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Embedding tetrahedra into quasirandom hypergraphs

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EMBEDDING TETRAHEDRA INTO QUASIRANDOM HYPERGRAPHS

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Abstract. We investigate extremal problems for quasirandom hypergraphs. We say that a 3-uniform hypergraph $H = (V, E)$ is $(d, \eta, \therefore)$-*quasirandom* if for any subset $X \subseteq V$ and every set of pairs $P \subseteq V \times V$ the number of pairs $(x, (y, z)) \in X \times P$ with $\{x, y, z\}$ being a hyperedge of H is in the interval $d\,|X|\,|P| \pm \eta\,|V|^3$. We show that for any $\varepsilon > 0$ there exists $\eta > 0$ such that every sufficiently large $(1/2 + \varepsilon, \eta, \therefore)$-quasirandom hypergraph contains a tetrahedron, i.e., four vertices spanning all four hyperedges. A known random construction shows that the density 1/2 is best possible. This result is closely related to a question of Erdős, whether every weakly quasirandom 3-uniform hypergraph H is with density bigger than 1/2, i.e., every large subset of vertices induces a hypergraph with density bigger than 1/2, contains a tetrahedron.

§1. Introduction

1.1. **Extremal problems for graphs and hypergraphs.** Given a fixed graph F a typical problem in extremal graph theory asks for the maximum number of edges that a (large) graph G on n vertices containing no copy of F can have. More formally, for a fixed graph F let the *extremal number* $\mathrm{ex}(n, F)$ be the number $|E|$ of edges of an F-free graph $G = (V, E)$ on $|V| = n$ vertices with the maximum number of edges. It is well known and not hard to observe that the sequence $\mathrm{ex}(n, F)/\binom{n}{2}$ is decreasing. Consequently one may define the *Turán density*

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}}$$

which describes the maximum density of large F-free graphs. The systematic study of these extremal parameters was initiated by Turán [24], who determined $\mathrm{ex}(n, K_k)$ for complete graphs K_k. Thanks to his work and the results from [6] by Erdős and Stone it is known that the Turán density of any graph F with at least one edge can be explicitly computed using

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the formula

$$\pi(F) = \tfrac{\chi(F)-2}{\chi(F)-1}\,. \tag{1}$$

Already in his original work [24] Turán asked for hypergraph extensions of these extremal problems. We restrict ourselves to 3-*uniform hypergraphs* $H = (V, E)$, where $V = V(H)$ is a finite set of *vertices* and the set of *hyperedges* $E = E(H) \subseteq \binom{V}{3}$ is a collection of 3-element sets of vertices. In this paper we shall only consider graphs and 3-uniform hypergraphs and when we are referring simply to a hypergraph we will always mean a 3-uniform hypergraph. Despite considerable effort no formula similar to (1) is known or conjectured to hold for general 3-uniform hypergraphs F. Determining the value of $\pi(F)$ is a well known and hard problem even for "simple" hypergraphs like the complete 3-uniform hypergraph $K_4^{(3)}$ on four vertices, which is also called the *tetrahedron*. Currently the best known bounds for its Turán density are

$$\frac{5}{9} \leqslant \pi(K_4^{(3)}) \leqslant 0.5616\,,$$

where the lower bounds is given by what is believed to be an optimal construction due to Turán (see, e.g., [4]). The upper bound is due to Razborov [14] and Baber and Talbot [1] and their proofs are based on the *flag algebra method* introduced by Razborov [13]. For a thorough discussion of Turán type results and problems for hypergraphs we refer to the recent survey of Keevash [10].

1.2. **Quasirandom graphs and hypergraphs.** We consider a variant of Turán type questions in connection with quasirandom hypergraphs. Roughly speaking, a quasirandom hypergraph "resembles" a random hypergraph of the same edge density, by sharing some of the key properties with it, i.e., properties that hold true for the random hypergraph with probability close to 1.

The investigation of quasirandom graphs was initiated with the observation that several such properties of randomly generated graphs are equivalent in a deterministic sense. This phenomenon turned out to be useful and had a number of applications in combinatorics. The systematic study of quasirandom graphs was initiated by Thomason [21, 22] and by Chung, Graham, and Wilson [2]. A pivotal feature of random graphs is the uniform edge distribution on "large" sets of vertices and a quantitative version of this property is used to define quasirandom graphs. More precisely, a graph $G = (V, E)$ is *quasirandom with density* $d > 0$ if for every subset of vertices $U \subseteq V$ the number $e(U)$ of edges contained in U satisfies

$$e(U) = d\tbinom{|U|}{2} + o(|V|^2)\,, \tag{2}$$

where $o(|V|^2)/|V|^2 \to 0$ as $|V(G)|$ tends to infinity. Strictly speaking, we consider here a sequence of graphs $G_n = (V_n, E_n)$ where the number of vertices $|V_n|$ tends to infinity, but for the sake of a simpler presentation we will suppress the sequence in the discussion here. The main result in [2] asserts, that satisfying (2) is deterministically equivalent to several other important properties of random graphs. In particular, it implies that for any fixed graph F with v_F vertices and e_F edges the number $N_F(G)$ of labeled copies of F in a quasirandom graph $G = (V, E)$ of density d satisfies

$$N_F(G) = d^{e_F}|V|^{v_F} + o(|V|^{v_F}). \tag{3}$$

In other words, the number of copies of F is close to the expected value in a random graph with edge density d.

The analogous statement for hypergraphs fails to be true and uniform edge distribution on vertex sets is not sufficient to enforce a property similar to (3) for all fixed 3-uniform hypergraphs F (see, e.g., Example 1.6 below). A stronger notion of quasirandomness for which such an embedding result actually is true, was considered in connection with the regularity method for hypergraphs (cf. Definition 1.4 and 2.3).

In fact below we consider four different notions of quasirandom hypergraphs. The first and weakest concept that we consider here is $(d, \eta, \therefore)$-quasirandomness.

Definition 1.1. A 3-uniform hypergraph $H = (V, E)$ on $n = |V|$ vertices is $(d, \eta, \therefore)$-*quasirandom* if for every triple of subsets $X, Y, Z \subseteq V$ the number $e_{\therefore}(X, Y, Z)$ of triples $(x, y, z) \in X \times Y \times Z$ with $\{x, y, z\} \in E$ satisfies

$$\big|e_{\therefore}(X, Y, Z) - d\,|X|\,|Y|\,|Z|\big| \leqslant \eta\, n^3 .$$

The central notion for the work presented here, however, is the following stronger concept of quasirandom hypergraphs, where we "replace" the two sets Y and Z by an arbitrary set of pairs P.

Definition 1.2. A 3-uniform hypergraph $H = (V, E)$ on $n = |V|$ vertices is $(d, \eta, \dot{\smile})$-*quasirandom* if for every subset $X \subseteq V$ of vertices and every subset of pairs of vertices $P \subseteq V \times V$ the number $e_{\dot{\smile}}(X, P)$ of pairs $(x, (y, z)) \in X \times P$ with $\{x, y, z\} \in E$ satisfies

$$\big|e_{\dot{\smile}}(X, P) - d\,|X|\,|P|\big| \leqslant \eta\, n^3 . \tag{4}$$

Since for any hypergraph $H = (V, E)$ and sets X, Y, $Z \subseteq V$ we have

$$e_{\therefore}(X, Y, Z) = e_{\dot{\smile}}(X, Y \times Z),$$

it follows from these definitions that any $(d, \eta, \dot{-})$-quasirandom hypergraph is also $(d, \eta, \therefore)$-quasirandom.

We also remark that the three vertices appearing in $\therefore$ (resp. the vertex and the edge depicted in $\dot{-}$) symbolically represent the possible choices for the sets X, Y, Z (resp. the set of vertices X and for the set of pairs P). Next we come to a notion where rather than a "set of vertices and a set of pairs" we consider "two sets of pairs".

Definition 1.3. A 3-uniform hypergraph $H = (V, E)$ on $n = |V|$ vertices is $(d, \eta, \wedge)$-*quasirandom* if for any two subsets of pairs P, $Q \subseteq V \times V$ the number $e_{\wedge}(P, Q)$ of pairs of pairs $((x, y), (x, z)) \in P \times Q$ with $\{x, y, z\} \in E$ satisfies

$$\big|e_{\wedge}(P, Q) - d\,|\mathcal{K}_{\wedge}(P, Q)|\big| \leqslant \eta\, n^3\,,$$

where $\mathcal{K}_{\wedge}(P, Q)$ denotes the set of pairs in $P \times Q$ of the form $((x, y), (x, z))$.

Finally we will introduce the following strongest notion of quasirandomness that plays an important rôle in the hypergraph regularity method.

Definition 1.4. A 3-uniform hypergraph $H = (V, E)$ on $n = |V|$ vertices is $(d, \eta, \triangle)$-*quasirandom* if for any three subsets P, Q, $R \subseteq V \times V$ the number $e_{\triangle}(P, Q, R)$ of triples $((x, y), (x, z), (y, z)) \in P \times Q \times R$ with $\{x, y, z\} \in E$ satisfies

$$\big|e_{\triangle}(P, Q, R) - d\,|\mathcal{K}_{\triangle}(P, Q, R)|\big| \leqslant \eta\, n^3\,,$$

where $\mathcal{K}_{\triangle}(P, Q, R)$ denotes the set of triples in $P \times Q \times R$ with $((x, y), (x, z), (y, z))$.

For a symbol $\star \in \{\therefore, \dot{-}, \wedge, \triangle\}$ we sometimes write a hypergraph H is $\star$-quasirandom to mean that it is $(d, \eta, \star)$-quasirandom for some positive d and some small η. More precisely, we imagine a sequence of $(d, \eta_n, \star)$-quasirandom hypergraphs with $\eta_n \to 0$ as $n \to \infty$. Moreover, for $\star \in \{\therefore, \dot{-}, \wedge, \triangle\}$ we denote by $\mathcal{Q}(d, \eta, \star)$ the class of all $(d, \eta, \star)$-quasirandom hypergraphs and one can observe that

$$\mathcal{Q}(d, \eta, \triangle) \subseteq \mathcal{Q}(d, \eta, \wedge) \subseteq \mathcal{Q}(d, \eta, \dot{-}) \subseteq \mathcal{Q}(d, \eta, \therefore) \tag{5}$$

holds for all $d \in [0, 1]$ and $\eta > 0$. We are interested in Turán densities for quasirandom hypergraphs given by the following functions.

Definition 1.5. Given a 3-uniform hypergraph F and a symbol $\star \in \{\therefore, \dot{-}, \wedge, \triangle\}$ we set

$$\begin{aligned}\pi_{\star}(F) = \sup\big\{d \in [0, 1]\colon\ &\text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}\\ &\text{an } F\text{-free 3-uniform hypergraph } H \in \mathcal{Q}(d, \eta, \star) \text{ with } |V(H)| \geqslant n\big\}\,.\end{aligned}$$

Due to the inclusions (5) we have

$$\pi_{\therefore}(F) \geqslant \pi_{\dot{-}}(F) \geqslant \pi_{\wedge}(F) \geqslant \pi_{\triangle}(F) \tag{6}$$

for any 3-uniform hypergraph F. The last among those four parameters is trivial because we have $\pi_{\triangle}(F) = 0$ for any graph F. This follows directly from the results in [11] (alternatively, it can be easily deduced by the regularity method for hypergraphs via a combined application of Theorems 2.2 and 2.3).

Moreover, when F is tripartite or linear it can be shown that all four of these quantities vanish (in fact, a full characterisation of this event is going to appear in [15]). For $F = K_4^{(3)-}$, the 3-uniform hypergraph consisting of three hyperedges on four vertices, it is known that

$$\pi_{\therefore}(K_4^{(3)-}) = \pi_{\dot{-}}(K_4^{(3)-}) = 1/4 \qquad \text{and} \qquad \pi_{\wedge}(K_4^{(3)-}) = 0\,.$$

In fact, Glebov, Kráľ, and Volec established $\pi_{\therefore}(K_4^{(3)-}) \leqslant 1/4$ in [8] (see also [16] for an alternative proof). On the other hand, one can check that the hypergraph corresponding to the cyclically oriented triangles of a random tournament, which provides $\pi_{\therefore}(K_4^{(3)-}) \geqslant 1/4$, is also $\dot{-}$-quasirandom, and by (6) we get

$$\frac{1}{4} \leqslant \pi_{\dot{-}}(K_4^{(3)-}) \leqslant \pi_{\therefore}(K_4^{(3)-}) \leqslant \frac{1}{4}\,,$$

which establishes our first claim. The second identity follows from $\pi_{\wedge}(K_4^{(3)}) = 0$, which will appear in [17].

The next case one might wish to study is the tetrahedron. The following random construction from [18] was used to show that $\pi_{\therefore}(K_4^{(3)}) \geqslant 1/2$ and Erdős [5] suggested that this might be best possible.

Example 1.6. Given any map $\varphi\colon \binom{[n]}{2} \to \{\text{red}, \text{green}\}$ we define the 3-uniform hypergraph H_φ with vertex set $[n] = \{1, \dots, n\}$ by putting a triple $\{i, j, k\}$ with $i < j < k$ into $E(H_\varphi)$ if and only if the colours of ij and ik differ.

Irrespective of the choice of the colouring φ, the hypergraph H_φ contains no tetrahedra: for if a, b, c, and d are any four distinct vertices, say with $a = \min(a, b, c, d)$, then it is impossible for all three of the pairs ab, ac, and ad to have distinct colours, whence not all three of the triples abc, abd, and acd can be hyperedges of H_φ.

It was noticed in [18] that if the colouring φ is chosen uniformly at random, then for any $\eta > 0$ the hypergraph H_φ is with high probability $(1/2, \eta, \therefore)$-quasirandom as n tends to infinity. This is easily checked using standard tail estimates for binomial distributions and similar arguments show that we may replace $\therefore$ by $\overset{\bullet}{\bullet\!\!-\!\!\bullet}$ in this observation. In other words, this example shows that

$$\pi_{\overset{\bullet}{\bullet\!-\!\bullet}}(K_4^{(3)}) \geqslant \tfrac{1}{2}\,.$$

holds.

Our main contribution here provides a matching upper bound and shows that for $K_3^{(4)}$-free $\overset{\bullet}{\bullet\!\!-\!\!\bullet}$-quasirandom hypergraphs the construction given in Example 1.6 is best possible.

Theorem 1.7 (Main result). *For every $\varepsilon > 0$ there exists an $\eta > 0$ and an integer n_0 such that every 3-uniform $(\frac{1}{2}+\varepsilon, \eta, \overset{\bullet}{\bullet\!\!-\!\!\bullet})$-quasirandom hypergraph H with at least n_0 vertices contains a tetrahedron.*

The proof of Theorem 1.7 will be based on the regularity method for 3-uniform hypergraphs which is summarised to the necessary extent in the following section. The details of the proof of Theorem 1.7 appear in Section 3. We close with a few remarks on extremal problems involving the tetrahedron for related notions of quasirandomness in Section 4.

§2. Hypergraph regularity method

A key tool in the proof of Theorem 1.7 is the regularity lemma for 3-uniform hypergraphs. We follow the approach from [19, 20] combined with the results from [9] and [12] and below we introduce the necessary notation.

For two disjoint sets X and Y we denote by $K(X,Y)$ the complete bipartite graph with that vertex partition. We say a bipartite graph $P = (X \dot\cup Y, E)$ ist (δ_2, d_2)-*regular* if for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ we have

$$\big|e(X',Y') - d_2|X'||Y'|\big| \leqslant \delta_2|X||Y|\,,$$

where $e(X',Y')$ denotes the number of edges of P with one vertex in X' and one vertex in Y'. Moreover, for $k \geqslant 2$ we say a k-partite graph $P = (X_1 \dot\cup \ldots \dot\cup X_k, E)$ is (δ_2, d_2)-regular, if all its $\binom{k}{2}$ naturally induced bipartite subgraphs $P[X_i, X_j]$ are (δ_2, d_2)-regular. For a tripartite graph $P = (X \dot\cup Y \dot\cup Z, E)$ we denote by $\mathcal{K}_3(P)$ the triples of vertices spanning a triangle in P, i.e.,

$$\mathcal{K}_3(P) = \{\{x,y,z\} \subseteq X \cup Y \cup Z\colon\ xy, xz, yz \in E\}\,.$$

If the tripartite graph P is (δ_2, d_2)-regular, then the so-called *triangle counting lemma* implies

$$|\mathcal{K}_3(P)| \leqslant d_2^3|X||Y||Z| + 3\delta_2|X||Y||Z|\,. \tag{7}$$

We say a 3-uniform hypergraph $H = (V, E_H)$ is regular w.r.t. a tripartite graph P if it matches approximately the same proportion of triangles for every subgraph $Q \subseteq P$. This we make precise in the following definition.

Definition 2.1. A 3-uniform hypergraph $H = (V, E_H)$ is (δ_3, d_3)*-regular w.r.t. a tripartite graph* $P = (X \dot\cup Y \dot\cup Z, E_P)$ with $V \supseteq X \cup Y \cup Z$ if for every tripartite subgraph $Q \subseteq P$ we have

$$\big||E_H \cap \mathcal{K}_3(Q)| - d_3|\mathcal{K}_3(Q)|\big| \leqslant \delta_3|\mathcal{K}_3(P)|\,.$$

Moreover, we simply say H *is* δ_3*-regular w.r.t.* P, if it is (δ_3, d_3)-regular for some $d_3 \geqslant 0$. We also define the *relative density* of H w.r.t. P by

$$d(H|P) = \frac{|E_H \cap \mathcal{K}_3(P)|}{|\mathcal{K}_3(P)|}\,,$$

where we use the convention $d(H|P) = 0$ if $\mathcal{K}_3(P) = \varnothing$. If H is not δ_3-regular w.r.t. P, then we simply refer to it as δ_3*-irregular*.

The regularity lemma for 3-uniform hypergraphs, introduced by Frankl and Rödl in [7], provides for every hypergraph H a partition of its vertex set and a partition of the edge sets of the complete bipartite graphs induced by the vertex partition such that for appropriate constants δ_3, δ_2, and d_2

(*1*) the bipartite graphs given by the partitions are (δ_2, d_2)-regular and

(*2*) H is δ_3-regular for "most" tripartite graphs P given by the partition.

In many proofs based on the regularity method it is convenient to "clean" the regular partition provided by the regularity lemma. In particular, we shall disregard hyperedges of H that belong to $\mathcal{K}_3(P)$ when H is not δ_3-regular or when $d(H|P)$ is very small. These properties are rendered in the following somewhat standard corollary of the regularity lemma.

Theorem 2.2. *For every $d_3 > 0$, $\delta_3 > 0$ and $m \in \mathbb{N}$, and every function $\delta_2\colon \mathbb{N} \to (0, 1]$, there exist integers T_0 and n_0 such that for every $n \geqslant n_0$ and every n-vertex 3-uniform hypergraph $H = (V, E)$ the following holds.*

There exists a subhypergraph $\hat{H} = (\hat{V}, \hat{E}) \subseteq H$, an integer $\ell \leqslant T_0$, a vertex partition $V_1 \dot\cup \dots \dot\cup V_m = \hat{V}$, and for all $1 \leqslant i < j \leqslant m$ there exists a partition

$$\mathcal{P}^{ij} = \{P^{ij}_\alpha = (V_i \dot\cup V_j, E^{ij}_\alpha)\colon\ 1 \leqslant \alpha \leqslant \ell\}$$

of $K(V_i, V_j)$ satisfying the following properties

(*i*) $|V_1| = \dots = |V_m| \geqslant (1 - \delta_3)n/T_0$,

(ii) *for every* $1 \leqslant i < j \leqslant m$ *and* $\alpha \in [\ell]$ *the bipartite graph* P^{ij}_{α} *is* $(\delta_2(\ell), 1/\ell)$*-regular,*

(iii) $\hat{H}$ *is* δ_3*-regular w.r.t.* $P^{ijk}_{\alpha\beta\gamma}$ *for all tripartite graphs (which will be later referred to as triads)*

$$P^{ijk}_{\alpha\beta\gamma} = P^{ij}_{\alpha} \dot\cup P^{ik}_{\beta} \dot\cup P^{jk}_{\gamma} = (V_i \dot\cup V_j \dot\cup V_k, E^{ij}_{\alpha} \dot\cup E^{ik}_{\beta} \dot\cup E^{jk}_{\gamma})\,, \tag{8}$$

with $1 \leqslant i < j < k \leqslant m$ *and* α, β, $\gamma \in [\ell]$, *where either* $d(H|P) = 0$ *or* $d(H|P) \geqslant d_3$, *and*

(iv) *for every* $1 \leqslant i < j < k \leqslant m$ *there are at most* $\delta_3\, \ell^3$ *triples* $(\alpha, \beta, \gamma) \in [\ell]^3$ *such that* H *is* δ_3*-irregular with respect to the triad* $P^{ijk}_{\alpha\beta\gamma}$.

The standard proof of Theorem 2.2 based on a refined version of the regularity lemma from [19, Theorem 2.3] can be found in [16, Corollary 3.3]. Actually the statement there differs from the one given here in the final clause, but the proof from [16] shows the present version as well. (In fact, the new version of *(iv)* is explicitly stated as clause (a) in the definition of the hypergraph R in [16, Proof of Corollary 3.3].)

We shall use a so-called *counting/embedding lemma*, which allows us to embed hypergraphs of fixed isomorphism type into appropriate and sufficiently regular and dense triads of the partition provided by Theorem 2.2. The following statement is a direct consequence of [12, Corollary 2.3].

Theorem 2.3 (Embedding Lemma)**.** *For every* 3*-uniform hypergraph* $F = (V_F, E_F)$ *with vertex set* $V_F = [f]$ *and every* $d_3 > 0$ *there exists* $\delta_3 > 0$*, and functions* $\delta_2 \colon \mathbb{N} \to (0,1]$ *and* $N \colon \mathbb{N} \to \mathbb{N}$ *such that the following holds for every* $\ell \in \mathbb{N}$.

Let $P = (V_1 \dot\cup \dots \dot\cup V_f, E_P)$ *be a* $(\delta_2(\ell), \frac{1}{\ell})$*-regular,* f*-partite graph with* $|V_1| = \dots = |V_f| \geqslant N(\ell)$ *and suppose* H *is an* f*-partite,* 3*-uniform hypergraph satisfying for every edge* $ijk \in E_F$

(a) H *is* δ_3*-regular w.r.t. to the tripartite graph* $P[V_i \dot\cup V_j \dot\cup V_k]$ *and*

(b) $d(H|P[V_i \dot\cup V_j \dot\cup V_k]) \geqslant d_3$

then H *contains a copy of* F*, where for every* $i \in [f] = V_F$ *the image of* i *is contained in* V_i.

In an application of Theorem 2.3 the tripartite graphs $P[V_i \dot\cup V_j \dot\cup V_k]$ in *(a)* and *(b)* will be given by triads $P^{ijk}_{\alpha\beta\gamma}$ from the partition given by Theorem 2.2.

For the proof of Theorem 1.7 we consider a $\therefore$-quasirandom hypergraph H of density $1/2+\varepsilon$. We will apply the regularity lemma in form of Theorem 2.2 to H. In the main part of the proof concerns the appropriate selection of dense and regular triads, that are ready for application of the embedding lemma with $F = K^{(3)}_4$. This will be the focus in Section 3.

§3. Embedding tetrahedra

In this section we deduce Theorem 1.7. The proof will be based on the regularity method for hypergraphs in form of Theorem 2.2 and the embedding lemma (Theorem 2.3). Below we reduce the proof of Theorem 1.7 to a lemma (see Lemma 3.1 below) which locates in a sufficiently regular partition of a $\therefore$-quasirandom hypergraph with density $> 1/2$ a collection of triads that are ready for an application of the embedding lemma for $K_4^{(3)}$.

Proof of Theorem 1.7. Given $\varepsilon > 0$ we have to find appropriate $\eta > 0$ and $n_0 \in \mathbb{N}$. For this purpose we start by choosing some auxiliary constants δ_3, d_3, δ, and m obeying the hierarchy

$$\delta_3 \ll d_3, \delta, m^{-1} \ll \varepsilon\,. \tag{9}$$

For these choices of δ_3 and d_3 and $F = K_4^{(3)}$ we appeal to Theorem 2.3 and obtain $\delta_2\colon \mathbb{N} \to \mathbb{N}$ and $N\colon \mathbb{N} \to \mathbb{N}$. Without loss of generality we may assume that for all $\ell \in \mathbb{N}$ we have

$$\delta_2(\ell) \ll \ell^{-1}, \varepsilon\,.$$

Applying Theorem 2.2 to d_3, δ_3, m, and δ_2 we get two integers T_0 and n_0'. Now we claim that any

$$\eta \ll T_0^{-1} \qquad \text{and} \qquad n_0 \gg \max(T_0 \cdot N(T_0), n_0')$$

are as desired.

To justify this, we let any $(1/2 + \varepsilon, \eta, \therefore)$-quasirandom hypergraph H on $n \geqslant n_0$ vertices be given. Since $n \geqslant n_0'$ holds as well, we may apply Theorem 2.2, thus getting a subhypergraph $\hat{H} \subseteq H$ with vertex partition $\hat{V} = V_1 \dot\cup \ldots \dot\cup V_m$ and edge partitions $\mathcal{P}^{ij} = \{P^{ij}_\alpha \colon \alpha \in [\ell]\}$ of $K(V_i, V_j)$ for $1 \leqslant i < j \leqslant m$.

In view of the embedding lemma (Theorem 2.3) the task that remains to be done is now reduced to the task of locating four vertex classes $V_{i_1}, \ldots, V_{i_4}$ with $i_1 < i_2 < i_3 < i_4$ and six bipartite graphs $P^{ab} \in \mathcal{P}^{i_a i_b}$ for $1 \leqslant a < b \leqslant 4$ from the regular partition, such that all triads

$$P^{abc} = P^{ab} \dot\cup P^{ac} \dot\cup P^{bc}$$

with $1 \leqslant a < b < c \leqslant 4$ are *dense* and *regular*, i.e., $d(H|P^{abc}) \geqslant d_3$ and H is δ_3-regular w.r.t. P^{abc}. For this purpose we reformulate our current situation in terms of a "reduced hypergraph" $\mathcal{A}$. The work we will then have to perform on $\mathcal{A}$ is deferred to Lemma 3.1 stated below.

The reduced hypergraph $\mathcal{A}$ is going to be 3-uniform and $\binom{m}{2}$-partite with vertex classes $\mathcal{P}^{ij}$ for $1 \leqslant i < j \leqslant m$. Among its $\binom{\binom{m}{2}}{3}$ naturally induced tripartite 3-uniform subhypergraphs only $\binom{m}{3}$ ones are inhabited by hyperedges: these are the hypergraphs $\mathcal{A}^{ijk}$ with vertex classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$ for $1 \leqslant i < j < k \leqslant m$. They are defined to have precisely those hyperedges $P^{ij}P^{ik}P^{jk}$ with $P^{ij} \in \mathcal{P}^{ij}$, $P^{ik} \in \mathcal{P}^{ik}$, and $P^{jk} \in \mathcal{P}^{jk}$ for which the triad

$$(V_i \dot\cup V_j \dot\cup V_k, P^{ij} \dot\cup P^{ik} \dot\cup P^{jk})$$

has $\hat{H}$-density at least d_3. To see that Lemma 3.1 is applicable (with $\varepsilon/2$ instead of ε), it is enough to verify that given $1 \leqslant i < j < k \leqslant m$ the following is true. There are at most $\delta\,|\mathcal{P}^{ij}|$ vertices $P^{ij} \in \mathcal{P}^{ij}$ whose degree in $\mathcal{A}^{ijk}$ is smaller than $(1/2 + \varepsilon/2)\ell^2$, and similarly for $\mathcal{P}^{ik}$ and $\mathcal{P}^{jk}$.

Since the proofs of all three of these statements are the same, we just deal with $\mathcal{P}^{ij}$ in the sequel. Let X^{ij}_k denote the set of all those $P^{ij} \in \mathcal{P}^{ij}$ for which there are more than $\delta\,\ell^2$ pairs $(P^{ik}, P^{jk}) \in \mathcal{P}^{ik} \times \mathcal{P}^{jk}$ such that H is δ_3-irregular with respect to the triad $\big(V_i \dot\cup V_j \dot\cup V_k, P^{ij} \dot\cup P^{ik} \dot\cup P^{jk}\big)$. Consequently the total number of triads involving V_i, V_j, and V_k with respect to which H is δ_3-irregular is on the one hand at least $\delta\,\ell^2\,|X^{ij}_k|$. On the other hand it is at most $\delta_3\ell^3$ by clause *(iv)* of Theorem 2.2. Consequently we have $|X^{ij}_k| \leqslant \delta_3\,\ell/\delta \leqslant \delta\ell$ (by the hierarchy given in (9)). It suffices to check that any $P^{ij} \in \mathcal{P}^{ij} \setminus X^{ij}_k$ belongs to at least $(1/2 + \varepsilon/2)\ell^2$ hyperedges of $\mathcal{A}^{ijk}$.

To verify this, we fix any $P^{ij} \in \mathcal{P}^{ij} \setminus X^{ij}_k$ for the remainder of the argument. Let us apply the $(1/2 + \varepsilon, \eta, \therefore)$-quasirandomness of H to V_k and the set of pairs

$$Q^{ij} = \{(x, y) \in V_i \times V_j \colon \{x, y\} \in E(P^{ij})\}$$

in the rôle of X and P of Definition 1.2. Concerning the number $e_{\therefore}(V_k, Q^{ij})$ of pairs $(v, (x, y)) \in V_k \times Q^{ij}$ with $\{v, x, y\} \in E(H)$ this tells us

$$e_{\therefore}(V_k, Q^{ij}) \geqslant \big(\tfrac{1}{2} + \varepsilon\big)|V_k|\,|E(P^{ij})| - \eta \cdot n^3\,.$$

Set $M = |V_i| = |V_j| = |V_k|$. Since

$$|E(P^{ij})| \geqslant \left(\frac{1}{\ell} - \delta_2(\ell)\right) M^2$$

follows from *(ii)*, we get

$$e_{\therefore}(V_k, Q^{ij}) \geqslant \left(\frac{1}{2} + \varepsilon\right)\left(\frac{1}{\ell} - \delta_2(\ell)\right) M^3 - \eta\,n^3\,.$$

As we have $M \geqslant \frac{n}{2T_0}$ by *(i)*, the hierarchy imposed on η leads to

$$e_{\therefore}(V_k, Q^{ij}) \geqslant \left(\frac{1}{2} + \frac{9\varepsilon}{10}\right) \cdot \frac{M^3}{\ell}\,.$$

On the other hand, we have

$$e_{\therefore}(V_k, Q^{ij}) = \sum_{(P^{ik},P^{jk})\in\mathcal{P}^{ik}\times\mathcal{P}^{jk}} \big|E(H)\cap\mathcal{K}_3(P^{ij}\cup P^{ik}\cup P^{jk})\big|\,. \tag{10}$$

The terms corresponding to triads $\big(V_i\dot\cup V_j\dot\cup V_k, P^{ij}\dot\cup P^{ik}\dot\cup P^{jk}\big)$ with respect to which H has at most the density d_3 contribute at most $d_3(\ell^{-3}+3\delta_2(\ell))M^3\ell^2$ (see (7)) and by $\delta_2(\ell)\ll\ell^{-1}$ this is at most $2d_3M^3/\ell$.

Further, by $P^{ij}\notin X_k^{ij}$ there are at most $\delta\,\ell^2$ terms on the right hand side of (10) corresponding to δ_3-irregular triads, and each of them contributes, for the same reason as above, at most $\frac{2M^3}{\ell^3}$ to the right hand side of (10).

The remaining terms from (10) satisfy $P^{ij}P^{ik}P^{jk}\in E(\mathcal{A}^{ijk})$ and each of them contributes at most $\big(1+\frac{\varepsilon}{5}\big)\cdot\frac{M^3}{\ell^3}$. So if $\deg(P^{ij})$ denotes the degree of P^{ij} in $\mathcal{A}^{ijk}$ we arrive at

$$e_{\therefore}(V_k, Q^{ij}) \leqslant \left(\Big(1+\frac{\varepsilon}{5}\Big)\frac{\deg(P^{ij})}{\ell^2}+2d_3+2\delta\right)\cdot\frac{M^3}{\ell}\,.$$

Comparing both estimates for $e_{\therefore}(V_k, Q^{ij})$ we deduce

$$\frac{1}{2}+\frac{9\varepsilon}{10}\leqslant\Big(1+\frac{\varepsilon}{5}\Big)\frac{\deg(P^{ij})}{\ell^2}+2d_3+2\delta\,.$$

We may assume $d_3,\delta\leqslant\frac{\varepsilon}{40}$, thus getting

$$\frac{1}{2}+\frac{4\varepsilon}{5}\leqslant\Big(1+\frac{\varepsilon}{5}\Big)\frac{\deg(P^{ij})}{\ell^2}$$

and by $\varepsilon\ll 1$ this implies

$$\frac{1+\varepsilon}{2}\cdot\ell^2\leqslant\deg(P^{ij})\,,$$

as desired. This concludes the reduction of Theorem 1.7 to Lemma 3.1. $\square$

Lemma 3.1. *For every $\varepsilon>0$ there exist $\delta>0$ and an integer m such that the following holds. If $\mathcal{A}$ is an $\binom{m}{2}$-partite 3-uniform hypergraph with*

- *(i) nonempty vertex classes $\mathcal{P}^{ij}$ for $1\leqslant i<j\leqslant m$ such that*
- *(ii) for each triple $1\leqslant i<j<k\leqslant m$ the restriction $\mathcal{A}^{ijk}$ of $\mathcal{A}$ to $\mathcal{P}^{ij}\cup\mathcal{P}^{ik}\cup\mathcal{P}^{jk}$ has the property that all but at most $\delta|\mathcal{P}^{ij}|$ vertices of $\mathcal{P}^{ij}$ are contained in at least $(1/2+\varepsilon)|\mathcal{P}^{ik}||\mathcal{P}^{jk}|$ hyperedges of $\mathcal{A}^{ijk}$ and the corresponding property holds for the vertex classes $\mathcal{P}^{ik}$ and $\mathcal{P}^{jk}$ as well,*

then there are four distinct indices i_1, i_2, i_3, and i_4 from $[m]$ together with six vertices $P^{ab}\in\mathcal{P}^{i_ai_b}$ for $1\leqslant a<b\leqslant 4$ such that all four $P^{12}P^{13}P^{23}$, $P^{12}P^{14}P^{24}$, $P^{13}P^{14}P^{34}$, and $P^{23}P^{24}P^{34}$ are triples of $\mathcal{A}$.

Proof. For a given $\varepsilon > 0$ we set

$$\delta = \frac{\varepsilon}{4} \qquad \text{and} \qquad m = 3 + 2^{1/\delta^3}\,.$$

We will find the desired configuration with $i_1 = 1$ and $i_2 = 2$. The argument splits into three steps. In the first step we select the indices i_3 and i_4. After that step it remains to select the six vertices $P^{i_a i_b}$ with $1 \leqslant a < b \leqslant 4$. In the second step we shall fix the three vertices $P^{i_a i_4}$ with $a = 1, 2, 3$ and in the third step we fix the remaining three vertices.

Step 1: Selecting i_3 and i_4. We commence by assigning a colour to each integer between 3 and m, the idea being that the colour of an index $i \in [3, m]$ encodes the sizes of holes (independent sets) in $\mathcal{A}^{12i}$. More precisely, for positive integers p, q, and r we say, that $\mathcal{A}^{12i}$ has a *(p, q, r)-hole* if there are three sets $I^{12} \subseteq \mathcal{P}^{12}$, $I^{1i} \subseteq \mathcal{P}^{1i}$, and $I^{2i} \subseteq \mathcal{P}^{2i}$ with

$$|I^{12}| \geqslant p \cdot \delta\,|\mathcal{P}^{12}|\,, \quad |I^{1i}| \geqslant q \cdot \delta\,|\mathcal{P}^{1i}|\,, \quad \text{and} \quad |I^{2i}| \geqslant r \cdot \delta\,|\mathcal{P}^{2i}|$$

such that $I^{12} \cup I^{1i} \cup I^{2i}$ is independent in $\mathcal{A}^{12i}$. Evidently, such a hole can only exist if $p, q, r \leqslant \delta^{-1}$.

Let Ξ_i be the set of all integer triples $(p, q, r) \in \left[1, \delta^{-1}\right]^3$ for which $\mathcal{A}^{12i}$ contains a (p, q, r)-hole. We think of the set Ξ_i as a colour that has been attributed to the index $i \in [3, m]$ and obviously there are at most $2^{1/\delta^3}$ possible colours. Since $m - 2$ exceeds the number of possible colours, the pigeonhole principle tells us that there exist two distinct integers i_3 and i_4 between 3 and m of the same colour. For the rest of the proof only the parts of $\mathcal{A}$ accessible via the indices 1, 2, i_3, and i_4 are relevant and so without loss of generality we may henceforth assume $i_3 = 3$, $i_4 = 4$, and

$$\Xi_3 = \Xi_4\,. \tag{11}$$

Step 2: Choosing P^{14}, P^{24}, and P^{34}. For any three distinct indices i, j, and k we denote the set of all vertices from $\mathcal{P}^{ij}$ whose degree in $\mathcal{A}^{ijk}$ is less than $\left(\frac{1}{2} + \varepsilon\right)|\mathcal{P}^{ik}|\,|\mathcal{P}^{jk}|$ by X^{ij}_k. In view of assumption *(ii)* of Lemma 3.1 we have

$$|X^{ij}_k| \leqslant \delta\,|\mathcal{P}^{ij}|\,. \tag{12}$$

Given two vertices P and P' of $\mathcal{A}$, we write $\mathrm{codeg}(P, P')$ for their *codegree*, i.e., for the size $|N(P, P')|$ of their *common neighbourhood*

$$N(P, P') = \left\{P'' \in V(\mathcal{A})\colon\ \{P, P', P''\} \in E(\mathcal{A})\right\}.$$

It follows from the partite structure of $\mathcal{A}$ that $N(P, P')$ is not empty only when $P \in \mathcal{P}^{ij}$ and $P' \in \mathcal{P}^{ik}$ for some distinct indices i, j, and k and in this case we have $N(P, P') \subseteq \mathcal{P}^{jk}$.

We fix P^{14} and P^{24} by selecting a pair

$$(P^{14}, P^{24}) \in (\mathcal{P}^{14} \setminus X_3^{14}) \times (\mathcal{P}^{24} \setminus X_3^{24})$$

with maximum codegree in $\mathcal{A}$. Let p be the largest integer such that

$$\operatorname{codeg}(P^{14}, P^{24}) \geqslant p \cdot \delta |\mathcal{P}^{12}|\,. \tag{13}$$

It follows from assumption *(ii)* that the average codegree among all pairs in $\mathcal{P}^{14} \times \mathcal{P}^{24}$ is at least

$$(1-\delta)(1/2+\varepsilon)|\mathcal{P}^{12}|$$

and, since X_3^{14} and X_3^{24} are small, a similar lower bound holds for the average (and hence the maximum) codegree in $(\mathcal{P}^{14} \setminus X_3^{14}) \times (\mathcal{P}^{24} \setminus X_3^{24})$. In fact, the number of hyperedges in $\mathcal{A}^{124}$ avoiding vertices in X_3^{14} and X_3^{24} is at least

$$\begin{aligned}(1/2+\varepsilon)|\mathcal{P}^{12}| \cdot (1-\delta)|\mathcal{P}^{14}||\mathcal{P}^{24}| - |\mathcal{P}^{12}||X_3^{14}||\mathcal{P}^{24}| - |\mathcal{P}^{12}||\mathcal{P}^{14}||X_3^{24}| \\ \geqslant \big((1/2+\varepsilon)(1-\delta) - 2\delta\big)|\mathcal{P}^{12}||\mathcal{P}^{14}||\mathcal{P}^{24}|\,.\end{aligned}$$

Note that we may assume that $\varepsilon \leqslant 1/2$, as otherwise the lemma is void. Consequently, the average codegree of the pairs in $(\mathcal{P}^{14} \setminus X_3^{14}) \times (\mathcal{P}^{24} \setminus X_3^{24})$ is at least $(1/2+\varepsilon-3\delta)|\mathcal{P}^{12}|$ and since (P^{14}, P^{24}) maximises the codegree we also have

$$p\,\delta \geqslant \tfrac{1}{2} + \varepsilon - 4\delta \geqslant \tfrac{1}{2}\,. \tag{14}$$

Having selected P^{14} and P^{24} we now select P^{34}. Due to $P^{14} \notin X_3^{14}$ and $P^{24} \notin X_3^{24}$ we have

$$\sum_{P \in \mathcal{P}^{34}} \operatorname{codeg}(P^{14}, P) \geqslant (\tfrac{1}{2}+\varepsilon)|\mathcal{P}^{13}|\,|\mathcal{P}^{34}|$$

as well as

$$\sum_{P \in \mathcal{P}^{34}} \operatorname{codeg}(P^{24}, P) \geqslant (\tfrac{1}{2}+\varepsilon)|\mathcal{P}^{23}|\,|\mathcal{P}^{34}|\,,$$

whence

$$\sum_{P \in \mathcal{P}^{34}} \left(\frac{\operatorname{codeg}(P^{14}, P)}{|\mathcal{P}^{13}|} + \frac{\operatorname{codeg}(P^{24}, P)}{|\mathcal{P}^{23}|} \right) \geqslant (1+2\varepsilon)|\mathcal{P}^{34}|\,.$$

For this reason, we may choose a vertex $P^{34} \in \mathcal{P}^{34}$ in such a way that

$$\frac{\operatorname{codeg}(P^{14}, P^{34})}{|\mathcal{P}^{13}|} + \frac{\operatorname{codeg}(P^{24}, P^{34})}{|\mathcal{P}^{23}|} \geqslant 1 + 2\varepsilon\,. \tag{15}$$

Because of $N(P^{14}, P^{34}) \subseteq \mathcal{P}^{13}$ and $N(P^{24}, P^{34}) \subseteq \mathcal{P}^{23}$ it follows that

$$|N(P^{14}, P^{34})| \geqslant 2\,\varepsilon\,|\mathcal{P}^{13}| \quad \text{and} \quad |N(P^{24}, P^{34})| \geqslant 2\,\varepsilon\,|\mathcal{P}^{23}|\,. \tag{16}$$

This concludes the selection of P^{14}, P^{24}, and P^{34}, which by (13), (14), and (16) guarantees that all three possible codegrees of these vertices are reasonable large. It is left to select P^{12}, P^{13}, and P^{23}. These three vertices have to form a hyperedge in $\mathcal{A}$ and each of them must be chosen from the common neighbourhood of two vertices chosen already, i.e., we have to make sure that there is a hyperedge $P^{12}P^{13}P^{23}$ of $\mathcal{A}$ with $P^{12} \in N(P^{14}, P^{24})$, $P^{13} \in N(P^{14}, P^{34})$, and $P^{23} \in N(P^{24}, P^{34})$. This is the content of the last step.

Step 3: Choosing P^{12}, P^{13}, and P^{23}. As mentioned above it suffices to find a hyperedge with each vertex coming from one of the common neighbourhoods

$$I^{12} = N(P^{14}, P^{24}), \quad I^{13} = N(P^{14}, P^{34}), \quad \text{and} \quad I^{23} = N(P^{24}, P^{34}),$$

since this would give rise to a choice of P^{12}, P^{13}, and P^{23} with the desired properties.

Suppose to the contrary that $\mathcal{A}^{123}[I^{12}, I^{13}, I^{23}]$ is independent, i.e., it gives rise to a (p, q, r)-hole in $\mathcal{A}^{123}$, where q and r are the largest integers such that

$$|I^{13}| \geqslant q \cdot \delta|\mathcal{P}^{13}| \qquad \text{and} \qquad |I^{23}| \geqslant r \cdot \delta|\mathcal{P}^{23}|$$

(for the definition of p see (13)). Owing to (15) and the maximality of q and r we have

$$(q + 1)\delta + (r + 1)\delta > 1 + 2\varepsilon .$$

Since $q\delta$ and $r\delta$ are bounded by 1 we also have

$$q\delta > 2(\varepsilon - \delta) \qquad \text{and} \qquad r\delta > 2(\varepsilon - \delta) . \tag{17}$$

Without loss of generality we may assume $q \geqslant r$ and since $\delta < \varepsilon$ we then have

$$q\delta > 1/2 . \tag{18}$$

Since the sets I^{12}, I^{13}, and I^{23} form a (p, q, r)-hole in $\mathcal{A}^{123}$, we have $(p, q, r) \in \Xi_3$ and owing to (11) we know that (p, q, r) is also in Ξ_4, i.e., $\mathcal{A}^{124}$ also contains a (p, q, r)-hole. This gives rise to an independent set in $\mathcal{A}^{124}$ formed by $J^{12} \subseteq \mathcal{P}^{12}$, $J^{14} \subseteq \mathcal{P}^{14}$, and $J^{24} \subseteq \mathcal{P}^{24}$ with

$$|J^{12}| \geqslant p \cdot \delta|\mathcal{P}^{12}|, \qquad |J^{14}| \geqslant q \cdot \delta|\mathcal{P}^{14}|, \qquad \text{and} \qquad |J^{24}| \geqslant r \cdot \delta|\mathcal{P}^{24}| .$$

We will use the fact that the chosen pair (P^{14}, P^{24}) maximises the codegree in $\mathcal{A}^{124}$ over all pairs in $(\mathcal{P}^{14} \backslash X_3^{14}) \times (\mathcal{P}^{24} \backslash X_3^{24})$. Owing to the maximal choice of p in (13) we have

$$\operatorname{codeg}(P^{14}, P^{24}) < (p + 1) \cdot \delta|\mathcal{P}^{12}| .$$

We consider the set

$$J_0^{24} = J^{24} \backslash (X_3^{24} \cup X_1^{24}) .$$

We will arrive at a contradiction by considering a vertex from J_0^{24}. It follows from (17) and (12) that

$$|J_0^{24}| \geqslant |J^{24}| - |X_3^{24}| - |X_1^{24}| \geqslant 2(\varepsilon - 2\delta)|\mathcal{P}^{24}| > 0\,.$$

Therefore, there exists at least a vertex $Q^{24} \in J_0^{24}$, which we fix for the rest of the proof. We estimate the degree $\deg(Q^{24})$ of Q^{24} in $\mathcal{A}^{124}$ in two ways. Since $Q^{24} \notin X_1^{24}$ we have

$$\deg(Q^{24}) \geqslant (1/2 + \varepsilon)|\mathcal{P}^{12}||\mathcal{P}^{14}|\,. \tag{19}$$

On the other hand, we write $\deg(Q^{24})$ as the sum of all codegrees of Q^{24} with a vertex from $\mathcal{P}^{14}$, i.e.,

$$\deg(Q^{24}) = \sum_{Q \in \mathcal{P}^{14}} \operatorname{codeg}(Q, Q^{24})$$

and consider three cases depending on Q. If $Q \in J^{14}$, then all common neighbours of Q^{24} and Q must lie outside J^{12}, as J^{12}, J^{14}, and J^{24} form a hole in $\mathcal{A}^{124}$. In particular, in this case we have

$$\operatorname{codeg}(Q, Q^{24}) \leqslant |\mathcal{P}^{12}| - |J^{12}| \leqslant (1 - p\delta)|\mathcal{P}^{12}|\,.$$

For the second case, we consider $Q \in X_3^{14}$ in this case we use the trivial upper bound

$$\operatorname{codeg}(Q, Q^{24}) \leqslant |\mathcal{P}^{12}|\,.$$

However, due to (12) we know that this will only contribute little to $\deg(Q^{24})$.

In the remaining case we have $Q \notin J^{14}$ and $Q \notin X_3^{14}$. Then we have $(Q, Q^{24}) \in (\mathcal{P}^{14} \backslash X_3^{14}) \times (\mathcal{P}^{24} \backslash X_3^{24})$ and by the maximal choice of (P^{14}, P^{24}) we infer

$$\operatorname{codeg}(Q, Q^{24}) \leqslant \operatorname{codeg}(P^{14}, P^{24}) < (p+1)\delta|\mathcal{P}^{12}|\,.$$

Putting the three cases together and we obtain

$$\deg(Q^{24}) \leqslant |J^{14}| \cdot (1 - p\delta)|\mathcal{P}^{12}| + |X_3^{14}| \cdot |\mathcal{P}^{12}| + (|\mathcal{P}^{14}| - |J^{14}|) \cdot (p+1)\delta|\mathcal{P}^{12}|$$

Let x, $y \in \mathbb{R}$ be given by $x = p\delta$ and $y = |J^{14}|/|\mathcal{P}^{14}|$. Recalling (14) and (18) we note that x, $y \geqslant 1/2$ and we can rewrite the last inequality as

$$\frac{\deg(Q^{24})}{|\mathcal{P}^{12}||\mathcal{P}^{14}|} \leqslant y(1-x) + \delta + (1-y)x + \delta\,.$$

Comparing this with (19) we arrive at

$$\frac{1}{2} + \varepsilon - 2\delta \leqslant y(1-x) + (1-y)x\,,$$

which due to x, $y \geqslant 1/2$ leads to the contradiction

$$0 \leqslant \tfrac{1}{2}(2x-1)(2y-1) \leqslant 2\delta - \varepsilon < 0$$

and concludes the proof of Lemma 3.1. □

§4. Concluding remarks

Continuing the discussion from the introduction we mention related concepts of quasirandom hypergraphs. In fact, for 3-uniform hypergraphs one can define for any antichain $\mathscr{A} \neq \{\{1,2,3\}\}$ from the power set of $\{1,2,3\}$ a notion of $\mathscr{A}$-quasirandom hypergraphs (see, e.g., [3, 23]) and these concepts differ for non-isomorphic antichains. Having this in mind, three more concepts of quasirandom hypergraphs arise, in addition to the four notions defined in Section 1. In view of our earlier notation, we depict these three new concepts by $\overset{\bullet}{\circ\circ}$, $\overset{\circ}{\bullet\bullet}$, and $\overset{\circ}{\bullet\!\!-\!\!\bullet}$.

Definition 4.1. A 3-uniform hypergraph $H = (V, E)$ on $n = |V|$ vertices is

(*i*) $(d, \eta, \overset{\bullet}{\circ\circ})$-*quasirandom* if for any subset $X \subseteq V$ the number $e_{\overset{\bullet}{\circ\circ}}(X)$ of triples $(x, v, v') \in X \times V \times V$ with $\{x, v, v'\} \in E$ satisfies
$$\big|e_{\overset{\bullet}{\circ\circ}}(X) - d\,|X|\,n^2\big| \leqslant \eta\, n^3\,.$$

(*ii*) $(d, \eta, \overset{\circ}{\bullet\bullet})$-*quasirandom* if for any subsets $Y, Z \subseteq V$ the number $e_{\overset{\circ}{\bullet\bullet}}(Y, Z)$ of triples $(v, y, z) \in V \times Y \times Z$ with $\{v, y, z\} \in E$ satisfies
$$\big|e_{\overset{\circ}{\bullet\bullet}}(Y, Z) - d\,|Y|\,|Z|\,n\big| \leqslant \eta\, n^3\,.$$

(*iii*) $(d, \eta, \overset{\circ}{\bullet\!\!-\!\!\bullet})$-*quasirandom* if for any subset $P \subseteq V \times V$ the number $e_{\overset{\circ}{\bullet\!\!-\!\!\bullet}}(P)$ of triples $(v, y, z) \in V \times P$ with $\{v, y, z\} \in E$ satisfies
$$\big|e_{\overset{\circ}{\bullet\!\!-\!\!\bullet}}(P) - d\,|P|\,n\big| \leqslant \eta\, n^3\,.$$

With these definitions at hand we may extend the notions $\mathcal{Q}(d, \eta, \star)$ and $\pi_\star$ to any symbol $\star \in \{\overset{\bullet}{\circ\circ}, \overset{\circ}{\bullet\bullet}, \overset{\circ}{\bullet\!\!-\!\!\bullet}\}$ (see Definition 1.5 and the paragraph before). It follows directly from the definitions that for any $d \in [0,1]$ and $\eta > 0$ we have
$$\mathcal{Q}(d,\eta,\overset{\bullet}{\bullet\!\!-\!\!\bullet}) \subseteq \mathcal{Q}(d,\eta,\overset{\circ}{\bullet\!\!-\!\!\bullet}) \subseteq \mathcal{Q}(d,\eta,\overset{\circ}{\bullet\bullet}) \subseteq \mathcal{Q}(d,\eta,\overset{\bullet}{\circ\circ}) \quad\text{and}\quad \mathcal{Q}(d,\eta,\overset{\bullet}{\bullet\bullet}) \subseteq \mathcal{Q}(d,\eta,\overset{\circ}{\bullet\bullet})\,. \tag{20}$$
However, there exist examples of hypergraphs that show that $\mathcal{Q}(d, \eta, \overset{\bullet}{\bullet\bullet})$ and $\mathcal{Q}(d, \eta, \overset{\circ}{\bullet\!\!-\!\!\bullet})$ incomparable in general. Consequently, we can extend (6) for every hypergraph F to
$$\pi(F) \geqslant \pi_{\overset{\bullet}{\circ\circ}}(F) \geqslant \pi_{\overset{\circ}{\bullet\bullet}}(F) \geqslant \pi_{\overset{\circ}{\bullet\!\!-\!\!\bullet}}(F) \geqslant \pi_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F) \geqslant \pi_{\wedge}(F) \geqslant \pi_{\triangle}(F) = 0$$
and
$$\pi_{\overset{\circ}{\bullet\bullet}}(F) \geqslant \pi_{\overset{\bullet}{\bullet\bullet}}(F) \geqslant \pi_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F)\,.$$

Note that in the hierachy given in (20) the weakest concept is $\therefore$-quasirandomness. It follows from its definition, that any 3-uniform $(d, \eta, \therefore)$-quasirandom hypergraph $H = (V, E)$ on n vertices has the property, that all but at most $2\sqrt{\eta}n$ vertices have its degree in the interval $d\,n^2/2 \pm \sqrt{\eta}\,n^2$. In fact, $\therefore$-quasirandom hypergraphs are the class of hypergraphs with approximately regular degree sequence. Owing to this, it is not hard to show that

$$\pi_{\therefore}(F) = \pi(F)$$

for any hypergraph F.

We conclude our discussion with an overview of the Turán densities of the tetrahedron. For that the well known Turán conjecture asserts

$$\pi(K_3^{(4)}) = \tfrac{5}{9}$$

and Theorem 1.7 and a result from [17] yield

$$\pi_{\therefore}(K_3^{(4)}) = \tfrac{1}{2} \qquad \text{and} \qquad \pi_{\wedge}(K_3^{(4)}) = 0\,.$$

Moreover, Example 1.6 implies

$$\pi_{\therefore}(K_3^{(4)}) \geqslant \tfrac{1}{2} \qquad \text{and} \qquad \pi_{\therefore}(K_3^{(4)}) \geqslant \tfrac{1}{2}$$

and it is tempting to conjecture for both cases that a matching upper bound holds. Maybe an interesting first step in that direction is to combine both incomparable assumptions given by $\therefore$- and $\therefore$-quasirandomness.

Question 4.1. Is it true that for every $\varepsilon > 0$ there exist $\eta > 0$ such that every sufficiently large hypergraph

$$H \in \mathcal{Q}(1/2 + \varepsilon, \eta, \therefore) \cap \mathcal{Q}(1/2 + \varepsilon, \eta, \therefore)$$

contains a tetrahedron?

In view of (5) and (20) a positive resolution of this question would strengthen our main result Theorem 1.7.

We close with the remark that concerning $\therefore$-quasirandomness to our knowledge it is only known that the validity of Turán's conjecture and Examaple 1.6 imply that $\pi_{\therefore}(K_4^{(3)})$ is in the interval $[1/2, 5/9]$.

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