CANONICAL TREE-DECOMPOSITIONS OF A GRAPH THAT DISPLAY ITS k-BLOCKS

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Abstract. A *k-block* in a graph G is a maximal set of at least k vertices no two of which can be separated in G by removing less than k vertices. It is *separable* if there exists a tree-decomposition of adhesion less than k of G in which this k-block appears as a part.

Carmesin, Diestel, Hamann, Hundertmark and Stein proved that every finite graph has a canonical tree-decomposition of adhesion less than k that distinguishes all its k-blocks and tangles of order k. We construct such tree-decompositions with the additional property that every separable k-block is equal to the unique part in which it is contained. This proves a conjecture of Diestel.

1. Introduction

Tangles in a graph G are orientations of the low order separations that consistently point towards some 'highly connected piece' of G. As a fundamental tool for their graph minors project, Robertson and Seymour [8] proved that every finite graph has a tree-decomposition that distinguishes every two maximal tangles.

More recently, *k-profiles* were introduced as a common generalisation of k-tangles and k-blocks [7]. Here, a *k-block* in a graph G is a maximal set of at least k vertices no two of which can be separated in G by removing less than k vertices. Carmesin, Diestel, Hamann and Hundertmark showed that every graph has a canonical tree-decomposition of adhesion less than k that distinguishes all its k-profiles [2].

In [3], these authors asked how one could improve the above tree-decompositions further so that they also display the structure of the k-blocks: it would be nice if we could compress any part containing a k-block so that it does not contain any 'junk'.

In this paper, we prove that this is possible simultaneously for all k-blocks that can be isolated at all in a tree-decomposition, canonical or not. More precisely, we call a k-block *separable* if it appears as a part in some tree-decomposition of adhesion less than k of G. We prove the following, which was conjectured by Diestel [5] (see also [3]).

Theorem 1. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ of adhesion less than k that distinguishes efficiently every two distinct k-profiles, and which has the further property that every separable k-block is equal to the unique part of $\mathcal{T}$ in which it is contained.*

We also prove the following related result:

Theorem 2. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ that distinguishes efficiently every two distinct maximal robust profiles, and which has the further property that every separable block inducing a maximal robust profile is equal to the unique part of $\mathcal{T}$ in which it is contained.*

See Section 2 for a definition of robust and [4] for an example showing that Theorem 2 fails if we leave out 'robust'. Theorem 2 without its description of the separable blocks is a result of Hundertmark and Lemanczyk [7], which implies the aforementioned theorem of Robertson and Seymour. In Section 4, we give an example showing that it is impossible to ensure that non-maximal robust separable blocks are also displayed by a tree-decomposition which distinguishes all the maximal robust profiles efficiently.

After recalling some preliminaries in Section 2, we develop the necessary tools in Section 3. Then we prove our main result in Section 4.

2. Preliminaries

Unless otherwise mentioned, G will always denote a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. Any graph-theoretic term and notation not defined here are explained in [6].

A vertex is called *central* in G if the greatest distance to any other vertex is minimal. It is well known that a finite tree T has either a unique central vertex or precisely two central adjacent vertices v and w. In the second case vw is called a *central edge*. For a vertex or edge to be central is obviously a property invariant under automorphisms of G.

Let us recall some notations from [2].

2.1. Separations.

An ordered pair (A, B) of subsets of $V(G)$ is a *separation* of G if $A \cup B = V(G)$ and if there is no edge $e = vw \in E(G)$ with $v \in A \setminus B$ and $w \in B \setminus A$. The cardinality $|A \cap B|$ of the *separator* $A \cap B$ of a separation (A, B) is the *order* of (A, B) and a separation of order k is a *k-separation*.

A separation (A, B) is *proper* if neither $A \subseteq B$ nor $B \subseteq A$. Otherwise (A, B) is *improper*. A separation (A, B) is *tight* if every vertex in $A \cap B$ has a neighbour in $A \setminus B$ and a neighbour in $B \setminus A$.

The set of separations of G is partially ordered via

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \ \wedge \ D \subseteq B.$$

For no two proper separations (A, B) and (C, D), the separation (A, B) is $\leq$-comparable with (C, D) and (D, C). In particular we obtain that (A, B) and (B, A) are not $\leq$-comparable.

A separation (A, B) is *nested* with a separation (C, D) if (A, B) is $\leq$-comparable with either (C, D) or (D, C). Since

$$(A, B) \leq (C, D) \iff (D, C) \leq (B, A),$$

being nested is symmetric and reflexive. Separations that are not nested are called *crossing*.

A separation (A, B) is *nested* with a set S of separations if (A, B) is nested with every $(C, D) \in S$. A set S of separations is *nested* with a set S' of separations if every $(A, B) \in S$ is nested with S' or equivalently every $(C, D) \in S'$ is nested with S.

A set N of separations is *nested* if its elements are pairwise nested. A set S of separations is *symmetric* if for every $(A, B) \in S$ it also contains its *inverse* separation (B, A). A symmetric set S of separations is also called a *separation system* or a *system of separations*, and if all its separations are proper, S is called a *proper separation system*. For a set S of separations the separation system *generated by* S is the separation system consisting of the separations in S and their inverses. A set S of separations is *canonical* if it is invariant under the automorphisms of G, i.e. for every $(A, B) \in S$ and for every $\varphi \in \mathrm{Aut}(G)$ we obtain $(\varphi[A], \varphi[B]) \in S$.

A separation (A, B) *separates* a vertex set $X \subseteq V(G)$ if X meets both $A \backslash B$ and $B \backslash A$. Given a set S of separations a vertex set $X \subseteq V(G)$ is *S-inseparable* if no separation $(A, B) \in S$ separates X. A maximal S-inseparable vertex set is an *S-block* of G.

For $k \in \mathbb{N}$ let $S_{<k}$ denote the set of separations of order less than k of G. The *$(< k)$-inseparable* sets are the $S_{<k}$-inseparable sets. So the *k-blocks* are exactly the $S_{<k}$-blocks of size at least k.

For two separations (A, B) and (C, D) not equal to $(V(G), V(G))$ consider a *cross-diagram* as in Figure 1. Every pair $(X, Y) \in \{A, B\} \times \{C, D\}$ denotes a *corner* of this cross-diagram, which we also denote by $\mathrm{cor}(X, Y)$. Let $\overline{X} \in \{A, B\} \backslash \{X\}$ and $\overline{Y} \in \{C, D\} \backslash \{Y\}$. In the diagram we consider the *center* $c := A \cap B \cap C \cap D$ and for a corner $\mathrm{cor}(X, Y)$ as above the *interior* $\mathrm{int}(X, Y) := (X \cap Y) \backslash (\overline{X} \cup \overline{Y})$ and the *links* $\ell_X := (X \cap Y \cap \overline{Y}) \backslash c$ and $\ell_Y := (Y \cap X \cap \overline{X}) \backslash c$. The vertex set $X \cap Y$ is the disjoint union of $\mathrm{int}(X, Y)$ with ℓ_X, ℓ_Y and c and thus can be associated with the corner $\mathrm{cor}(X, Y)$.

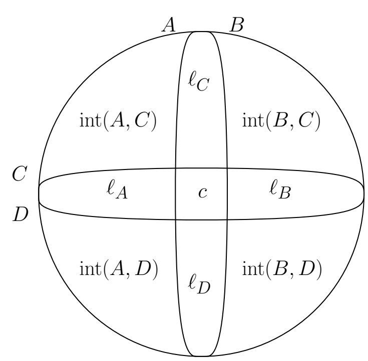


FIGURE 1. cross-diagram for (A, B) and (C, D)

Remark 2.1. Two separations (A, B) and (C, D) are nested, if and only if for one of their corners $\mathrm{cor}(X, Y)$ the interior $\mathrm{int}(X, Y)$ and its links ℓ_X and ℓ_Y are empty. □

For a corner $\mathrm{cor}(X,Y)$ there is a *corner separation* $(X\cap Y, \overline{X}\cup\overline{Y})$, which is again a separation of G.

Lemma 2.2. [4, Lemma 2.2] *For two crossing separations (A,B) and (C,D) any of its corner separation is nested with every separation that is nested with both (A,B) and (C,D).*

In particular a corner separation is nested with (A,B), (C,D) and all corner separations. A double counting argument yields:

Remark 2.3. For any two separations (A,B) and (C,D), the orders of the separations $(A\cap C, B\cup D)$ and $(B\cap D, A\cup C)$ sum to $|A\cap B|+|C\cap D|$. □

2.2. **Tree-decompositions.** Recall that a *tree-decomposition* $\mathcal{T}$ of G is a pair $\big(T,(P_t)_{t\in V(T)}\big)$ of a tree T and a family of vertex sets $P_t\subseteq V(G)$ for every node $t\in V(T)$, such that

- (T1) $V(G)=\bigcup_{t\in V(T)} P_t$;
- (T2) for every edge $e\in E(G)$ there is a node $t\in V(T)$ such that both end vertices of e lie in P_t;
- (T3) whenever t_2 lies on the t_1-t_3 path in T we obtain $P_{t_1}\cap P_{t_3}\subseteq P_{t_2}$.

The sets P_t are the *parts* of $\mathcal{T}$. For an edge $tt'\in E(T)$ the intersection $P_t\cap P_{t'}$ is the corresponding *adhesion set* and the maximum size of an adhesion set of $\mathcal{T}$ is the *adhesion* of $\mathcal{T}$. A node $t\in V(T)$ is a *hub node* if the corresponding part P_t is a subset of $P_{t'}$ for some neighbour t' of t. If t is a hub node, then P_t is a *hub*. A tree-decomposition $\mathcal{T}=\big(T,(P_t)_{t\in V(T)}\big)$ of G and a tree-decomposition $\mathcal{T}'=\big(T',(P'_t)_{t\in V(T')}\big)$ of G' are *isomorphic* if there is an isomorphism $\varphi: G\to G'$ and an isomorphism $\psi: T\to T'$ such that for every part P_t of $\mathcal{T}$ we obtain $\varphi[P_t]=P'_{\psi(t)}$. We say φ *induces* an isomorphism between $\mathcal{T}$ and $\mathcal{T}'$. A tree-decomposition $\mathcal{T}$ is *canonical* if it is invariant under the automorphisms of G, i.e. every automorphism of G induces an automorphism of $\mathcal{T}$.

Let $\big(T,(P_t)_{t\in V(T)}\big)$ be a tree-decomposition of G. For $t\in V(T)$ the *torso* H_t is the graph obtained from $G[P_t]$ by adding all edges joining two vertices in a common adhesion set $P_t\cap P_u$ for any $tu\in E(T)$. A separation (A,B) of $G[P_t]$ is a separation of H_t if and only if it does not separate any adhesion set $P_t\cap P_{t'}$ for $tt'\in E(T)$. A separation (A,B) of G with $A\cap B\subseteq P_t$ for some node $t\in V(T)$ that does not separate any adhesion set $P_t\cap P_{t'}$ for $tt'\in E(T)$ *induces* the separation $(A\cap P_t, B\cap P_t)$ of H_t.

Every oriented edge $\vec{e}=t_1t_2$ of T divides $T-e$ in two components T_1 and T_2 with $t_1\in V(T_1)$ and $t_2\in V(T_2)$. By [6, Lemma 12.3.1] $\vec{e}$ *induces* the separation $\big(\bigcup_{t\in V(T_1)} P_t, \bigcup_{t\in V(T_2)} P_t\big)$ of G such that the separator coincides with the adhesion set $P_{t_1}\cap P_{t_2}$. We say a separation is *induced* by $\mathcal{T}$ if it is induced by an oriented edge of T.

The set of separations induced by a tree-decomposition $\mathcal{T}$ (of adhesion less than k) is a nested system $N(\mathcal{T})$ of separations (of order less than k). We say $N(\mathcal{T})$ is *induced* by $\mathcal{T}$. Clearly if $\mathcal{T}$ is canonical, then so is $N(\mathcal{T})$.

Conversely, as proven in [4], every nested separation system N *induces* a tree-decomposition $\mathcal{T}(N)$:

Theorem 2.4. [4, Theorem 4.8] *Let N be a canonical nested separation system of G. Then there is a canonical*[1] *tree-decomposition $\mathcal{T}(N)$ of G such that*

(i) *every N-block of G is a part of $\mathcal{T}(N)$;*
(ii) *every part of $\mathcal{T}(N)$ is either an N-block of G or a hub;*
(iii) *the separations of G induced by $\mathcal{T}(N)$ are precisely those in N;*
(iv) *every separation in N is induced by a unique oriented edge of $\mathcal{T}(N)$.*

2.3. **Profiles.** Let S be a separation system. A subset $O \subseteq S$ is an *orientation* of S if for every $(A,B) \in S$ exactly one of (A,B) and (B,A) is an element of O. An orientation O of S is *consistent* if for every (A,B), $(C,D) \in S$ with $(A,B) \in O$ and $(C,D) \leq (A,B)$ we obtain $(C,D) \in O$ as well. A consistent orientation P of $S_{<k}$ is called a *k-profile* if it satisfies

(P) for all (A,B), $(C,D) \in P$ we have $(B \cap D, A \cup C) \notin P$.

In particular if the order $|(A \cup C) \cap (B \cap D)|$ of this corner separation is less than k, we have $(A \cup C, B \cap D) \in P$. Sometimes we omit the k and call P a *profile*.

It is easy to check that every k-block b *induces* a k-profile via

$$P_k(b) := \{(A,B) \in S_{<k} \mid b \subseteq B\}.$$

Also *tangles* of order k (or *k-tangles*), as introduced by Robertson and Seymour [8], are k-profiles. For more background on profiles, see [7].

For $r \in \mathbb{N}$, a k-profile P is *r-robust* if for any $(A,B) \in P$ and any $(C,D) \in S_{<r+1}$ one of $(A \cup C, B \cap D)$, $(A \cup D, B \cap C)$ either has order at least $k-1$, or is in P. If P is r-robust for all $r \in \mathbb{N}$, then we call P *robust.*

A robust k-profile P is *maximal* if there does not exist a robust ℓ-profile Q with $P \subsetneq Q$ and $\ell > k$. Then P is just called a *maximal robust profile.*

Remark 2.5. (i) Every k-profile is ℓ-robust for all $\ell < k$;
(ii) if a k-block b contains a complete graph on k vertices, then the induced k-profile $P_k(b)$ is robust. □

The next lemma basically states that every k-profile induces a k-haven, as introduced by Seymour and Thomas [9].

Lemma 2.6. *Let $X \subseteq V(G)$ with $|X| < k$ and let Q be a k-profile. Then there exists a component C of $G - X$ such that $(V(G) \setminus C, C \cup X) \in Q$.*

Furthermore, $(V(G) \setminus C, C \cup N(C)) \in Q$ as well.

Proof. Let $C_1, \ldots, C_n$ denote the components of $G - X$ and for $i \in \{1, \ldots, n\}$ let $(A_i, B_i) := (V(G) \setminus C_i, C_i \cup X)$. To reach a contradiction suppose that $(B_i, A_i) \in Q$ for all $i \in \{1, \ldots, n\}$. Then (P) yields inductively for all $m \leq n$

[1] In the original paper this theorem is stated without the canonicity since it holds in a greater generality. But it is clear from the proof that if N is canonical, then so is $\mathcal{T}(N)$.

that $\big(\bigcup_{i\le m} B_i, \bigcap_{i\le m} A_i\big) \in Q$, since their separators all equal X. Hence for $m = n$, we obtain $(V(G), X) \in Q$, contradicting the consistency of Q with $(X, V(G)) \le (V(G), X)$. Thus there is a component C of $G - X$ such that $(A, B){:=}(V(G)\backslash C, C \cup X) \in Q$.

Now suppose $(C \cup N(C), V(G)\backslash C) \in Q$. Then (P) with (A, B) yields that $\big((V(G)\backslash C) \cup C \cup N(C), (C \cup X) \cap (V(G)\backslash C)\big) = (V(G), X) \in Q$, contradicting the consistency of Q again. □

A k-profile Q *inhabits* a part P_t of a tree-decomposition $\big(T, (P_t)_{t\in V(T)}\big)$ if for every $(A, B) \in Q$ we obtain that $(B\backslash A) \cap P_t$ is not empty. Note that if for a node t every separation induced by an oriented edge ut of T has order less than k, then Q inhabits P_t if and only if all those separations are in Q.

Corollary 2.7. *Let $\big(T, (P_t)_{t\in V(T)}\big)$ be a tree-decomposition and let Q be a k-profile. If Q inhabits a part P_t, then $|P_t| \ge k$.*

Proof. Our aim is to show that if $|P_t| < k$, then any k-profile Q does not inhabit P_t. By Lemma 2.6 there is a component C of $G - P_t$ such that $(V(G)\backslash C, C \cup P_t) \in Q$. Since $(C \cup P_t)\backslash(V(G)\backslash C) = C$ and since $C \cap P_t$ is empty, we obtain that Q does not inhabit P_t. □

A set $\mathcal{P}$ of profiles is *canonical* if for every $P \in \mathcal{P}$ and every automorphism φ of G the profile $\big\{\big(\varphi[A], \varphi[B]\big) \mid (A, B) \in P\big\}$ is also in $\mathcal{P}$.

Two profiles P and Q are *distinguishable* if there is a separation (A, B) with $(A, B) \in P$ and $(B, A) \in Q$. Such a separation *distinguishes* P and Q. It is said to distinguish P and Q *efficiently* if its order $|A \cap B|$ is minimal among all separations distinguishing P and Q. A set $\mathcal{P}$ of profiles is *distinguishable* if every two distinct $P, Q \in \mathcal{P}$ are distinguishable. A tree-decomposition $\mathcal{T}$ *distinguishes* two profiles P and Q (efficiently) if some (A, B) induced by $\mathcal{T}$ distinguishes them (efficiently).

For our main result of this paper, we will build on the following theorem.

Theorem 2.8. [7, Theorem 2.6][2] *Every graph G has a canonical tree-decomposition of adhesion less than k that distinguishes every two distinguishable $(k-1)$-robust ℓ-profiles of G for some values $\ell \le k$ efficiently.*

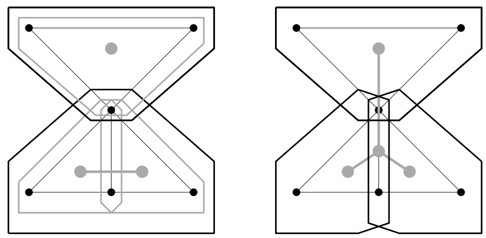
Moreover, every separation induced by the tree-decomposition distinguishes some of those profiles efficiently.

3. Construction methods

3.1. **Sticking tree-decompositions together.** Given a tree-decomposition $\mathcal{T}$ of G and for each torso H_t a tree-decomposition $\mathcal{T}^t$, our aim is to construct a new tree-decomposition $\overline{\mathcal{T}}$ of G by gluing together the tree-decompositions $\mathcal{T}^t$ of the torsos along $\mathcal{T}$ in a canonical way.

[2]Since [7] is unpublished, see also [4, Theorem 6.3] for a version just concerning robust blocks or [1, Theorem 9.2] for a version also dealing with infinite graphs.

Example 3.1. First we shall give the construction of $\overline{\mathcal{T}}$ for a particular example: G is obtained from three edge-disjoint triangles intersecting in a single vertex by identifying two other vertices of distinct triangles. The tree-decomposition $\mathcal{T}$ of G and the tree-decompositions of the torsos are depicted in Figure 2 (a). In order to stick the tree-decompositions of the torsos together in a canonical way, we first have to refine them, see Figure 2 (b).



(a) (b)

FIGURE 2. (a) shows the tree-decomposition $\mathcal{T}$ of G, drawn in black, and the tree-decompositions of the torsos, drawn in grey. (b) shows the canonically glued tree-decomposition $\overline{\mathcal{T}}$.

Before we can construct $\overline{\mathcal{T}}$, we need some preparation.

Construction 3.2. Given a tree-decomposition $\mathcal{T} = \big(T, (P_t)_{t\in V(T)}\big)$ of G, we construct a new tree-decomposition $\widetilde{\mathcal{T}} = \big(\widetilde{T}, (P_t)_{t\in V(\widetilde{T})}\big)$ of G by contracting every edge tu of T where $P_t = P_u$.[3] In this tree-decomposition two adjacent nodes never have the same part. Let $F \subseteq E(\widetilde{T})$ be the set of edges tu where neither $P_t \subseteq P_u$ nor $P_u \subseteq P_t$. By subdividing every edge $tu \in F$ and assigning to the subdivided node x the part $P_x{:=}P_t \cap P_u$, we obtain a new tree-decomposition $\widehat{\mathcal{T}} = \big(\widehat{T}, (P_t)_{t\in V(\widehat{T})}\big)$.

Remark 3.3. $\widehat{\mathcal{T}}$ defined as in Construction 3.2 satisfies the following:

(i) every separation induced by $\widehat{\mathcal{T}}$ is also induced by $\mathcal{T}$;
(ii) for every edge $tu \in E(T)$ that induces a separation not induced by $\widehat{\mathcal{T}}$ we have $P_t = P_u$;
(iii) for every edge $tu \in E(\widehat{T})$ precisely one of P_t or P_u is a proper subset of the other;
(iv) if $\mathcal{T}$ distinguishes two profiles Q_1 and Q_2 efficiently, then so does $\widehat{\mathcal{T}}$;
(v) if $\mathcal{T}$ is canonical, then $\widehat{\mathcal{T}}$ is canonical as well. □

Lemma 3.4. *Let K be a complete subgraph of G and $\widehat{\mathcal{T}}$ as in Construction 3.2. Then there is a node t of $\widehat{T}$ with $V(K) \subseteq P_t$ such that P_t is fixed by every automorphism of G fixing K.*

[3]Here we understand the nodes of $\widehat{T}$ to be nodes of T, where a node obtained through the contraction of an edge tu to be identified with either t or u.

Proof. As K is complete, there is a node $u \in V(\widehat{T})$ with $V(K) \subseteq P_u$.

Let W be the subforest of nodes w with $K \subseteq P_w$, which is connected as $\widehat{\mathcal{T}}$ is a tree-decomposition. Now W either has a central vertex t or a central edge tu such that P_u is a proper subset of P_t (cf Remark 3.3 (iii)). In both cases P_t is fixed by the automorphisms of G that fix K. □

Construction 3.5. Let $\mathcal{T} = \big(T, (P_t)_{t\in V(T)}\big)$ be a tree-decomposition of G. For each $t \in V(T)$ let $\mathcal{T}^t = \big(T^t, (P^t_u)_{u\in V(T_t)}\big)$ be a tree-decomposition of the torso H_t. For each $\mathcal{T}^t$ let $\widehat{\mathcal{T}}^t$ be as in Construction 3.2. For $e = tu \in E(T)$ let A_e denote the adhesion set $P_t \cap P_u$. Since $H_t[A_e]$ is complete, we can apply Lemma 3.4: there is a node $\gamma(t,u)$ of $\widehat{T}^t$ with $A_e \subseteq P^t_{\gamma(t,u)}$ such that $P^t_{\gamma(t,u)}$ is fixed by every automorphism of H_t fixing K.

We obtain a tree $\overline{T}$ from the disjoint union of the trees $\widehat{T}^t$ for all $t \in V(T)$ by adding the edges $\gamma(t,u)\gamma(u,t)$ for each $tu \in E(T)$. Let $\overline{P}_u$ be P^t_u for the unique $t \in V(T)$ with $u \in V(\widehat{T}^t)$. Then $\overline{\mathcal{T}}{:=}\big(\overline{T}, (\overline{P}_t)_{t\in V(\overline{T})}\big)$ is a tree-decomposition of G.

Two torsos H_t and H_u of $\mathcal{T}$ are *similar*, if there is an automorphism of G that induces an isomorphism between H_t and H_u. The family $\big(\mathcal{T}^t\big)_{t\in V(T)}$ is *canonical* if all the $\mathcal{T}^t$ are canonical and for any two similar torsos H_t and H_u of $\mathcal{T}$ every automorphism of G that witnesses the similarity of H_t and H_u induces an isomorphism between $\mathcal{T}^t$ and $\mathcal{T}^u$.

Lemma 3.6. *The tree-decomposition $\overline{\mathcal{T}}$ as in Construction 3.5 satisfies the following:*

(i) *for $t \in V(T)$ every node $u \in V(T^t)$ is also a node of $\overline{T}$ and $\overline{P}_u = P^t_u$;*
(ii) *every node $u \in V(\overline{T})$ that is not a node of any T^t is a hub node;*
(iii) *every separation (A,B) induced by $\overline{\mathcal{T}}$ is either induced by $\mathcal{T}$ or there is a node $t \in V(T)$ such that $(A \cap P_t, B \cap P_t)$ is induced by $\mathcal{T}^t$;*
(iv) *every separation induced by $\mathcal{T}$ is also induced by $\overline{\mathcal{T}}$;*
(v) *for every separation (C,D) induced by $\widehat{\mathcal{T}}^t$ there is a separation (A,B) induced by $\overline{\mathcal{T}}$ such that $A \cap B \subseteq P_t$ and $(A \cap P_t, B \cap P_t) = (C,D)$;*
(vi) *if $\mathcal{T}$ and the family of the $\mathcal{T}^t$ are canonical, then $\overline{\mathcal{T}}$ is canonical.*

Proof. Whilst (i) is true by construction, the nodes added in the construction of $\widehat{\mathcal{T}}^t$ are hub nodes by definition, yielding (ii). Furthermore, (iii), (iv) and (v) follow by construction with Remark 3.3 (i) and the observation that for all $tu \in E(T)$ the adhesion sets $\overline{P}_{\gamma(t,u)} \cap \overline{P}_{\gamma(u,t)}$ and $P_t \cap P_u$ are equal. Finally, (iv) follows with Remark 3.3 (v) and Lemma 3.4 from the construction. □

3.2. **Obtaining tree-decompositions from almost nested sets of separations.** Theorem 2.4 gives a way how to transform a nested set of separations into a tree-decomposition. In this subsection, we extend this to sets of ‘almost nested’ separations.

For a separation (A,B) of G and $X \subseteq V(G)$, the pair $(A \cap X, B \cap X)$ is a separation of $G[X]$, which we call the *restriction* $(A,B)\restriction X$ of (A,B) to X. Note that $(A,B)\restriction X$ is proper if and only if (A,B) separates X. The *restriction* $S\restriction X$ to X of a set S of separations of G to X consists of the proper separations $(A,B)\restriction X$ with $(A,B) \in S$.

For a set S of separations of G let $\min_{\mathrm{ord}}(S)$ denote the set of those separations in S with minimal order. Note that if S is non-empty, then so is $\min_{\mathrm{ord}}(S)$, and that $\min_{\mathrm{ord}}$ commutes with graph isomorphisms.

A finite sequence $(\beta_0, \ldots, \beta_n)$ of vertex sets of G is called an *S-focusing sequence* if

(F1) $\beta_0 = V(G)$;
(F2) for all $i < n$, the separation system N_{β_i} generated by $\min_{\mathrm{ord}}(S\restriction\beta_i)$ is non-empty and is nested with the set $S\restriction\beta_i$;
(F3) β_{i+1} is an N_{β_i}-block of $G[\beta_i]$.

An S-focusing sequence $(\beta_0, \ldots, \beta_n)$ is *good* if

(F∗) the separation system N_{β_n} generated by $\min_{\mathrm{ord}}(S\restriction\beta_n)$ is nested with the set $S\restriction\beta_n$.

Note that for an S-focusing sequence $(\beta_0, \ldots, \beta_n)$ we obtain $\beta_n \subseteq \beta_{n-1} \subseteq \ldots \subseteq \beta_0$. The set of all S-focusing sequences is partially ordered by extension, where $(V(G))$ is the smallest element. The subset $\mathcal{F}_S$ of all good S-focusing sequences is downwards closed in this partial order.

Lemma 3.7. *Let $(\beta_0, \ldots, \beta_n) \in \mathcal{F}_S$ and let $(A,B) \in S$. If $(A,B)\restriction\beta_n$ is proper, then $A \cap B \subseteq \beta_n$.*

Proof. By assumption $(A,B)\restriction\beta_n$ is proper, hence there are $a \in (\beta_n \cap A) \setminus B$ and $b \in (\beta_n \cap B) \setminus A$. Since $\beta_n \subseteq \beta_i$ for all $i \leq n$ the separations $(A,B)\restriction\beta_i$ are proper as well. Suppose for a contradiction there is a vertex $v \in (A \cap B) \setminus \beta_n$. Let $j < n$ be maximal with $v \in \beta_j$. Since β_{j+1} is an N_{β_j}-block of $G[\beta_j]$, there is a separation $(C,D) \in N_{\beta_j}$ with $v \in C \setminus D$ and $\{a,b\} \subseteq \beta_n \subseteq \beta_{j+1} \subseteq D$.

Now a, b and v witness that $(A,B)\restriction\beta_j$ and (C,D) are not nested: Indeed, a witnesses that D is not a subset of $B \cap \beta_j$. Similarly, b witnesses that D is not a subset of $A \cap \beta_j$. But v witnesses that neither $A \cap \beta_j$ nor $B \cap \beta_j$ is a subset of D. Thus we get a contradiction to the assumption that N_{β_j} is nested with the set $S\restriction\beta_j$. □

A set S of separations of G is *almost nested* if all S-focusing sequences are good. In this case the maximal elements of $\mathcal{F}_S$ in the partial order are exactly the S-focusing sequences $(\beta_0, \ldots, \beta_n)$ with $N_{\beta_n} = \emptyset$, and hence $S\restriction\beta_n = \emptyset$.

Lemma 3.8. *Let S be an almost nested set of separations of G.*

(i) *If $(\beta_0, \ldots, \beta_n) \in \mathcal{F}_S$ is maximal, then β_n is an S-block.*
(ii) *If b is an S-block, there is a maximal $(\beta_0, \ldots, \beta_n) \in \mathcal{F}_S$ with $\beta_n = b$.*

Proof. Let $(\beta_0, \ldots, \beta_n) \in \mathcal{F}_S$ be maximal. Then $S\restriction\beta_n$ is empty, i.e. no $(A,B) \in S$ induces a proper separation of $G[\beta_n]$. Hence β_n is S-inseparable.

For every $v \in V(G)\setminus\beta_n$ there is an $i < n$ and a separation in N_{β_i} separating v from β_n. Hence β_n is an S-block.

Conversely given an S-block b, let $(\beta_0, \dots, \beta_n) \in \mathcal{F}_S$ be maximal with the property $b \subseteq \beta_n$, which exists since $(V(G)) \in \mathcal{F}_S$ and since $\mathcal{F}_S$ is finite. Since b is N_{β_n}-inseparable, there is some N_{β_n}-block β_{n+1} containing b. The choice of $(\beta_0, \dots, \beta_n)$ implies that $(\beta_0, \dots, \beta_{n+1}) \notin \mathcal{F}_S$ and hence $N_{\beta_n} = \emptyset$, i.e. $(\beta_0, \dots, \beta_n)$ is a maximal element of $\mathcal{F}_S$. Thus β_n is an S-block with $b \subseteq \beta_n$ and hence $b = \beta_n$. $\square$

Construction 3.9. Let S be an almost nested set of separations of G. We recursively construct for every S-focusing sequence $(\beta_0, \dots, \beta_n)$ a tree-decomposition $\mathcal{T}^{\beta_n}$ of $G[\beta_n]$ so that the tree-decomposition $\mathcal{T}^{V(G)}{=:}\mathcal{T}(S)$ for the smallest S-focusing sequence $(V(G))$ is a tree-decomposition of G.

For each maximal S-focusing sequence $(\beta_0, \dots, \beta_m)$ we take the trivial tree-decomposition of $G[\beta_m]$ with only a single part. Suppose that $\mathcal{T}^{\beta}$ has already been defined for every successor $(\beta_0, \dots, \beta_n, \beta)$ of $(\beta_0, \dots, \beta_n)$. To define $\mathcal{T}^{\beta_n}$ we start with the tree-decomposition $\mathcal{T}(N_{\beta_n})$ of $G[\beta_n]$ as given by Theorem 2.4. For each hub node h we take the trivial tree-decomposition of H_h and for each node t whose part is an N_{β_n}-block β, we take $\mathcal{T}^{\beta}$ given from the S-focusing sequence $(\beta_0, \dots, \beta_n, \beta)$. This is indeed a tree-decomposition of the torso H_t, which we will show in Theorem 3.10. Hence we can apply Construction 3.5 to $\mathcal{T}(N_{\beta_n})$ and the family of tree-decompositions of the torsos to get $\mathcal{T}^{\beta_n}$.

Given an S-focusing sequence $(\beta_0, \dots, \beta_n)$, any two separations in N_{β_n} have the same order ℓ. We call ℓ the *rank* of $(\beta_0, \dots, \beta_n)$. If N_{β_n} is empty, we set the rank to be ∞.

For an almost nested set S of separations of G two S-focusing sequences $(\beta_0, \dots, \beta_n)$ and $(\alpha_0, \dots \alpha_m)$ are *similar* if there is an automorphism ψ of G inducing an isomorphism between $G[\beta_n]$ and $G[\alpha_m]$. Similar S-focusing sequences clearly have the same rank. If S is canonical, then ψ induces an isomorphism between $\mathcal{T}(N_{\beta_n})$ and $\mathcal{T}(N_{\alpha_m})$ as obtained from Theorem 2.4.

Theorem 3.10. *The tree-decomposition $\mathcal{T}(S)$ as in Construction 3.9 is well-defined and satisfies the following:*

(i) *every S-block of G is a part of $\mathcal{T}(S)$;*
(ii) *every part of $\mathcal{T}(S)$ is either an S-block of G or a hub;*
(iii) *for every separation (A, B) induced by $\mathcal{T}(S)$ there is a separation $(A', B') \in S$ such that $A \cap B = A' \cap B'$;*
(iv) *if S is canonical, then so is $\mathcal{T}(S)$.*

Proof. We show inductively that for any S-focusing sequence $(\beta_0, \dots, \beta_n)$ the tree-decomposition $\mathcal{T}^{\beta_n}$ has the following properties:

(a) every S-block included in β_n is a part of $\mathcal{T}^{\beta_n}$;
(b) every part of $\mathcal{T}^{\beta_n}$ is either an S-block or a hub;
(c) every separation (A, B) induced by $\mathcal{T}^{\beta_n}$ is proper;

(d) and for every separation (A, B) induced by $\mathcal{T}^{\beta_n}$ there is a separation $(A', B') \in S$ and an S-focusing sequence $(\beta_0, \dots, \beta) \geq (\beta_0, \dots, \beta_n)$ such that $(A', B') \restriction \beta = (A, B)$.

Furthermore we show for canonical S by induction, that

(e) if $(\alpha_0, \dots, \alpha_m)$ and $(\beta_0, \dots, \beta_n)$ are similar, then $\mathcal{T}^{\alpha_m}$ and $\mathcal{T}^{\beta_n}$ are isomorphic;

(f) $\mathcal{T}^{\beta_n}$ is canonical.

The tree-decompositions for the maximal S-focusing sequences satisfy (a) and (b) by Lemma 3.8, and (c) and (d) since their trees do not have any edges. If for two S-blocks b_1 and b_2 there is an isomorphism between $G[b_1]$ and $G[b_2]$ induced by an automorphism of G, then clearly the tree-decompositions are isomorphic. Hence (e) and (f) hold for all S-focusing sequences of rank ∞.

Suppose for our induction hypothesis that for every S-focusing sequence $(\alpha_0, \dots, \alpha_m)$ with rank greater than r the tree-decomposition $\mathcal{T}^{\alpha_m}$ of $G[\alpha_m]$ has the desired properties. Let $(\beta_0, \dots, \beta_n)$ be an S-focusing sequence of rank r. Then for each successor $(\beta_0, \dots, \beta_n, \beta)$ the tree-decomposition $\mathcal{T}^{\beta}$ is indeed a tree-decomposition of the corresponding torso: for a separation (A, B) induced by $\mathcal{T}^{\beta}$ consider (A', B') as given in (d). By (F$*$) we obtain that $(A', B') \restriction \beta_n = (A, B)$ is nested with N_{β_n}, hence (A, B) does not separate any adhesion set in H_t. Hence $\mathcal{T}^{\beta_n}$ is indeed well-defined.

Lemma 3.6 (i), (ii) and (iii) and the induction hypothesis yield (a), (b) and (c) for $\mathcal{T}^{\beta_n}$. Also by Lemma 3.6 (iii) for a separation (A, B) induced by $\mathcal{T}^{\beta_n}$ either $(A, B) \in N_{\beta_n} \subseteq S \restriction \beta_n$ or (A, B) induces a separation in $\mathcal{T}^{\beta}$ for an N_{β_n}-block β on the corresponding torso. In the first case $(\beta_0, \dots, \beta_n)$ is the desired S-focusing sequence for (d) and in the second case the induction hypothesis yields $(A', B') \in S$ and the desired S-focusing sequence extending $(\beta_0, \dots, \beta_n, \beta)$. Hence (d) holds for $\mathcal{T}^{\beta_n}$.

Suppose S is canonical. Let $(\alpha_0, \dots, \alpha_m)$ be similar to $(\beta_0, \dots, \beta_n)$. Then every automorphism of G that witnesses the similarity also witnesses that $\mathcal{T}(N_{\alpha_m})$ and $\mathcal{T}(N_{\beta_n})$ are isomorphic. Hence any torso of $\mathcal{T}(N_{\alpha_m})$ is similar to the corresponding torso of $\mathcal{T}(N_{\beta_n})$ and by induction hypothesis the tree-decompositions of the torsos are isomorphic. Therefore following Construction 3.5 yields (e). If two torsos H_t and H_u of $\mathcal{T}(N_{\beta_n})$ are similar, then either $V(H_t)$ and $V(H_u)$ are $N(\beta_n)$-blocks whose corresponding S-focusing sequences are similar and have rank greater than r, or they are hubs. If they are N_{β_n}-blocks, the chosen tree-decompositions are isomorphic by the induction hypothesis. If they are hubs, the chosen trivial tree-decompositions are isomorphic as witnessed by every automorphism of G witnessing the similarity of H_t and H_u. Hence this family of tree-decompositions of the torsos of $\mathcal{T}(N_{\beta_n})$ is canonical and with Lemma 3.6 (vi) we get (f).

Inductively the tree-decomposition $\mathcal{T}^{V(G)} = \mathcal{T}(S)$ of G satisfies (i), (ii) and (iv) by (a), (b) and (f). Finally, (iii) follows from (c), (d) and Lemma 3.7. $\square$

3.3. **Extending a nested set of separations.** In this subsection we give a condition for when we can extend a nested set of separations so that it distinguishes any two distinguishable profiles in a given set $\mathcal{P}$ efficiently.

Let N be a nested separation system of G and $\mathcal{T}(N) = \big(T, (P_t)_{t \in V(T)}\big)$ be the tree-decomposition of G as in Theorem 2.4. Recall that a separation (A, B) of G nested with N *induces* a separation $(A \cap P_t, B \cap P_t)$ of each torso H_t. An ℓ-profile $\widetilde{Q}$ of H_t is *induced* by a k-profile Q of G if for every $(A', B') \in \widetilde{Q}$ there is an $(A, B) \in Q$ which induces (A', B') on H_t.

Construction 3.11. Let $t \in V(T)$ and let Q be a k-profile of G. We construct a profile $\widetilde{Q}^t$ of the torso H_t which is induced by Q.
Case 1: Q inhabits P_t.
Let (A, B) be a proper separation of H_t of order less than k. By Lemma 2.6, there is a unique component C of $G-(A \cap B)$ with $(V(G) \setminus C, C \cup N(C)) \in Q$. As Q is consistent and inhabits P_t, the set $C \cap P_t$ is non-empty and either a subset of $A \setminus B$ or $B \setminus A$, but not both. If $(C \cap P_t) \subseteq (B \setminus A)$, then we let $(A, B) \in \widetilde{Q}^t$. Otherwise we let $(B, A) \in \widetilde{Q}^t$.
Case 2: Q does not inhabit P_t and $(V(G) \setminus C, C \cup N(C)) \notin Q$ for all components C of $G - P_t$.
Let (A, B) be a proper separation of H_t of order less than k. By Lemma 2.6, there is a unique component C of $G-(A \cap B)$ with $(V(G) \setminus C, C \cup N(C)) \in Q$. Since C is not a component of $G - P_t$, the set $C \cap P_t$ is non-empty by assumption, and we define $\widetilde{Q}^t$ as above.
Case 3: Q does not inhabit P_t and there is a component C of $G - P_t$ such that $(V(G) \setminus C, C \cup N(C)) \in Q$.
Let m denote the size of the neighbourhood of C. Let b be the m-block of H_t containing $N(C)$. For $\widetilde{Q}^t$ we take the m-profile induced by b.

The following is straightforward to check:

Remark 3.12. The set $\widetilde{Q}^t$ as in Construction 3.11 is a profile of H_t induced by Q. Moreover, if Q is r-robust, then so is $\widetilde{Q}^t$. □

The next remark is a direct consequence of the relevant definitions.

Remark 3.13. Let Q_1 and Q_2 be profiles of G.

- (i) If a separation (A, B) of G nested with N distinguishes Q_1 and Q_2 efficiently, then the induced separation $(A \cap P_t, B \cap P_t)$ of H_t distinguishes $\widetilde{Q}_1^t$ and $\widetilde{Q}_2^t$ efficiently for any part P_t where it is proper;
- (ii) if a separation (A, B) of some torso H_t distinguishes $\widetilde{Q}_1^t$ and $\widetilde{Q}_2^t$, then any separation of G that induces (A, B) on H_t distinguishes Q_1 and Q_2. □

Lemma 3.14. *Let Q_1 and Q_2 be profiles of G which are not already distinguished efficiently by N. Let (A, B) distinguish them efficiently such that it is nested with N. Then there is a part P_t of $\mathcal{T}(N)$ such that the induced separation $(A \cap P_t, B \cap P_t)$ of the torso H_t is proper.*

Proof. Since (A, B) is nested with N, there is a part P_t such that $A \cap B \subseteq P_t$. Suppose that $(A \cap P_t, B \cap P_t)$ is not proper. Without loss of generality let $(B \backslash A) \cap P_t$ be empty and let $(A, B) \in Q_1$.

By Lemma 2.6 we obtain a component K of $G - (A \cap B)$ such that $(A, B) \leq (V(G) \backslash K, K \cup N(K)) \in Q_1$. By consistency of Q_2 the separation $(V(G) \backslash K, K \cup N(K))$ still distinguishes Q_1 and Q_2, and since (A, B) distinguishes Q_1 and Q_2 efficiently, the neighbourhood of K is $A \cap B$. Let u be the neighbour of t such that the by tu induced separation $(C_t, D_t) \in N$ satisfies $K \cup N(K) \subseteq D_t$. If $(B \backslash A) \cap P_u$ is empty, we obtain $(C_u, D_u) \in Q_1$ as before and by construction we obtain $(C_t, D_t) < (C_u, D_u)$.

Among all parts P_t containing $A \cap B$ such that $(B \backslash A) \cap P_t$ is empty, we choose a part P_x such that (C_x, D_x) is maximal with respect to the ordering of separations. Let y denote the neighbour of x such that xy induces (C_x, D_x). There is a vertex $v \in (C_x \cap D_x) \backslash (A \cap B)$, since otherwise (C_x, D_x) would distinguish Q_1 and Q_2 efficiently. Since we assumed that $(B \backslash A) \cap P_x$ is empty, we deduce that $v \in A \backslash B$. Therefore $(A \backslash B) \cap P_y$ is not empty. Hence if $(A \cap P_y, B \cap P_y)$ on H_y were improper, then $(B \backslash A) \cap P_y$ would be empty and (C_y, D_y) would contradict the maximality of (C_x, D_x). □

For a nested separation system N let $S^N_{<k}$ be the set of separations of order less than k of G nested with N.

Construction 3.15. Let $N \subseteq S_{<r+1}$ be a nested separation system of G and let $\mathcal{P}$ be a set r-robust ℓ-profiles of G for some values $\ell \leq r + 1$, such that $S^N_{<r+1}$ distinguishes any two distinguishable profiles in $\mathcal{P}$ efficiently.

Let $\mathcal{T}(N) = \big(T, (P_t)_{t \in V(T)}\big)$ be as in Theorem 2.4 and let $\mathcal{P}^t$ be the set of profiles $\widetilde{Q}^t$ of H_t for $Q \in \mathcal{P}$. Applying Theorem 2.8 to the graphs H_t and the maximal k of any k-profile in $\mathcal{P}^t$, we get a tree-decomposition $\mathcal{T}^t$ of H_t that distinguishes every two distinguishable profiles in $\mathcal{P}^t$ efficiently. Note that if $\mathcal{P}$ is canonical, then the family $(\mathcal{T}^t)_{t \in V(T)}$ is canonical as well. By applying Lemma 3.6 we obtain a tree-decomposition $\overline{\mathcal{T}}$ and the corresponding nested system $\overline{N}$ of separations of order at most r induced by $\overline{\mathcal{T}}$.

Theorem 3.16. *The nested separation system $\overline{N}$ as in Construction 3.15 satisfies the following.*

(i) $N \subseteq \overline{N}$;
(ii) $\overline{N}$ *distinguishes every two distinguishable profiles in $\mathcal{P}$ efficiently;*
(iii) *if N and $\mathcal{P}$ are canonical, then so is $\overline{N}$.*

Proof. Lemma 3.6 (iv) yields (i). For (ii), consider two distinguishable profiles $Q_1, Q_2 \in \mathcal{P}$ not already distinguished efficiently by N. By assumption, there is some $(A, B) \in S^N_{<r+1}$ distinguishing Q_1 and Q_2 efficiently.

By Lemma 3.14 and Remark 3.13 (i) there is a part P_t of $\mathcal{T}(N)$ such that $\widetilde{Q}^t_1$ and $\widetilde{Q}^t_2$ are distinguished efficiently by $(A \cap P_t, B \cap P_t)$. Hence Theorem 2.8, Remark 3.3 (iv), Lemma 3.6 (v) and Remark 3.13 (ii) yield a separation of order $|A \cap B|$ in $\overline{N}$ distinguishing Q_1 and Q_2, yielding (ii).

Finally, (iii) holds by construction. □

4. Proof of the main result

Given a k-block b and a component C of $G-b$, then $(C\cup N(C), V(G)\setminus C)$ is a separation. By $S_k(b)$ we denote the set of all those separations. Note that $S_k(b)$ is a nested set of separations, while for different (r-robust) k-blocks b, b' the union $S_k(b) \cup S_k(b')$ need not to be nested [3].

Lemma 4.1. *Let b be a k-block of G. Then b is separable if and only if every separation in $S_k(b)$ has order less than k.*

Proof. For the 'only if'-implication, let $\mathcal{T} = \big(T, (P_t)_{t\in V(T)}\big)$ be a tree-decomposition of adhesion less than k of G with $P_t = b$ for some $t \in V(T)$. Let C be a component of $G - b$. There is a separation (A, B) induced by $\mathcal{T}$ with $C\subseteq A\setminus B$ and $b\subseteq B$. Hence $N(C)\subseteq A \cap B$, and so has less than k vertices.

For the 'if'-implication, just consider the star-decomposition induced by $S_k(b)$, whose central part is b. This tree-decomposition has adhesion less than k if and only if all separations in $S_k(b)$ have order less than k. □

Remark 4.2. Let b be a k-block of G. For all $(A, B) \in S_k(b)$ the separator $A \cap B$ is a subset of b. □

Given some $r \in \mathbb{N}$ and a set $\mathcal{B}$ of distinguishable[4] r-robust k-blocks for some values $k \leq r + 1$, we define

$$S(\mathcal{B}){:=}\bigcup\big\{S_k(b) \cap S_{<k} \;\big|\; b \text{ is a } k\text{-block in } \mathcal{B}\big\}.$$

Note that if the set of profiles induced by $\mathcal{B}$ is canonical, then so is $S(\mathcal{B})$.

Lemma 4.3. *Every separable k-block $b \in \mathcal{B}$ is an $S(\mathcal{B})$-block.*

Proof. Suppose for a contradiction there is a k'-block $b' \in \mathcal{B}$ and a separation $(A, B) \in S_{k'}(b') \cap S_{<k'} \subseteq S(\mathcal{B})$ separating b. Consider a separation (C, D) distinguishing b and b' efficiently with $b \subseteq C$ and $b' \subseteq D$. Since $|C \cap D| < k$, there is a vertex $v \in b\setminus(C \cap D)$. And since $(A \cap B) \subseteq b' \subseteq D$, the link ℓ_C is empty. Therefore we deduce that either $v \in A\setminus B$ or $v \in B\setminus A$. Let w denote a vertex of b such that (A, B) separates v and w. Both the corner separations $(A\cap C, B\cup D)$ and $(B\cap C, A\cup D)$ have order at most $|C\cap D| < k$. But one of them separates v from w, contradicting the $(< k)$-inseparability of b. Hence b is $S(\mathcal{B})$-inseparable.

Let X be an $S(\mathcal{B})$-inseparable set including b and let $v \in V(G)\setminus b$. Then there is some $(A, B) \in S_k(b)$ separating b from v. Lemma 4.1 implies that $(A, B) \in S_k(b) \cap S_{<k} \subseteq S(\mathcal{B})$ and thus v is not in X. Hence $X = b$. □

Lemma 4.4. *Let (A, B) and (C, D) be tight separations of G such that $A\setminus B$ is connected and the link ℓ_A is empty. Then (A, B) and (C, D) are nested.*

[4]A set of blocks is *distinguishable* if the set of induced profiles is distinguishable.

Proof. Since $A\backslash B$ is connected, either $\operatorname{int}(A,C)$ or $\operatorname{int}(A,D)$ is empty, say $\operatorname{int}(A,C)$. Thus there cannot be a vertex in the link ℓ_C because it would have a neighbour in $A\backslash B$, which is impossible. Hence (A,B) and (C,D) are nested by Remark 2.1. □

Lemma 4.5. *Let $(A,B),(C,D) \in S(\mathcal{B})$ be crossing. Then the links ℓ_B and ℓ_D are empty.*

Moreover, the separation $(K \cup N(K), V(G)\backslash K)$ for every component K of $G[\operatorname{int}(B,D)]$ is in $S(\mathcal{B})$ and its order is strictly less than the orders of both (A,B) and (C,D).

Proof. Let b_1 and b_2 be blocks in $\mathcal{B}$ such that $(A,B) \in S_{k_1}(b_1) \cap S_{<k_1}$ and $(C,D) \in S_{k_2}(b_2) \cap S_{<k_2}$. We may assume that the order k_2 of b_2 is at most the order k_1 of b_1. By Lemma 4.4, there are vertices $v_A \in \ell_A$ and $v_C \in \ell_C$. By Remark 4.2, $v_C \in b_1$. As (C,D) cannot separate b_1, the block b_1 is contained in $B \cap C$. In particular, the link ℓ_D is empty.

Let X be a component of $G - C \cap D$ that contains a vertex w of b_2. Note that X is unique as b_2 is a k_2-block. As ℓ_D is empty, X must be contained in $D \cap A$ or $D \cap B$. Since b_2 contains v_A, it must be contained in $D \cap A$. Indeed, otherwise the corner separation of $B \cap D$ would separated w from v_A. Hence ℓ_B is empty.

Let K be an arbitrary component of $G[\operatorname{int}(B,D)]$. Let $E{:=}K \cup N(K)$ and $F{:=}V(G)\backslash K$. Since the center c is a subset of $b_1 \cap b_2$ and since $K \cap (b_1 \cup b_2)$ is empty, K is a component of both $G - b_1$ and $G - b_2$. Hence (E,F) is in both $S_{k_1}(b_1)$ and $S_{k_2}(b_2)$. And since $E \cap F \subseteq c$ and since ℓ_A and ℓ_C are not empty, we deduce that $|E \cap F| < \min\{|A \cap B|, |C \cap D|\}$. □

Lemma 4.6. *$S(\mathcal{B})$ is almost nested.*

Proof. We have to show that every $S(\mathcal{B})$-focusing sequence $(\beta_0, \dots, \beta_n)$ is good, i.e. N_{β_n} is nested with $S(\mathcal{B})\restriction\beta_n$. Let $(\beta_0, \dots, \beta_n)$ be an $S(\mathcal{B})$-focusing sequence. Let $(A,B)\restriction\beta_n \in N_{\beta_n}$ and $(C,D)\restriction\beta_n \in S(\mathcal{B})\restriction\beta_n$. If (A,B) and (C,D) are nested, then so are $(A,B)\restriction\beta_n$ and $(C,D)\restriction\beta_n$. Suppose (A,B) and (C,D) are crossing. By Lemma 4.5 ℓ_B and ℓ_D are empty. If $\operatorname{int}(B,D) \cap \beta_n$ is empty, then by Remark 2.1 $(A,B)\restriction\beta_n$ and $(C,D)\restriction\beta_n$ are nested. Hence by Lemma 4.5 it suffices to show that $(E\backslash F) \cap \beta_n$ is empty for every $(E,F) \in S(\mathcal{B})$ with $E \subseteq B \cap D$ whose order is strictly smaller than the order of (A,B).

Since $(A,B)\restriction\beta_n$ is proper, there is a $v \in \beta_n\backslash B \subseteq \beta_n\backslash E \subseteq (F\backslash E) \cap \beta_n$. Since $(A,B)\restriction\beta_n$ has minimal order among all separations in $S(\mathcal{B})\restriction\beta_n$, we deduce that $(E,F)\restriction\beta_n$ is improper and hence either $(F\backslash E) \cap \beta_n$ or $(E\backslash F) \cap \beta_n$ is empty. Now v witnesses that $(E\backslash F) \cap \beta_n$ is empty, as desired. □

Lemma 4.7. *Given $r \in \mathbb{N}$, let $\mathcal{P}$ be a set of r-robust distinguishable k-profiles for some values $k \leq r+1$. Let N be a nested separation system such that for every $(C,D) \in N$, there is some ℓ-profile in $\mathcal{P}$ induced by an ℓ-block b with $(C \cap D) \subseteq b$. Then any two distinct $P, Q \in \mathcal{P}$ are distinguished efficiently by a separation nested with N.*

Proof. Let (A,B) distinguish $P,Q \in \mathcal{P}$ efficiently such that the number of separations in N nested with (A,B) is maximal. Without loss of generality let $(A,B) \in P$. Let $k{:=}|A\cap B|$. We prove that (A,B) is nested with N.

Suppose for a contradiction that there is some $(C,D) \in N$ not nested with (A,B). Let b be an $(\ell+1)$-block such that $(C\cap D){\subseteq}b$ whose induced profile $P_{\ell+1}(b)$ is in $\mathcal{P}$.

Case 1: $k \leq \ell$**.** Remark 4.2 implies that $C\cap D$ is $(\leq \ell)$-inseparable and hence one of the links ℓ_A or ℓ_B is empty. Without loss of generality let ℓ_B be empty. The orders of the corner separations $(A\cup D, B\cap C)$ and $(A\cup C, B\cap D)$ are less or equal than $|A\cap B|$. Hence they are oriented by P and Q. Applying Lemma 2.6 to $X{:=}A\cap B$ and P yields a component K of $G-X$ with $(V(G)\backslash K, K\cup N(K)) \in P$. In particular we get $K\subseteq B\backslash A$ by consistency. Since ℓ_B is empty and K is connected, we obtain $K\subseteq C\backslash D$ or $K\subseteq D\backslash C$. Therefore either $(A\cup D, B\cap C)$ or $(A\cup C, B\cap D)$ is in P by consistency to $(V(G)\backslash K, K\cup N(K))$, and not in Q by consistency to (B,A).

Hence there is a corner separation of (A,B) and (C,D) distinguishing P and Q efficiently. By Lemma 2.2 it is nested with every separation in N that is also nested with (A,B), as well as with (C,D). Hence it crosses strictly less separations of N than (A,B), contradicting the choice of (A,B). Thus (A,B) is nested with N.

Case 2: $k \geq \ell$**.** We prove this case by induction on k with Case 1 as the base case. By the efficiency of (A,B), the separation (C,D) does not distinguish P and Q. Thus we may assume that (C,D) is in both P and Q. If one of the corner separations $(A\cap D, B\cup C)$ or $(B\cap D, A\cup C)$ had order at most k, then it would violate the maximality of (A,B) by Lemma 2.2. Indeed, it would be nested with every separation in N that is also nested with (A,B), as well as with (C,D).

Hence we may assume that both these corner separations have order larger than k and therefore both links ℓ_A and ℓ_B are not empty. By Remark 2.3, the opposite corner separations $(A\cap C, B\cup D)$ and $(B\cap C, A\cup D)$ have order strictly less than $|C\cap D|$ and are in $P_{\ell+1}(b)$ since $C\cap D{\subseteq}b$. As b is r-robust, $(C,D)\in P_{\ell+1}(b)$. Hence (C,D) distinguishes P and $P_{\ell+1}(b)$.

By the induction hypothesis, there is a separation (E,F) of order at most ℓ distinguishing P and $P_{\ell+1}(b)$ efficiently that is nested with N. We may assume that $(E,F)\in P_{\ell+1}(b)$ and $(F,E)\in P$. Furthermore, (E,F) does not distinguish P and Q, since $|E\cap F| < |A\cap B|$. We claim that $(C,D)\leq(F,E)$. Indeed, since (C,D) and (F,E) are nested and P contains both of them, either $(C,D)\leq(F,E)$ or $(F,E)\leq(C,D)$. By consistency of $P_{\ell+1}(b)$, we can conclude that $(C,D)\leq(F,E)$.

If the order of $(E\cap B, F\cup A)$ is at most k, then it would distinguish P and Q efficiently. It would violate the maximality of (A,B) by Lemma 2.2 since it is nested with every separation in N that is also nested with (A,B), as well as with (C,D) itself as $(C,D)\geq(E,F)\geq(E\cap B, F\cup A)$. Thus we may assume that $(E\cap B, F\cup A)$ has order larger than k. Similarly we may assume that $(E\cap A, F\cup B)$ has order larger than k.

Again by Remark 2.3, the opposite corner separations $(F \cap A, E \cup B)$ and $(F \cap B, E \cup A)$ have order less than $|E \cap F|$. But by construction they separate ℓ_A and ℓ_B and hence b, contradicting the fact that b is $(\leq \ell)$-inseparable. □

Theorem 4.8. *Let G be a finite graph, $r \in \mathbb{N}$ and let $\mathcal{P}$ be a canonical set of r-robust distinguishable ℓ-profiles for some values $\ell \leq r+1$.*

Then G has a canonical tree-decomposition $\mathcal{T}$ that distinguishes efficiently every two distinct profiles in $\mathcal{P}$, and which has the further property that every separable block whose induced profile is in $\mathcal{P}$ is equal to the unique part of $\mathcal{T}$ in which it is contained.

Proof. Let $\mathcal{B}$ be the set of blocks whose induced profiles are in $\mathcal{P}$. We consider $S(\mathcal{B})$ as above. Lemma 4.6 and Construction 3.9 yield a canonical tree-decomposition $\mathcal{T}(S(\mathcal{B}))$ where by Lemma 4.3 and Theorem 3.10 (i) every separable $b \in \mathcal{B}$ is equal to the unique part in which it is contained.

Let N be the nested separation system induced by $\mathcal{T}(S(\mathcal{B})))$. With Lemma 4.7 we can apply Construction 3.15 to obtain $\overline{N}$, which by Theorem 3.16 (ii) distinguishes the profiles in $\mathcal{P}$ efficiently.

It is left to show that no separation $(A, B) \in \overline{N} \setminus N$ separates a separable k-block $b \in \mathcal{B}$. Suppose for a contradiction that $(A, B) \in \overline{N} \setminus N$ separates b. Let P_t be the part of $\mathcal{T}(S(\mathcal{B}))$ with $P_t = b$. Note that since the adhesion sets $P_t \cap P_u$ for any edge tu have size strictly smaller than k and since the only profile in $\mathcal{P}$ inhabiting P_t is $P_k(b)$, no profile in $\mathcal{P}$ induces an ℓ-profile for some $\ell \geq k+1$ on the torso H_t. Then by Construction 3.15 and Lemma 3.6 (iii) the induced separation $(A \cap P_t, B \cap P_t)$ is a proper separation of H_t distinguishing two $(\leq k)$-profiles of H_t efficiently. But since H_t has no proper $(< k)$-separation, it has no two distinguishable $(\leq k)$-profiles.

Hence Theorem 2.4 yields a tree-decomposition $\mathcal{T}(\overline{N})$ with the desired properties. □

Corollary 4.9. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ that distinguishes efficiently every two distinct maximal robust profiles, and which has the further property that every separable block inducing a maximal robust profile is equal to the unique part of $\mathcal{T}$ in which it is contained.*

Proof. Since the set of maximal robust profiles is by definition distinguishable, we can apply Theorem 4.8. □

Corollary 4.10. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ of adhesion less than k that distinguishes efficiently every two distinct k-profiles, and which has the further property that every separable k-block is equal to the unique part of $\mathcal{T}$ in which it is contained.*

Proof. By Remark 2.5 (i) any k-profile is $(k-1)$-robust. Since the set of all k-profiles is by definition distinguishable, we can apply Theorem 4.8. □

Theorem 4.8 fails if we do not require that $\mathcal{P}$ is distinguishable:

Example 4.11. Consider the graph obtained by two cliques K_1 and K_2 of size at least $k+1 \geq 7$ sharing $k-1$ vertices, together with a vertex v joined to two vertices of $K_1 - K_2$ and to two vertices of $K_2 - K_1$, see Figure 3.

Then $K_1 \cup K_2$ is a separable 5-block, as witnessed by the separation $(\{v\} \cup N(v), K_1 \cup K_2)$. But the two $(k+1)$-blocks K_1 and K_2 are only distinguished efficiently by $(K_1 \cup \{v\}, K_2 \cup \{v\})$. Since this separation crosses any separation separating v from $K_1 \cup K_2$, there is no tree-decomposition that distinguishes K_1 and K_2 efficiently such that there is a part equal to $K_1 \cup K_2$. Moreover, even the union of the parts inhabited by $P_5(K_1 \cup K_2)$ in any tree-decomposition that distinguishes K_1 and K_2 efficiently contains with v a vertex outside the block.

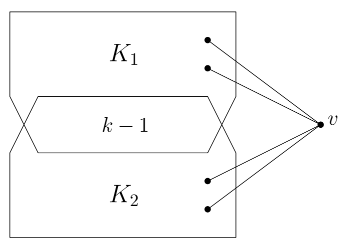


Figure 3. The graph of Example 4.11

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