DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 91-113
October 1991

On the General Theory of Quantized Fields

K. Fredenhagen

II. Institut für Theoretische Physik, Universität Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX,
send them to the following (if possible by air mail):

DESY
Bibliothek
Notkestrasse 85
D-2000 Hamburg 52
Germany

DESY 91-113
October 1991

ISSN 0418-9833

On the General Theory of Quantized Fields*

dedicated to the memory of Res Jost (†1990)

Klaus Fredenhagen

II. Institut für Theoretische Physik, Universität Hamburg

Content

1 The Aim of the General Theory of Quantized Fields

The term "General Theory of Quantized Fields", replacing the synonymous but somewhat misleading term "Axiomatic Field Theory", is to my knowledge due to Res Jost. He was one of the great pioneers in our field, and I dedicate this lecture to his memory. What is the aim of the general theory of quantized fields?

An answer may be found in Jost's famous book with the same title [1]. Jost formulates it as follows: "The general theory of quantized fields analyzes the notions which are at the basis of all previously analyzed specific models". So the general theory of quantized fields is concerned with model independent aspects of quantum field theory. Actually, the richness of the general theory of quantized fields is intimately connected with the notorious difficulties in constructing and evaluating examples of quantum field theories.

There has been a lot of progress in quantum field theory since the time the project of the general theory of quantized fields was started by the work of Wightman, Haag, Lehmann, Symanzik, Zimmermann and others. Nonabelian gauge theories became the most promising models for elementary

* to appear in the Proceedings of the IAMP conference Leipzig 1991

particle physics, renormalization has much better been understood by now, interacting models in low dimensions were constructed, numerical approximation on lattices and semiclassical arguments supplement the perturbation theoretical approach, and experiments in elementary particle physics have supported much of the theoretical predictions. More recently, a rich class of interesting models was detected in 2 dimensional conformal field theory, integrable models with exact S-matrices were found, and string theory and topological field theory broadened the range of quantum field theory.

In spite of all these successes, quantum field theory remains a very difficult subject, and the general analysis of structural properties is often the most convenient way to derive predictions in combination with some input from the models or even directly from experiment.

In my lecture I want to describe the present stage of the general theory of quantized fields on the example of 5 subjects. They are ordered in the direction from large to small distances.

The first one is the by now classical problem of the structure of superselection sectors. It involves the behavior of the theory at spacelike infinity and is directly connected with particle statistics and internal symmetries. It has become popular in recent years by the discovery of a lot of nontrivial models in 2d conformal field theory, by connections to integrable models and critical behavior in statistical mechanics and by the relations to the Jones' theory of subfactors in von Neumann algebras and to the corresponding geometrical objects (braids, knots, 3d manifolds,...).

At large timelike distances the by far most important feature of quantum field theory is the particle structure. This will be the second subject of my lecture. It follows the technically most involved part which is concerned with the behavior at finite distances. Two aspects, nuclearity which emphasizes the finite density of states in phase space, and the modular structure which relies on the infinite number of degrees of freedom present even locally, and their mutual relations will be treated .

The next point, involving the structure at infinitesimal distances, is the connection between the Haag-Kastler framework of algebras of local and the framework of Wightman fields. Finally, problems in approaches to quantum gravity will be discussed, as far as they are accessible by the methods of the general theory of quantized fields.

2 Sectors, Statistics and Symmetry

The theory of superselection sectors has ever been a challenge since the observation of Wick, Wightman and Wigner [2] that the superposition principle in quantum physics does not hold unrestrictedly. It was formulated as a classification problem on the representations of the algebra of observables by Haag and Kastler [3], was initiated by Borchers [4] and reached a certain degree of completeness in the work of Doplicher, Haag and Roberts [5].

The basic structure is a net of von Neumann algebras

$$\mathcal{A} = (\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}, \qquad \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0) \tag{2.1}$$

where $\mathcal{K}$ is the set of open double cones ("diamonds") in Minkowski space. This net is assumed to satisfy the usual Haag-Kastler axioms,

$$(i) \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \text{for} \quad \mathcal{O}_1 \subset \mathcal{O}_2 \quad \text{(isotony)} \tag{2.2}$$

$$(ii) \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \quad \text{for} \quad \mathcal{O}_1 \subset \mathcal{O}_2' \quad \text{(locality)} \tag{2.3}$$

($\mathcal{A}(\mathcal{O})'$ is the commutant of $\mathcal{A}(\mathcal{O})$ and $\mathcal{O}'$ the spacelike complement of $\mathcal{O}$, $\mathcal{O} \in \mathcal{K}$)

$$(iii) \bigcap_{\mathcal{O}\in\mathcal{K}} \mathcal{A}(\mathcal{O})' = \mathbb{C}\mathbb{1} \quad \text{(irreducibility)}. \tag{2.4}$$

One then looks for representations π fulfilling the DHR selection criterion

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \pi_0|_{\mathcal{A}(\mathcal{O}')}, \quad \mathcal{O} \in \mathcal{K} \tag{2.5}$$

where $\pi_0(A) = A$ is the defining representation of $\mathcal{A}$ ("vacuum representation") and $\mathcal{A}(\mathcal{O}')$ is the algebra generated by $\mathcal{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}', \mathcal{O}_1 \in \mathcal{K}$.

Crucial for the analysis is a strengthened version of locality which is supposed to hold in the vacuum representation

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')' \quad \text{(Haag duality)}. \tag{2.6}$$

Haag duality is known to hold in many cases; there are, however, also situations where Haag duality fails, e.g. in the case of spontaneous breakdown of symmetry, a subject treated in the poster session by Roberts.

Provided Haag duality holds, there exists an endomorphism $\rho \in End(\mathcal{A})$ such that

$$(i) \quad \pi \simeq \pi_0 \circ \rho \tag{2.7}$$

$$(ii) \quad \rho(A) = A \quad \text{for} \quad A \in \mathcal{A}(\mathcal{O}'). \tag{2.8}$$

Let $\Delta_\pi(\mathcal{O})$ be the set of all endomorphisms of $\mathcal{A}$ satisfying (i) and (ii), and let $\rho, \rho' \in \Delta_\pi(\mathcal{O})$. Then again by Haag duality there exists a unitary $U \in \mathcal{A}(\mathcal{O})$ intertwining ρ and ρ',

$$U\rho(A) = \rho'(A)U \quad , A \in \mathcal{A}. \tag{2.9}$$

(The set of all operators satisfying (2.9) will be denoted by (ρ, ρ').) Thus, composition of endomorphisms induces a composition law for unitary equivalence classes of representations satisfying (2.5),

$$[\pi_1] \times [\pi_2] := [\pi_0 \circ \rho_1\rho_2] \quad , \quad \rho_i \in \Delta_{\pi_i}(\mathcal{O}). \tag{2.10}$$

In cases where ρ is an automorphism, the irreducibility of π_0 implies the irreducibility of $\pi_o \circ \rho$; such sectors are called simple, and the product of simple sectors is again simple. If, however, the corresponding endomorphism is not surjective, the product of irreducible representations is, in general, reducible. This phenomenon is related to the violation of Haag duality in the pertaining representation. As a measure of the degree of violation one may use the Jones Index [6]

$$Ind[\pi(\mathcal{A}(\mathcal{O}'))' : \pi(\mathcal{A}(\mathcal{O}))] \tag{2.11}$$

which coincides with the square $d(\pi)^2$ of the statistical dimension $d(\pi)$ of π in the DHR theory [7].

The Jones Index may be illustrated most easily in the finite dimensional example of the inclusion

$$\begin{cases} M_n(\mathbb{C}) \to M_{nm}(\mathbb{C}), & n, m \in \mathbb{N}, \quad n > m \\ A \quad \mapsto A \otimes \mathbb{1}_m \end{cases} \tag{2.12}$$

where $M_k(\mathbb{C})$ denotes the algebra of $(k \times k)$-matrices over $\mathbb{C}, k \in \mathbb{N}$. In this case one has

$$Ind[M_{nm}(\mathbb{C}) : M_n(\mathbb{C})] = \frac{dim(M_{nm}(\mathbb{C}))}{dim(M_n(\mathbb{C}))} = m^2. \tag{2.13}$$

In the infinite dimensional situation of quantum field theory an analytic characterization of the index due to Pimsner and Popa [8] is more useful. There is a conditional expectation $\mathcal{E}_\pi$ from $\pi(\mathcal{A}(\mathcal{O}'))'$ onto $\pi(\mathcal{A}(\mathcal{O}))$ (in the finite dimensional example above this is the partial trace) such that

$$\mathcal{E}_\pi(A^*A) \geq d(\pi)^{-2} A^*A \quad , \quad A \in \mathcal{A} \tag{2.14}$$

and $d(\pi)$ is the smallest positive number for which the inequality is true. Inequality (2.14) already appears in [5]. Representations with $d(\pi) < \infty$ were said to have finite statistics.

The nice feature of such representations is that they are completely reducible. Moreover, one finds

$$(i) \quad d(\pi_1 \times \pi_2) = d(\pi_1)d(\pi_2) \tag{2.15}$$

$$(ii) \quad d(\oplus_i \pi_i) = \sum_i d(\pi_i) \tag{2.16}$$

hence the statistical dimensions behave like dimensions of vector spaces under tensor products and direct sums.

The main mean for the decomposition into irreducible components is the notion of statistics, very similar to the role permutation symmetry plays in the reduction of tensor products of group representations. Let (DHR) denote the class of all representations of $\mathcal{A}$ satisfying (2.5) and having finite statistics, and let

$$\Delta(\mathcal{O}) = \bigcup_{\pi\in(\mathrm{DHR})} \Delta_\pi(\mathcal{O}) \quad . \tag{2.17}$$

As a consequence of locality, endomorphisms $\rho_i \in \Delta(\mathcal{O}_i), i = 1,2$ with $\mathcal{O}_1 \subset \mathcal{O}_2'$ commute. Now let $\rho \in \Delta(\mathcal{O})$ and $\rho_1 \in \Delta_{\pi_0\circ\rho}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}'$, and choose a unitary $U \in (\rho,\rho_1)$. Then one defines the "statistics operator"

$$\epsilon_\rho = \rho(U)^{-1}U \quad \in (\rho^2,\rho^2) = \rho^2(\mathcal{A})' \subset \mathcal{A}(\mathcal{O}). \tag{2.18}$$

ϵ_ρ turns out to be independent of the choice of

$$\begin{aligned} &(i) \quad U \in (\rho,\rho_1) \\ &(ii) \quad \rho_1 \in \Delta_{\pi_0\circ\rho}(\mathcal{O}_1) \\ &(iii) \quad \mathcal{O}_1 \text{ in a connected component of } \mathcal{O}'. \end{aligned} \tag{2.19}$$

Hence in $D \geq 3$ space time dimensions ϵ_ρ is uniquely determined, and in $D = 2$ dimensions there are at most two different operators ϵ_ρ dependent on whether $\mathcal{O}_1$ is in the right or left component of the spacelike complement of $\mathcal{O}$.

ϵ_ρ satisfies the braid (i.e. constant Yang Baxter) relation

$$\epsilon_\rho\rho(\epsilon_\rho)\epsilon_\rho = \rho(\epsilon_\rho)\epsilon_\rho\rho(\epsilon_\rho) \tag{2.20}$$

which implies that

$$\epsilon^{(\rho)} : \sigma_i \mapsto \rho^{i-1}(\epsilon_\rho) \tag{2.21}$$

induces a unitary representation of the braid group B_∞. (σ_i denotes the transposition of the i-th and (i+1)-st string.) In $D \geq 3$ dimensions one finds the additional relation

$$\epsilon_\rho^2 = \mathbb{1} \tag{2.22}$$

which says that $\epsilon^{(\rho)}$ factors through a representation of the symmetric group.

For an analysis of the occuring representations [5,9]one uses a so called left inverse of ρ. Let $\rho \in \Delta(\mathcal{O})$ be irreducible. Then there exists a unique left inverse

$$\phi : \mathcal{A} \to \mathcal{A} \tag{2.23}$$

with $\phi\circ\rho = id$ and with $\rho\circ\phi$ being a conditional expectation from $\mathcal{A}$ onto $\rho(\mathcal{A})$ (which coincides essentially with the expectation $\mathcal{E}_\pi$ considered in the definition of the statistical dimension (2.14)).

Now for ρ irreducible, $\phi(\epsilon_\rho)$ is a multiple of the identity, and

$$\phi(\epsilon_\rho) = \frac{\kappa(\rho)}{d(\rho)}\mathbb{1}, \quad |\kappa(\rho)| = 1 \tag{2.24}$$

where $\kappa(\rho)$ is called the statistical phase and $d(\rho) = d(\pi_0\circ\rho)$ is the statistical dimension defined above. By iteration one obtains a function φ of positive type on the braid group B_∞,

$$\varphi(b)\mathbb{1} = \lim_{n\to\infty} \phi^n(\epsilon^{(\rho)}(b)). \tag{2.25}$$

φ is a trace on B_∞,

$$\varphi(b_1b_2) = \varphi(b_2b_1) \quad , \quad b_1,b_2 \in B_\infty \tag{2.26}$$

and fulfils

$$\varphi(b\sigma_n^{\pm1}) = \frac{\kappa(\rho)^{\pm1}}{d(\rho)}\varphi(b) \quad , \quad b \in B_n; \tag{2.27}$$

such traces were called Markov traces by Jones [6], since due to Markovs Theorem [10] they induce ribbon invariants and, after renormalization, link invariants.

A partial classification can be performed according to the number of different eigenvalues of ϵ_ρ.

In the simplest case ϵ_ρ is a multiple of the identity. This happens if and only if ρ is an automorphism or, equivalently, if and only if ρ^2 is irreducible. In the next simple case ϵ_ρ has two different eigenvalues. This is always true when (2.22) holds [5] or when ρ^2 is a direct sum of two irreducible representations [9]. Then $\epsilon^{(\rho)}$ induces a representation of the Hecke algebra, hence is quasiequivalent to the braid group representations found by Jones, Ocneanu and Wenzl [7,11,12] tensored by a one dimensional representation, and φ leads to the Jones- and HOMFLY-invariants [7,13].

In the presence of three different eigenvalues a general analysis was not yet possible. Only the case where ρ^2 is a direct sum of three irreducible representations one of which is a simple sector was completely analyzed [14,15]. One obtains the Birman-Wenzl algebra [16], the Murakami-Wenzl representation of B_∞ [17,18] and the Kauffman link invariant [19].

According to the general analysis each (DHR)-representation of a 2d quantum field theory satisfying Haag duality leads to a ribbon invariant. It is, however, not to be expected that every representation which describes particles is in the (DHR)-class. What can be proved is that (in $D \geq 3$ dimensions) every irreducible representation π with an isolated mass shell in the energy momentum spectrum satisfies the following weaker version of (DHR) [20,21]:

(i) there exists a unique vacuum representation π_0 such that

$$\pi|_{\mathcal{A}(\mathcal{C}')} \simeq \pi_0|_{\mathcal{A}(\mathcal{C}')} \tag{2.28}$$

for all spacelike cones $\mathcal{C} = a + \bigcup_{\lambda>0}\lambda\mathcal{O} \quad , \mathcal{O} \in \mathcal{K}, \mathcal{O} \subset \{0\}'$.

(ii) provided π_0 satifies Haag duality, π has finite statistics.

One then can perform an analogous analysis of the set of sectors satisfying (2.28). The relation $\epsilon_\rho^2 = 1$ can be shown only in $D \geq 4$ dimensions, hence one may have braid group statistics in $D = 3$ dimensions [22,23]. In $D = 2$ dimensions the mentioned result leads to the possible existence of two different vacua corresponding to the two different directions in which spacelike cones can point ("solitons"); composition of soliton sectors was defined in [24] but not yet analyzed in detail (see [25] for an earlier approach to this problem).

A finer classification of the set of sectors may be done in terms of a generalized internal symmetry. The most directly visible structure is the following category. Its objects are the endomorphisms $\rho \in \Delta(\mathcal{O})$, its arrows are the intertwiners $T \in (\rho, \rho')$, $\rho, \rho' \in \Delta(\mathcal{O})$, and the composition of arrows is the product of intertwiners. In addition this category has a tensor (monoidal) structure,

$$\begin{cases} \rho, \rho' \mapsto \rho\rho' \\ S, T \mapsto S \times T := S\rho(T) \end{cases} \tag{2.29}$$

$(S \in (\rho, \rho'), T \in (\rho_1, \rho_1'), S \times T \in (\rho\rho', \rho_1\rho_1'))$ with the statistics operators describing the permutation of factors in the tensor product. This category has all the properties postulated by Moore and Seiberg as "axioms for rational conformal field theory" [26], so this structure turns out to be a generic feature of quantum field theory and is by no means restricted to conformal invariant theories.

In that cases where permutation statistics holds, Doplicher and Roberts [27,28] showed that this category is equivalent to the category of finite dimensional representations of a uniquely determined compact group G which may in turn be considered as the group of internal symmetries. Moreover, they construct an embedding of the observable algebra $\mathcal{A}$ into a so called field algebra $\mathcal{F} = (\mathcal{F}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ which satisfies graded locality and on which G acts by automorphisms such that $\mathcal{A}(\mathcal{O})$ is the set of fixed points of $\mathcal{F}(\mathcal{O})$.

In the braid group case examples of these categories can be obtained from quantum groups at roots of unity. In the approach of Fröhlich and Kerler [29] one looks at the ring of representations and factors out the ideal of degenerate representations. In an alternative approach due to Todorov et al. [30] fields transforming covariantly under a quantum group act on an indefinite metric space from which the physical Hilbert space may be recovered. A recent construction of Mack and Schomerus [31] , explained in more detail in the poster session by Schomerus, avoids the use of indefinite metrics. Covariant fields were constructed which act on the physical Hilbert space, but the symmetry is not the quantum group itself but the quotient with respect to some ideal resulting in a modification of the Hopf algebra structure ("weak quasi quasitriangular Hopf algebra").

There is also a proposal for the general case with a finite number of sectors ("rational theories"). Rehren [32] constructs a field algebra similar

to the field algebra of Doplicher and Roberts where now the role of the symmetry group is played by a bimodule over the hyperfinite von Neumann factor $\mathcal{R}$ of type II_1 which is the standard algebra in the Jones theory of subfactors. There are fields $\psi_i, i = 1, ..n$ and a homomorphism

$$\rho : \begin{cases} \mathcal{R} \to M_n(\mathcal{R}) \\ r \mapsto (\rho_{ij}(r)) \end{cases} \tag{2.30}$$

into the algebra of $(n \times n)$-matrices with entries in $\mathcal{R}$ such that covariance takes the form

$$r\psi_i = \sum_j \psi_j \rho_{ji}(r) \quad , r \in \mathcal{R}. \tag{2.31}$$

For more details one may consult the contribution of Rehren to these proceedings.

A classification of symmetries may also be done directly in terms of the Jones theory of subfactors. So Longo analyzed the inclusion

$$\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O}) \quad , \rho \in \Delta(\mathcal{O}) \tag{2.32}$$

and found additional restrictions on the statistical dimensions [14].

3 Particles and Infraparticles

At large timelike distances the dominant structure of quantum field theory is the occurence of particles. The relation between particles and fields is most elegantly expressed in terms of the LSZ reduction formulae; there is, however, in general no 1-1 correspondence between particles and fields.

According to Wigner particles are characterized as eigenstates of the mass operator. The Haag-Ruelle theory [33,34] then succeeds in constructing the corresponding incoming and outgoing multiparticle scattering states provided the mass shell of the particle is isolated in the energy momentum spectrum and the particle is uncharged. This result has been generalized in several directions. Doplicher, Haag and Roberts performed the same construction for particles in (DHR)- representations [5], Buchholz proved (by different methods) corresponding results for massless particles [35], Buchholz and myself did the construction for charged particles in general massive 4d theories, Fröhlich, Gabbiani and Marchetti [36] and Rüger [37]generalized this construction to the case of braid group statistics in 3 dimensions ("anyons" (simple sectors), "plektons" (general case), resp.). The existence of scattering states could be proven even in euclidean lattice field theories [38] where the local structure of quantum field theory is strongly disturbed.

As a resumee the mathematically rather nontrivial fact of existence of scattering states is by now well understood and may be considered as the easy part of the problem. There are, however, really hard problems where progress has been very slow.

The first one to mention is the famous problem of asymptotic completeness. Even in quantum mechanics this problem is by no means simple, but has nevertheless been solved recently (see the contribution of Graf to these proceedings). In the case that all particles are massive and uncharged one may approach this question in terms of the analyticity properties of Wightman functions (see the contribution of Bros to these Proceedings).

A more general approach is due to Buchholz. It takes into account that Wigner's characterization of particles is too narrow since it excludes the so called infraparticles occuring in theories with massless particles which cannot be characterized by normalizable eigenvectors of the mass operator due to the presence of a cloud of massless particles. Also the usual description by improper momentum eigenstates meets problems since states with different momenta typically cannot be coherently superimposed. There is, however, a new proposal by Buchholz, Porrmann and Stein [39] (presented in the poster session by Buchholz) where improper momentum eigenstates are defined as linear mappings from a left ideal $\mathcal{L}$ of almost local vacuum annihilators into the physical Hilbert space $\mathcal{H}$,

$$|p\rangle : \begin{cases} \mathcal{L} \to \mathcal{H} \\ L \mapsto L|p\rangle \end{cases} \tag{3.1}$$

where $|p\rangle$ represents the bare particle ("particle weight") and $L|p\rangle$ is a normalizable state describing the particle together with some infrared cloud. In this framework, multiparticle scattering weights can be constructed as linear functionals on an algebra of particle counters, and cross sections can be directly defined without any infrared regularization.

Wigner's particle definition seems also to be too restrictive for situations at finite temperature. As first observed by Narnhofer, Requardt and Thirring [40], eigenstates of the mass operator at finite temperature describe noninteracting particles. In which sense the concept of interacting particles at finite temperature is meaningful is unclear at present; an interpretation in terms of resonances similar to the Stark effect in atomic physics was presented in the poster session by Landsman [41].

4. Nuclearity and Modular Structure

At finite distances, two competing principles are to be taken into account. The first one emphasizes that locally one has a situation similar to quantum mechanics with a finite number of degrees of freedom. Semiclassically, the density of states in phase space for n degrees of freedom is h^{-n}, h denoting Planck's constant. In order to incorporate this structure into quantum field theory, Haag and Swieca [42] introduced the compactness criterion, i.e.

$$\{P_E A\Omega, A \in \mathcal{A}(\mathcal{O}), ||A|| \leq 1\} \tag{4.1}$$

where P_E denotes the spectral projection onto energies $\leq E$ and Ω is the vacuum is supposed to be a relatively compact subset of Hilbert space (w.r.t. the strong topology).

The Haag-Swieca criterion was sharpened to nuclearity by Buchholz and Wichmann [43]. Namely, the set

$$\{e^{-\beta H} A\Omega, A \in \mathcal{A}(\mathcal{O}), ||A|| \leq 1\} \tag{4.2}$$

shall be nuclear (which roughly means that it is contained in the image of the unit ball under a trace class operator T) for $\beta > 0$, and the nuclear size $Z(\beta, \mathcal{O})$ of this set (roughly the trace norm of T) which plays a role similar to a partition function is supposed to satisfy the bound

$$Z(\beta, \mathcal{O}) \leq const\ \exp\{const\ \beta^{-n}\}, n > 0. \tag{4.3}$$

Nuclearity implies the existence of product states, i.e. states having no correlations between observables in regions separated by a finite distance, equivalently, that inclusions $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}_1)$ are split if the closure of $\mathcal{O}$ is contained in $\mathcal{O}_1$, i.e. there is a von Neumann factor $\mathcal{N}$ of type I (i.e. isomorphic to $\mathcal{B}(\mathcal{H})$) such that $\mathcal{A}(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{A}(\mathcal{O}_1)$ and, moreover, the existence of KMS-states for all $\beta > 0$ [44].

The contrary principle, that quantum field theory, strictly speaking, even locally has an infinite number of degrees of freedom relies on the Reeh-Schlieder theorem [45] which states that every vector which is in the image of $e^{-\beta H}$ for some $\beta > 0$ (in particular the vacuum Ω) is cyclic and separating for each algebra $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$. Therefore the modular operator Δ and the modular conjugation J of the Tomita-Takesaki theory may be defined for the pair $(\mathcal{A}(\mathcal{O}), \Omega)$,

$$\begin{aligned} (\Omega, A\Delta B\Omega) &= (\Omega, BA\Omega) \\ J\Delta^{\frac{1}{2}} A\Omega &= A^* \Omega \quad , \quad A, B \in \mathcal{A}(\mathcal{O}). \end{aligned} \tag{4.4}$$

A geometrical interpretation of the modular operators was found by Bisognano and Wichmann [46] for the case where $\mathcal{O}$ is replaced by the so called wedge $W = \{x \in \mathbb{R}^D, |x^o| < x^1\}$.Under the additional assumption that the local algebras are generated by Wightman fields they show that Δ^{it} is the Lorentz boost with velocity $\tanh 2\pi t$ in the 1-direction and J is the PCT operator with parity P replaced by the reflection on the plane $x^1 = 0$.

Recently Borchers proved a remarkable result [47](presented in the poster session) which shows that the structure discovered by Bisognano and Wichmann does in no means depend on special features of quantum field theory. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, let $\Omega \in \mathcal{H}$ be cyclic and separating for $\mathcal{M}$ and denote by Δ and J the operators of the Tomita-Takesaki theory associated to the pair $(\mathcal{M}, \Omega)$. Let in addition $U(x)$

be a 1-parameter group of unitaries acting on $\mathcal{H}$ with a positive generator such that Ω is invariant and $U(x)\mathcal{M}U(x)^{-1} \subset \mathcal{M}$ for $x \geq 0$. Then Borchers proves the commutation relation

$$\begin{aligned} \Delta^{it}U(x)\Delta^{-it} &= U(e^{-2\pi t}x) \\ JU(x)J &= U(-x) \end{aligned} \tag{4.5}$$

which are precisely the commutation relations of Lorentz boosts with light-like translations.

There are mutual relations between nuclearity and modularity [48,49]. In particular, it is possible to replace the Hamiltonian H in the nuclearity condition by a function of the modular operator of a slightly larger region. This is based on the following inequality between the modular Hamiltonian $K = -\ln \Delta$ and H. As is easily seen the modular Hamiltonian increases with the algebra, hence $K \leq K_W$ for wedges W containing $\mathcal{O}$. Using the fact that the latter are boost generators Buchholz, D'Antoni and Longo find the relation [48]

$$K \leq \frac{\tan \pi\lambda}{\lambda} dH + \frac{\ln 2}{\lambda} \quad ,0 < \lambda < \frac{1}{4} \tag{4.6}$$

where d is the timelike extension of $\mathcal{O}$.

5 Haag-Kastler Nets versus Wightman Fields

There are two main frameworks for quantum field theory; Haag-Kastler nets and Wightman fields. The formulation in terms of Haag-Kastler nets is very well suited for the discussion of structural problems. It is, however, difficult to write down Haag-Kastler nets explicitely.

In this respect, Wightman fields are much better behaved, since they may be directly extracted from models, they have a simple connection to an euclidean formulation and they can be obtained in terms of formal perturbation series. Usually, one believes that both frameworks are physically equivalent. There is however no proof. Actually, there are some structural problems for Haag-Kastler nets for which the relations to Wightman fields are important. Such problems are

(i) the geometrical meaning of modular operators (see however the mentioned result of Borchers)
(ii) a general proof of Haag duality, a problem connected to (i)
(iii) the notion of a scaling limit.

The relations between both frameworks have been studied in several papers. For the construction of Haag-Kastler nets conditions were found by Borchers and Zimmermann [50](Ω analytic vector for smeared field),

Driessler and Fröhlich [51](linear H bounds), Driessler, Summers and Wichmann [52], Buchholz [53] and Borchers and Yngvason [54](see below). The construction of Wightman fields associated to a given Haag-Kastler net was first investigated by Hertel and myself [55](fields with polynomial H bounds) and then generalized by Summers [56](generalized H bounds) and Rehberg and Wollenberg [57,58] (pre-Wightman fields). Recently, Jörß[59] succeeded in deriving fields from a conformally covariant Haag-Kastler net.

Let me describe the result of Borchers and Yngvason in more detail. Let φ be a hermitean scalar Wightman field satisfying a generalized H bound

$$||\varphi(f)e^{-H^\alpha}|| < \infty \quad \text{for some } \alpha, 0 < \alpha < 1. \tag{5.1}$$

Under this condition the weak commutants

$$\mathcal{A}_w(\mathcal{O}) = \{A \in \mathcal{B}(\mathcal{H}), (\Phi, A\varphi(f)\Psi) = (\varphi(f)^*\Phi, A\Psi), f \in \mathcal{D}(\mathcal{O}), \Phi, \Psi \in \bigcup_{\alpha>0} e^{-H^\alpha}\mathcal{H}\} \tag{5.2}$$

are von Neumann algebras [52]. Now, following Powers [60], Borchers and Yngvason define a state ω on a *-algebra $\mathcal{A}$ to be centrally positive w.r.t. $A \in \mathcal{A}, A = A^*$ if

$$\omega(\sum_n B_n A^n) \geq 0 \tag{5.3}$$

whenever $\sum_n B_n \lambda^n \geq 0$ for all $\lambda \in \mathbb{R}$.

Theorem [54]. *There exists a Haag Kastler net $(\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ such that for each real test function $f \in \mathcal{D}(\mathcal{O})$ $\varphi(f)$ has a selfadjoint extension affiliated to $\mathcal{A}(\mathcal{O})$ if and only if for each f, ω_o is centrally positive with respect to $\varphi(f)$ on the algebra generated by $\varphi(f)$ and $\varphi(g)$, supp $g \subset \mathcal{O}'$.*

A nice feature of this criterion is that it leads to testable predictions. Look for instance at the polynomial $p(x,y) = x^2y^2(x^2+y^2-1) + \frac{1}{27}$. p assumes positive values for real arguments, hence for each $x \in \mathbb{R}$ it is a sum of absolute squares of polynomials in y. Thus when φ is associated to a Haag Kastler net, ω_o must satisfy the inequality

$$\omega_0(\varphi(f)^2\varphi(g)^2) \leq \omega_0(\varphi(f)^4\varphi(g)^2) + \omega_0(\varphi(f)^2\varphi(g)^4) + \frac{1}{27}, \tag{5.3}$$

where f and g are real test functions, $supp\ f \subset \mathcal{O}$, $supp\ g \subset \mathcal{O}'$.

6. Conceptual Problems in Approaches to Quantum Gravity

The impact of quantum gravity on the framework of the general theory of quantized fields may be discussed on three levels. The first one is the semiclassical level where gravity is treated classically and quantum field theory lives on a curved space time. On a first sight this situation does not seem to lead to conceptual problems, since all local aspects of quantum field theory can be easily generalized. There is, however, one important global aspect of quantum field theory, namely the spectrum condition, and it is a major problem to find the appropriate generalization of this condition in a curved background. In general, there is no timelike Killing vector which could play the role of the generator of time translations; but even in stationary spacetimes where by definition a timelike Killing vector field exists the positivity of the corresponding generator is not always the good property required for a physical state. For instance, for a Schwarzschild black hole, it is generally believed that not the ground state with respect to the time translations of the Schwarzschild space time ("Boulware vacuum") is the right state but states which exhibit Hawking radiation. The problem was first observed by Fulling [61] and found a conceptually satisfactory formulation by Haag, Narnhofer and Stein [62] and, independently, by Kay [63]. According to these authors, in order to define the theory, one must in addition to the field equations and the commutation relations specify the set of admissible states such that the local algebras have a unique completion to von Neumann factors (principle of "local definiteness" [62]). It is, however, not known in general how the admissible states can be characterized. Physically most appealing is the idea of "local stability" [62,64] which says that the Wightman functions should have a short distance scaling limit which then as a theory on Minkowski space should satisfy the spectrum condition. Local stability is a condition which is applicable also in interacting theories, unfortunately it does not fix the local algebras uniquely, in general. A related condition is the so called Hadamard condition [65] which can be formulated only for linear field theories; it states that the singularity of the 2-point function is the singularity of Hadamard's fundamental solution of the wave equation. The Hadamard condition implies local stability, and Kay and Wald [66] showed that for stationary states on the Kruskal space time it implies that the state is a KMS state with the Hawking temperature. It is to be expected that the Hadamard condition fixes the local von Neumann algebra but a proof is not yet available (see however [67]). A third approach to the determination of the local von Neumann algebras is the construction of "adiabatic vacuum states". This idea was carried out by Lüders and Roberts [68] for Robertson-Walker space times; it amounts to an approximation of a nonstationary space time by a stationary one with a nonsingular timelike Killing vector and to a construction of the admissible

states in terms of corrections to the ground state of the stationary problem. Lüders and Roberts could show that this procedure fixes the local von Neumann algebras uniquely and that they are factors in accordance with the principle of local definiteness.

The second level in an approach to quantum gravity is the treatment of backreaction. As shown in [69], the scaling behavior of the state near the horizon of a spherically symmetric collapsing star is in 1-1 correspondence to the long time asymptotic radiation. One has tried to treat the change of the metric due the mass loss by radiation in terms of a semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi\kappa\langle T_{\mu\nu,\mathrm{ren}}\rangle \qquad (6.1)$$

where $T_{\mu\nu,\mathrm{ren}}$ means the energy momentum tensor after subtraction of some diverging term which depends on the space time metric. The procedure looks somewhat arbitrary and actually the equation has no solution. Recently, a different approach related to ideas of Haag has been proposed by Salehi [70]. Salehi considers the singular part of $T_{\mu\nu}$ as that part which is responsible for the gravitational field and introduces a cutoff at length $\kappa^{\frac{1}{2}}$. His attempt results in an effective nonlinear equation for the matter field φ; whether his approach is free from inconsistencies is not known at the moment.

At the end of my lecture I want to add some remarks on full quantum gravity (level 3 of the above classification). There are several directions in which quantum gravity is approached at present; string theory, gravity in 2 and 3 dimensions (leading to topological field theories), noncommutative geometry and Ashtekar's Hamiltonian formulation of classical gravity. Unfortunately, general quantum field theory has not much to offer at the moment to these problems. The main obstacle in incorporating gravity into the framework of general quantum field theory is the fact that classical space time plays a distinguished role in quantum field theory. This becomes especially clear in the algebraic formulation in terms of Haag-Kastler nets, but is implicit in every formulation of field theory. The main role played by space time is in the interpretation of the theory. One possible route to quantum gravity is therefore the derivation of a space time structure from observables.An interesting idea how this could be done is due to Bannier [71]. Bannier starts from a family of von Neumann algebras $(\mathcal{A}_i)_{i\in I}$ representing the observable algebras of physical subsystems. The crucial ingredient is now the notion of a "good subsystem" $i < j,\ i,j \in I$ which is a partial ordering stronger than the set theoretical inclusion $\mathcal{A}_i \subset \mathcal{A}_j$. A subset $J \subset I$ is called overlapping if any finite subset $J_0 \subset J$ has a lower bound in I. Let M be the set of maximal overlapping subsets of I. Then M gets the structure of a topological Hausdorff space by defining as open sets in M those which contain with every $x \in M$ for some finite subset $J_0 \subset x$ all $y \in M$ with $J_0 \subset y$. If one applies this construction to the local net $(\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ and uses the partial ordering

$$\mathcal{O}_1 < \mathcal{O}_2 \iff \text{the inclusion } \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \text{ is split} \qquad (6.2)$$

one recovers Minkowski space [71]. In order to apply this idea in a more general situation one has to find an intrinsic characterization of the algebras $\mathcal{A}_i$ (candidates: hyperfinite factors of type III_1 [72]) and an ordering relation (candidate: split inclusion). The result will be a topological space which can be used as a device for an interpretation of the theory.

References

1. R. Jost: The General Theory of Quantized Fields. AMS 1965, Providence, Rhode Island
2. G.C. Wick, A.S. Wightman, E.P. Wigner: The Intrinsic Parity of Elementary Particles. Phys.Rev.**88**, 101(1952)
3. R. Haag, D. Kastler: An Algebraic Approach to Quantum Field Theory. J.Math.Phys.**5**,848(1964)
4. H.-J. Borchers: Local Rings and the Connection of Spin with Statistics. Commun.Math.Phys. **1**,281(1965)
5. S. Doplicher, R. Haag, J.E. Roberts: Local Observables and Particle Statistics. Commun.Math.Phys.**23**,199(1971) and **35**,49(1974)
6. V.F. Jones: Index of Subfactors. Invent.Math.**72**,1 (1983)
7. R. Longo: Index for Subfactors and Statistics of Quantum Fields. Commun.Math.Phys.**126**,217(1989) and **130**,285(1990)
8. M. Pimsner, S. Popa: Entropy and Index for Subfactors. Ann. Sci. Ec. Norm. Sup. **19**,57(1986)
9. K. Fredenhagen, K.H. Rehren, B. Schroer: Superselection Sectors with Braid Group Statistics and Exchange Algebras. Commun. Math. Phys. **125**,201(1989)
10. J. Birman: Braids, Links and Mapping Class Groups. Ann. Math.Studies **82**, Princeton University Press 1974
11. A. Ocneanu: Quantized Groups, String Algebras and Galois Theory for Algebras. Lond.Math.Soc., Lect. Notes Vol. **136**, Evans, Takesaki (eds.),119-172(1989)
12. H. Wenzl: Hecke Algebras of Type A_n and Subfactors. Invent. Math. **92**,349(1988)
13. P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millet, A. Ocneanu: A New Polynomial Invariant for Knots and Links. Bull.AMS **12**,103(1985)
14. R. Longo: Minimal Index and Braided Subfactors (preprint)
15. K.H. Rehren: Braid Group Statistics and their Superselection Rules. In "The Algebraic Theory of Superselection Sectors. Introduction and Recent Results", D. Kastler (ed.), World Scientific 1990
16. J. Birman, H. Wenzl: Braids, Link Polynomials and a New Algebra. Trans. AMS **313**,249(1989)
17. J. Murakami: The Kauffman Polynomial of Links and Representation Theory. Osaka J.Math. **24**,745(1987)
18. H. Wenzl: Quantum Groups and Subfactorsof Type B,C and D. Commun. Math. Phys.**133**,383(1990)

19. L.H. Kauffman: On Knots. Annals of Math. Study **115**, Princeton University Press (1987)
20. D. Buchholz, K. Fredenhagen: Locality and the Structure of Particle States. Commun. Math. Phys.**84**,1(1982)
21. K. Fredenhagen: On the Existence of Antiparticles. Commun. Math. Phys. **79**,141(1981)
22. J. Fröhlich, F. Gabbiani: Braid Statistics in Local Quantum Theory. Rev. Math. Phys. **2**, 251 (1990)
23. K. Fredenhagen: Structure of Superselection Sectors in Low Dimensional Quantum Field Theory. In "Differential Geometric Methods in Theoretical Physics", L.L. Chau, W. Nahm (eds.), Plenum Press 1990
24. K. Fredenhagen: Generalizations of the Theory of Superselection Sectors. In "The Algebraic Theory of Superselection Sectors. Introduction and Recent Results", D. Kastler (ed.), World Scientific 1990
25. J. Fröhlich: New Superselection Sectors ("Soliton States") in Two Dimensional Bose Quantum Field Theory Models. Commun. Math. Phys. **47**,269(1976)
26. G. Moore, N. Seiberg: Polynomial Equations for Rational Conformal Field Theories. Phys.Lett. **212 B**,451(1988)
27. S. Doplicher, J.E. Roberts: Endomorphisms of C^*-algebras, Cross Products and Duality for Compact Groups. Ann.Math.**130**,75(1989)
28. S. Doplicher, J.E. Roberts: Why there is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics. Commun. Math. Phys.**131**,51(1990)
29. J.Fröhlich, T.Kerler (to be published)
30. L.K. Hadjiivanov, R.R. Paunov, I.T. Todorov:Quantum Group Extended Chiral p-Models. INRNE-TH-90-7 (preprint)
31. G. Mack, V. Schomerus: Quasi Quantum Group Symmetry and Local Braid Relations in the Conformal Ising Model. DESY 91-060 (preprint)
32. K.H. Rehren: Field Operators for Anyons and Plektons. DESY 91-043 (preprint)
33. R. Haag: Quantum Field Theories with Composite Particles and Asymptotic Conditions. Phys. Rev. **112**, 669 (1958)
34. D. Ruelle: On the Asymptotic Condition in Quantum Field Theory. Helv. Phys. Acta **35**, 147 (1962)
35. D. Buchholz: Collision Theory for Massless Bosons. Commun. Math. Phys. **52**, 147 (1977)
36. J. Fröhlich, P. Marchetti: Spin-Statistics Theorem and Scattering in Planar Quantum Field Theory with Braid Statistics. Nucl. Phys. **B356**, 533 (1991)
37. S. Rüger: Streutheorie für Teilchen mit Zopfgruppenstatistik. Diplomarbeit FU Berlin 1990
38. J.C.A. Barata, K. Fredenhagen: Particle Scattering in Euclidean Lattice Field Theories. Commun. Math. Phys.**138**, 507 (1991)
39. D. Buchholz, M. Porrmann, U. Stein: Dirac versus Wigner: Towards a Universal Particle Concept in Local Quantum Field Theory. DESY 91-062 (preprint)(to appear in Phys.Lett.B)
40. H. Narnhofer, H. Requardt, W. Thirring: Quasi-Particles at Finite Temperature. Commun. Math. Phys. **92**, 247 (1983)

41. N.P.Landsman: The Inherent Non-Perturbativeness of Thermal Field Theories (and a Possible Perturbativization). Nucl. Phys. A **525**, 397(1991)
42. R. Haag, J.A. Swieca: When does a Quantum Field Theory Describe Particles. Commun. Math. Phys.**1**, 308 (1965)
43. D. Buchholz, E. Wichmann: Causal Independence andthe Energy Level Density of States in Local Quantum Field Theory. Commun. Math. Phys. **106**, 106 (1986)
44. D. Buchholz, P. Junglas: Local Properties of Equilibrium States and the Particle Spectrum in Quantum Field Theory. Lett. Math. Phys. **11**, 51 (1986)
45. H. Reeh, S. Schlieder: Bemerkungen zur Unitäräquivalenz von lorentzinvarianten Feldern. Nuovo Cim. X **22**, 1051 (1961)
46. J.J. Bisognano, E.H. Wichmann: On the Duality Condition for a Hermitean Scalar Field. J. Math. Phys **17**, 303 (1975)
47. H.J. Borchers: The CPT Theorem in Two-Dimensional Theories of Local Observables (preprint 1991)
48. D. Buchholz, C. D'Antoni, R. Longo: Nuclear Maps and Modular Structures. J. Funct. Anal. **88**, 233 (1990) and Commun. Math. Phys. **129**, 115 (1990)
49. D. Buchholz, J. Yngvason: Generalized Nuclearity Conditions and the Split Property in Quantum Field Theory (preprint)
50. H.J. Borchers, W. Zimmermann. Nuovo Cim. **31**, 1047 (1964)
51. W. Driessler, J. Fröhlich. Ann. Inst. Henri Poinc. **27**, 221 (1977)
52. W. Driessler, S.J. Summers, E.H. Wichmann: On the Connection Between Quantum Fields and Von Neumann Algebras of Local Operators. Commun. Math. Phys. **105**, 49 (1986)
53. D. Buchholz: On Quantum Fields that Generate Local Algebras . J. Math. Phys. **31**, 1839 (1990)
54. H.J. Borchers, J. Yngvason: Positivity of Wightman functions and the Existence of Local Nets. Commun. Math. Phys. **127**, 607 (1990)
55. K. Fredenhagen, J. Hertel: Local Algebras of Observables and Pointlike Localized Fields. Commun. Math. Phys. **80**, 555 (1981)
56. S.J. Summers. Helv. Phys. Acta **60**, 1004 (1987)
57. H. Rehberg, M. Wollenberg. Math. Nachr. **125**, 1 (1986)
58. M. Wollenberg. Rep. Math. Phys. **22**, 409 (1985) and Math. Nachr. **128**, 169 (1986)
59. M. Jörß: Lokale Netze auf dem eindimensionalen Lichtkegel. Diplomarbeit FU Berlin 1991
60. R.T. Powers. Commun. Math. Phys.**21**, 85 (1971) and Trans. AMS **187**, 261 (1974)
61. S.A. Fulling. Phys. Rev. D**7**, 2850 (1973)
62. R. Haag, H. Narnhofer, U. Stein: On Quantum Field Theory in Gravitational Background. Commun. Math. Phys.**94**, 219 (1984)
63. B.S. Kay: Quantum Field Theory in Curved Space Time. In "Differential Geometrical Methods in Theoretical Physics"**373**, K. Bleuler, M. Werner (eds.), Kluwer 1988 , references therein and contribution to these proceedings

64. K. Fredenhagen, R. Haag: Generally Covariant Quantum Field Theory and Scaling Limits. Commun. Math. Phys. **108**, 91 (1987)
65. S. Adler, J. Liebermann, Y.J. Ng, Ann. Phys. **106**, 279 (1978)
66. B.S. Kay, R.M. Wald: Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasi-Free States on Spacetime with a Bifurcate Killing Horizon (preprint)
67. D. Bernard: Hadamard Singularity and Quantum States in Bianchi Type-I Space Time. Phys. Rev. D**33**, 3581 (1986)
68. C. Lüders, J.E. Roberts: Local Quasiequivalence and Adiabatic Vacuum States. Commun. Math. Phys. **134**, 29 (1990)
69. K. Fredenhagen, R. Haag: On the Derivation of Hawking Radiation Associated with the Formation of a Black Hole. Commun. Math. Phys. **127**, 273 (1990)
70. H. Salehi. Ph.D.Thesis Hamburg 1991
71. U. Bannier. Ph.D.Thesis Hamburg 1987
72. D. Buchholz, C. D'Antoni, K. Fredenhagen:The Universal Structure of Local Algebras. Commun. Math. Phys. **111**, 123 (1987)