Randomization for Markov chains with applications to networks in a random environment

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We[1] develop randomized modifications of Markov chains and apply these modifications to the routing chains of customers in Jacksonian stochastic networks. The aim of our investigations is to find new rerouting schemes for non standard Jackson networks which hitherto resist computing explicitly the stationary distribution.

The non standard properties we can handle by suitable algorithms encompass several modifications of Jackson networks known in the literature, especially breakdown and repair of nodes with access modification for customers to down nodes, finite buffers with control of buffer overflow. The rerouting schemes available in the literature for these situations are special cases of our rerouting schemes, which can deal also with partial degrading of service capacities and even with speed up of service.

In any case we require our algorithms to react on such general changes in the network with the aim to maintain the utilization of the nodes. To hold this invariant under change of service speeds (intensities) our algorithms not only adapt the routing probabilities but decrease automatically the overall arrival rate to the network if necessary.

Our main application is for stochastic networks in a random environment. The impact of the environment on the network is by changing service speeds (by upgrading and/or degrading, breakdown, repair) and we implement the randomization algorithms to react to the changes of the environment. On the other side, customers departing from the network may enforce the environment to jump immediately. So our environment is not Markov for its own.

The main result is to compute explicitly the joint stationary distribution of the queue lengths vector and the environment which is of product form: Environment and queue lengths vector, and the queue lengths over the network are decomposable.

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Contents

1 Introduction

Queueing networks with product form steady state distribution have found many fields of applications, e.g. production systems, telecommunications, and computer system modeling. The success of this class of models and its relatives stems mainly from the simple structure of the steady state distribution which provides access to easy performance evaluation procedures.

Starting from the work of Jackson [Jac57] various generalizations have been developed. A special branch of research which recently has found some interest are product form models for queueing networks in a random environment with product form steady state distributions.

For single service stations (in isolation) there is a long history with investigations on the behaviour of the stations under external influences, which are subsumed under the term of an environment. Similarly birth-death processes (as generalizations of classical $M/M/1/\infty$ queues) in a random environment are well investigated. Most of this work resulted in complex steady state distributions, see e.g., [CT81], [Cog80], [Yec73], [KY12], [Fal96]. A special branch

of related research is concerned with service systems under external influences which cause the service process to break down or decreases availability of servers, see e.g. [Zol66], and for recent survey [KPC12]. The results in these articles most often lack the elegance of Jackson's product form steady state and the simplicity of the birth-death steady states.

Some exceptions are e.g. [SSD$^+$06], [KN13], and [SAH13], where the environment of a production model (queue) is an associated inventory, [SD03], where the influence of the environment on the queue results in randomly occurring breakdowns of the production facility, and [KD14], where the environment encompasses e.g. the node's neighbours, their status, etc. In these papers on queueing-environment processes it is shown that the two-dimensional steady state distribution factorizes into the product of the marginal one-dimensional steady state distributions, shortly: In steady state and in the long run the steady state distribution for the queue and the environment decouples (for fixed time points).

Related work is by Yamazaki and Miyazawa [YM95] where the environment of a queue is called the "set of background states" which determines transition rates of the queue and on the other side is influenced by state changes of the queue. Yamazaki and Miyazawa prove a decomposition property (which is in the spirit of decoupling as in Jackson's theorem) **not** for the queue and the environment but for the joint (QUEUEING/ENVIRONMENT PROCESS) and some supplementary variables which are introduced for Markovisation of the process in case of non exponential service times and non exponential holding times for the background states.

A first approach to find product form steady state distributions for Jackson networks in a random environment seemingly was the work of Zhu [Zhu94]. Economou [Eco05], Balsamo and Marin [BM13], and Tsitsiashvili, Osipova, Koliev, Baum [TOKB02] continued the investigations for partly different settings. The general procedure in these papers for a network of single exponential servers is as follows (explained in terms of Zhu's notation).

The key stochastic ingredients in the **classical Jackson network** for node i of the network are an external Poisson-λ_i arrival stream, exponential-μ_i service times, and a Markovian routing scheme, which produces an overall arrival rate η_i and a local marginal stationary distribution π_i with

$$\pi_i(n) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^n, \quad n \in \mathbb{N}_0. \tag{1.1}$$

In [Zhu94] these fundamental parameters depend on the state of the external environment. If the environment is in state k the parameters are $\lambda_i(k), \mu_i(k), \eta_i(k)$, and with the additional assumption that the utilizations $\rho_i(k) := \eta_i(k)/\mu_i(k)$ do not depend on k, i.e.

$$\rho_i(k) = \eta_i(k)/\mu_i(k) = \eta_i/\mu_i =: \rho_i, \tag{1.2}$$

say, the local marginal stationary distribution π_i is (1.1) again. Zhu and his followers do not explain how this independence of k for the utilizations emerges, i.e. no (physical) control mechanism is described which holds $\eta_i(k)/\mu_i(k)$ invariant during changes of the environment.

Tsitsiashvili, Osipova, Koliev, Baum [TOKB02] argue for the invariance $\eta_i(k)/\mu_i(k) = \eta_i/\mu_i$ by pointing on the fact that there exist natural control mechanisms of technical and biological systems to react on changes of the environment, but no explicit rule to maintain the utilization is given.

The quest for such rules has on the other side a long history in related fields, especially in the control of communications networks, where the term "rerouting schemes" describes the necessity to react, e.g. to buffer overflow, broken down nodes, (partial) degrading of transmission

lines. Examples are described in [SD03] under the heading of "skipping", "repeated service–random destination", "stalling". The first regime (skipping) is called "jump over protocol" by other authors, see [Dij88].

To put it into a more concrete example: These schemes try to mimic by stylized policies in communications networks an exchange of routing tables or the rules for dynamic traffic allocation to paths as reaction to changes of the environment or the network's load situation, for a discussion see [NSB$^+$03] or [GKK95].

Our present investigation originates from the observation often made in stochastic networks with blocking or with unreliable servers, that it is possible to obtain explicit product form stationary distributions by implementing a clever rerouting regime for customers who find at a node, selected for his next entrance, the buffer full or the node broken down, see e.g. [SD03] and the literature cited there. A bulk of examples can be found throughout the book [Dij93] and in Chapters 1 and 9 of [BvD10]. These rerouting schemes maintain the utilization as in (1.2) for those nodes which are not blocked, resp. not broken down. We remark that these results are not covered by the model and results of Zhu, where the $\eta_i(k)$ are required to be strictly positive for all nodes i and for all environment states k.

Similar product form network models for randomized medium access control protocols have found interest recently because of tractability of the model, see [SS12].

(I) Our first contribution is that we complement the results of Zhu by providing physically meaningful rerouting schemes which maintain in his setting the utilizations (1.2).

(II) Our second contribution is to extend Zhu's results in a way that the mentioned results on networks with breakdowns in [SD03] are covered and generalized.

We start as Zhu [Zhu94] with a Jackson network with locally queue length (n_i) dependent service intensities, which may depend additionally on the actual state k of the network's environment $(\mu_i(n_i, k))$. The external arrival rates are $\lambda_i(k)$, and the overall arrival rates are $\eta_i(k)$. We will construct explicit control schemes which adapt the routing to the changes in the parameters, when the environment changes from k to m and the service intensities from $\mu_i(n_i, k)$ to $\mu_i(n_i, m)$. We will prove that under these new control schemes the respective ratios are maintained constant: $\overline{\eta_i(k)/\mu_i(n_i, k)} = \eta_i/\mu_i(n_i)$ will be independent of k (but not of n_i) as long as $\eta_i(k) > 0$ holds. Our theorems will cover the case $\eta_i(k) = 0$ as well, which is in force e.g., if an unreliable node i is down and therefore does not accept new customers.

The most important consequence will be that for Jackson networks in a random environment we obtain a product form stationary distribution, similar to [Zhu94][Theorem 1], but under much more general assumptions. That is, the joint network-environment process decouples and additionally the joint queue length distribution of the network processes decouples as well.

(III) Our third contribution is an extension of the environment structures found in the mentioned previous literature because in our setting the network process influences the environment process as well. We emphasize that different from the mentioned work in [Zhu94], [Eco05], [BM13], and Tsitsiashvili, Osipova, Koliev, Baum [TOKB02], our environment process is not Markov for its own because changes in the queue length processes may enforce the environment to immediate changes as well. There will be a two-way interaction between the service systems and the environment in our model.

We point out that the modifications of the random walks which we construct encompass the rerouting schemes which are often found in the literature and are called e.g. "jump over protocol" or "skipping" or "blocking after service and retrial". This implies that our results

generalize those in [Zhu94], [Eco05], [BM13], and [TOKB02], and as well some of [SD03], which are not covered by the results of the previous papers.

(IV) We start our presentation in Section 2 with a detailed study of the routing chains for the selection of individual customers' itineraries in the network and suitable modifications of these chains in terms of general Markov chains, resp. random walks. The modification is realized in analogy to principles occurring in MCMC algorithms by attaching to any state of the chain a (state dependent) "acceptance probability" to operate via Bernoulli experiments on the original transition matrix of the chain, which is considered as "candidate-generating matrix" (see [Bre99][Section 7.1]).

The jump is realized, if accepted, and the chain settles down at the selected state for the next time slot. But other than in MCMC algorithms, "not accepted" in our modification means not necessarily for the random walk to stay on at the departure state. We consider additionally to the standard one the policy that from the selected, but not accepted state the chain tries again to find a next state, now with probabilities from the "candidate-generating matrix" determined by the row of the not accepted state. Thereafter acceptance is tested again, and so on.

We will show that the steady state distributions of the original chain and the modified chains are intimately connected and that we can express the new steady state easily in terms of the steady state of the old chain and the acceptance probabilities.

(V) Although our research started with a quest for new rerouting schemes for Jackson networks in case of (partial) non-availability of servers, the developed schemes seem to be of interest for their own.

From an abstract point of view our modification of the original chain can be considered as a complicated change of measure for the process distribution, see [AG07][Chapter V, 1c, Example 8], which results in a surprisingly simple explicit change of measure for the stationary distribution.

The modification algorithms which lead to this change of measure can be distinguished according to the property to be local or global (as discussed in [SS12]) with respect to the one-step transition graph of the original chain.

In Section 2.3 we discuss the connections of our randomization algorithm to MCMC algorithm and to von Neumann's acceptance-rejection scheme for sampling from a complicated distribution, and furthermore compare the different modifications of the random walk with respect to Peskun ordering, and the consequences thereof.

(VI) The applications of the randomized random walk algorithms to stochastic networks in the present paper is in the spirit of "Performability" as stated in [Mey84][p. 648]: "If computing system performance is degradable, then . . . system evaluation must deal simultaneously with aspects of both performance and reliability." A more recent compendium on these topics is [HMRT01]. We start with a standard Jackson network and consider the situation where the service capacities of nodes are degraded and by some network control the utilization of the nodes should be maintained. (This is the situation of Zhu et al. in the mentioned papers as long as no node is completely broken down.)

The network controller's policy is in Section 3.2.1 to reject a portion of the load offered to the network and to redistribute the admitted load. This is organized by applying our modification algorithms for random walks to the routing chains for the customers. Additionally to partial degrading we can handle with our algorithm complete breakdown of nodes, and in Section 3.2.2 we allow even to speed up service at some nodes, while others are degraded or broken down. In any case the utilization of the nodes which are not completely down is maintained

by the control policy.

While these sections are concerned with transforming a Jackson network into another one where service capacities are degraded and/or upgraded and routing is adapted, maintaining the nodes' utilization and therefore the joint product form stationary distribution, we utilize the obtained transformation rules in Section 3.3 to adapt routing by different algorithms to the impact of a dynamically changing environment. The environment's changes cause the nodes' service capacities to degrade and/or upgrade, even complete breakdown with following repair or partial repair can be handled.

The most surprising result is an iterated product form stationary distribution of the system process, which is a multidimensional Markov process recording jointly the environment's status and the joint queue lengths vector for the network process. The product form which is obtained says that

1. the queue length vector and the environment status, and

2. inside of the joint queue lengths the local queue lengths

asymptotically and in equilibrium decouple (are decomposable in the sense of [YM95]). This result is even more remarkable as we do not require that the environment process is a Markov process of its own, as it is necessary for the results proved in [Zhu94], [Eco05], [BM13], and [TOKB02]. We can handle a two-way interaction between environment which enforces changes of the nodes' service capacity when it changes, and the queueing network process, which triggers immediate jumps of the environment, when a customer departs from the network. Such (more complicated) two-way interaction is investigated in [YM95] as well, but due to the more complicated structure they loose in their results decoupling of the (single) queue length and the environment status.

Notation and conventions (noch zu ueberarbeiten):

- $\mathbb{R}_0^+ = [0, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{N} = 1, 2, 3, \ldots$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$

- For sets A, B we write $A \subset B$ for A which is a subset of B or equals B, and we write $A \subsetneq B$ for A which is a subset of B but does not equal B.

- Throughout, the node set of our graphs (networks) are denoted by $\overline{J} := \{1, \ldots, J\}$, and the "extended node set" is $\overline{J}_0 := \{0, 1, \ldots, J\}$, where "0" refers to the external source and sink of the network.

- $\mathbf{e}_j$ is the standard j-th base vector in $\mathbb{N}_0^{\overline{J}}$ if $1 \leq j \leq J$.

- $\mathbf{n} = (n_j : j \in \overline{J})$.

- For any finite index set $\overline{F} = \{0, 1, \ldots, F\}$ and any $\boldsymbol{\alpha} = (\alpha_j : j \in \overline{F})$ we define a matrix $I_{\boldsymbol{\alpha}}$ which is the diagonal matrix indexed by $\overline{F}$ with α_i on its diagonal, i.e.

$$I_{\boldsymbol{\alpha}} := \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_F \end{pmatrix},$$

and similarly we define $I_{(1-\boldsymbol{\alpha})}$ indexed by $\overline{F}$ as

$$I_{(1-\boldsymbol{\alpha})} := \begin{pmatrix} 1 - \alpha_0 & & & \\ & 1 - \alpha_1 & & \\ & & \ddots & \\ & & & 1 - \alpha_F \end{pmatrix}.$$

.

- Here and elsewhere we agree that empty entries in a matrix are read to be zero.

- For $\mathbf{x} = (x_j : j \in \overline{F})$ we define $||\mathbf{x}||_\infty := \sup_{j \in \overline{F}} |x_j|$.

- $1_{[expression]}$ is the indicator function which is 1 if *expression* is true and 0 otherwise.

- All random variables occurring in the sequel are defined on a common underlying probability space $(\Omega, \mathcal{F}, P)$.

2 Randomized random walks

Let $X = (X_n : n \in \mathbb{N}_0)$ be a homogeneous irreducible Markov chain on a finite state space $\overline{F}$ with one-step transition probability matrix $r = (r(i,j) : i,j \in \overline{F})$ and (unique) steady state distribution $\eta = (\eta_i : i \in \overline{F})$. This chain represents in our network applications the homogeneous Markov chain that describes the random walk (routing) of the customers on the nodes of the network. In this application scenario the routing will be modified by a network controller as a reaction to changes of the network's parameter due to the impact of the environment.

The general principle is: The transition matrix r will be used as a "candidate generating matrix" for the next state of the random walk. The candidate state will be accepted with state dependent probabilities and we develop different policies to continue when the proposed state is rejected.

2.1 Randomized skipping

The following modification of X with prescribed $B \subsetneq \overline{F}$ is known under the terms of "Markov chains with taboo set B", or "jump-over-protocol for B", or "skipping B". In the realm of queueing network theory this principle for modifying a Markov chain seems to be introduced independently several times and was used to resolve blocking, see e.g. [Dij88], [EF98] and [Ser99][Chapter 3.6], (where it is called blocking and rerouting) and the references therein. As a general methodology skipping was already introduced by Schassberger [Sch84] and later on, based on Schassberger's result, it was used in [DS96] to construct general abstract network processes.

An intuitive description of the skipping principle can be given in terms of a random walk on $\overline{F}$ governed by r: If the random walker's (RW) path governed by the Markov chain is restricted by a taboo set $B \subsetneq \overline{F}$, RW applies
Skipping B: If RW is in state $i \in \overline{F}$ and selects (with probability $r(i,j)$) its destination $j \in \overline{F} \setminus B$, this jump is allowed and immediately performed. If (with probability $r(i,k)$) he decides to jump to state $k \in B$, he only performs an imaginary jump to k, spending no time there, but jumping on immediately governed by the matrix r, i.e. with probability $r(k,l)$ he

selects another possible successor state l; if $l \in \overline{F} \setminus B$, then the jump is performed immediately, but if $l \in B$, RW has to perform another random choice as if he would depart from l; and so on.

Our extended modification scheme for X and r is a randomized generalization of that skipping scheme. For the states in $\overline{F}$ are given "acceptance probabilities" by some vector $\boldsymbol{\alpha} = (\alpha_j \in [0,1] : j \in \overline{F})$. The random walker (RW) selects his itinerary under r and the constraints $\boldsymbol{\alpha}$ by

Randomized skipping with acceptance probabilities $\boldsymbol{\alpha}$: If RW is in state $i \in \overline{F}$ and selects (with probability $r(i,j)$) its destination $j \in \overline{F}$, a Bernoulli experiment is performed with success (acceptance) probability α_j, independent of the past, given j. If the experiment is successful ($= 1$), this jump is accepted, immediately performed, and RW settles down at j for at least one time slot. If the experiment is not successful ($= 0$), this jump is not accepted and RW only performs an imaginary jump to j, spending no time there, but jumping on immediately governed by the matrix r, i.e. with probability $r(j,l)$ he selects another possible successor state l; thereafter a Bernoulli experiment is performed with success (acceptance) probability α_l, independent of the past, given l. If the experiment is successful ($= 1$), this jump is accepted, immediately performed, and RW settles down at l for at least one time slot. If the experiment is not successful ($= 0$), this jump is not accepted and RW only performs an imaginary jump to l, spending no time there, but jumps on immediately according to the matrix r; and so on.

Example 2.1. *If for $B \subsetneq \overline{F}$*

$$\alpha_j = \begin{cases} 0 & if \quad j \in B, \\ 1 & if \quad j \in \overline{F} \setminus B, \end{cases}$$

then we have exactly the classical skipping over taboo set B as described above, because a jump to $j \in B$ is never accepted, but whenever a jump to $j \in \overline{F} \setminus B$ is proposed, this will be accepted with probability 1.

If we have some general $\boldsymbol{\alpha}$ then the set $B(\boldsymbol{\alpha}) = \{j \in \overline{F} : \alpha_j = 0\}$ is a taboo set for the "randomized skipping process".

2.1.1 Transition matrix

It is easy to see that this construction of a modified chain by randomized skipping generates a new homogeneous Markov chain, the transition matrix of which will be denoted by $r^{(\boldsymbol{\alpha})}$. For simplicity of presentation we will denote a Markov chain under this regime by $X^{(\boldsymbol{\alpha})}$.

To determine the transition probabilities $r^{(\boldsymbol{\alpha})}(i,j)$ we construct an auxiliary absorbing Markov chain $(X^{(A)}, Y^{(A)})$ with state space $\overline{F} \times \{0,1\}$ such that

- $X^{(A)}$ records the itinerary of RW during his (possibly many) imaginary jumps until its candidate state is accepted - if this happened RW settles down there forever, because the chain $(X^{(A)}, Y^{(A)})$ is absorbed,

- $Y^{(A)}$ indicates whether a candidate state is accepted or not,

- the states in $\overline{F} \times \{1\}$ are absorbing,

- initial states for $(X^{(A)}, Y^{(A)})$ are restricted to $\overline{F} \times \{0\}$, and therefore $Y^{(A)}$ stays at 0 until absorption of $(X^{(A)}, Y^{(A)})$ in $\overline{F} \times \{1\}$.

The transition probabilities for $(X^{(A)}, Y^{(A)})$ are for $i, j \in \overline{F}$ as long as at time n the candidate state is not accepted

$$P(X_{n+1}^{(A)} = j, Y_{n+1}^{(A)} = 1 | X_n^{(A)} = i, Y_n^{(A)} = 0) = r(i,j)\alpha_j = (r \cdot I_{\boldsymbol{\alpha}})_{ij}, \tag{2.1}$$

$$P(X_{n+1}^{(A)} = j, Y_{n+1}^{(A)} = 0 | X_n^{(A)} = i, Y_n^{(A)} = 0) = r(i,j)(1 - \alpha_j) = (r \cdot I_{(1-\boldsymbol{\alpha})})_{ij}, \tag{2.2}$$

and thereafter

$$P(X_{n+1}^{(A)} = j, Y_{n+1}^{(A)} = 1 | X_n^{(A)} = i, Y_n^{(A)} = 1) = \delta_{ij}, \tag{2.3}$$

$$P(X_{n+1}^{(A)} = j, Y_{n+1}^{(A)} = 0 | X_n^{(A)} = i, Y_n^{(A)} = 1) = 0. \tag{2.4}$$

We denote the first entrance time into $\overline{F} \times \{1\}$ of $(X^{(A)}, Y^{(A)})$ by

$$\tau^{(A)} := \inf\{n \geq 1 | (X_n^{(A)}, Y_n^{(A)}) \in F \times \{1\}\} = \inf\{n \geq 1 | Y_n^{(A)} = 1\}, \tag{2.5}$$

which is P-a.s. finite.

Theorem 2.2. *For the Markov chain $X = (X_n : n \in \mathbb{N}_0)$ with transition matrix $r = (r(i,j) : i, j \in \overline{F})$ and a non zero vector $\boldsymbol{\alpha} = (\alpha_j : j \in \overline{F})$ of acceptance probabilities denote by $X^{(\boldsymbol{\alpha})}$ the Markov chain modification of X under randomized skipping and by $r^{(\boldsymbol{\alpha})}$ the transition matrix of $X^{(\boldsymbol{\alpha})}$, and by $B(\boldsymbol{\alpha}) = \{j \in \overline{F} : \alpha_j = 0\} \subsetneq \overline{F}$ the taboo set for $X^{(\boldsymbol{\alpha})}$. Then from the auxiliary chain $(X^{(A)}, Y^{(A)})$ we obtain*

$$r^{(\boldsymbol{\alpha})}(i,j) = P\left(X^{(A)}(\tau^{(A)}) = j | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right), \quad i, j \in \overline{F}. \tag{2.6}$$

The Markov chain $X^{(\boldsymbol{\alpha})}$ with state space $\overline{F}$ and transition matrix $r^{(\boldsymbol{\alpha})}$ is irreducible on $\overline{F} \setminus B(\boldsymbol{\alpha})$, and it holds

$$r^{(\boldsymbol{\alpha})} = \sum_{k=0}^{\infty} \left(r I_{(1-\boldsymbol{\alpha})}\right)^k r I_{\boldsymbol{\alpha}} = \left(I - r I_{(1-\boldsymbol{\alpha})}\right)^{-1} r I_{\boldsymbol{\alpha}}. \tag{2.7}$$

Proof. Irreducibility of $X^{(\boldsymbol{\alpha})}$ on $\overline{F} \setminus B(\boldsymbol{\alpha})$ and the first statement (2.6) follows directly from the construction.

For $i \in \overline{F}, j \in B(\boldsymbol{\alpha})$ we obtain from the definition of acceptance probability $\alpha_j = 0$

$$r^{(\boldsymbol{\alpha})}(i,j) = 0. \tag{2.8}$$

Note, that from (2.2) follows

$$P\left(X_n^{(A)} = k, Y_n^{(A)} = 0, Y_{n-1}^{(A)} = 0, \ldots \right. \tag{2.9}$$
$$\left. \ldots, Y_2^{(A)} = 0, Y_1^{(A)} = 0, | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right) = \left(\left(r I_{(1-\boldsymbol{\alpha})}\right)^n\right)_{ik}$$

It follows for $i \in \overline{F}, j \in \overline{F} \setminus B(\boldsymbol{\alpha})$

$$
\begin{aligned}
r_{ij}^{(\boldsymbol{\alpha})} &= P\left(X^{(A)}(\tau^{(A)}) = j | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right) \\
&= P\left(X^{(A)}(\tau^{(A)}) = j, Y^{(A)}(\tau^{(A)}) = 1 | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right) \\
&= \sum_{n=1}^{\infty} P\left(X^{(A)}(\tau^{(A)}) = j, Y^{(A)}(\tau^{(A)}) = 1, \tau^{(A)} = n | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right) \\
&= \sum_{n=1}^{\infty} \sum_{k \in \overline{F}} P\left(X_n^{(A)} = j, Y_n^{(A)} = 1, X_{n-1}^{(A)} = k, Y_{n-1}^{(A)} = 0, Y_{n-2}^{(A)} = 0, \dots \right. \\
&\qquad\qquad\qquad \left. \dots, Y_2^{(A)} = 0, Y_1^{(A)} = 0 | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right) \\
&\overset{(1)}{=} \sum_{n=1}^{\infty} \sum_{k \in \overline{F}} P\left(X_n^{(A)} = j, Y_n^{(A)} = 1 | X_{n-1}^{(A)} = k, Y_{n-1}^{(A)} = 0\right) \\
&\qquad\qquad \cdot P\left(X_{n-1}^{(A)} = k, Y_{n-1}^{(A)} = 0, Y_{n-2}^{(A)} = 0, \dots \right. \\
&\qquad\qquad\qquad \left. \dots, Y_2^{(A)} = 0, Y_1^{(A)} = 0 | (X_0^{(A)}, Y_0^{(A)}) = (i, 0)\right) \\
&\overset{(2)}{=} \sum_{k \in \overline{F}} \sum_{n=1}^{\infty} (r I_{\boldsymbol{\alpha}})_{kj} \left(\left(r I_{(1-\boldsymbol{\alpha})}\right)^{n-1}\right)_{ik} \overset{(3)}{=} \sum_{k \in \overline{F}} (r I_{\boldsymbol{\alpha}})_{kj} \left(\sum_{n=1}^{\infty} \left(r I_{(1-\boldsymbol{\alpha})}\right)^{n-1}\right)_{ik} \\
&= \left(\left(I - r I_{(1-\boldsymbol{\alpha})}\right)^{-1} r I_{\boldsymbol{\alpha}}\right)_{ij},
\end{aligned}
$$

which is (2.7). Here (1) utilizes the Markov property of $(X^{(A)}, Y^{(A)})$, (2) follows from (2.9) and (2.1), and in (3) convergence of $\sum_{n=1}^{\infty} \left(r I_{(1-\boldsymbol{\alpha})}\right)^{n-1}$ follows from irreducibility of r and from the substochasticity of $r I_{(1-\boldsymbol{\alpha})}$, which is strict for at least one row. $\qquad\square$

We remark that we can see the stochasticity of $r^{(\boldsymbol{\alpha})}$ from

$$
\begin{aligned}
r^{(\boldsymbol{\alpha})} \mathbf{e} \overset{(2.7)}{=} &\; (I - r I_{(1-\boldsymbol{\alpha})})^{-1} \big[r \underbrace{I_{\boldsymbol{\alpha}}}_{=(I - I_{(1-\boldsymbol{\alpha})})} \big] \mathbf{e} = (I - r I_{(1-\boldsymbol{\alpha})})^{-1} \left[r(I - I_{(1-\boldsymbol{\alpha})}) \right] \mathbf{e} \\
= &\; (I - r I_{(1-\boldsymbol{\alpha})})^{-1} \big[\underbrace{r\mathbf{e}}_{=I\mathbf{e}} - r I_{(1-\boldsymbol{\alpha})} \mathbf{e} \big] = (I - r I_{(1-\boldsymbol{\alpha})})^{-1} (I - r I_{(1-\boldsymbol{\alpha})}) \mathbf{e} = \mathbf{e}.
\end{aligned}
$$

Furthermore, we note that the states in $B(\boldsymbol{\alpha})$ are inessential, because the $B(\boldsymbol{\alpha})$-columns $(r_{ij}^{(\boldsymbol{\alpha})} : i \in \overline{F})$ for $j \in B(\boldsymbol{\alpha})$ are zero.

Example 2.3. *Consider a set $\overline{F} = \{0, 1, 2, 3, 4\}$ and a routing matrix*

$$
r = \begin{pmatrix}
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & & 1 & & & \\
1 & & & 1 & & \\
2 & & & & 0.6 & 0.4 \\
3 & 1 & & & & \\
4 & & & 1 & &
\end{array}
\end{pmatrix}.
$$

After applying the skipping rule to r with availability vector $\boldsymbol{\alpha} = (1, 1, 0.5, 1, 1)$ we get

$$
r^{(\boldsymbol{\alpha})} =
\begin{pmatrix}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & & 1 & & & \\
1 & & & 0.5 & 0.3 & 0.2 \\
2 & & & & 0.6 & 0.4 \\
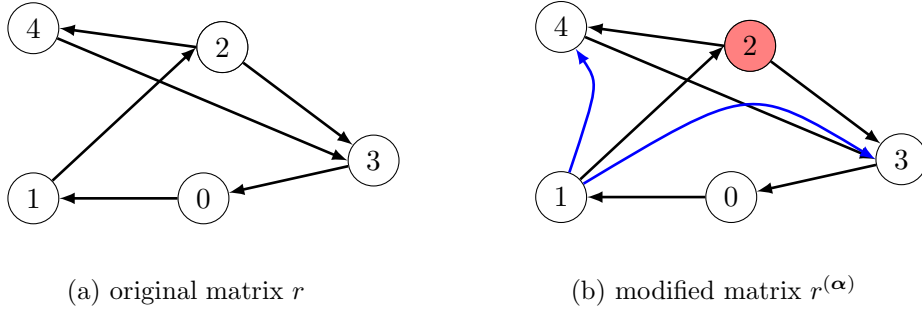3 & 1 & & & & \\
4 & & & 1 & &
\end{pmatrix} .
$$

See Figure 1.

(a) original matrix r (b) modified matrix $r^{(\boldsymbol{\alpha})}$

Figure 1: Matrix r on $\overline{F} = \{0, 1, 2, 3, 4\}$ and $r^{(\boldsymbol{\alpha})}$ with $\boldsymbol{\alpha} = (1, 1, 0.5, 1, 1)$ from Example 2.3.

2.1.2 Stationary distribution

Recall that X is irreducible with transition matrix r on the finite state space $\overline{F}$ and has the unique stationary distribution $\eta = (\eta_j : j \in \overline{F})$. For a non zero vector $\boldsymbol{\alpha} = (\alpha_j : j \in \overline{F})$ the chain $X^{(\boldsymbol{\alpha})}$ with state space $\overline{F}$ and transition matrix $r^{(\boldsymbol{\alpha})}$ is irreducible on $\overline{F} \setminus B(\boldsymbol{\alpha})$, with $B(\boldsymbol{\alpha}) = \{j \in \overline{F} : \alpha_j = 0\}$. We denote its stationary distribution by $\eta^{(\boldsymbol{\alpha})} = (\eta^{(\boldsymbol{\alpha})}(j) : j \in \overline{F})$, which has support $\overline{F} \setminus B(\boldsymbol{\alpha})$. We will study the relation between η and $\eta^{(\boldsymbol{\alpha})}$.

Proposition 2.4. *Let x be a solution of the balance equation $x \cdot r = x$ then*

$$
y = x \cdot I_{\boldsymbol{\alpha}}
$$

solves the steady state equation $y \cdot r^{(\boldsymbol{\alpha})} = y$ of the modified Markov chain with randomized skipping.

Proof. From $x \cdot r = x$ we obtain

$$
xr \underbrace{\left(I_{\boldsymbol{\alpha}} + I_{(1-\boldsymbol{\alpha})} \right)}_{=I} = x \iff xrI_{\boldsymbol{\alpha}} = x - xrI_{1-\boldsymbol{\alpha}} \iff xrI_{\boldsymbol{\alpha}} = x(I - rI_{(1-\boldsymbol{\alpha})})
$$

$$
\iff x \underbrace{(I - rI_{(1-\boldsymbol{\alpha})})(I - rI_{(1-\boldsymbol{\alpha})})^{-1}}_{=I} rI_{\boldsymbol{\alpha}} = x(I - rI_{(1-\boldsymbol{\alpha})})
$$

and with $y := x(I - rI_{(1-\boldsymbol{\alpha})})$ follows that a required solution of $y \cdot r^{(\boldsymbol{\alpha})} = y$ is

$$
y = x(I - rI_{(1-\boldsymbol{\alpha})}) = x - xI_{(1-\boldsymbol{\alpha})} = xI_{\boldsymbol{\alpha}} .
$$

$\square$

Proposition 2.5. *Let y be a solution of the balance equation $y \cdot r^{(\boldsymbol{\alpha})} = y$ then*

$$x = y(I - rI_{(1-\boldsymbol{\alpha})})^{-1} \tag{2.10}$$

is a solution of the steady state equation $x \cdot r = x$ and it holds

$$y = (\alpha_j x_j : j \in \overline{F}). \tag{2.11}$$

Proof. From the definition of $r^{(\boldsymbol{\alpha})}$ follows

$$y \underbrace{(I - rI_{(1-\boldsymbol{\alpha})})^{-1}rI_{\boldsymbol{\alpha}}}_{=r^{(\boldsymbol{\alpha})}} \quad = \quad y \underbrace{(I - rI_{(1-\boldsymbol{\alpha})})^{-1}(I - rI_{(1-\boldsymbol{\alpha})})}_{=I}$$

So $x = y(I - rI_{(1-\boldsymbol{\alpha})})^{-1}$ fulfils

$$xrI_{\boldsymbol{\alpha}} = x(I - rI_{(1-\boldsymbol{\alpha})}) \iff xr \underbrace{\left(I_{\boldsymbol{\alpha}} + I_{(1-\boldsymbol{\alpha})}\right)}_{=I} = x$$

which is $x \cdot r = x$. (2.11) follows from $x(I - rI_{(1-\boldsymbol{\alpha})}) = y$ as in Proposition 2.4. $\qquad\square$

Theorem 2.6. *If η is the unique steady state distribution of the irreducible Markov chain X on the finite state space $\overline{F}$, then the unique steady state distribution $\eta^{(\boldsymbol{\alpha})}$ of the Markov chain $X^{(\boldsymbol{\alpha})}$ is*

$$\eta^{(\boldsymbol{\alpha})} = \left(C^{(\boldsymbol{\alpha})}\right)^{-1} (\eta_j \alpha_j : j \in \overline{F}) \tag{2.12}$$

with normalization constant

$$C^{(\boldsymbol{\alpha})} = (\eta I_{\boldsymbol{\alpha}}\mathbf{e}) = \langle \eta, \boldsymbol{\alpha} \rangle. \tag{2.13}$$

$\eta^{(\boldsymbol{\alpha})}$ *has support $\overline{F} \setminus B(\boldsymbol{\alpha})$.*

Proof. From Propositions 2.4 and 2.5 we know that there is a one-to-one connection between all solutions of $x \cdot r = x$ and $y \cdot r^{(\boldsymbol{\alpha})} = y$.

From the assumptions on X we know that η is the unique stochastic solution of $x \cdot r = x$. Its normalized companion is therefore (2.12) by (2.11). $\qquad\square$

2.2 Randomized reflection

An important problem in the control of transmission networks is to react to full buffers at receiver stations by the network provider. There are many detailed control regimes to resolve blocking which occurs when buffers overflow. It turned out that it is usually difficult to construct analytical network models for this problem which admit closed form solutions for main performance metrics.

The following control principle is common to resolve blocking situations in networks with blocking of stations due to full buffers or blocking due to resource sharing, and is called blocking principle REPETITIVE SERVICE - RANDOM DESTINATION (RS–RD).

For applications in modeling of communication protocols in systems with finite buffers or for ALOHA-type protocols see [Kle76, Section 5.11], Within the abstract framework of reversible processes it occurs in [Lig85, Proposition II.5.10].

The fundamental idea is that whenever a packet is to be send from some node i to node j and it is observed that the buffer for incoming packets at j is full the packet is rejected

(lost) and node i who has saved a copy tries to resend this packet (REPETITIVE SERVICE), but not necessarily to j but selects another destination node k (RANDOM DESTINATION). This reflection procedure is iterated until the packet is sent to a node with free buffer places.

We will apply this scheme to modify the homogeneous irreducible Markov chain $X = (X_n : n \in \mathbb{N}_0)$ on the finite state space $\overline{F}$ with one-step transition probability matrix $r = (r(i,j) : i, j \in \overline{F})$ and (unique) steady state distribution $\eta = (\eta_i : i \in \overline{F})$.

In terms of a random walk on $\overline{F}$ governed by r an intuitive description is: If the random walker's (RW) path governed by the Markov chain is restricted by a taboo set $B \subsetneq \overline{F}$, RW applies

Reflection at B: If RW is in state $i \in \overline{F}$ and selects (with probability $r(i,j)$) its destination $j \in \overline{F} \setminus B$, this jump is allowed and immediately performed. If (with probability $r(i,k)$) he decides to jump to state $k \in B$, he is reflected at k and spends at least one further time slot at node i. Thereafter he restarts the procedure possibly with another destination node, i.e. with probability $r(i,l)$ he selects successor state l; if $l \in \overline{F} \setminus B$ then the jump is performed immediately, but if $l \in B$ RW is reflected again; and so on.

Our extended modification scheme for X and r is a randomized generalization of that reflection scheme. For the states in $\overline{F}$ are given "acceptance probabilities" by some vector $\boldsymbol{\alpha} = (\alpha_j \in [0,1] : j \in \overline{F})$. The random walker (RW) selects his itinerary under r and the constraints $\boldsymbol{\alpha}$ by

Randomized reflection with acceptance probabilities $\boldsymbol{\alpha}$: If RW is in state $i \in \overline{F}$ and selects (with probability $r(i,j)$) its destination $j \in \overline{F}$, a Bernoulli experiment is performed with success (acceptance) probability α_j, independent of the past, given j. If the experiment is successful ($= 1$), this jump is accepted, immediately performed, and RW settles down at j for at least one time slot. If the experiment is not successful ($= 0$), this jump is not accepted and RW stays on at i for at least one further time slot. If this slot expires with probability $r(i,l)$ RW selects another possible successor state l; thereafter a Bernoulli experiment is performed with success (acceptance) probability α_l, independent of the past, given l. If the experiment is successful ($= 1$), this jump is accepted, immediately performed, and RW settles down at l for at least one time slot. If the experiment is not successful ($= 0$), this jump is not accepted and RW stays on at i for at least one further time slot; and so on.

Example 2.7. *If for $B \subsetneq \overline{F}$*

$$\alpha_j = \begin{cases} 0 & if \quad j \in B, \\ 1 & if \quad j \in \overline{F} \setminus B, \end{cases}$$

then we have exactly the classical reflection at taboo set B as described above, because a jump to $j \in B$ is never accepted, but whenever a jump to $j \in \overline{F} \setminus B$ is proposed, this will be accepted with probability 1.

If we have some general $\boldsymbol{\alpha}$ then the set $B(\boldsymbol{\alpha}) = \{j \in \overline{F} : \alpha_j = 0\}$ is a taboo set for the "randomized reflection process".

2.2.1 Transition matrix and stationary distribution

It is easy to see that this construction of a modified chain by randomized reflection generates a new homogeneous Markov chain.

For the Markov chain $X = (X_n : n \in \mathbb{N}_0)$ with transition matrix $r = (r(i,j) : i,j \in \overline{F})$ and a non zero vector $\boldsymbol{\alpha} = (\alpha_j : j \in \overline{F})$ of acceptance probabilities denote by $X^{(\boldsymbol{\alpha})}$ the Markov chain modification of X under randomized reflection and by $r^{(\boldsymbol{\alpha})}$ the transition matrix of $X^{(\boldsymbol{\alpha})}$, and by $B(\boldsymbol{\alpha}) = \{j \in \overline{F} : \alpha_j = 0\} \subsetneq \overline{F}$ the taboo set for $X^{(\boldsymbol{\alpha})}$.

Then we obtain

$$r^{(\boldsymbol{\alpha})}(i,j) = \begin{cases} r(i,j)\alpha_j, & i,j \in \overline{F},\ i \neq j, \\ r(i,i) + \sum_{k \in \overline{F}} r(i,k)(1 - \alpha_j), & i \in \overline{F},\ i = j. \end{cases} \tag{2.14}$$

The Markov chain $X^{(\boldsymbol{\alpha})}$ with state space $\overline{F}$ and transition matrix $r^{(\boldsymbol{\alpha})}$ may be reducible even on $\overline{F} \setminus B(\boldsymbol{\alpha})$.

A standard assumption in the literature for applying this reflection principle as rerouting scheme is that the original routing Markov chain X, resp. its transition matrix r is reversible for some probability $\eta = (\eta_i : i \in \overline{F})$. We shall set this assumption always in force when investigating this protocol. To determine the appropriate modified transition matrix $r^{(\boldsymbol{\alpha})}$ and the associated stationary distribution $\eta^{(\boldsymbol{\alpha})}$ is direct in this case.

Proposition 2.8. *If η is the unique steady state distribution of the reversible irreducible Markov chain X on the finite state space $\overline{F}$, then the Markov chain $X^{(\boldsymbol{\alpha})}$ is reversible as well with steady state distribution $\eta^{(\boldsymbol{\alpha})}$ with*

$$\eta^{(\boldsymbol{\alpha})} = \left(C^{(\boldsymbol{\alpha})} \right)^{-1} (\eta_j \alpha_j : j \in \overline{F}) \tag{2.15}$$

with normalization constant

$$C^{(\boldsymbol{\alpha})} = (\eta I_{\boldsymbol{\alpha}} \mathbf{e}) = \langle \eta, \boldsymbol{\alpha} \rangle \tag{2.16}$$

$\eta^{(\boldsymbol{\alpha})}$ *has support* $\overline{F} \setminus B(\boldsymbol{\alpha})$.

Proof. The proof is by directly checking the detailed balance equations for $r^{(\boldsymbol{\alpha})}$.

$$(\eta_j \alpha_j) r^{(\boldsymbol{\alpha})}(j,i) = (\eta_i \alpha_j) r^{(\boldsymbol{\alpha})}(i,j), \qquad i,j \in \overline{F}$$
$$\Leftrightarrow \quad (\eta_j \alpha_j) r(j,i)\alpha_i = (\eta_i \alpha_i) r(i,j)\alpha_j, \qquad i,j \in \overline{F}.$$

<div align="right">□</div>

2.3 Discussion of randomization algorithms

A closer look on our randomization procedures, considered as algorithms to manipulate a random walk and its stationary distribution, reveals close connections to standard simulation procedures. This will be discussed in this section and we will compare furthermore the randomization procedures.

2.3.1 General sampling schemes

Our starting point is an ergodic random walk X on $\overline{F}$ with stationary distribution η. X is modified according to an acceptance regime $\boldsymbol{\alpha}$. Our aim is to study the impact of this modification on the stationary distribution. This is different to the standard problem of simulation, i.e. to generate samples from a given distribution, which is known in principle, but not easy to access. We mention two simulation algorithms which are structurally related to our procedures.

(I) Markov Chain Monte Carlo algorithms start from a target distribution η which is usually not directly accessible for which e.g.

$$\int_{\overline{F}} f(x)\eta(dx) = \sum_{x \in \overline{F}} f(x)\eta_x \, ,$$

has to be computed. The idea is to construct a homogeneous Markov chain with stationary distribution η and to sample from the Markov chains' state distribution after a sufficiently long time horizon.

The construction of the one-step transition kernel following Hastings or Metropolis (see [Bre99][Section 7.7.1]) yields a two-step scheme:

1. from the present state generate a proposal for the next state and decide about acceptance of the proposed state,

2. if the proposed state is rejected stay on at the present state and restart after a time slot.

This is a procedure as descibed in Proposition 2.8. The conclusion is that the rerouting scheme **rs-rd** used to model a transmission network's reaction on full buffers and its generalization, our randomized reflection, can be considered as MCMC processes.

(II) Von Neumann's acceptance-rejection method for sampling from a complicated distribution η (see [Bre99][p. 292]) can be connected to our modification of a Markov chain by randomized skipping. If the random walk of Section 2.1 is an i.i.d. sequence the randomized skipping procedure is exactly sampling from $(\eta_j \alpha_j : j \in \overline{F})$ by sampling from η with possible rejection.

We remark that in MCMC algorithms usually reversible chains are constructed with acceptance probabilities which may depend on the departure state and the proposal. Such generalization is easily constructed here as well, in our notation we would incorporate success probabilities α_{ij}. Proposition 2.8 can be modified directly to incorporate this feature.

A similar property and the proof thereof for randomized skipping for non-reversible Markov chains seems to be not possible without loosing the simple to evaluate steady state distribution.

2.3.2 Importance sampling

Our randomization procedures resemble obviously importance sampling procedures in simulations, because we

1. produce a weighted version of the probability distribution η on $\overline{F}$, and

2. define a modified random walk that generates this weighted distribution as its limiting distribution,

and this can therefore be considered as construction of a change of measure. In importance sampling such change of measure is used to make the "more important states" more often visited by the random walker. An important problem is to compute expectations of the form

$$\int_{\overline{F}} f(x)\eta(dx) = \sum_{x \in \overline{F}} f(x)\eta_x \, , \tag{2.17}$$

15

where $X = (X_n : n \in \mathbb{N}_0)$ with transition matrix r is ergodic with limiting distribution η, where η is not known or not easily accessible.

For simplicity we assume $\alpha_j > 0$ for all $j \in \overline{F}$ and use the notation of the previous sections. Recall that $C^{(\boldsymbol{\alpha})} = \langle \eta, \boldsymbol{\alpha} \rangle$ is the normalization constant of $\eta^{(\boldsymbol{\alpha})}$. By simple formal manipulation

$$\sum_{x \in \overline{F}} f(x) \eta_x \;=\; \sum_{x \in \overline{F}} \frac{f(x)}{\langle \eta, \boldsymbol{\alpha} \rangle^{-1}} \cdot \frac{\eta_x}{\langle \eta, \boldsymbol{\alpha} \rangle} \;=\; \frac{\sum_{x \in \overline{F}} \frac{f(x)}{\alpha_x} \cdot \frac{\eta_x \alpha_x}{\langle \eta, \boldsymbol{\alpha} \rangle}}{\sum_{x \in \overline{F}} \frac{1}{\alpha_x} \cdot \frac{\eta_x \alpha_x}{\langle \eta, \boldsymbol{\alpha} \rangle}} \;.$$

Nominator and denominator are integrals with respect to the stationary distribution of $X^{(\boldsymbol{\alpha})}$ derived from X with acceptance probability $\boldsymbol{\alpha} = (\alpha_x : x \in \overline{F})$. X is ergodic and $\alpha_x > 0 \; \forall x \in \overline{F}$ implies that $X^{(\boldsymbol{\alpha})}$ is ergodic on $\overline{F}$ as well.

Denote by $T(i)$ the first entrance time into i, then from the regenerative structure of $X^{(\boldsymbol{\alpha})}$ we have

$$\sum_{x \in \overline{F}} \frac{f(x)}{\alpha_x} \cdot \frac{\eta_x \alpha_x}{\langle \eta, \boldsymbol{\alpha} \rangle} = \frac{E^{(\boldsymbol{\alpha})} \left[\sum_{n=0}^{T(i)-1} \frac{f(X_n^{(\boldsymbol{\alpha})})}{\alpha_{X_n^{(\boldsymbol{\alpha})}}} \middle| X_0^{(\boldsymbol{\alpha})} = i \right]}{E^{(\boldsymbol{\alpha})} \left[T(i) \middle| X_0^{(\boldsymbol{\alpha})} = i \right]}$$

and

$$\sum_{x \in \overline{F}} \frac{1}{\alpha_x} \cdot \frac{\eta_x \alpha_x}{\langle \eta, \boldsymbol{\alpha} \rangle} = \frac{E^{(\boldsymbol{\alpha})} \left[\sum_{n=0}^{T(i)-1} \frac{1}{\alpha_{X_n^{(\boldsymbol{\alpha})}}} \middle| X_0^{(\boldsymbol{\alpha})} = i \right]}{E^{(\boldsymbol{\alpha})} \left[T(i) \middle| X_0^{(\boldsymbol{\alpha})} = i \right]} \;.$$

This yields eventually

$$\sum_{x \in \overline{F}} f(x) \eta_x = \frac{E^{(\boldsymbol{\alpha})} \left[\sum_{n=0}^{T(i)-1} \frac{f(X_n^{(\boldsymbol{\alpha})})}{\alpha_{X_n^{(\boldsymbol{\alpha})}}} \middle| X_0^{(\boldsymbol{\alpha})} = i \right]}{E^{(\boldsymbol{\alpha})} \left[\sum_{n=0}^{T(i)-1} \frac{1}{\alpha_{X_n^{(\boldsymbol{\alpha})}}} \middle| X_0^{(\boldsymbol{\alpha})} = i \right]} \;.$$

To obtain an estimator for $E^{(\boldsymbol{\alpha})} \left[\sum_{n=0}^{T(i)-1} g \left(X_n^{(\boldsymbol{\alpha})} \right) \middle| X_0^{(\boldsymbol{\alpha})} = i \right]$ denote $T^0(i) \equiv 0$, and $T^{(n+1)}(i) := \inf(k > T^{(n)}(i) : X_k^{(\boldsymbol{\alpha})} = i)$, $n \geq 0$, and take the sample means of the independent replications

$$\frac{1}{K} \sum_{k=1}^{K} \left(\sum_{n=T^{(k-1)}(i)}^{T^{(k)}(i)} g \left(X_n^{(\boldsymbol{\alpha})} \right) \middle| X_0^{(\boldsymbol{\alpha})} = i \right) ,$$

which converge a.s. to the target expression for $K \to \infty$.

So, if it is possible to simulate without too much effort the system which is described by the Markov chain $X^{(\boldsymbol{\alpha})}$, we can approximate the integral by the quotient of the two time averages over the regeneration periods.

We remark that by our simple construction we not only can make important states more probable to visit, but for a given initial state i we can control the expected length of the regeneration period.

If X is reversible, the one step transition matrix of the chain modified by randomized reflection is directly accessible, see (2.14).

In the general case when X is modified by randomized skipping the one step transition matrix of the modified chain is possibly not directly available, see (2.7). We can remedy this drawback by simulating instead the auxiliary chain $(X^{(A)}, Y^{(A)})$ (with obvious modifications) which is used to find the one step transition matrix for randomized skipping in Section 2.1.1. The modification of $(X^{(A)}, Y^{(A)})$ is not to make $\overline{F} \times \{1\}$ absorbing. The transition rates (2.3) and (2.4) are replaced by

$$P(X_{n+1}^{(A)} = j, Y_{n+1}^{(A)} = 1 | X_n^{(A)} = i, Y_n^{(A)} = 1) \;=\; r(i,j)\alpha_j, \tag{2.18}$$

$$P(X_{n+1}^{(A)} = j, Y_{n+1}^{(A)} = 0 | X_n^{(A)} = i, Y_n^{(A)} = 1) \;=\; r(i,j)(1 - \alpha_j). \tag{2.19}$$

In the time average over a regeneration period we then have to sum instead of $f(X_n^{(\boldsymbol{\alpha})}(\omega))$ the values $f(X_n^{(A)}(\omega)) \cdot Y_n^{(A)}(\omega)$ and instead of $\frac{1}{\alpha_{X_n^{(\boldsymbol{\alpha})}(\omega)}}$ the values $\frac{Y_n^{(A)}(\omega)}{\alpha_{X_n^{(\boldsymbol{\alpha})}(\omega)}}$ and to divide by the expected sojourn time in $\overline{F} \times \{1\}$ until return to $(i,1)$ given start in $(i,1)$ (which cancels out).

2.3.3 Comparison of randomized skipping and randomized reflection

We can compare transition matrices having the same dimension and stationary distribution by Peskun ordering (see [Pes73]) which is standard ordering in MCMC simulations.

Definition 2.9. *Let $r = (r(i,j) : i, j \in \overline{F})$ and $r' = (r'(i,j) : i, j \in \overline{F})$ be transition matrices on a finite set $\overline{F}$ such that $\xi r = \xi r' = \xi$ for a probability vector ξ.*

We say that r' is smaller than r in the Peskun order, $r' \prec_P r$, if for all $j, i \in \overline{F}$ with $i \neq j$ holds $r'(j,i) \leq r(j,i)$.

Peskun used this order to compare reversible transition matrices with the same stationary distribution and their asymptotic variance. A useful interpretation is that in case of $r' \prec_P r$ a random walker under r explores his state space faster than a random walker under r'.

Proposition 2.10. *Let X be irreducible with transition matrix r on the finite state space $\overline{F}$ which is reversible for the stationary distribution $\eta = (\eta_j : j \in \overline{F})$, and $\boldsymbol{\alpha} = (\alpha_j : j \in \overline{F})$ be a non zero vector.*
Denote by $r_{skip}^{(\boldsymbol{\alpha})}$ the modification of r by randomized skipping and by $r_{refl}^{(\boldsymbol{\alpha})}$ the modification of r by randomized reflection. Then it holds $r_{refl}^{(\boldsymbol{\alpha})} \prec_P r_{skip}^{(\boldsymbol{\alpha})}$.

Proof. The proof is obvious because for all $j, i \in \overline{F}$ with $i \neq j$ holds $r_{refl}^{(\boldsymbol{\alpha})}(i,j) = r(i,j) \cdot \alpha_j$ and $r_{skip}^{(\boldsymbol{\alpha})}(i,j) = r(i,j) \cdot \alpha_j +$ possible further terms, which are all non-negative. $\qquad\square$

Consequences of $r_{refl}^{(\boldsymbol{\alpha})} \prec_P r_{skip}^{(\boldsymbol{\alpha})}$ in Proposition 2.10 are

1. the asymptotic variance in the central limit theorem for a Markov chain driven by $r_{skip}^{(\boldsymbol{\alpha})}$ is smaller than for a Markov chain driven by $r_{refl}^{(\boldsymbol{\alpha})}$ although both chains have the same limiting distribution (see [Tie98]).

2. the spectral gap of a Markov chain driven by $r_{refl}^{(\boldsymbol{\alpha})}$ is smaller than for a Markov chain driven by $r_{skip}^{(\boldsymbol{\alpha})}$ although both chains have the same limiting distribution. This means that the modification of **X** by randomized skipping converges faster to equilibrium than the modification of **X** by randomized reflection.

A further distinction between modifications of $\mathbf{X}$ by randomized skipping and by randomized reflection is visible if we consider for a reversible Markov chain $\mathbf{X}$ with transition matrix r the associated transition graph $G = (\overline{F}, \mathcal{E})$ where $(i, j) \in \mathcal{E} \Leftrightarrow r(i, j) > 0$.

Note that from reversibility follows $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. Denote by $\mathcal{N}(i) := \{j \in \overline{F} : (i, j) \in \mathcal{E}\}$ the one-step neighbourhood of i.

If we consider randomized skipping and by randomized reflection as algorithms to produce modifications of the random walk $\mathbf{X}$, then the algorithm for randomized reflection is local with respect to G because transitions out of states i are determined only on the basis of decisions in $\mathcal{N}(i)$.

On the other side, randomized skipping is global with respect to G because for a transition out of i we possibly must consider random experiments anywhere on the graph.

To construct local algorithms for control, scheduling, and development of complex systems is important for key applications, e.g. wireless sensor networks or autonomous interacting robotic systems.

For mathematical investigations of this problem we refer to [SS12], where an optimal local control algorithm for wireless sensor networks is determined by local approximation of an optimal global algorithm.

3 Stochastic networks in a random environment

3.1 Jackson networks

We consider a Jackson network [Jac57] with node set $\overline{J} := \{1, \dots, J\}$. Customers arrive in independent external Poisson streams, at node j with finite intensity $\lambda_j \geq 0$, we set $\lambda = \lambda_1 + \dots + \lambda_J > 0$. Customers are indistinguishable and follow the same rules. Customers request for service which is exponentially distributed with mean 1 at all nodes, all these requests constitute an independent family of variables which are independent of the arrival streams.

Nodes are exponential single servers with state dependent service rates and with an infinite waiting room under first–come–first–served (FCFS) regime. If at node j there are $n_j > 0$ customers present, either in service or waiting, then service is provided there with intensity $\mu_j(n_j) > 0$. (Therefore, in general, the obtained service time is not exponential-1.)

Routing is Markovian, a customer departing from node i immediately proceeds to node j with probability $r(i, j) \geq 0$, and departs from the network with probability $r(j, 0)$. Taking $r(0, j) = \lambda_j / \lambda$, $r(0, 0) = 0$, we assume that the (extended) routing matrix $r = (r(i, j) : i, j = 0, \dots, J)$ is irreducible. This ensures that the traffic equations

$$\eta_j = \lambda_j + \sum_{i=1}^{J} \eta_i r(i, j), \qquad j = 1, \dots, J, \tag{3.1}$$

have a unique solution which we denote by $\eta = (\eta_j : j = 1, \dots, J)$. We extend the traffic equation (3.1) to a steady state equation for a routing Markov chain by

$$\eta_j = \sum_{i=0}^{J} \eta_i r(i, j), \qquad j = 0, 1, \dots, J, \tag{3.2}$$

which is solved by $\eta = (\eta_j : j = 0, 1, \dots, J)$, where $\eta_0 := \lambda$ and the other η_j are from (3.1). We use η in both meanings and emphasize the latter one by *extended traffic solution η. η is

usually not a stochastic vector. Let $\mathbf{X} = (X(t) : t \geq 0)$ denote the vector process recording the queue lengths in the network for time t. $X(t) = (X_1(t), \ldots, X_J(t)) \in \mathbb{N}_0^{\overline{J}}$ reads: at time t there are $X_j(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that $\mathbf{X}$ is a strong Markov process on state space $\mathbb{N}_0^{\overline{J}}$ with generator $Q^{\mathbf{X}} = (q^{\mathbf{X}}(\mathbf{n}, \mathbf{n}') : \mathbf{n}, \mathbf{n}' \in \mathbb{N}_0^{\overline{J}})$. The strict positive transition rates of $Q^{\mathbf{X}}$ which describe the physical actions of the network are for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} + \mathbf{e}_i) = \lambda r(0, i), \qquad\qquad i \in \overline{J}, \qquad (3.3)$$
$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_i) = 1_{[n_j > 0]} \mu_j(n_j) r(j, i), \qquad i \neq j,\, i, j \in \overline{J},$$
$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = 1_{[n_j > 0]} \mu_j(n_j) r(j, 0), \qquad\qquad j \in \overline{J}.$$

For an ergodic network process $\mathbf{X}$ Jackson's theorem [Jac57] states that the unique steady state and limiting distribution ξ on $\mathbb{N}_0^{\overline{J}}$ is

$$\xi(\mathbf{n}) = \xi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \qquad (3.4)$$

with normalizing constants $C(j)$ for the marginal (over nodes) distributions of $\mathbf{X}$.

3.2 Modified Jackson networks

We consider in this section the problem to adjust routing to changes of service capacities of the network and assume that the routing regime is constructed to meet some optimization criterion with respect to the local queue lengths X_j for the fixed $\mu_j(n_j)$, $j \in \overline{J}$. This means that the ratios $\eta_j / \mu_j(n_j)$ reflect a locally optimal behaviour of the nodes and should be maintained when service capacities change.

3.2.1 Degraded service and adapted routing

If the service intensities $\mu_i(n_i)$ at node i are degraded by a factor $\gamma_i \in [0, 1]$ for $i \in \overline{J}$, we have to readjust the routing regime in a way to meet these ratios again, at least approximately. It is possible that some nodes can break down completely, i.e. $\gamma_\ell = 0$ for such node ℓ. Clearly, the broken down nodes should not be visited any longer, and from the side constraint to maintain at least approximately the ratios "overall arrival rate"/"service rates", it is tempting to try a rerouting by randomized skipping or reflection to some suitably selected "acceptance probability vector" according to the $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\gamma})$ (If there is no ambiguity we will shortly write only $\boldsymbol{\alpha}$.)
Then the new routing will have two components:

1. Part of the total external arrival rate will be rejected, and

2. the admitted load will be redistributed among the nodes which are not completely broken down in a way to meet exactly the old ratios.

We define $\alpha_0 = 1$ and take the given γ_i, $i \in \overline{J}$, to define the vector $\boldsymbol{\alpha}(\boldsymbol{\gamma}) = (\alpha_i, i \in \overline{J}_0)$ of acceptance probabilities as $\alpha_i := \gamma_i$ and modify the extended routing matrix $r = (r(i, j) :$

$i, j \in \overline{J}_0$) according to the randomized skipping, respectively randomized reflection principle from Sections 2.1.1 and 2.2.1.

It was surprising to us that we have to choose $\alpha_j = \gamma_j$ to adjust the acceptance probability of node j precisely to the service rate factor $\gamma_j \in [0, 1]$. This will no longer be possible if we later on will allow more general factors $\gamma_j \in [0, \infty)$, which may lead to acceptance probabilities $\alpha_j \neq \gamma_j$.

The modified routing matrix is in both cases (randomized skipping, resp. reflection) denoted by $r^{(\alpha)} = (r^{(\alpha)}(i, j) : i, j \in \overline{J}_0)$, and is given alternatively according to Theorem 2.2, respectively Proposition 2.4. Note, that from $(\alpha_0 = 1)$ we obtain $\eta^{(\alpha)}(0) = \eta(0)$.

Definition 3.1. *The set of "blocked nodes" $B(\boldsymbol{\gamma}) \subset \overline{J}$ is defined by*

$$j \in B(\boldsymbol{\gamma}) \Longleftrightarrow \gamma_j = 0,$$

and its complement in $\overline{J}$, the set of the "working nodes" $W(\boldsymbol{\gamma}) \subset \overline{J}$, is defined by

$$j \in W(\boldsymbol{\gamma}) \Longleftrightarrow \gamma_j > 0.$$

When the service rates of a Jackson network are modified according to $\boldsymbol{\gamma}$ and the routing is adjusted according to $\boldsymbol{\alpha}(\boldsymbol{\gamma})$ we obtain a new network process $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$ as the vector process recording the queue lengths in the network. $X^{(\gamma)}(t) = (X_1^{(\gamma)}(t), \ldots, X_J^{(\gamma)}(t)) \in \mathbb{N}_0^{\overline{J}}$ reads: at time t there are $X_j^{(\gamma)}(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that $\mathbf{X}$ is a strong Markov process on state space $\mathbb{N}_0^{\overline{J}}$ with generator $Q^{\mathbf{X}^{(\gamma)}} =: Q^{(\gamma)} = (q^{(\gamma)}(\mathbf{n}, \mathbf{n}') : \mathbf{n}, \mathbf{n}' \in \mathbb{N}_0^{\overline{J}})$. The strict positive transition rates of $Q^{(\gamma)}$ are under both rerouting regimes for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$\begin{aligned} q^{(\gamma)}(\mathbf{n}, \mathbf{n} + \mathbf{e}_i) &= \lambda r^{(\alpha)}(0, i), & i &\in \overline{J}_0, \qquad (3.5) \\ q^{(\gamma)}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_i) &= 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\alpha)}(j, i) & i, j &\in \overline{J}, \, i \neq j, \\ q^{(\gamma)}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) &= 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\alpha)}(j, 0), & j &\in \overline{J}. \end{aligned}$$

That the construction is successful in maintaining the ratios (overall arrival rate/service rates) shows the next theorems, which will be proved almost simultaneously. Recall $0/0 := 0$.

Theorem 3.2. [MODIFIED JACKSON NETWORKS: RANDOMIZED SKIPPING AND DEGRADING NODES] *Let $\mathbf{X}$ be an ergodic Jackson network process as described in Section 3.1 with stationary distribution ξ from (3.4). If the service intensities $\mu_i(n_i)$ at node i are degraded by a factor $\gamma_i \in [0, 1]$ for $i \in \overline{J}$, and routing is changed to $r^{(\alpha)}$, with $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\gamma}) = (\alpha_i : i \in \overline{J}_0)$ where $\alpha_0 = 1$ and $\alpha_j = \gamma_j$ for $j \in \overline{J}$, by randomized skipping according to Theorem 2.2, then ξ is a stationary distribution for $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$ as well.*
Moreover, if $B(\boldsymbol{\gamma}) = \emptyset$, then $\mathbf{X}^{(\gamma)}$ is ergodic.
If $B(\boldsymbol{\gamma}) \neq \emptyset$, then $\mathbf{X}^{(\gamma)}$ is not irreducible on $\mathbb{N}_0^{\overline{J}}$ and its state space is divided into an infinite set of closed subspaces

$$\mathbb{N}_0^{W(\boldsymbol{\gamma})} \times \{(n_j : j \in B(\boldsymbol{\gamma}))\} \quad \forall (n_j : j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\boldsymbol{\gamma})}.$$

For any probability distribution φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ there exists a stationary distribution $\xi_\varphi^{(\gamma)}$ for $\mathbf{X}^{(\gamma)}$, which is for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$\xi_\varphi^{(\gamma)}(\mathbf{n}) = \xi_\varphi^{(\gamma)}(n_1, \ldots, n_J) = \prod_{j \in W(\boldsymbol{\gamma})} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma})).$$

Example 3.3. *Consider a Jackson network on a node set $\overline{J} = \{1, 2, 3, 4\}$ with a routing matrix r as in Example 2.3 with service rate factors $\gamma = (1, 0.5, 1, 1)$. Then the availability vector is $\boldsymbol{\alpha} = (1, 1, 0.5, 1, 1)$ and a new modified routing matrix $r^{(\boldsymbol{\alpha})}$ is as in Example 2.3.*
See Figure 2 on page 21.

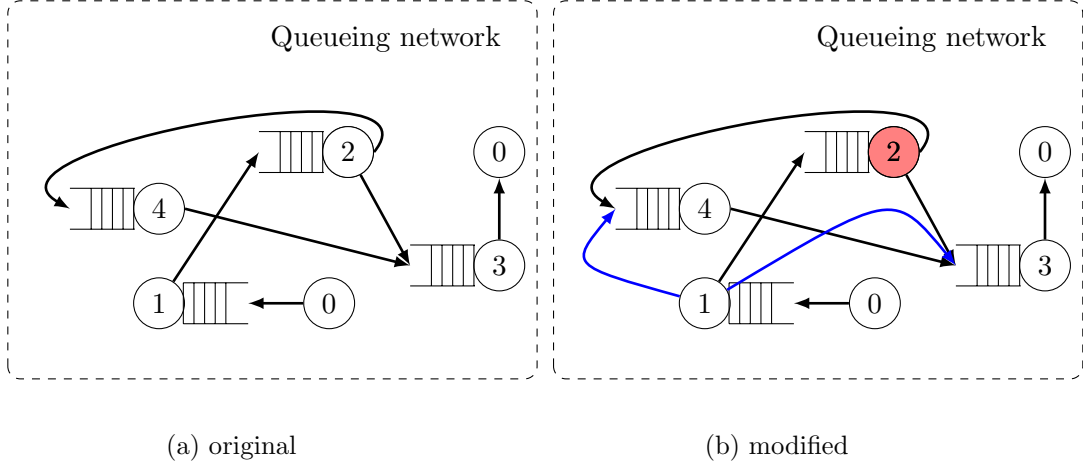(a) original   (b) modified

Figure 2: Original and modified according to the skipping rule Jackson networks from Example 3.3.

Theorem 3.4. [MODIFIED JACKSON NETWORKS: RANDOMIZED REFLECTION AND DEGRADING NODES] *Let $\mathbf{X}$ be an ergodic Jackson network process as described in Section 3.1 with reversible routing matrix r and with stationary distribution ξ from (3.4). If the service intensities $\mu_i(n_i)$ at node i are degraded by a factor $\gamma_i \in [0, 1]$ for $i \in \overline{J}$, and routing is changed to $r^{(\boldsymbol{\alpha})}$, with $\boldsymbol{\alpha} = \boldsymbol{\alpha}\boldsymbol{\gamma} = (\alpha_i : i \in \overline{J}_0)$ where $\alpha_0 = 1$ and $\gamma_j = \alpha_j$ for $j \in \overline{J}$, by randomized reflection according to Proposition 2.8, then ξ is a stationary distribution for $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$ as well.*

Moreover, if $B(\boldsymbol{\gamma}) = \emptyset$, then $\mathbf{X}^{(\gamma)}$ is ergodic.

If $B(\boldsymbol{\gamma}) \neq \emptyset$, then $\mathbf{X}^{(\gamma)}$ is not irreducible on $\mathbb{N}_0^{\overline{J}}$ and its state space is divided into an infinite set of closed subspaces

$$\mathbb{N}_0^{W(\gamma)} \times \{(n_j : j \in B(\boldsymbol{\gamma}))\} \quad \forall (n_j : j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\gamma)}.$$

For any probability distribution φ on $\mathbb{N}_0^{B(\gamma)}$ there exists a stationary distribution $\xi_\varphi^{(\gamma)}$ for $\mathbf{X}^{(\gamma)}$, which is for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$\xi_\varphi^{(\gamma)}(\mathbf{n}) = \xi_\varphi^{(\gamma)}(n_1, \ldots, n_J) = \prod_{j \in W(\gamma)} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma})).$$

Proof of Theorem 3.2 and Theorem 3.4. The global balance equation $x \cdot Q^{(\gamma)} = 0$ for the joint

queue length process $X^{(\gamma)}$ of the modified system is in both settings for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$x(\mathbf{n}) \left(\sum_{j \in \overline{J}} \lambda r^{(\alpha)}(0, j) + \sum_{j \in \overline{J}} 1_{[n_j > 0]} \gamma_j \mu_j(n_j)(1 - r^{(\alpha)}(j, j)) \right)$$

$$= \sum_{i \in \overline{J}} x(\mathbf{n} - \mathbf{e}_i) 1_{[n_i > 0]} \lambda r^{(\alpha)}(0, i) + \sum_{j \in \overline{J}} \sum_{i \in \overline{J} \setminus \{j\}} x(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) 1_{[n_i > 0]} \gamma_j \mu_j(n_j + 1) r^{(\alpha)}(j, i)$$

$$+ \sum_{j \in \overline{J}} x(\mathbf{n} + \mathbf{e}_j) \gamma_j \mu_j(n_j + 1) r^{(\alpha)}(j, 0) . \tag{3.6}$$

We first consider the case $B(\gamma) \neq \emptyset$. Then for $i \in B(\gamma)$ we have $\gamma_i = \alpha_i = 0$ and $r^{(\alpha)}(j, i) = 0$ for all $j \in \overline{J}_0$ and (3.6) reduces to

$$x(\mathbf{n}) \left(\sum_{j \in W(\gamma)} \lambda r^{(\alpha)}(0, j) + \sum_{j \in W(\gamma)} 1_{[n_j > 0]} \gamma_j \mu_j(n_j)(1 - r^{(\alpha)}(j, j)) \right)$$

$$= \sum_{i \in W(\gamma)} x(\mathbf{n} - \mathbf{e}_i) 1_{[n_i > 0]} \lambda r^{(\alpha)}(0, i)$$

$$+ \sum_{j \in W(\gamma)} \sum_{i \in W(\gamma) \setminus \{j\}} x(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) 1_{[n_i > 0]} \gamma_j \mu_j(n_j + 1) r^{(\alpha)}(j, i)$$

$$+ \sum_{j \in W(\gamma)} x(\mathbf{n} + \mathbf{e}_j) \gamma_j \mu_j(n_j + 1) r^{(\alpha)}(j, 0) . \tag{3.7}$$

Inserting $x(n_1, \ldots, n_J) = \prod_{j \in W(\gamma)} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\gamma))$ for any probability density φ on $\mathbb{N}_0^{B(\gamma)}$ we see that immediately $\prod_{j \in W(\gamma)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\gamma))$ cancels. Multiplication with $\left(\prod_{j \in W(\gamma)} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right)^{-1}$ yields

$$\left(\sum_{j \in W(\gamma)} \lambda r^{(\alpha)}(0, j) + \sum_{j \in W(\gamma)} 1_{[n_j > 0]} \gamma_j \mu_j(n_j)(1 - r^{(\alpha)}(j, j)) \right)$$

$$= \sum_{i \in W(\gamma)} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \lambda r^{(\alpha)}(0, i)$$

$$+ \sum_{j \in W(\gamma)} \sum_{i \in W(\gamma) \setminus \{j\}} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \frac{\eta_j}{\mu_j(n_j + 1)} \gamma_j \mu_j(n_j + 1) r^{(\alpha)}(j, i)$$

$$+ \sum_{j \in W(\gamma)} \frac{\eta_j}{\mu_j(n_j + 1)} \gamma_j \mu_j(n_j + 1) r^{(\alpha)}(j, 0) .$$

We replace γ_j by α_j which are equal for all $j \in \overline{J}$ and get

$$
\left(\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j>0]} \alpha_j \mu_j(n_j)(1 - r^{(\boldsymbol{\alpha})}(j,j)) \right)
$$
$$
= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i>0]} \lambda r^{(\boldsymbol{\alpha})}(0,i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma}) \setminus \{j\}} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i>0]} \frac{\eta_j}{\mu_j(n_j+1)} \alpha_j \mu_j(n_j+1) r^{(\boldsymbol{\alpha})}(j,i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \frac{\eta_j}{\mu_j(n_j+1)} \alpha_j \mu_j(n_j+1) r^{(\boldsymbol{\alpha})}(j,0) . \tag{3.8}
$$

Reordering and canceling this yields

$$
\left(\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j>0]} \alpha_j \mu_j(n_j) \right)
$$
$$
= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i>0]} \lambda r^{(\boldsymbol{\alpha})}(0,i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i>0]} \frac{\eta_j}{\mu_j(n_j+1)} \alpha_j \mu_j(n_j+1) r^{(\boldsymbol{\alpha})}(j,i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \frac{\eta_j}{\mu_j(n_j+1)} \alpha_j \mu_j(n_j+1) r^{(\boldsymbol{\alpha})}(j,0) , \tag{3.9}
$$

and so

$$
\left(\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j>0]} \alpha_j \mu_j(n_j) \right) \tag{3.10}
$$
$$
= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i>0]} \lambda r^{(\boldsymbol{\alpha})}(0,i) + \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i>0]} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j,i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \eta_j \alpha_j r^{(\boldsymbol{\alpha})}(j,0) .
$$

The first term on the left side and the last term on the right side equate

$$
\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) = \lambda(1 - r^{(\boldsymbol{\alpha})}(0,0)), \quad \text{and}
$$

$$
\underbrace{\left(\sum_{j \in W(\boldsymbol{\gamma})} \eta_j \alpha_j r^{(\boldsymbol{\alpha})}(j,0) + \eta_0 \alpha_0 r^{(\boldsymbol{\alpha})}(0,0) \right)}_{= \eta_0 \alpha_0} - \eta_0 \alpha_0 r^{(\boldsymbol{\alpha})}(0,0) = \lambda(1 - r^{(\boldsymbol{\alpha})}(0,0)) ,
$$

where we used $\eta_0 = \lambda$, $\alpha_0 = 1$, and that $(\eta_j \alpha_j : j \in \overline{J}_0)$ is an invariant measure for $r^{(\boldsymbol{\alpha})}$. So

(3.10) reduces to

$$\sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j > 0]} \alpha_j \mu_j(n_j) \tag{3.11}$$

$$= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i).$$

Take any $i \in W(\boldsymbol{\gamma})$ with $n_i > 0$ and consider the summands with this i:

$$\alpha_i 1_{[n_i > 0]} \mu_i(n_i) = \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i),$$

which is

$$\alpha_i \eta_i = \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i),$$

and recalling $\eta_0 = \lambda$, $\alpha_0 = 1$ and $\alpha_j = 0$ for $j \in B(\boldsymbol{\gamma})$ this is

$$\alpha_i \eta_i = \sum_{j \in W(\boldsymbol{\gamma}) \cup \{0\}} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i). \tag{3.12}$$

Note, that any i will occur in such a procedure for some suitable state vector with $n_i > 0$. Therefore, if (3.12) would be true, we would eventually arrive at

$$(\alpha_j \eta_j : j \in \overline{J}_0) \cdot r^{(\boldsymbol{\alpha})} = (\alpha_j \eta_j : j \in \overline{J}_0).$$

Now (3.12) is true for the setting of Theorem 3.2 by Theorem 2.2 and for the setting of Theorem 3.4 by Proposition 2.8, which finishes the parts for $B(\boldsymbol{\gamma}) \neq \emptyset$ of the proofs.

The case $B(\boldsymbol{\gamma}) = \emptyset$ is proved similarly. $\square$

3.2.2 Upgraded and/or degraded service and adapted routing

We allow now more general changes of service rates by non-negative factors $\gamma_j \in [0, \infty)$. This means that we either speed up service at node j, if $\gamma_j > 1$ or have a degraded server at node j, if $\gamma_j < 1$. The case $\gamma_j \leq 1 \ \forall j \in \overline{J}$ was considered in the previous section. When at least one service rate increases, i.e. when $||\boldsymbol{\gamma}||_\infty > 1$, we introduce a new mechanism to adapt the network's load and routing. In this case we increase the total network input by a factor $\beta = ||\boldsymbol{\gamma}||_\infty > 1$ and choose as acceptance probability vector $\boldsymbol{\alpha}$ the relative service rate changes $\alpha_j = \frac{\gamma_j}{||\boldsymbol{\gamma}||_\infty}$.

We will proceed with the general case $\gamma_j \in [0, \infty)$ in a way that the previous case is covered by our general formalism.

Remark: If $\gamma_j > 1$, node j can process more load without being overloaded, which is easily seen by considering a single $M/M/1/\infty$ queue. In a network however, this additional load departing from j can cause overloaded other nodes. It will be therefore possible that some of the offered new total input of rate $\beta \cdot \lambda$ will not be accepted after readjusting the routing. Our randomized random walk algorithms form Section 2 will automatically compute the rejection rates for the external arrivals at all nodes.

The new network process will be denoted as in Section 3.2.1 as $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$, the vector process recording the queue lengths in the network. $X_t^{(\gamma)} = (X_1^{(\gamma)}(t), \ldots, X_J^{(\gamma)}(t)) \in \mathbb{N}_0^{\overline{J}}$ reads: at time t there are $X_j^{(\gamma)}(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that $\mathbf{X}$ is a strong Markov process on state space $\mathbb{N}_0^{\overline{J}}$ with generator $Q^{\mathbf{X}^{(\gamma)}} =: Q^{(\gamma)} = (q^{(\gamma)}(\mathbf{n}, \mathbf{n}') : \mathbf{n}, \mathbf{n}' \in \mathbb{N}_0^{\overline{J}})$. The strict positive transition rates of $Q^{(\gamma)}$ are under both rerouting regimes for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

The generator $Q^{(\gamma)}$ of this process is

$$q^{(\gamma)}(\mathbf{n}, \mathbf{n} + \mathbf{e}_i) = \beta \lambda r^{(\boldsymbol{\alpha})}(0, i), \qquad\qquad i \in \overline{J}_0 , \qquad (3.13)$$
$$q^{(\gamma)}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_i) = 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\boldsymbol{\alpha})}(j, i) \qquad i, j \in \overline{J}, \ i \neq j ,$$
$$q^{(\gamma)}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\boldsymbol{\alpha})}(j, 0), \qquad\qquad j \in \overline{J} .$$

Theorem 3.5. [MODIFIED JACKSON NETWORKS: RANDOMIZED SKIPPING AND GENERAL CHANGE OF SERVICE] *Let $\mathbf{X}$ be an ergodic Jackson network process as described in Section 3.1 with stationary distribution ξ from (3.4), where the service intensities $\mu_i(n_i)$ at node i are changed by a factor $\gamma_i \in [0, \infty)$ for $i \in \overline{J}$. Denote*

$$\begin{aligned}\alpha_0 &= 1, \\\alpha_j &= \begin{cases} \gamma_j & \text{if } ||\boldsymbol{\gamma}||_\infty \leq 1, \\ \frac{\gamma_j}{||\boldsymbol{\gamma}||_\infty} & \text{if } ||\boldsymbol{\gamma}||_\infty > 1, \end{cases} \quad \forall j \in \overline{J},\end{aligned} \qquad (3.14)$$

and

$$\beta := \begin{cases} 1 & \text{if } ||\gamma||_\infty \leq 1, \\ ||\gamma||_\infty & \text{if } ||\gamma||_\infty > 1, \end{cases} \qquad (3.15)$$

and change routing by randomized skipping with $\boldsymbol{\alpha} = (\alpha_i : i \in \overline{J}_0)$ according to Theorem 2.2, and change the total network input by factor β. Then ξ is a stationary distribution for $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$ as well.
Moreover, if $B(\boldsymbol{\gamma}) = \emptyset$, then $\mathbf{X}^{(\gamma)}$ is ergodic.
If $B(\boldsymbol{\gamma}) \neq \emptyset$, then $\mathbf{X}^{(\gamma)}$ is not irreducible on $\mathbb{N}_0^{\overline{J}}$ and its state space is divided into an infinite set of closed subspaces

$$\mathbb{N}_0^{W(\boldsymbol{\gamma})} \times \{(n_j : j \in B(\boldsymbol{\gamma}))\} \quad \forall (n_j : j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\boldsymbol{\gamma})} .$$

For any probability distribution φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ there exists a stationary distribution $\xi_\varphi^{(\gamma)}$ for $\mathbf{X}^{(\gamma)}$, which is for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$\xi_\varphi^{(\gamma)}(\mathbf{n}) = \xi_\varphi^{(\gamma)}(n_1, \ldots, n_J) = \prod_{j \in W(\boldsymbol{\gamma})} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma})) .$$

Theorem 3.6. [MODIFIED JACKSON NETWORKS: RANDOMIZED REFLECTION AND GENERAL CHANGE OF SERVICE] *Let $\mathbf{X}$ be an ergodic Jackson network process as described in Section 3.1 with reversible routing matrix r and with stationary distribution ξ from (3.4), where the service intensities $\mu_i(n_i)$ at node i are changed by a factor $\gamma_i \in [0, \infty)$ for $i \in \overline{J}$.*
Take $\boldsymbol{\alpha} = (\alpha_i : i \in \overline{J}_0)$ and β as defined in Theorem 3.5, and change routing by randomized reflection according to Proposition 2.8 with $\boldsymbol{\alpha} = (\alpha_i : i \in \overline{J}_0)$, and the total network input by factor β.

Then ξ is a stationary distribution for $\mathbf{X}^{(\boldsymbol{\gamma})} = (X_t^{(\boldsymbol{\gamma})} : t \geq 0)$ as well.
Moreover, if $B(\boldsymbol{\gamma}) = \emptyset$, then $\mathbf{X}^{(\boldsymbol{\gamma})}$ is ergodic.
If $B(\boldsymbol{\gamma}) \neq \emptyset$, then $\mathbf{X}^{(\boldsymbol{\gamma})}$ is not irreducible on $\mathbb{N}_0^{\overline{J}}$ and its state space is divided into an infinite set of closed subspaces

$$\mathbb{N}_0^{W(\boldsymbol{\gamma})} \times \{(n_j : j \in B(\boldsymbol{\gamma}))\} \quad \forall (n_j : j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\boldsymbol{\gamma})}.$$

For any probability distribution φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ there exists a stationary distribution $\xi_\varphi^{(\boldsymbol{\gamma})}$ for $\mathbf{X}^{(\boldsymbol{\gamma})}$, which is for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\bar{J}}$

$$\xi_\varphi^{(\boldsymbol{\gamma})}(\mathbf{n}) = \xi_\varphi^{(\boldsymbol{\gamma})}(n_1, \ldots, n_J) = \prod_{j \in W(\boldsymbol{\gamma})} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma})).$$

Proof of Theorem 3.5 and Theorem 3.6. The global balance equation $x \cdot Q^{(\boldsymbol{\gamma})} = 0$ for the joint queue length process $X^{(\boldsymbol{\gamma})}$ of the modified system is in both settings for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\bar{J}}$

$$
x(\mathbf{n}) \left(\sum_{j \in \overline{J}} \beta \lambda r^{(\boldsymbol{\alpha})}(0, j) + \sum_{j \in \overline{J}} 1_{[n_j > 0]} \gamma_j \mu_j(n_j)(1 - r^{(\boldsymbol{\alpha})}(j, j)) \right)
$$
$$
= \sum_{i \in \overline{J}} x(\mathbf{n} - \mathbf{e}_i) 1_{[n_i > 0]} \beta \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in \overline{J}} \sum_{i \in \overline{J} \setminus \{j\}} x(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) 1_{[n_i > 0]} \gamma_j \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j, i)
$$
$$
+ \sum_{j \in \overline{J}} x(\mathbf{n} + e_j) \gamma_j \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j, 0). \tag{3.16}
$$

We first consider the case $B(\boldsymbol{\gamma}) \neq \emptyset$. Then for $i \in B(\boldsymbol{\gamma})$ we have $\alpha_i = 0$ and $r^{(\boldsymbol{\alpha})}(j, i) = 0$ for all $j \in \overline{J}_0$ and (3.16) reduces to

$$
x(\mathbf{n}) \left(\sum_{j \in W(\boldsymbol{\gamma})} \beta \lambda r^{(\boldsymbol{\alpha})}(0, j) + \sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j > 0]} \gamma_j \mu_j(n_j)(1 - r^{(\boldsymbol{\alpha})}(j, j)) \right)
$$
$$
= \sum_{i \in W(\boldsymbol{\gamma})} x(\mathbf{n} - \mathbf{e}_i) 1_{[n_i > 0]} \beta \lambda r^{(\boldsymbol{\alpha})}(0, i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma}) \setminus \{j\}} x(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) 1_{[n_i > 0]} \gamma_j \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j, i)
$$
$$
+ \sum_{j \in W(\boldsymbol{\gamma})} x(\mathbf{n} + \mathbf{e}_j) \gamma_j \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j, 0). \tag{3.17}
$$

Inserting $x(n_1, \ldots, n_J) = \prod_{j \in W(\boldsymbol{\gamma})} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma}))$ for any probability density φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ we see that immediately $\prod_{j \in W(\boldsymbol{\gamma})} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma}))$ cancels.

Multiplication with $\left(\beta \prod_{j \in W(\boldsymbol{\gamma})} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)}\right)^{-1}$ yields

$$
\left(\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j > 0]} \frac{\gamma_j}{\beta} \mu_j(n_j)(1 - r^{(\boldsymbol{\alpha})}(j,j))\right)
$$

$$
= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \lambda r^{(\boldsymbol{\alpha})}(0,i)
$$

$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma}) \setminus \{j\}} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \frac{\eta_j}{\mu_j(n_j + 1)} \frac{\gamma_j}{\beta} \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j,i)
$$

$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \frac{\eta_j}{\mu_j(n_j + 1)} \frac{\gamma_j}{\beta} \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j,0) \,. \tag{3.18}
$$

Using the fact that $\gamma_j/\beta = \gamma_j/\|\boldsymbol{\gamma}\|_\infty = \alpha_j$ for all $j \in \overline{J}$ we get the equation

$$
\left(\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j \in W(\boldsymbol{\gamma})} 1_{[n_j > 0]} \alpha_j \mu_j(n_j)(1 - r^{(\boldsymbol{\alpha})}(j,j))\right)
$$

$$
= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \lambda r^{(\boldsymbol{\alpha})}(0,i)
$$

$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma}) \setminus \{j\}} \frac{\mu_i(n_i)}{\eta_i} 1_{[n_i > 0]} \frac{\eta_j}{\mu_j(n_j + 1)} \alpha_j \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j,i)
$$

$$
+ \sum_{j \in W(\boldsymbol{\gamma})} \frac{\eta_j}{\mu_j(n_j + 1)} \alpha_j \mu_j(n_j + 1) r^{(\boldsymbol{\alpha})}(j,0) \,. \tag{3.19}
$$

The last equation is exactly (3.8). The rest of the proof is as in the proof of Theorem 3.2 and Theorem 3.4. □

Corollary 3.7. *If in the setting of Theorem 3.5 or of Theorem 3.6 the Jackson network process* $\mathbf{X}$ *is ergodic with stationary distribution* ξ *from* (3.4), *and if after modification all nodes are still working, possibly with degraded capacity, i.e.* $B(\boldsymbol{\gamma}) = \emptyset$, *then in both cases* $\mathbf{X}^{(\boldsymbol{\gamma})} = (X_t^{(\boldsymbol{\gamma})} : t \geq 0)$ *is ergodic as well with* ξ *as unique stationary and limiting distribution.*

Corollary 3.8. *If in the framework of Theorem 3.5, respectively of Theorem 3.6 we have* $r^{(\boldsymbol{\alpha})}(0,0) > 0$ *then the* effective arrival rate *after modification is* $\beta\lambda(1 - r^{(\boldsymbol{\alpha})}(0,0))$.

The following result summarizes the content of Theorem 3.6 and Theorem 3.5 and extends both to an abstract framework.

Corollary 3.9. [MODIFIED JACKSON NETWORKS: GENERAL CHANGE OF ROUTING AND GENERAL CHANGE OF SERVICE] *Let* $\mathbf{X}$ *be an ergodic Jackson network process as described in Section 3.1 with stationary distribution* ξ *from* (3.4), *where the service intensities* $\mu_i(n_i)$ *at node i are changed by a factor* $\gamma_i \in [0, \infty)$ *for* $i \in \overline{J}$. *We change routing to follow some matrix* $r^{(\boldsymbol{\alpha})}$ *with invariant measure* $y = (\alpha_j \eta_j : j \in \overline{J}_0)$ *and increase the total network input by a factor* β, *where* $\alpha_0 = 1$, α_j *and* β *are defined as in* (3.14) *and* (3.15).

We denote the resulting Markovian state process on $\mathbb{N}_0^{\overline{J}}$ by $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$.

Then ξ is a stationary distribution for $\mathbf{X}^{(\gamma)} = (X^{(\gamma)}(t) : t \geq 0)$ as well.

Moreover, we define $B(\boldsymbol{\gamma}) = \{j \in \overline{J} : \gamma_j = 0\}$.

If $B(\boldsymbol{\gamma}) = \emptyset$, then $\mathbf{X}^{(\gamma)}$ is ergodic.

If $B(\boldsymbol{\gamma}) \neq \emptyset$, then $\mathbf{X}^{(\gamma)}$ is not irreducible on $\mathbb{N}_0^{\overline{J}}$ and its state space is divided into an infinite set of closed subspaces

$$\mathbb{N}_0^{W(\boldsymbol{\gamma})} \times \{(n_j : j \in B(\boldsymbol{\gamma}))\} \quad \forall (n_j : j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\boldsymbol{\gamma})} \, .$$

For any probability distribution φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ there exists a stationary distribution $\xi_\varphi^{(\gamma)}$ for $\mathbf{X}^{(\gamma)}$, which is for $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\overline{J}}$

$$\xi_\varphi^{(\gamma)}(\mathbf{n})) = \xi_\varphi^{(\gamma)}(n_1, \ldots, n_J) = \prod_{j \in W(\boldsymbol{\gamma})} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1} \cdot \varphi(n_j : j \in B(\boldsymbol{\gamma})) \, .$$

3.3 Jackson networks in a random environment

We consider a classical Jackson network as described in Section 3.1 and assume that its development is influenced by the time varying status of its external environment. On the other hand the network may trigger the environment to change its status. We therefore have a twofold interacting dynamic which is determined on one side by the environment as a jump process $\mathbf{Y} = (Y(t) : t \geq 0)$ in continuous time, changes of which result in changes of the network's parameter, and on the other side by the network process $\mathbf{X} = (X(t) : t \geq 0)$ as jump process where some jumps of $\mathbf{X}$ enforce the environment to immediately react to this jump. To be more precise:

The state (status) of the environment is recorded in a countable environment state space K and whenever the environment at time t is in state $Y(t) = k \in K$ it changes its status to $m \in K$ with rate $\nu(k, m)$.

The network process $\mathbf{X}$ records the joint queue length vector, as in Section 3.1, and $X_j(t) = n_j \in \mathbb{N}_0$ is the local queue length at node $j \in \overline{J}$. Whenever the environment is in state $k \in K$ and at node j a customer is served and leaves the network, then this jump of the local queue length triggers with probability $R_j(k, m)$ the environment to jump immediately from state k to $m \in K$.

Associated with each environment state $k \in K$ is a vector $\boldsymbol{\gamma}(k) \in [0, \infty)^{\overline{J}}$ which determines the factor by which the service capacities at the nodes are changed, similar to the vector $\boldsymbol{\gamma} \in [0, \infty)^{\overline{J}}$ in Section 3.2. This results in a state dependent service rate $\mu_j(n_j, k) = \gamma_j(k) \cdot \mu_j(n_j)$ if the queue length at node j is n_j and the environment status is k.

The network reacts to the impact of the environment when this is in state k by modifying the routing according to different strategies, which we have described in Sections 2.1 and 2.2 and possibly with admitting more customers into the network. The latter part of the control strategy is set in force whenever in environment state k there exist some $\gamma_j(k) > 1$. In such state $k \in K$ the overall arrival rate to the network is increased by $\beta(\boldsymbol{\gamma}(k)) = ||\boldsymbol{\gamma}(k)||_\infty$ from λ to $\lambda \cdot \beta(\boldsymbol{\gamma}(k))$.

A schematic example of this kind of system is shown on Figure 3 on page 29.
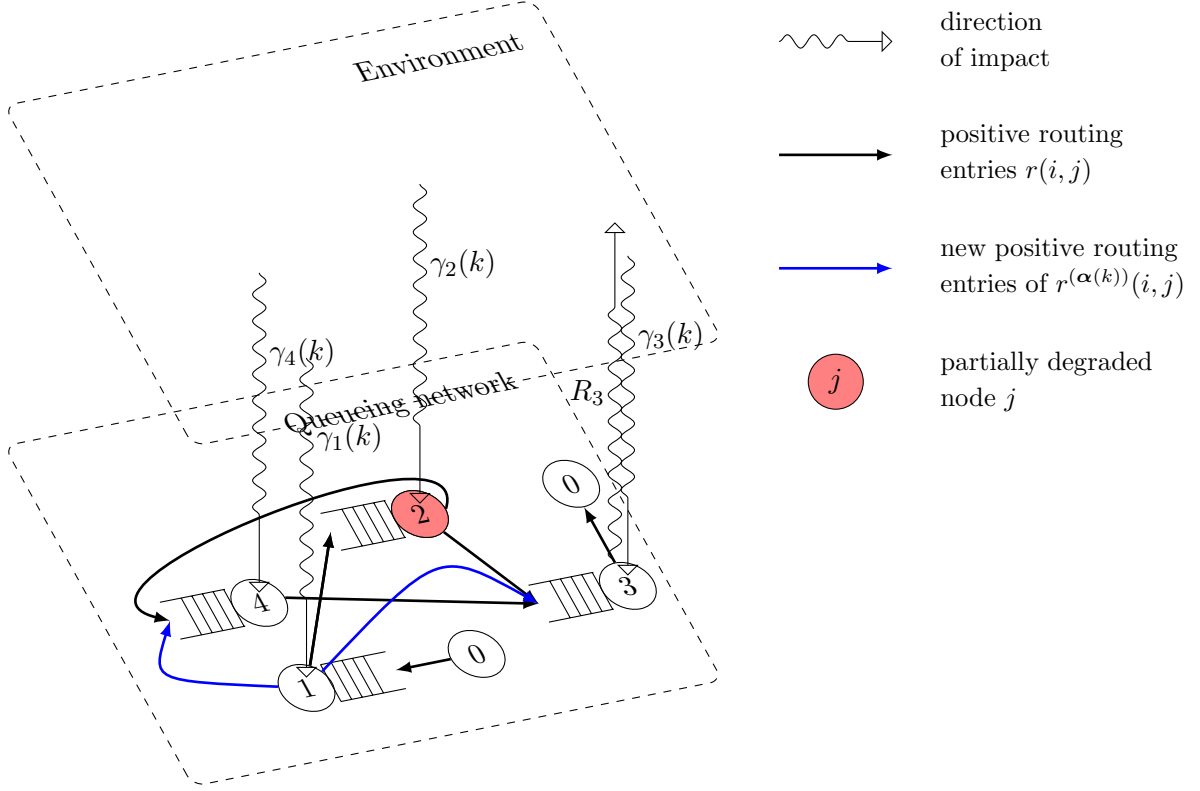
Figure 3: Jackson network and random environment with two-way interaction, The service rates are modified by the environment. The routing is adopted according to the skipping rule. The customers, who leave the node 3, modify the environment.

3.3.1 Rerouting according to randomized skipping

In this section we consider the case that the modification of routing in reaction to the servers' change of capacity is by randomized skipping according to Section 2.1. We will investigate this case in full detail, while other modifications thereafter can be described with less details.

We need here environment dependent rerouting with acceptance probabilities $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\gamma(k))$, modified rerouting matrices $r^{(\boldsymbol{\alpha}(\gamma(k)))}$, and overall load factors $\beta(\gamma(k))$.

To keep notation short we will write $\boldsymbol{\alpha}(k) = (\alpha_j(k) : j \in \overline{J}_0)$, instead of $\boldsymbol{\alpha}(\gamma(k))$, $r^{(\boldsymbol{\alpha}(k))}$ instead of $r^{(\boldsymbol{\alpha}(\gamma(k)))}$ and $\beta(k)$ instead of $\beta(\gamma(k))$.

The randomized skipping according to Section 2.1 yields a routing regime $r^{(\boldsymbol{\alpha}(k))}$ according to Theorem 2.2, and the total service input rate is changed by a factor $\beta(k)$. $\boldsymbol{\alpha}$ and β are defined similar to (3.14) and (3.15), i.e. for $k \in K$:

$$
\begin{aligned}
\alpha_0(k) &= 1, \\
\alpha_j(k) &= \begin{cases} \gamma_j(k) & \text{if } ||\boldsymbol{\gamma}(k)||_\infty \le 1, \\ \frac{\gamma_j(k)}{||\boldsymbol{\gamma}(k)||_\infty} & \text{if } ||\boldsymbol{\gamma}(k)||_\infty > 1, \end{cases} \quad \forall j \in \overline{J}
\end{aligned} \tag{3.20}
$$

and

$$
\beta(k) := \begin{cases} 1 & \text{if } ||\gamma(k)||_\infty \le 1, \\ ||\gamma(k)||_\infty & \text{if } ||\gamma(k)||_\infty > 1. \end{cases} \tag{3.21}
$$

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We further define $B(\boldsymbol{\gamma}(k))$ and $W(\boldsymbol{\gamma}(k))$ similar to Definition 3.1 as set of completely broken down nodes, resp. as set of nodes which, although possibly being degraded or upgraded, can still serve customers under environment condition k.

Definition 3.10. *We denote the coupled process* (QUEUE LENGTHS, ENVIRONMENT) *over time by* $\mathbf{Z} = (\mathbf{X}, \mathbf{Y}) = (Z(t) : t \geq 0) = ((X(t), Y(t)) : t \geq 0)$ *with state space* $E := \mathbb{N}_0^{\overline{J}} \times K$.

As described above the dynamics of $\mathbf{Z}$ relies for the environment process $\mathbf{Y}$ especially on a generator matrix $V = (\nu(k, m) : k, m \in K)$ and stochastic matrices $R_j = (R_j(k, m) : k, m \in K), j \in \overline{J}$. Recall that the original extended routing matrix $r = (r(i, j) : i, j \in \overline{J}_0)$ is irreducible and that $r^{(\boldsymbol{\alpha}(k))}$ in the considered case of randomized skipping is irreducible on $W(\gamma(k)) \cup \{0\}$.

With the standard independence assumptions for the inter-arrival and service times and the conditional independence assumptions for the routing process and the jumps of the environment triggered by departing customers the following statement is obvious.

Proposition 3.11. *The queue lengths-environment process* $\mathbf{Z}$ *is a homogeneous Markov process on state space* $E := \mathbb{N}_0^{\overline{J}} \times K$ *with generator* $Q^{\mathbf{Z}} = (q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n}', k')) : (\mathbf{n}, k), (\mathbf{n}', k') \in E)$. *The strict positive transition rates of* $Q^{\mathbf{Z}}$, *which describe the physical actions of the network and the environment, are for* $(\mathbf{n}, k) = ((n_1, \ldots, n_J), k) \in \mathbb{N}_0^{\overline{J}} \times K$

$$q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n} + \mathbf{e}_i, k)) = \beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i), \qquad\qquad i \in \overline{J},$$

$$q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n} - \mathbf{e}_j + \mathbf{e}_i, k)) = 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, i), \qquad j, i \in \overline{J}, \ i \neq j,$$

$$q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n} - \mathbf{e}_j, m)) = 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, 0)R_j(k, m), \qquad j \in \overline{J},$$

$$q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n}, m)) = \nu(k, m), \qquad\qquad m \in K. \quad (3.22)$$

Theorem 3.12. *Assume the queue lengths-environment process* $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$ *from Definition 3.10. to be ergodic and assume further that the pure Jackson network process* $\mathbf{X}$ *without environment is ergodic with stationary and limiting distribution* ξ *on* $\mathbb{N}_0^{\overline{J}}$ *from* (3.4)

$$\xi(\mathbf{n}) = \xi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1}, \quad \mathbf{n} \in \mathbb{N}_0^{\overline{J}},$$

with normalizing constants $C(j)$ *for the marginal (over nodes) distributions of* $\mathbf{X}$.
Then the queue lengths-environment process $\mathbf{Z}$ *has the unique steady state distribution* $\pi = (\pi(\mathbf{n}, k) : \mathbf{n} \in \mathbb{N}_0^{\overline{J}}, k \in K)$ *of product form:*

$$\pi(\mathbf{n}, k) = \xi(\mathbf{n})\theta(k), \quad \mathbf{n} \in \mathbb{N}_0^{\overline{J}}, k \in K,$$

where θ *is the unique stochastic solution of the following reduced generator equation* $\theta \cdot Q_{red} = 0$ *with*

$$Q_{red} := \left[V + \sum_{j \in \overline{J}} \eta_j I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0))}(R_j - I) \right]. \quad (3.23)$$

Here we denote for $j \in \overline{J}$ *the real valued functions* γ_j *and* $r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ *on* K, *and by* $\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ *their point wise multiplication, the result of which we interpret as vector to obtain the diagonal matrix* $I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0))}$.

Proof. The global balance equation of the process $\mathbf{Z}$ is

$$\pi(\mathbf{n}, k)\left(\sum_{i \in \overline{J}} \beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i) + \underbrace{\sum_{m \in K \setminus \{k\}} \nu(k, m)}_{-\nu(k,k)} + \sum_{j \in \overline{J}} 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)(1 - r^{(\boldsymbol{\alpha}(k))}(j, j)) \right)$$

$$= \sum_{i \in \overline{J}} \pi(\mathbf{n} - \mathbf{e}_i, k)1_{[n_i > 0]}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i))$$

$$+ \sum_{i \in \overline{J}} \sum_{j \in \overline{J} \setminus \{i\}} \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, k)1_{[n_i > 0]}\gamma_j(k)\mu_j(n_j + 1)r^{(\boldsymbol{\alpha}(k))}(j, i)$$

$$+ \sum_{j \in \overline{J}} \sum_{m \in K} \pi(\mathbf{n} + \mathbf{e}_j, m)\gamma_j(m)\mu_j(n_j + 1)r^{(\boldsymbol{\alpha}(m))}(j, 0)R_j(m, k)$$

$$+ \sum_{m \in K \setminus \{k\}} \pi(\mathbf{n}, m)\nu(m, k)$$

Inserting $\pi(\mathbf{n}, k) = \xi(\mathbf{n})\theta(k)$ and adding $\xi(\mathbf{n})\theta(k)\left((\nu(k, k) + \sum_{j \in \overline{J}} 1_{n_j > 0)}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, j) \right)$ on both sides we obtain

$$\xi(\mathbf{n})\theta(k)\left(\sum_{i \in \overline{J}} \beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i) + \sum_{j \in \overline{J}} 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j) \right)$$

$$= \sum_{i \in \overline{J}} \xi(\mathbf{n} - \mathbf{e}_i)\theta(k)1_{[n_i > 0]}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i))$$

$$+ \sum_{i \in \overline{J}} \sum_{j \in \overline{J}} \xi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)\theta(k)1_{[n_i > 0]}\gamma_j(k)\mu_j(n_j + 1)r^{(\boldsymbol{\alpha}(k))}(j, i)$$

$$+ \sum_{j \in \overline{J}} \sum_{m \in K} \xi(\mathbf{n} + \mathbf{e}_j)\theta(m)\gamma_j(m)\mu_j(n_j + 1)r^{(\boldsymbol{\alpha}(m))}(j, 0)R_j(m, k)$$

$$+ \sum_{m \in K} \xi(\mathbf{n})\theta(m)\nu(m, k)\,.$$

Rearranging terms and blowing up this is

$$\theta(k)\left[\xi(\mathbf{n})\left(\sum_{i\in\overline{J}}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0,i)+\sum_{j\in\overline{J}}1_{[n_j>0]}\gamma_j(k)\mu_j(n_j)\right)\right] \qquad (3.24)$$

$$\begin{aligned}
=\ &\theta(k)\left[\sum_{i\in\overline{J}}\xi(\mathbf{n}-\mathbf{e}_i)1_{[n_i>0]}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0,i))\right.\\
&\quad+\sum_{i\in\overline{J}}\sum_{j\in\overline{J}}\xi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)1_{[n_i>0]}\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,i)\\
&\quad\left.+\sum_{j\in\overline{J}}\xi(\mathbf{n}+\mathbf{e}_j)\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,0)\right]\\
&-\theta(k)\sum_{j\in\overline{J}}\xi(\mathbf{n}+\mathbf{e}_j)\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,0)\\
&+\sum_{j\in\overline{J}}\sum_{m\in K}\xi(\mathbf{n}+\mathbf{e}_j)\theta(m)\gamma_j(m)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(m))}(j,0)R_j(m,k)\\
&+\sum_{m\in K}\xi(\mathbf{n})\theta(m)\nu(m,k)\,.
\end{aligned}$$

For each fixed environment state k the terms in squared brackets equate from Theorem 3.5, where for $B(\boldsymbol{\gamma}(k))$ we set in modified notation ($\varphi\to\varphi(k)$) from that theorem the specific probabilities

$$\varphi(k)(n_j:j\in B(\boldsymbol{\gamma}(k))):=\prod_{j\in B(\boldsymbol{\gamma}(k))}\prod_{k=1}^{n_j}\frac{\eta_j}{\mu_j(k)}C(j)^{-1},\quad(n_j:j\in B(\boldsymbol{\gamma}(k)))\in\mathbb{N}_0^{B(\boldsymbol{\gamma}(k))}.\quad(3.25)$$

Dividing by $\xi(\mathbf{n})$ and canceling $\mu_j(n_j+1)$ we arrive at

$$\begin{aligned}
0\ =\ &-\theta(k)\sum_{j\in\overline{J}}\eta_j\gamma_j(k)r^{(\boldsymbol{\alpha}(k))}(j,0)\\
&+\sum_{j\in\overline{J}}\sum_{m\in K}\theta(m)\eta_j\gamma_j(m)r^{(\boldsymbol{\alpha}(m))}(j,0)R_j(m,k)\\
&+\sum_{m\in K}\theta(m)\nu(m,k)\,.
\end{aligned}$$

Rearranging terms we see

$$\begin{aligned}
&\theta(k)\sum_{j\in\overline{J}}\eta_j\gamma_j(k)r^{(\boldsymbol{\alpha}(k))}(j,0)\\
=\ &\sum_{m\in K}\theta(m)\left(\nu(m,k)+\sum_{j\in\overline{J}}\eta_j\gamma_j(m)r^{(\boldsymbol{\alpha}(m))}(j,0)R_j(m,k)\right),
\end{aligned}$$

and

$$\theta(k)\left(-\nu(k,k)+\sum_{j\in\overline{J}}\eta_j\gamma_j(k)r^{(\boldsymbol{\alpha}(k))}(j,0)(1-R_j(k,k))\right)$$

$$=\sum_{m\in K\setminus\{k\}}\theta(m)\left(\nu(m,k)+\sum_{j\in\overline{J}}\eta_j\gamma_j(m)r^{(\boldsymbol{\alpha}(m))}(j,0)R_j(m,k)\right),$$

which finally leads for any prescribed $k\in K$ to

$$0=\sum_{m\in K}\theta(m)\left(\nu(m,k)+\sum_{j\in\overline{J}}\eta_j\gamma_j(m)r^{(\boldsymbol{\alpha}(m))}(j,0)(R_j(m,k)-\delta_{mk})\right).\qquad(3.26)$$

Denote for $j\in\overline{J}$ the real valued functions γ_j and $r^{(\boldsymbol{\alpha}(\cdot))}(j,0)$ on K, and by $\gamma_j\bullet r^{(\boldsymbol{\alpha}(\cdot))}(j,0)$ the point wise multiplication the result of which we interpret as vector to obtain the diagonal matrix $I_{(\gamma_j\bullet r^{(\boldsymbol{\alpha}(\cdot))}(j,0))}$. Then (3.26) can be written in matrix form as

$$0=\theta\underbrace{\left[V+\sum_{j\in\overline{J}}\eta_j I_{(\gamma_j\bullet r^{(\boldsymbol{\alpha}(\cdot))}(j,0))}(R_j-I)\right]}_{=:Q_{red}}.\qquad(3.27)$$

So we have identified (3.23) and because the non diagonal elements of the matrix Q_{red} on the right are non-negative whereas the row sum is zero, Q_{red} is a generator matrix of some Markov process.

If the equation (3.27) has no stochastic solution the global balance equation of $\mathbf{Z}$ would have a non-trivial non-negative solution which cannot be normalized. This would contradict ergodicity. The same argument shows that the solution of (3.27) must be unique. $\qquad\square$

Remark: Although in the original Jackson network the service (rate $\mu_j(n_j)$) and routing (probabilities $r(i,j)$) are locally determined with respect to the transition graph of r, the network control may in general be by global algorithms due to the applied randomized skipping by $r^{(\boldsymbol{\alpha})}$.

3.3.2 Rerouting according to randomized reflection

In this section we assume that the modification of routing in reaction to the servers' change of capacities is by randomized reflection according to Section 2.2, which yields a routing regime $r^{(\boldsymbol{\alpha}(k))}$ according to Proposition 2.8. We use $\boldsymbol{\alpha}(k)$ and $\beta(k)$ as defined in (3.20) and (3.21), and take $B(\boldsymbol{\gamma}(k))$ and $W(\boldsymbol{\gamma}(k))$ as in Definition 3.1.

Recall, that the dynamics of the environment process $\mathbf{Y}$ is driven by a generator matrix $V=(\nu(k,m):k,m\in K)$ and stochastic matrices $R_j=(R_j(k,m):k,m\in K),j\in\overline{J}$ as described on p. 28. Note, that the original extended routing matrix $r=(r(i,j):i,j\in\overline{J}_0)$ is irreducible but under randomized reflection $r^{(\boldsymbol{\alpha}(k))}$ may be reducible even on $W(\boldsymbol{\gamma}(k))\cup\{0\}$, which does not destroy the ergodicity of the system process $\mathbf{Z}=(\mathbf{X},\mathbf{Y})$. We then have a formally similar statement as in Proposition 3.11.

Proposition 3.13. *The queue lengths-environment process* $\mathbf{Z} = (\mathbf{X}, \mathbf{Y}) = (Z(t) : t \geq 0) = ((X(t), Y(t)) : t \geq 0)$ *is a homogeneous Markov process on state space* $E := \mathbb{N}_0^{\overline{J}} \times K$ *with generator* $Q^{\mathbf{Z}} = (q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n}', k')) : (\mathbf{n}, k), (\mathbf{n}', k') \in E)$. *The strict positive transition rates of* $Q^{\mathbf{Z}}$, *are for* $(\mathbf{n}, k) = ((n_1, \ldots, n_J), k) \in \mathbb{N}_0^{\overline{J}} \times K$

$$q((\mathbf{n}, k), (\mathbf{n} + \mathbf{e}_i, k)) = \beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i), \qquad\qquad i \in \overline{J},$$

$$q((\mathbf{n}, k), (\mathbf{n} - \mathbf{e}_j + \mathbf{e}_i, k)) = 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, i), \qquad j, i \in \overline{J}, \ i \neq j,$$

$$q((\mathbf{n}, k), (\mathbf{n} - \mathbf{e}_j, m)) = 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, 0)R_j(k, m), \qquad j \in \overline{J},$$

$$q((\mathbf{n}, k), (\mathbf{n}, m)) = \nu(k, m), \qquad\qquad m \in K.$$

As was pointed out in Section 2.2 a necessary ingredient for successfully applying randomized reflection as rerouting regime is reversibility of $r = (r(i, j) : i, j \in \overline{J}_0)$, which we will now set in force.

Theorem 3.14. *Consider the queue lengths-environment process* $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$ *from Proposition 3.13 and assume that the extended routing matrix* $r = (r(i, j) : i, j \in \overline{J}_0)$ *is reversible for* $\eta = (\eta_j : j \in \overline{J}_0)$.

Assume $\mathbf{Z}$ *to be ergodic and assume further that the pure Jackson network process* $\mathbf{X}$ *without environment is ergodic with stationary and limiting distribution* ξ *on* $\mathbb{N}_0^{\overline{J}}$ *from (3.4)*

$$\xi(\mathbf{n}) = \xi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1}, \quad \mathbf{n} \in \mathbb{N}_0^{\overline{J}}.$$

Then the queue lengths-environment process $\mathbf{Z}$ *has the unique steady state distribution* $\pi = (\pi(\mathbf{n}, k) : \mathbf{n} \in \mathbb{N}_0^{\overline{J}}, k \in K)$ *of product form:*

$$\pi(\mathbf{n}, k) = \xi(\mathbf{n})\theta(k), \quad \mathbf{n} \in \mathbb{N}_0^{\overline{J}}, k \in K,$$

where θ *is the unique stochastic solution of the following reduced generator equation* $\theta \cdot Q_{red} = 0$ *with*

$$Q_{red} := \left[V + \sum_{j \in \overline{J}} \eta_j I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j,0))}(R_j - I) \right]. \tag{3.28}$$

Here we denote for $j \in \overline{J}$ *the real valued functions* γ_j *and* $r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ *on* K, *and by* $\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ *their point wise multiplication, the result of which we interpret as vector to obtain the diagonal matrix* $I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j,0))}$.

The proof of the theorem is along the lines of the proof of Theorem 3.12, where we used almost completely the general abstract notation $r^{(\boldsymbol{\alpha}(k))}$ for the rerouting regime. Only when manipulating (3.24) we had to refer to the specific Theorem 3.5, which would be substituted now by referring to Theorem 3.6. The residual derivations are similar.

Remark: The control of the customers' routing under randomized reflection by $r^{(\boldsymbol{\alpha})}$ is by local decisions with respect to the transition graph of r. So the network process is locally determined as well.

3.3.3 Rerouting by general randomization

This last observation of the previous section clearly suggests to extract the general principle for randomized rerouting. We consider modification of routing in reaction to the servers' change of capacities by environment dependent factors $\boldsymbol{\gamma}(k) \in [0, \infty)^{\overline{J}}$ to $\mu_j(n_j, k) = \gamma_j(k)\mu_j(n_j)$. We use the notation introduced in Section 3.3.1 and $\boldsymbol{\alpha}(k)$ and $\beta(k)$ as defined in (3.20) and (3.21), and take $B(\boldsymbol{\gamma}(k))$ and $W(\boldsymbol{\gamma}(k))$ as in Definition 3.1.

We have the usual dynamics of the environment process $\mathbf{Y}$ with $V = (\nu(k, m) : k, m \in K)$ and $R_j = (R_j(k, m) : k, m \in K), j \in \overline{J}$. For the general rerouting regimes $r^{(\boldsymbol{\alpha}(k))}, k \in K$, with $\boldsymbol{\alpha}(k)$ with $\alpha_0(k) = 1$ and $\boldsymbol{\alpha}(k) \in [0, 1]^{\overline{J}_0}$, we only require the properties described in Corollary 3.9 and have the following generalization in the spirit of Zhu's main theorem [Zhu94][p. 12], without specifying explicitly the control regimes for rerouting. Note, that our environment process is not Markov because of the two-way interaction of environment and service process, while Zhu's theorem requires the environment to be Markov for its own.

Corollary 3.15. *The queue lengths-environment process* $\mathbf{Z} = (\mathbf{X}, \mathbf{Y}) = (Z(t) : t \geq 0) = ((X(t), Y(t)) : t \geq 0)$ *is a homogeneous Markov process on state space* $E := \mathbb{N}_0^{\overline{J}} \times K$ *with generator* $Q^{\mathbf{Z}} = (q^{\mathbf{Z}}((\mathbf{n}, k), (\mathbf{n}', k')) : (\mathbf{n}, k), (\mathbf{n}', k') \in E)$. *The strict positive transition rates of* $Q^{\mathbf{Z}}$, *are for* $(\mathbf{n}, k) = ((n_1, \ldots, n_J), k) \in \mathbb{N}_0^{\overline{J}} \times K$

$$q((\mathbf{n}, k), (\mathbf{n} + \mathbf{e}_i, k)) = \beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i), \qquad\qquad i \in \overline{J},$$

$$q((\mathbf{n}, k), (\mathbf{n} - \mathbf{e}_j + \mathbf{e}_i, k)) = 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, i), \qquad j, i \in \overline{J}, \ i \neq j,$$

$$q((\mathbf{n}, k), (\mathbf{n} - \mathbf{e}_j, m)) = 1_{[n_j > 0]}\gamma_j(k)\mu_j(n_j)r^{(\boldsymbol{\alpha}(k))}(j, 0)R_j(k, m), \qquad j \in \overline{J},$$

$$q((\mathbf{n}, k), (\mathbf{n}, m)) = \nu(k, m), \qquad\qquad\qquad m \in K.$$

Assume that the rerouting regimes $r^{(\boldsymbol{\alpha}(k))}, \ k \in K$, *have invariant measures*
$y(k) = (\alpha_j(k) \cdot \eta_j : j \in \overline{J}_0)$.

Assume $\mathbf{Z}$ *to be ergodic and assume further that the pure Jackson network process* $\mathbf{X}$ *without environment is ergodic with stationary and limiting distribution* ξ *on* $\mathbb{N}_0^{\overline{J}}$ *from* (3.4)

$$\xi(\mathbf{n}) = \xi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1}, \quad \mathbf{n} \in \mathbb{N}_0^{\overline{J}}.$$

Then the queue_lengths-environment process $\mathbf{Z}$ *has the unique steady state distribution* $\pi = (\pi(\mathbf{n}, k) : \mathbf{n} \in \mathbb{N}_0^{\overline{J}}, k \in K)$ *of product form:*

$$\pi(\mathbf{n}, k) = \xi(\mathbf{n})\theta(k), \quad \mathbf{n} \in \mathbb{N}_0^{\overline{J}}, k \in K,$$

where θ *is the unique stochastic solution of the following* reduced generator equation
$\theta \cdot Q_{red} = 0$ *with*

$$Q_{red} := \left[V + \sum_{j \in \overline{J}} \eta_j I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0))}(R_j - I) \right]. \tag{3.29}$$

Here we denote for $j \in \overline{J}$ *the real valued functions* γ_j *and* $r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ *on* K, *and by* $\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ *their point wise multiplication, the result of which we interprete as vector to obtain the diagonal matrix* $I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j, 0))}$.

Note on figures

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