\square$

Example 3.16. The Connes-Landi spheres $\mathbb{S}^n_\theta$ are extensions of the noncommutative tori $\mathbb{T}^n_\theta$ (cf. [29]). We are in particularly interested in the case $n = 7$. In this case there is a continuous action of the 2-torus $\mathbb{T}^2$ on the 7-sphere $\mathbb{S}^7 \subseteq \mathbb{C}^4$ given by

$$\sigma : \mathbb{S}^7 \times \mathbb{T}^2 \to \mathbb{S}^7, \quad ((z_1, z_2, z_3, z_4), (t_1, t_2)) \mapsto (t_1 z_1, t_1 z_2, t_2 z_3, t_2 z_4).$$

Let $(C(\mathbb{S}^7), \mathbb{T}^2, \alpha_\sigma)$ be the corresponding C*-dynamical system. Furthermore, let $(\mathbb{T}^2_\theta, \mathbb{T}^2, \alpha^2_\theta)$ be the free C*-dynamical system associated to the gauge action on the noncommutative 2-torus $\mathbb{T}^2_\theta$ (see Example 5.2 below). The Connes–Landi sphere $\mathbb{S}^7_\theta$ is defined as the fixed point algebra of the tensor product action $\alpha_\sigma \otimes \alpha^2_\theta$ of $\mathbb{T}^2$ on $C(\mathbb{S}^7, \mathbb{T}^2_\theta) = C(\mathbb{S}^7) \otimes_{\min} \mathbb{T}^2_\theta$, i. e.,

$$\mathbb{S}^7_\theta := C(\mathbb{S}^7, \mathbb{T}^2_\theta)^{\mathbb{T}^2}.$$

Our intention is to use Theorem 3.15 (b) to endow $\mathbb{S}^7_\theta$ with a free $\mathrm{SU}_2(\mathbb{C})$-action. For this purpose, we consider the free and continuous $\mathrm{SU}_2(\mathbb{C})$-action on the 7-sphere $\mathbb{S}^7$ given by

$$\mu : \mathbb{S}^7 \times \mathrm{SU}_2(\mathbb{C}) \to \mathbb{S}^7, \quad ((z_1, z_2, z_3, z_4), M) \mapsto (z_1, z_2, z_3, z_4) \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

It follows from Theorem 3.4 that the corresponding C*-dynamical system $(C(\mathbb{S}^7), \mathrm{SU}_2(\mathbb{C}), \alpha_\mu)$ is free. Moreover, it is easily verified that the actions α_μ and α_σ commute. Therefore, Theorem 3.15 (b) implies that the C*-dynamical system $(\mathbb{S}^7_\theta, \mathrm{SU}_2(\mathbb{C}), \alpha_\mu \otimes \mathrm{id}_{\mathbb{T}^2_\theta})$ is free.

4 Hilbert Modules Associated to Free Actions

Throughout this section let $(\mathcal{A}, G, \alpha)$ be a fixed C*-dynamical system. Our intention is to study Hilbert module structures on modules which are in a natural way associated to $(\mathcal{A}, G, \alpha)$. In particular, we obtain results which lay the foundation for our attempt to classify free and compact group actions on C*-algebras.

Definition 4.1. (*Generalized spaces of sections*). Let (π, V) be a finite-dimensional unitary representation of G. We consider the space $\mathcal{A} \otimes V$ equipped with the continuous linear action $\alpha \otimes \pi$ of G and denote by $\Gamma_{\mathcal{A}} V$ the corresponding fixed point space, i. e.,

$$\Gamma_{\mathcal{A}} V := \{ s \in \mathcal{A} \otimes V \mid (\alpha_g \otimes \pi_g)(s) = s \ \text{ for all } \ g \in G \}.$$

We point out that $\Gamma_{\mathcal{A}} V$ comes equipped with a natural $\mathcal{B} - \mathcal{B}$-bimodule structure.

Remark 4.2. Let (P, X, G, q, σ) be a locally trivial principal bundle and (π, V) be a finite-dimensional unitary representation of G. Furthermore, let $\mathbb{V}_\pi := P \times_G V$ be the associated vector bundle over X with bundle projection $q_\pi : \mathbb{V}_\pi \to X$ given by $q_\pi([p, v]) := q(p)$. Then it is not hard to see that the map

$$T_\pi : C(P, V)^G \to \Gamma \mathbb{V}_\pi, \quad T_\pi(f)(q(p)) := [p, f(p)]$$

is a topological isomorphism of $C(X) - C(X)$-bimodules. Since $\Gamma_{C(P)} V \cong C(P, V)^G$ holds as $C(X) - C(X)$-bimodules, the previous discussion justifies to interpret the spaces from Definition 4.1 as noncommutative vector bundles which are associated to the C*-dynamical system (A, G, α).

We proceed with two finite-dimensional unitary representations (π, V) and (ρ, W) of G. For $s = \sum_i x_i \otimes v_i \in \mathcal{A} \otimes V$ and $t = \sum_j y_j \otimes w_j \in \mathcal{A} \otimes W$ we define

$$m_{V,W}(s \otimes_{\mathcal{B}} t) := \sum_{i,j} x_i y_j \otimes v_i \otimes w_j \quad \text{and} \quad i_V(s) := \sum_i x_i^* \otimes \bar{v}_i$$

which give rise to maps

$$m_{V,W} : (\mathcal{A} \otimes V) \otimes_{\mathcal{B}} (\mathcal{A} \otimes W) \to \mathcal{A} \otimes V \otimes W \quad \text{and} \quad i_V : \mathcal{A} \otimes V \to \mathcal{A} \otimes \bar{V}.$$

Lemma 4.3. *Restricting the maps $m_{V,W}$ and i_V to the subspaces $\Gamma_{\mathcal{A}}V \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}}W$ and $\Gamma_{\mathcal{A}}V$, respectively, provides well-defined maps*

$$m_{V,W} : \Gamma_{\mathcal{A}}V \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}}W \to \Gamma_{\mathcal{A}}(V \otimes W) \quad \textit{and} \quad i_V : \Gamma_{\mathcal{A}}V \to \Gamma_{\mathcal{A}}\bar{V}.$$

Proof. We first show that the map $m_{V,W}$ is well-defined. In fact, for $g \in G$, $s \in \Gamma_{\mathcal{A}}V$ and $t \in \Gamma_{\mathcal{A}}W$ we obtain

$$(\alpha_g \otimes \pi_g \otimes \rho_g)\big(m_{V,W}(s \otimes_{\mathcal{B}} t)\big) = m_{V,W}\big((\alpha_g \otimes \pi_g)(s) \otimes_{\mathcal{B}} (\alpha_g \otimes \pi_g)(t)\big) = m_{V,W}(s \otimes_{\mathcal{B}} t).$$

It immediately follows that $m_{V,W}(s \otimes_{\mathcal{B}} t) \in \Gamma_{\mathcal{A}}(V \otimes W)$. To verify that the map i_V is well-defined we choose $g \in G$ and $s \in \Gamma_{\mathcal{A}}V$. Then a short calculation gives

$$(\alpha_g \otimes \bar{\pi}_g)\big(i_V(s)\big) = i_V\big((\alpha_g \otimes \pi_g)(s)\big) = i_V(s)$$

from which we conclude that $i_V(s) \in \Gamma_{\mathcal{A}}\bar{V}$. □

Our next goal is to equip the generalized isotypic components with Hilbert module structures. For this purpose, we fix a finite-dimensional unitary representation (π, V) of G and point out that $\mathcal{A} \otimes V$ carries a natural Morita equivalence $\mathcal{A} \otimes \mathcal{L}(V) - \mathcal{A}$-bimodule structure with inner products given by

$$_{\mathcal{A}\otimes\mathcal{L}(V)}\langle x \otimes v, y \otimes w\rangle := xy^* \otimes \theta_{v,w} \quad \text{and} \quad \langle x \otimes v, y \otimes w\rangle_{\mathcal{A}} := x^*y \cdot {}_V\langle w, v\rangle \qquad (1)$$

for $x, y \in \mathcal{A}$ and $v, w \in V$. In particular, the induced norms $_{\mathcal{A}\otimes\mathcal{L}(V)}\|\cdot\|$ and $\|\cdot\|_{\mathcal{A}}$ are equal.

Lemma 4.4. *The space $\Gamma_{\mathcal{A}}V$ is a closed subspace of the Banach space $(\mathcal{A} \otimes V, \|\cdot\|_{\mathcal{A}})$.*

Proof. To verify the assertion it suffices to note that the action $\alpha \otimes \pi$ of G on $\mathcal{A} \otimes V$ is strongly continuous with respect $\|\cdot\|_{\mathcal{A}}$. □

Proposition 4.5. *The following definitions make $\Gamma_{\mathcal{A}}V$ into a Hilbert $\mathcal{C}(\pi) - \mathcal{B}$-bimodule:*

(i) *$c.s$ for $c \in \mathcal{C}(\pi)$ and $s \in \Gamma_{\mathcal{A}}V$.*

(ii) *$s.b$ for $s \in \Gamma_{\mathcal{A}}V$ and $b \in \mathcal{B}$.*

(iii) *$_{\mathcal{C}(\pi)}\langle s, t\rangle := {}_{\mathcal{A}\otimes\mathcal{L}(V)}\langle s, t\rangle$ for $s, t \in \Gamma_{\mathcal{A}}V$.*

(iv) *$\langle s, t\rangle_{\mathcal{B}} := \langle s, t\rangle_{\mathcal{A}}$ for $s, t \in \Gamma_{\mathcal{A}}V$.*

Proof. We first verify that the defining operations (i)-(iv) are well-defined.

(i) For $g \in G$, $c \in \mathcal{C}(\pi)$ and $s \in \Gamma_{\mathcal{A}}V$ we obtain

$$(\alpha_g \otimes \pi_g)(c.s) = \big((\alpha_g \otimes \mathrm{Ad}[\pi_g])(c).(\alpha_g \otimes \pi_g)(s)\big) = c.s$$

(ii) For $g \in G$, $s \in \Gamma_{\mathcal{A}} V$ and $b \in \mathcal{B}$ we obtain

$$(\alpha_g \otimes \pi_g)(s.b) = \big((\alpha_g \otimes \pi_g)(s)\big).\alpha_g(b) = s.b.$$

(iii) For $g \in G$ and $s, t \in \Gamma_{\mathcal{A}} V$ we obtain

$$(\alpha_g \otimes \mathrm{Ad}[\pi_g])\big({}_{\mathcal{C}(\pi)}\langle s, t\rangle\big) = {}_{\mathcal{C}(\pi)}\langle(\alpha_g \otimes \pi_g)(s), (\alpha_g \otimes \pi_g)(t)\rangle = {}_{\mathcal{C}(\pi)}\langle s, t\rangle.$$

(iv) For $g \in G$ and $s, t \in \Gamma_{\mathcal{A}} V$ we obtain

$$\begin{aligned} \alpha_g(\langle s, t\rangle_{\mathcal{B}}) &= \langle(\alpha_g \otimes \mathrm{id}_V)(s), (\alpha_g \otimes \mathrm{id}_V)(t)\rangle_{\mathcal{A}} \\ &= \langle(\mathrm{id}_A \otimes \pi_g^*)(s), (\mathrm{id}_A \otimes \pi_g^*)(t)\rangle_{\mathcal{A}} \\ &= \langle s, t\rangle_{\mathcal{A}} = \langle s, t\rangle_{\mathcal{B}}. \end{aligned}$$

Moreover, we note that the algebraic properties that need to be checked follow from the corresponding properties of the Morita equivalence $\mathcal{A} \otimes \mathcal{L}(V) - \mathcal{A}$-bimodule $\mathcal{A} \otimes V$. Finally, we show that $\Gamma_{\mathcal{A}} V$ is complete with respect to the induced norms ${}_{\mathcal{C}(\pi)}\|\cdot\|$ and $\|\cdot\|_{\mathcal{B}}$. Indeed, the construction of the inner products implies that the induced norms are restrictions of the induced norms ${}_{\mathcal{A}\otimes\mathcal{L}(V)}\|\cdot\|$ and $\|\cdot\|_{\mathcal{A}}$ on $\mathcal{A} \otimes V$. In particular, we conclude that ${}_{\mathcal{C}(\pi)}\|\cdot\|$ and $\|\cdot\|_{\mathcal{B}}$ are equal. The assertion is therefore a consequence of Lemma 4.4. ☐

Corollary 4.6. *The space $\Gamma_{\mathcal{A}} V$ carries a natural right Hilbert $\mathcal{B} - \mathcal{B}$-bimodule structure.*

Adapting the operations in (1) leads to the following statement.

Corollary 4.7. *Let $(\bar{\pi}, \bar{V})$ be the dual representation of (π, V). Then the space $\Gamma_{\mathcal{A}} \bar{V}$ carries a natural Hilbert $\mathcal{B} - \mathcal{C}(\pi)$-bimodule structure and therefore also a natural right Hilbert $\mathcal{B} - \mathcal{B}$-bimodule structure.*

We continue with studying the maps $m_{V,W}$ defined in Lemma 4.3. Our intention is to show that these maps are actually isomorphisms of right Hilbert $\mathcal{B} - \mathcal{B}$-bimodules in the case the C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free. To this end, we first need the following two results.

Lemma 4.8. *The pointwise multiplication and the following inner product makes $L^1(G, \mathcal{A})$ into a right pre-Hilbert $\mathcal{B} - \mathcal{B}$-bimodule:*

$$\langle f_1, f_2\rangle_{\mathcal{B}} := \int_{G\times G} \alpha_g(f_1^*(h) f_2(h))_{\mathcal{B}} \, dg dh$$

for $f_1, f_2 \in L^1(G, \mathcal{A})$.

Proof. The proof of this statement is straightforward by using the faithfulness and positivity of the Haar measure. ☐

Proposition 4.9. *Consider $\mathcal{A}$ equipped with the right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure given in Proposition 3.1 (4) and $L^1(G,\mathcal{A})$ equipped with the right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure given in Lemma 4.8. Then the map*

$$\Phi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to L^1(G,\mathcal{A}), \quad \Phi(x \otimes_{\mathcal{B}} y)(g) := x\alpha_g(y)$$

is an isometry of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules.

Proof. It is obvious that the map Φ preserves the $\mathcal{B}-\mathcal{B}$-bimodule structure. Moreover, an explicit computation using the invariance of the Haar measure shows that

$$\begin{aligned}\langle \Phi(x_1 \otimes_{\mathcal{B}} y_1), \Phi(x_2 \otimes_{\mathcal{B}} y_2)\rangle_{\mathcal{B}} &= \int_{G\times G} \alpha_g(\alpha_h(y_1^*)x_1^*x_2\alpha_h(y_2))\, dgdh \\ &= \int_{G\times G} \alpha_{gh}(y_1^*)\alpha_g(x_1^*x_2)\alpha_{gh}(y_2))\, dgdh \\ &= \int_G \alpha_g(y_1^*)\Big(\int_G \alpha_h(x_1^*x_2)\, dh\Big)\alpha_g(y_2)\, dg.\end{aligned}$$

holds for all $x_1, x_2, y_1, y_2 \in \mathcal{A}$ and it is not hard to see that the last expression of this equation is equal to

$$\langle y_1, \langle x_1, x_2\rangle_{\mathcal{B}}.y_2\rangle_{\mathcal{B}} = \langle x_1 \otimes_{\mathcal{B}} y_1, x_2 \otimes_{\mathcal{B}} y_2\rangle_{\mathcal{B}}.$$

From this observation we immediately conclude that Φ is an isometry of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. ☐

As already mentioned in the beginning, the generalized isotypic components can be interpreted as noncommutative vector bundles which are associated to the C*-dynamical system (A, G, α). Given two finite-dimensional unitary representations (π, V) and (ρ, W) of G, it is a well-known fact from the classical theory of free and compact group actions that the space $\Gamma_{\mathcal{A}}(V \otimes W)$ is isomorphic as a $\mathcal{B}-\mathcal{B}$-bimodule to the balanced tensor product of $\Gamma_{\mathcal{A}}V$ and $\Gamma_{\mathcal{A}}W$ over the algebra of the base space (cf. [22, Proposition 2.6]). In the C*-algebraic setting we obtain a similar statement.

Theorem 4.10. *Let (π, V) and (ρ, W) be two finite-dimensional unitary representations of G. Then the map*

$$m_{V,W} : \Gamma_{\mathcal{A}}V\widehat{\otimes}_{\mathcal{B}}\Gamma_{\mathcal{A}}W \to \Gamma_{\mathcal{A}}(V \otimes W)$$

is an isometry of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. Moreover, if the C-dynamical $(\mathcal{A}, G, \alpha)$ is free, then $m_{V,W}$ is an isomorphism of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules.*

Proof. We first note that a straightforward computation shows that the map $m_{V,W}$ is an isometry with respect to the $\mathcal{B}$-valued inner products on $\Gamma_{\mathcal{A}}V\widehat{\otimes}_{\mathcal{B}}\Gamma_{\mathcal{A}}W$ and $\Gamma_{\mathcal{A}}(V \otimes W)$, respectively. To verify the second assertion we have to make a small detour. Indeed, we consider $\mathcal{A} \otimes V$ and $\mathcal{A} \otimes W$ equipped with their natural right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure induced by right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure on $\mathcal{A}$ given in Proposition 3.1. Furthermore, we consider $L^1(G, \mathcal{A} \otimes V \otimes W)$ equipped with the natural right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure induced by the right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure on on $L^1(G,\mathcal{A})$ as described in Lemma 4.8. Then it follows from Proposition 4.9 that the map

$$\Phi_{V,W} : (\mathcal{A} \otimes V) \otimes_{\mathcal{B}} (\mathcal{A} \otimes W) \to L^1(G, \mathcal{A} \otimes V \otimes W)$$
$$(x \otimes v) \otimes_{\mathcal{B}} (y \otimes w) \mapsto \Phi(x \otimes_{\mathcal{B}} y) \otimes v \otimes w,$$

is an isometry of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. Moreover, it is easily checked that $\Phi_{V,W}$ is $G \times G$-equivariant with respect to the unitary action of $G \times G$ on $(\mathcal{A} \otimes V) \otimes_{\mathcal{B}} (\mathcal{A} \otimes W)$ given for $g_1, g_2 \in G$, $x, y \in \mathcal{A}$, $v \in V$ and $w \in W$ by

$$(g_1, g_2).((x \otimes v) \otimes_{\mathcal{B}} (y \otimes w)) := (\alpha_{g_1}(x) \otimes \pi_{g_1}(v)) \otimes_{\mathcal{B}} (\alpha_{g_2}(y) \otimes \rho_{g_2}(w))$$

and the unitary action of $G \times G$ on $L^1(G, \mathcal{A} \otimes V \otimes W)$ given for $g, g_1, g_2 \in G$ and $f \in L^1(G, \mathcal{A} \otimes V \otimes W)$ by

$$((g_1, g_2).f)(g) := (\alpha_{g_1} \otimes \pi_{g_1} \otimes \rho_{g_2})(f(g_1^{-1} g g_2)).$$

The $G \times G$-equivariance now implies that $\Phi_{V,W}$ induces an isometry of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules between the corresponding fixed point spaces, i. e., between the pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules $\Gamma_{\mathcal{A}} V \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}} W$ and $L^1(G, \mathcal{A})^{G \times G}$. Since the map

$$L^1(G, \mathcal{A})^{G \times G} \to \Gamma_{\mathcal{A}}(V \otimes W), \quad f \mapsto f(1)$$

is an isomorphism of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules, we conclude that their composition provides an isometry of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules between $\Gamma_{\mathcal{A}} V \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}} W$ and $\Gamma_{\mathcal{A}}(V \otimes W)$. This composition is precisely $m_{V,W}$. We point out that this observation gives an alternative proof for the assertion that $m_{V,W}$ is an isometry of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. Suppose finally that the C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free. Then it is a consequence of Theorem 3.10 (a) $\Rightarrow$ (b) that the map $\Phi_{V,W}$ has dense image in $L^1(G, \mathcal{A} \otimes V \otimes W)$, which in turn shows by the previous discussion that $m_{V,W}$ has dense image in $\Gamma_{\mathcal{A}}(V \otimes W)$. Therefore, extending $m_{V,W}$ to the completion $\Gamma_{\mathcal{A}} V \widehat{\otimes}_{\mathcal{B}} \Gamma_{\mathcal{A}} W$ gives rise to the desired isomorphism of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. $\square$

As an important consequence we obtain the following result about the Morita equivalence bimodule structure of the generalized isotypic components.

Corollary 4.11. *Suppose that the C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free and let (π, V) be a finite-dimensional unitary representation of G. Then the space $\Gamma_{\mathcal{A}} V$ carries the structure of a $\mathcal{C}(\pi)-\mathcal{B}$-Morita equivalence bimodule.*

Proof. By Proposition 4.5 it remains to verify that the inner products ${}_{\mathcal{C}(\pi)}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ are full. Indeed, we first note that the $\mathcal{C}(\pi)$-valued inner product ${}_{\mathcal{C}(\pi)}\langle\cdot,\cdot\rangle$ satisfies

$$ {}_{\mathcal{C}(\pi)}\langle s,t\rangle = \big(\big(\operatorname{id}_{\mathcal{A}} \otimes \varphi_{V,\bar{V}}\big)\circ m_{V,\bar{V}}\big)\big(s \otimes_{\mathcal{B}} i_V(t)\big)$$

for all $s,t \in \Gamma_{\mathcal{A}}V$. Therefore, Theorem 4.10 applied to the representations (π,V) and $(\bar{\pi},\bar{V})$ implies that the range of ${}_{\mathcal{C}(\pi)}\langle\cdot,\cdot\rangle$ is dense in $\mathcal{C}(\pi)$ which is equivalent saying that ${}_{\mathcal{C}(\pi)}\langle\cdot,\cdot\rangle$ is full. Next, we examine the inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$. In this situation a similar observation shows that

$$\langle s,t\rangle_{\mathcal{B}} := \big(\big(\operatorname{id}_{\mathcal{A}} \otimes \operatorname{tr}_V\big)\circ\big(\operatorname{id}_{\mathcal{A}} \otimes \varphi_{\bar{V},V}\big)\circ m_{\bar{V},V}\big)\big(i_V(s) \otimes_{\mathcal{B}} t\big)$$

holds for all $s,t \in \Gamma_{\mathcal{A}}V$. Since $\mathcal{B} = (\operatorname{id}_{\mathcal{A}} \otimes \operatorname{tr}_V)(\mathcal{C}(\pi))$, the assertion that $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ is full follows from Theorem 4.10 applied to the representations $(\bar{\pi},\bar{V})$ and (π,V). $\square$

Hilbert Module Structures on the Generalized Isotypic Components

We now turn our attention back to the generalized isotypic components described in Definition 3.7. Given a finite-dimensional unitary representation (π,V) of G, it is easily seen that $A_2(\pi)$ carries a natural $\mathcal{B}\otimes\mathcal{L}(V)-\mathcal{C}(\pi)$-Hilbert bimodule structure with inner products given by

$$ {}_{\mathcal{B}\otimes\mathcal{L}(V)}\langle s,t\rangle := st^* \quad \text{and} \quad \langle s,t\rangle_{\mathcal{C}(\pi)} := s^*t$$

for $s,t \in A_2(\pi)$. Moreover, the operations restrict to a natural right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure with right $\mathcal{B}$-valued inner product given by

$$\langle s,t\rangle_{\mathcal{B}} := (\operatorname{id}_{\mathcal{A}} \otimes \operatorname{tr}_V)(s^*t)$$

for $s,t \in A_2(\pi)$. Indeed, since $s^*t \in \mathcal{C}(\pi)$ for all $s,t \in A_2(\pi)$, the well-definedness of the right $\mathcal{B}$-valued inner product is a consequence of $\mathcal{B} = (\operatorname{id}_{\mathcal{A}} \otimes \operatorname{tr}_V)(\mathcal{C}(\pi))$ and the fact that the map $\operatorname{id}_{\mathcal{A}} \otimes \operatorname{tr}_V$ is faithful and positive. That $A_2(\pi)$ is also complete with respect to the induced norm is, for example, shown in [15, Corollary 2.6 (3)]. As a matter of fact, the completeness of $A_2(\pi)$ also follows from Proposition 4.13 below. Our goal is to obtain a characterization of freeness in terms of the generalized isotypic components and their natural right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure. But before pursuing this goal, we make a small detour and recall the topological isomorphism between the spaces $A(\pi)$ and $A_2(\pi)$ from Remark 3.8. The next proposition shows that this isomorphism actually respects their right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure in case the right $\mathcal{B}$-valued inner product on $A_2(\pi)$ is normalized by the factor $\frac{1}{\dim V}$.

Proposition 4.12. *Let (π,V) be an irreducible unitary representation of G and consider the isotypic component $A(\pi)$ equipped with the right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure induced by Proposition 3.1. Then $A(\pi)$ is complete with respect to the corresponding norm and the map*

$$q : A(\pi) \to A_2(\pi), \quad q(x) := \dim V \cdot \int_G \alpha_g(x) \otimes \pi_g^* \, dg$$

from Remark 3.8 is an isomorphism of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules.

Proof. In view of Remark 3.8 it suffices to show that ${}_{\mathcal{B}}\langle q(x), q(y)\rangle = {}_{\mathcal{B}}\langle x, y\rangle$ holds for all $x, y \in \mathcal{A}$. This can easily be done by using the formula

$$\alpha_h(x) = \dim V \cdot \int_G \mathrm{tr}_V(\pi_g^* \pi_h) \cdot \alpha_g(x)\, dg$$

describing the action of G on $\mathcal{A}(\pi)$. In fact, an explicit computation shows that

$$\begin{aligned}
\langle q(x), q(y)\rangle_{\mathcal{B}} &= \dim V \cdot \int_{G\times G} \mathrm{tr}_V(\pi_g \pi_h^*) \cdot \alpha_g(x^*)\alpha_h(y)\, dg dh \\
&= \int_G \alpha_g(x^*)\Big(\dim V \cdot \int_G \mathrm{tr}_V(\pi_g \pi_h^*) \cdot \alpha_h(y)\, dh\Big)\, dg \\
&= \int_G \alpha_g(x^*)\Big(\dim V \cdot \int_G \mathrm{tr}_V(\pi_h^* \pi_g) \cdot \alpha_h(y)\, dh\Big)\, dg \\
&= \int_G \alpha_g(x^*)\alpha_g(y)\, dg = \int_G \alpha_g(x^* y)\, dg = \langle x, y\rangle_{\mathcal{B}}.
\end{aligned}$$

□

We proceed with two finite-dimensional unitary representations (π, V) and (ρ, W) of G. For elements $s = \sum_i x_i \otimes S_i \in \mathcal{A} \otimes \mathcal{L}(V)$ and $t = \sum_j y_j \otimes T_j \in \mathcal{A} \otimes \mathcal{L}(W)$ we define

$$m_{\pi,\rho}(s \otimes_{\mathcal{B}} t) := \sum_{i,j} x_i y_j \otimes S_i \otimes T_j \quad \text{and} \quad i_\pi(s) := \sum_i x_i^* \otimes \bar{T}_i$$

which similar as before give rise to well-defined maps

$$m_{\pi,\rho} : A_2(\pi) \otimes_{\mathcal{B}} A_2(\rho) \to A_2(\pi \otimes \rho) \quad \text{and} \quad i_\pi : A_2(\pi) \to A_2(\bar{\pi}).$$

The next result establishes an identification between the spaces $\Gamma_{\mathcal{A}}(\bar{V}) \otimes V$ and $A_2(\pi)$ which can be used to obtain statements involving $A_2(\pi)$ by applying the results of the first part of this section to $\Gamma_{\mathcal{A}}(\bar{V}) \otimes V$. The arguments are consist of straightforward computations and therefore we omit the proofs.

Lemma 4.13. *The map $\varphi_\pi : \mathcal{A} \otimes \bar{V} \otimes V \to \mathcal{A} \otimes \mathcal{L}(V)$ given on simple tensors by*

$$\varphi_\pi(x \otimes \bar{v_1} \otimes v_2) := x \otimes \theta_{v_2, v_1}$$

restricts to an isomorphism of Hilbert $B \otimes \mathcal{L}(V) - \mathcal{C}(\pi)$-bimodules between the spaces $\Gamma_{\mathcal{A}}(\bar{V}) \otimes V$ and $A_2(\pi)$, and, therefore, also to an isomorphism of right Hilbert $\mathcal{B} - \mathcal{B}$-bimodules.

Corollary 4.14. *The map $\varphi_{\bar{\pi},\pi} : \mathcal{A} \otimes \mathcal{L}(\bar{V} \otimes V) \to \mathcal{A} \otimes \mathcal{L}(V) \otimes \mathcal{L}(V)$ given on simple tensors and rank one operators by*

$$\varphi_{\bar{\pi},\pi}(x \otimes \theta_{\bar{v_1}\otimes v_2, \bar{v_3}\otimes v_4}) := x \otimes \theta_{v_3, v_4} \otimes \theta_{v_2, v_1}$$

restricts to an isomorphism of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules between the spaces $A_2(\bar{\pi} \otimes \pi)$ and $\mathcal{C}(\pi) \otimes \mathcal{L}(V)$.

We continue with pointing out a multiplicative relationship between the generalized isotypic components. It follows from Theorem 4.10 and Lemma 4.13.

Corollary 4.15. *Let (π, V) and (ρ, W) be two finite-dimensional unitary representations of G. Then the map*

$$m_{\pi,\rho} : A_2(\pi)\widehat{\otimes}_{\mathcal{B}}A_2(\rho) \to A_2(\pi \otimes \rho) \tag{2}$$

is an isometry of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. Moreover, if $(\mathcal{A}, G, \alpha)$ is free, then $m_{\pi,\rho}$ is an isomorphism of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules.

We are finally ready to give a characterization of freeness in terms of the Hilbert $\mathcal{B}-\mathcal{B}$-bimodule maps defined in Corollary 4.15 (2).

Corollary 4.16. *The following statements are equivalent:*

(*a*) *The C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free.*

(*b*) *For each $\pi \in \widehat{G}$ the map $m_{\bar{\pi},\pi}$ defined in Corollary 4.15 is an isomorphism of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules.*

(*c*) *For each $\pi \in \widehat{G}$ the space $A_2(\pi)$ carries the structure of a Morita equivalence $B \otimes \mathcal{L}(V) - \mathcal{C}(\pi)$-bimodule.*

Proof. The implication (a) $\Rightarrow$ (b) is a direct consequence of Corollary 4.15. Moreover, the implication (b) $\Rightarrow$ (c) follows from an adaption of the argument in Corollary 4.11. Indeed, we first note that the $\mathcal{C}(\pi)$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{C}(\pi)}$ satisfies

$$\langle s, t\rangle_{\mathcal{C}(\pi)} = s^*t = \big((\,\mathrm{id}_{\mathcal{A}\otimes\mathcal{L}(V)} \otimes \mathrm{tr}_V) \circ \varphi_{\bar{\pi},\pi} \circ m_{\bar{\pi},\pi}\big)(i_\pi(s) \otimes_{\mathcal{B}} t)$$

for all $s, t \in A_2(\pi)$. Therefore, Corollary 4.14 and Corollary 4.15 imply that the range of $\langle \cdot, \cdot \rangle_{\mathcal{C}(\pi)}$ is dense in $\mathcal{C}(\pi)$, i. e., $\langle \cdot, \cdot \rangle_{\mathcal{C}(\pi)}$ is full. A similar observation shows that also the $\mathcal{B}\otimes\mathcal{L}(V)$-valued inner product ${}_{\mathcal{B}\otimes\mathcal{L}(V)}\langle \cdot, \cdot \rangle$ is full. To verify the last implication, we only point out that the assumption in (c) means that the $\mathcal{C}(\pi)$-valued inner products $\langle \cdot, \cdot \rangle_{\mathcal{C}(\pi)}$, $\pi \in \widehat{G}$, are full which is equivalent saying that $\mathrm{Sp}(\alpha) = \widehat{G}$. The assertion therefore follows from Theorem 3.10 (c) $\Rightarrow$ (a). $\square$

5 On Free Actions of Compact Abelian Groups: Some General Theory

In the remaining part of this article we assume that G is a compact abelian group if not mentioned otherwise. In this situation, given a C*-dynamical system $(\mathcal{A}, G, \alpha)$ and a character $\pi \in \widehat{G} = \mathrm{Hom}_{\mathrm{gr}}(G, \mathbb{T})$, we have

$$A(\pi) = \{a \in \mathcal{A} \mid \alpha_g(a) = \pi_g \cdot a \ \text{ for all } \ g \in G\}.$$

Moreover, it is easily seen that $A_2(\pi) = A(\pi)$ and that $\mathcal{C}(\pi) = \mathcal{B}$. Hence, Corollary 4.16 simplifies to the following statement.

Corollary 5.1. *Let $(\mathcal{A}, G, \alpha)$ is a C^*-dynamical system. Then the following conditions are equivalent:*

(a) *The C^*-dynamical system $(\mathcal{A}, G, \alpha)$ is free.*

(b) *For each $\pi \in \widehat{G}$ the space $A(\pi)$ is a Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodule.*

(c) *For each $\pi \in \widehat{G}$ the multiplication map on $\mathcal{A}$ induces an isomorphism of Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodules between $A(-\pi)\widehat{\otimes}_{\mathcal{B}}A(\pi)$ and $\mathcal{B}$.*

As we will soon see below, Corollary 5.1 gives rise to a first invariant for free actions of compact abelian groups. For the time being, we continue with some examples to get more comfortable with free actions of compact abelian groups.

Example 5.2. Let θ be a real skew-symmetric $n \times n$ matrix. The *noncommutative n-torus* $\mathbb{T}^n_\theta$ is the universal unital C*-algebra generated by unitaries $U_1, \dots, U_n$ with

$$U_r U_s = \exp(2\pi i \theta_{rs}) U_s U_r \quad \text{for all} \quad 1 \leq r, s \leq n.$$

It carries a continuous action α of the n-dimensional torus $\mathbb{T}^n$ by algebra automorphisms which is on generators defined by

$$\alpha_z(U^{\mathbf{k}}) := z^{\mathbf{k}} \cdot U^{\mathbf{k}},$$

where $z^{\mathbf{k}} := z_1^{k_1} \cdots z_n^{k_n}$ and $U^{\mathbf{k}} := U_1^{k_1} \cdots U_n^{k_n}$ for $z = (z_1, \dots, z_n) \in \mathbb{T}^n$ and $\mathbf{k} := (k_1, \dots, k_n) \in \mathbb{Z}^n$. The isotypic component $(\mathbb{T}^n_\theta)(\mathbf{k})$ corresponding to the character $\mathbf{k} \in \mathbb{Z}^n$ is given by $\mathbb{C} \cdot U^{\mathbf{k}}$. In particular, each isotypic component contains invertible elements from which we conclude that the C*-dynamical system $(\mathbb{T}^n_\theta, \mathbb{T}^n, \alpha^n_\theta)$ is free.

Example 5.3. The group C*-algebra $C^*(H)$ of the discrete (three-dimensional) *Heisenberg group H* is the universal C*-algebra generated by unitaries U, V and W satisfying

$$UW = WU, \quad VW = WV \quad \text{and} \quad UV = WVU.$$

It carries a continuous action α of the 2-dimensional torus $\mathbb{T}^2$ by algebra automorphisms which is on generators defined by

$$\alpha_{(z,w)}(U^k V^l W^m) := z^k w^l \cdot U^k V^l W^m,$$

where $(z, w) \in \mathbb{T}$ and $k, l, m \in \mathbb{Z}$. The corresponding fixed point algebra $\mathcal{B}$ is the centre of $C^*(H)$ which is equal to the group C^*-algebra $C^*(Z)$ of the center $Z \cong \mathbb{Z}$ of H. Moreover, the isotypic component $C^*(H)_{(k,l)}$ corresponding to the character $(k, l) \in \mathbb{Z}^2$ is given by $\mathcal{B} \cdot U^k V^l$. In particular, each isotypic component $C^*(H)_{(k,l)}$ contains invertible elements from which we conclude that the C*-dynamical system $(C^*(H), \mathbb{T}^2, \alpha)$ is free. We point out that $C^*(H)$ serves as a “universal” noncommutative principal $\mathbb{T}^2$-bundle in [18] and that its K-groups are isomorphic to $\mathbb{Z}_3$.

Example 5.4. For $q \in [-1, 1]$ consider the C*-algebra $\mathrm{SU}_q(2)$ from [59]. We recall that it is the universal C*-algebra generated by two elements a and c subject to the five relations

$$a^*a + cc^* = 1, \quad aa^* + q^2cc^* = 1, \quad cc^* = c^*c, \quad ac = qca \quad \text{and} \quad ac^* = qc^*a.$$

It carries a continuous action α of the one-dimensional torus $\mathbb{T}$ by algebra automorphisms, which is on the generators defined by

$$\alpha_z(a) := z \cdot a \quad \text{and} \quad \alpha_z(c) := z \cdot c, \quad z \in \mathbb{T}.$$

The fixed point algebra of this action is the *quantum 2-sphere* S_q^2 and we call the corresponding C*-dynamical system $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ the *quantum Hopf fibration.* It is free according to [49, Corollary 3]. In fact, the author shows that if E is a locally finite graph with no sources and no sinks, then the natural gauge action on the graph C*-algebra $C^*(E)$ is free.

Remark 5.5. We recall that Example 5.2 and Example 5.3 are special cases of so-called *trivial noncommutative principal bundles* as discussed in [51, 52, 54]. In fact, it is not hard to see that each trivial noncommutative principal bundle is free (cf. [47]).

We now turn our attention back to Corollary 5.1 which suggests the relevance of Morita equivalence $\mathcal{B} - \mathcal{B}$ bimodules. These objects have a natural interpretation as noncommutative line bundles "over" $\mathcal{B}$. In particular, just like in the classical theory of line bundles, the set of their equivalence classes carry a natural group structure.

Definition 5.6. Let $\mathcal{B}$ be a C*-algebra. Then the set of equivalence classes of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules form a group with respect to the internal tensor product of Hilbert $\mathcal{B}-\mathcal{B}$-bimodules, which is called the *Picard group* of $\mathcal{B}$ and is denoted by $\mathrm{Pic}(\mathcal{B})$.

Example 5.7. Let $\mathcal{B}$ be a C*-algebra and $\alpha \in \mathrm{Aut}(\mathcal{B})$. Furthermore, let M_α be the vector space $\mathcal{B}$ endowed with the canonical left Hilbert $\mathcal{B}$-module structure, but with the right action of $\mathcal{B}$ on M_α given by $m.b := m\alpha(b)$ for $m \in M_\alpha$ and $b \in \mathcal{B}$, and the right $\mathcal{B}$-valued inner product given by $\langle m_1, m_2\rangle_{\mathcal{B}} = \alpha^{-1}(m_1^* m_2)$ for $m_1, m_1 \in M_\alpha$. It is straightforward from the construction that M_α is a Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule, and if $\beta \in \mathrm{Aut}(\mathcal{B})$ is another automorphism, then $M_\alpha \widehat{\otimes}_{\mathcal{B}} M_\beta$ is $\mathcal{B}-\mathcal{B}$-Morita equivalent to $M_{\alpha\circ\beta}$. Hence, we obtain an group homomorphism from $\mathrm{Aut}(\mathcal{B})$ to $\mathrm{Pic}(\mathcal{B})$.

For $u \in U(\mathcal{B})$ let $\mathrm{Ad}_u : \mathcal{B} \to \mathcal{B}$, $b \mapsto ubu^*$ be the corresponding conjugation map. Then Ad_u is an inner automorphism of $\mathcal{B}$, i. e., $\mathrm{Ad}_u \in \mathrm{Inn}(\mathcal{B})$ and it is easily seen that the map $\mathcal{B} \to M_{\mathrm{Ad}_u}$, $b \mapsto bu$ is an isomorphism of Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodules. Summarizing we have the following exact sequence

$$1 \longrightarrow \mathrm{Inn}(\mathcal{B}) \longrightarrow \mathrm{Aut}(\mathcal{B}) \longrightarrow \mathrm{Pic}(\mathcal{B}).$$

In particular, we have an injection $\mathrm{Out}(\mathcal{B}) \to \mathrm{Pic}(\mathcal{B})$ which turns out to be an isomorphism in the case $\mathcal{B}$ is separable and stable (cf. [8, Section 3]).

Example 5.8. ([8, Section 3]). Let $\mathcal{B}$ be a finite-dimensional C*-algebra. Then $\operatorname{Pic}(\mathcal{B})$ is isomorphic to the group of permutations of the spectrum of $\mathcal{B}$.

Example 5.9. ([8, 9]). Let $\mathcal{B} = C(X)$ for some compact space X. Then $\operatorname{Pic}(\mathcal{B})$ is isomorphic to the semi-direct product $\operatorname{Pic}(X) \rtimes \operatorname{Homeo}(X)$, where $\operatorname{Pic}(X)$ denotes the set of equivalence classes of complex line bundles over X and $\operatorname{Homeo}(X)$ the group of homeomorphisms of X.

Example 5.10. ([27]). Let θ be an irrational number in $[0,1]$ and $\mathbb{T}^2_\theta$ the corresponding quantum 2-torus. Then $\operatorname{Pic}(\mathbb{T}^2_\theta)$ is isomorphic to $\operatorname{Out}(\mathbb{T}^2_\theta)$ in case θ is quadratic and to $\operatorname{Out}(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}$ otherwise.

The next statement is a first step towards finding invariants, i. e., classification data, for free C*-dynamical systems with a prescribed fixed point algebra.

Proposition 5.11. *Each free C*-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathcal{B}$ gives rise to a group homomorphism $\varphi_{\mathcal{A}} : \widehat{G} \to \operatorname{Pic}(\mathcal{B})$ given by $\varphi_{\mathcal{A}}(\pi) := [A(\pi)]$.*

Proof. To verify the assertion we choose $\pi, \rho \in \widehat{G}$ and use Corollary 5.1 to compute

$$\varphi_{\mathcal{A}}(\pi + \rho) = [A(\pi + \rho)] = [A(\pi)\widehat{\otimes}_{\mathcal{B}} A(\rho)] = [A(\pi)][A(\rho)] = \varphi_{\mathcal{A}}(\pi)\varphi_{\mathcal{A}}(\rho).$$

This shows that the map $\varphi_{\mathcal{A}}$ is a group homomorphism. □

Remark 5.12. Let $\mathbb{K}$ be the algebra of compact operators on some separable Hilbert space. The C*-algebra $\mathrm{SU}_q(2)$ is described in [16] as an extension of $C(\mathbb{T})$ by $C(\mathbb{T}) \otimes \mathbb{K}$, i. e., by a short exact sequence

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathbb{K} \longrightarrow \mathrm{SU}_q(2) \to C(\mathbb{T}) \longrightarrow 0 \tag{3}$$

of C*-algebras. If we consider $C(\mathbb{T})$ endowed with the canonical $\mathbb{T}$-action induced by right-translation, then a short observation shows that the sequence (3) is in fact $\mathbb{T}$-equivariant. In particular, it induces the following short exact sequence of C*-algebras:

$$0 \longrightarrow (C(\mathbb{T}) \otimes \mathbb{K}) \rtimes \mathbb{T} \longrightarrow \mathrm{SU}_q(2) \rtimes_\alpha \mathbb{T} \longrightarrow C(\mathbb{T}) \rtimes \mathbb{T} \longrightarrow 0. \tag{4}$$

Since

$$(C(\mathbb{T}) \otimes \mathbb{K}) \rtimes \mathbb{T} \cong (C(\mathbb{T}) \rtimes \mathbb{T}) \otimes \mathbb{K} \cong \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$$

and $C(\mathbb{T}) \rtimes \mathbb{T} \cong \mathbb{K}$ by the well-known Stone-von Neumann Theorem, we conclude from [45, Proposition 6.12] that the crossed product $\mathrm{SU}_q(2) \rtimes_\alpha \mathbb{T}$ is stable. Moreover, the fact that the C*-dynamical system $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ is free implies that the crossed product $\mathrm{SU}_q(2) \rtimes \mathbb{T}$ is Morita equivalent to the corresponding fixed-point algebra S^2_q and thus they are also stably isomorphic by a famous result of Brown, Green and Rieffel (cf. [8, Theorem 1.2] or [43, Section 5.5]), i. e., $\mathrm{SU}_q(2) \rtimes \mathbb{T} \cong S^2_q \otimes \mathbb{K}$. The previous result affirms a question of the second author in the context of a notion of freeness which is related to Green's Theorem (cf. [21, Corollary 15] and [18]).

6 Construction of Free Actions of Compact Abelian Groups

Throughout this section let $\mathcal{B}$ be a fixed unital C*-algebra and G a fixed compact abelian group. Our goal is to construct a free C*-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathcal{B}$ from data which is only associated to the pair $(\mathcal{B}, G)$. Before we step into the details, let us briefly review the adaption of the GNS-representation for C*-dynamical systems with arbitrary fixed point algebra. The constructions in later sections will essentially reconstruct this representation from the isotypic components and the structure among them. So let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system and $P_0 : \mathcal{A} \to \mathcal{A}$ the conditional expectation onto the fixed point algebra $\mathcal{B}$. Then $\mathcal{A}$ can be equipped with the structure of a right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule with respect to the usual multiplication and the inner product given by $\langle x, y\rangle_{\mathcal{B}} := P_0(x^*y)$ for $x, y \in \mathcal{A}$. Since P_0 is faithful, this inner product on $\mathcal{A}$ is definite and we may take the completion of $\mathcal{A}$ with respect to the norm $\|x\|_2 := \|P_0(x^*x)\|^{1/2}$. This provides a right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule $L^2(\mathcal{A})$ with $\mathcal{A}$ as a dense subset. For each $\pi \in \hat{G}$ the projection P_π onto the isotypic component $A(\pi)$ can be continuously extended to a self-adjoint projection on $L^2(\mathcal{A})$. In particular, the sets $A(\pi)$ are closed, pairwise orthogonal right Hilbert $\mathcal{B}-\mathcal{B}$-subbimodules of $L^2(\mathcal{A})$ and $L^2(\mathcal{A})$ can be decomposed into

$$L^2(\mathcal{A}) = \overline{\bigoplus_{\pi\in\hat{G}} A(\pi)}^{\|\cdot\|_2}$$

as right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules. For each element $a \in \mathcal{A}$ the left multiplication operator $\lambda_a : \mathcal{A} \to \mathcal{A}$, $x \mapsto ax$, then extends to an adjointable operator on $L^2(\mathcal{A})$. The arising representation

$$\lambda : \mathcal{A} \to \mathcal{L}(L^2(\mathcal{A})), \quad a \mapsto \lambda_a$$

is called the *left regular representation* of $\mathcal{A}$. For each $g \in G$ the automorphism α_g extends from $\mathcal{A}$ to an automorphism $U_g : L^2(\mathcal{A}) \to L^2(\mathcal{A})$ of right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules and the strongly continuous group $(U_g)_{g\in G}$ implements α_g in the sense that

$$\alpha_g(\lambda_a) = U_g\, \lambda_a\, U_g^+$$

for all $a \in \mathcal{A}$. The vector $\mathbb{1}_{\mathcal{B}} = \mathbb{1}_{\mathcal{A}} \in L^2(\mathcal{A})$ is obviously cyclic and separating for this representation. In particular, the left regular representation is faithful and we may identify $\mathcal{A}$ with the subalgebra $\lambda(\mathcal{A})$. Since the sum of the isotypic components is dense in $\mathcal{A}$, the C*-algebra $\lambda(\mathcal{A})$ is in fact generated by the operators λ_a with $a \in A(\pi)$, $\pi \in \hat{G}$. For such elements a in some fixed isotypic component $A(\pi)$, $\pi \in \hat{G}$, the operator λ_a maps each subset $A(\rho) \subseteq L^2(\mathcal{A})$, $\rho \in \hat{G}$, into $A(\pi + \rho)$ and therefore it is determined by the multiplication map

$$m_{\pi,\rho} : A(\pi) \otimes A(\rho) \to A(\pi + \rho), \quad m_{\pi,\rho}(x, y) := xy = \lambda_x(y).$$

It is easily verified that $m_{\pi,\rho}$ factors to an isometry of the right Hilbert $\mathcal{B}-\mathcal{B}$-bimodules $A(\pi) \otimes_{\mathcal{B}} A(\rho)$ and $A(\pi + \rho)$.

6.1 Associativity and Factor Systems

In what follows we consider a fixed group homomorphism $\varphi : \widehat{G} \to \operatorname{Pic}(\mathcal{B})$. Furthermore, we choose for each $\pi \in \widehat{G}$ a Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule M_π in the isomorphism class $\varphi(\pi) \in \operatorname{Pic}(\mathcal{B})$. Since φ is a group homomorphism, the Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules $M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho$ and $M_{\pi+\rho}$ must be isomorphic for all $\pi, \rho \in \widehat{G}$ with respect to some isomorphism

$$\Psi_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_{\pi+\rho}$$

of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules. We additionally choose for each pair $\pi, \rho \in \widehat{G}$ such an isomorphism $\Psi_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_{\pi+\rho}$, which in turn provides a bilinear map

$$m_{\pi,\rho} : M_\pi \times M_\rho \to M_{\pi+\rho}, \quad m_{\pi,\rho}(x, y) := \Psi_{\pi,\rho}(x \otimes_{\mathcal{B}} y).$$

The family of all such maps $(m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ now gives rise to a multiplication map m on the algebraic vector space

$$A = \bigoplus_{\pi\in\widehat{G}} M_\pi.$$

Proposition 6.1. *The following statements are equivalent:*

(a) *m is associative, i. e., A is an algebra.*

(b) *For all $\pi, \rho, \sigma \in \widehat{G}$ we have*

$$\Psi_{\pi+\rho,\sigma} \circ (\Psi_{\pi,\rho} \otimes_{\mathcal{B}} \mathrm{id}_\sigma) = \Psi_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{\rho,\sigma}). \tag{5}$$

Proof. For given $\pi, \rho, \sigma \in \widehat{G}$ we explicitly compute for all $x \in \mathrm{M}_\pi$, $y \in M_\rho$, $z \in M_\sigma$:

$$\begin{aligned} m(x, m(y, z)) &= m(x, \Psi_{\rho,\sigma}(y \otimes_{\mathcal{B}} z)) = \Psi_{\pi,\rho+\sigma}(x, \Psi_{\rho,\sigma}(y \otimes_{\mathcal{B}} z)) \\ m(m(x, y), z) &= m(\Psi_{\pi,\rho}(x \otimes_{\mathcal{B}} y), z) = \Psi_{\pi+\rho,\sigma}(\Psi_{\pi,\rho}(x \otimes_{\mathcal{B}} y), z). \end{aligned}$$

Therefore, m is associative if and only if equation (5) holds for all $\pi, \rho, \sigma \in \widehat{G}$. □

Definition 6.2. We call a family $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ a *factor system* for φ if it satisfies $(M_0, \Psi_{0,0}) = (\mathcal{B}, \mathrm{id}_{\mathcal{B}})$ and the condition in Proposition 6.1.

Remark 6.3. The normalization condition implies $\Psi_{\pi,0} = \Psi_{0,\pi} = \mathrm{id}_\pi$ and

$$\Psi_{\pi,-\pi} \otimes_{\mathcal{B}} \mathrm{id}_\pi = \mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{-\pi,\pi}. \tag{6}$$

for all $\pi \in \widehat{G}$. In particular, it follows that the algebra A equipped with the multiplication map m is unital with unit $\mathbb{1}_{\mathcal{B}} \in \mathcal{B} \subseteq A$. It also assures that on each subspace $M_\pi \subseteq A$ the multiplication with elements of $\mathcal{B}$ coincides with the usual action of $\mathcal{B}$.

6.2 Construction of an Involution

We continue with a fixed factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ for φ and we write A for the associated algebra. Our goal is to turn A into a *-algebra and right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule. For this purpose we will involve the right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule structure of each direct summands M_π of A, i. e., the right $\mathcal{B}$-valued inner products $\langle\cdot,\cdot\rangle_\pi$ on M_π.

Lemma 6.4. *The map $\langle\cdot,\cdot\rangle : A \times A \to \mathcal{B}$ defined for $x = \bigoplus_\pi x_\pi, y = \bigoplus_\pi y_\pi \in A$ by*

$$\langle x, y\rangle_{\mathcal{B}} := \sum_{\pi\in\widehat{G}} \langle x_\pi, y_\pi\rangle_\pi$$

turns A into a right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodule and satisfies

$$\langle m(b,x), m(b,x)\rangle_{\mathcal{B}} \leq \|b\|^2 \langle x, x\rangle_{\mathcal{B}} \tag{7}$$

for all $x \in A$ and $b \in \mathcal{B}$.

Proof. The necessary computations are straightforward and thus left to the reader. We only point out that the inequality (7) is a consequence of the corresponding inequalities satisfied by the right $\mathcal{B}$-valued inner products $\langle\cdot,\cdot\rangle_\pi$. ☐

Lemma 6.5. *For each $y \in M_\pi$ and every $\rho \in \widehat{G}$ the left multiplication operator*

$$\ell_y : M_\rho \to M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho, \quad x \mapsto y \otimes_{\mathcal{B}} x$$

is adjointable (and hence bounded) with adjoint given by

$$\ell_y^+ : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_\rho, \quad z \otimes_{\mathcal{B}} x \mapsto \langle y, z\rangle_{\mathcal{B}}.x.$$

Proof. To verify the assertion we first note that the linear span of elements $z \otimes_{\mathcal{B}} z'$ with $z \in M_\pi$ and $z' \in M_\rho$ is dense in $M_{\pi+\rho}$. For such an element and $x \in M_\rho$ we obtain

$$\langle \ell_y(x),\ z \otimes_{\mathcal{B}} z'\rangle_{\mathcal{B}} = \langle y \otimes_{\mathcal{B}} x, z \otimes_{\mathcal{B}} z'\rangle_{\mathcal{B}} = \langle x, \langle y, z\rangle.z'\rangle_{\mathcal{B}} = \langle x, \ell_y^+(z \otimes_{\mathcal{B}} z')\rangle_{\mathcal{B}}$$

which implies that ℓ_y is adjointable with adjoint given by the map ℓ_y^+. ☐

Proposition 6.6. *For each $y \in M_\pi$ and every $\rho \in \widehat{G}$ the left multiplication operator*

$$\lambda_y : M_\rho \to M_{\pi+\rho}\ \ \lambda_y(x) := m(y, x)$$

is adjointable (and hence bounded) and satisfies

$$\langle \lambda_y(x),\ \lambda_y(x)\rangle_{\mathcal{B}} \leq \|\langle y, y\rangle_{\mathcal{B}}\| \cdot \langle x, x\rangle_{\mathcal{B}} \tag{8}$$

for all $x \in M_\rho$.

Proof. That the left multiplication operator $\lambda_y : M_\rho \to M_{\pi+\rho}$ is adjointable for each $y \in M_\pi$ and every $\rho \in \widehat{G}$ is an immediate consequence of Lemma 6.5 and the unitarity of the map $\Psi_{\pi,\rho}$ because $\lambda_y = \Psi_{\pi,\rho} \circ \ell_y$. The asserted inequality (8) then easily follows from a short computation involving inequality (7). Indeed, we obtain

$$\begin{aligned}\langle \lambda_y(x),\ \lambda_y(x)\rangle_{\mathcal{B}} &= \langle \Psi_{\pi,\rho}(y \otimes_{\mathcal{B}} x), \Psi_{\pi,\rho}(y \otimes_{\mathcal{B}} x)\rangle_{\mathcal{B}} = \langle y \otimes_{\mathcal{B}} x, y \otimes_{\mathcal{B}} x\rangle_{\mathcal{B}} \\ &= \langle x, \langle y, y\rangle_{\mathcal{B}}.x\rangle_{\mathcal{B}} \leq \|\langle y, y\rangle_{\mathcal{B}}\| \langle x, x\rangle_{\mathcal{B}}\end{aligned}$$

for all $x \in M_\rho$. □

Corollary 6.7. *For each $a \in A$ the left multiplication operator*

$$\lambda_a : A \to A, \quad \lambda_a(x) := m(a, x)$$

is adjointable and bounded.

We are now ready to introduce an involution on A which turns A into a *-algebra. Here we use the fact that the involution is determined by the inner product if we impose that the inner product on A takes it canonical form.

Definition 6.8. The adjoint map $i : A \to A$, $a \mapsto i(a)$ is given by

$$i(a) := \lambda_a^+(\mathbb{1}_{\mathcal{B}}).$$

It is clearly antilinear and maps the subspace M_π, $\pi \in \widehat{G}$, into $M_{-\pi}$. Moreover, on the subspace $\mathcal{B} \subseteq A$ the imap coincides with the usual adjoint, i. e., we have $i(b) = b^*$.

The following lemma shows that the adjoints of left multiplication operators commute with right multiplication operators.

Lemma 6.9. *For all $x \in M_\pi$, $y \in M_\rho$ and $z \in M_{\pi+\sigma}$ with $\pi, \rho, \sigma \in \widehat{G}$ we have*

$$m(\lambda_x^+(z), y) = \lambda_x^+(m(z, y)).$$

Proof. It suffices to note that equation (5) implies that

$$\lambda_x \circ \Psi_{\sigma,\rho} = \Psi_{\pi+\sigma,\rho} \circ (\lambda_x \otimes_{\mathcal{B}} \mathrm{id}_\rho).$$

Indeed, taking adjoints then leads to

$$\Psi_{\sigma,\rho}^+ \circ \lambda_x^+ = (\lambda_x^+ \otimes_{\mathcal{B}} \mathrm{id}_\rho) \circ \Psi_{\pi+\sigma,\rho}^+$$

which verifies the asserted formula since the maps $\Psi_{\sigma,\rho}$ and $\Psi_{\pi+\sigma,\rho}$ are unitary. □

Theorem 6.10. *For all $x \in A$ we have $\lambda_x^+ = \lambda_{i(x)}$.*

Proof. It suffices to show the assertion for elements in individual direct summands. For this, let $x \in M_\pi$, $y \in M_\rho$, and $z \in M_\sigma$ with $\pi, \rho, \sigma \in \widehat{G}$. Then using Lemma 6.9 gives

$$\begin{aligned}\langle \lambda_{i(x)}(y), z\rangle_{\mathcal{B}} &= \langle m(i(x), y), z\rangle_{\mathcal{B}} = \langle m(\lambda_x^+(\mathbb{1}_{\mathcal{B}}), y), z\rangle_{\mathcal{B}} = \langle \lambda_x^+(m(\mathbb{1}_{\mathcal{B}}, y), z\rangle_{\mathcal{B}} \\ &= \langle m(\mathbb{1}_{\mathcal{B}}, y), m(x, z)\rangle_{\mathcal{B}} = \langle y, m(x, z)\rangle_{\mathcal{B}} = \langle \lambda_x^+(y), z\rangle_{\mathcal{B}}.\end{aligned}$$ □

We conclude this section with a bunch of useful corollaries, e. g., we finally verify that the map $i : A \to A$ from Definition 6.8 actually defines an involution.

Corollary 6.11. *Let $P_0 : A \to A$ be the canonical projection onto the subalgebra $\mathcal{B}$. Then for all $x, y \in A$ we have*

$$\langle x, y\rangle_{\mathcal{B}} = P_0(m(i(y), x)).$$

Proof. Since the element $\mathbb{1}_{\mathcal{B}}$ is fixed by P_0 we conclude from Theorem 6.10 that

$$\langle x, y\rangle_{\mathcal{B}} = \langle m(i(y), x), \mathbb{1}_{\mathcal{B}}\rangle_{\mathcal{B}} = \langle P_0(m(i(y), x)), \mathbb{1}_{\mathcal{B}}\rangle_{\mathcal{B}} = P_0(m(i(y), x)).$$ □

Corollary 6.12. *The algebra A is involutive, i. e., for all $x, y \in A$ we have*

$$i(i(x)) = x \quad \textit{and} \quad i(m(x, y)) = m(i(y), i(x)).$$

Proof. Applying Theorem 6.10 twice gives

$$\begin{gathered}\langle i(i(x)), z\rangle_{\mathcal{B}} = \langle \mathbb{1}_{\mathcal{B}}, m(i(x), z)\rangle_{\mathcal{B}} = \langle x, z\rangle_{\mathcal{B}}, \\ \langle i(m(x, y)), z\rangle_{\mathcal{B}} = \langle \mathbb{1}_{\mathcal{B}}, m(m(x, y), z)\rangle_{\mathcal{B}} = \langle i(x), m(y, z)\rangle_{\mathcal{B}} = \langle m(i(y), i(x)), z\rangle_{\mathcal{B}}\end{gathered}$$

for all $z \in A$ which in turn implies that $i(i(x)) = x$ and $i(m(x, y)) = m(i(y), i(x))$. □

6.3 Construction of a Free Action

In the last subsection we turned $A = \bigoplus_{\pi \in \widehat{G}} M_\pi$ into a *-algebra and right pre-Hilbert $\mathcal{B} - \mathcal{B}$-bimodule. We denote by $\bar{A}$ the corresponding completion of A with respect to the norm

$$\|x\|_2 := \|\langle x, x\rangle_{\mathcal{B}}\|^{1/2} = \|P_0(m(i(x), x))\|^{1/2}$$

(cf. Corollary 6.11). By Corollary 6.7, left multiplication with an element $a \in A$ extends to an adjointable linear map on $\bar{A}$. Therefore, the map

$$\lambda : A \to \mathcal{L}(\bar{A}), \quad a \mapsto \lambda_a$$

is well-defined. Moreover, the characterization of the norm implies that the vector $\mathbb{1}_{\mathcal{B}} \in \bar{A}$ is separating the operators $\lambda(A) \subseteq \mathcal{L}(\bar{A})$, i. e., if $\lambda_a(\mathbb{1}_{\mathcal{B}}) = 0$ for some $a \in A$ then $a = 0$. The intention of this section is to finally construct a free C*-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathcal{B}$.

Proposition 6.13. *The map $\lambda : A \to \mathcal{L}(\bar{A})$, $a \mapsto \lambda_a$ is a faithful representation of the *-algebra A by adjointable operators on the right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule $\bar{A}$. Moreover, its restriction to each M_π, $\pi \in \widehat{G}$, is isometric.*

Proof. The necessary algebraic conditions are easily checked using Corollary 6.12. Moreover, the injectivity of the map λ is a consequence of the previous discussion about the separating vector $\mathbb{1}_{\mathcal{B}} \in \bar{A}$. To verify that the restriction of λ to each M_π, $\pi \in \widehat{G}$, is isometric, we fix $\pi \in \widehat{G}$ and use inequality (8) of Proposition 6.6 which implies that $\|\lambda_y\|_{\mathrm{op}}^2 \leq \|y\|_2^2$ holds for all $y \in M_\pi$. On the other hand, the inequality

$$\|y\|_2^2 = \|\lambda_y(\mathbb{1}_{\mathcal{B}})\|_2^2 \leq \|\lambda_y\|_{\mathrm{op}}^2$$

follows from the observation that $\mathbb{1}_{\mathcal{B}} \in \bar{A}$ satisfies $\|\mathbb{1}_{\mathcal{B}}\|_2 = 1$. We conclude that $\|\lambda_y\|_{\mathrm{op}} = \|y\|_2$ holds for each $y \in M_\pi$, which finally shows that the restriction of λ to M_π is isometric and thus completes the proof. ☐

Definition 6.14. We denote by $\mathcal{A}$ the C*-algebra which is generated by the image of λ, i. e., the closure of $\lambda(A)$ with respect to the operator norm on $\mathcal{L}(\bar{A})$. In particular, we point out that $\mathcal{A}$ contains A as a dense *-subalgebra.

To proceed we need to endow the C*-algebra $\mathcal{A}$ with a continuous action of G by *-automorphisms. For this purpose we first construct a strongly continuous unitary representation of G on the right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule $\bar{A}$.

Lemma 6.15. *For each $\pi \in \widehat{G}$ the map $U_\pi : G \to U(M_\pi)$, $g \mapsto (U_\pi)_g$ given by*

$$(U_\pi)_g(x) := \pi_g \cdot x$$

is a strongly continuous unitary representation of G on the right $\mathcal{B}-\mathcal{B}$-Hilbert bimodule M_π. Moreover, taking direct sums and continuous extensions then gives rise to a strongly continuous unitary representation $U : G \to U(\bar{A})$, $g \mapsto U_g$ of G on the right Hilbert $\mathcal{B}-\mathcal{B}$-bimodule $\bar{A}$.

Proof. The necessary computations are straightforward using the right $\mathcal{B}$-valued inner products $\langle \cdot, \cdot \rangle_\pi$ and $\langle \cdot, \cdot \rangle$ of the spaces M_π and $\bar{A}$, respectively. ☐

Lemma 6.16. *The map $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, $g \mapsto \alpha_g$ given by*

$$\alpha_g(\lambda_a) := U_g \, \lambda_a \, U_g^+$$

is a continuous action of G on $\mathcal{A}$ by *-automorphisms.

Proof. The action property is obviously satisfied by the definition of the map α. Moreover, its continuity follows from the strong continuity of the map U from Lemma 6.15. ☐

We are finally ready to present the main result of this section.

Theorem 6.17. *The C*-dynamical system* $(\mathcal{A}, G, \alpha)$ *associated to the factor system* $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ *is free and satisfies* $A(\pi) = M_\pi$ *for all* $\pi \in \widehat{G}$*. In particular, its fixed point algebra is given by* $\mathcal{B}$*.*

Proof. (i) Let $\pi \in \widehat{G}$. We first check that the corresponding isotypic component $A(\pi)$ is equal to M_π. Indeed, using the separating vector shows that $\alpha_g(a) = U_g(a)$ holds for elements $a \in A$. In particular, the elements of $M_\pi \subseteq \mathcal{A}$ are contained in $A(\pi)$. Moreover, the continuity of the projection $P_\pi : \mathcal{A} \to \mathcal{A}$ onto $A(\pi)$ implies that $M_\pi = P_\pi(A) \subseteq \mathcal{A}$ is dense in $A(\pi)$. Since the restriction of λ to M_π is isometric (cf. Lemma 6.13), we conclude that M_π is closed in $A(\pi)$ and hence that $A(\pi) = M_\pi$ as claimed. In particular, the fixed point algebra of $(\mathcal{A}, G, \alpha)$ is given by $\mathcal{B}$.

(ii) Next we show that the C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free. For this purpose, we again fix $\pi \in \widehat{G}$. Since $A(\pi) = M_\pi$ holds by part (i) and

$$M_{-\pi} \cdot M_\pi := \operatorname{span}\{m(x, y) \mid x \in M_{-\pi}, y \in M_\pi\}$$

is dense in $\mathcal{B}$ by construction, it follows that the multiplication map on $\mathcal{A}$ induces an isomorphism of $\mathcal{B} - \mathcal{B}$-Morita equivalence bimodules between $A(-\pi)\widehat{\otimes}_{\mathcal{B}}A(\pi)$ and $\mathcal{B}$. We therefore conclude from Corollary 5.1 that the C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free. ☐

Remark 6.18. To emphasize the dependence on the factor system $(M, \Psi) := (M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$, we will occasionally write $\big(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)}\big)$ for the associated free C*-dynamical system from Theorem 6.17.

We conclude this section with an "inverse construction", i. e., we show how to associate a factor system to a given free C*-dynamical system. Moreover, we show that the corresponding C*-dynamical system is equivalent to the original one in the following sense:

Definition 6.19. Let $\mathcal{B}$ be a unital C*-algebra. Moreover, let $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}', G, \alpha')$ be two free C*-dynamical systems such that $\mathcal{A}^G = (\mathcal{A}')^G = \mathcal{B}$. We call $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}', G, \alpha')$ *equivalent* if there is a G-equivariant *-isomorphism $T : \mathcal{A} \to \mathcal{A}'$ satisfying $T_{|\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$.

Proposition 6.20. *Each free C*-dynamical system* $(\mathcal{A}, G, \alpha)$ *with fixed point algebra* $\mathcal{B}$ *gives rise to a factor system* $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ *for* $\varphi_{\mathcal{A}}$ *(cf. Proposition 5.11) with*

$$m_{\pi,\rho} : A(\pi)\widehat{\otimes}_{\mathcal{B}}A(\rho) \to A(\pi + \rho), \quad x \otimes_{\mathcal{B}} y \mapsto xy.$$

Moreover, its associated C-dynamical system is equivalent to* $(\mathcal{A}, G, \alpha)$*.*

Proof. That the pair $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ defines a factor system for $\varphi_{\mathcal{A}}$ is a consequence of Corollary 5.1 and the associativity of the multiplication map of $\mathcal{A}$. Moreover, its associated C*-dynamical system is equivalent to $(\mathcal{A}, G, \alpha)$, since they are isomorphic on the dense *-subalgebra $\bigoplus_{\pi\in\widehat{G}} A(\pi)$. ☐

7 Classification of Free Actions of Compact Abelian Groups

In the previous section we have seen how a factor system gives rise to a free C*-dynamical system and vice versa. In this section we finally establish a classification theory for free actions of compact abelian groups. If not mentioned otherwise, $\mathcal{B}$ denotes a fixed unital C*-algebra and G a fixed compact abelian group.

Definition 7.1. We write $\mathrm{Ext}(\mathcal{B}, G)$ for the set of equivalence classes of free actions with fixed point algebra $\mathcal{B}$.

For a C*-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathcal{B}$ we write $[(\mathcal{A}, G, \alpha)]$ for its equivalence class in $\mathrm{Ext}(\mathcal{B}, G)$. The group homomorphism in Proposition 5.11 then provides an invariant

$$I : \mathrm{Ext}(\mathcal{B}, G) \to \mathrm{Hom}_{\mathrm{gr}}(\widehat{G}, \mathrm{Pic}(\mathcal{B})), \quad I([(\mathcal{A}, G, \alpha)]) := \varphi_{\mathcal{A}}$$

In particular, we may partition $\mathrm{Ext}(\mathcal{B}, G)$ into the subsets

$$\mathrm{Ext}(\mathcal{B}, G, \varphi) := I^{-1}(\varphi) = \{[(\mathcal{A}, G, \alpha)] \in \mathrm{Ext}(\mathcal{B}, G) \mid \varphi_{\mathcal{A}} = \varphi\}$$

for a group homomorphism $\varphi : \hat{G} \to \mathrm{Pic}(\mathcal{B})$. For a fixed φ, set $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ may be empty. We postpone this problem until the end of the section and concentrate first on characterizing the set $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ and its C*-dynamical systems. We start with a useful statement about automorphisms of Morita equivalence bimodules. Although it might be well-known to experts, we have not found such a statement explicitly discussed in the literature.

Proposition 7.2. *Let T be an automorphism of the Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule M. Then there exists a unique unitary element u of the center of $\mathcal{B}$, i. e., an element $u \in UZ(\mathcal{B})$, such that $T(m) = u.m$ for all $m \in M$. In particular, the map*

$$\psi : UZ(\mathcal{B}) \to \mathrm{Aut}_{\mathrm{ME}}(M), \quad \psi(u)(m) := u.m,$$

is an isomorphism of groups.

Proof. We divide the proof of this statement into two steps:

(i) In the first step we show that the assertion holds for the canonical Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule $\mathcal{B}$. To see that this is true, we choose $u \in UZ(\mathcal{B})$ and note that the map $T_u : \mathcal{B} \to \mathcal{B}$, $b \mapsto u \cdot b$ defines an automorphism of the Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule $\mathcal{B}$. In particular, the assignment

$$\psi_1 : UZ(\mathcal{B}) \to \mathrm{Aut}_{\mathrm{ME}}(\mathcal{B}), \quad u \mapsto T_u$$

is an isomorphism of groups. In fact, given $T \in \mathrm{Aut}_{\mathrm{ME}}(\mathcal{B})$, a short calculation shows that T is uniquely determined by $T(\mathbb{1}_{\mathcal{B}})$ which is an element in $UZ(\mathcal{B})$.

(ii) In the second step we show that Morita equivalence automorphisms of $\mathcal{B}$ are in one-to-one correspondence with automorphisms of M. To begin with, we denote by $\overline{M}$ the conjugate module and recall that the map

$$\Psi : M\widehat{\otimes}_{\mathcal{B}}\overline{M} \to \mathcal{B}, \quad \Psi(m \otimes_{\mathcal{B}} \overline{m'}) := {}_{\mathcal{B}}\langle m, m'\rangle$$

for $m, m' \in M$ defines an isomorphism of Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodules (cf. [43, Proposition 3.28]). Therefore, given an element $T \in \mathrm{Aut}_{\mathrm{ME}}(M)$, a short observation shows that the composition map $T_\Psi := \Psi \circ (T \otimes \mathrm{id}_{\overline{M}}) \circ \Psi^{-1}$ defines an automorphism of the Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodule $\mathcal{B}$. Next, we show that the map

$$\psi_2 : \mathrm{Aut}_{\mathrm{ME}}(\mathcal{B}) \to \mathrm{Aut}_{\mathrm{ME}}(M), \quad \phi_2(T)(m) := T(\mathbb{1}_{\mathcal{B}}).m$$

is an isomorphism of groups. In fact, we first note that ϕ_2 is a well-defined and injective group homomorphism. The surjectivity of ϕ_2 is a consequence of the equation

$$\psi_2(T_\Psi)(m) = T_\Psi(\mathbb{1}_{\mathcal{B}}).m = T(m)$$

which can be verified for all $m \in M$ by using a sequence $\{m_n\}_{n\in\mathbb{N}}$ in M such that $\sum_{n=1}^{\infty} {}_{\mathcal{B}}\langle m_n, m_n\rangle$ converges to $\mathbb{1}_{\mathcal{B}}$. The assertion thus follows from $\psi = \psi_2 \circ \psi_1$. ☐

Corollary 7.3. *Let M be a Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodule and $u \in UZ(\mathcal{B})$. Then there exists a unique element $\Phi_M(u) \in UZ(\mathcal{B})$ such that $\Phi_M(u).m = m.u$ holds for all $m \in M$. Furthermore, the map*

$$\Phi_M : UZ(\mathcal{B}) \to UZ(\mathcal{B}), \quad u \mapsto \Phi_M(u)$$

is an automorphism of groups.

Proof. The first assertion is an immediate consequence of Proposition 7.2 applied to the automorphism of M defined by $m \mapsto m.u$. That the map Φ_M is an automorphism of groups follows from a short calculation. ☐

Proposition 7.4. *The map*

$$\Phi : \mathrm{Pic}(\mathcal{B}) \to \mathrm{Aut}(UZ(\mathcal{B})), \quad [M] \mapsto \Phi_M$$

is a group homomorphism.

Proof. (i) We first show that Φ is well-defined. Therefore let $\Psi : M \to N$ be an isomorphism of Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodules and $u \in UZ(\mathcal{B})$. Then

$$\Phi_M(u).\Psi(m) = \Psi(\Phi_M(u).m) = \Psi(m.u) = \Psi(m).u = \Phi_N(u).\Psi(m)$$

holds for all $m \in M$ which implies that $\Phi_M = \Phi_N$.

(ii) To see that Φ is a group homomorphism, let M and N be Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules and $u \in UZ(\mathcal{B})$. Then

$$\begin{aligned}
&\Phi_{M\widehat{\otimes}_{\mathcal{B}}N}(u).(m \otimes_{\mathcal{B}} n) = (m \otimes_{\mathcal{B}} n).u = m \otimes_{\mathcal{B}} (n.u) = m \otimes_{\mathcal{B}} (\Phi_N(u).n) \\
&= (m.\Phi_N(u)) \otimes_{\mathcal{B}} n = (\Phi_M(\Phi_N(u)).m) \otimes_{\mathcal{B}} n = (\Phi_M \circ \Phi_N)(u).(m \otimes_{\mathcal{B}} n).
\end{aligned}$$

holds for all $m \in M$ and $n \in N$ which shows that $\Phi_{M\widehat{\otimes}_{\mathcal{B}}N} = \Phi_M \circ \Phi_N$. ☐

Remark 7.5. We point out that the map Φ from Proposition 7.4 induces a map

$$\Phi_* : \mathrm{Hom}_{\mathrm{gr}}(\widehat{G}, \mathrm{Pic}(\mathcal{B})) \to \mathrm{Hom}_{\mathrm{gr}}(\widehat{G}, \mathrm{Aut}(UZ(\mathcal{B})), \quad \Phi_*(\varphi) := \Phi \circ \varphi.$$

In particular, each $\varphi \in \mathrm{Hom}_{\mathrm{gr}}(\widehat{G}, \mathrm{Pic}(\mathcal{B}))$ determines a $\widehat{G}$-module structure on $UZ(\mathcal{B})$ which enables us to make use of classical group cohomology. In fact, given an element $\varphi \in \mathrm{Hom}_{\mathrm{gr}}(\widehat{G}, \mathrm{Pic}(\mathcal{B}))$, the cohomology groups

$$H^n_\varphi(\widehat{G}, UZ(\mathcal{B})) := H^n_{\Phi\circ\varphi}(\widehat{G}, UZ(\mathcal{B}))$$

are at our disposal (cf. [31, Chapter IV]).

We now return to our main goal and continue with treating the question which factor systems give rise to equivalent free C*-dynamical systems.

Definition 7.6. Let $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ be a group homomorphism. We call two factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ for φ *equivalent* if there exists a family $(T_\pi : M_\pi \to M'_\pi)_{\pi\in\widehat{G}}$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule isomorphisms satisfying

$$\Psi'_{\pi,\rho} \circ (T_\pi \otimes_{\mathcal{B}} T_\rho) = T_{\pi+\rho} \circ \Psi_{\pi,\rho} \tag{9}$$

for all $\pi, \rho \in \widehat{G}$.

Theorem 7.7. *Let $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ be a group homomorphism. Furthermore, let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ be two factor systems for φ. Then the following statements are equivalent:*

(a) *The factor systems are equivalent.*

(b) *The associated free C*-dynamical systems are equivalent.*

Proof. (a) Suppose first that the factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ are equivalent and let $(T_\pi : M_\pi \to M'_\pi)_{\pi\in\widehat{G}}$ be a family of $\mathcal{B}-\mathcal{B}$-Morita equivalence bimodule isomorphisms such that equation (9) holds for all $\pi, \rho \in \widehat{G}$. Furthermore, let

$$A := \bigoplus_{\pi\in\widehat{G}} M_\pi \qquad \text{and} \qquad A' := \bigoplus_{\pi\in\widehat{G}} M_\pi$$

be the corresponding *-algebras with involutions given by i and i', respectively. Then a short observation shows that the direct sum of the maps $T_\pi : M_\pi \to M'_\pi$, $\pi \in \widehat{G}$, provides a G-equivariant *-isomorphism $T : A \to A'$ of algebras. In fact, the map T is clearly a G-equivariant isomorphism of right pre-Hilbert $\mathcal{B}-\mathcal{B}$-bimodules by construction. Moreover, the assumption that equation (9) holds for all $\pi, \rho \in \widehat{G}$ implies that T is multiplicative. That it is also *-preserving, i. e., that $T(i(x)) = i'(T(x))$ holds for all $x \in A$, now follows from Theorem 6.10. Passing over to the continuous extension of T provides a G-equivariant isomorphism $\bar{T} : \bar{A} \to \bar{A}'$ of right Hilbert $\mathcal{B} - \mathcal{B}$-bimodules and it is easily checked with the help of the previous discussion that the relation

$$\mathrm{Ad}[\bar{T}] \circ \lambda = \lambda' \circ T$$

holds, where $\lambda : A \to \mathcal{L}(\bar{A})$ and $\lambda' : A' \to \mathcal{L}(\bar{A}')$ denote the faithful *-representations from Proposition 6.13. In particular, we conclude that the map $\mathrm{Ad}[\bar{T}] : \mathcal{L}(\bar{A}) \to \mathcal{L}(\bar{A}')$ restricts to a G-equivariant *-isomorphism between the associated free C*-dynamical systems $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}', G, \alpha')$ which completes the first part of the proof.

(b) Suppose, conversely, that the associated free C*-dynamical systems $((\mathcal{A}, m), G, \alpha)$ and $((\mathcal{A}', m'), G, \alpha')$ are equivalent and let $T : \mathcal{A} \to \mathcal{A}'$ be a G-equivariant *-isomorphism. Then it is a consequence of the G-equivariance of the map T that the corresponding restriction maps $T_\pi := T_{|M_\pi} : M_\pi \to M'_\pi$, $\pi \in \widehat{G}$, are well-defined and $\mathcal{B} - \mathcal{B}$ bimodule isomorphisms. Moreover, the multiplicativity of T implies that equation (9) holds for all $\pi, \rho \in \widehat{G}$. Hence it remains to show that the family $(T_\pi : M_\pi \to M'_\pi)_{\pi \in \widehat{G}}$ preserves the $\mathcal{B}$-valued inner products. To see that this is true, we first conclude from the *-invariance of T that $T_\pi(x)^* = T_{-\pi}(x^*)$ holds for all $\pi \in \widehat{G}$ and all $x \in M_\pi$. It follows from a short computation involving equation (9) that

$$\langle T_\pi(x), T_\pi(y)\rangle_{\mathcal{B}} = m'(T_\pi(x), T_\pi(y)^*) = m'(T_\pi(x), T_{-\pi}(y^*)) = m(x, y^*) = \langle x, y\rangle_{\mathcal{B}}$$

holds for all $\pi \in \widehat{G}$ and all $x, y \in M_\pi$. The corresponding computation for the left $\mathcal{B}$-valued inner products can be verified in a similar way and completes the proof. □

Theorem 7.8. *Let $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ be a group homomorphism with $\mathrm{Ext}(\mathcal{B}, G, \varphi) \neq \emptyset$. Furthermore, choose for all $\pi \in \widehat{G}$ a Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodule $M_\pi \in \varphi(\pi)$ such that $M_0 = \mathcal{B}$. Then the following assertions hold:*

(a) *Each class in $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ can be represented by a free C*-dynamical system of the form $\big(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)}\big)$.*

(b) *Any other free C*-dynamical system $(\mathcal{A}_{(M,\Psi')}, G, \alpha_{(M,\Psi')})$ representing an element of $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ satisfies $\Psi' = \omega\Psi$ with $(\omega\Psi)_{\pi,\rho} := \omega(\pi, \rho)\Psi_{\pi,\rho}$ for all $\pi, \rho \in \widehat{G}$ for some 2-cocycle*

$$\omega \in Z^2_\varphi(\widehat{G}, UZ(B)).$$

(c) *The free C*-dynamical systems* $\big(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)}\big)$ *and* $\big(\mathcal{A}_{(M,\omega\Psi)}, G, \alpha_{(M,\omega\Psi)}\big)$ *are equivalent if and only if*

$$\omega \in B^2_\varphi(\widehat{G}, UZ(B)).$$

Proof. (a) Let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system representing an element in $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ and recall that $(\mathcal{A}, G, \alpha)$ gives rise to a factor system for φ of the form $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ (cf. Proposition 6.20). Then the assumption implies that there is a family $(T_\pi : M_\pi \to A(\pi))_{\pi\in\widehat{G}}$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule isomorphisms which can be used to define another family $(\Psi''_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\widehat{G}}$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule isomorphisms by

$$\Psi''_{\pi,\rho} := T^+_{\pi+\rho} \circ m_{\pi,\rho} \circ (T_\pi \otimes_{\mathcal{B}} T_\rho).$$

In particular, it is not hard to see that the later family gives rise to a factor system $(M_\pi, \Psi''_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ for φ which is equivalent to $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$. Therefore, the assertion is finally a consequence of Theorem 7.7.

(b) Let $\big(\mathcal{A}_{(M,\Psi')}, G, \alpha_{(M,\Psi')}\big)$ be any other free C*-dynamical system representing an element of $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ and choose $\pi, \rho \in \widehat{G}$. Then Proposition 7.2 implies that the automorphism $\Psi'_{\pi,\rho} \circ \Psi^{-1}_{\pi,\rho}$ of the Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule $M_{\pi+\rho}$ provides a unique element $\omega(\pi,\rho) \in UZ(\mathcal{B})$ satisfying

$$\Psi'_{\pi,\rho} = \omega(\pi,\rho)\Psi_{\pi,\rho}.$$

Moreover, it is easily seen that the corresponding map $\omega : \widehat{G}\times\widehat{G} \to UZ(\mathcal{B})$ is a normalized 2-cochain. To see that ω actually defines a 2-cocycle, i. e., an element in $Z^2_\varphi(\widehat{G}, UZ(\mathcal{B}))$, we repeatedly use the factor system condition equation (5) and Proposition 7.4. For example, we find that

$$\begin{aligned}
\Psi'_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi'_{\rho,\sigma}) &= \Psi'_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \omega(\rho,\sigma)\Psi_{\rho,\sigma}) \\
&= \Psi'_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi\, \omega(\rho,\sigma) \otimes_{\mathcal{B}} \Psi_{\rho,\sigma}) \\
&= \Psi'_{\pi,\rho+\sigma} \circ (\Phi_\pi(\omega(\rho,\sigma))\, \mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{\rho,\sigma}) \\
&= \Phi_\pi(\omega(\rho,\sigma))\Psi'_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{\rho,\sigma}) \\
&= \Phi_\pi(\omega(\rho,\sigma))\omega(\pi,\rho+\sigma)\Psi_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{\rho,\sigma})
\end{aligned}$$

holds for all $\pi, \rho, \sigma \in \widehat{G}$, where $\mathrm{id}_\pi\, \omega(\rho,\sigma) = \Phi_\pi(\omega(\rho,\sigma))\, \mathrm{id}_\pi$ is understood in the sense of Corollary 7.3.

(c) If $\omega = d_\varphi h$ holds for some element $h \in C^1(\widehat{G}, UZ(\mathcal{B}))$, then the factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ are equivalent. Hence, the assertion follows from Theorem 7.7. If, on the other hand, $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ are equivalent, then we conclude from Proposition 7.2 that there exists an element $h \in C^1(\widehat{G}, UZ(\mathcal{B}))$ which implements the equivalence given by a family $(T_\pi)_{\pi\in\widehat{G}}$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule isomorphisms $T_\pi : M_\pi \to M_\pi$, i. e., we have $T_\pi = T_{h(\pi)}$ for all $\pi \in \widehat{G}$. Moreover, a short observation shows that $\omega = d_\varphi h \in B^2_\varphi(\widehat{G}, UZ(\mathcal{B}))$. □

Corollary 7.9. *Let $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ be a group homomorphism with $\mathrm{Ext}(\mathcal{B}, G, \varphi) \neq \emptyset$. Then the map*

$$H^2_\varphi(\widehat{G}, UZ(B)) \times \mathrm{Ext}(\mathcal{B}, G, \varphi) \to \mathrm{Ext}(\mathcal{B}, G\varphi)$$
$$\Big([\omega], \big[(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})\big]\Big) \mapsto \big[(\mathcal{A}_{(M,\omega\Psi)}, G, \alpha_{(M,\omega\Psi)})\big]$$

is a well-defined simply transitive action.

We conclude with a remark which shows that our constructions from Section 6 are, up to isomorphisms, inverse to each other.

Remark 7.10. It is easily seen that Proposition 6.20 applied to the free C*-dynamical system $(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})$ in Theorem 6.17 recovers the original factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ for φ. Indeed, Theorem 6.17 shows that $\mathcal{A}_{(M,\Psi)}(\pi) = M_\pi$ for all $\pi \in \widehat{G}$. Moreover, the multiplication map of $\mathcal{A}_{(M,\Psi)}$ is by construction uniquely determined by the factor system, i. e., $m_{\pi,\rho} = \Psi_{\pi,\rho}$ for all $\pi, \rho \in \widehat{G}$. It follows that our constructions, i. e., the procedure of associating a free C*-dynamical system to a factor system and vice versa, are, up to isomorphisms, inverse to each other:

$$(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}} \xmapsto{\mathrm{C}^*} (\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})$$
$$(A, G, \alpha) \xmapsto{\mathrm{F.S.}} (A(\pi), m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}.$$

Non-emptiness of $\mathrm{Ext}(\mathcal{B}, G, \varphi)$

As we have already discussed before, each group homomorphism $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ gives rise to both a family $(M_\pi)_{\pi\in\widehat{G}}$ of Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodules and a family

$$(\Psi_{\pi,\rho} : M_\pi \widehat{\otimes}_\mathcal{B} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\widehat{G}}$$

of Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodules isomorphisms. Given a group homomorphism $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ and such a family $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ satisfying $M_0 = \mathcal{B}$, $\Psi_{0,0} = \mathrm{id}_\mathcal{B}$ and $\Psi_{\pi,0} = \Psi_{0,\pi} = \mathrm{id}_\pi$ for all $\pi \in \widehat{G}$ (which need not be a factor system), we can examine for all $\pi, \rho, \sigma \in \widehat{G}$ the automorphism

$$d_M\Psi(\pi, \rho, \sigma) := \Psi_{\pi+\rho,\sigma} \circ (\Psi_{\pi,\rho} \otimes_\mathcal{B} \mathrm{id}_\sigma) \circ (\mathrm{id}_\pi \otimes_\mathcal{B} \Psi^+_{\rho,\sigma}) \circ \Psi^+_{\pi,\rho+\sigma}$$

of the Morita equivalence $\mathcal{B} - \mathcal{B}$-bimodule $M_{\pi+\rho+\sigma}$. The family of all such maps $(d_M\Psi(\pi, \rho, \sigma))_{\pi,\rho,\sigma\in\widehat{G}}$ can be interpreted as an obstruction to the associativity of the multiplication (cf. Proposition 6.1). On the other hand, it follows from the construction and from Proposition 7.2 that the map $d_M\Psi$ can also be considered as a normalized $UZ(\mathcal{B})$-valued 3-cochain on $\widehat{G}$, i. e., as an element in $C^3(\widehat{G}, UZ(\mathcal{B}))$. In fact, even more is true:

Lemma 7.11. *The map $d_M\Psi$ defines an element in $Z^3_\varphi(\widehat{G}, UZ(\mathcal{B}))$.*

Proof. For the sake of brevity we omit the lengthy calculation at this point and refer instead to [37, Lemma 1.10 (5)]. $\square$

Lemma 7.12. *The class $[d_M\Psi]$ in $H^3_\varphi(\widehat{G}, UZ(\mathcal{B}))$ is independent of all choices made.*

Proof. (i) We first show that the class $[d_M\Psi]$ is independent of the choice of the family $(\Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$. Therefore, let $(\Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ be another choice and note that Proposition 7.2 implies that there exists an element $h \in C^2(\widehat{G}, UZ(\mathcal{B}))$ satisfying $\Psi'_{\pi,\rho} = h(\pi,\rho)\Psi_{\pi,\rho}$ for all $\pi,\rho \in \widehat{G}$. A short calculation then shows that

$$\Psi'_{\pi+\rho,\sigma} \circ (\Psi'_{\pi,\rho} \otimes_{\mathcal{B}} \mathrm{id}_\sigma) = h(\pi+\rho,\sigma)h(\pi,\rho)d_M\Psi(\pi,\rho,\sigma)(\Psi_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{\rho,\sigma}))$$

holds for all $\pi, \rho\sigma \in \widehat{G}$. On the other hand, it follows from Proposition 7.4 that

$$\begin{aligned}&d_M\Psi'(\pi,\rho,\sigma)(\Psi'_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi'_{\rho,\sigma}))\\ =&d_M\Psi'(\pi,\rho,\sigma)h(\pi,\rho+\sigma)\pi.h(\rho,\sigma)(\Psi_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi_{\rho,\sigma}))\end{aligned}$$

holds for all $\pi,\rho,\sigma \in \widehat{G}$. From these observations we can now easily conclude that the 3-cocycles $d_M\Psi'$ and $d_M\Psi$ are cohomologous.

(ii) As a second step, we show that the class $[d_M\Psi]$ does not dependent on the choice of the family $(M_\pi)_{\pi\in\widehat{G}}$. For this purpose, let $(M'_\pi)_{\pi\in\widehat{G}}$ be another choice and note that the construction implies that there is a family $(T_\pi : M_\pi \to M'_\pi)_{\pi\in\widehat{G}}$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule isomorphisms. This family can now be used to define another family $(\Psi''_{\pi,\rho} : M'_\pi \widehat{\otimes}_{\mathcal{B}} M'_\rho \to M'_{\pi+\rho})_{\pi,\rho\in\widehat{G}}$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodule isomorphisms by

$$\Psi'_{\pi,\rho} := T_{\pi+\rho} \circ \Psi_{\pi,\rho} \circ (T^+_\pi \otimes_{\mathcal{B}} T^+_\rho).$$

Then an explicit computation shows that

$$\begin{aligned}d_M\Psi'(\pi,\rho,\sigma) &= \Psi'_{\pi+\rho,\sigma} \circ (\Psi'_{\pi,\rho} \otimes_{\mathcal{B}} \mathrm{id}_\sigma) \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi'^{+}_{\rho,\sigma}) \circ \Psi'^{+}_{\pi,\rho+\sigma}\\ &= T_{\pi+\sigma+\rho} \circ \Psi_{\pi+\rho,\sigma} \circ (T^+_{\pi+\rho} \otimes_{\mathcal{B}} T^+_\sigma)\\ &\qquad \circ (T_{\pi+\rho} \otimes_{\mathcal{B}} \mathrm{id}_\sigma) \circ (\Psi_{\pi,\rho} \otimes_{\mathcal{B}} \mathrm{id}_\sigma) \circ (T^+_\pi \otimes_{\mathcal{B}} T^+_\rho \otimes_{\mathcal{B}} \mathrm{id}_\sigma)\\ &\qquad \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} T_\rho \otimes_{\mathcal{B}} T_\sigma) \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} \Psi^+_{\rho,\sigma}) \circ (\mathrm{id}_\pi \otimes_{\mathcal{B}} T^+_{\rho+\sigma})\\ &\qquad \circ (T_\pi \otimes_{\mathcal{B}} T_{\rho+\sigma}) \circ \Psi^+_{\pi,\rho+\sigma} \circ T^+_{\pi+\rho+\sigma}\\ &= d_M\Psi(\pi,\rho,\sigma)\end{aligned}$$

holds for all $\pi,\rho,\sigma \in \widehat{G}$. We conclude that $d_M\Psi' = d_M\Psi$, i. e., that the 3-cocycle $d_M\Psi$ is unchanged. $\square$

Definition 7.13. Let $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ be a group homomorphism. We call

$$\chi(\varphi) := [d_M \Psi] \in H^3_\varphi(\widehat{G}, UZ(\mathcal{B}))$$

the *characteristic class* of φ.

The following result provides a group theoretic criterion for the non-emptiness of the set $\mathrm{Ext}(\mathcal{B}, G, \varphi)$.

Theorem 7.14. *Let $\varphi : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ be a group homomorphism. Then $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ is non-empty if and only if the class $\chi(\varphi) \in H^3_\varphi(\widehat{G}, UZ(\mathcal{B}))$ vanishes.*

Proof. ($\Rightarrow$) Suppose first that $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ is non-empty and let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system representing an element in $\mathrm{Ext}(\mathcal{B}, G, \varphi)$. Then $(\mathcal{A}, G, \alpha)$ gives rise to a factor system for φ of the form $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ (cf. Proposition 6.20) and the associativity of the multiplication implies that the corresponding characteristic class $\chi(\varphi) \in H^3_\varphi(\widehat{G}, UZ(\mathcal{B}))$ vanishes.

($\Leftarrow$) Let $(M_\pi)_{\pi\in\widehat{G}}$ be a family of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules and

$$(\Psi_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\widehat{G}}$$

a family of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules isomorphisms as described in the introduction. Furthermore, suppose, conversely, that the class

$$\chi(\varphi) = [d_M \Psi] \in H^3_\varphi(\widehat{G}, UZ(\mathcal{B}))$$

vanishes. Then there exists an element $h \in C^2(\widehat{G}, UZ(B))$ with $d_M\Psi = d_\varphi h^{-1}$ which can be use to define a family $(\Psi'_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\widehat{G}}$ of Morita equivalence bimodule $\mathcal{B}-\mathcal{B}$-isomorphism by

$$\Psi'_{\pi,\rho} := h(\pi,\rho)\Psi_{\pi,\rho}.$$

The construction implies that $d_M\Psi' = \mathbb{1}_{\mathcal{B}}$. In particular, it follows that $(M_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ is a factor system for φ and we can finally conclude from Theorem 6.17 that the set $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ is non-empty. □

Remark 7.15. The intention of this remark is to describe the elements of the set $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ for a given group homomorphism $\varphi : \widehat{G} \to \mathrm{Out}(\mathcal{B})$ (cf. Example 5.7). For this purpose, let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system representing an element of $\mathrm{Ext}(\mathcal{B}, G, \varphi)$. Then it is not hard to see that each isotypic component contains an invertible element. In fact, it follows from Corollary 5.1 that the map

$$A(-\pi)\widehat{\otimes}_{\mathcal{B}} A(\pi) \to \mathcal{B}, \quad x \otimes_{\mathcal{B}} y \mapsto xy \tag{10}$$

is an isomorphism of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules for all $\pi \in \widehat{G}$. Moreover, the assumption on φ implies that for each $\pi \in \widehat{G}$ there is an automorphism $S(\pi) \in \mathrm{Aut}(\mathcal{B})$

and an isomorphism $T_\pi : M_{S(\pi)} \to A(\pi)$ of Morita equivalence $\mathcal{B}-\mathcal{B}$-bimodules. If we now define $u_\pi := T_\pi(\mathbb{1}_{\mathcal{B}})$, then a short observations shows that

$$A(\pi) = u_\pi \mathcal{B} = \mathcal{B} u_\pi,$$

from which we conclude together with equation (10) that

$$u_\pi \mathcal{B} u_{-\pi} = u_{-\pi} \mathcal{B} u_\pi = \mathcal{B}.$$

Consequently, the element $u_\pi \in A(\pi)$ is invertible (in $\mathcal{A}$). Conversely, let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system such that each isotypic component contains an invertible element. Then it is easily verified that $(\mathcal{A}, G, \alpha)$ is free and that the corresponding group homomorphism $\varphi_{\mathcal{A}} : \widehat{G} \to \mathrm{Pic}(\mathcal{B})$ from Proposition 5.11 takes values in $\mathrm{Out}(\mathcal{B})$. In particular, $(\mathcal{A}, G, \alpha)$ represents an element in $\mathrm{Ext}(\mathcal{B}, G, \varphi_{\mathcal{A}})$.

C*-dynamical systems with the property that each isotypic component contains invertible elements have been studied, for example, in [46, 52, 54, 58] and may be considered as a noncommutative version of trivial principal bundles (cf. Example 5.5 and [47]).

Remark 7.16. The aim of the following discussion is to explain how to classify the C*-dynamical systems described in Remark 7.15. Indeed, let $(\mathcal{A}, G, \alpha)$ be a C*-dynamical system such that each isotypic component contains an invertible element. Furthermore, let $(u_\pi)_{\pi \in \widehat{G}}$ be a family of unitaries with $u_\pi \in A(\pi)$ and $u_0 = \mathbb{1}_{\mathcal{B}}$. Then the maps

$$S : \widehat{G} \to \mathrm{Aut}(\mathcal{B}), \quad S(\pi)(b) := u_\pi b u_\pi^*$$

and

$$\omega : \widehat{G} \times \widehat{G} \to U(\mathcal{B}), \quad \omega(\pi, \sigma) := u_\pi u_\rho u_{\pi+\rho}^*,$$

give rise to an element

$$(S, \omega) \in C^1(\widehat{G}, \mathrm{Aut}(\mathcal{B})) \times C^2(\widehat{G}, U(\mathcal{B}))$$

satisfying

$$S(\pi)(\omega(\rho, \sigma))\omega(\pi, \rho + \sigma) = \omega(\pi + \rho, \sigma)\omega(\pi, \rho) \tag{11}$$

for all $\pi, \rho, \sigma \in \widehat{G}$ and

$$S(\pi)(S(\rho)(b)) = \omega(\pi, \rho) S(\pi + \rho)(b) \omega(\pi, \rho)^* \tag{12}$$

for all $\pi, \rho \in \widehat{G}$ and $b \in \mathcal{B}$. The corresponding families $(M_\pi)_{\pi \in \widehat{G}}$ and $(\Psi_{\pi,\rho})_{\pi,\rho \in \widehat{G}}$ given by $M_\pi := M_{S(\pi)}$ and

$$\Psi_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathcal{B}} M_\rho \to M_{\pi+\rho}, \quad b \otimes_{\mathcal{B}} b' \mapsto b S(\pi)(b') \omega(\pi, \rho)$$

are easily seen to provide a factor system for $\varphi_{\mathcal{A}}$ whose associated free C*-dynamical system is equivalent to $(\mathcal{A}, G, \alpha)$. Conversely, each element

$$(S, \omega) \in C^1(\widehat{G}, \mathrm{Aut}(\mathcal{B})) \times C^2(\widehat{G}, U(\mathcal{B}))$$

satisfying equation (11) for all $\pi, \rho, \sigma \in \widehat{G}$ and equation (12) for all $\pi, \rho \in \widehat{G}$ and $b \in \mathcal{B}$ leads in the described way to a free C*-dynamical system representing an element of $\mathrm{Ext}(\mathcal{B}, G, \varphi)$ with

$$\varphi := \mathrm{pr}_{\mathcal{B}} \circ S : \widehat{G} \to \mathrm{Out}(\mathcal{B}),$$

where $\mathrm{pr}_{\mathcal{B}} : \mathrm{Aut}(\mathcal{B}) \to \mathrm{Out}(\mathcal{B})$ denotes the canonical quotient homomorphism. It is worth pointing out that in this situation, the involution can be expressed explicitly in terms of the pair (S, ω) (cf. [54, Construction A24]).

8 Principlal Bundles and Group Cohomology

Each locally trivial principal bundle (P, X, G, q, σ) can be considered as a geometric object that is glued together from local pieces which are trivial, i. e., which are of the form $U \times G$ for some small open subset U of X. This approach immediately leads to the concept of G-valued cocycles and therefore to a cohomology theory, called the Čech cohomology for the pair (X, G). This cohomology theory gives a complete classification of locally trivial principal bundles with structure group G and base space X (cf. [50]). On the other hand, Theorem 3.4 implies that each locally trivial principal bundle (P, X, G, q, σ) gives rise to a free C*-dynamical system $(C(P), G, \alpha_\sigma)$ and it is therefore natural to ask how the Čech cohomology for the pair (X, G) is related to our previous classification theory. But since our construction in Section 6 is global in nature, it is not obvious how to encode local triviality in our factor system approach (though we recall that in the smooth category there is a one-to-one correspondence between free and proper actions and locally trivial principal bundles). For this reason we now focus our attention on topological principal bundles, a notion of principal bundles which need not be locally trivial.

Remark 8.1. Let P be a locally compact space and G a locally compact group. Furthermore, let $\sigma : P \times G \to P$ be a continuous action which is free and proper and write $\mathrm{pr} : P \to P/G$ for the corresponding quotient map. Then each pair $(p_1, p_2) \in P \times P$ with $\mathrm{pr}(p_1) = \mathrm{pr}(p_2)$ determines a unique element $\tau(p_1, p_2) \in G$ such that $p_1.\tau(p_1, p_2) = p_2$ and it follows from [43, Lemma 4.63] that the surjective map

$$\tau : P \times_{P/G} P := \{(p_1, p_2) \in P \times P : \mathrm{pr}(p_1) = \mathrm{pr}(p_2)\} \to G$$

is continuous. It is called the *translation map* and is part of the definition of principal bundles proposed in [26]. A short observation shows that each free and continuous action $\sigma : P \times G \to P$ has a continuous translation map is automatically proper. Therefore the class of principal bundles in [26] coincides with the class of free and proper G-spaces. For this reason, it makes sense to call a continuous action $\sigma : P \times G \to P$ which is free and proper a *topological principal bundle*. As mentioned before these principal bundles are, in general, not locally trivial (cf. [43, Remark 4.68]).

We now come back to our original setting. Let P be a compact space and G a compact abelian group. Furthermore, let $\sigma : P \times G \to P$ be a topological principal bundle and $\pi \in \widehat{G}$. Then the corresponding isotypic component $C(P)(\pi)$ is finitely generated and projective as a $C(P/G)$-module according to Theorem 3.5. Therefore, the Theorem of Serre and Swan (cf. [48]) gives rise to a (locally trivial) complex line bundle $\mathbb{V}_\pi$ over P/G such that $C(P)(\pi)$ is isomorphic as a right $C(P/G)$-module to the corresponding space $\Gamma\mathbb{V}_\pi$ of continuous sections. We point out that this isomorphism can be extended to an isomorphism between Morita equivalence $C(P/G) - C(P/G)$-bimodules. In what follows we identify the Morita equivalence $C(P/G) - C(P/G)$-bimodule $C(P)(\pi)$ with the corresponding complex line bundle $\mathbb{V}_\pi$ defining an element in $\mathrm{Pic}(P/G)$. Then the group homomorphism $\widehat{G} \to \mathrm{Pic}(C(P/G))$ induced by the topological principal bundle $\sigma : P \times G \to P$ is given by

$$\pi \mapsto [\mathbb{V}_\pi] \in \mathrm{Pic}(P/G) \subseteq \mathrm{Pic}(C(P/G))$$

(cf. Example 5.9 and Proposition 5.11). In particular, the $\widehat{G}$-module structure on $U(C(P/G)) = C(P/G, \mathbb{T})$ induced by this group homomorphism (cf. Remark 7.5) is trivial since the left and right action of $C(P/G)$ on $C(P)(\pi)$ commute. Summarizing, we have shown the following statement.

Lemma 8.2. *Let X be a compact space. For a group homomorphism $\varphi : \widehat{G} \to \mathrm{Pic}(C(X))$ to define a topological principal G-bundle over X it is necessary that $\mathrm{im}(\varphi) \subseteq \mathrm{Pic}(X)$ and, therefore, that the induced module structure on $C(X, \mathbb{T})$ is trivial.*

We continue with a fixed group homomorphism $\varphi : \widehat{G} \to \mathrm{Pic}(X) \subseteq \mathrm{Pic}(C(X))$ and a factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho \in \widehat{G}}$ for φ. In this situation, the canonical flip

$$\mathrm{fl}_{\pi,\rho} : M_\pi \widehat{\otimes}_{C(X)} M_\rho \to M_\rho \widehat{\otimes}_{C(X)} M_\pi, \quad x \otimes_{C(X)} y \mapsto y \otimes_{C(X)} x$$

defines an isomorphism of Morita equivalence $C(X) - C(X)$-bimodules for all $\pi, \rho \in \widehat{G}$. In particular, we can examine for all $\pi, \rho \in \widehat{G}$ the automorphism

$$\mathcal{C}_M\Psi(\pi,\rho) := \Psi_{\rho,\pi} \circ \mathrm{fl}_{\pi,\rho} \circ \Psi^+_{\pi,\rho} \tag{13}$$

of the Morita equivalence $C(X) - C(X)$-bimodule $M_{\pi+\rho}$. According to Proposition 7.2, the map $\mathcal{C}_M\Psi$ can also be considered as a normalized $C(X, \mathbb{T})$-valued 2-cochain on $\widehat{G}$, i. e., as an element in $C^2(\widehat{G}, C(X, \mathbb{T}))$. In fact, even more is true:

Lemma 8.3. *The map $\mathcal{C}_M\Psi$ defines an antisymmetric element in $Z^2(\widehat{G}, C(X, \mathbb{T}))$.*

Proof. It is obvious that the map $\mathcal{C}_M\Psi$ satisfies $\mathcal{C}_M\Psi(\rho,\pi) = \mathcal{C}_M\Psi(\pi,\rho)^*$ for all $\pi,\rho \in \widehat{G}$. In order to verify that $\mathcal{C}_M\Psi$ defines an element in $Z^2(\widehat{G}, C(X,\mathbb{T}))$ we have to show that

$$\mathcal{C}_M\Psi(\rho,\sigma)\mathcal{C}_M\Psi(\pi,\rho+\sigma) = \mathcal{C}_M\Psi(\pi+\rho,\sigma)\mathcal{C}_M\Psi(\pi,\rho)$$

holds for all $\pi,\rho,\sigma \in \widehat{G}$. Indeed, explicit computations using the factor system property (cf. equation (5)) and (13) show that

$$\Psi_{\sigma+\rho,\pi} \circ (\Psi_{\sigma,\rho} \otimes_{C(X)} \mathrm{id}_\pi) = \mathcal{C}_M\Psi(\rho,\sigma)\mathcal{C}_M\Psi(\pi,\rho+\sigma)\Psi_{\pi,\sigma+\rho} \circ (\mathrm{id}_\pi \otimes_{C(X)} \Psi_{\rho,\sigma})$$

and

$$\Psi_{\sigma,\rho+\pi} \circ (\mathrm{id}_\sigma \otimes_{C(X)} \Psi_{\rho,\pi}) = \mathcal{C}_M\Psi(\pi,\rho)\mathcal{C}_M\Psi(\pi+\rho,\sigma)\Psi_{\pi+\rho,\sigma} \circ (\Psi_{\pi,\rho} \otimes_{C(X)} \mathrm{id}_\pi)$$

hold for all $\pi,\rho,\sigma \in \widehat{G}$. Again using equation (5) to the right-hand side of the later expression then leads to the desired 2-cocycle condition

$$\mathcal{C}_M\Psi(\rho,\sigma)\mathcal{C}_M\Psi(\pi,\rho+\sigma) = \mathcal{C}_M\Psi(\pi+\rho,\sigma)\mathcal{C}_M\Psi(\pi,\rho)$$

for $\pi,\rho,\sigma \in \widehat{G}$. □

Lemma 8.4. *Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ be two equivalent factor systems for φ. Then we have*

$$\mathcal{C}_{M'}\Psi' = \mathcal{C}_M\Psi.$$

Proof. By assumption there is a family $(T_\pi : M_\pi \to M'_\pi)_{\pi\in\widehat{G}}$ of Morita equivalence $C(X)-C(X)$-bimodule isomorphisms satisfying

$$\Psi'_{\pi,\rho} := T_{\pi+\rho} \circ \Psi_{\pi,\rho} \circ (T^+_\pi \otimes_{\mathcal{B}} T^+_\rho).$$

Therefore, an explicit computation shows that

$$\begin{aligned}
\mathcal{C}_{M'}\Psi'(\pi,\rho) &= \Psi'_{\rho,\pi} \circ \mathrm{fl}'_{\pi,\rho} \circ \Psi'^{+}_{\pi,\rho} \\
&= T_{\rho+\pi} \circ \Psi_{\rho,\pi} \circ (T^+_\rho \otimes_{\mathcal{B}} T^+_\pi) \circ \mathrm{fl}'_{\pi,\rho} \circ (T_\pi \otimes_{\mathcal{B}} T_\rho) \circ \Psi^+_{\pi,\rho} \circ T^+_{\pi+\rho} \\
&= T_{\rho+\pi} \circ \Psi_{\rho,\pi} \circ \mathrm{fl}_{\pi,\rho} \circ \Psi^+_{\pi,\rho} \circ T^+_{\pi+\rho} = \mathcal{C}_M\Psi(\pi,\rho)
\end{aligned}$$

holds for all $\pi,\rho \in \widehat{G}$. □

Lemma 8.5. *Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ be two factor systems for φ. Then there exists an element $\omega \in Z^2(\widehat{G}, C(X,\mathbb{T}))$ satisfying*

$$\mathcal{C}_{M'}\Psi'(\pi,\rho) = \omega(\pi,\rho)\omega(\rho,\pi)^*\mathcal{C}_M\Psi(\pi,\rho)$$

for all $\pi,\rho \in \widehat{G}$.

Proof. To verify the assertion we use Theorem 7.8 which implies that the factor system $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ is equivalent to a factor system of the form $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ for some 2-cocycle $\omega \in Z^2(\widehat{G}, C(X,\mathbb{T}))$. In particular, we can now conclude from Lemma 8.4 and a short calculation that

$$\mathcal{C}_{M'}\Psi'(\pi,\rho) = \mathcal{C}_M(\omega\Psi)(\pi,\rho) = (\omega\Psi)_{\rho,\pi} \circ \mathrm{fl}_{\pi,\rho} \circ (\omega\Psi)^+_{\pi,\rho} = \omega(\rho,\pi)\omega(\pi,\rho)^*\mathcal{C}_M\Psi(\pi,\rho)$$

holds for all $\pi, \rho \in \widehat{G}$. $\square$

In what follows we denote the set of all biadditive maps $\widehat{G} \times \widehat{G} \to C(X,\mathbb{T})$ that vanish on the diagonal by $\mathrm{Alt}^2(\widehat{G}, C(X,\mathbb{T}))$. A short observation shows that each element $\omega \in Z^2(\widehat{G}, C(X,\mathbb{T}))$ gives rise to an element $\lambda_\omega \in \mathrm{Alt}^2(\widehat{G}, C(X,\mathbb{T}))$ defined by

$$\lambda_\omega(\pi,\rho) := \omega(\pi,\rho)\omega(\rho,\pi)^*$$

which only depends on the class $[\omega] \in H^2(\widehat{G}, C(X,\mathbb{T}))$. In particular, we obtain a group homomorphism

$$\lambda : H^2(\widehat{G}, C(X,\mathbb{T})) \to \mathrm{Alt}^2(\widehat{G}, C(X,\mathbb{T})), \quad [\omega] \mapsto \lambda_\omega$$

whose kernel is given by the subgroup $H^2_{\mathrm{ab}}(\widehat{G}, C(X,\mathbb{T}))$ of $H^2(\widehat{G}, C(X,\mathbb{T}))$ describing the abelian extensions of $\widehat{G}$ by $C(X,\mathbb{T})$. We recall from [38, Proposition II.3] that the corresponding short exact sequence

$$0 \longrightarrow H^2_{\mathrm{ab}}(\widehat{G}, C(X,\mathbb{T})) \longrightarrow H^2(\widehat{G}, C(X,\mathbb{T})) \xrightarrow{\lambda} \mathrm{Alt}^2(\widehat{G}, C(X,\mathbb{T})) \longrightarrow 0$$

is split. Moreover, we write $\mathrm{pr}_{\mathrm{ab}} : H^2(\widehat{G}, C(X,\mathbb{T})) \to H^2_{\mathrm{ab}}(\widehat{G}, C(X,\mathbb{T}))$ for the induced projection map.

Proposition 8.6. *The class* $\mathrm{pr}_{\mathrm{ab}}([\mathcal{C}_M\Psi]) \in H^2_{\mathrm{ab}}(\widehat{G}, C(X,\mathbb{T}))$ *does not depend on the choice of the factor system and is therefore an invariant for the set* $\mathrm{Ext}(C(X), G, \varphi)$.

Proof. Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ be two factor systems for φ. Then it follows from Lemma 8.5 and the construction of the map $\mathrm{pr}_{\mathrm{ab}}$ that

$$\mathrm{pr}_{\mathrm{ab}}([\mathcal{C}_{M'}\Psi']) = \mathrm{pr}_{\mathrm{ab}}([\lambda_\omega\mathcal{C}_M\Psi]) = \mathrm{pr}_{\mathrm{ab}}([\lambda_\omega][\mathcal{C}_M\Psi]) = \mathrm{pr}_{\mathrm{ab}}([\mathcal{C}_M\Psi]). \qquad \square$$

Definition 8.7. Let $\varphi : \widehat{G} \to \mathrm{Pic}(X)$ be a group homomorphism. Then we call

$$\chi_2(\varphi) := \mathrm{pr}_{\mathrm{ab}}([\mathcal{C}_M\Psi]) \in H^2_{\mathrm{ab}}(\widehat{G}, C(X,\mathbb{T}))$$

the *secondary characteristic class* of φ.

The following result provides a group theoretic criterion for the existence of topological principal G-bundle over X.

Theorem 8.8. *Let $\varphi : \widehat{G} \to \mathrm{Pic}(X)$ be a group homomorphism. Then the following statements are equivalent:*

(*a*) *The map φ defines a topological principal bundle $\sigma : P \times G \to P$ over X, i. e., the set $\mathrm{Ext}(C(X), G, \varphi)$ contains an element which can be represented by $(C(P), G, \alpha_\sigma)$.*

(*b*) *The map φ satisfies the following two conditions in the indicated order:*

(b_1) *The class $\chi(\varphi) \in H^3(\widehat{G}, C(X, \mathbb{T}))$ vanishes.*

(b_2) *Furthermore, the class $\chi_2(\varphi) \in H^2_{\mathrm{ab}}(\widehat{G}, C(X, \mathbb{T}))$ vanishes.*

Proof. (a) $\Rightarrow$ (b): Suppose first that the map φ defines a topological principal bundle $\sigma : P \times G \to P$ over X. Then the corresponding C*-dynamical system $(C(P), G, \alpha_\sigma)$ is free according to Theorem 3.4 and we conclude that $\mathrm{Ext}(C(X), G, \varphi) \neq \emptyset$. This observation is by Theorem 7.14 equivalent to the vanishing of the characteristic class $\chi(\varphi) \in H^3(\widehat{G}, C(X, \mathbb{T}))$. To verify that the class $\chi_2(\varphi) \in H^2_{\mathrm{ab}}(\widehat{G}, C(X, \mathbb{T}))$ also vanishes, we note that the canonical associated factor system is given by

$$\Psi_{\pi,\rho} : C(P)(\pi)\widehat{\otimes}_{C(X)}C(P)(\rho) \to C(P)(\pi + \rho), \quad f \otimes_{C(X)} g \mapsto fg.$$

Therefore, the claim follows from the commutativity of $C(P)$ since we have

$$\Psi_{\pi,\rho}(f \otimes_{C(X)} g) = fg = gf = \Psi_{\rho,\pi}(g \otimes_{C(X)} f)$$

for all $f \in C(P)(\pi)$ and $g \in C(P)(\rho)$, i. e., $\mathcal{C}_M\Psi = \mathbb{1}_{C(X)}$.

(b) $\Rightarrow$ (a): If condition (b_1) is satisfied, then it follows from Theorem 7.14 that there is a free C*-dynamical system $(\mathcal{A}, G, \alpha)$ representing an element of $\mathrm{Ext}(C(X), G, \varphi)$. Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ be its associated factor system (for φ). We then use condition (b_2) to find an element $\omega \in Z^2(\widehat{G}, C(X, \mathbb{T}))$ such that

$$\lambda_{\omega^*} = \mathcal{C}_M\Psi \in \mathrm{Alt}^2(\widehat{G}, C(X, \mathbb{T})).$$

Consequently, the corresponding factor system $(M_\pi, \omega(\pi, \rho)\Psi_{\pi,\rho})_{\pi,\rho\in\widehat{G}}$ for φ satisfies $\mathcal{C}_M(\omega\Psi) = \mathbb{1}_{C(X)}$ meaning that its associated free C*-dynamical system is equivalent to one of the form $(C(P), G, \alpha_\sigma)$ induced by some topological principal bundle $\sigma : P \times G \to P$ over X. ◻

9 Examples

In the last section of this paper we present some examples.

Example 9.1. Let $\mathbb{M}_m(\mathbb{C})$ be the C*-algebra of $m\times m$ matrices and recall that its natural representation on $\mathbb{C}^m$ is, up to equivalence, the only irreducible representation of $\mathbb{M}_m(\mathbb{C})$. Therefore, it follows from Example 5.8 that $\mathrm{Pic}(\mathbb{M}_m(\mathbb{C}))$ is trivial. In particular, there is only the trivial group homomorphism from $\widehat{G}$ to $\mathrm{Pic}(\mathbb{M}_m(\mathbb{C}))$ and a possible realization is given by the free C*-dynamical system

$$(C(G,\mathbb{M}_m(\mathbb{C})),G,\mathrm{rt}\otimes\mathrm{id}_{\mathbb{M}_n(\mathbb{C})}),$$

where

$$\mathrm{rt}:G\times C(G)\to C(G),\quad \mathrm{rt}(g,f)(h):=f(hg)$$

denotes the right-translation action by G. Moreover, we conclude from Corollary 7.9 that all free actions of G with fixed point algebra $\mathbb{M}_m(\mathbb{C})$ are parametrized by the cohomology group $H^2(\widehat{G},\mathbb{T})$. In the case $G=\mathbb{T}^n$, $n\in\mathbb{N}$, this cohomology group is isomorphic to $\mathbb{T}^{\frac{1}{2}n(n-1)}$ and parametrizes the free actions given by tensor products of the noncommutative n-tori endowed with their natural $\mathbb{T}^n$-action (cf. Example 5.2) and the C*-algebra $\mathbb{M}_m(\mathbb{C})$.

Example 9.2. Consider the 2-fold direct sum $\mathbb{M}_2(\mathbb{C})\oplus\mathbb{M}_2(\mathbb{C})$ and notice that the group $UZ(\mathbb{M}_2(\mathbb{C})\oplus\mathbb{M}_2(\mathbb{C}))$ is isomorphic to $\mathbb{T}^2$. Since the spectrum of $\mathbb{M}_2(\mathbb{C})\oplus\mathbb{M}_2(\mathbb{C})$ contains two elements, it follows from Example 5.8 that $\mathrm{Pic}(\mathbb{M}_2(\mathbb{C})\oplus\mathbb{M}_2(\mathbb{C}))$ is isomorphic to $\mathbb{Z}_2$. If $\varphi:\mathbb{Z}\to\mathbb{Z}_2$ denote the canonical group homomorphism with kernel $2\mathbb{Z}$, then it is a consequence of [31, Chapter VI.6] that the cohomology groups $H^2_\varphi(\mathbb{Z},\mathbb{T}^2)$ and $H^3_\varphi(\mathbb{Z},\mathbb{T}^2)$ are trivial. Therefore, Theorem 7.14 implies that the set $\mathrm{Ext}(\mathbb{M}_2(\mathbb{C})\oplus\mathbb{M}_2(\mathbb{C}),\mathbb{T},\varphi)$ is non-empty and contains according to Corollary 7.9 exactly one element, namely the class of the trivial system

$$(C(\mathbb{T},\mathbb{M}_2(\mathbb{C})\oplus\mathbb{M}_2(\mathbb{C}),\mathbb{T},\mathrm{rt}\otimes\mathrm{id}).$$

Example 9.3. For the following discussion we recall the notation from Example 5.2. Let $\mathbb{T}^n_\theta$ be the noncommutative n-torus defined by the real skew-symmetric $n\times n$ matrix θ and let ω_n be the corresponding $\mathbb{T}$-valued 2-cocycle on $\mathbb{Z}^n$ given for $\mathbf{k},\mathbf{k}'\in\mathbb{Z}^n$ by

$$\omega_n(\mathbf{k},\mathbf{k}'):=\exp({}_{\mathbb{C}^n}\langle\theta\mathbf{k},\mathbf{k}'\rangle).$$

Furthermore, let $S:\mathbb{Z}^m\to\mathrm{Aut}(\mathbb{T}^n_\theta)$ be a group homomorphism leaving the isotypic components of $\mathbb{T}^n_\theta$ (with respect to the canonical gauge action by $\mathbb{T}^n$) invariant, i.e., such that for all $\mathbf{l}\in\mathbb{Z}^m$ and $\mathbf{k}\in\mathbb{Z}^n$

$$S(\mathbf{l})U_{\mathbf{k}}=c_{\mathbf{l},\mathbf{k}}U_{\mathbf{k}}$$

for some $c_{\mathbf{l},\mathbf{k}}\in\mathbb{T}$. Then, given another $\mathbb{T}$-valued 2-cocycle ω_m on $\mathbb{Z}^m$, it follows from Remark 7.16, that the pair (S,ω_m) gives rise to a factor system for the group homomorphism $\varphi:=\mathrm{pr}_{\mathbb{T}^n_\theta}\circ S:\mathbb{Z}^m\to\mathrm{Pic}(\mathbb{T}^n_\theta)$. Moreover, it is easily seen that the associated free C*-dynamical system is equivalent to the free C*-dynamical system $(\mathbb{T}^{n+m}_{\theta'}, \mathbb{T}^m, \alpha)$, where $\mathbb{T}^{n+m}_{\theta'}$ denotes the noncommutative $(n+m)$-torus determined by the $\mathbb{T}$-valued 2-cocycle on $\mathbb{Z}^{n+m}$ given for $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^n$ and $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^m$ by

$$\omega_{n+m}((\mathbf{k}, \mathbf{l}), (\mathbf{k}', \mathbf{l}')) := c_{\mathbf{l}, \mathbf{k}'} \omega_n(\mathbf{k}, \mathbf{k}') \omega_m(\mathbf{l}, \mathbf{l}')$$

and α is the restriction of the gauge action $\alpha^{n+m}_{\theta'}$ to the closed subgroup $\mathbb{T}^m$ of $\mathbb{T}^{n+m}$. That $(\mathbb{T}^{n+m}_{\theta'}, \mathbb{T}^m, \alpha)$ is actually free is a consequence of Proposition 3.11. In particular, it represents an element in $\operatorname{Ext}(\mathbb{T}^n_\theta, \mathbb{T}^m, \varphi)$ which is according to Corollary 7.9 parametrized by the cohomology group

$$H^2_\varphi(\mathbb{Z}^m, UZ(\mathbb{T}^n_\theta)).$$

Remark 9.4. The previous example can be used to construct free C*-dynamical systems which are not C*-algebraic bundles (cf. [18, 43]) over the fixed point algebra.

Example 9.5. Let θ be an irrational number in $[0,1]$ and $\mathbb{T}^2_\theta$ the corresponding noncommutative 2-torus from Example 5.2. We recall that in this case $UZ(\mathbb{T}^2_\theta)$ is isomorphic to $\mathbb{T}$. Furthermore, let $\varphi : \mathbb{Z}^2 \to \operatorname{Pic}(\mathbb{T}^2_\theta)$ be any group homomorphism (note that $\mathbb{T}^2 \subseteq \operatorname{Aut}(\mathbb{T}^2_\theta)$ to apply the construction in Example 9.3). Then it is a consequence of [31, Chapter VI.6] that the cohomology group $H^3_\varphi(\mathbb{Z}^2, \mathbb{T})$ is trivial. Therefore, Theorem 7.14 implies that the set $\operatorname{Ext}(\mathbb{T}^2_\theta, \mathbb{T}^2, \varphi)$ is non-empty and according to Corollary 7.9 parametrized by the cohomology group $H^2_\varphi(\mathbb{Z}^2, \mathbb{T})$. For its computation we refer, for example, to [52, Proposition 6.2].

Example 9.6. Let H be the discrete (three-dimensional) Heisenberg group and let $(C^*(H), \mathbb{T}^2, \alpha)$ the corresponding free C*-dynamical system from Example 5.3. If

$$\varphi : \mathbb{Z}^2 \to \operatorname{Pic}(C(\mathbb{T})) \cong \operatorname{Pic}(\mathbb{T}) \rtimes \operatorname{Homeo}(\mathbb{T})$$

denotes the associated group homomorphism, then the class of $(C^*(H), \mathbb{T}^2, \alpha)$ is contained in the set $\operatorname{Ext}(C(\mathbb{T}), \mathbb{T}^2, \varphi)$ of equivalence classes of realizations of φ, which is by Corollary 7.9 parametrized by the cohomology group $H^2_\varphi(\mathbb{Z}^2, C(\mathbb{T}, \mathbb{T}))$. For its computation we refer, again, to [52, Proposition 6.2].

Example 9.7. For $q \in [-1,1]$ let $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ be the quantum Hopf fibration from Example 5.4 and $L_q(1)$ the isotypic component corresponding to $1 \in \mathbb{Z}$. If

$$\varphi : \mathbb{Z} \to \operatorname{Pic}(S_q(2)), \quad 1 \mapsto [L_q(1)]$$

denotes the associated group homomorphism, then the class of $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ is contained in the set $\operatorname{Ext}(S_q(2), \mathbb{T}, \varphi)$ of equivalence classes of realizations of φ, which is by Corollary 7.9 parametrized by the cohomology group $H^2_\varphi(\mathbb{Z}, UZ(S_q(2)))$. It follows, for example, from [31, Chapter IV.6] that this cohomology group is trivial, i. e., up to isomorphism the quantum Hopf fibration $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ is the unique realization of the group homomorphism φ.

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