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A HYPERBOLIC LIE ALGEBRA FROM SUPERGRAVITY

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Abstract

It is shown that the hyperbolic extension of $SL(2, \mathbf{R})$ can be realized non-linearly in the chiral reduction of simple ($N = 1$) supergravity from four dimensions to one dimension. Remarkably, it does not appear to be possible to obtain a non-trivial realization of this symmetry without fermions.

It has been known for a long time that higher-dimensional theories of gravity exhibit unexpected symmetries upon reduction to lower dimensions [1]. In 1971, Geroch was able to show that there is an infinite-dimensional symmetry group acting on the solutions of Einstein's equations with two (commuting) Killing vectors [2] (this group is nowadays referred to as the "Geroch group"). This result result was considerably elaborated and further developed by the general relativists in the following years [3]. An important intermediate step was the discovery of a linear system (or Lax pair), demonstrating explicitly the integrability of Einstein's theory after dimensional reduction to two dimensions [4]. With the advent of supergravity [5] and the remarkable discovery of "hidden symmetries" in dimensionally reduced supergravities [6], particle physicists became also interested in (supersymmetric) Kaluza Klein theories in their quest for a unified description of the fundamental interactions (see e.g. [7]). The connection between these developments and the work of the general relativists was apparently first realized by Julia [8], who emphasized the importance of group theoretical concepts for the investigation of the structural properties of dimensionally reduced gravity and supergravity theories, and showed quite explicitly that the Geroch group in infinitesimal form is nothing but the affine Kac Moody algebra $A_1^{(1)}$. He also demonstrated the presence of a central term, which had gone unnoticed by the general relativists. The underlying group theoretic structure and the connection with the σ-models encountered in particle physics were further elucidated by Breitenlohner and Maison [9]. These results were subsequently generalized to two-dimensional supergravities [10]. Through this work it has become clear that the emergence of infinite-dimensional symmetries in the reduction to two dimensions is a generic phenomenon; for matter coupled theories the general result is that, in two dimensions, the symmetry is enlarged to the affine extension of the (finite dimensional) Lie group present in three dimensions, with a central charge acting on the conformal factor through a constant rescaling [8,9,11].

It is the purpose of this letter to show that a further enlargement of symmetry takes place upon reduction to *one* dimension. In [12] it was already pointed out that a consideration of extended Dynkin diagrams suggests the emergence of hyperbolic Lie algebras in this reduction. Our results confirm this conjecture, however with an important (and perhaps surprising) modification: the desired enhancement of symmetry cannot be realized without fermions, but apparently requires the locally supersymmetric extension of Einstein's theory, i.e. supergravity [5]. Quite independently of its physical significance, the concrete realization of a hyperbolic Lie algebra in a physics inspired model would constitute a major step forward in view of how little is known about such algebras. Generally speaking, hyperbolic Lie algebras are special Kac Moody algebras which are associated with indefinite Cartan matrices; in addition, one imposes the constraint that any regular subalgebra (obtained by deleting a point from the Dynkin diagram) be either finite or affine [13]. The latter requirement implies the existence of a "maximally extended" Kac Moody algebra of this type, the (so far elusive) E_{10}

algebra extending the simply laced exceptional Lie algebras. The general characterization of hyperbolic algebras is an open mathematical problem. For instance, no concrete realization parallelling the characterization of affine algebras in terms of two-dimensional current algebra has been found so far. It is to be hoped that the results described here represent a first step towards such a realization of hyperbolic algebras and will pave the way for a better understanding of their structure. The fact that Einstein's theory and its supersymmetric extension may provide some essential clues in this search is probably quite significant in itself.

For the sake of simplicity, I will here concentrate on ordinary (super)gravity in four dimensions. Reducing this theory to three dimensions reveals a hidden $SL(2,\mathbf{R})$; further reduction to two dimensions leads to the affine extension of this group as already mentioned above. A crucial role in realizing the infinite-dimensional symmetry group on the components of the gravitational fields after dimensional reduction to two dimensions is played by duality rotations akin to those leaving invariant Maxwell's equations in vacuum. The importance of generalized duality invariance was, of course, already emphasized in [6], but two dimensions are distinguished by the fact that the dual of a scalar field is again a scalar field. For ordinary gravity, duality leads to the appearance of *two* $SL(2;\mathbf{R})$ groups, the Ehlers and the Matzner Misner groups, whose interplay engenders the infinite-dimensional Geroch group $\widehat{SL(2,\mathbf{R})}$, with the associated Lie algebra $A_1^{(1)}$. We will show that, in the final step of the dimensional reduction to *one* dimension, a hyperbolic Lie algebra emerges which can be realized in terms of non-linear and non-local transformations acting on the components of the vierbein and the gravitino. It is characterized by the following generalized Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{1}$$

(the generators are labeled by $i,j = 1,0,-1$). The generating (Serre) relations read [13]

$$[h_i, h_j] = 0 \ , \quad [e_i, f_j] = \delta_{ij} h_j \ , \quad [h_i, e_j] = A_{ij} e_j \ , \quad [h_i, f_j] = -A_{ij} f_j \tag{2a}$$

$$(\mathrm{ad}\, e_i)^{1-A_{ij}}(e_j) = 0 \ , \quad (\mathrm{ad}\, f_i)^{1-A_{ij}}(f_j) = 0 \quad (i \neq j) \tag{2b}$$

The full hyperbolic algebra is spanned by all multiple commutators which do not vanish by virtue of the above relations. The Cartan matrix (1) describes the simplest example of such an algebra and was already investigated in [14]. Inspection of (1) reveals two important subalgebras: the upper two-by-two block gives the Cartan matrix of the Lie algebra $A_1^{(1)}$ associated with $\widehat{SL(2,\mathbf{R})}$ (the Geroch group), while the lower two-by-two block yields the Cartan matrix of the Lie algebra A_2 corresponding to the group $SL(3,\mathbf{R})$. A central result of this paper* is that this $SL(3,\mathbf{R})$ group can be explicitly realized on the components of

* *This result was already anticipated in discussions with P. Breitenlohner and D. Maison.*

the vierbein; it is the natural extension of the Matzner Misner group and will henceforth be referred to as the "Matzner Misner $SL(3,\mathbf{R})$ group". The hyperbolic algebra can then be manufactured out of these two building blocks; all that remains to be done is to assemble the pieces and to make sure that the Ehlers $SL(2,\mathbf{R})$ commutes with the second $SL(2,\mathbf{R})$ contained in the Matzner-Misner $SL(3,\mathbf{R})$*.

To describe the dimensional reduction in somewhat more detail we need some conventions and notations (such as, for instance, the labeling of curved and flat indices); these will be taken over for the most part from [11] or simply stated as we go along. In addition, I will assume some acquaintance with the basics of supergravity in the following (see e.g. [16] for an introduction and further references). It is convenient to consider the reduction from $d = 4$ to $d = 3$ first, dropping the dependence on the third (space-like) coordinate x^3. The vierbein is decomposed as follows

$$E_M^{\ A} = \begin{pmatrix} \Delta^{-1/2} e_m^{\ a} & B_m \Delta^{1/2} \\ 0 & \Delta^{1/2} \end{pmatrix} \tag{3}$$

with the dreibein $e_m^{\ a}$ carrying no physical (i.e. propagating) degrees of freedom in three dimensions. Observe that partial use has been made of the local Lorentz group to fix a triangular gauge for the vierbein. A similar decomposition is necessary for the gravitino in the case of supergravity. For lack of space, I will not discuss the dimensional reduction in detail here (which is standard anyhow), but rather state the result. After reduction to three dimensions, the theory contains a gravitino ψ_a (a complex vector spinor), which does not propagate, and a complex (two-component) spinor χ describing the physical states of helicity $s = \pm\frac{3}{2}$ (see [11], where, however two real spinors were used instead of one complex spinor). Already in three dimensions, the Kaluza Klein vector B_m can be replaced on shell by a scalar field B through a duality transformation; for simple supergravity, the relevant equation reads

$$\tfrac{1}{2}\epsilon^{mnp}\partial_p B = \tfrac{1}{2}\Delta^2 B^{mn} + \Delta\epsilon^{mnp}\Big(3\bar{\chi}\gamma_p\chi + \tfrac{1}{2}i\epsilon_{prq}\bar{\psi}^r\psi^s - \bar{\psi}_a\gamma_p\gamma^a\chi - \bar{\chi}\gamma^a\gamma_p\psi_a\Big) \tag{4}$$

This equation is consistent because the divergence of the right hand side vanishes by the equation of motion for B_m. The complex field $\Delta \pm iB$ then represents the two ($s = \pm 2$) helicity states of the graviton. The above equation and its dimensional reduction will play an important role below.

Next, we descend from $d = 3$ to $d = 2$, dropping the dependence on one more (space-like)

* *This suggests that the present construction can be generalized only to those hyperbolic algebras which contain a Matzner Misner $SL(3,\mathbf{R})$ subalgebra corresponding to the extended roots of the Dynkin diagram; the rest of the Dynkin diagram would then give rise to the generalization of the Ehlers group. Algebras of this type are called "superaffine" and have been completely classified [15]. The most interesting example in this class is, of course, E_{10}, which is associated with the dimensional reduction of $N = 8$ supergravity to one dimension.*

coordinate x^2. Using local Lorentz invariance again, we can bring the dreibein into the form

$$e_m^{\ a} = \begin{pmatrix} e_\mu^{\ \alpha} & \rho A_\mu \\ 0 & \rho \end{pmatrix} \tag{5}$$

It is advantageous at this point to employ light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ to parametrize the dependence on the remaining two coordinates, with similar notation for the two-dimensional tensor indices. It is convenient to choose a diagonal gauge for the zweibein

$$e_\mu^{\ \alpha} = \begin{pmatrix} e_{\dot{+}}^{\ +} & e_{\dot{+}}^{\ -} \\ e_{\dot{-}}^{\ +} & e_{\dot{-}}^{\ -} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_{\dot{-}} \end{pmatrix} \tag{6}$$

where the dots temporarily serve to distinguish curved from flat indices; I will abandon this convention, when there is no more danger of confusing the two kinds of indices. Note that, in contrast to [10,11], we do not assume the zweibein to be proportional to the unit matrix (with a conformal factor λ). This is because under the new symmetry transformations to be introduced below, the lower component $\lambda_{\dot{-}}$ will mix with the the other fields, whereas the Geroch group acts on it only through the central term, scaling it by a constant factor. Notice that the upper component $\lambda_{\dot{+}}$ has been put equal to unity by a local $SO(1,1)$ rotation (which scales $e_{\dot{+}}^{\ +}$ and $e_{\dot{-}}^{\ -}$ oppositely). There is a residual invariance under conformal coordinate transformations which must, however, be accompanied by a compensating $SO(1,1)$ rotation to maintain the gauge $\lambda_{\dot{+}} = 1$. By means of such a transformation the field ρ can in principle be identified with one of the two-dimensional coordinates by a conformal coordinate transformation, as is common practice in the study of stationary axisymmetric solutions of Einstein's equations [3]. The vector A_μ is auxiliary: without fermionic matter couplings, it can be gauged away completely (in the absence of topological excitations). For this reason, it has been largely ignored in the literature so far. In the presence of fermionic matter, it can be eliminated in terms of fermionic bilinears [10,11]. In any case, this field can no longer be discarded if one wants to enlarge the symmetry to a hyperbolic Lie algebra. This point will be further elaborated below. For the gravitino we adopt the gauge

$$\psi_a = (\gamma_\alpha \psi, \psi_2) \tag{7}$$

This was referred to as the "superconformal" gauge in [10], because the condition (7) is preserved under residual superconformal transformations.

Finally, the reduction to $d = 1$ must now be described. Obviously, the duality equation (4) involves the Levi Civita tensor also after the dimensional reduction to two dimensions. Therefore simply dropping the dependence on either x^0 or x^1 will not do because with this truncation the duality equation collapses to the trivial statement $0 = 0$. Rather one must perform the truncation with respect to one of the *light-like* coordinates $x^\pm$, in terms of which the duality transformation becomes diagonal*. Thus we put

$$\phi(x^0, x^1) \longrightarrow \phi(x^+) \tag{8}$$

* *A chiral truncation of this type was already considered in [11]*

for all (bosonic and fermionic) fields. Furthermore, we require the negative chirality components of χ and ψ_2 to vanish, i.e.

$$(\chi)_- \equiv \tfrac{1}{2}\gamma_+\gamma_-\chi = 0 \quad , \quad (\psi_2)_- \equiv \tfrac{1}{2}\gamma_+\gamma_-\psi_2 = 0 \tag{9}$$

while we retain both chirality components for ψ with

$$\varphi \equiv \tfrac{1}{2}\gamma_-\gamma_+\psi \quad , \quad \theta \equiv \tfrac{1}{2}\gamma_+\gamma_-\psi \tag{10}$$

Let us also put $A_+ = B_+ = 0$, fixing the ordinary Kaluza Klein gauge invariances (note that A_- and B_- are gauge-invariant in the chiral truncation). To see that these choices are not entirely arbitrary and do not adversely affect the physical content of the theory, we must examine in somewhat more detail the "equations of motion" in the chiral truncation. For the physical fields Δ, B_2 (or; equivalently, Δ and B), and χ, which incorporate the propagating supergravitational degrees of freedom, these are automatically satisfied by virtue of the truncation; hence, these fields are arbitrary functions of x^+. This is not the case for the remaining fields, however, which are constrained by the higher-dimensional equations of motion.

The equation relating the field B_2 to its dual field B reads

$$-\tfrac{1}{2}\Delta^{-1}\partial_+ B = \tfrac{1}{2}\rho^{-1}\Delta\partial_+ B_2 + i\bar{\chi}\gamma_+\psi_2 - i\bar{\psi}_2\gamma_+\chi + \tfrac{1}{2}i\bar{\varphi}\gamma_+\psi_2 - \tfrac{1}{2}i\bar{\psi}_2\gamma_+\varphi - 3\bar{\chi}\gamma_+\chi \tag{11}$$

and follows directly from (4) by putting $m = -$. Setting $m = 2$ on the other hand, we obtain

$$\tfrac{1}{2}\Delta\lambda_-^{-1}\Big(\partial_+ B_- - A_-\partial_+ B_2\Big) = i\bar{\theta}\varphi - i\bar{\varphi}\theta + 2i\bar{\theta}\chi - 2i\bar{\chi}\theta \tag{12}$$

The equation for A_- reads

$$\tfrac{1}{2}\rho\lambda_-^{-1}\partial_+ A_- = \bar{\varphi}\theta + \bar{\theta}\varphi + i\bar{\theta}\psi_2 - i\bar{\psi}_2\theta \tag{13}$$

These equations clearly demonstrate the need for the "wrong" chirality component θ: without it, the right hand sides of (12) and (13) would vanish, and we would be right back to the bosonic theory. Incidentally, the last two equations are equivalent to the vanishing of the following components of the dimensionally reduced four-dimensional supercovariant spin-connection

$$\widehat{\omega}_{+-2} = \widehat{\omega}_{+-3} = 0 \tag{14}$$

The equations governing the gravitino components are obtained by similarly analyzing the Rarita Schwinger equation of the four-dimensional theory. After some work, we arrive at the following equations, neglecting cubic spinor terms,

$$\Big(\partial_+ + \tfrac{1}{4}i\Delta^{-1}\partial_+ B\Big)\theta = 0 \tag{15}$$

and

$$\rho^{-1}\partial_+(\rho\psi_2)+\tfrac{1}{4}i\Delta^{-1}\partial_+B\,\psi_2 = \\ = \lambda_-^{-1}\partial_+\lambda_-\,\psi_2 + i\rho^{-1}\partial_+\rho\varphi + i\Delta^{-1}\partial_+(\Delta - iB)\chi \tag{16}$$

The first of these equations tells us that θ is covariantly constant and hence can be taken different from zero. This is, of course, crucial for the right hand sides of (12) and (13) not to vanish. The second equation (16) allows us to solve for φ in terms of the physical fields; this is most easily seen in the superconformal gauge $\psi_2 = 0$ where the equation can be solved algebraically. Observe that through the the chiral reduction, the number of physical degrees of freedom has been halved, since only the right-moving degrees of freedom are retained; the unphysical fields can either be gauged away or solved for in terms of the other fields.

After these preparations we are now ready to list the transformation rules for all the field components, deferring a complete derivation to another paper. The explicit determination of the variations involves various compensating rotations needed to restore the gauge conditions introduced above; these are, however, necessary only for the variations with f_i. We first give the transformations of the bosonic fields (with the notational convention $e_1 \equiv \delta_{e_1}$ etc. for the infinitesimal variations).

For the Ehlers transformations, one finds

$$e_1(B) = -1 \quad , \quad e_1(\Delta) = e_1(B_2) = e_1(A_-) = e_1(B_-) = e_1(\rho) = e_1(\lambda_-) = 0 \tag{17a}$$

$$h_1(B) = -2B \quad , \quad h_1(\Delta) = -2\Delta \quad , \quad h_1(B_2) = 2B_2 \quad , \quad h_1(B_-) = 2B_- \\ h_1(A_-) = h_1(\lambda_-) = h_1(\rho) = 0 \tag{17b}$$

$$f_1(\Delta) = 2\Delta B \quad , \quad f_1(B) = B^2 - \Delta^2 \quad , \quad f_1(A_-) = f_1(\rho) = f_1(\lambda_-) = 0 \tag{17c}$$

The variations $f_1(B_-)$ and $f_1(B_2)$, which follow from the duality equations, are more complicated. They can be deduced by substituting the above variations into the duality equation (11). In this way we obtain the non-local transformations

$$\partial_+ f_1(B_2) = -2B\partial_+B_2 + 2\rho\Delta^{-1}\partial_+\Delta + 4\rho\left(\bar\psi_2\gamma_+\chi + \bar\chi\gamma_+\psi_2\right) \tag{18}$$

and

$$\partial_+ f_1(B_-) = -2B\partial_+B_- + 2\rho A_-\Delta^{-1}\partial_+\Delta + \\ + 8\lambda_-(\bar\theta\chi + \bar\chi\theta) + 4\rho A_-(\bar\chi\gamma_+\psi_2 + \bar\psi_2\gamma_+\chi) \tag{19}$$

For the Matzner Misner transformations, we get

$$e_0(B_2) = -1 \quad , \quad e_0(\Delta) = e_0(B) = e_0(A_-) = e_0(B_-) = e_0(\rho) = e_0(\lambda_-) = 0 \tag{20a}$$

$$h_0(B) = 2B \quad , \quad h_0(\Delta) = 2\Delta \quad , \quad h_0(B_2) = -2B_2 \quad , \quad h_0(A_-) = A_- \\ h_0(B_-) = -B_- \quad , \quad h_0(\lambda_-) = \lambda_- \quad , \quad h_0(\rho) = 0 \tag{20b}$$

$$f_0(\Delta) = -2\Delta B_2 \quad , \quad f_0(B_2) = B_2^2 - \rho^2\Delta^{-2} \quad , \quad f_0(A_-) = B_- - B_2A_- \\ f_0(B_-) = B_-B_2 - \rho^2\Delta^{-2}A_- \quad , \quad f_0(\lambda_-) = -B_2\lambda_- \quad , \quad f_0(\rho) = 0 \tag{20c}$$

Again, $f_0(B)$ is more complicated. Invoking the duality equation once more, we get

$$\partial_+ f_0(B) = -2B_2\partial_+B + 2\Delta\partial_+(\rho\Delta^{-1}) - 4\rho\left(\bar\chi\gamma_+\psi_2 + \bar\psi_2\gamma_+\chi\right) \tag{21}$$

Note that $\delta\rho = 0$ under both sets of transformations, and that λ_- is inert under the Ehlers group, whereas $\lambda_-^{-1}\delta\lambda_- = \frac{1}{2}\Delta^{-1}\delta\Delta$ for the Matzner Misner group. In determining the action of the Ehlers and the Matzner Misner groups on the fermions, we must keep in mind that these groups (as well as the new $SL(2,\mathbf{R})$ group to be introduced below) act on the fermions only via the induced compensating Lorentz rotations. This implies that the fermion fields are inert with respect to the generators e_i and h_i. Their transformations under the operators f_1 and f_0 are given by

$$f_1(\chi) = -\tfrac{3}{2}i\Delta\chi \quad , \quad f_1(\psi_2) = +\tfrac{1}{2}i\Delta\psi_2 \quad , \quad f_1(\varphi) = +\tfrac{1}{2}i\Delta\varphi \quad , \quad f_1(\theta) = +\tfrac{1}{2}i\Delta\theta \tag{22}$$

and

$$f_0(\chi) = -\rho\Delta^{-1}\left(\tfrac{3}{2}i\chi + \psi_2\right) \quad , \quad f_0(\psi_2) = +\tfrac{1}{2}i\rho\Delta^{-1}\psi_2 \quad , \quad f_0(\theta) = -\tfrac{1}{2}i\rho\Delta^{-1}\theta \\ f_0(\varphi) = +\rho\Delta^{-1}\left(\tfrac{1}{2}i\varphi + 2i\chi + \psi_2\right) \tag{23}$$

The "off-diagonal" terms in (23) are a consequence of the fact that the relevant compensating Lorentz rotation acts not only on the spinor components but also on the vector components of the gravitino field in four dimensions. From previous work [2,8,9,10], we know already that the operators e_i, h_i, f_i for $i = 0,1$ obey the generating relations (2a) and (2b) of the Kac Moody algebra $A_1^{(1)}$. Although this is guaranteed by the existence of linear systems for gravity and supergravity, a direct proof of this assertion by evaluation of the relevant commutators is perhaps more convincing. For the generators e_i and h_i this is a rather trivial exercise, but the computations become progressively more involved as the number of generators f_i increases. The most tedious part of the proof is the verification of the quadrilinear relations $[f_0,[f_0,[f_0,f_1]]] = [f_1,[f_1,[f_1,f_0]]] = 0$*. As one can see from (18) and (21), an infinity of dual potentials is needed to realize the full algebra, as already observed by Geroch.

* *This may explain why, to the best of my knowledge, these relations have never been explicitly checked in literature! Readers willing to try their mettle should note the following relations:*

$$(f_1)^3(B_-) = (f_1)^3(B_2) = (f_1)^3(\rho\Delta^{-1}) = 0 \quad , \quad (f_0)^3(B) = (f_0)^3(\Delta) = 0$$

Also, from (18) and (21),

$$f_1(B_2) + f_0(B) = -2BB_2 + 2\rho$$

It has already been mentioned that there is a central charge which acts non-trivially on the conformal factor [8,9]. In the present formulation, the central charge is given by $c = h_0 + h_1$ [14], and its expected action on the physical fields and on λ_- is easily verified. As is evident from (17) and (20), however, it also acts non-trivially on the new fields A_- and B_- (with $c(A_-) = A_-$ and $c(B_-) = B_-$). This already indicates that the central charge will be deprived of its special status in the full hyperbolic algebra. Inspection of (1) and (2) shows that c does not commute with those elements of the hyperbolic algebra involving the generators e_{-1} and f_{-1} since, for instance, $[c, e_{-1}], [c, f_{-1}] \neq 0$.

Let us now turn to the new transformations, which extend $A_1^{(1)}$ to the hyperbolic algebra characterized by the Cartan matrix (1). These are obtained by performing an $SL(2,\mathbf{R})$ rotation on the "$-$" and the "2"-components of the vierbein, or equivalently the dreibein in (5) (of course with appropriate compensating rotations for f_{-1}). These transformations do not act on the fields Δ and B. On the remaining field components, they are given by

$$e_{-1}(A_-) = -1 \quad , \quad e_{-1}(B_-) = -B_2 \quad , \quad e_{-1}(B_2) = e_{-1}(\lambda_-) = e_{-1}(\rho) = 0 \tag{24a}$$

$$\begin{aligned} &h_{-1}(B_2) = B_2 \quad , \quad h_{-1}(B_-) = -B_- \quad , \quad h_{-1}(A_-) = -2A_- \\ &h_{-1}(\rho) = \rho \quad , \quad h_{-1}(\lambda_-) = -\lambda_- \end{aligned} \tag{24b}$$

$$\begin{aligned} &f_{-1}(B_2) = -B_- \quad , \quad f_{-1}(\rho) = -\rho A_- \quad , \quad f_{-1}(\lambda_-) = \lambda_- A_- \\ &f_{-1}(A_-) = A_-^2 \quad , \quad f_{-1}(B_-) = 0 \end{aligned} \tag{24c}$$

As is well known, one can now introduce the operator $d = h_1 + h_0 + h_{-1}$, which together with the central charge operator c is conventionally used to complete the Cartan subalgebra of the finite-dimensional Lie algebra to that of its hyperbolic extension (see e.g. [17]). Finally, the fermionic transformations read

$$f_{-1}(\psi_2) = -\rho^{-1}\lambda_-\gamma_-\theta \quad , \quad f_{-1}(\chi) = f_{-1}(\varphi) = f_{-1}(\theta) = 0 \tag{25}$$

Their derivation requires a compensating local supersymmetry transformation in addition to the usual Lorentz rotation.

One can now check that the enlarged algebra satisfies all the relations (2a) and (2b), and, in particular, the trilinear relations $[f_0,[f_0,f_{-1}]] = [f_{-1},[f_{-1},f_0]] = 0$ (the relations involving the generators e_0 and e_{-1} are again trivial). Of course, it is crucial here that the new $SL(2,\mathbf{R})$ group introduced in (24) and (25) commutes with the Ehlers group; this is, in fact, the only part of the calculation that was not guaranteed to work beforehand. Actually, most of the relevant commutators vanish trivially, but a little more work is required to establish the vanishing of the commutator $[f_1, f_{-1}]$ on the fields B_- and B_2. In addition, I have verified that the "equations of motion" (11–13), (15) and (16) are transformed into one another or simply annihilated by the action of all generators and hence covariant. It is noteworthy that, at least formally, these checks also work for the purely bosonic theory.

However, putting $\theta = 0$ in (13), it is immediately obvious that $\partial_+ A_- = 0$, and therefore $A_- = const$ (or $= 0$ with appropriate boundary conditions). Hence, in order to obtain non-trivial transformations, we must take into account matter couplings. It is not difficult to convince oneself that bosonic matter fields (at least those of the type encountered here, which arise from dimensional reduction of supergravity) are of no help because they do not couple to the relevant component of the spin-connection (i.e. to $\partial_+ A_-$). It is here that fermionic fields become unavoidable. This result corroborates earlier speculations [11] according to which fermionic fields are needed in order to further enlarge the (infinite-dimensional) coset space of the bosonic theory.

One may wonder where exactly the difficulties in characterizing the hyperbolic algebra reside from the point of view taken in this paper. After all, the major problem here is to somehow control the multitude of generators arising through multiple commutators of the basic generators e_i or f_i. Sample calculations involving the generators f_i quickly reveal the complications. It is evident already from (19) that new dual potentials must now be introduced over and above those already necessary for the realization of the Geroch group. In the latter case, the required dual potentials could be compactly assembled into the solution of a linear system through the introduction of a suitable spectral parameter. Characterizing the hyperbolic algebra will require a similarly compact description of the full set of new dual potentials. Presumably, this can only be achieved through a linearization of the hyperbolic transformations given above and by generalizing the notion of maximally compact subalgebras known from finite-dimensional and affine algebras.

A more detailed account of the results presented here is in preparation.

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