

(Re)constructing Lie groupoids from their bisections and
applications to prequantisation

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Abstract

This paper is about the relation of the geometry of Lie groupoids over a fixed compact manifold M and the geometry of their (infinite-dimensional) bisection Lie groups. In the first part of the paper we investigate the relation of the bisections to a given Lie groupoid, where the second part is about the construction of Lie groupoids from candidates for their bisection Lie groups. The procedure of this second part becomes feasible due to some recent progress in the infinite-dimensional Frobenius theorem, which we heavily exploit. The main application to the prequantisation of (pre)symplectic manifolds comes from an integrability constraint of closed Lie subalgebras to closed Lie subgroups. We characterise this constraint in terms of a modified discreteness conditions on the periods of that manifold.

Keywords: global analysis, Lie groupoid, infinite-dimensional Lie group, mapping space, bisection, prequantisation, homotopy groups of diffeomorphism groups

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Introduction

The Lie group structure on bisection Lie groups was established in [Ryb02, SW15b], along with a smooth action of the bisections on the arrow manifold of a Lie groupoid. In this paper, we develop a tight relation between Lie groupoids and their bisection Lie groups by making use of this action. As a first step, we show how this action can be used to reconstruct a Lie groupoid from its bisections. In general, this will not be possible, since there may not be enough bisections for a reconstruction to work. However, under mild assumptions on the Lie groupoid, e.g., the groupoid being source connected, the reconstruction works quite well. It is worthwhile to note that in this chapter the analytical tools we need are of moderate complexity, all results follow from a thorough usage of the concept of a submersion.

In an intermediate step, we analyse in some more detail the structure that bisections have in addition to merely being Lie groups. For instance, they act on M and the stabilisers are closely related to the vertex groups of the Lie groupoid. To this end, we use the results of Glöckner on submersion properties in infinite-dimensions [Glö06, Glö15a].

The next step is then to take the insights from the previous sections in order to formulate structures and conditions on a Lie group that turn it into a Lie group of bisections of a Lie groupoid. This becomes analytically more challenging, and the procedure is only possible due to a heavy usage of the recent results from [Glö15a], building on a generalised Frobenius Theorem of integrable co-Banach distributions [Eyn14, Hil00]. The result is the concept of a transitive pair, which describes in an efficient way the necessary structure that is needed on a Lie group in order to relate it in a natural way to the bisection group of a Lie groupoid. However, we restrict here to transitive (respectively locally trivial) Lie groupoids, the general theory will be part of future research.

In the final section we then apply the integration theory for abelian extensions from [Nee04] in order to derive a transitive pair from an integration of an extension of $\mathcal{V}(M)$, given by a closed 2-form $\omega \in \Omega^2(M)$ to an extension of $\text{Diff}(M)_0$. The crucial point here is the integration of a certain Lie subalgebra to a closed Lie subgroup of the integrated extension, for which we derive a new discreteness condition in terms of the associated period groups.

We now go into some more detail and explain the main results. Suppose $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid. This means that G, M are smooth manifolds, equipped with submersions $\alpha, \beta: G \rightarrow M$ and an associative and smooth multiplication $G \times_{\alpha, \beta} G \rightarrow G$ that admits a smooth identity map $1: M \rightarrow G$ and a smooth inversion $\iota: G \rightarrow G$. Then the bisections $\text{Bis}(\mathcal{G})$ of \mathcal{G} are the sections $\sigma: M \rightarrow G$ of α such that $\beta \circ \sigma$ is a diffeomorphism of M . This becomes a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M.$$

This group is an (infinite-dimensional) Lie group (cf. [SW15b]) if M is compact, G is modelled on a metrisable space and the groupoid \mathcal{G} admits a local addition adapted to the source projection α , i.e. it restricts to a local addition on each fibre $\alpha^{-1}(x)$ for $x \in M$ (cf. [Mic80, SW15b]).

By construction of the Lie group structure, we obtain a natural action $\text{Bis}(\mathcal{G}) \times M \rightarrow M$, $(\sigma, m) \mapsto \beta(\sigma(m))$ of $\text{Bis}(\mathcal{G})$ on M . This action gives rise to the bisection action groupoid $\mathcal{B}(\mathcal{G}) := (\text{Bis}(\mathcal{G}) \times M \rightrightarrows M)$. Observe that the action is constructed from the joint evaluation map

$$\text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow G, \quad (\sigma, m) \mapsto \sigma(m)$$

and the target projection of the groupoid \mathcal{G} . While the target projection is a feature of the groupoid \mathcal{G} , the evaluation map yields a groupoid morphism over M from $\mathcal{B}(\mathcal{G})$ to \mathcal{G} . Thus information on the groupoid can be recovered from the group of bisections and the base manifold via the joint evaluation map. The idea is to recover the smooth structure of the arrow manifold from the evaluation map based on the following result:

Theorem A. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, where M is compact, G is modelled on a metrisable space and \mathcal{G} admits an α -adapted local addition. Then the joint evaluation ev is a submersion. If \mathcal{G} is in addition source-connected, i.e. the source fibres are connected manifolds, then ev is surjectiv. ■*

Note that as an interesting consequence of Theorem A we obtain also information on the evaluation of smooth maps from a compact manifold M into a manifold N modelled on a metrisable space. In this case, the evaluation map

$$\text{ev}: C^\infty(M, N) \times M \rightarrow N, \quad (f, m) \mapsto f(m)$$

is a surjective submersion.

A crucial point of our approach will be that the joint evaluation map is a surjective submersion. Unfortunately, this will not be the case in general as there are Lie groupoids without enough bisections (see Remark 2.18 b) for an example). In this case there is no hope to recover the manifold of arrows, whence not all information on the groupoid is contained in its group of bisections. However, it turns out that at least the identity subgroupoid can always be reconstructed. Moreover, Theorem A still gives a sufficient criterion for the surjectivity of ev . It is sufficient that the Lie groupoid is source connected, and in general this condition can not be dispensed with. As a byproduct, we obtain generalisations of some results about the existence of global bisections through each point (cf. [ZCL09]) to infinite dimensions. Hence we can reconstruct the groupoid from its bisections and the base manifold for a fairly broad class of Lie groupoids. Namely, we obtain the following reconstruction result.

Theorem B. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, where M is compact, G is modelled on a metrisable space and \mathcal{G} admits an α -adapted local addition. If the joint evaluation map ev is surjective, e.g. \mathcal{G} is source-connected, then the Lie groupoid morphism*

$$\text{ev}: \mathcal{B}(\mathcal{G}) = (\text{Bis}(\mathcal{G}) \times M \rightrightarrows M) \rightarrow \mathcal{G}$$

is the groupoid quotient of $\mathcal{B}(\mathcal{G})$ by a normal Lie subgroupoid. In this case, the Lie group of bisections and the manifold M completely determine the Lie groupoid \mathcal{G} . ■

Note that Theorem B really is a reconstruction theorem, i.e., we need a Lie groupoid to begin with as a candidate for the quotient. Hence, the original groupoid is needed and Theorem B does not provide a way to construct \mathcal{G} without knowing it beforehand. The problem here is twofold. At first, we need to know the kernel of ev , or some equivalent information, that allows us to determine the groupoid \mathcal{G} as a quotient of $(\text{Bis}(\mathcal{G}) \times M \rightrightarrows M)$. The other problem is an analytical problem: quotients of (infinite-dimensional) Lie groupoids and Lie groups usually do not admit a suitable smooth structure. In particular, it is not known if the familiar tools, e.g. Godement's criterion, carry over to the infinite-dimensional setting beyond the Banach setting.

Nevertheless, one can extract some information on the quotient from Theorem B. The groupoid quotient is controlled by certain subgroups of the group of bisections which arise from the kernel of the joint evaluation. Namely, the source fibre over m in the kernel corresponds to the Lie subgroup $\text{Bis}_m(\mathcal{G}) = \{\sigma \in \text{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\}$ of all bisections which take m to the corresponding unit. Observe that $\text{Bis}_m(\mathcal{G})$ sits inside the Lie subgroup $\text{Loop}_m(\mathcal{G}) := \{\sigma \in \text{Bis}(\mathcal{G}) \mid \beta(\sigma(m)) = m\}$ of all elements whose image at m is an element in the vertex group. Both subgroups will later turn out to be important in the construction of Lie groupoids from their bisections. In this context Lie theoretic properties, like regularity in the sense of Milnor, of these subgroups are crucial to our approach. Thus we first study them in a separate section. Regularity (in the sense of Milnor) of a Lie group roughly means that a certain class of differential equations can be solved on the Lie group. Many familiar results from finite-dimensional Lie theory carry over only to regular Lie groups (cf. [Glö15b] for a survey). Our results then subsume the following theorem.

Theorem C. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Banach Lie groupoid, then for each $m \in M$ the inclusions*

$$\text{Bis}_m(\mathcal{G}) \subseteq \text{Loop}_m(\mathcal{G}) \subseteq \text{Bis}(\mathcal{G})$$

turn $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ into split Lie subgroups which are regular in the sense of Milnor. As submanifolds these subgroups are even co-Banach submanifolds. ■

These Lie subgroups are of interest, as the quotient $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ reconstructs the source fibre of \mathcal{G} over the point m and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ reconstructs the vertex group at m . Moreover, Theorem C enables

us to construct a natural smooth structure on these quotients which coincides a posteriori with the manifold structure on the fibre and the vertex group, respectively.

We now use the results obtained so far to turn the reconstruction result given Theorem B into a construction result, at least in the locally trivial case. This means that we start with Lie groups, some extra structure on them and then produce a Lie groupoid such that the groups are related to the Lie group of bisections. Recall that a locally trivial Lie groupoid is completely determined by its source fibre and the vertex group over a given point. Hence, if we fix a point $m \in M$, the problem to construct a Lie groupoid reduces in the locally trivial case to reconstructing a manifold (modelling the source fibre) and a Lie group (modelling the vertex group). This motivates the notion of a *transitive pair* (cf. Definition 4.1). A transitive pair (θ, H) consists of a transitive Lie group action $\theta: K \times M \rightarrow M$ and a normal subgroup H of the m -stabiliser K_m of θ , such that H is a regular and co-Banach Lie subgroup of K_m . The guiding example is here the transitive pair $(\beta_{\mathcal{G}} \circ \text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow M, \text{Bis}_m(\mathcal{G}))$ induced by the bisections of a locally trivial Lie groupoid \mathcal{G} with connected base M . The notion of transitive pair can be thought of as a generalisation of a Klein geometry for fibre-bundles (see Remark 4.19 for further information). We then obtain the following construction principle.

Theorem D. *Let $(\theta: K \times M \rightarrow M, H)$ be a transitive pair. Then there is a locally trivial Banach Lie groupoid $\mathcal{R}(\theta, H)$ together with a Lie group morphism $a_{\theta, H}: K \rightarrow \text{Bis}(\mathcal{R}(\theta, H))$. If $(\theta, H) = (\beta_{\mathcal{G}} \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ for some locally trivial Banach Lie groupoid \mathcal{G} , then $a_{\theta, H}$ is an isomorphism. ■*

One should think of the construction principle from Theorem D as an analogue of the reconstruction result in Theorem B for locally trivial Lie groupoids. The crucial difference here is that Theorem D makes no reference to the Lie groupoid, but constructs it purely from the given transitive pair. Note that the Lie group morphism $a_{\theta, H}$ in Theorem D will in general not be an isomorphism. However, the morphisms are interesting in their own right due to the fact that the definition of a transitive pair is quite flexible. It allows us to construct for a wide range of Lie groups with transitive actions on M Lie group morphisms into the Lie group of bisections $\text{Bis}(\mathcal{R}(\theta, H))$. Moreover, these Lie group morphisms carry geometric information and thus connect the group actions of both Lie groups.

In the last section we then invoke the integration theory of abelian extensions of infinite-dimensional Lie algebras from [Nee04] to construct transitive pairs from a closed 2-form ω on a 1-connected and compact manifold M . We formulate the results here for $\text{Diff}(M)_0$, whereas in the text we allow for more general $K \leq \text{Diff}(M)$. If ω is prequantisable, then the prequantisation provides a gauge groupoid and thus an integration of the Lie algebroid extension. By the results from [SW15b] and [Nee04], the associated Lie algebra extension also integrates to an extension of Lie groups. In the other direction, we show that the integration of the extensions of Lie algebras to transitive pairs is in fact a two-step process. The first step is concerned with the integration of the extension

$$C^\infty(M) \rightarrow C^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}(M) \rightarrow \mathcal{V}(M) \tag{1}$$

of Lie algebras to an extension of Lie groups, where $\overline{\omega}$ is the abelian cocycle $(X, Y) \mapsto \omega(X, Y)$. By the results of [Nee04], the integration of (1) is governed by the discreteness of the *primary* periods, i.e., the periods of the Lie group $\text{Diff}(M)_0$ for the equivariant extension $\overline{\omega}^{\text{eq}}$ of $\overline{\omega}$. The second step is then the integration of the Lie subalgebra of $C^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}(M)$, that corresponds to the vector fields vanishing in the base-point, to a closed Lie subgroup. This is governed by the discreteness of the *secondary* periods, i.e., the periods of (M, ω) modulo the periods of $(\text{Diff}(M)_0, \overline{\omega}^{\text{eq}})$ (see Remark 5.5 and Remark 5.10 for a precise definition).

Theorem E. *Let M be a compact and 1-connected manifold with base-point m and $\omega \in \Omega^2(M)$ be closed. If the extension (1) integrates to an extension of Lie groups, then (M, ω) is prequantisable if and only if the secondary periods are discrete. The latter is equivalent to the integrability of the subalgebra $C_m^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}_m(M)$ to a closed Lie subgroup, where $C_m^\infty(M)$ and $\mathcal{V}_m(M)$ denote the functions (respectively vector fields) that vanish in m . ■*

Finally, we would like to remark that the constructions of Lie groupoids given in the present paper yield functors on suitable categories of Lie groups and Lie groupoids. These functors are closely connected to the bisection functor. However, there is no need for these results in the present paper as we are only concerned with the (re-)construction of Lie groupoids. Thus we will largely avoid categorical language and postpone a detailed investigation of these functors to [SW15a].

1 Locally convex Lie groupoids and the Lie group of bisections

In this section we recall basic facts and conventions on Lie groupoids and bisections used in this paper. We refer to [Mac05] for an introduction to (finite-dimensional) Lie groupoids and the associated group of bisections. The notation for Lie groupoids and their structural maps also follows [Mac05]. However, we do not restrict our attention to finite dimensional Lie groupoids. Hence, we have to augment the usual definitions with several comments. Note that we will work all the time over a fixed base manifold M .

Definition 1.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a groupoid over M with source projection $\alpha: G \rightarrow M$ and target projection $\beta: G \rightarrow M$. Then \mathcal{G} is a *(locally convex and locally metrisable) Lie groupoid over M* ¹ if

- the objects M and the arrows G are locally convex and locally metrisable manifolds,
- the smooth structure turns α and β into surjective submersions, i.e., they are locally projections²
- multiplication $m: G \times_{\alpha, \beta} G \rightarrow G$, object inclusion $1: M \rightarrow G$ and inversion $\iota: G \rightarrow G$ are smooth. ■

Definition 1.2. The *group of bisections* $\text{Bis}(\mathcal{G})$ of \mathcal{G} is given as the set of sections $\sigma: M \rightarrow G$ of α such that $\beta \circ \sigma: M \rightarrow M$ is a diffeomorphism. This is a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M. \quad (2)$$

The object inclusion $1: M \rightarrow G$ is then the neutral element and the inverse element of σ is

$$\sigma^{-1}(x) := \iota(\sigma((\beta \circ \sigma)^{-1}(x))) \text{ for } x \in M. \quad (3)$$

In [SW15b] we have established a Lie group structure on the group of bisections of a (locally convex) Lie groupoid which admits a certain type of local addition. To understand the Lie group structure on $\text{Bis}(\mathcal{G})$ one uses local additions (cf. Definition B.7) which respect the fibres of a submersion. This is an adaptation of the construction of manifold structures on mapping spaces [Woc13, KM97, Mic80] (see also Appendix B).

Definition 1.3. We say that that a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ admits an *adapted local addition* if G admits a local addition which is adapted to the source projection α (or, equivalently, to the target projection β). ■

Recall from [SW15b, Section 3] the following facts on the Lie group structure of the group of bisections:

Theorem 1.4. *Suppose M is compact and $\mathcal{G} = (G \rightrightarrows M)$ is a locally convex and locally metrisable Lie groupoid over M which admits an adapted local addition. Then $\text{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$ (with the manifold structure from Theorem B.9). Moreover, the induced manifold structure and the group multiplication*

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M$$

turn $\text{Bis}(\mathcal{G})$ into a Lie group modelled on $E_1 := \{\gamma \in C^\infty(M, TG) \mid \forall x \in M, \gamma(x) \in T_{1_x}\alpha^{-1}(x)\}$. ■

As the Lie group $\text{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$ the exponential law for smooth maps B.9 c) applies to maps defined on $\text{Bis}(\mathcal{G})$. In particular, as in [SW15b, Proposition 3.11] one easily concludes that the natural action of the bisections on the arrows and the evaluation of bisections are smooth:

Proposition 1.5. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid such that M is a compact manifold and \mathcal{G} admits an adapted local addition. Then*

- a) *the natural action $\gamma: \text{Bis}(\mathcal{G}) \times G \rightarrow G, (\psi, g) \mapsto \psi(\beta(g)) \cdot g$ is smooth and for $g \in G$ the restricted action*

$$\gamma_g: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(\alpha(g)), \psi \mapsto \psi(\beta(g)) \cdot g$$

is smooth.

- b) *the evaluation map $\text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow G, (\sigma, m) \mapsto \sigma(m)$ is smooth and satisfies*

$$\text{ev}(\sigma, m) = \sigma(m) = \gamma(\sigma, 1_m). \quad (4)$$

Furthermore, the map $\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m), \sigma \mapsto \sigma(m)$ is smooth for each $m \in M$.

¹See Appendix B for references on differential calculus in locally convex spaces.

²This implies that the occurring fibre-products are submanifolds of the direct products, see [Woc13, Appendix C].

2 Reconstruction of the Lie groupoid from its group of bisections

In this section, the close link between Lie groupoids and their Lie groups of bisections is established. More precisely, we will show that each Lie groupoid is the quotient of its Lie group of bisections. In the end, we discuss some examples and applications of this link.

Throughout this section assume that $\mathcal{G} = (G \rightrightarrows M)$ is a locally metrisable Lie groupoid that admits an adapted local addition and that has a compact space of objects M .

Definition 2.1. The Lie group $\text{Bis}(\mathcal{G})$ acts on M , via the action induced by $(\beta_{\mathcal{G}})_* : \text{Bis}(\mathcal{G}) \rightarrow \text{Diff}(M)$ and the natural action of $\text{Diff}(M)$ on M . This gives rise to the action Lie groupoid $\mathcal{B}(\mathcal{G}) := \text{Bis}(\mathcal{G}) \times M$, with source and target projections defined by $\alpha_{\mathcal{B}}(\sigma, m) = m$ and $\beta_{\mathcal{B}}(\sigma, m) = \beta_{\mathcal{G}}(\sigma(m))$. The multiplication on $\mathcal{B}(\mathcal{G})$ is defined by

$$(\sigma, \beta_{\mathcal{G}}(\tau(m))) \cdot (\tau, m) := (\sigma \star \tau, m).$$

Clearly, any morphism $f: \mathcal{G} \rightarrow \mathcal{H}$ of Lie groupoids over M induces a morphism $f_* \times \text{id}_M: \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$ of Lie groupoids. ■

Remark 2.2. The Lie groupoid $\mathcal{B}(\mathcal{G})$ admits an adapted local addition (for $\alpha_{\mathcal{B}}$ and thus also for $\beta_{\mathcal{B}}$). In fact, this is the case for the Lie group $\text{Bis}(\mathcal{G})$ and the finite-dimensional manifold M separately (cf. [KM97, p. 441]), and on $\text{Bis}(\mathcal{G}) \times M$ one can simply take the product of these local additions. In addition, the evaluation map $\text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow G$ is a morphism of Lie groupoids over M

$$\left(\begin{array}{c} \text{Bis}(\mathcal{G}) \times M \\ \alpha_{\mathcal{B}} \downarrow \quad \downarrow \beta_{\mathcal{B}} \\ M \end{array} \right) \xrightarrow{\text{ev}} \left(\begin{array}{c} G \\ \alpha_{\mathcal{G}} \downarrow \quad \downarrow \beta_{\mathcal{G}} \\ M \end{array} \right). \quad \blacksquare$$

We will now study the analytic properties of the morphism $\text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow G$. It will turn out that under mild assumptions on \mathcal{G} the groupoid is a groupoid quotient of the action groupoid $\mathcal{B}(\mathcal{G})$ via the evaluation map. The key point to establish this result is to prove that ev actually induces a quotient map, i.e. we need ev to be a surjective submersion. Let us first deal with some preparations:

Remark 2.3. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and fix $m \in M$ and $\tau \in \text{Bis}(\mathcal{G})$. Our goal is to split the space of sections $\Gamma(\tau^*T^\alpha G)$ into a product of (closed) subspaces. To this end define

$$\Gamma(\tau^*T^\alpha G)_m := \{X \in \Gamma(\tau^*T^\alpha G) \mid X(m) = 0_{\tau(m)}\}.$$

Choose a bundle trivialisaton $\lambda: M_\lambda \rightarrow \tau^*\pi^\alpha(M_\lambda) \times E$ of the bundle $\tau^*\pi^\alpha: \tau^*T^\alpha G \rightarrow M$ such that $m \in \tau^*\pi^\alpha(M_\lambda)$. Since M is compact, we can choose a smooth cut-off function $\rho: M \rightarrow [0, 1]$ with $\rho_\lambda(m) = 1$ and $\rho|_{M \setminus \tau^*\pi^\alpha(M_\lambda)} \equiv 0$. Then we obtain a (non-canonical) isomorphism of topological vector spaces

$$I_\lambda: \Gamma(\tau^*T^\alpha G) \rightarrow \Gamma(\tau^*T^\alpha G)_m \times T_{\tau(m)}^\alpha G, \quad X \mapsto (X - (\rho \circ \tau^*\pi^\alpha) \cdot \lambda^{-1}(\tau^*\pi^\alpha, \text{pr}_2 \circ \lambda(X(m))), X(m)). \quad (5)$$

its inverse is given by

$$I_\lambda^{-1}(X_0, y) := X_0 + (\rho \circ \tau^*\pi^\alpha) \cdot \lambda^{-1}(\tau^*\pi^\alpha, \text{pr}_2 \circ \lambda(y)) \quad (6)$$

This turns $\Gamma(\tau^*T^\alpha G)_m$ into a complemented subspace of $\Gamma(\tau^*T^\alpha G)$. Moreover, if \mathcal{G} is a Banach-Lie groupoid then $T_{\tau(m)}^\alpha G$ is a Banach space and thus $\Gamma(\tau^*T^\alpha G)_m$ turns into a co-Banach subspace of $\Gamma(\tau^*T^\alpha G)$. ■

Let us establish the submersion property for the restricted action of the group of bisections on the manifold of arrows.

Proposition 2.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $g \in G$. Then the restricted action*

$$\gamma_g: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(\alpha(g)), \sigma \mapsto \gamma(\sigma, g) = \sigma(\beta(g)) \cdot g$$

is a submersion.

Proof. From Proposition 1.5 a) we infer that γ_g is smooth. Hence, we only have to prove that γ_g is locally a projection. To see this fix $\tau \in \text{Bis}(\mathcal{G})$ and a chart $\kappa: U_\kappa \rightarrow V_\kappa \subseteq E$ of the manifold $\alpha^{-1}(\alpha(g))$ such that $\gamma_g(\tau) \in U_\kappa$. Furthermore, choose a bundle trivialisaton λ of $\tau^*T^\alpha G$ and construct the vector space isomorphism (5) for λ .

Now consider the canonical chart (O_τ, φ_τ) of $\text{Bis}(\mathcal{G})$. The set $I_\lambda \circ \varphi_\tau(O_\tau)$ is an open zero-neighbourhood in $\Gamma(\tau^*T^\alpha G)_{\beta(g)} \times T_{\tau(\beta(g))}^\alpha G$. Shrinking O_τ , we can assume that $O_\tau \subseteq \gamma_g^{-1}(U_\kappa)$ and that there are open zero-neighbourhoods $U \subseteq \Gamma(\tau^*T^\alpha G)_{\beta(g)}$ and $W \subseteq T_{\tau(\beta(g))}^\alpha G$ such that $U \times W = I_\lambda \circ \varphi_\tau(O_\tau)$. In conclusion, we obtain a commutative diagram

$$\begin{array}{ccccc} \text{Bis}(\mathcal{G}) \supseteq O_\tau & \xrightarrow{\gamma_g} & \alpha^{-1}(\alpha(g)) & \xrightarrow{\kappa} & E \\ I_\lambda \circ \varphi_\tau \downarrow & & & & \parallel \\ U \times W & \xrightarrow{\tilde{\gamma}_g} & & & E \end{array}$$

with $\tilde{\gamma}_g := \kappa \circ \gamma_g \circ (I_\lambda \circ \varphi_\tau)^{-1}|_{U \times W}$. Denote by Σ the adapted local addition of \mathcal{G} and consider the right translation $R_g: \alpha^{-1}(\beta(g)) \rightarrow \alpha^{-1}(\alpha(g)), h \mapsto hg$. Then (6) and the definition of φ_τ^{-1} yield

$$\begin{aligned} \tilde{\gamma}_g(X, y) &= \kappa(\gamma_g(\Sigma(X + (\rho \circ \pi^\alpha) \cdot \lambda^{-1}(\pi^\alpha, \text{pr}_2 \lambda(y)))))) = \kappa(\Sigma(\underbrace{X(\beta(g))}_{=0} + \underbrace{(\rho(\beta(g)))}_{=1} \cdot y) \cdot g) \\ &= \kappa(\Sigma(y) \cdot g) = \kappa \circ R_g \circ \Sigma(y). \end{aligned} \quad (7)$$

Note that by (7) the map $\tilde{\gamma}_g$ does neither depend on the choice of the trivialisaton λ nor on the the cut-off function ρ_λ .

By definition of the adapted local addition, Σ restricts to a diffeomorphism $W \rightarrow \Sigma(W) \subseteq \alpha^{-1}(\beta(g))$. Moreover, κ is a chart and the right translation R_g is a diffeomorphism. Hence $\psi := \kappa \circ R_g \circ \Sigma|_W: W \rightarrow E$ is a diffeomorphism onto its (open) image, mapping $W \subseteq T_{\tau(\beta(g))}^\alpha G$ to an open subset in E . Now (7) yields a commutative diagram

$$\begin{array}{ccccc} \text{Bis}(\mathcal{G}) \supseteq O_\tau & \xrightarrow{\gamma_g} & \alpha^{-1}(\alpha(g)) & \xrightarrow{\kappa} & E \\ (\text{id}_U, \psi) \circ I_\lambda \circ \varphi_\tau \downarrow & & & & \parallel \\ U \times \psi(W) & \xrightarrow{\text{pr}_2} & & & E \end{array}$$

from which we conclude that γ_g is on O_τ a projection. As $\tau \in \text{Bis}(\mathcal{G})$ was chosen arbitrarily, γ_g is a submersion. ■

The following corollary is now an immediate consequence of (4) and Proposition 2.4:

Corollary 2.5. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, then for each $m \in M$ the evaluation map*

$$\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m), \quad \sigma \mapsto \sigma(m)$$

is a submersion.

Furthermore, we can establish the existence of bisections through certain arrows which coincide with the object inclusion outside of pre-chosen open sets.

Lemma 2.6. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $m \in M$ fixed. For any m -neighbourhood $U \subseteq M$ there exists a 1_m -neighbourhood $W \subseteq \alpha^{-1}(m) \cap \beta^{-1}(U)$ such that for each $g \in W$ there is a bisection $\sigma_g \in \text{Bis}(\mathcal{G})$ with $\sigma_g(m) = g$ and $\sigma_g(y) = 1_y$ for all $y \in M \setminus U$.*

Proof. Choose a C^∞ -function $\lambda: M \rightarrow [0, 1]$ such that $\lambda(m) = 1$ and $\lambda|_{M \setminus U} \equiv 0$. We denote by

$$\varphi_1: \Gamma(1^*T^\alpha G) \supseteq \Omega \rightarrow \text{Bis}(\mathcal{G}), X \mapsto \Sigma \circ X$$

the canonical manifold chart of $\text{Bis}(\mathcal{G})$ (see Theorem 1.4). Computing with local representatives, it is easy to see that the map $m_\lambda: \Gamma(1^*T^\alpha G) \rightarrow \Gamma(1^*T^\alpha G), X \mapsto (x \mapsto \lambda(x) \cdot X(x))$ is continuous linear. Hence, there is an open zero-neighbourhood $P \subseteq \Omega \subseteq \Gamma(1^*T^\alpha G)$ with $m_\lambda(P) \subseteq \Omega$.

Now define $W := \text{ev}_m(\varphi_1(P)) \cap \beta^{-1}(U)$. Observe that W is an open 1_m -neighbourhood in $\alpha^{-1}(m) \cap \beta^{-1}(U) \subseteq \alpha^{-1}(m)$ since P is open and ev_m is a submersion by Corollary 2.5. Moreover, for $g \in W$ we have $g = s_g(m)$ for some $s_g \in \varphi_1(P)$. Define $X_g := m_\lambda \circ \varphi_1^{-1}(s_g)$ for each $g \in W$. By construction X_g is contained in Ω since m_λ takes the section $\varphi_1^{-1}(s_g) \in P$ to Ω . Hence $\sigma_g := \varphi_1(X_g)$ makes sense and is a bisection of \mathcal{G} . From $\lambda(m) = 1$ we derive that $\sigma_g(m) = s_g(m) = g$. Moreover, for $x \in M \setminus U$ we obtain by definition of a local addition

$$\sigma_g(x) = \Sigma(\underbrace{\lambda(x)}_{=0} \varphi_1^{-1}(s_g)(x)) = \Sigma(0_{T_x^\alpha G}) = 1_x.$$

Hence $\sigma_g(x) = 1_x$ for all $x \in M \setminus U$. \blacksquare

We can now prove a variant of [ZCL09, Theorem 3.2] for infinite-dimensional Lie groupoids over compact base. The proof of loc.cit. carries over verbatim if one uses Lemma 2.6, whence we omit it.

Proposition 2.7. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $g \in \alpha^{-1}(m)$ for $m \in M$. Suppose that $W \subseteq \alpha^{-1}(m)$ is connected and contains g and 1_m and there is $U \subseteq M$ with $\beta(W) \subseteq U$. Then there exists $\sigma_g \in \text{Bis}(\mathcal{G})$ with $\sigma_g(m) = g$ and $\sigma_g(x) = 1_x$ for all $x \in M \setminus U$.*

Having dealt with the pointwise evaluation, we will now use the results obtained so far to also show that the joint evaluation is a submersion.

Proposition 2.8. *Let M be a compact manifold, Q be a locally metrisable manifold and $s: Q \rightarrow M$ be a submersion such that Q admits a local addition that is adapted to s . Then the joint evaluation map*

$$\text{ev}: \Gamma(M \overset{s}{\leftarrow} Q) \times M \rightarrow Q, \quad (\sigma, m) \mapsto \sigma(m) \tag{8}$$

is a submersion. Here, $\Gamma(M \overset{s}{\leftarrow} Q) := \{\sigma \in C^\infty(M, Q) \mid s \circ \sigma = \text{id}_M\}$ is the space of sections of s , which is a submanifold of $C^\infty(M, Q)$ by [SW15b, Proposition 3.6].

Proof. Let $(\sigma_0, m_0) \in \Gamma(M \overset{s}{\leftarrow} Q) \times M$ be arbitrary, but fixed from now on. Set $q_0 := \sigma_0(m_0)$. Then there exist open neighbourhoods $O \subseteq M$ of m_0 and $P \subseteq Q$ of q_0 such that $\sigma_0(M) \subseteq P$ and $s|_{s^{-1}(O)} \cong \text{pr}_1$. Indeed, there exist open neighbourhoods P' of q_0 and O' of m_0 such that $s|_{P'} \cong \text{pr}_1$ and $s(P') = O'$. After shrinking O' if necessary, we may assume that $\sigma(O') \subseteq P'$. Then we choose O to be an open neighbourhood O with $\overline{O} \subseteq O'$ and set $P := s^{-1}(M \setminus \overline{O}) \cup P'$.

From this it follows that $C^\infty(M, P) \cap \Gamma(M \overset{s}{\leftarrow} Q)$ is an open neighbourhood of M . Observe that $s|_P: P \rightarrow M$ is also a submersion and that the adapted local addition on Q restricts to an adapted local addition on P . Consequently, the manifold structure on

$$C^\infty(M, P) \cap \Gamma(M \overset{s}{\leftarrow} Q) = \Gamma(M \overset{s|_P}{\leftarrow} P)$$

that is induced from $\Gamma(M \overset{s}{\leftarrow} Q)$ on the left hand side coincides with the manifold structure on the right hand side that is induced from applying [SW15b, Proposition 3.6] to the submersion $s|_P$. Thus it suffices to consider the case where $Q = P$ and $s = s|_P$.

We now reduce the claimed submersion property of (8) to the case of the evaluation in m_0 . Since the evaluation map $\text{ev}_{m_0}: \text{Diff}(M) \rightarrow M$ is also a submersion, there exists a local smooth section of it, i.e., an open neighbourhood $U_0 \subseteq M$ of m_0 and a smooth map $\beta: U_0 \rightarrow \text{Diff}(M)$ such that $\beta(m_0) = \text{id}_M$ and $\beta(m)(m_0) = m$ for all $m \in U_0$. Moreover, we may assume that $U_0 \subseteq O$ and β takes values in the identity neighbourhood $\varphi_{\text{id}}(\Omega)$ where φ_{id} is the chart of $\text{Diff}(M)$ from Theorem B.9 a).

Consider now $\varphi_{\text{id}} \circ \beta: U_0 \rightarrow \Gamma(TM)$. We choose a and a compact m_0 -neighbourhood K and a neighbourhood $\Omega \subseteq TO$ of the zero-section which is mapped by the local addition on M (which was used to define φ_{id}) to O . For later use, we shrink Ω to achieve that $\Omega \cap T_x M$ is convex for each $x \in O$. Then the open set $[K, \Omega] := \{X \in \Gamma(TM) \mid X(K) \subseteq \Omega\} \subseteq \Gamma(TM)$ is a zero-neighbourhood. Further, φ_{id}^{-1} maps $[K, \Omega] \cap (\varphi_{\text{id}}^{-1})^{-1}(\text{Diff}(M))$ to the set of diffeomorphisms which map K into O . Shrinking U_0 we can achieve that $\varphi_{\text{id}} \circ \beta$ takes its image in $[K, \Omega]$, i.e. the diffeomorphisms in the image of β map K into O .

Apply now the exponential law [Woc13, Corollary 7.5] (cf. Theorem B.9 c) to obtain a smooth map

$$\gamma := (\varphi_{\text{id}} \circ \beta)^\vee : U_0 \times M \rightarrow TM, (u, m) \mapsto \varphi_{\text{id}} \circ \beta(u)(m) \text{ with } \gamma(\cdot, m) \in T_m M \quad \forall m \in M.$$

Note that $\gamma(m_0, \cdot)$ coincides with the zero-section as $\beta(m_0) = \text{id}_M$. Choose a smooth cutoff function $\rho : M \rightarrow [0, 1]$ which maps a m_0 -neighbourhood $N \subseteq K$ to 1 and vanishes near the boundary ∂K . Multiplying fibre-wise we obtain a smooth map $\tilde{\gamma} : U_0 \times M \rightarrow TM, (u, m) \mapsto \rho(m) \cdot \gamma(u, m)$ which vanishes near the boundary of K . Apply the exponential law in reverse to obtain a smooth map $\tilde{\gamma}^\wedge : U_0 \rightarrow \Gamma(TM)$.

Now recall that $(\varphi_{\text{id}}^{-1})^{-1}(\text{Diff}(M))$ is an open set in the compact open C^1 -topology on $\Gamma(TM)$ (see [Mic80, 4.3] and [KM97, proof of Theorem 43.1]). Hence we can choose suitable convex zero-neighbourhoods which control only the values of $X \in \Gamma(TM)$ and TX on suitable compact sets, such that their intersection is contained in $(\varphi_{\text{id}}^{-1})^{-1}(\text{Diff}(M))$. Since $\gamma(m_0, \cdot)$ is the zero-section, we can thus shrink U_0 to achieve that $\tilde{\gamma}^\wedge$ is still contained in $[K, \Omega] \cap (\varphi_{\text{id}}^{-1})^{-1}(\text{Diff}(M))$. As $\tilde{\gamma}^\wedge$ vanishes near the boundary of K (and outside of K), we can replace β with a map which satisfies $\beta(m)(O) \subseteq O$ for all $m \in U_0$.

Choose another smooth cutoff function δ which vanishes near the boundary ∂U_0 and takes an open m_0 -neighbourhood $U \subseteq N$ to 1. Multiplying (in local charts) $\varphi_{\text{id}} \circ \beta$ with δ , we can assume that $\beta(m) = \text{id}_M$ for all m near the boundary ∂U_0 . Thus we can extend β to a smooth function $\beta : M \rightarrow \text{Diff}(M)$ by setting $\beta(m) = \text{id}_M$ if $m \notin U_0$. By construction this map satisfies $\beta(m)(O) \subseteq O$ for all $m \in M$. Moreover, since ρ and δ take the m_0 -neighbourhood $U \cap N$ to 1, β satisfies $\beta(m)(m_0) = m$ for all $m \in U \cap N$.

Choose a diffeomorphism $\xi : s^{-1}(O) \rightarrow O \times W$ that makes

$$\begin{array}{ccc} s^{-1}(O) & \xrightarrow{\xi} & O \times W \\ & \searrow s & \swarrow \text{pr}_1 \\ & & O \end{array}$$

commute, we may lift each $\beta(m)$ to a diffeomorphism

$$\tilde{\beta}_m : Q \rightarrow Q, \quad q \mapsto \begin{cases} \xi^{-1}(\beta(m)(o), w) & \text{if } q = \xi^{-1}(o, w) \in s^{-1}(O) \\ q & \text{if } q \notin s^{-1}(O) \end{cases}.$$

From this explicit construction it follows in particular, that the map

$$Q \times M \rightarrow Q, \quad (q, m) \mapsto \tilde{\beta}(m)(q)$$

is smooth. This then gives rise to the diffeomorphism

$$B : \Gamma(M \overset{s}{\leftarrow} Q) \times M \rightarrow \Gamma(M \overset{s}{\leftarrow} Q) \times M, \quad (\sigma, m) \mapsto ((\tilde{\beta}(m))^{-1} \circ \sigma \circ \beta(m), m).$$

with inverse $(\sigma, m) \mapsto (\tilde{\beta}(m) \circ \sigma \circ (\beta(m))^{-1}, m)$. Moreover, $\beta(m_0) = \text{id}_M$ implies $B(\sigma_0, m_0) = (\sigma_0, m_0)$.

For the next step, let $\Sigma : TQ \supseteq \Omega \rightarrow Q$ be an s -adapted local addition with corresponding open σ_0 -neighbourhood $O_{\sigma_0} \subseteq C^\infty(M, Q)$ and let $\varphi_{\sigma_0} : O_{\sigma_0} \rightarrow \Gamma(\sigma_0^*(TQ))$ be the chart from Theorem B.9 a). Recall from the proof of [SW15b, Proposition 3.6] that φ_{σ_0} also is a submanifold chart for $\Gamma(M \overset{s}{\leftarrow} Q)$ that maps the open neighbourhood $O_{\sigma_0} \cap \Gamma(M \overset{s}{\leftarrow} Q)$ onto an open neighbourhood of the zero section in the closed subspace $\{\tau \in \Gamma(\sigma_0^*(TQ)) \mid \tau(m) \in T_{\sigma_0(m)} s^{-1}(m) \text{ for all } m \in M\}$ of $\Gamma(\sigma_0^*(TQ))$.

Denote $R_0 := \varphi_{\sigma_0}(O_{\sigma_0} \cap \Gamma(M \overset{s}{\leftarrow} Q))$. Then we define the diffeomorphism

$$\Xi : B^{-1}(O_{\sigma_0} \cap \Gamma(M \overset{s}{\leftarrow} Q)) \times O \rightarrow R_0 \times O, \quad (f, m) \mapsto (\varphi_{\sigma_0}((\tilde{\beta}(m))^{-1} \circ f \circ \beta(m)), m).$$

By shrinking $B^{-1}(O_{\sigma_0} \times O)$ if necessary, we may assume that ev maps the open neighbourhood $(B^{-1}(O_{\sigma_0} \times O))$ of (σ_0, m_0) into the open neighbourhood $s^{-1}(O)$ of q_0 . Moreover, the following diagram commutes by the definitions of φ_{σ_0} and of $\tilde{\beta}(m)$

$$\begin{array}{ccc} B^{-1}(O_{\sigma_0} \times O) & \xrightarrow{\text{ev}} s^{-1}(O) & \xrightarrow{\xi} W \times O \\ \Xi \downarrow & & \uparrow \Sigma \times \text{id}_O \\ R_0 \times O & \xrightarrow{\text{ev}_{m_0} \times \text{id}_O} & T_{q_0} s^{-1}(m_0) \times O. \end{array}$$

Since W is diffeomorphic to $s^{-1}(m_0)$ and Σ restricts to a local diffeomorphism of a zero-neighbourhood in $T_{q_0}s^{-1}(m_0)$ onto an open neighbourhood of q_0 in $s^{-1}(m_0)$, it follows that the arrow on the right is a local diffeomorphism. Hence each morphism except ev and $\text{ev}_{m_0} \times \text{id}_O$ in this diagram is a local diffeomorphism. It will thus follow from showing that ev_{m_0} is a submersion and [Glö15a, Lemma 1.6] that ev is one.

The map ev_{m_0} is defined on an open zero neighbourhood of the space

$$E_{\sigma_0} := \{\tau \in \Gamma(\sigma_0^*(TQ)) \mid \tau(m) \in T_{\sigma_0(m)}s^{-1}(m) \text{ for all } m \in M\}$$

and takes values in $T_{q_0}s^{-1}(m_0)$.

By using the diffeomorphism $\xi: s^{-1}(O) \rightarrow W \times O$ we have the following identifications

$$T_{q_0}s^{-1}(m_0) \cong T_{(w_0, m_0)}\xi(s^{-1}(m_0)) \cong F$$

where F is the modelling space of W and $(w_0, m_0) := \xi(q_0)$. Using these and a cutoff function, one can build as in Remark 2.3 a continuous inverse to ev_{m_0} that takes $T_{q_0}s^{-1}(m_0)$ into E_{σ_0} . Thus ev_{m_0} is a submersion, finishing the proof. ■

Corollary 2.9. *Let M be a compact manifold and N be a locally metrisable manifold that admits a local addition. Then the joint evaluation map*

$$\text{ev}: C^\infty(M, N) \times M \rightarrow N, \quad (f, m) \mapsto f(m) \tag{9}$$

is a submersion.

Proof. We have the natural identification $C^\infty(M, N) \cong \Gamma(M \xleftarrow{\text{pr}_2} N \times M)$. Thus Proposition 2.8 shows that

$$C^\infty(M, N) \times M \rightarrow N \times M, \quad (f, m) \mapsto (f(m), m)$$

is a submersion. Now (9) is a submersion, since it is the composition to two submersions. ■

Corollary 2.10. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then the joint evaluation map*

$$\text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow G, \quad (\sigma, m) \mapsto \sigma(m)$$

is a submersion.

Proof. This is implied by Proposition 2.8 since $\text{Bis}(\mathcal{G})$ is an open submanifold of $\Gamma(M \xleftarrow{\alpha} G)$. ■

Corollary 2.11. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then the division morphism*

$$\delta: \text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \times M \rightarrow G, \quad (\sigma, \tau, m) \mapsto (\sigma \star \tau^{-1})(m)$$

and for $m \in M$ the restricted division $\delta_m: \text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$, $\delta_m(\sigma, \tau) := \delta(\sigma, \tau, m)$ are submersions.

Proof. Note that we can write $\delta(\sigma, \tau, m) = \text{ev}(\sigma \star \tau^{-1}, m)$ and $\delta_m(\sigma, \tau) = \text{ev}_m(\sigma \star \tau^{-1})$. Since ev and ev_m are submersions by Corollary 2.5 and Corollary 2.10, it suffices to prove that the map

$$f: \text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \rightarrow \text{Bis}(\mathcal{G}), \quad (\sigma, \tau) \mapsto \sigma \star \tau^{-1}$$

is a submersion. However, as $(\text{Bis}(\mathcal{G}), \star)$ is a Lie group the map f is a submersion. ■

We have now established that the evaluation map from the bisections onto the manifold of arrows is a submersion. However, to completely determine the manifold of arrows, we need ev to be surjective. Note that this means that there is a (global) bisection through each point in G . Consider first an easy example

Example 2.12. Let H be a Lie group modelled on a metrisable space which acts on the compact manifold M , i.e. the associated action groupoid $H \ltimes M$ admits an adapted local addition by Remark 2.2. Then the evaluation $\text{ev}: \text{Bis}(H \ltimes M) \times M \rightarrow H \times M$, $(\sigma, m) \mapsto (\sigma(m), m)$ is a surjective submersion. We already know from Corollary 2.10 that ev is a submersion and thus have to establish only surjectivity. For each pair $(h, m) \in H \times M$ we can define the constant bisection $c_h: M \rightarrow H \times M, n \mapsto (h, n)$ which is contained in $\text{Bis}(H \ltimes M)$. Hence $\text{ev}(c_h, m) = (h, m)$ and thus ev is surjective. In particular, for each arrow $g \in H \times M$ in the action groupoid \mathcal{G} there is a global bisection σ_g with $\sigma_g(\alpha(g)) = g$.

The structure of $\text{Bis}(H \ltimes M)$ is interesting in its own. Since the second component of a bisection $\sigma: M \rightarrow H \times M$ has to be the identity, $\text{Bis}(H \ltimes M)$ can be identified with the subset

$$\{\gamma \in C^\infty(M, H) \mid m \mapsto \gamma(m).m \text{ is a diffeomorphism of } M\}$$

of $C^\infty(M, H)$. Since $C^\infty(M, H) \rightarrow C^\infty(M, M)$, $\gamma \mapsto (m \mapsto \gamma(m).m)$ is smooth and $\text{Diff}(M) \subseteq C^\infty(M, M)$ is open, it follows that $\text{Bis}(M \ltimes H)$ is an open submanifold of $C^\infty(M, H)$ that contains the constant maps. However, the group structure on the functions from M to H is not given by the pointwise multiplication, but by

$$(\gamma \star \eta)(m) := \gamma(\eta(m).m) \cdot \eta(m).$$

One effect of this is that $\gamma \mapsto (m \mapsto \gamma(m).m)$ is a homomorphism $\text{Bis}(H \ltimes M) \rightarrow \text{Diff}(M)$. ■

In general there will not be a bisection through each arrow of a given groupoid (see Remark 2.18 b) below). Nevertheless, for source connected finite-dimensional Lie groupoids it is known (see [ZCL09]) that bisections through each arrow exist. We will now generalise this result to our infinite-dimensional setting.

Definition 2.13. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid.

- a) Denote for $m \in M$ by C_m the connected component of 1_m in $\alpha^{-1}(m)$. Then we define the subset $C(\mathcal{G}) := \bigcup_{m \in M} C_m$. By [Mac05, Proposition 1.5.1] we obtain a wide subgroupoid $C(\mathcal{G}) \rightrightarrows M$ of \mathcal{G} , called the *identity-component subgroupoid* of \mathcal{G} .³
- b) The groupoid \mathcal{G} is called α - or *source connected* if for each $m \in M$ the fibre $\alpha^{-1}(m)$ is connected.

Observe that for an α -connected groupoid \mathcal{G} we have $C(\mathcal{G}) = \mathcal{G}$. ■

Note that Proposition 2.7 implies that for α -connected Lie groupoids there is for every arrow a bisection whose image contains the given arrow. However, we give an alternative proof in the following theorem, which also yields more information:

Theorem 2.14. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally metrisable Lie groupoid with compact M that admits an adapted local addition.

- a) The image of the evaluation ev is an open and wide Lie subgroupoid which contains the identity subgroupoid $C(\mathcal{G})$.
- b) The identity subgroupoid $C(\mathcal{G})$ coincides with $\text{ev}(\text{Bis}(\mathcal{G})_0 \times M)$, where $\text{Bis}(\mathcal{G})_0$ is the identity component of $\text{Bis}(\mathcal{G})$. Hence, $C(\mathcal{G})$ is an open Lie subgroupoid of \mathcal{G} .

Assume in addition that \mathcal{G} is α -connected, then

- c) For each $g \in G$ there is a bisection $\sigma_g \in \text{Bis}(\mathcal{G})_0$ with $\sigma_g(\alpha(g)) = g$. In particular, ev is surjective.

Proof. a) The image $\mathcal{U} := \text{ev}(\text{Bis}(\mathcal{G}) \times M)$ contains the image of the object inclusion $1: M \rightarrow G$, i.e. $1_m \in \mathcal{U}$ for all $m \in M$. Define for $m \in M$ the set $\mathcal{U}_m = \mathcal{U} \cap \alpha^{-1}(m)$ and note that $\mathcal{U}_m = \text{ev}_m(\text{Bis}(\mathcal{G}))$. As $\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ is a submersion by Corollary 2.5 we infer that \mathcal{U}_m is an open subset of $\alpha^{-1}(m)$.

³Note that at this stage we do not know that $C(\mathcal{G})$ is a Lie subgroupoid. In particular, the proof for finite dimensional Lie groupoids (see [Mac05, Proposition 1.5.1]) does not carry over to our setting. Compare however Theorem 2.14 b).

By construction of the group operations of $\text{Bis}(\mathcal{G})$ the set \mathcal{U} yields a wide subgroupoid $\mathcal{U} \rightrightarrows M$ of \mathcal{G} such that \mathcal{U}_m is open in $\alpha^{-1}(m)$. Hence the image of ev is an open and wide Lie subgroupoid. Now [Mac05, Proposition 1.5.7]⁴ shows that \mathcal{U}_m is also closed in $\alpha^{-1}(m)$. Thus the clopen set \mathcal{U}_m contains the connected component of $1_m \in \alpha^{-1}(m)$. Since this holds for each m , we see that $C(\mathcal{G}) \subseteq \mathcal{U}$.

b) Set $B_0 := \text{Bis}(\mathcal{G})_0$. As B_0 is an open subgroup, an argument as in a) shows that $\text{ev}(B_0 \times M)$ is a subgroupoid of \mathcal{G} which contains $C(\mathcal{G})$.

Furthermore, for each $m \in D_i$ the set $\text{ev}_m(B_0) \subseteq \alpha^{-1}(m)$ is connected and contains 1_m . Thus by definition of the connected set $C_m \subseteq \alpha^{-1}(m)$ we have $\text{ev}_m(B_0) \subseteq C_m \subseteq C(\mathcal{G})$. Hence, $\text{ev}(B_0 \times M) = C(\mathcal{G})$ and since ev is a submersion by Corollary 2.10, $C(\mathcal{G})$ is open in G . In particular, $C(\mathcal{G})$ is an open subgroupoid of \mathcal{G} , i.e. it is an open Lie subgroupoid.

c) If \mathcal{G} is α -connected then $C(\mathcal{G}) = G$ whence the assertion follows from b). ■

Definition 2.15. We say that a for a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ there exists a *bisection through each arrow* if \mathcal{G} satisfies the condition of Theorem 2.14 c), i.e. for each $g \in G$ there exists $\sigma_g \in \text{Bis}(\mathcal{G})$ with $\sigma_g(\alpha(g)) = g$. ■

Note that Part c) of Theorem 2.14 yields [ZCL09, Theorem 3.1] as a corollary for (finite-dimensional) Lie groupoids over a compact base.

Corollary 2.16. *In a finite-dimensional source-connected Lie groupoid with compact space of objects there exist bisections through each arrow.*

We also obtain the following well known result on the natural action of $\text{Diff}(M)$ on M (cf. [Ban97, MV94]).

Corollary 2.17. *If M is a compact and connected manifold, then $\text{Diff}(M)_0$ acts transitively on M .*

Remark 2.18. a) Note that [ZCL09, Theorem 3.1] holds for arbitrary finite-dimensional α -connected Lie groupoids whereas Corollary 2.16 is limited to groupoids over compact base.

b) The assumption on \mathcal{G} to be source connected cannot be dispensed with. For instance, if N, N' are non-diffeomorphic compact manifolds (of the same dimension), then the pair groupoid $\mathcal{P}(M) := (M \times M \rightrightarrows M)$ of $M := N \sqcup N'$ has $\text{Bis}(\mathcal{P}(M)) \cong \text{Diff}(M)$ and the action of $\text{Bis}(\mathcal{P}(M))$ on the source fibre naturally identifies with the natural action of $\text{Diff}(M)$ on M . But since N, N' are not diffeomorphic, there cannot exist a diffeomorphism of M that interchanges the points n and n' if $n \in N$ and $n' \in N'$. Consequently, there cannot exist a bisection through the morphism $((n, n'), (n', n))$ of $\mathcal{P}(M)$.

However, as we have seen in Example 2.12, there exist non-source connected Lie groupoids, for which there exist bisections through each point. For another example consider the gauge groupoid $\mathcal{G} = \text{Gauge}(M \times K)$ of the trivial principal bundle $M \times K \rightarrow M$, then $\text{Bis}(\text{Gauge}(M \times K)) \cong \text{Aut}(M \times K) \cong C^\infty(M, K) \rtimes \text{Diff}(M)$. If M is connected, then $\text{Diff}(M)$ acts transitively on M and $C^\infty(M, K)$ always acts transitively on K . Thus there exist a bisection through each arrow of $\text{Gauge}(M \times K)$, even if K is not connected. ■

Before we continue with our investigation of the bisection action groupoid, note the following interesting consequences of Corollary 2.5.

Lemma 2.19. *Let \mathcal{G} be locally trivial and denote by $\theta: \text{Bis}(\mathcal{G}) \times M \rightarrow M, (\sigma, x) \mapsto \beta_{\mathcal{G}} \circ \sigma(m)$ the canonical Lie group action. Then*

a) θ restricts for each $m \in M$ to a submersion $\theta_m: \text{Bis}(\mathcal{G}) \times \{m\} \rightarrow M$.

If in addition \mathcal{G} admits bisections through each arrow or M is connected, then

b) $\mathcal{B}(\mathcal{G})$ is locally trivial and θ is transitive. ■

⁴Loc.cit. considers only finite-dimensional Lie groupoids. However, the proof of this result carries over verbatim to the infinite-dimensional setting.

Proof. As \mathcal{G} is locally trivial $\beta_{\mathcal{G}}|_{\alpha^{-1}(m)} : \alpha^{-1}(m) \rightarrow M$ is a surjective submersion. Since the $\alpha_{\mathcal{B}}$ -fibre is $\text{Bis}(\mathcal{G})$ and $\text{ev}_m : \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ is a submersion by Corollary 2.5 we see that $\beta_{\mathcal{B}}|_{\alpha_{\mathcal{B}}^{-1}(m)} = \beta_{\mathcal{G}} \circ \text{ev}_m$ is a submersion. We claim that $\beta_{\mathcal{B}}$ is surjective. If this is true then $\mathcal{B}(\mathcal{G})$ is locally trivial by [Mac05, Proposition 1.3.3] which carries over verbatim to our infinite-dimensional setting. Moreover, we derive that the canonical action of $\text{Bis}(\mathcal{G})$ is transitive and restricts to a submersion $\text{Bis}(\mathcal{G}) \times \{m\} \rightarrow M$ for all $m \in M$

To prove the claim we have to treat both cases separately. Assume first that \mathcal{G} admits bisections through each arrow, then ev_m is surjective and thus $\beta_{\mathcal{B}}$ is surjective. On the other hand let now M be connected. Then we note that $\beta_{\mathcal{B}} = \widetilde{\text{ev}}_m \circ (\beta_{\mathcal{G}})_*$, where $(\beta_{\mathcal{G}})_* : \text{Bis}(\mathcal{G}) \rightarrow \text{Diff}(M), \sigma \mapsto \beta_{\mathcal{G}} \circ \sigma$ and $\widetilde{\text{ev}}_m : \text{Diff}(M) \rightarrow M, \varphi \mapsto \varphi(m)$. Now the image of $(\beta_{\mathcal{G}})_*$ contains the identity component $\text{Diff}(M)_0$ of $\text{Diff}(M)$ (by [SW15b, Example 3.16]) and $\text{Diff}(M)_0$ acts transitively on the connected manifold M by Corollary 2.17. Thus $\beta_{\mathcal{B}}$ is surjective.

We conclude that in both cases the assertion holds. \blacksquare

Remark 2.20. Quotient constructions for Lie groupoids (and already for Lie groups) are quite tricky. In fact, Lie groupoids are a tool to circumvent badly behaved quotients (for instance for non-free group actions). However, each category carries a natural notion of quotient object for an internal equivalence relation. If \mathcal{C} is a category with finite products and $R \subseteq E \times E$ is an internal equivalence relation, then the quotient $E \rightarrow E/R$ in \mathcal{C} (uniquely determined up to isomorphism) is, if it exists, the coequaliser of the diagram

$$R \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} E. \quad (10)$$

If, in the case that the quotient exists, (10) is also the pull-back of $E \rightarrow E/R$ along itself, then the quotient $E \rightarrow E/R$ is called *effective* (see [MLM94, Appendix.1] for details). We want to apply this to the category LieGroupoids_M , whose objects are locally convex and locally metrisable Lie groupoids over M and whose morphisms are smooth functors that are the identity on M . Note that the product of two Lie groupoids $(G \rightrightarrows M)$ and $(H \rightrightarrows M)$ is given by restricting the product Lie groupoid $(G \times H \rightrightarrows M \times M)$ to the diagonal $M \cong \Delta M \subseteq M \times M$. \blacksquare

Theorem 2.21. *If $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid with a bisection through each arrow in G , e.g. \mathcal{G} is source connected, then the morphism $\text{ev} : \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{G}$ is the quotient of $\mathcal{B}(\mathcal{G})$ in LieGroupoids_M by*

$$R = \{(\sigma, m), (\tau, m) \in \text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \times M \mid \sigma(m) = \tau(m)\}.$$

Proof. We first note that R is isomorphic to $K \times \text{Bis}(\mathcal{G})$, where

$$K := \{(\sigma, m) \in \text{Bis}(\mathcal{G}) \times M \mid \sigma(m) = 1_m\} = \text{ev}^{-1}(M) \quad (11)$$

is the kernel of ev . To see this note that as $M \subseteq G$ is a closed submanifold and ev is a submersion by Corollary 2.10, it follows that K is a closed submanifold of $\text{Bis}(\mathcal{G}) \times M$. Now

$$\text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \times M \rightarrow \text{Bis}(\mathcal{G}) \times M \times \text{Bis}(\mathcal{G}), \quad ((\sigma, m), (\tau, m)) \mapsto (\sigma \star \tau^{-1}, m, \tau)$$

is a diffeomorphism which maps R onto $K \times \text{Bis}(\mathcal{G})$. Consequently, R is a closed submanifold of $\text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \times M$.

We now argue that R is in fact an internal equivalence relation. It is clear that the relation is reflexive and symmetric. After applying the diffeomorphism (11), the second projection $\text{pr}_2 : R \rightarrow \text{Bis}(\mathcal{G}) \times M, ((\sigma, m), (\tau, m)) \mapsto (\tau, m)$ is an actual projection. So pr_2 is a surjective submersion. Thus the pull-back

$$R * R := R \times_{(\text{Bis}(\mathcal{G}) \times M)} R = \{((\sigma, m), (\tau, m)), ((\sigma', m'), (\tau', m')) \mid m = m', \tau = \sigma'\}$$

is a submanifold of $R \times R$ and

$$R * R \rightarrow \text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) \times M, \quad (((\sigma, m), (\tau, m)), ((\tau, m), (\tau', m))) \mapsto ((\sigma, m), (\tau', m))$$

clearly factors through R . Consequently, R is an internal equivalence relation.

Finally, if $f: \text{Bis}(\mathcal{G}) \times M \rightarrow H$ is a morphism of Lie groupoids that satisfies $f \circ \text{pr}_1 = f \circ \text{pr}_2$ (for $\text{pr}_i: R \rightarrow \text{Bis}(\mathcal{G}) \times M$ the canonical maps), then we construct a smooth map $\varphi: G \rightarrow H$ by taking a local smooth section of ev and composing it with f . Since $f \circ \text{pr}_1 = f \circ \text{pr}_2$, two possible pre-images of an element from G in $\text{Bis}(\mathcal{G}) \times M$ are mapped to the same element in H , and thus φ is well-defined and smooth by construction. One directly checks that it also defines a morphism of Lie groupoids (i.e., φ is compatible with the structure maps, see also [Mac05, Proposition 2.2.3]). ■

Remark 2.22. We have seen in Theorem 2.21 that a source connected Lie groupoid \mathcal{G} with compact base is the quotient of its associated bisection action Lie groupoid $\mathcal{B}(\mathcal{G})$. However, Theorem 2.21 already uses that a candidate for the quotient, namely \mathcal{G} exists. Thus the theorem does not provide the existence of the quotient without using \mathcal{G} . ■

In the proof of Theorem 2.21 it is visible that the quotient of $\mathcal{B}(\mathcal{G})$ was taken with respect to the kernel⁵ of the base-preserving morphism ev . We refer to [Mac05, 2.2] for details on the groupoid quotient by a normal subgroupoid. Since ev is a surjective submersion in the situation of Theorem 2.21, its kernel is a Lie subgroupoid of $\mathcal{B}(\mathcal{G})$. Finally, the α -fibre of the kernel over $m \in M$ is given by

$$\text{ev}(\cdot, m)^{-1}\{1_m\} = \{\sigma \in \text{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\}.$$

In the next section we will study the subgroups of $\text{Bis}(\mathcal{G})$ which arise from this construction. Later on these information will allow us to investigate the groupoid quotients in more detail.

3 Subgroups and quotients associated to the bisection group

In this section we study subgroups of the bisections which are associated to a fixed point in the base manifold. These subgroups are closely related to the reconstruction result outlined in Theorem 2.21 and will become an important tool to study locally trivial Lie groupoids in Section 4 and 5.

As before, (unless stated explicitly otherwise) we shall assume that $\mathcal{G} = (G \rightrightarrows M)$ is a locally metrisable Lie groupoid over a compact base M which admits an adapted local addition.

Definition 3.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid (we require neither that M is compact or finite dimensional nor that G admits a local addition). Fix $m \in M$ and denote by $\text{Vert}_m(\mathcal{G})$ the vertex subgroup of the groupoid \mathcal{G} . There are now two subsets of $\text{Bis}(\mathcal{G})$ whose elements are characterised by their value at m

$$\begin{aligned} \text{Loop}_m(\mathcal{G}) &:= \{\sigma \in \text{Bis}(\mathcal{G}) \mid \sigma(m) \in \text{Vert}_m(\mathcal{G}) = \alpha^{-1}(m) \cap \beta^{-1}(m)\}, \\ \text{Bis}_m(\mathcal{G}) &:= \{\sigma \in \text{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\}. \end{aligned}$$

Clearly $\text{Bis}_m(\mathcal{G}) \subseteq \text{Loop}_m(\mathcal{G})$ and both are subgroups of $\text{Bis}(\mathcal{G})$ by definition of the group operation (see (2) and (3)). ■

Note that $\text{Bis}_m(\mathcal{G})$ is a normal subgroup of $\text{Loop}_m(\mathcal{G})$ as for $\sigma \in \text{Bis}_m(\mathcal{G})$ and $\tau \in \text{Loop}_m(\mathcal{G})$ we have

$$\begin{aligned} \tau \star \sigma \star \tau^{-1}(m) &= \tau \star (\sigma \circ \beta \circ \tau)(m) = \tau(\underbrace{\beta \circ \sigma \circ \beta \circ \tau(m)}_{=m}) \cdot \underbrace{(\sigma(\beta \circ \tau(m)))}_{=m} \cdot \tau(m) \\ &= \tau(m) \cdot 1_m \cdot \tau^{-1}(m) = 1_m. \end{aligned}$$

We will now investigate the subgroups from Definition 3.1 in the case that $\text{Bis}(\mathcal{G})$ is a Lie group. Thus M will be assumed to be compact, whence Lemma A.4 implies that the vertex group $\text{Vert}_m(\mathcal{G})$ of $\mathcal{G} = (G \rightrightarrows M)$ is a submanifold of G and in particular a Lie group.

⁵i.e. the set $\{g \in \mathcal{B}(\mathcal{G}) \mid \text{ev}(g) = 1_x \text{ for some } x \in M\}$ which is a wide subgroupoid of the inner subgroupoid, called a normal Lie subgroupoid [see [Mac05, Definition 2.2.2)].

Proposition 3.2. *Fix some $m \in M$.*

- a) *The group $\text{Loop}_m(\mathcal{G})$ is a Lie subgroup of $\text{Bis}(\mathcal{G})$ and as a submanifold in $\text{Bis}(\mathcal{G})$ it is of finite codimension.*
- b) *The map $\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ restricts to a Lie group morphism $\psi_m: \text{Loop}_m(\mathcal{G}) \rightarrow \text{Vert}_m(\mathcal{G})$ whose kernel is $\text{Bis}_m(\mathcal{G})$. Moreover, ψ_m is a submersion.*
- c) *The group $\text{Bis}_m(\mathcal{G})$ is a split Lie subgroup of $\text{Loop}_m(\mathcal{G})$ and a split Lie subgroup of $\text{Bis}(\mathcal{G})$. It is modelled on $\Gamma(1^*T^\alpha G)_m = \{X \in \Gamma(1^*T^\alpha G) \mid X(1_m) = 0_{1_m}\}$.*
- d) *If \mathcal{G} is a Banach-Lie groupoid then $\text{Bis}_m(\mathcal{G})$ is a co-Banach submanifold in $\text{Loop}_m(\mathcal{G})$ and also in $\text{Bis}(\mathcal{G})$.*

Proof. a) Recall that by Corollary 2.5 the map $\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ is a submersion. Moreover, the vertex group $\text{Vert}_m(\mathcal{G})$ is a submanifold of finite codimension of $\alpha^{-1}(m)$ by Lemma A.4, Thus $\text{Loop}_m(\mathcal{G}) = \text{ev}_m^{-1}(\text{Vert}_m(\mathcal{G}))$ is a submanifold of finite codimension in $\text{Bis}(\mathcal{G})$ by [Glö15a, Theorem C]. Now $\text{Loop}_m(\mathcal{G})$ is a subgroup and a split submanifold of $\text{Bis}(\mathcal{G})$ whence a split Lie subgroup.

b) To see that ψ_m is also a group homomorphism we pick $\sigma, \tau \in \text{Loop}_m(\mathcal{G})$ and compute

$$\psi_m(\sigma \star \tau) = \psi_m((\sigma \circ \beta \circ \tau) \cdot \tau) = \sigma(\beta(\tau(m))) \cdot \tau(m) = \sigma(m) \cdot \tau(m) = \psi_m(\sigma) \cdot \psi_m(\tau).$$

As $\text{ev}_m(\sigma) = 1_m$ if and only if $\sigma(m) = 1_m$, we see that $\text{Bis}_m(\mathcal{G})$ is the kernel of ψ_m . Having applied [Glö15a, Theorem C] in part a), we observe that this also entails that ψ_m is a submersion.

c) By part b) the subgroup $\text{Bis}_m(\mathcal{G})$ is the preimage of a singleton under a submersion, whence a split submanifold of $\text{Loop}_m(\mathcal{G})$ by the regular value theorem [Glö15a, Theorem D]. In particular, $\text{Bis}_m(\mathcal{G})$ becomes a split Lie subgroup of $\text{Loop}_m(\mathcal{G})$. Since $\text{Loop}_m(\mathcal{G})$ is a split submanifold in $\text{Bis}(\mathcal{G})$, [Glö15a, Lemma 1.4] yields that $\text{Bis}_m(\mathcal{G})$ is a split submanifold of $\text{Bis}(\mathcal{G})$ and thus a split Lie subgroup of $\text{Bis}(\mathcal{G})$.

Recall that by the regular value theorem the tangent space of $\text{Bis}_m(\mathcal{G})$ at 1 is the kernel $\ker T_1 \text{ev}_m \subseteq T_1 \text{Bis}(\mathcal{G}) \cong \Gamma(1^*T^\alpha G)$. Taking identifications we compute $\ker T_1 \text{ev}_m$ as a subspace of $\Gamma(1^*T^\alpha G)$. On the level of isomorphism classes of curves the isomorphism $\varphi_{\mathcal{G}}: T_1 \text{Bis}(\mathcal{G}) \rightarrow \Gamma(1^*T^\alpha G)$ is given by $\varphi_{\mathcal{G}}([t \mapsto c(t)]) = (m \mapsto [t \mapsto c^\wedge(t, m)])$ (cf. [SW15b, Remark 4.1]). Hence $\varphi_{\mathcal{G}}$ identifies $T_1 \text{ev}_m$ with $\Gamma(1^*T^\alpha G) \rightarrow G, X \mapsto X(m)$ whose kernel is $\Gamma(1^*T^\alpha G)_m$.

d) If \mathcal{G} is a Banach-Lie groupoid then $\alpha^{-1}(m)$ and thus also $\text{Vert}_m(\mathcal{G})$ are manifolds modelled on Banach-spaces. In this situation the regular value theorem implies that $\text{Bis}_m(\mathcal{G})$ is a co-Banach submanifold of $\text{Loop}_m(\mathcal{G})$. Since $\text{Loop}_m(\mathcal{G})$ is of finite codimension in $\text{Bis}(\mathcal{G})$, we deduce that $\text{Bis}_m(\mathcal{G})$ is a co-Banach submanifold of $\text{Bis}(\mathcal{G})$. ■

An important property of the Lie subgroups constructed in Proposition 3.2 is that they are regular as Lie groups (we recall the definition of regularity for Lie groups in Appendix B). Namely, the subgroups $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ will be regular Lie groups if $\text{Bis}(\mathcal{G})$ is a regular Lie group. Let us first recall when bisection groups are regular.⁶

Remark 3.3. In [SW15b, Section 5] we established C^k -regularity of $\text{Bis}(\mathcal{G})$ if \mathcal{G} is either

- a) a Banach-Lie groupoid (then $\text{Bis}(\mathcal{G})$ is C^0 -regular).
- b) or a locally-trivial Lie groupoid whose vertex groups are locally exponential C^k -regular Lie groups (then $\text{Bis}(\mathcal{G})$ is C^k -regular) ■

We will now prove that $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ inherits the regularity properties from $\text{Bis}(\mathcal{G})$.

Proposition 3.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and fix $m \in M$. Assume that $\text{Bis}(\mathcal{G})$ is C^k -regular for some $k \in \mathbb{N}_0 \cup \{\infty\}$, e.g. in the situation of Remark 3.3. Let H be either $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ and consider $\eta \in C^k([0, 1], \mathbf{L}(H))$.*

⁶At the moment, no example of a non-regular Lie group modelled on a space with suitable completeness properties (i.e. Mackey completeness) is known.

a) The solution γ_η of the initial value problem

$$\begin{cases} \gamma'(t) &= \gamma(t) \cdot \eta(t) \quad \forall t \in [0, 1] \\ \gamma(0) &= 1 \end{cases} \quad (12)$$

in G takes its image in H .

b) The Lie groups $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ are C^k -regular

Proof. a) Fix $\eta \in C^k([0, 1], \mathbf{L}(H))$ together with the evolution $\gamma_\eta: [0, 1] \rightarrow \text{Bis}(\mathcal{G})$ of η . The composition $\text{ev}_m \circ \gamma_\eta$ yields a smooth curve in $\alpha^{-1}(m)$. As γ_η solves (12) we already know that $\text{ev}_m \circ \gamma_\eta(0) = 1_m$. Now we consider both subgroups separately:

Case $H = \text{Bis}_m(\mathcal{G})$. By definition, γ_η will take its image in $\text{Bis}_m(\mathcal{G})$ if $\text{ev}_m \circ \gamma_\eta(t) = 1_m, \forall t \in [0, 1]$, i.e. we have to prove that $(\text{ev}_m \circ \gamma_\eta)'(t) = 0 \in T_{\gamma_\eta(t)}^\alpha G, \forall t \in [0, 1]$. To this end fix $t \in [0, 1]$ and a curve $c_{t,\eta}:]-\varepsilon, \varepsilon[\rightarrow \text{Bis}_m(\mathcal{G})$ such that $c_{t,\eta}(0) = 1$ and $\eta(t)$ (as an element in the tangent space $T_1 \text{Bis}_m(\mathcal{G})$) coincides with the equivalence class $[s \mapsto c_{t,\eta}(s)]$. Now we compute

$$\begin{aligned} (\text{ev}_m \circ \gamma_\eta)'(t) &= T \text{ev}_m(\gamma_\eta'(t)) \stackrel{(12)}{=} T \text{ev}_m(\gamma_\eta(t) \cdot \eta(t)) = T(\text{ev}_m \circ \lambda_{\gamma_\eta(t)})(\eta(t)) \\ &= [s \mapsto \text{ev}_m(\gamma_\eta(t) \star c_{t,\eta}(s))] = [s \mapsto \gamma_\eta(t)(\beta \circ c_{t,\eta}(s)(m)) \cdot c_{t,\eta}(s)(m)] \\ &= [s \mapsto \gamma_\eta(t)(m) \cdot 1_m] = 0 \in T_{\gamma_\eta(t)(m)}^\alpha G \end{aligned} \quad (13)$$

In passing from the second to the last line, we have used $c_{t,\eta}(s) \in \text{Bis}_m(\mathcal{G})$, whence $c_{t,\eta}(s)(m) = 1_m$. We finally conclude that $\gamma_\eta(t) \in \text{Bis}_m(\mathcal{G})$ for all $t \in [0, 1]$ if $\eta \in C^k([0, 1], \mathbf{L}(\text{Bis}_m(\mathcal{G})))$.

Case $H = \text{Loop}_m(\mathcal{G})$. We need to show that for all $t \in [0, 1]$ we have $\gamma_\eta(t) \in \text{Vert}_m(\mathcal{G})$, i.e. that $\beta \circ \text{ev}_m \circ \gamma_\eta(t) = m$. Since $\text{ev}_m(\gamma_\eta(0)) = 1_m$ this will follow from $(\beta \circ \text{ev}_m \circ \gamma_\eta)'(t) = 0$ for all $t \in [0, 1]$. To this end fix again $t \in [0, 1]$ and a curve $c_{t,\eta}:]-\varepsilon, \varepsilon[\rightarrow \text{Loop}_m(\mathcal{G})$ with $c_{t,\eta}(0) = 1$ and $\eta(t) = [s \mapsto c_{t,\eta}(s)]$. Computing as in (13) we obtain

$$(\beta \circ \text{ev}_m \circ \gamma_\eta)'(t) \stackrel{(13)}{=} [s \mapsto \beta(\gamma_\eta(t)(m) \cdot c_{t,\eta}(s)(m))] = [s \mapsto \beta(\gamma_\eta(t)(m))] = 0 \in T_{\beta(\gamma_\eta(t)(m))} M.$$

We can thus conclude that $\gamma_\eta(t) \in \text{Loop}_m(\mathcal{G})$ for all $t \in [0, 1]$ if $\eta \in C^k([0, 1], \mathbf{L}(\text{Loop}_m(\mathcal{G})))$.

b) Proposition 3.2 c) asserts that the subgroups $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ are closed subgroups of the C^k -regular Lie group $\text{Bis}(\mathcal{G})$. By part (a), the Lie groups $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ are C^k -semiregular. Thus Lemma B.5 proves that $\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})$ are C^k -regular. ■

Example 3.5. Let M be a compact manifold and consider the pair groupoid $\mathcal{P}(M) = (M \times M \rightrightarrows M)$. The vertex group $\text{Vert}_m(\mathcal{P}(M))$ for $m \in M$ is just $\{(m, m)\}$, whence $\text{Loop}_m(\mathcal{P}(M)) = \text{Bis}(\mathcal{P}(M))_m$. Then the map $(\text{pr}_2)_*: \text{Bis}(\mathcal{P}(M)) \rightarrow \text{Diff}(M), \sigma \mapsto \text{pr}_2 \circ \sigma$ is an isomorphism of Lie groups. By construction, this restricts to an isomorphism

$$\text{Loop}_m(\mathcal{P}(M)) = \text{Bis}(\mathcal{P}(M))_m \cong \text{Diff}_m(M) := \{\varphi \in \text{Diff}(M) \mid \varphi(m) = m\}.$$

In particular, we infer from Proposition 3.2 and Proposition 3.4 that $\text{Diff}_m(M)$ is a regular and split Lie subgroup of $\text{Diff}(M)$. ■

Proposition 3.6. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. For $m \in M$ we endow the right coset space $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ with the quotient topology induced by $\text{Bis}(\mathcal{G})$. Then the map ev_m induces homeomorphisms

- a homeomorphism $\widetilde{\text{ev}}_m$ of $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ onto $\text{ev}_m(\text{Bis}(\mathcal{G})) \subseteq \alpha^{-1}(m)$,
- an isomorphism of topological groups e_m of $\Lambda_m := \text{Loop}_m(\mathcal{G})/\text{Bis}(\mathcal{G})$ onto an open subgroup of $\text{Vert}_m(\mathcal{G})$,

Moreover, $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ carry unique manifold structures turning the canonical quotient maps into submersions.

If there is a bisection through every arrow in G , e.g. \mathcal{G} is α -connected, then $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \cong \alpha^{-1}(m)$ as manifolds and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \cong \text{Vert}_m(\mathcal{G})$ as Lie groups.

Proof. a) By definition, the quotient topology turns $q_m: \text{Bis}(\mathcal{G}) \rightarrow \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}), \sigma \mapsto \sigma \text{Bis}_m(\mathcal{G})$ into a quotient map. Recall from Corollary 2.5 that $\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ is a submersion, whence its image in the α -fibre is open. For $\tau \in \text{Bis}(\mathcal{G})_m$ we observe $\text{ev}_m(\sigma \star \tau) = \sigma \star \tau(m) = \sigma(\beta(\tau(m)))\tau(m) = \sigma(\beta(1_m))1_m = \sigma(m) = \text{ev}_m(\sigma)$. Hence ev_m is constant on right cosets and ev_m factors through $h_m: \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \rightarrow \text{im ev}_m \subseteq \alpha^{-1}(m), \sigma \text{Bis}_m(\mathcal{G}) \mapsto \text{ev}_m(\sigma)$. Now h_m is continuous since $\text{ev}_m = h_m \circ q_m$ is continuous.

Let us prove that for $\sigma, \tau \in \text{Bis}(\mathcal{G})$ with $\text{ev}_m(\sigma) = \text{ev}_m(\tau)$ we have $\sigma^{-1} \star \tau \in \text{Bis}_m(\mathcal{G})$. Using the formulae (2) and (3) for the group operations of $\text{Bis}(\mathcal{G})$ we obtain

$$\sigma^{-1} \star \tau(m) = \sigma^{-1}(\beta(\tau(m)))\tau(m) = \sigma^{-1}(\beta(\sigma(m)))\sigma(m) = \iota(\sigma(m))\sigma(m) = 1_m.$$

Hence $\sigma^{-1} \star \tau \in \text{Bis}_m(\mathcal{G})$ if $\sigma(m) = \tau(m)$ and in this case we see $\sigma \text{Bis}_m(\mathcal{G}) = \tau \text{Bis}_m(\mathcal{G})$. This implies that h_m is a bijection onto the open set im ev_m .

We deduce from Corollary 2.5 that $\text{ev}_m|_{\text{im ev}_m}: \text{Bis}(\mathcal{G}) \rightarrow \text{im ev}_m \subseteq \alpha^{-1}(m)$ is a surjective submersion. In particular, ev_m is open, whence a quotient map and thus $q_m = h_m^{-1} \circ \text{ev}_m$ implies that h_m^{-1} is continuous.

b) By Proposition 3.2 b) we know that ev_m induces a Lie group morphism $\psi_m: \text{Loop}_m(\mathcal{G}) \rightarrow \text{Vert}_m(\mathcal{G})$ which is a submersion. Its kernel is the normal Lie subgroup $\text{Bis}_m(\mathcal{G})$. Thus $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ with the quotient topology becomes a topological group such that ψ_m descends to an isomorphism of topological groups onto $\text{ev}_m(\text{Loop}_m(\mathcal{G})) \subseteq \text{Vert}_m(\mathcal{G})$. Endow the quotient with the manifold structure turning the isomorphism into an isomorphism of Lie groups. Then the canonical quotient map becomes a submersion as a composition of a diffeomorphism and the submersion ψ_m .

The manifold structure on the open submanifolds $\text{im ev}_m \subseteq \alpha^{-1}(m)$ and $\text{ev}_m(\text{Loop}_m(\mathcal{G})) \subseteq \text{Vert}_m(\mathcal{G})$ is uniquely determined up to diffeomorphism by the property that ev_m is a submersion. This is due to [Glö15a, Lemma 1.9].

The last assertion follows from part a) and b), since then $\text{im ev}_m = \alpha^{-1}(m)$ and $\text{im } \psi_m = \text{Vert}_m(\mathcal{G})$ hold. ■

Example 3.7. Suppose $\pi: P \rightarrow M$ is a principal K -bundle for some locally exponential Lie group K with connected P . Then the gauge groupoid $\text{Gauge}(P) := ((P \times P)/K \rightrightarrows M)$ admits an adapted local addition [SW15b, Proposition 3.14] and $\text{Bis}(\text{Gauge}(P))$ is naturally isomorphic to $\text{Aut}(P)$. Assume that K and P are locally metrisable, i.e. $\text{Gauge}(P)$ is locally metrisable. The source fibre $\alpha^{-1}(m) = (P_m \times P)/K$ of $\text{Gauge}(P)$ is diffeomorphic to P by choosing $o \in P_m$ and mapping $\langle p, q \rangle$ to $q.(p^{-1} \cdot o)$. Here we use $p^{-1} \cdot o$ as the suggestive notation for the element $k \in K$ that satisfies $p.k = o$. With respect to these identification the evaluation ev_m turns into the evaluation map

$$\text{ev}_o: \text{Aut}(P) \rightarrow P, \quad \varphi \mapsto \varphi(o).$$

Consequently,

$$\text{Aut}_o(P) := \text{ev}^{-1}(o) = \{f \in \text{Aut}(P) \mid f(o) = o\}$$

is a Lie subgroup of $\text{Aut}(P)$ and by Proposition 3.6, $\text{Aut}(P)/\text{Aut}_o(P)$ carries a unique smooth structure turning the induced map $[\varphi] \mapsto \varphi(p)$ into a diffeomorphism. So we may view P as a homogeneous space for its automorphism group. In particular, this applies to the trivial bundle, yielding a smooth structure on $\text{Diff}(M)/\text{Diff}_m(M)$ and a diffeomorphism $\text{Diff}(M)/\text{Diff}_m(M) \cong M$. ■

By now, the quotients $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ carry a manifold structure which was derived from the manifold structure of the α -fibre to the quotient. However, if the Lie groupoid \mathcal{G} is a Banach-Lie groupoid then the homogeneous space $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ already carries a natural manifold structure as a homogeneous space. This is a consequence of Glöckner's inverse function theorem (see the next Lemma for references and details). Again this manifold structure turns the canonical quotient map into a submersion.

Lemma 3.8. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Banach-Lie groupoid and $m \in M$. Then the homogeneous spaces $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ are manifolds and these manifold structures coincide with the ones obtained in Proposition 3.6.*

Proof. Combining Proposition 3.4 and Remark 3.3 we see that $\text{Bis}_m(\mathcal{G})$ is a C^0 -regular closed Lie subgroup of $\text{Bis}(\mathcal{G})$ and of $\text{Loop}_m(\mathcal{G})$. Moreover, $\text{Bis}_m(\mathcal{G})$ is a co-Banach submanifold in both $\text{Loop}_m(\mathcal{G})$ and $\text{Bis}(\mathcal{G})$. Thus the homogeneous spaces $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ carry manifold structures by [Glö15a, Theorem G (a)]. Furthermore, since $\text{Bis}_m(\mathcal{G})$ is a normal Lie subgroup, the manifold $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ becomes a Lie group. In both cases this manifold structure turns the canonical quotient map into a submersion. By the uniqueness assertion in Proposition 3.6 the manifold structures on the homogeneous spaces must coincide. ■

The interesting feature of the manifold structures obtained on the homogeneous spaces $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ for Banach-Lie groupoids is exactly that it coincides with the structure induced by the fibre. Hence, under some assumptions, we can endow the quotient $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ with a manifold structure which does not a priori use the manifold structure on the α -fibre. We will apply these results in the next section after we compile some more facts on natural group actions on the quotient $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$.

Lemma 3.9. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, fix $m \in M$ and define $\Lambda_m := \text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$. Then the quotient $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ admits left- and right group actions*

$$\begin{aligned} \lambda_{\text{Bis}(\mathcal{G})} : \text{Bis}(\mathcal{G}) \times (\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})) &\rightarrow \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}), (\sigma, \tau \text{Bis}_m(\mathcal{G})) \mapsto (\sigma \star \tau) \text{Bis}_m(\mathcal{G}) \\ \rho_{\Lambda_m} : (\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})) \times \Lambda_m &\rightarrow \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}), (\sigma \text{Bis}_m(\mathcal{G}), \gamma \text{Bis}_m(\mathcal{G})) \mapsto (\sigma \star \gamma) \text{Bis}_m(\mathcal{G}) \end{aligned}$$

which commute, i.e. $\lambda_{\text{Bis}(\mathcal{G})}(\sigma, \cdot) \circ \rho_{\Lambda_m}([\tau], \cdot) = \rho_{\Lambda_m}([\tau], \cdot) \circ \lambda_{\text{Bis}(\mathcal{G})}(\sigma, \cdot)$ for all $\sigma \in \text{Bis}(\mathcal{G})$ and $[\tau] \in \Lambda_m$.

Proof. Observe that $\text{Bis}(\mathcal{G})$ acts via left translation on itself and this action descends to a group action on the quotient $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$. Moreover, $\text{Loop}_m(\mathcal{G})$ acts via right translation on $\text{Bis}(\mathcal{G})$ and this action descends to a $\Lambda_m = \text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ -action on the quotient. The left action by left translation on $\text{Bis}(\mathcal{G})$ commutes with the right translation with elements in $\text{Loop}_m(\mathcal{G})$, whence the induced actions on the quotient commute. ■

Lemma 3.10. *Let \mathcal{G} be a transitive Lie groupoid which admits bisections through each arrow. Then $\lambda_{\text{Bis}(\mathcal{G})}$ is an effective group action, i.e. $\lambda_{\text{Bis}(\mathcal{G})}(\sigma, \cdot) = \text{id}_{\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})}$ implies $\sigma = 1$.*

Proof. The prerequisites imply that ev_m is a surjective map. Consider $\sigma \in \text{Bis}(\mathcal{G}) \setminus \{1\}$ and choose $n \in M$ such that $\sigma(n) \neq 1_n$. Now \mathcal{G} is transitive, whence there is $g_n \in \alpha^{-1}(m)$ with $\beta(g_n) = n$. As ev_m is surjective we can choose $\tau \in \text{Bis}(\mathcal{G})$ with $\tau(m) = g_n$. Arguing indirectly, we assume that $[\sigma \star \tau] = [\tau]$, i.e. there is $s \in \text{Bis}_m(\mathcal{G})$ with $\sigma \star \tau = \tau \star s$. Evaluating in m , we use $s \in \text{Bis}_m(\mathcal{G})$ to obtain

$$\sigma(n) \cdot g = \sigma(\beta(\tau(m))) \cdot \tau(m) = (\sigma \star \tau)(m) = (\tau \star s)(m) = \tau(m) \cdot 1_m = g$$

Hence, $\sigma(n) = g \cdot g^{-1} = 1_n$ follows, contradicting our choice of n . We conclude that $\lambda_{\text{Bis}(\mathcal{G})}$ is effective. ■

Remark 3.11. In general, the left action $\Lambda_{\text{Bis}(\mathcal{G})}$ will not be effective. To see this, we return to the example given in Remark 2.18 b):

Let $M = N \sqcup N'$ be the disjoint union of two non-isomorphic smooth manifolds. Then the pair groupoid $\mathcal{P}(M)$ is locally trivial, but there are arrows which are not contained in the image of any bisection. Fix $m \in N$ and recall that

$$\text{Bis}(\mathcal{G}) \cong \text{Diff}(M) \cong \text{Diff}(N) \times \text{Diff}(N') \text{ and } \text{Bis}_m(\mathcal{G}) \cong \text{Diff}_m(N) \times \text{Diff}(N').$$

Thus $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \cong \text{Diff}(N)/\text{Diff}_m(N)$. Hence, for $\varphi \in \text{Diff}(N')$ the bisection $\text{id}_N \times \varphi$ acts trivially on $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$. If N' is not the singleton manifold, choose $\varphi \neq \text{id}_{N'}$, to see that $\lambda_{\text{Bis}(\mathcal{G})}$ can not be effective. ■

Lemma 3.12. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, fix $m \in M$ and define $\Lambda_m := \text{Loop}_m(\mathcal{G}) / \text{Bis}_m(\mathcal{G})$. Then the map*

$$\widetilde{\text{ev}}_m : \text{Bis}(\mathcal{G}) / \text{Bis}_m(\mathcal{G}) \rightarrow \widetilde{\text{ev}}_m(\text{Bis}(\mathcal{G}) / \text{Bis}_m(\mathcal{G})) \cong \alpha^{-1}(m), \sigma \text{Bis}_m(\mathcal{G}) \mapsto \sigma(m)$$

(cf. Proposition 3.6) is equivariant with respect to the right Λ_m -action ρ_{Λ_m} and the right $\text{Vert}_m(\mathcal{G})$ -action, i.e. for $\tau \in \text{Loop}_m(\mathcal{G})$ and $\sigma \in \text{Bis}(\mathcal{G})$ we obtain the formula

$$\widetilde{\text{ev}}_m((\sigma \star \tau) \text{Bis}_m(\mathcal{G})) = \widetilde{\text{ev}}_m(\rho_{\Lambda_m}(\sigma \text{Bis}_m(\mathcal{G}), \tau \text{Bis}_m(\mathcal{G}))) = \widetilde{\text{ev}}_m(\sigma \text{Bis}_m(\mathcal{G})) \cdot \tau(m)$$

Proof. Fix $\sigma \in \text{Bis} \mathcal{G}$ and $\tau \in \text{Loop}_m(\mathcal{G})$ and compute

$$\widetilde{\text{ev}}_m((\sigma \star \tau) \text{Bis}_m(\mathcal{G})) = (\sigma \star \tau)(m) = \sigma(\underbrace{\beta(\tau(m))}_{=m}) \cdot \tau(m) = \sigma(m) \cdot \tau(m) = \widetilde{\text{ev}}_m(\sigma \text{Bis}_m(\mathcal{G}))\tau(m). \quad \blacksquare$$

4 Locally trivial Lie groupoids and transitive group actions

Our aim is now to study the construction of groupoids from their groups of bisections for locally trivial Lie groupoids. Again we consider in this section only Lie groupoids over a compact manifold M that admit an adapted local addition. Moreover, we choose and fix a point $m \in M$.

For locally trivial Lie groupoids the α -fibre over any point already determines the manifold of arrows. Hence, the groupoid quotient discussed in Theorem 2.21 of $\mathcal{B}(\mathcal{G})$ is determined by a quotient of the α -fibre. To construct the quotient, one needs to construct the fibre over a point and the vertex group from the group action of $\text{Bis}(\mathcal{G})$ on M and the subgroup $\text{Bis}_m(\mathcal{G})$. Following Proposition 3.6 these objects can be obtained as certain quotients of the group of bisections. The idea is now to study similar situation for abstract Lie groups and relate these Lie groups to groups of bisections. To this end, we define the central notion of this section:

Definition 4.1. Let $\theta: K \times M \rightarrow M$ be a transitive (left-)Lie group action of a Lie group K modelled on a metrisable space and H be a subgroup of K .

Then we call (θ, H) a *transitive pair* (over M with base point m) if the following conditions are satisfied:

- (P1) the action is smoothly transitive, i.e., the orbit map $\theta_m := \theta(\cdot, m)$ is a surjective submersion,
- (P2) H is a normal Lie subgroup of the stabiliser K_m of m and this structure turns H into a regular Lie group which is co-Banach as a submanifold in K_m .

The largest subgroup of H which is a normal subgroup of K is called *kernel* of the transitive pair.⁷ If the action of K on M is also n -fold transitive, then we call (θ, H) an *n -fold transitive pair*. \blacksquare

Transitive pairs are closely related to Klein geometries [Sha97, Chapter 3]. Indeed, they can be understood as infinite-dimensional Klein geometries for principal bundles. This view motivates the notion of the kernel of a transitive pair. We will come back to this perspective in Remark 4.19.

A transitive pair will allow us to construct a locally trivial Lie groupoid which is related to the group action on M . Before we begin with this construction, let us first exhibit two examples of transitive pairs.

Example 4.2. a) Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach-Lie groupoid over a compact manifold M . By [SW15b, Proposition 3.12] \mathcal{G} admits an adapted local addition, whence $\text{Bis}(\mathcal{G})$ becomes a Lie group. Then the Lie group action $\theta: \text{Bis}(\mathcal{G}) \times M \rightarrow M, \theta := \beta \circ \text{ev}$ induces a submersion $\theta_m = \beta|_{\alpha^{-1}(m)} \circ \text{ev}_m$ by Lemma 2.19. Observe that $K_m = \text{Loop}_m(\mathcal{G})$ and $\text{Bis}_m(\mathcal{G}) \subseteq \text{Loop}_m(\mathcal{G})$ is a normal subgroup. Combining Proposition 3.2 and Proposition 3.4 we see that $(\theta, \text{Bis}_m(\mathcal{G}))$ is a transitive pair if the action θ is transitive.

In general θ will not be transitive. However, under some mild assumptions, e.g., M being connected or if \mathcal{G} admits bisections through each arrow (see Lemma 2.19), the action will be transitive and we obtain a transitive pair.

The preceding example motivated the definition of a transitive pair. However, one has considerable freedom in choosing the ingredients for such a pair (see also Remark 4.3):

⁷We will see in Proposition 4.16 that there exists a kernel for each transitive pair. By standard arguments for topological groups, the kernel is a closed subgroup. In general this will not entail that the kernel is a closed Lie subgroup (of the infinite-dimensional Lie group K).

- b) Consider the diffeomorphism group $\text{Diff}(M)$ of a compact and connected manifold M . Choose a Lie group B modelled on a Banach space and define $K := \text{Diff}(M) \times B$. Then K becomes a Lie group which acts transitively via $\theta: K \times M \rightarrow M, ((\varphi, b), m) \mapsto \varphi(m)$. Fix $m \in M$ and observe that θ_m is a submersion as $\theta_m = \text{ev}_m \circ \text{pr}_1$ and $\text{ev}_m: \text{Diff}(M) \rightarrow M$ is a submersion. By construction $K_m = \text{Diff}_m(M) \times B$ and $H := K_m$ is a regular (and normal) Lie subgroup of K_m by Proposition 3.4.⁸ We conclude that $(\text{ev}_m \circ \text{pr}_1, \text{Diff}_m(M) \times B)$ is a transitive pair. ■

Remark 4.3. The conditions (P1) and (P2) in Definition 4.1 are quite weak. We illustrate this by rewriting the conditions for finite-dimensional Lie groups:

If K is a finite-dimensional Lie group, the conditions (P1) and (P2) are equivalent to

(Pfin) H is a normal closed subgroup of the stabiliser K_m of m under the action θ .

Proof. To see this note that $K_m = \theta_m^{-1}(m)$ is a closed subgroup of the finite-dimensional Lie group K , whence it is a Lie subgroup of K . Then θ_m factors through $K/K_m \cong M$ and thus (P1) holds as $K \rightarrow K/K_m$ is a submersion. Note that H is a closed subgroup of K_m and every Lie subgroup of a finite-dimensional Lie group is co-Banach as a submanifold and a regular Lie group. Hence (Pfin) implies (P2). ■

Remark 4.4. From a transitive pair $(\theta: K \times M \rightarrow M, H)$ we can construct the following normal subgroupoid $N(\theta, H)$ of the action groupoid $K \times_{\theta} M$. For each $n \in M$, we choose some $k_n \in K$ with $\theta(k_n, m) = n$ and set $H_n := k_n \cdot H \cdot k_n^{-1}$. Then H_n is a normal subgroup of K_m that does not depend on the choice of k_n . Indeed, if $\theta(k'_n, m) = n$, then we have $k_n^{-1}k'_n \in K_m$ and thus

$$k_n H k_n^{-1} = k_n k_n^{-1} k'_n H (k'_n)^{-1} k_n k_n^{-1} = k'_n H (k'_n)^{-1},$$

since H is normal in K_m . Moreover,

$$N(\theta, H) := \bigcup_{n \in M} H_n \times \{n\}$$

is a closed submanifold of $K \times M$, which can be seen as follows: by (P1) there exist for each $n \in M$ an open neighbourhood $U \subseteq M$ of n such that we can choose k_p to depend smoothly on p for $p \in U$. Then

$$G \times U \rightarrow G \times U, \quad (g, p) \mapsto (k_p^{-1} g k_p, p)$$

is a diffeomorphism that maps $N(\theta, H) \cap (G \times U)$ to the submanifold $H \times U$ of $G \times M$. Thus $N(\theta, H)$ is a normal Lie subgroupoid of $K \times_{\theta} M$.

On the other hand, given a transitive action of K on M , each normal Lie subgroupoid of $K \times M$ gives rise to a normal subgroup H of K_m , and one easily sees that these two constructions are inverse to each other. Thus normal Lie subgroupoids of action groupoids are the equivalent reformulation of transitive pairs that do not require to fix a point $m \in M$. However, it will be analytically much easier to work with transitive pairs (see Remark 4.10) and to have in mind that the choice of a base point does not matter. ■

Now we associate a locally trivial Lie groupoid to a transitive pair. To this end, we will first construct a principal bundle which will then give rise to the desired locally trivial Lie groupoid.

Proposition 4.5. *Let (θ, H) be a transitive pair. Then the quotients K/H and $\Lambda_m := K_m/H$ are Banach manifolds. Moreover, the map θ_m induces a Λ_m -principal bundle $\pi: K/H \rightarrow M, kH \mapsto \theta(k, m)$.*

To prove Proposition 4.5, the following Lemma deals with some needed technical details first (compare [Woc13, Example D.4]).

⁸Here we have used that for the pair groupoid $\mathcal{P}(M)$ the Lie group $\text{Diff}(M) = \text{Bis}(\mathcal{P}(M))$ is regular and $\text{Bis}_m(\mathcal{P}(M)) = \text{Diff}_m(M)$. Moreover, B is regular as a Banach Lie group and $\mathbf{L}(\text{Diff}(M) \times B) \cong \mathbf{L}(\text{Diff}(M)) \times \mathbf{L}(B)$.

Lemma 4.6. *Let (θ, H) be a transitive pair. The group action θ induces a $K_m = \theta_m^{-1}(m)$ -principal bundle $\theta_m: K \rightarrow M$. Canonical bundle trivialisations for this bundle are given by*

$$\theta_m^{-1}(U_i) \rightarrow U_i \times K_m, \quad g \mapsto (\theta_m(g), \sigma_i(\theta_m(g))^{-1} \cdot g) \quad (14)$$

where $(\sigma_i: U_i \rightarrow \theta_m^{-1}(U_i))_{i \in I}$ is a section atlas of θ_m .

Proof. Note that θ_m is a surjective submersion. Thus [Glö15a, Theorem D] implies that K_m is a split Lie subgroup in K and θ_m descends to a homeomorphism $K/K_m \cong M$. Identify K/K_m with the manifold M . Clearly by conjugation $K_m \cong \theta_m^{-1}(n)$ for all $n \in M$. It is now trivial to check that (14) yields bundle trivialisations whose trivialisation changes are K_m -torsor isomorphisms. ■

Proof (of Proposition 4.5). The manifold M is finite-dimensional, whence K_m is a Lie subgroup of finite codimension (as a submanifold). By (P2) H is a co-Banach submanifold in K_m and thus H is also a co-Banach submanifold of K by [Glö15a, Lemma 1.4]. In particular, H is a Lie subgroup of K . As H is a regular Lie group by (P2), we can apply [Glö15a, Theorem G (a)] to obtain a manifold structure on the quotients

$$p_m: K \rightarrow K/H \text{ and } q_m: K_m \rightarrow K_m/H =: \Lambda_m$$

turning the projections into submersions. Moreover, we deduce from [Glö15a, Theorem G] that K/H is a Banach manifold, Λ_m is a Banach-Lie group, q_m is a morphism of Lie groups and Λ_m acts on K/H via

$$\rho_{\Lambda_m}: K/H \times \Lambda_m \rightarrow K/H, (gH, \lambda H) \mapsto (g \cdot \lambda)H.$$

The subgroup H is contained in K_m , whence θ_m induces a map $\pi: K/H \rightarrow M$ which satisfies $\pi \circ p_m = \theta_m$. Now p_m and θ_m are submersions, whence π is smooth with surjective tangent map at every point (cf. [Glö15a, p.2 and Lemma 1.8]). Since M is finite-dimensional, [Glö15a, Theorem A] implies that π is a submersion. The action θ is transitive and thus π is surjective submersion. Note that the π -fibre over a point $n \in M$ is given by

$$\pi^{-1}(n) = p_m(\theta_m^{-1}(n)) = \{gH \in K/H \mid \theta(g, m) = n\}.$$

Now it is easy to see that the π -fibres coincide with the orbits of the action ρ_{Λ_m} and the ρ_{Λ_m} -action on the fibres is free, i.e. $\pi^{-1}(n)$ is a Λ_m -torsor for each $n \in M$.

To turn $\pi: K/H \rightarrow M$ into a Λ_m -principal bundle we will now prove that the change of trivialisations induce Λ_m -torsor isomorphisms. Recall that by Lemma 4.6 the bundle $\theta_m: K \rightarrow M$ is a K_m -principal bundle. The trivialisations (14) descent to K/H via

$$\kappa_i: \pi^{-1}(U_i) \rightarrow U_i \times \Lambda_m, \quad (gH) \mapsto (\pi(gH), ((\sigma_i(\pi(gH)))^{-1} \cdot g)H). \quad (15)$$

For each $i \in I$ we obtain a commutative diagram

$$\begin{array}{ccccc} K & \xleftarrow{\cong} & \theta_m^{-1}(U_i) & \xrightarrow{(14)} & U_i \times K_m \\ & \searrow p_m & \downarrow & \swarrow \text{id}_{U_i} \times q_m & \downarrow \text{pr}_1 \\ & & K/H & \xleftarrow{\cong} \pi^{-1}(U_i) & \xrightarrow{\kappa_i} U_i \times \Lambda_i \\ & \searrow \theta_m & & \downarrow & \downarrow \text{pr}_1 \\ M & \xleftarrow{\cong} & U_i & \xrightarrow{\cong} & U_i \end{array}$$

and the sets $\pi^{-1}(U_i), i \in I$ cover K/H . Note that the trivialisation changes descent to Λ_m -torsor isomorphisms, whence the κ_i form an atlas of Λ_m -principal bundle trivialisations for K/H . Summing up, we have constructed a principal Λ_m -bundle $\pi: K/H \rightarrow M$. ■

Example 4.7. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach-Lie groupoid such that there is a bisection through each element in G . Consider the transitive pair $(\beta \circ \text{ev}_m, \text{Bis}_m(\mathcal{G}))$ discussed in Example 4.2 a). The m -stabiliser of $\beta \circ \text{ev}$ coincides with $\text{Loop}_m(\mathcal{G})$ and Lemma 4.6 yields the $\text{Loop}_m(\mathcal{G})$ -bundle $\beta \circ \text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow M$. Moreover, the Λ_m -principal bundle constructed in Proposition 4.5 is $\pi: \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \rightarrow M, \sigma \text{Bis}_m(\mathcal{G}) \mapsto \beta(\sigma(m))$ with $\Lambda_m = \text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$. Then the α -fibre through m yields a $\text{Vert}_m(\mathcal{G})$ -principal bundle $\beta|_{\alpha^{-1}(m)}: \alpha^{-1}(m) \rightarrow M$. Now Proposition 3.6 allows us to identify Λ_m and $\text{Vert}_m(\mathcal{G})$ and Lemma 3.12 shows that $\text{ev}_m: \text{Bis}(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ descends to a Λ_m -principal bundle isomorphism

$$\begin{array}{ccc} \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) & \xrightarrow{\widetilde{\text{ev}}_m} & \alpha^{-1}(m) \\ & \searrow \pi & \swarrow \beta|_{\alpha^{-1}(m)} \\ & & M \end{array} \quad \blacksquare$$

Definition 4.8. Let (θ, H) be a transitive pair with associated Λ_m -principal bundle $\pi: K/H \xrightarrow{\Lambda_m} M$. As principal bundles correspond to locally trivial groupoids, this allows us to construct a gauge groupoid

$$\mathcal{R}(\theta, H) := \left(\begin{array}{c} \frac{K/H \times K/H}{\Lambda_m} \\ \alpha_{\mathcal{R}} \downarrow \downarrow \beta_{\mathcal{R}} \\ M \end{array} \right)$$

with $\alpha_{\mathcal{R}}(\langle gH, kH \rangle) = \pi(kH)$ and $\beta_{\mathcal{R}}(\langle gH, kH \rangle) = \pi(gH)$. \blacksquare

Note that we will work with the gauge groupoid $\mathcal{R}(\theta, H)$ associated to the transitive pair (θ, H) and not with the principal bundle (although many constructions will be carried out in the context of principal bundles). The reason for this is that many interesting maps considered later can not be described as morphisms of principal bundles (with fixed structure group). However, one can treat these maps as morphisms of (locally trivial) Lie groupoids over the fixed base M . Hence we prefer the groupoid perspective.

Before we continue, let us record some technical details on the construction of the Lie groupoid $\mathcal{R}(\theta, H)$.

Remark 4.9. Let (θ, H) be a transitive pair with $\theta: K \times M \rightarrow M$.

- Observe that as K/H is a Banach manifold, the gauge groupoid $\mathcal{R}(\theta, H)$ is a Banach-Lie groupoid and thus $\mathcal{R}(\theta, H)$ admits an adapted local addition by [SW15b, Proposition 3.12]. Moreover, the gauge groupoid $\mathcal{R}(\theta, H)$ is source connected if and only if K/H is connected.
- Choose a section atlas $(\sigma_i, U_i)_{i \in I}$ of $\theta_m: K \xrightarrow{K_m} M$ as in (14). This atlas induces a section atlas $s_i := p_m \circ \sigma_i: U_i \rightarrow \pi^{-1}(U_i) \subseteq K/H$ of the bundle $\pi: K/H \xrightarrow{\Lambda_m} M$ (cf. the proof of Proposition 4.5). Using these section, we identify the bisections of $\mathcal{R}(\theta, H)$ with bundle automorphisms via the Lie group isomorphism from [SW15b, Example 3.16]

$$\text{Aut}(\pi: K/H \rightarrow M) \rightarrow \text{Bis}(\mathcal{R}(\theta, H)), \quad f \mapsto (m \mapsto \langle f(s_i(m)), s_i(m) \rangle), \text{ if } m \in U_i. \quad (16)$$

For later use we recall that the bundle trivialisations (15) induce charts for the manifold $\frac{K/H \times K/H}{\Lambda_m}$ via

$$\frac{\pi^{-1}(U_i) \times \pi^{-1}(U_j)}{\Lambda_m} \rightarrow U_i \times U_j \times \Lambda_m, \quad \langle p_1, p_2 \rangle \mapsto (\pi(p_1), \pi(p_2), \delta(\sigma_i(\pi(p_1)), p_1) \delta(\sigma_j(\pi(p_2)), p_2)^{-1}). \quad (17)$$

where $\delta: K/H \times_{\pi} K/H \rightarrow \Lambda_m$ is the smooth map mapping a pair (p, q) to the element $p^{-1} \cdot q \in \Lambda_m$ which maps p to q (via the Λ_m -right action). \blacksquare

Remark 4.10. In the base point free version of transitive pairs from Remark 4.4, one constructs the groupoid $\mathcal{R}(\theta, H)$ by taking the quotient

$$(K \times_{\theta} M)/N(\theta, H).$$

Indeed, we have the isomorphisms of Lie groupoids over M

$$K \times_{\theta} M := \left(\begin{array}{c} K \times M \\ \downarrow \downarrow \\ M \end{array} \right) \cong \left(\begin{array}{c} \frac{(K \times K)}{K_m} \\ \downarrow \downarrow \\ M \end{array} \right) \cong \left(\begin{array}{c} \frac{(K \times K)/H}{K_m} \\ \downarrow \downarrow \\ M \end{array} \right).$$

Then $N(\theta, H)$ corresponds exactly to the normal subgroupoid $\frac{(H \times H)}{H} / \frac{H}{K_m}$ on the right hand side.

However, it is much harder to construct the smooth structure on the quotient $(K \times_{\theta} M)/N(\theta, H)$ directly. For instance, in the finite-dimensional case (or also in the case of K being a Banach-Lie group) one can use Godement's criterion for the existence of the quotient $(K \times_{\theta} M)/N(\theta, H)$ in the category of smooth manifolds (cf. [Mac05, Theorem 2.2.4]). It is not known to the authors whether Godement's criterion extends beyond Banach manifolds, whereas the construction of $\mathcal{R}(\theta, H)$ as in 4.8 is possible in the full generality of our definition of a transitive pair. \blacksquare

So far we have constructed a locally trivial groupoid $\mathcal{R}(\theta, H)$ associated to a transitive pair (θ, H) . Let us now analyse how the Lie group K (i.e. the group acting via θ on M) is related to the Lie group $\text{Bis}(\mathcal{R}(\theta, H))$. To this end, we study a natural Lie group morphism $K \rightarrow \text{Bis}(\mathcal{R}(\theta, H))$ which is closely related to the action of the transitive pair.

Lemma 4.11. *Let (θ, H) be a transitive pair. Then the action of K on K/H by left multiplication gives rise to a group homomorphism $K \rightarrow \text{Aut}(\pi: K/H \xrightarrow{\Lambda_m} M)$. With respect to the canonical isomorphism $\text{Aut}(\pi: K/H \xrightarrow{\Lambda_m} M) \cong \text{Bis}(\mathcal{R}(\theta, H))$ of Lie groups from (16) this gives rise to the group homomorphism*

$$a_{\theta, H}: K \rightarrow \text{Bis}(\mathcal{R}(\theta, H)), \quad k \mapsto (x \mapsto \langle k \cdot s_i(x), s_i(x) \rangle), \text{ for } x \in U_i,$$

where $s_i = p_m \circ \sigma_i, i \in I$ are the sections from 4.9 b). Moreover, $a_{\theta, H}$ is smooth and thus a morphism of Lie groups.

Proof. Consider the smooth group action $\lambda_K: K \times K/H \rightarrow K/H, (k, gH) \mapsto \lambda_k(gH) := (kg)H$. By Lemma 3.9 this group action commutes with the right action ρ_{Λ_m} on K/H . Hence for each $k \in K$ the map $\lambda_K(k): K/H \rightarrow K/H$ is a bundle automorphism of the Λ_m -principal bundle. Now $a_{\theta, H}(k)$ is the image of the bundle automorphism $\lambda_K(k)$ under the Lie group isomorphism (16). Since $\lambda_K(kk') = \lambda_K(k)\lambda_K(k')$, we derive that $a_{\theta, H}$ is a group homomorphism.

Let us now prove that $a_{\theta, H}$ is smooth. To this end recall that K and $\frac{K/H \times K/H}{\Lambda_m}$ are modelled on metrisable spaces and $\text{Bis}(\mathcal{R}(\theta, H)) \subseteq C^\infty(M, \frac{K/H \times K/H}{\Lambda_m})$. Since M is compact, we apply the exponential law Theorem B.9 c) to see that $a_{\theta, H}$ will be smooth if the map

$$a_{\theta, H}^\vee: K \times M \rightarrow \frac{K/H \times K/H}{\Lambda_m}, (k, x) \mapsto \langle k \cdot s_i(x), s_i(x) \rangle, \text{ for } x \in U_i$$

is smooth. We work locally around $(k, x) \in K \times M$. Fix $i \in I$ such that $x \in U_i$ and recall that $s_i = p_m \circ \sigma_i$ and $\theta_m(\sigma_i) = \text{id}_{U_i}$ hold. Then we have

$$\beta_{\mathcal{R}}(\langle k \cdot s_i(x), s_i(x) \rangle) = \pi(k \cdot s_i(x)) = \theta_m(k \cdot \sigma_i(x)) = \theta(k, \theta_m(\sigma_i(x))) = \theta(k, x). \quad (18)$$

Now we choose $j \in I$ with $\theta(k, x) \in U_j$ and denote by κ_{ji} the manifold charts (17) defined in Remark 4.9. Then the composition $\kappa_{ji} \circ a_{\theta, H}$ which is defined at least on the pair (k, x) and we compute:

$$\begin{aligned} \kappa_{ji} \circ a_{\theta, H}^\vee(k, x) &\stackrel{(17)}{=} (\pi(k \cdot s_i(x)), \pi(s_i(x)), \delta(s_j(\pi(k \cdot s_i(x))), k \cdot s_i(x)) \delta^{-1}(\underbrace{s_i \circ \pi \circ s_i(x)}_{=x}, s_i(x))) \\ &\stackrel{(18)}{=} (\theta(k, x), x, \delta(s_j(\theta(k, x)), k \cdot s_i(x))) = (\theta(k, x), x, \delta(s_j(\theta(k, x)), \lambda_K(k, s_i(x)))) \end{aligned}$$

Note that the above formula did not depend on (k, x) , whence it is valid for all (g, y) with $a_{\theta, H}^{\vee}(g, y) \in \frac{\pi^{-1}(U_j) \times \pi^{-1}(U_i)}{\Lambda_m}$. In particular, we see that $\kappa_{ji} \circ a_{\theta, H}^{\vee}$ is smooth as a composition of the smooth maps θ, δ and λ_K . Since $\frac{K/H \times K/H}{\Lambda_m}$ carries the identification topology with respect to the atlas $(K_{ji})_{i,j \in I}$, we deduce that $a_{\theta, H}^{\vee}$ is smooth. Summing up, this proves that $a_{\theta, H}$ is smooth and thus a Lie group morphism. ■

Before we clarify the relation of $a_{\theta, H}$ and θ let us return briefly to the problem of (re-)constructing a Lie groupoid from its group of bisections (see Theorem 2.21 and Remark 2.22). To obtain a construction principle for Lie groupoids, we would like $a_{\theta, H}$ to be an isomorphism of Lie groups. Then $a_{\theta, H}$ would identify the Lie group with the group of bisections of $\mathcal{R}(\theta, H)$ and thus transitive pairs would induce (up to isomorphism) unique locally trivial Lie groupoids. However, in general for an arbitrary transitive pair (θ, H) the Lie group morphism $a_{\theta, H}$ will neither be injective nor surjective. We illustrate this with two examples:

Example 4.12. a) Let K be a compact finite-dimensional Lie group. Then K acts on itself transitively via left multiplication $\lambda: K \times K \rightarrow K$. Take $m = 1_K$ and $H = \{1_K\}$ to obtain the principal bundle $\text{id}_K: K \xrightarrow{H} K$. The associated gauge groupoid is the pair groupoid $K \times K \rightrightarrows K$ whose bisections are given by $\text{Diff}(K)$. Taking this identification, $a_{\theta, H}$ becomes the map $K \rightarrow \text{Diff}(K), k \mapsto \lambda(k, \cdot)$ which will only be surjective in trivial cases.

b) We return to Example 4.2 b): Let B be a Banach Lie group, M a compact connected manifold and $m \in M$. Then $(\text{ev}_m \circ \text{pr}_1, \text{Diff}_m(M) \times B)$ is a transitive pair. Set $H := \text{Diff}_m(M) \times B$ and observe

$$(\text{Diff}(M) \times B)/H \cong \text{Diff}(M)/\text{Diff}_m(M) \cong M.$$

Moreover, since H is the m -stabiliser of the action $\text{ev}_m \circ \text{pr}_1$, we deduce that $\mathcal{R}(\text{ev}_m \circ \text{pr}_1, H)$ is isomorphic to the pair groupoid $\mathcal{P}(M)$. With respect to these identifications, the map $a_{\text{ev}_m \circ \text{pr}_1, H}$ becomes

$$\text{Diff}(M) \times B \rightarrow \text{Bis}(\mathcal{P}(M)) \cong \text{Diff}(M), (\varphi, b) \mapsto \varphi$$

which is surjective but can not be injective for non-trivial B . Note that this example arose from enlarging $\text{Diff}(M) \cong \text{Bis}(\mathcal{P}(M))$. Moreover, we record that the action of $\text{Diff}(M) \times B$ by left multiplication on $(\text{Diff}(M) \times B)/H$ is not effective and this causes $a_{\text{ev}_m \circ \text{pr}_1, H}$ to be not injective (cf. Lemma 4.15 below). ■

As we have already pointed out, transitive pairs are quite flexible and more general than groups of bisections (of locally trivial Lie groupoids) However, transitive pairs are a source of Lie group morphisms from Lie groups with transitive actions on M into the bisections of suitable locally trivial Lie groupoids over M . In particular, the morphism $a_{\theta, H}$ is closely related to the action θ as the following Lemma shows.

Lemma 4.13. *For a transitive pair (θ, H) the Lie group morphism $a_{\theta, H}$ makes the diagram*

$$\begin{array}{ccc} K & \xrightarrow{a_{\theta, H}} & \text{Bis}(\mathcal{R}(\theta, H)) \\ & \searrow^{\theta^\wedge} & \downarrow (\beta_{\mathcal{R}})_* \\ & & \text{Diff}(M) \end{array}$$

commutative. If $a_{\theta, H}$ is an isomorphism of Lie groups, then the map θ^\wedge is a submersion.

Proof. Observe first that for $x \in M$ we have (after choosing an appropriate section s_i) the formula (18)

$$\beta_{\mathcal{R}} \circ a_{\theta, H}(k)(x) = \beta_{\mathcal{R}}(\langle k \cdot s_i(x), s_i(x) \rangle) = \theta(k, x) = \theta^\wedge(k)(x).$$

Hence $(\beta_{\mathcal{R}})_* \circ a_{\theta, H} = \theta^\wedge$ and the diagram commutes. This also entails $a_{\theta, H}(H) \subseteq \text{Loop}_m(\mathcal{R}(\theta, H))$.

If $a_{\theta, H}$ is a Lie group isomorphism, θ^\wedge is a submersion if $(\beta_{\mathcal{R}})_*: \text{Bis}(\mathcal{R}(\theta, H)) \rightarrow \text{Diff}(M)$ is a submersion. However, since $\mathcal{R}(\theta, H)$ is locally trivial, the map $(\beta_{\mathcal{R}})_*$ is a submersion by [SW15b, Example 3.16]. ■

Remark 4.14. That θ^\wedge must be a submersion if $a_{\theta,H}$ is an isomorphism of Lie groups can be understood as the statement that the Lie group K needs to be large enough to be eligible to be the bisection group. In particular, if M is not a zero-dimensional manifold, this condition rules out every transitive pair which arises by a group action of a finite-dimensional Lie group. ■

Building on the observation in Example 4.12 b) we will now develop a simple criterion which ensures that for a given transitive pair (θ, H) the morphism $a_{\theta,H}$ is injective.

Lemma 4.15. *Let (θ, H) be a transitive pair. The Lie group morphism*

$$a_{\theta,H}: K \rightarrow \text{Bis}(\mathcal{R}(\theta, H)), g \mapsto (kH \mapsto (g \cdot k)H)$$

is injective if and only if the action $\lambda_H: H \times K/H \rightarrow K/H, (h, gH) \mapsto (hg)H$ is effective.

Proof. The isomorphism $\text{Aut}(\pi: K/H \xrightarrow{\Lambda_m} M) \cong \text{Bis}(\mathcal{R}(\theta, H))$ allows us to rewrite $a_{\theta,H}$ as the left multiplication $\lambda_K: K \rightarrow \text{Aut}(\pi: K/H \xrightarrow{\Lambda_m} M)$, where $\lambda_K(k): K/H \rightarrow K/H, gH \mapsto (kg)H$. Assume first that $a_{\theta,H}$ (and thus also λ_K) is injective. Now consider $k \in H$ such that $(hg)H = gH$ for all $g \in G$. This implies $\lambda_K(h)(gH) = (hg)H = gH = \text{id}_{K/H}(gH)$, whence $h = 1_K$ as the group homomorphism λ_K is injective.

Conversely assume that the action $H \times K/H \rightarrow K/H, (h, gH) \mapsto (hg)H$ is effective and consider $g \in \ker \lambda_K$, i.e. $\lambda_K(g) = \text{id}_{G/H}$. Then $(g \cdot k)H = kH$ holds for all $kH \in G/H$. As this entails $gH = 1_G H$, we deduce $g \in H$. Now the left action of H on G/H by multiplication is effective, forcing $g \in H$ to be the identity 1_K . ■

Proposition 4.16. *Consider a transitive pair (θ, H) and denote by λ_H the left action on the quotient K/H as in Lemma 4.15. Then the kernel of $a_{\theta,H}$ is given by*

$$\ker a_{\theta,H} = \{h \in H \mid \lambda_H(h, \cdot) \equiv \text{id}_{K/H}\}$$

and coincides with the kernel of the transitive pair (θ, H) . In particular every transitive pair admits a unique kernel.

Proof. Recall from the proof of Lemma 4.15 that $\ker a_{\theta,H}$ is contained in H and consists of all elements of H which act trivially by left multiplication on K/H . Thus we obtain the first description of $\ker a_{\theta,H}$.

As a kernel of a Lie group morphism, $\ker a_{\theta,H}$ is a closed and normal subgroup of K . Let us now prove that every subgroup G of H which is normal in K is contained in $\ker a_{\theta,H}$. Then for $k \in K$ and $g \in G \subseteq H$ we derive from G being normal in K that $gk = kg'$ for $g' \in H$, i.e. $gkH = kH$ for all $k \in K$. Thus elements in G act trivially on K/H , whence $G \subseteq \ker a_{\theta,H}$. We conclude that $\ker a_{\theta,H}$ is the kernel of the transitive pair (θ, H) . ■

Definition 4.17. A transitive pair (θ, H) is called *effective* if H acts effectively on the quotient K/H by left multiplication, i.e. the kernel of the transitive pair is trivial. ■

The characterisation of the kernel of a transitive pair in Proposition 4.16 can be used to compute it. For the examples considered in this section we obtain:

Example 4.18. The transitive pairs in Example 4.2 a) and 4.12 a) are effective, whence the kernel is trivial. For the transitive pair $(\text{ev}_m \circ \text{pr}_1, \text{Diff}(M)_m \times B)$ from Example 4.2 b) the kernel is $\{\text{id}_M\} \times B$ (by Example 4.12 b)). In Theorem 5.15 below the kernel of a class of transitive pairs arising from extensions of diffeomorphism groups is computed. ■

Although the criterion for the injectivity of $a_{\theta,H}$ gives rise to a very natural condition on the transitive pair, the question of surjectivity is much more subtle.

Remark 4.19. We now describe the relation between Klein geometries [Sha97, Chapter 3] and transitive pairs. First note that our setting is infinite-dimensional, and the notion of a transitive pair takes the additional analytical issues caused by this into account.

Recall that a *Klein geometry* is a pair (K, H) , where K is a finite-dimensional Lie group (called the *principal group*) and H is a closed subgroup such that the manifold K/H is connected. The *kernel* of a Klein geometry is the largest subgroup L of H which is normal in K . A Klein geometry is called *effective* if L is trivial. Klein geometries are constructed to model geometry via the principal H -bundle $K \rightarrow K/H$.

Note that the principal group of a Klein geometry is finite-dimensional. Hence the quotient K/H inherits a canonical manifold structure turning $K \rightarrow K/H$ into a submersion. In our infinite-dimensional setting the quotient does not automatically inherit a manifold structure, whence a transitive pair has to guarantee this behaviour via extra assumptions (cf. Proposition 4.5).

In studying a transitive pair (θ, H) , we are interested in the principal K_m/H -bundle $K/H \rightarrow K/K_m = M$. Thus a transitive pair encodes more information than a Klein geometry, as the principal H -bundle $K \rightarrow K/H$ is obtained as additional information. To some extent, one can interpret a transitive pair as a ‘‘Klein geometry for principal bundles’’. In particular, the notion of a transitive pair also covers the concept of an (infinite-dimensional) Klein geometry in the case that $K_m = H$ and the quotient K/H is connected.

Finally, there is a close connection between effective transitive pairs and effective Klein geometries. Namely, the kernel of the transitive pair (i.e. the kernel of the Lie group morphism $a_{\theta, H}: K \rightarrow \mathcal{R}(\theta, H)$) is by definition the largest closed subgroup of H which is normal in K . Hence the kernel of the transitive pair (θ, H) corresponds to the kernel of a Klein geometry. Let us stress again that contrary to the finite dimensional case, the kernel of a transitive pair will only be a closed subgroup and not automatically a closed Lie subgroup. Summing up, if the pair (θ, H) is effective and $H = K_m$ and K/H is connected, then the transitive pair corresponds to an (infinite-dimensional) effective Klein geometry. ■

Let us now return to Example 4.2 a) and consider $\mathcal{R}(\theta, H)$ and $a_{\theta, H}$ for the action of a bisection group on M . We will see that the constructions given in this section are in a certain sense inverse to computing the bisections of a locally trivial groupoid.

Example 4.20. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach-Lie groupoid such that there is a bisection through every $g \in G$. Consider the group action $\beta \circ \text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow M, (\tau, m) \mapsto \beta \circ \tau(m)$. We have seen in Example 4.2 a) that $(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ is a transitive pair. Moreover, since every locally trivial Lie groupoid is transitive, Lemma 3.10 shows that $(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ is an effective transitive pair. For this special effective transitive pair we note the following consequences (a detailed verification for the claims made in this example can be found in Lemma 4.21 below):

The groupoid \mathcal{G} is isomorphic to $\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$. We have already seen in Example 4.7 that the vertex bundles associated to the locally trivial Lie groupoids \mathcal{G} and $\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ are isomorphic. Moreover, from Proposition 3.6 we recall that $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \cong \alpha^{-1}(m)$ and $\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \cong \text{Vert}_m(\mathcal{G})$. Hence the gauge groupoid $\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ is canonically isomorphic to the gauge groupoid of a vertex bundle of \mathcal{G} . Consider the canonical map over M from the gauge groupoid $\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ to the locally trivial Lie groupoid \mathcal{G}

$$\chi_{\mathcal{G}}: \frac{\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \times \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})}{\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})} \rightarrow G, \langle \sigma \text{Bis}_m(\mathcal{G}), \tau \text{Bis}_m(\mathcal{G}) \rangle \mapsto \sigma(m) \cdot (\tau(m))^{-1}. \quad (19)$$

Since \mathcal{G} admits bisections through each arrow and \mathcal{G} is locally trivial, Proposition 3.6 shows that the image of $\chi_{\mathcal{G}}$ coincides with G . One then proves that $\chi_{\mathcal{G}}$ is an isomorphism of Lie groupoids over M . We can thus recover the locally trivial Lie groupoid \mathcal{G} (up to isomorphism depending on m) as the groupoid $\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$.

The map $a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}: \text{Bis}(\mathcal{G}) \rightarrow \text{Bis}(\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})))$ is a Lie group isomorphism. For this special transitive pair, an inverse of $a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}$ is given by $\text{Bis}(\chi_{\mathcal{G}}): \text{Bis}(\mathcal{R}(\theta, H)) \rightarrow \text{Bis}(\mathcal{G}), \sigma \mapsto \chi_{\mathcal{G}} \circ \sigma$. ■

Lemma 4.21. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach-Lie groupoid such that the action $\beta \circ \text{ev}$ of the bisections on M is transitive, e.g. \mathcal{G} is source-connected. Then*

- a) *the Lie groupoid morphism $\chi_{\mathcal{G}}$ from (19) induces an isomorphism onto the open subgroupoid $\text{ev}(\text{Bis}(\mathcal{G}) \times M)$ of \mathcal{G} . Hence ev is surjective if and only if $\chi_{\mathcal{G}}: \mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})) \rightarrow \mathcal{G}$ is an isomorphism.*
- b) *the Lie group morphism $\text{Bis}(\chi_{\mathcal{G}}): \text{Bis}(\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))) \rightarrow \text{Bis}(\mathcal{G})$ is an isomorphism with inverse $a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}$.*

Proof. a) Clearly $\chi_{\mathcal{G}}$ is injective and after restricting it to its image, it becomes a bijection. Now consider the commutative diagram

$$\begin{array}{ccc}
 \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \times \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) & \xrightarrow{\widetilde{\text{ev}}_m \times \widetilde{\text{ev}}_m} & \alpha^{-1}(m) \times \alpha^{-1}(m) \\
 \downarrow q & & \downarrow (\xi, \eta) \mapsto \xi \cdot \eta^{-1} \\
 \frac{\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \times \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})}{\text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})} & \xrightarrow{\chi_{\mathcal{G}}} & G
 \end{array} \quad (20)$$

Here q is the canonical quotient map which is a submersion. The map $\widetilde{\text{ev}}_m \times \widetilde{\text{ev}}_m$ is the map induced on the quotient via $\widetilde{\text{ev}}_m \times \widetilde{\text{ev}}_m \circ (p_m \times p_m) = \text{ev}_m \times \text{ev}_m$ where $p_m: \text{Bis}(\mathcal{G}) \rightarrow \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ is the canonical quotient map. Now $\text{ev}_m \times \text{ev}_m$ is a submersion by [Glö15a, Lemma 1.6] and Corollary 2.5). Since \mathcal{G} is a Banach-Lie groupoid and q_m is a submersion, we deduce with [Glö15a, Lemma 1.10] that $\widetilde{\text{ev}}_m \times \widetilde{\text{ev}}_m$ is a submersion. Finally, [Mac05, Proposition 1.3.3] shows that the division map $\alpha^{-1}(m) \times \alpha^{-1}(m) \rightarrow G$, $(\xi, \eta) \mapsto \xi \cdot \eta^{-1}$ is a surjective submersion as \mathcal{G} is a locally trivial Lie groupoid.

Note that the division map restricts to a map $\text{im}(\text{ev}_m \times \text{ev}_m) \rightarrow \text{ev}(\text{Bis}(\mathcal{G}) \times M)$ because for $\sigma, \tau \in \text{Bis}(\mathcal{G})$ we have $\sigma(m) \cdot (\tau(m))^{-1} = (\sigma \star \tau^{-1})(\beta(\tau(m))) \in \text{ev}(\text{Bis}(\mathcal{G}) \times M)$. In particular $\text{im}(\chi_{\mathcal{G}}) \subseteq \text{ev}(\text{Bis}(\mathcal{G}) \times M)$. We claim that the image of $\chi_{\mathcal{G}}$ coincides with $\text{ev}(\text{Bis}(\mathcal{G}) \times M)$. If the claim is true, then the proof can be finished as follows: The set $\text{ev}(\text{Bis}(\mathcal{G}) \times M)$ is an open and wide subgroupoid of \mathcal{G} by Theorem 2.14. Hence (20) proves that $\chi_{\mathcal{G}}$ induces a Lie groupoid isomorphism from $\mathcal{R}(\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G}))$ onto the subgroupoid $\text{ev}(\text{Bis}(\mathcal{G}) \times M)$ of \mathcal{G} .

Proof of the claim: Choose a bisection $\gamma \in \text{Bis}(\mathcal{G})$ and $x \in M$ and let us show that $\gamma(x) \in \text{im} \chi_{\mathcal{G}}$. The action $\beta \circ \text{ev}: \text{Bis}(\mathcal{G}) \times M \rightarrow M$, $(\sigma, y) \mapsto \beta(\sigma(y))$ is transitive. Hence there are $\sigma, \tau \in \text{Bis}(\mathcal{G})$ with $\beta(\tau(m)) = x$ and $\beta(\sigma(m)) = \beta(\gamma(x))$. We compute

$$(\gamma(x))^{-1} \cdot \sigma(m) \cdot (\tau(m))^{-1} = (\gamma(x))^{-1} \cdot (\sigma \star \tau^{-1})(x) = \underbrace{\gamma^{-1} \star \sigma \star \tau^{-1}}_{=: l_x \in \text{Bis}(\mathcal{G})}(x).$$

By construction $\gamma \star l_x(x) = \sigma(m) \cdot (\tau(m))^{-1}$ and thus $\gamma(x) = \sigma(m) \cdot ((l_x \star \tau)(m))^{-1} \in \text{im} \chi_{\mathcal{G}}$. As γ and x were arbitrary this establishes $\text{ev}(\text{Bis}(\mathcal{G}) \times M) = \text{im} \chi_{\mathcal{G}}$.

b) Set $\Lambda_m := \text{Loop}_m(\mathcal{G})/\text{Bis}_m(\mathcal{G})$. To compute on the Λ_m -principal bundle $\beta \circ \text{ev}_m: \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \rightarrow M$ we choose a section atlas $(s_i, U_i)_{i \in I}$ and obtain

$$\begin{aligned}
 \text{Bis}(\chi_{\mathcal{G}}) \circ a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}(\sigma) &= \chi_{\mathcal{G}} \circ a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}(\sigma) = \chi_{\mathcal{G}} \circ (x \mapsto \langle \sigma \star s_i(x), s_i(x) \rangle, x \in U_i) \\
 &= (x \mapsto (\sigma \star s_i(x))(m) \cdot (s_i(x)(m))^{-1}, x \in U_i) \\
 &= (x \mapsto \sigma(\underbrace{\beta(s_i(x)(m))}_{=x}) \cdot (s_i(x)(m)) \cdot (s_i(x)(m))^{-1}, x \in U_i) = \sigma.
 \end{aligned}$$

Hence $\text{Bis}(\chi_{\mathcal{G}}) \circ a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})} = \text{id}_{\text{Bis}(\mathcal{G})}$. Let us show $a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})} \circ \text{Bis}(\chi_{\mathcal{G}}) = \text{id}_{\text{Bis}(\mathcal{R}(\theta, H))}$. Denote by $\psi: \text{Aut}(\beta \circ \text{ev}_m: \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \rightarrow M) \rightarrow \text{Bis}(\mathcal{R}(\theta, H))$ the isomorphism from (16). We fix a Λ_m -principal bundle automorphism f and compute the image of $\psi(f)$ under $a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})} \circ \text{Bis}(\chi_{\mathcal{G}})$ as

$$\begin{aligned}
 a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})} \circ \text{Bis}(\chi_{\mathcal{G}})(\psi(f)) &= a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}(\chi_{\mathcal{G}}(x \mapsto \langle f(s_i(x)), s_i(x) \rangle, x \in U_i) \\
 &= \psi \circ \psi^{-1} \circ a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}(x \mapsto (f(s_i(x))(m)) \cdot (s_i(x)(m))^{-1}, x \in U_i) \\
 &= \psi(\tau \text{Bis}_m(\mathcal{G}) \mapsto (((f(s_i(\bullet)))(m)) \cdot (s_i(\bullet)(m))^{-1}) \star \tau) \text{Bis}_m(\mathcal{G}). \quad (21)
 \end{aligned}$$

In passing from the second to the third line we have used that ψ^{-1} takes $a_{\theta, H}$ to the left action $\lambda_{\text{Bis}(\mathcal{G})}$ discussed in Lemma 3.9. We will now prove that the argument of ψ in (21) coincides with the principal bundle automorphism f . If this is true, then $\text{Bis}(\chi_{\mathcal{G}})$ is an isomorphism with inverse $a_{\beta \circ \text{ev}, \text{Bis}_m(\mathcal{G})}$.

The equivariant map $\widetilde{\text{ev}}_m: \text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G}) \rightarrow \alpha^{-1}(m)$ from Lemma 3.12 identifies $\text{Bis}(\mathcal{G})/\text{Bis}_m(\mathcal{G})$ with the open subset $\text{ev}_m(\text{Bis}(\mathcal{G}))$ of $\alpha^{-1}(m)$. Recall from Proposition 3.6 that Λ_m is isomorphic to an

open subgroup $e_m(\Lambda_m) \subseteq \text{Vert}_m(\mathcal{G})$ and $e_m(\Lambda_m) = \{\tau(m) \mid \tau \in \text{Loop}_m(\mathcal{G})\}$. Since $\beta \circ \text{ev}$ is transitive, the $\text{Vert}_m(\mathcal{G})$ -principal bundle $\beta|_{\alpha^{-1}(m)}: \alpha^{-1}(m) \rightarrow M$ restricts on $\text{ev}_m(\text{Bis}(\mathcal{G}))$ to a $e_m(\Lambda_m)$ -principal bundle. Identifying the groups, we obtain a Λ_m -principal bundle $\beta|_{\text{ev}_m(\text{Bis}(\mathcal{G}))}: \text{ev}_m(\text{Bis}(\mathcal{G})) \rightarrow M$ and $\widetilde{\text{ev}}_m$ induces an isomorphism of Λ_m -principal bundles (cf. Example 4.7).

The principal bundle isomorphism $\widetilde{\text{ev}}_m$ allows us to associate to the automorphism f a $e_m(\Lambda_m)$ -bundle automorphism $\tilde{f}: \text{ev}_m(\text{Bis}(\mathcal{G})) \rightarrow \text{ev}_m(\text{Bis}(\mathcal{G}))$ via $\tilde{f} \circ \widetilde{\text{ev}}_m = \widetilde{\text{ev}}_m \circ f$. Evaluating the argument of ψ from (21) at $\tau \text{Bis}_m(\mathcal{G})$, our preparations allow us to compute as follows.

$$\begin{aligned} \widetilde{\text{ev}}_m^{-1}(((f(s_i(\bullet)))(m)) \cdot (s_i(\bullet)(m))^{-1}) \star \tau(m) &= \widetilde{\text{ev}}_m^{-1}((f(s_i(\beta(\tau(m))))(m)) \cdot (s_i(\beta(\tau(m))))(m))^{-1} \cdot \tau(m)) \\ &= \widetilde{\text{ev}}_m^{-1}(\tilde{f}(s_i(\beta(\tau(m))))(m)) \cdot \underbrace{(s_i(\beta(\tau(m))))(m))^{-1} \cdot \tau(m)}_{\in e_m(\Lambda_m) \subseteq \text{Vert}_m(\mathcal{G})}) \\ &= \widetilde{\text{ev}}_m^{-1}(\tilde{f}(\underbrace{s_i(\beta(\tau(m))))(m))^{-1} \cdot \tau(m)}_{=1_{\beta(\tau(m))}}) \\ &= \widetilde{\text{ev}}_m^{-1}(\tilde{f}(\tau(m))) = f(\tau \text{Bis}_m(\mathcal{G})) \quad \blacksquare \end{aligned}$$

Problem 4.22. a) It would be interesting to develop a notion/theory of infinitesimal transitive pairs, i.e., an infinitesimal action $\mathfrak{k} \rightarrow \mathcal{V}(M)$ together with an ideal \mathfrak{h} of \mathfrak{k}_m . In particular, a derivation of the Lie algebra morphism $\mathbf{L}(a_{\theta,H})$ directly from these data in case of an effective transitive pair would be a valuable tool.

b) It would also be interesting to develop the theory of this section for not necessarily locally trivial Lie groupoids (analogously perhaps to the notion of Lie-Rinehart algebras in the infinitesimal setting, see [Hue90]). ■

5 Integrating extensions of Lie groups to transitive pairs

We now study the application of the previously developed theory to the integration theory of extensions of Lie algebroids and Lie groupoids. Throughout this section, M denotes a compact and 1-connected manifold for which we choose some fixed base-point $m \in M$. We will heavily use the integration theory of abelian extensions of infinite-dimensional Lie groups, for which we refer to [Nee04] (see also the Appendix in [Nee04] for some of the terminology that we are using).

Throughout the rest of this section, K will be a connected and C^∞ -regular Lie subgroup of $\text{Diff}(M)$ such that (θ, K_m) is a 2-fold transitive pair, where $\theta: K \times M \rightarrow M$, is the natural action of $K \leq \text{Diff}(M)$ on M and K_m is the stabiliser of the base-point m in K . In particular, $\text{ev}_m: K \rightarrow M$ then factors through a diffeomorphism $K/K_m \rightarrow M$ and K_m acts transitively on $M \setminus \{m\}$.

The Lie group $\text{Diff}(M)$ acts naturally and smoothly on $C^\infty(M) := C^\infty(M, \mathbb{R})$ via $\varphi.f := f \circ \varphi^{-1}$. This turns $\mathfrak{a} := C^\infty(M)$ into a K -module containing the constant functions $\mathfrak{a}_0 = \mathfrak{a}^K \cong \mathbb{R}$. If we set $\mathfrak{k} = \mathbf{L}(K)$ and $\mathfrak{k}_m = \mathbf{L}(K_m)$, then \mathfrak{a} is also a \mathfrak{k} -module for the derived action

$$(X.f)(n) := df_n(X(n)).$$

Moreover, $\mathfrak{a}_m := C_m^\infty(M)$ is a \mathfrak{k}_m -submodule for

$$C_m^\infty(M) := \{f \in C^\infty(M) \mid f(m) = 0\}.$$

Clearly, \mathfrak{a}_0 is a module complement for this \mathfrak{k}_m -submodule, so that we have $\mathfrak{a} \cong \mathfrak{a}_0 \oplus \mathfrak{a}_m$ as \mathfrak{k}_m -modules.

The last piece of information that we choose is a closed 2-form $\omega \in \Omega^2(M) := \Omega^2(M, \mathbb{R})$. This gives rise to the abelian cocycle

$$\overline{\omega}: \mathfrak{k} \times \mathfrak{k} \rightarrow \mathfrak{a}, \quad (X, Y) \mapsto (n \mapsto \omega_n(X(n), Y(n))),$$

i.e.,

$$[(f, X), (g, Y)] := (X.g - Y.f + \bar{\omega}(X, Y), [X, Y])$$

defines on $\mathfrak{a} \oplus \mathfrak{k}$ the structure of a Lie algebra. We denote this Lie algebra by $\mathfrak{a} \oplus_{\bar{\omega}} \mathfrak{k}$, and the canonical maps give rise to an abelian extension

$$\mathfrak{a} \rightarrow \mathfrak{a} \oplus_{\bar{\omega}} \mathfrak{k} \rightarrow \mathfrak{k}$$

of topological Lie algebras. Moreover, as elements of \mathfrak{k}_m vanish when they are evaluated in m , we have $\bar{\omega}(\mathfrak{k}_m \times \mathfrak{k}_m) \subseteq \mathfrak{a}_m$. Hence there is a subalgebra

$$\mathfrak{a}_m \oplus_{\bar{\omega}_m} \mathfrak{k}_m \leq \mathfrak{a} \oplus_{\bar{\omega}} \mathfrak{k},$$

where $\bar{\omega}_m: \mathfrak{k}_m \times \mathfrak{k}_m \rightarrow \mathfrak{a}_m$ denotes the restriction (and corestriction) of $\bar{\omega}$ to $\mathfrak{k}_m \times \mathfrak{k}_m$ (and \mathfrak{a}_m).

Before we go on we give the two examples of the above situations that we have in mind.

Example 5.1. a) The easiest example is that where $K = \text{Diff}(M)_0$ and $\mathfrak{a} = C^\infty(M)$. Then we have seen in Example 3.5 that $\text{Diff}_m(M)$ is a regular Lie subgroup of $\text{Diff}(M)$. Moreover, $\text{Diff}(M)$ acts smoothly transitively on M by Corollary 2.17 and Corollary 2.5. Finally, $\text{Diff}(M)$ acts 2-fold transitively on M [MV94]. In this case $\omega \in \Omega^2(M)$ can be an arbitrary closed 2-form on M (cf. [Nee04, Section 9]).

b) For the next example, let $\omega \in \Omega^2(M)$ be symplectic and $\dim(M) = 2d$. Then

$$\text{Symp}(M, \omega) := \{\varphi \in \text{Diff}(M) \mid \varphi^*\omega - \omega = 0\}$$

is a closed Lie subgroup of $\text{Diff}(M)$ [KM97, Theorem 43.12] with Lie algebra

$$\mathfrak{symp}(M, \omega) := \{X \in \mathcal{V}(M) \mid \mathcal{L}_X\omega = d(i_X\omega) = 0\}$$

the symplectic vector fields on M . This Lie algebra has the Hamiltonian vector fields

$$\mathfrak{ham}(M, \omega) := \{X \in \mathfrak{symp}(M) \mid i_X\omega \text{ is exact}\}$$

as closed subalgebra. Moreover, the chart constructed in the proof of [KM97, Theorem 43.12] maps

$$\mathfrak{k}_m := \mathfrak{symp}_m(M, \omega) := \{X \in \mathfrak{symp}(M, \omega) \mid X(m) = 0\}$$

to

$$\text{Symp}_m(M, \omega) := \{\varphi \in \text{Symp}(M, \omega) \mid \varphi(m) = m\}.$$

Thus $K := \text{Symp}(M, \omega)_0$ has $K_m = \text{Symp}_m(M, \omega)_0$ as Lie subgroup with Lie algebra \mathfrak{k}_m . That (θ, K_m) is indeed a transitive pair will follow from the following two propositions. Finally, K acts 2-fold transitively by [MV94]. ■

Proposition 5.2. *Let (M, ω) be a compact symplectic manifold and $x \in M$. Then the evaluation map*

$$\text{ev}_x: \text{Symp}(M, \omega)_0 \rightarrow M, \quad \varphi \mapsto \varphi(x)$$

is a submersion. If, moreover, M is connected, then ev_x is also surjective.

Proof. We show that ev_x is a submersion by showing that $T_{\text{id}} \text{ev}_x: T_{\text{id}} \text{Symp}(M, \omega) \rightarrow T_x M$ is surjective. This suffices by [Glö15a, Theorem A]. To this end, note that $T_{\text{id}} \text{ev}_x$ is given with respect to the identification $\mathfrak{symp}(M, \omega) \cong T_{\text{id}} \text{Symp}(M, \omega)$ by

$$\text{ev}_x: \mathfrak{symp}(M, \omega) \rightarrow T_x M, \quad X \mapsto X(x)$$

(cf. [KM97, Corollary 42.18] or [SW15b, Theorem 7.9]). Thus the claim follows from observing that for each $v \in T_x M$ there exists a function $H \in C^\infty(M)$ with $dH_x = \omega_x(v, \cdot)$, and thus a Hamiltonian vector field X (which is then in particular symplectic) with $X(x) = v$.

Since the point $x \in M$ in the above argument was arbitrary, this shows in particular that each orbit of the natural action of $\text{Symp}(M, \omega)$ on M is open and thus consist of unions of path components. In particular, ev_x is surjective if M is connected. \blacksquare

Proposition 5.3. *Let (M, ω) be a compact symplectic manifold and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\text{Symp}(M, \omega)$ and*

$$\text{Symp}_m(M, \omega) := \{\varphi \in \text{Symp}(M, \omega) \mid \varphi(m) = m\}$$

are C^k -regular Lie subgroups of $\text{Diff}(M)$.

Proof. By [KM97, Theorem 43.12] the Lie group $\text{Symp}(M, \omega)$ is a closed Lie subgroup of $\text{Diff}(M)$. Recall from [SW15b, Theorem 5.5] that $\text{Diff}(M)$ is C^k -regular. Arguing as in the proof of [KM97, Theorem 43.12] (where C^∞ -(semi)regularity for $\text{Symp}(M, \omega)$ was established), one proves that $\text{Symp}(M, \omega)$ is C^k -semiregular. Hence Lemma B.5 implies that $\text{Symp}(M, \omega)$ is C^k -regular.

Fix some $\eta \in C^k([0, 1], \mathfrak{symp}_m(M, \omega))$ and let γ_η be a solution in $\text{Symp}(M, \omega)$ of the corresponding initial value problem from Definition B.3. Note that γ_η is also a solution of the corresponding initial value problem in $\text{Diff}(M)$. By Proposition 3.4, η_γ takes its image in $\text{Symp}_m(M, \omega)$, whence $\text{Symp}_m(M, \omega)$ is C^k -semiregular. Again by Lemma B.5, $\text{Symp}_m(M, \omega)$ is also C^k -regular. \blacksquare

For later reference we also record the following fact (see also [JV15, Corollary 3.5]).

Proposition 5.4. *If (M, ω) is a compact and connected symplectic manifold, then the restriction of the abelian cocycle*

$$\bar{\omega}: \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow C^\infty(M), \quad (X, Y) \mapsto \omega(X, Y)$$

to $\mathfrak{ham}(M, \omega)$ is a coboundary. This also applies to the restriction of $\bar{\omega}$ to $\mathfrak{symp}(M, \omega)$ if M is 1-connected.

Proof. We consider the continuous linear surjection $\xi: C^\infty(M) \rightarrow \mathfrak{ham}(M, \omega)$, $f \mapsto X_f$, where X_f is the unique vector field such that $i_{X_f} \omega = df$. Since $\ker(\xi) \cong \mathbb{R}$ is finite-dimensional and thus complemented, ξ has a continuous linear section $\sigma: \mathfrak{ham}(M, \omega) \rightarrow C^\infty(M)$. We claim that

$$b: \mathfrak{ham}(M, \omega) \rightarrow C^\infty(M), \quad X \mapsto \int_M \sigma(X) \omega^d - \sigma(X)$$

satisfies $d_{\text{CE}b} = \omega$, where $d = \frac{\dim(M)}{2}$ and we consider \mathbb{R} as constant functions on M . To this end we note that we have the equality

$$\omega(X, Y) - \int_M \omega(X, Y) \omega^d = \sigma([X, Y]) - \int_M \sigma([X, Y]) \omega^d. \quad (22)$$

Indeed, both sides of (22) are uniquely determined by the property of being mapped to $[X, Y]$ under ξ and having vanishing integral over M . If $X = X_f$ and $Y = X_g$ for some $f, g \in C^\infty(M)$, then we also have

$$\int_M \omega(X, Y) \omega^d = \int_M \{f, g\} \omega^d = \int_M (\mathcal{L}_{X_f} g) \omega^d = \int_M \mathcal{L}_{X_f}(g \omega^d) = \int_M d(i_{X_f} g \omega^n) = 0. \quad (23)$$

Thus we conclude from (23) and (22) that

$$d_{\text{CE}b}(X, Y) = -X.\sigma(Y) + Y.\sigma(X) - \left(\int_M \sigma([X, Y]) \omega^d - \sigma([X, Y]) \right) = \omega(X, Y).$$

If M is also simply connected, then $\mathfrak{ham}(M, \omega) = \mathfrak{symp}(M, \omega)$ and the assertion follows. \blacksquare

Remark 5.5. We shortly fix our conventions about the periods and prequantisation of presymplectic manifolds and integration of abelian extensions of Lie algebras to Lie group extensions. Let N be an arbitrary

manifold, V be a locally convex space and $\omega \in \Omega^2(M, V)$ be a V -valued closed 2-form. Then the *period homomorphism* associated to the cohomology class $[\omega] \in H_{\text{dR}}^2(N, V)$ is the homomorphism

$$\text{per}_{[\omega]}: \pi_2(N) \rightarrow V, \quad [\sigma] \mapsto \int_{S^2} \sigma^* \omega,$$

where $\sigma: S^2 \rightarrow N$ is a smooth representative of $[\sigma]$ (cf. [Nee02, Appendix A.3] or [Woc09, Corollary 14]). Moreover, let $\Gamma \subseteq \mathbb{R}$ an arbitrary but fixed discrete subgroup and set $A_\Gamma := \mathfrak{a}/\Gamma$ denote by $q_\Gamma: \mathfrak{a} \rightarrow A_\Gamma$ the canonical quotient homomorphism. In addition, we assume that V comes along with a distinguished embedding $\mathbb{R} \hookrightarrow V$.

We now consider two special cases of this. At first, assume that $V = \mathbb{R}$ and that N is finite-dimensional (or more generally smoothly paracompact) and 1-connected. Then the subgroup $\text{per}_{[\omega]}(\pi_2(M))$ is called the group of *periods* (or shortly just the periods) of $(M, [\omega])$. Moreover, we say that (M, ω) is Γ -*prequantisable* if there a principal $\mathbb{T}_\Gamma := \mathbb{R}/\Gamma$ -bundle $P \rightarrow N$ that admits a connection with curvature ω . By the general theory (see, e.g., [Woo92, Chapter 8] or [Kos70]) this is the case if and only if the periods $\text{per}_{[\omega]}(\pi_2(M))$ are contained in Γ (or more generally if $[\omega]$ is contained in the image of $H^2(N, \Gamma) \rightarrow H^2(M, V)$ if N is not simply connected).

The other case is that $N = K$ and $V = \mathfrak{a}$ for K and \mathfrak{a} . Then we consider the equivariant extension $\overline{\omega}^{\text{eq}} \in \Omega^2(K, \mathfrak{a})$ of $\overline{\omega}$, which is given by

$$(\overline{\omega}^{\text{eq}})_k: T_k K \times T_k K \rightarrow \mathfrak{a}, \quad (X, Y) \mapsto k \cdot \overline{\omega}(X \cdot k^{-1}, Y \cdot k^{-1}).$$

Note that we have here used the right trivialisation of the tangent bundle of a Lie group, which is more adapted to our setting than the left trivialisation (cf. [SW15b, Remarks 2.4 and 2.6]). Then we call $\text{per}_{[\overline{\omega}^{\text{eq}}]}(\pi_2(K))$ the group of *primary periods* (or shortly just the primary periods) of $(K, [\omega])$. Note that they really only depend on the cohomology class of $[\omega]$ in $H_{\text{dR}}^2(M, \mathbb{R})$ by the following lemma. Moreover, the primary periods $\text{per}_{[\overline{\omega}^{\text{eq}}]}(\pi_2(K))$ are contained in the fixed points \mathfrak{a}^K [Nee04, Lemma 4.2]. If they are also contained in Γ , then there exists an extension of Lie groups

$$A_\Gamma \rightarrow K^\sharp \xrightarrow{q^\sharp} \tilde{K}, \quad (24)$$

whose underlying extension of Lie algebras is equivalent to $\mathfrak{a} \rightarrow \mathfrak{a} \oplus_\omega \mathfrak{g} \rightarrow \mathfrak{g}$ [Nee04, Theorem 6.7]. Here and in the sequel, $q_{\pi_1(K)}: \tilde{K} \rightarrow K$ denotes the universal covering morphism. The extension (24) is uniquely determined (up to equivalence) by the associated Lie algebra extension [Nee04, Theorem 7.2]. Moreover, $K^\sharp \rightarrow \tilde{K}$ is a principal A_Γ -bundle that admits a connection with curvature ω^{eq} , so ω^{eq} is Γ -prequantisable on \tilde{K} in this case (cf. the proof of Lemma 5.13). The question, whether $A_\Gamma \rightarrow K^\sharp \rightarrow \tilde{K}$ factors to an extension of K by A_Γ is then controlled by the flux homomorphism

$$F_{[\overline{\omega}]}: \pi_1(K) \rightarrow H_{\text{dR}}^1(M, \mathbb{R}) \subseteq H_c^1(\mathfrak{k}, \mathfrak{a}).$$

(cf. [Nee04, Lemma 6.2, Proposition 6.3 and Proposition 9.13]). Since M is assumed to be 1-connected, $F_{[\overline{\omega}]}$ vanishes automatically. Almost all flux phenomena will be irrelevant for this paper since we only work with 1-connected M . The flux will only occur shortly in Remark 5.12. One main observation of this section is that the integration of the Lie algebra extension $\mathfrak{a} \rightarrow \mathfrak{a} \oplus_\omega \mathfrak{k} \rightarrow \mathfrak{k}$ to a transitive pair is *not* governed by the flux, but rather by the *secondary* periods that we will introduce in Remark 5.10. ■

Lemma 5.6. *The map*

$$H_{\text{dR}}^2(M, \mathbb{R}) \rightarrow H_c^2(\mathcal{V}(M), C^\infty(M)), \quad [\omega] \mapsto [\overline{\omega}]. \quad (25)$$

is injective.

Proof. We may assume without loss of generality that $\dim(M) \geq 2$. Let $H_{c,\Delta}^2(\mathcal{V}(M), C^\infty(M))$ be the diagonal cohomology of $\mathcal{V}(M)$ with coefficients in $C^\infty(M)$, i.e., the cohomology of the subcomplex of \mathbb{R} -linear alternating cochains

$$\xi: \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow C^\infty(M)$$

for which $\xi(X, Y)(m)$ only depends on the germs of X and Y at m . Since $\omega \in \Omega^2(M)$ is C^∞ -linear in both arguments, $\overline{\omega}$ is a cocycle of this kind, so that the map (25) factors as

$$H_{\text{dR}}^2(M, \mathbb{R}) \rightarrow H_{c, \Delta}^2(\mathcal{V}(M), C^\infty(M)) \rightarrow H_c^2(\mathcal{V}(M), C^\infty(M)). \quad (26)$$

Now the first map in (26) is injective by [Los70, Corollary 2] and the second map in (26) is injective by [Fuk86, Theorem 2.4.10], and the assertion follows. \blacksquare

The following is the only result on the flux homomorphism that we shall need in the sequel.

Lemma 5.7. *The flux*

$$F_{[\overline{\omega}_m]}: \pi_1(K_m) \rightarrow H_c^1(\mathfrak{k}_m, \mathfrak{a}_m)$$

also vanishes. More generally, if M is only assumed to be compact, then $F_{[\overline{\omega}_m]}$ vanishes if $F_{[\overline{\omega}]}$ does so.

Proof. This follows immediately from the commuting diagram

$$\begin{array}{ccccccc} \widetilde{K}_m & \longrightarrow & \widetilde{K} & \xrightarrow{F_{[\overline{\omega}]}} & H_{\text{dR}}^1(M, \mathbb{R}) & \hookrightarrow & H_c^1(\mathfrak{k}, \mathfrak{a}) \xrightarrow{\text{res}_{\mathfrak{k}_m}} H_c^1(\mathfrak{k}_m, \mathfrak{a}) \\ \parallel & & & & & & \downarrow \text{pr}_m \\ \widetilde{K}_m & \xrightarrow{F_{[\overline{\omega}_m]}} & & & & & H_c^1(\mathfrak{k}_m, \mathfrak{a}_m), \end{array}$$

where $\widetilde{K}_m \rightarrow \widetilde{K}$ is induced by the inclusion $K_m \hookrightarrow K$, $\text{res}_{\mathfrak{k}_m}$ is induced by the inclusion $\mathfrak{k}_m \hookrightarrow \mathfrak{k}$ and pr_m is induced by the morphism $\mathfrak{a} \rightarrow \mathfrak{a}_m$, $f \mapsto f - f(m)$ of \mathfrak{k}_m -modules. \blacksquare

The previous remark introduces the most important concepts from the integration theory of abelian extensions of Lie groups that occur in the sequel, for the rest see [Nee04]. From Proposition 5.4 we immediately obtain the following

Corollary 5.8. *If $K = \text{Symp}(M, \omega)_0$ for $\omega \in \Omega^2(M)$ symplectic, then $\text{per}_{[\overline{\omega}]}(\pi_2(K)) = 0$.*

Remark 5.9. Keeping the notation as in Remark 5.5, we can also integrate the restricted extension

$$\mathfrak{a}_m \rightarrow \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \rightarrow \mathfrak{k}_m$$

to a unique extension

$$\mathfrak{a}_m \rightarrow K_m^\sharp \rightarrow \widetilde{K}_m.$$

Indeed, $\text{per}_{\overline{\omega}_m}(\pi_2(K_m))$ is contained in the fixed-points of K_m . Since K_m is assumed to act transitively on $M \setminus \{m\}$, it follows that

$$(\mathfrak{a}_m)^{K_m} = \{f \in C^\infty(M) \mid f \text{ is constant on } M \setminus \{m\} \text{ and } f(m) = 0\} = \{0\}$$

by the continuity of f . Consequently, $\text{per}_{\overline{\omega}_m}(\pi_2(K_m)) = \{0\}$ and [Nee04, Theorem 6.7] implies that there exists an extension $\mathfrak{a}_m \rightarrow K_m^\sharp \rightarrow \widetilde{K}_m$ whose derived extension is equivalent to $\mathfrak{a}_m \rightarrow \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \rightarrow \mathfrak{k}_m$. \blacksquare

Now we have the extensions $K^\sharp \rightarrow \widetilde{K}$ from Remark 5.9 and $K_m^\sharp \rightarrow \widetilde{K}_m$ from Remark 5.9 at hand, and we want to build our transitive pair from them. The question is now how these two extensions are related to one another. The crucial information about this is contained in the modification of the period homomorphism $\text{per}_{[\omega]}^\flat$ and the associated secondary periods that we introduce and analyse now.

Remark 5.10. From the transitive action of K on M we obtain the principal K_m -bundle $\text{ev}_m: K \rightarrow M$. This gives in particular rise to the long exact sequence

$$\cdots \rightarrow \pi_2(K_m) \rightarrow \pi_2(K) \xrightarrow{\text{ev}_m} \pi_2(M) \xrightarrow{\delta} \pi_1(K_m) \rightarrow \pi_1(K) \rightarrow \cdots$$

At $\pi_2(M)$, this induces a short exact sequence

$$0 \rightarrow \Lambda \xrightarrow{\text{ev}_m} \pi_2(M) \xrightarrow{\delta} \Delta \rightarrow 0$$

with $\Delta := \ker(\pi_1(K_m) \rightarrow \pi_1(K))$, $\Lambda := \text{im}(\pi_2(K) \rightarrow \pi_2(M))$. From [Nee11, Theorem 3.18] it follows that $\text{per}_{[\varpi]}([\gamma]) = \text{per}_{[\omega]}([\text{ev}_m \circ \gamma])$ holds for $\gamma \in C_*^\infty(\mathbb{S}^2, K)$ and thus $\text{per}_{[\varpi]}$ factors through $\text{per}_{[\omega]} \Big|_{\Lambda}$.

If now $\Gamma \leq \mathbb{R}$ is a (not necessarily discrete) subgroup that contains $\text{per}_{[\varpi]}(\pi_2(K))$ and $q_\Gamma: \mathbb{R} \rightarrow \mathbb{R}/\Gamma$ is the quotient map, then $q_\Gamma \circ \text{per}_{[\omega]}$ factors through δ and a homomorphism $\text{per}_{[\omega]}^b: \Delta \rightarrow \mathbb{R}/\Gamma$. This gives rise to a morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \pi_2(M) & \longrightarrow & \Delta & \longrightarrow & 0 \\ & & \downarrow \text{per}_{[\varpi]} & & \downarrow \text{per}_{[\omega]} & & \downarrow \text{per}_{[\omega]}^b & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\Gamma & \longrightarrow & 0 \end{array}$$

of short exact sequences. ■

Definition 5.11. If, in the setting of the previous remark, $\Gamma := \text{per}_{[\varpi]}(\pi_2(K))$ is the group of primary periods (cf. Remark 5.5), then we call

$$\text{per}_{[\omega]}^b(\pi_1(K_m)) \subseteq \mathbb{R}/\Gamma$$

the group of *secondary periods* (or shortly the secondary periods) of $(K, [\omega])$. ■

Remark 5.12. We keep the notation from Remark 5.5 and Remark 5.9. The inclusion $\mathfrak{a}_m \oplus_{\varpi_m} \mathfrak{k}_m \hookrightarrow \mathfrak{a} \oplus_{\omega} \mathfrak{k}$ induces a unique homomorphism

$$\varphi_m: K_m^\# \rightarrow K$$

(cf. [Mil84, Theorem 8.1]) that makes the diagram

$$\begin{array}{ccccc} \mathfrak{a}_m & \longrightarrow & K_m^\# & \xrightarrow{q_m^\#} & \widetilde{K}_m \\ \downarrow & & \downarrow \varphi_m & & \downarrow q_\Delta \\ A_\Gamma & \longrightarrow & K^\# & \longrightarrow & \widetilde{K} \end{array} \quad (27)$$

commute. Here, $\mathfrak{a}_m \hookrightarrow A_\Gamma = \mathfrak{a}/\Gamma$ is the morphism induced from the embedding $\mathfrak{a}_m \hookrightarrow \mathfrak{a}$ and the quotient $\mathfrak{a} \rightarrow \mathfrak{a}/\Gamma$ (note that $\mathfrak{a}_m \cap \Gamma = \{0\}$). The morphism $q_\Delta: \widetilde{K}_m \rightarrow \widetilde{K}$ in (27) is induced from the inclusion $K_m \hookrightarrow K$. The kernel $\ker(q_\Delta)$ can be identified with $\Delta = \ker(\pi_1(K_m) \rightarrow \pi_1(K))$ and the image with \widetilde{K}_m/Δ . Then the latter is a closed Lie subgroup of \widetilde{K} . This gives rise to an extension

$$A_m^\# \longrightarrow K_m^\# \xrightarrow{q_\Delta \circ q_m^\#} \widetilde{K}_m/\Delta$$

of \widetilde{K}_m/Δ by $A_m^\# := (q_m^\#)^{-1}(\Delta)$. Now $F_{[\omega_m]}$ vanishes by Lemma 5.7, whence there exists a homomorphism $\sigma: \Delta \rightarrow Z(K_m^\#)$ with $q_m \circ \sigma = \text{id}_\Delta$ [Nee04, Corollary 6.6]. ■

Lemma 5.13. *In the situation of Remark 5.12 and Remark 5.10 we have*

$$\varphi_m(\sigma(x)) + \mathfrak{a}_m = \text{per}_{[\omega]}^b(x) + \mathfrak{a}_m$$

for each $x \in \Delta$.

Proof. We first recall the following fact about abelian extensions of Lie groups. If $A \rightarrow \widehat{G} \xrightarrow{q} G$ is such an extension and if $\tau: \mathfrak{g} := \mathbf{L}(G) \rightarrow \widehat{\mathfrak{g}} := \mathbf{L}(\widehat{G})$ is a continuous linear splitting of the induced extension

$\mathbf{L}(A) \rightarrow \mathbf{L}(\widehat{G}) \rightarrow \mathbf{L}(G)$, then we obtain a connection on the principal right A -bundle $q: \widehat{G} \rightarrow G$, induced for each $\widehat{g} \in \widehat{G}$ by the horizontal lift of tangent vectors

$$\tau_{\widehat{g}}: T_g G \rightarrow T_{\widehat{g}} \widehat{G}, \quad v \mapsto \tau(v \cdot g^{-1}) \cdot \widehat{g}$$

where $g := q(\widehat{g})$. The curvature of this connection is given by ω_τ^{eq} , where $\omega_\tau: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbf{L}(A)$ is the abelian Lie algebra cocycle $(x, y) \mapsto \tau([x, y]) - [\tau(x), \tau(y)]$. Note again that we have here used the right trivialisation of the tangent bundle.

For each $x \in \Delta$ we now choose a representative $\gamma_x \in C_*^\infty(\mathbb{S}^1, K_m)$ in $\pi_1(K_m)$ such that there exists a smooth map $F_x: [0, 1] \times \mathbb{S}^1 \rightarrow K$ with $F_x(0, \cdot) \equiv e_K$, $F_x(1, \cdot) = \gamma_x$ and $F_x(t, *) = \text{id}_M$ for all $t \in [0, 1]$. If γ_x^\sharp denotes the horizontal lift of γ_x , induced by the canonical linear splitting $\mathfrak{k}_m \rightarrow \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ and the Lie algebra isomorphism $\mathbf{L}(K_m^\sharp) \cong \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$, then there exists some $b_\sigma: \Delta \rightarrow \mathfrak{a}_m$ such that $\sigma(x) = \gamma_x^\sharp(1) + b_\sigma(x)$ (cf. [Nee04, Corollary 6.6]).

By the construction of the morphism φ_m (cf. [Mil84, Theorem 8.1]),

$$\varphi_m(\sigma(x)) = \varphi_m(\gamma_x^\sharp(1)) + \varphi_m(b_\sigma(x)) = \Gamma_x(1) + \varphi_m(b_\sigma(x)),$$

where $\Gamma_x \in C^\infty([0, 1], K^\sharp)$ is the solution of the initial value problem in K^\sharp with $\Gamma_x(0) = e_{K^\sharp}$ and $\delta^l(\Gamma_x) = \iota \circ \delta^l(\gamma_x^\sharp)$ for $\iota: \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \rightarrow \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ the canonical embedding. Note that Γ_x exists as K^\sharp is C^∞ -regular as an extension of C^∞ -regular Lie groups (cf. [NS12, Appendix B]). This implies in particular that Γ_x is a horizontal lift of the smooth loop $\gamma'_x := q_\Delta \circ \gamma_x$ for the connection on the principal A_Γ -bundle $K^\sharp \rightarrow \widetilde{K}$ that is induced by the canonical linear splitting $\mathfrak{k} \rightarrow \mathfrak{a} \oplus \mathfrak{k}$ and the Lie algebra isomorphism $\mathbf{L}(K^\sharp) \cong \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$. Consequently, $\varphi_m(\gamma_x^\sharp(1)) = \Gamma_x(1)$ equals the holonomy of the loop γ'_x for this connection.

If we take the restriction $K^\sharp|_{\widetilde{K}_m/\Delta}$ of the A_Γ -bundle $K^\sharp \rightarrow K$ to the Lie subgroup \widetilde{K}_m/Δ , then the curvature of the above connection takes values in the subspace \mathfrak{a}_m of $A_\Gamma \cong \mathfrak{a}_m \times \mathbb{T}_\Gamma$. Consequently, the connection is flat modulo \mathfrak{a}_m and thus the \mathbb{T}_Γ -component of the holonomy can be computed as the integral of the curvature over any filler of the loop γ'_x . This implies

$$\Gamma_x(1) + \mathfrak{a}_m = \text{hol}(\gamma'_x) + \mathfrak{a}_m = \int_F \omega^{\text{eq}} + \mathfrak{a}_m = \text{per}_{[\omega]}^b(x) + \mathfrak{a}_m.$$

Since $\varphi_m(b_\sigma(x)) \subseteq \varphi_m(\mathfrak{a}_m) \subseteq \mathfrak{a}_m$, this establishes the claim. \blacksquare

The following example illustrates the rôle of $\text{per}_{[\omega]}^b$ pretty well.

Example 5.14. Let ω be the standard volume form on $M := \mathbb{S}^2$ with total volume 1. Then we consider the subgroup $K := \text{Diff}(\mathbb{S}^2)_0$. The action of $\text{SO}_3(\mathbb{R})$ on \mathbb{S}^2 by rotations induces a map $\text{SO}_3(\mathbb{R}) \rightarrow \text{Diff}(\mathbb{S}^2)_0$, which is a homotopy equivalence [Sma59]. Consequently, $\pi_2(K) = 0$ and the primary periods vanish. Thus we may take $\Gamma = 0$ to integrate the extension

$$C^\infty(\mathbb{S}^2) \rightarrow C^\infty(\mathbb{S}^2) \oplus_{\overline{\omega}} \mathcal{V}(\mathbb{S}^2) \rightarrow \mathcal{V}(\mathbb{S}^2)$$

to an extension

$$C^\infty(\mathbb{S}^2) \rightarrow K^\sharp \rightarrow \widetilde{K}$$

of Lie groups (cf. Remark 5.5). From the five lemma and the long exact sequence in homotopy groups of the fibrations $K_m \rightarrow K \rightarrow \mathbb{S}^2$ and $\text{SO}_2(\mathbb{R}) \rightarrow \text{SO}_3(\mathbb{R}) \rightarrow \mathbb{S}^2$ it follows that the induced map $\text{SO}_2(\mathbb{R}) \rightarrow K_m$ is also a homotopy equivalence. From this it follows that the exact sequence

$$\pi_2(K) \rightarrow \pi_2(\mathbb{S}^2) \rightarrow \pi_1(K_m) \rightarrow \pi_1(K) \rightarrow \pi_1(\mathbb{S}^2)$$

identifies with

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

and with respect to this identification we have $\Delta = 2\mathbb{Z}$. Since $\text{per}_{[\omega]}$ is given by the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$, it follows that $\text{per}_{[\omega]}^b$ is given by

$$\text{per}_{[\omega]}^b : \Delta \cong 2\mathbb{Z} \rightarrow \mathbb{R}, \quad 2x \mapsto x.$$

Thus the secondary periods coincide with \mathbb{Z} and $\varphi_m(A_m^\sharp) = C_m^\infty(\mathbb{S}^2) \times \mathbb{Z}$. \blacksquare

We now put all the bits and pieces that we have collected so far together.

Theorem 5.15. *Let M be a compact and 1-connected manifold, $\omega \in \Omega^2(M)$ be closed and let $K \leq \text{Diff}(M)$ be a connected and C^∞ -regular Lie subgroup such that $\text{ev}_m : \widetilde{K} \rightarrow M$ is a subjective submersion and K acts 2-fold transitively on M . Let Δ be the kernel of the map $\widetilde{K}_m \rightarrow \widetilde{K}$ induced on the universal cover by the inclusion $K_m \hookrightarrow K$. Suppose $\Gamma, \Pi \subseteq \mathbb{R}$ are discrete subgroups with $\text{per}_{[\omega]} \subseteq \Gamma \subseteq \Pi$, set $A_\Gamma := C^\infty(M, \mathbb{R})/\Gamma$ and identify A_Γ with $C_m^\infty(M) \times \mathbb{T}_\Gamma$ (as abelian Lie groups or as K_m -modules). Then the following assertions are equivalent:*

a) $\text{per}_{[\omega]}^b(\Delta) \subseteq \Pi/\Gamma$.

b) Let

$$A_\Gamma \rightarrow K^\sharp \xrightarrow{q^\sharp} \widetilde{K}$$

be the unique extension of \widetilde{K} by A_Γ whose Lie algebra extension is equivalent to $\mathfrak{a} \rightarrow \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k} \rightarrow \mathfrak{k}$ with $\mathfrak{a} := C^\infty(M)$. If $\mathfrak{a}_m := C_m^\infty(M)$, then the closed Lie subalgebra $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ of $\mathfrak{a} \oplus \mathfrak{k}$ integrates to a closed Lie subgroup I_m of K^\sharp such that $I_m \cap A_\Gamma \subseteq \mathfrak{a}_m \times \Pi/\Gamma$.

c) Let

$$\mathfrak{a}_m \rightarrow K_m^\sharp \xrightarrow{q_m^\sharp} \widetilde{K}_m$$

be the unique extension of \widetilde{K}_m by \mathfrak{a}_m whose Lie algebra extension is equivalent to $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ and set $A_m^\sharp := (q_m^\sharp)^{-1}(\Delta)$. Then the image of the Lie group morphism $\varphi_m : K_m^\sharp \rightarrow K^\sharp$ induced by the canonical embedding $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \rightarrow \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ is a closed Lie subgroup and $\varphi_m(A_m^\sharp) \subseteq C_m^\infty(M, \mathbb{R}) \times \Pi/\Gamma$.

If one (and thus all) of these conditions is satisfied, then the composition of the maps

$$\theta^\wedge : K^\sharp \xrightarrow{q^\sharp} \widetilde{K} \xrightarrow{q_{\pi_1(K)}} K \hookrightarrow \text{Diff}(M)$$

gives rise to a transitive pair $(\theta, (\varphi_m(K_m^\sharp) \cdot \Pi/\Gamma))$ with kernel Π/Γ . Moreover, the associated principal \mathbb{T}_{Π} -bundle $K^\sharp / (\varphi_m(K_m^\sharp) \cdot \Pi/\Gamma) \rightarrow M$ admits a connection whose curvature equals ω and thus is (together with the choice of such a connection) a Π -prequantisation of ω .

Proof. We will use throughout that C^∞ -regularity is an extension property [NS12, Appendix B], so that all extensions that appear will automatically be C^∞ -regular (which suffices to integrate Lie algebra morphisms to 1-connected Lie groups).

a) \Rightarrow c): Consider \widetilde{K}_m/Δ as a closed Lie subgroup of \widetilde{K} and obtain from this the extension

$$A_\Gamma \rightarrow K^\sharp|_{\widetilde{K}_m/\Delta} \rightarrow \widetilde{K}_m/\Delta.$$

Since $A_\Gamma \cong \mathfrak{a}_m \times \mathbb{T}_\Gamma$ is a decomposition of A_Γ as a \widetilde{K}_m/Δ -module and $\sigma : \Delta \rightarrow K^\sharp$ induces the isomorphism $A_m^\sharp \cong \mathfrak{a}_m \times \Delta$, we obtain from Remark 5.12 and Lemma 5.13 the morphism

$$\begin{array}{ccccc} \mathfrak{a}_m \times \Delta & \longrightarrow & K_m^\sharp & \longrightarrow & \widetilde{K}_m/\Delta \\ \text{id}_{\mathfrak{a}_m} \times \text{per}_{[\omega]}^b \downarrow & & \downarrow \varphi_m & & \downarrow \\ A_\Gamma & \longrightarrow & K^\sharp & \longrightarrow & \widetilde{K} \end{array}$$

of extensions of Lie groups. Since $\widetilde{K}_m/\Delta \hookrightarrow \widetilde{K}$ is injective and $\text{im}(\text{per}_{[\omega]}^\flat) \subseteq \Pi/\Gamma$ is discrete in \mathbb{T}_Γ , this implies that $K^\sharp|_{\widetilde{K}_m/\Delta}$ reduces to an extension of \widetilde{K}_m/Δ by the closed Lie subgroup $\mathfrak{a}_m \times \text{im}(\text{per}_{[\omega]}^\flat)$ of $\mathfrak{a}_m \times \mathbb{T}_\Gamma$. This reduction is itself a closed Lie subgroup which equals $\varphi_m(K_m^\sharp)$ by construction. Moreover, $\varphi_m(A_m^\sharp) \subseteq C_m^\infty(M) \times \Pi/\Gamma$ follows from $\mathfrak{a}_m \times \text{im}(\text{per}_{[\omega]}^\flat) \subseteq \mathfrak{a}_m \times \Pi/\Gamma$.

c) \Rightarrow b): As above we see that $\varphi_m(K_m^\sharp) \subseteq K^\sharp$ is a reduction of $K^\sharp|_{\widetilde{K}_m/\Delta}$ to an extension of \widetilde{K}_m/Δ by $\mathfrak{a}_m \times \text{im}(\text{per}_{[\omega]}^\flat)$, and thus in particular a closed Lie subgroup. Thus we may take $I_m := \varphi_m(K_m^\sharp)$.

b) \Rightarrow a): By construction we have that $\widetilde{I}_m \cong K_m^\sharp$, so that φ_m factors through the inclusion $I_m \hookrightarrow K^\sharp$ and the universal covering map $K_m^\sharp \rightarrow I_m$. Thus $\text{per}_{[\omega]}^\flat(\Delta) \subseteq \Pi/\Gamma$ follows from $I_m \cap A_\Gamma \subseteq \mathfrak{a}_m \times \Pi/\Gamma$.

It remains to show the assertion that if **c)** is satisfied, then $(\theta, \varphi_m(K_m^\sharp) \cdot \Pi/\Gamma)$ is a transitive pair with kernel Π/Γ and that $K^\sharp/(\varphi_m(K_m^\sharp) \cdot \Pi/\Gamma) \rightarrow M$ is a Π -prequantisation of ω .

Set $H_m := \varphi_m(K_m^\sharp) \cdot \Pi/\Gamma$. To show that (θ, H_m) is a transitive pair, we first show that H_m is in fact a normal subgroup of $(\theta^\wedge)^{-1}(K_m)$. In fact, each $g \in (\theta^\wedge)^{-1}(K_m)$ may be written as a product $a_0 \cdot g_m$ for $g_m \in \varphi_m(K_m^\sharp)$ and $a_0 \in \mathbb{T}_\Gamma$. This is due to the fact that $\varphi_m(K_m^\sharp)$ is a reduction of $K^\sharp|_{\widetilde{K}_m/\Delta}$ to an extension by $C_m^\infty(M, \mathbb{R}) \times \text{im}(\text{per}_{[\omega]}^\flat)$ and that $C_m^\infty(M, \mathbb{R}) \cdot \mathbb{T}_\Gamma \cong C^\infty(M, \mathbb{R})/\Gamma = A_\Gamma$. Since the action of $k \in \widetilde{K}$ on $C^\infty(M, \mathbb{T}_\Gamma)$ coincides with conjugation action of an arbitrary lift of k to K^\sharp it follows that $a_0 \in \mathbb{T}_\Gamma = C^\infty(M, \mathbb{T}_\Gamma)^{\widetilde{K}} \subseteq Z(K^\sharp)$ and thus $\text{Ad}(a_0) = \text{id}_{K^\sharp}$. Consequently, $\text{Ad}(g) = \text{Ad}(g_m)$ and thus $\text{Ad}(g)$ preserves the subalgebra $\mathbf{L}(\varphi_m(K_m^\sharp))$. Furthermore, $\text{Ad}(g) = \mathbf{L}(c_g)$ (where c_g is conjugation by g and the groups K^\sharp and $\varphi_m(K_m^\sharp)$ are connected and regular Lie groups. Thus Lemma B.6 implies that conjugation by g preserves $\varphi_m(K_m^\sharp)$. Since $H_m = \varphi_m(K_m^\sharp) \cdot \Pi/\Gamma$ and $\Pi/\Gamma \subseteq \mathbb{T}_\Gamma \subseteq Z(K^\sharp)$ follows as above, it also follows that conjugation by g preserves H_m .

Since $g \in (\theta^\wedge)^{-1}(K_m)$ was arbitrary, this shows that H_m is normal in $(\theta^\wedge)^{-1}(K_m)$. To conclude that (θ, H_m) is a transitive pair it thus suffices to observe that $\theta(\cdot, m) = \text{ev}_m \circ q_{\pi_1(K)} \circ q^\sharp$ clearly is a submersion, and $\mathbf{L}(\widehat{G})/\mathbf{L}(H_m)$ is finite-dimensional, so H_m in particular co-Banach.

It remains to show that $P := K^\sharp/(H_m) \rightarrow M$ is a Π -prequantisation. First note that H_m is an extension of \widetilde{K}_m/Δ by $\mathfrak{a}_m \times \Pi/\Gamma$ and $(\theta^\wedge)^{-1}(K_m)$ is an extension of \widetilde{K}_m/Δ by $\mathfrak{a}_m \times \mathbb{T}_\Gamma$, so that the morphism of Lie groups

$$\mathbb{T}_\Pi \rightarrow ((\theta^\wedge)^{-1}(K_m))/H_m, \quad (28)$$

induced by mapping an element of \mathbb{T}_Π to the respective constant function, is an isomorphism. With respect to this isomorphism we endow $P \rightarrow M$ with the structure of a \mathbb{T}_Π -bundle over M .

We now construct a connection on $P \rightarrow M$ with curvature ω as follows. Let $H \leq T\widetilde{K}$ be a horizontal distribution on the bundle $\text{ev}_m: \widetilde{K} \rightarrow M$, i.e., we have each $k \in \widetilde{K}$ a subspace $\widetilde{H}_k \leq T_k\widetilde{K}$ such that $\widetilde{H}_{k \cdot k'} = \widetilde{H}_k \cdot k'$ for $k' \in \widetilde{K}_m/\Delta$ and that $T\text{ev}_m: \widetilde{H}_k \rightarrow T_{k(m)}M$ is a linear isomorphism. Denote by $\tilde{\sigma}: \mathcal{V}(M) \rightarrow \mathcal{V}(\widetilde{K})^{\widetilde{K}_m/\Delta}$ the corresponding horizontal lift of vector fields. On the bundle $q^\sharp: K^\sharp \rightarrow \widetilde{K}$ we have the connection which is induced by the isomorphisms $TR_{k^{-1}}: T_k\widetilde{K} \rightarrow T_e\widetilde{K} \cong \mathfrak{k}$, $TR_{\bar{k}^{-1}}: T_{\bar{k}}K^\sharp \rightarrow T_eK^\sharp \cong \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ and the canonical linear splitting $\sigma: \mathfrak{k} \rightarrow \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$. Denote the corresponding horizontal lift by $\sigma_{\overline{\omega}}: \mathcal{V}(\widetilde{K}) \rightarrow \mathcal{V}(K^\sharp)^{A_\Gamma}$. If we now set

$$H_{\bar{k}}^\sharp := \sigma(\widetilde{H}_k \cdot k^{-1}) \cdot \bar{k}$$

for $\bar{k} \in \widetilde{K}$ and $k := q^\sharp(\bar{k})$ (where we suppressed the isomorphisms $T_k\widetilde{K} \cong \mathfrak{k}$ and $T_{\bar{k}}K^\sharp \cong \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$), then $T_{\bar{k}}(\text{ev}_m \circ q^\sharp)$ also restricts to a linear isomorphism on $H_{\bar{k}}^\sharp$ and we have

$$H_{\bar{k}\bar{k}'}^\sharp = \sigma(\widetilde{H}_{kk'}(kk')^{-1}) \cdot \bar{k}\bar{k}' = H_{\bar{k}}^\sharp \cdot \bar{k}'$$

for each $\bar{k}' \in (q^\sharp)^{-1}(\widetilde{K}_m/\Delta)$ and $k' := q^\sharp(\bar{k}')$. Consequently, H^\sharp defines a horizontal distribution on the bundle $\text{ev}_m \circ q^\sharp: K^\sharp \rightarrow M$. If $\sigma^\sharp: \mathcal{V}(M) \rightarrow \mathcal{V}(K^\sharp)^{(\theta^\wedge)^{-1}(K_m)}$ denotes the corresponding horizontal lift of vector fields, then we clearly have $\sigma^\sharp = \sigma_{\overline{\omega}} \circ \tilde{\sigma}$.

From this we obtain a connection

$$\sigma_P: \mathcal{V}(M) \rightarrow \mathcal{V}(P)^{\mathbb{T}\Pi}, \quad X \mapsto TQ_*(\sigma^\sharp(X))$$

on $P \rightarrow M$, where $Q: K^\sharp \rightarrow P = K^\sharp/(H_m)$ is the canonical quotient morphism. For the curvature of the connection σ_P we then have

$$F_{\sigma_P}(X, Y) := \sigma_P([X, Y]) - [\sigma_P(X), \sigma_P(Y)] = TQ_*(\sigma^\sharp([X, Y]) - [\sigma^\sharp(X), \sigma^\sharp(Y)]) = TQ_*(F_{\sigma^\sharp}(X, Y))$$

for $X, Y \in \mathcal{V}(M)$, and, furthermore

$$F_{\sigma^\sharp}(X, Y) = \sigma_{\overline{\omega}}(\tilde{\sigma}([X, Y])) - [\sigma_{\overline{\omega}}(\tilde{\sigma}(X)), \sigma_{\overline{\omega}}(\tilde{\sigma}(Y))] = \sigma_{\overline{\omega}}(\tilde{\sigma}([X, Y])) - \sigma_{\overline{\omega}}([\tilde{\sigma}(X), \tilde{\sigma}(Y)]) + F_{\sigma_{\overline{\omega}}}(\tilde{\sigma}(X), \tilde{\sigma}(Y)) = \sigma_{\overline{\omega}}(F_{\tilde{\sigma}}(X, Y)) + F_{\sigma_{\overline{\omega}}}(\tilde{\sigma}(X), \tilde{\sigma}(Y)).$$

Since $F_{\sigma_{\overline{\omega}}} = \overline{\omega}^{\text{eq}}$ (cf. the proof of Lemma 5.13) and since $\sigma_{\overline{\omega}}(F_{\tilde{\sigma}}(X, Y))$ is at each point tangential to the fibre H_m of Q , it follows that

$$\begin{aligned} TQ(F_{\sigma^\sharp}(X, Y)(k(m))) &= TQ(F_{\sigma_{\overline{\omega}}}(\tilde{\sigma}(X)(k), \tilde{\sigma}(Y)(k))) = TQ(\overline{\omega}^{\text{eq}}(\tilde{\sigma}(X)(k), \tilde{\sigma}(Y)(k))) \\ &= \text{ev}_m(k \cdot \overline{\omega}(\tilde{\sigma}(X)(k) \cdot k^{-1}, \tilde{\sigma}(Y)(k) \cdot k^{-1})) = \omega_{k(m)}(X(k(m)), Y(k(m))) \end{aligned}$$

for each $\bar{k} \in K^\sharp$ and $k := q^\sharp(\bar{k}) \in \tilde{K}$. Thus $F_{\sigma_P} = \omega$.

It remains to check that the kernel actually coincides with Π/Γ . To this end we define $q := q_{\pi_1(K)} \circ q^\sharp$ and consider the diagram

$$\begin{array}{ccc} \ker(q^\sharp) & \xrightarrow{a_{\theta, H_m}|_{\ker(q^\sharp)}} & \ker((\beta_{\mathcal{R}})_*) \\ \downarrow & & \downarrow \\ K^\sharp & \xrightarrow{a_{\theta, H_m}} & \text{Bis}(\mathcal{R}(\theta, H_m)) \\ \downarrow q^\sharp & \searrow \theta^\wedge & \downarrow (\beta_{\mathcal{R}})_* \\ K & \xrightarrow{\quad \quad \quad} & \text{Diff}(M), \end{array}$$

which commutes by Lemma 4.13 and the construction of θ^\wedge . From this it follows that the kernel of a_{θ, H_m} is contained in $\ker(q^\sharp)$. Moreover, we have $\ker(q^\sharp) = A_\Gamma = C^\infty(M, \mathbb{T}\Gamma)$ by definition. To determine $a_{\theta, H_m}(\gamma)$ for $\gamma \in \ker(q)$, we first note that the element in $\text{Aut}(K^\sharp/H_m \rightarrow M)$ corresponding to $a_{\theta, H_m}(\gamma)$ is given by $\bar{k}H_m \mapsto (\gamma \cdot \bar{k})H_m$ (cf. Lemma 4.11). On the other hand, an element $\eta \in \ker((\beta_{\mathcal{R}})_*) \cong C^\infty(M, \mathbb{T}\Pi)$ acts on K^\sharp/H_m by

$$\bar{k}H_m \mapsto \bar{k}H_m \cdot \eta(\theta(\bar{k}, m)) = (\bar{k} \cdot \eta(\theta(\bar{k}, m)))H_m,$$

since the bundle projection $K^\sharp/H_m \rightarrow M$ is given by $\bar{k}H_m \mapsto \theta(\bar{k}, m)$ and $\mathbb{T}\Gamma$ is contained in $Z(K^\sharp)$ (cf. (28)). From this it follows that the value of $a_{\theta, H_m}(\gamma)$ in $\theta(\bar{k}, m)$ has to satisfy

$$(\bar{k}^{-1} \cdot \gamma \cdot \bar{k})H_m = a_{\theta, H_m}(\gamma)(\theta(\bar{k}, m))H_m.$$

Since $\bar{k}^{-1} \cdot \gamma \cdot \bar{k} = \gamma \circ \theta^\wedge(\bar{k})$ follows from the fact that $K^\sharp \rightarrow K$ is an abelian extension for the natural action of K on $C^\infty(M, \mathbb{T}\Gamma)$, we conclude that $a_{\theta, H_m}|_{\ker(q)}$ coincides with the map that is induced by the projection $\mathbb{T}\Gamma \rightarrow \mathbb{T}\Pi = \mathbb{T}\Gamma/(\Pi/\Gamma)$ and the isomorphisms $C^\infty(M, \mathbb{T}\Gamma) \cong \ker(q^\sharp)$ and $C^\infty(M, \mathbb{T}\Pi) \cong \ker((\beta_{\mathcal{R}})_*)$. Consequently, $\ker(a_{\theta, H_m}) = \ker(a_{\theta, H_m}) \cap \ker(q^\sharp) \cong \Pi/\Gamma$ (cf. Proposition 4.16). \blacksquare

Corollary 5.16. *With the notation and under the assumptions of Theorem 5.15 the following assertions are equivalent:*

- a) *The primary periods $\text{per}_{\overline{\omega}}(\pi_2(M)) \subseteq \mathbb{R}$ are discrete.*

b) The extension $\mathfrak{a} \rightarrow \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k} \rightarrow \mathfrak{k}$ integrates to an extension $A_{\Gamma} \rightarrow K^{\sharp} \rightarrow \widetilde{K}$ of Lie groups.

If one (and thus both) of these conditions is satisfied, then the following assertions are equivalent:

i) The secondary periods $\text{per}_{[\omega]}(\pi_1(K_m)) \subseteq \mathbb{R}$ are discrete.

ii) The closed subalgebra $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}$ of $\mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ integrates to a closed Lie subgroup of K^{\sharp} .

Corollary 5.17. *If M is a compact and 1-connected manifold and $\omega \in \Omega^2(M)$ is closed, then the extension $\mathbb{R} \rightarrow \mathbb{R} \oplus_{\omega} TM \rightarrow TM$ of Lie algebroids integrates to an extension of Lie groupoids if and only if the extension of Lie algebras $C^{\infty}(M) \rightarrow C^{\infty}(M) \oplus_{\overline{\omega}} \mathcal{V}(M) \rightarrow \mathcal{V}(M)$ integrates to an extension $A \rightarrow \widehat{K} \rightarrow \text{Diff}(M)_0$ of Lie groups and $C_m^{\infty}(M) \oplus_{\overline{\omega}_m} \mathcal{V}_m(M)$ integrates to a closed Lie subgroup in \widehat{K} .*

Example 5.18. This is a continuation of Example 5.14. Of course, (\mathbb{S}^2, ω) is \mathbb{Z} -prequantisable, a (suitably normalised) prequantisation is the Hopf fibration $\mathbb{S}^3 \rightarrow \mathbb{S}^2$, viewed as a $U(1) := T_{\mathbb{Z}}$ -principal bundle (together with the standard contact form on \mathbb{S}^3). From this we obtain the extension

$$C^{\infty}(\mathbb{S}^2, U(1)) \rightarrow \text{Aut}(\mathbb{S}^3 \rightarrow \mathbb{S}^2) \rightarrow \text{Diff}(\mathbb{S}^2)_0$$

(cf. Example 3.7) whose Lie algebra extension is equivalent to

$$C^{\infty}(\mathbb{S}^2, \mathbb{R}) \rightarrow C^{\infty}(\mathbb{S}^2, \mathbb{R}) \oplus_{\overline{\omega}} \mathcal{V}(\mathbb{S}^2) \rightarrow \mathcal{V}(\mathbb{S}^2). \quad (29)$$

Since the primary periods vanish, the extension (29) integrates to an extension (unique up to equivalence)

$$C^{\infty}(\mathbb{S}^2, \mathbb{R}) \rightarrow K^{\sharp} \rightarrow \widetilde{\text{Diff}(\mathbb{S}^2)_0}$$

and the identity on $C^{\infty}(\mathbb{S}^2, \mathbb{R}) \oplus_{\overline{\omega}} \mathcal{V}(\mathbb{S}^2)$ integrates to a Lie group morphism $\psi: K^{\sharp} \rightarrow \text{Aut}(\mathbb{S}^3 \rightarrow \mathbb{S}^2)$. This morphism makes

$$\begin{array}{ccccc} C^{\infty}(\mathbb{S}^2, \mathbb{R}) & \longrightarrow & K^{\sharp} & \longrightarrow & \widetilde{\text{Diff}(\mathbb{S}^2)_0} \\ \downarrow q_{\mathbb{Z}} & & \downarrow \psi & & \downarrow q_{\pi_1} \\ C^{\infty}(\mathbb{S}^2, U(1)) & \longrightarrow & \text{Aut}(\mathbb{S}^3 \rightarrow \mathbb{S}^2) & \longrightarrow & \text{Diff}(\mathbb{S}^2)_0 \end{array}$$

commute, where $q_{\mathbb{Z}}$ is induced by the quotient map $\mathbb{R} \rightarrow U(1) = \mathbb{R}/\mathbb{Z}$ and q_{π_1} is the universal covering morphism of $\text{Diff}(\mathbb{S}^2)_0$. If now $o \in \mathbb{S}^3$ is mapped to the base-point $m \in \mathbb{S}^2$, then $\text{Aut}_o(\mathbb{S}^3 \rightarrow \mathbb{S}^2)$ is a closed Lie subgroup of $\text{Aut}(\mathbb{S}^3 \rightarrow \mathbb{S}^2)$ (cf. Example 3.7) and since q_{π_1} is a covering morphism, $\psi^{-1}(\text{Aut}_o(\mathbb{S}^3 \rightarrow \mathbb{S}^2))$ is also a closed Lie subgroup of K^{\sharp} . Since $\psi|_{C^{\infty}(\mathbb{S}^2, \mathbb{R})} = q_{\mathbb{Z}}$, we have

$$\psi^{-1}(C^{\infty}(\mathbb{S}^2, U(1)) \cap \text{Aut}_o(\mathbb{S}^3 \rightarrow \mathbb{S}^2)) = C_m^{\infty}(\mathbb{S}^2, \mathbb{R}) \times \mathbb{Z},$$

and thus $\psi^{-1}(\text{Aut}_o(\mathbb{S}^3 \rightarrow \mathbb{S}^2))$ gives rise to an extension

$$C_m^{\infty}(\mathbb{S}^2, \mathbb{R}) \times \mathbb{Z} \rightarrow \psi^{-1}(\text{Aut}_o(\mathbb{S}^3 \rightarrow \mathbb{S}^2)) \rightarrow q_{\pi_1}^{-1}(\text{Diff}_m(\mathbb{S}^2)_0).$$

If we identify $q_{\pi_1}^{-1}((\text{Diff}(\mathbb{S}^2)_0)_m)$ with $\widetilde{\text{Diff}_m(\mathbb{S}^2)_0}/\Delta$ (for Δ as in Remark 5.10), then we deduce from Example 5.14 and Theorem 5.15 that $\psi^{-1}(\text{Aut}_o(\mathbb{S}^3 \rightarrow \mathbb{S}^2))$ is precisely the Lie subgroup $\varphi_m((\text{Diff}(\mathbb{S}^2)_0)_m^{\sharp})$. \blacksquare

Example 5.19. An example where the conditions of Theorem 5.15 are not fulfilled is the following (cf. [TZ06, Example 1]). Let η be the standard volume form on \mathbb{S}^2 with total volume 1. On $M := \mathbb{S}^2 \times \mathbb{S}^2$, consider the form $\omega \in \Omega^2(\mathbb{S}^2 \times \mathbb{S}^2)$

$$\omega_{(p,q)}: T_{(p,q)}\mathbb{S}^2 \times \mathbb{S}^2 \cong T_p\mathbb{S}^2 \times T_q\mathbb{S}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto \eta(x) + \lambda\eta(y).$$

Then $\pi_2(\mathbb{S}^2 \times \mathbb{S}^2) = \mathbb{Z} \times \mathbb{Z}$ and we have $\text{per}_{[\omega]}((1, 0)) = 1$ and $\text{per}_{[\omega]}((0, 1)) = \lambda$. Thus $\text{per}_{[\omega]}(\pi_2(\mathbb{S}^2 \times \mathbb{S}^2))$ is the subgroup of \mathbb{R} which is generated by 1 and λ . If $\lambda \notin \mathbb{Q}$, then this is not contained in any discrete subgroup.

If we take $K = \text{Symp}(M, \omega)$, then the primary periods vanish by Corollary 5.8 and we may take $\Gamma = \{0\}$ in Theorem 5.15. Consequently, the secondary periods $\text{per}_{[\omega]}^b(\pi_1(K_m))$ are not contained in any discrete subgroup of \mathbb{R} and the subalgebra $C_m^\infty(M) \oplus_{\bar{\omega}_m} \mathfrak{k}_m$ does not integrate to a closed Lie subgroup in the extension

$$C^\infty(M, \mathbb{R}) \rightarrow K^\sharp \rightarrow \tilde{K}. \quad \blacksquare$$

Problem 5.20. If one takes the results of this section, then the following questions seem to be natural and interesting.

- How do the primary and secondary periods $\text{per}_{[\bar{\omega}]}(\pi_2(K))$ and $\text{per}_{[\omega]}^b(\pi_1(K_m))$ vary if one varies the subgroup K ? It is clear that for K, K' with $K \leq K'$ we have $\text{per}_{[\bar{\omega}]}(\pi_2(K)) \leq \text{per}_{[\bar{\omega}]}(\pi_2(K'))$ and $\text{per}_{[\omega]}^b(\pi_1(K_m)) \leq \text{per}_{[\omega]}^b(\pi_1(K'_m))$, but under which assumptions does one have equality here? In particular, it would be interesting to have a symplectic manifold with non-vanishing primary periods for $K = \text{Diff}(M)_0$ (since for $K = \text{Symp}(M, \omega)$ the primary periods always vanish by Proposition 5.8).
- It would be interesting to develop an integration theory for infinitesimal transitive pairs (cf. Problem 4.22). In particular, this should shed some further light on the precise relation between the integration theory of Lie algebroids, Lie algebras (of sections) and the associated obstructions.
- What is the interplay between the primary and secondary periods and the flux group

$$F_{[\omega]}(\pi_1(\text{Symp}(M, \omega))) \subseteq H^1(M, \mathbb{R})$$

in the case that M is only assumed to be connected? Conjecturally, there might be a relation of the long exact homotopy sequence of the evaluation fibration and the one induced by $\Gamma \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\Gamma$

$$\begin{array}{ccccccc} \pi_2(\text{Symp}(M, \omega)) & \longrightarrow & \pi_2(M) & \longrightarrow & \pi_1(\text{Symp}_m(M, \omega)) & \longrightarrow & \pi_1(\text{Symp}(M, \omega)) \\ \downarrow \text{per}_{[\bar{\omega}]} & & \downarrow \text{per}_{[\omega]} & & \downarrow ? & & \downarrow F_{[\omega]} \\ H^0(M, \Gamma) & \longrightarrow & H^0(M, \mathbb{R}) & \longrightarrow & H^0(M, \mathbb{R}/\Gamma) & \longrightarrow & H^1(M, \Gamma) \end{array}$$

in case that the primary periods $\text{per}_{[\bar{\omega}]}(\pi_2(\text{Symp}(M, \omega))) \leq \mathbb{R}$ are contained in the discrete subgroup $\Gamma \leq \mathbb{R}$ and the flux group is contained in $H^1(M, \Gamma)$. Note that the flux group is known to be discrete by the proof of the flux conjecture [Ono06] and that both, the secondary periods and the flux subgroup are related to the integrability of Lie subalgebras to closed Lie subgroups (cf. [MS98, Proposition 10.20]). The conjectural homomorphism $\pi_1(\text{Symp}_m(M, \omega)) \dashrightarrow H^0(M, \mathbb{R}/\Gamma)$ should be related to the fluxes $F_{[\bar{\omega}]}$, $F_{[\bar{\omega}_m]}$ and the homomorphism $\varphi_m: K_m^\sharp \rightarrow K^\sharp$, restricted to the pre-image $(q_m^\sharp)^{-1}(\pi_1(\text{Symp}_m(M, \omega)))$. However, this involves the (continuous) Lie algebra cohomology of \mathfrak{k}_m with coefficients in $C_m^\infty(M)$, a topic that goes beyond the scope of the present paper. \blacksquare

A Local bisections for infinite-dimensional Lie groupoids

In this appendix we prove that (infinite-dimensional) Lie groupoids over a finite-dimensional manifold admit local bisections through each point. Consequently, we are able to derive that their vertex groups are in a natural way Lie groups. These results are standard for finite-dimensional Lie groupoids. We repeat them here for the readers convenience since some details of proofs need to be adapted for our infinite-dimensional setting.

Definition A.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid. For $U \subseteq M$, a *local bisection of \mathcal{G} on U* is a smooth map $\sigma: U \rightarrow G$ such that $\alpha \circ \sigma = \text{id}_U$ and $\beta \circ \sigma: U \rightarrow (\beta \circ \sigma)(U) \subseteq M$ is a diffeomorphism. ■

Lemma A.2 ([Mac05, Proposition 1.4.9]). *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid such that M is a finite dimensional manifold. For each $g \in G$ there exists an $\alpha(g)$ -neighborhood $U \subseteq M$ and a local bisection σ of \mathcal{G} on U such that $\sigma(\alpha(g)) = g$.*

Proof. Note that in [Mac05] only finite-dimensional Lie groupoids are discussed. However, to prove the assertion, one can copy the proof of [Mac05, Proposition 1.4.9] verbatim to our setting, since the Lie groupoid is assumed to have a finite-dimensional base. The crucial point here is that the algebraic argument used in the proof of [Mac05, Proposition 1.4.9] carries over to subspaces of finite codimension of arbitrary locally convex spaces. Assuming that the base M is finite-dimensional ensures exactly this property. ■

For mappings into finite-dimensional manifolds (whose domain is an infinite-dimensional manifold), one can define maps of constant rank analogously to the finite-dimensional case (cf. [Glö15a, Theorem F]).

Corollary A.3 ([Mac05, Corollary 1.4.10]). *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid over a finite dimensional manifold M . Then for each $m \in M$ the maps*

$$\beta|_{\alpha^{-1}(m)}: \alpha^{-1}(m) \rightarrow M, g \mapsto \beta(g) \text{ and } \alpha|_{\beta^{-1}(m)}: \beta^{-1}(m) \rightarrow M, g \mapsto \alpha(g)$$

are maps of constant rank.

The next proof follows [Mac05, Corollary 1.4.11] but we need to adapt the arguments.

Lemma A.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid over a finite dimensional manifold M . Then for all $m, n \in M$, $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold (of finite codimension) of $\alpha^{-1}(m)$, of $\beta^{-1}(n)$ and of G . In particular, each vertex group $\text{Vert}_m(\mathcal{G}) := \alpha^{-1}(m) \cap \beta^{-1}(m)$ is a closed submanifold of G and this structure turns it into a Lie group.*

Proof. The set $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is the preimage of n under the constant rank map $\beta|_{\alpha^{-1}(m)}$. As M is a finite dimensional manifold we can apply Glöckner's constant rank theorem [Glö15a, Theorem F]. Thus $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold of finite codimension in $\alpha^{-1}(m)$. Moreover, $\alpha^{-1}(m)$ is a split submanifold of G (by the regular value theorem [Glö15a, Theorem D]) and since M is finite-dimensional, $\alpha^{-1}(m)$ is even of finite-codimension in G . Thus [Glö15a, Lemma 1.4] implies that also $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold (of finite-codimension) of G . Analogously one shows that $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold (of finite-codimension) of $\beta^{-1}(n)$.

Groupoid multiplication and inversion induce a group structure on $\text{Vert}_m(\mathcal{G})$. By the universal property of the pullback, the inclusion $\text{Vert}_m(\mathcal{G}) \times \text{Vert}_m(\mathcal{G}) \rightarrow G \times_{\alpha, \beta} G$ is smooth, whence this group structure turns $\text{Vert}_m(\mathcal{G})$ into a Lie group. ■

B Locally convex manifolds, Lie groups and spaces of smooth maps

In this appendix we collect the necessary background on the theory of manifolds and Lie groups that are modelled on locally convex spaces and how spaces of smooth maps can be equipped with such a structure. Let us first recall some basic facts concerning differential calculus in locally convex spaces. We follow [Glö02, BGN04].

Definition B.1. Let E, F be locally convex spaces, $U \subseteq E$ be an open subset, $f: U \rightarrow F$ a map and $r \in \mathbb{N}_0 \cup \{\infty\}$. If it exists, we define for $(x, h) \in U \times E$ the directional derivative

$$df(x, h) := D_h f(x) := \lim_{t \rightarrow 0} t^{-1}(f(x + th) - f(x)).$$

We say that f is C^r if the iterated directional derivatives

$$d^{(k)}f(x, y_1, \dots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$ and $y_1, \dots, y_k \in E$ and define continuous maps $d^{(k)}f: U \times E^k \rightarrow F$. If f is C^∞ it is also called smooth. We abbreviate $df := d^{(1)}f$. From this definition of smooth map there is an associated concept of locally convex manifold, i.e., a Hausdorff space that is locally homeomorphic to open subsets of locally convex spaces with smooth chart changes. Accordingly, a locally convex Lie group is a manifold, equipped with a group structure such that all group operations are smooth. See [Woc13, Nee06, Glö02] for more details. ■

Definition B.2. Let M be a smooth manifold. Then M is called *Banach* (or *Fréchet*) manifold if all its modelling spaces are Banach (or Fréchet) spaces. The manifold M is called *locally metrisable* if the underlying topological space is locally metrisable (equivalently if all modelling spaces of M are metrisable). It is called *metrisable* if it is metrisable as a topological space (equivalently locally metrisable and paracompact). ■

Definition B.3. Let H be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the right translation $\rho_h: H \rightarrow H$, $x \mapsto xh$ by $h \in H$ to define $v.h := T_{\mathbf{1}}\rho_h(v) \in T_h H$ for $v \in T_{\mathbf{1}}(H) =: \mathbf{L}(H)$. Following [Glö15b], H is called C^r -semiregular if for each C^r -curve $\gamma: [0, 1] \rightarrow \mathbf{L}(H)$ the initial value problem

$$\begin{cases} \eta'(t) &= \gamma(t). \eta(t) \\ \eta(0) &= \mathbf{1} \end{cases}$$

has a (necessarily unique) C^{r+1} -solution $\text{Evol}(\gamma) := \eta: [0, 1] \rightarrow H$. If in addition the map

$$\text{evol}: C^r([0, 1], \mathbf{L}(H)) \rightarrow H, \quad \gamma \mapsto \text{Evol}(\gamma)(1)$$

is smooth, we call H C^k -regular. ■

Remark B.4. If H is C^r -regular and $r \leq s$, then H is also C^s -regular. A C^∞ -regular Lie group H is called *regular* (in the sense of Milnor) – a property first defined in [Mil84]. Every finite dimensional Lie group is C^0 -regular (cf. [Nee06]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [Mil84], [Nee06], [Glö15a], cf. also [KM97] and the references therein). ■

Lemma B.5. Let G be a C^k -regular Lie group for $k \in \mathbb{N}_0 \cup \{\infty\}$ and H be a closed Lie subgroup of G . If H is C^k -semiregular, then H is C^k -regular.

Proof. Denote by $i_H: H \rightarrow G$ and $I_H: C^{k+1}([0, 1], H) \rightarrow C^{k+1}([0, 1], G)$ the canonical inclusions. Then $\mathbf{L}(i_H): \mathbf{L}(H) \rightarrow \mathbf{L}(G)$ allows us to identify curves $\eta \in C^k([0, 1], \mathbf{L}(H))$ with C^k -curves $\mathbf{L}(i_H) \circ \eta$ in $\mathbf{L}(G)$. As G and H are C^k -semiregular, we obtain maps $\text{Evol}_J: C^k([0, 1], \mathbf{L}(J)) \rightarrow C^{k+1}([0, 1], J)$ for $J \in \{G, H\}$ which map a curve to the solution of the initial value problem

$$\begin{cases} \gamma'(t) &= \gamma(t). \eta(t) \quad \forall t \in [0, 1] \\ \gamma(0) &= \mathbf{1} \end{cases}$$

in the respective group. Consider the map $\mathbf{L}(i_H)_* : C^k([0, 1], \mathbf{L}(H)) \rightarrow C^k([0, 1], \mathbf{L}(\text{Bis}(\mathcal{G})))$, $\eta \mapsto \mathbf{L}(i_H) \circ \eta$, which is smooth by [GN12, Lemma 1.2]. By [Glö15b, 1.16] we have

$$\text{Evol}_G \circ \mathbf{L}(i_H) = I_H \circ \text{Evol}_H$$

As H is a closed subgroup of G , the same holds for $C^{k+1}([0, 1], H) \subseteq C^{k+1}([0, 1], G)$ (cf. [Glö15b, 1.8 and 1.10]) Hence Evol_H is smooth if $I_H \circ \text{Evol}_H$ is smooth. Observe that since G is C^k -regular, Evol_G and thus $\text{Evol}_G \circ \mathbf{L}(i_H)_* = I_H \circ \text{Evol}_H$ is smooth. We deduce that Evol_H is smooth and [Glö15b, Lemma 3.1] shows that H is C^k -regular. ■

Lemma B.6. *Suppose H, K are Lie groups with Lie subgroups $H_* \leq H$, $K_* \leq K$ such that K and K_* are regular and H_* is connected. Let $\varphi: H \rightarrow K$ be a morphism of Lie groups. If $\mathbf{L}(\varphi)(\mathbf{L}(H_*)) \subseteq \mathbf{L}(K_*)$, then $\varphi(H_*) \subseteq K_*$.⁹*

Proof. Since H_* is connected, it is contained in the identity component of H . Restricting to that component we may assume that also H is connected. Then [Nee06, Proposition II.4.1] implies that $\mathbf{L}(\varphi)$ has at most one integration to a morphism of Lie groups, which thus has to coincide with φ . Recall that the left logarithmic derivative $\delta^l(\gamma): [0, 1] \rightarrow \mathbf{L}(H)$ of a C^1 -curve $\gamma: [0, 1] \rightarrow H$ in a Lie group is defined via $\delta^l(\gamma)(t) := \gamma(t)^{-1} \cdot \gamma'(t)$. Thus $\text{Evol}(\eta) = \gamma$ if and only if $\delta^l(\gamma) = \eta$ and $\gamma(0) = \mathbf{1}_H$. The integration of $\mathbf{L}(\varphi)$ is constructed by taking a smooth path $\gamma: [0, 1] \rightarrow H$ with $\gamma(0) = \mathbf{1}_H$, applying $\mathbf{L}(\varphi)$ to $\delta^l(\gamma)$, solving the initial value problem

$$\begin{cases} \delta^l(\eta) &= \mathbf{L}(\varphi) \circ \delta^l(\gamma) \\ \eta(0) &= \mathbf{1}_K \end{cases} \quad (30)$$

in K and setting $\varphi(\gamma(1)) = \eta(1)$ (which is possible since K is regular, cf. [Mil84, Theorem 8.1]). Under the assumptions made, this only depends on $\gamma(1)$.

Since H_* is assumed to be connected, we can choose for each $h \in H_*$ a path γ with $\gamma(0) = e_H$ and $\gamma(1) = h$ such that $\gamma(t) \in H_*$ for all $t \in [0, 1]$. Consequently, $\delta^l(\gamma)(t) \in \mathbf{L}(H_*)$, and thus $\mathbf{L}(\varphi)(\delta^l(\gamma)(t)) \in \mathbf{L}(K_*)$ for all $t \in [0, 1]$. Now K_* is regular and $\mathbf{L}(\varphi) \circ \delta^l(\gamma)$ takes its image in $\mathbf{L}(K_*)$ by assumption. Thus we can solve (30) in K_* . From [Glö15b, Lemma 10.1], we deduce that the solution to the initial value problem (30) for η in K coincides with the solution in K_* , and thus takes its values in K_* . Summing up, $\varphi(h) = \eta(1) \in K_*$ for each $h \in H_*$. ■

Definition B.7. Suppose M is a smooth manifold. Then a *local addition* on M is a smooth map $\Sigma: U \subseteq TM \rightarrow M$, defined on an open neighbourhood U of the submanifold $M \subseteq TM$ such that

- $\pi \times \Sigma: U \rightarrow M \times M$, $v \mapsto (\pi(v), \Sigma(v))$ is a diffeomorphism onto an open neighbourhood of the diagonal $\Delta M \subseteq M \times M$ and
- $\Sigma(0_m) = m$ for all $m \in M$.

We say that M admits a local addition if there exist a local addition on M . ■

Definition B.8. (cf. [Mic80, 10.6]) Let $s: Q \rightarrow N$ be a surjective submersion. Then a *local addition adapted to s* is a local addition $\Sigma: U \subseteq TQ \rightarrow Q$ such that the fibres of s are additively closed with respect to Σ , i.e. $\Sigma(v_q) \in s^{-1}(s(q))$ for all $q \in Q$ and $v_q \in T_q s^{-1}(s(q))$ (note that $s^{-1}(s(q))$ is a submanifold of Q). ■

An important tool will be the following excerpt from [Woc13, Theorem 7.6].

Theorem B.9. *Let M be a compact manifold and N be a locally convex and locally metrisable manifold that admits a local addition $\Sigma: U \subseteq TN \rightarrow N$. Set $V := (\pi \times \Sigma)(U)$, which is an open neighbourhood of the diagonal ΔN in $N \times N$. For each $f \in C^\infty(M, N)$ we set*

$$O_f := \{g \in C^\infty(M, N) \mid (f(x), g(x)) \in V\}.$$

Then the following assertions hold.

⁹The authors believe that this result is well-known to experts in the field but were unable to locate a reference.

- a) The set O_f contains f , is open in $C^\infty(M, N)$ and the formula $(f(x), g(x)) = (f(x), \Sigma(\varphi_f(g)(m)))$ determines a homeomorphism

$$\varphi_f: O_f \rightarrow \{h \in C^\infty(M, TN) \mid \pi(h(x)) = f(x)\} \cong \Gamma(f^*(TN))$$

from O_f onto the open subset $\{h \in C^\infty(M, TN) \mid \pi(h(x)) = f(x)\} \cap C^\infty(M, U)$ of $\Gamma(f^*(TN))$.

- b) The family $(\varphi_f: O_f \rightarrow \varphi_f(O_f))_{f \in C^\infty(M, N)}$ is an atlas, turning $C^\infty(M, N)$ into a smooth locally convex and locally metrisable manifold. The manifold structure is independent of the choice of the local addition.
- c) If L is another locally convex and locally metrisable manifold, then a map $f: L \times M \rightarrow N$ is smooth if and only if $f^\wedge: L \rightarrow C^\infty(M, N)$ is smooth. In other words,

$$C^\infty(L \times M, N) \rightarrow C^\infty(L, C^\infty(M, N)), \quad f \mapsto f^\wedge$$

is a bijection (which is even natural).

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