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The Average Action for Scalar Fields near Phase Transitions

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Abstract

We compute the average action for scalar fields in two, three and four dimensions, including the effects of wave function renormalization. A study of the one loop evolution equations for the scale dependence of the average action gives a unified picture of the qualitatively different behaviour in various dimensions for discrete as well as abelian and nonabelian continuous symmetry. The different phases and the phase transitions can be infered from the evolution equation.

1 Introduction

Theories with massless particles often have a complicated long distance behaviour. Perturbation theory (the saddle point expansion of the functional integral) is plagued bysave infrared singularities. Logarithmic singularities can be handled relatively easily since the precise definition of a physical infrared cutoff is not very important. For a logarithmic singularity the infrared and ultraviolet behaviour are connected. The usual renormalization group equations for dimensionless couplings apply. The problem becomes much harder for power singularities where naive perturbation theory fails.

Among the well known examples for theories with infrared power singularities are scalar theories with spontaneously broken continuous symmetry (e.g. φ^4-theory)save in two or three dimensions. The singularities arise from the fluctuations of the Goldstone bosons. It is perhaps less well known that even in four dimensional scalar theories the wave function renormalization for the radial excitation exhibits a quadratic infrared singularity in the spontaneously broken phase. In spontaneously broken gauge theories the Goldstone bosons disappear from the spectrum and the gauge boson masses M act as a physical infrared regulator. In models with a gauge hierachy M is much smaller than other mass scales. The gauge bosons are effectively massless at distances smaller than M^{-1}. Any extrapolation between the physics at short distances and distances of the order of the inverse gauge boson mass has again to deal with the infrared problems. This applies, in particular, to the computation of the scale of spontaneous symmetry breaking or the gauge boson mass. Another class of severe infrared problems in four dimensional field theories appears for physics at finite temperature T. At momentum scales smaller than T the infrared behaviour corresponds to a three dimensional field theory.

In order to deal with the long distance physics in statistical mechanics Kadanoff and Wilson [1] have introduced an effective action for blockspins, i.e. averages of fields on the sites of a coarse grain (block) lattice. This elegant idea becomes in practice difficult to handle if the long distance physics is invariant under space rotations and translations. The lack of symmetry of the block lattice has to be compensated by a very complicated structure of the effective action. These problems are overcome by formulating the average action [2, 3] in continuous space. An extrapolation from short to long distances requires a study of the dependence of the average action Γ_k on the volume $\sim k^{-d}$ over which the average is taken. The average action retains the full symmetries of the model. For field theories it can be expanded in the number of derivatives, i.e. potential, kinetic term and higher derivatives. Many invariants appear which may not be present in the classical (short distance) action, as for example φ^6 and φ^8 terms in the scalar potential. This is the prize to pay for integrating out the modes with momenta $q^2 \gtrsim k^2$. The use of Γ_k instead of the classical action for a description of the long distance physics has to deal with the complications due to these (infinitely) many invariants. This is the aim of this paper. We find that the rich structure of Γ_k may actually result in an important gain: The loop expansion is not necessarily an expansion in the small coupling constant anymore. It may converge much more rapidly than the series in the coupling constant!

This paper is devoted to a study of the $SO(N)$ symmetric scalar theory in two, three and four dimensions. We omit the effects of terms with more than two derivatives in Γ_k and study the most general form of the average potential and kinetic terms. We derive the scale dependence of the minimum of the potential and the quartic coupling (sect. 4) as well as the wave function renormalization (sect. 5) in this general setting. As a first step we truncate the corresponding evolution equations and retain only the standard kinetic term and the φ^4 coupling. (The effects of the neglected contributions will be added in a subsequent paper.) We also concentrate on potentials where the minimum occurs for a nonvanishing scalar field and deal mainly with the case of continuous symmetry ($N \geq 2$). We find that the truncated evolution equations for the average action describe well the phase structure of scalar theories in two, three and four dimensions.

2 Evolution equations for the average action

We consider the most general Euclidean action for N real scalar fields χ^a which is invariant under translations, rotations and internal $O(N)$ transformations ($O(1) = Z_2$ for $N = 1$) and contains no more than two derivatives

$$\begin{aligned} S[\chi] &= \int d^dx\{V(\xi) + \frac{1}{2}\bar{Z}(\xi)\partial_\mu\chi^a\partial^\mu\chi_a \\ &+ \frac{1}{4}\bar{Y}(\xi)(\chi^a\partial_\mu\chi_a)(\chi^b\partial^\mu\chi_b)\} \end{aligned} \tag{2.1}$$

$$\xi = \frac{1}{2}\chi^a\chi_a, \quad a = 1\ldots N \tag{2.2}$$

The kinetic terms involve two different functions $\bar{Z}(\xi)$ and $\bar{Y}(\xi)$. (For $N = 1$ there is only one independent invariant with two derivatives and we set $\bar{Y} \equiv 0$.) For N even, we may alternatively use a formulation with $M = N/2$ complex scalar fields

$$\chi^a_{(C)}(x) = \frac{1}{\sqrt{2}}\left(\chi^a_{(R)}(x) + i\chi^{M+a}_{(R)}(x)\right), \quad a = 1\ldots M \tag{2.3}$$

$$\begin{aligned} S &= \int d^dx\Big\{V(\xi) + \bar{Z}(\xi)\partial_\mu\chi^\dagger\partial^\mu\chi + \frac{1}{4}\bar{Y}(\xi)\partial_\mu\xi\partial^\mu\xi\Big\} \\ \xi &= \chi^\dagger\chi \end{aligned} \tag{2.4}$$

In case of continuous symmetry ($N \geq 2$) it is instructive to split χ into a radial mode σ and 'Goldstone modes' π_a

$$\chi(x) = \frac{1}{\sqrt{2}}\sigma(x)\exp\left(i\lambda^a\pi_a(x)\right) \tag{2.5}$$

(for complex $\chi(x)$, hermitean generators λ^a, $a = 1\ldots N-1$, and $\xi = \frac{1}{2}\sigma^2$). For $N = 2$, the action reads

$$S = \int d^dx\left\{\frac{1}{2}\left(\bar{Z} + \bar{Y}\xi\right)\partial_\mu\sigma\partial^\mu\sigma + \bar{Z}\xi\partial_\mu\pi\partial^\mu\pi + V(\xi)\right\} \tag{2.6}$$

whereas for $N \geq 3$ the kinetic term for π^a is generalized to a nonabelian nonlinear σ-model. One notices that $\bar{Z}$ determines the kinetic term for the Goldstone bosons π. The kinetic term for the radial mode σ is in general different, namely proportional to $\bar{Z} + \bar{Y}\xi$. The action for the N-component φ^4-theory obtains as a special case for

$$V = -\bar{\mu}^2\xi + \frac{1}{2}\bar{\lambda}\xi^2, \quad \bar{Z} = const., \quad \bar{Y} = 0 \tag{2.7}$$

with 'bare coupling' $\lambda_0 = \bar{\lambda}\bar{Z}^{-2}$.

The average action $\Gamma_k[\varphi]$ is a functional of the average field $\varphi(x)$. It is defined by

$$\exp(-\Gamma_k[\varphi]) = \int \mathcal{D}\chi P_k[\varphi,\chi]\exp(-S[\chi]) \tag{2.8}$$

The constraint P_k enforces the average of χ over a volume with size k^{-d} to equal the average field φ up to small fluctuations [3]. We write the constraint as a Gaussian in momentum space, with

$$\begin{aligned}\varphi(x) &= \sum_q \varphi(q)\exp(-iq_\mu x^\mu) \\ \varphi(-q) &= \varphi^*(q) \text{ for real } \varphi(x)\end{aligned} \tag{2.9}$$

and similar for χ, namely

$$P_k[\varphi,\chi] = \exp\left\{-\frac{1}{2}\Omega\sum_q \frac{q^2}{1-f_k^2(q)}\left(\varphi^\dagger(q) - f_k(q)\chi^\dagger(q)\right)\left(\varphi(q) - f_k(q)\chi(q)\right)\right\} \tag{2.10}$$

Here Ω denotes the total volume of space and should be taken to infinity at the end. The factor $\frac{1}{2}$ should be omitted for complex $\varphi(x)$ and $\chi(x)$. The function $f_k(q)$ determines the details of the averaging procedure: We define the average of χ over a volume $\sim k^{-d}$ as

$$\begin{aligned}\phi_k(x) &= \int d^dy f_k(x-y)\chi(y) \\ \phi_k(q) &= f_k(q)\chi(q)\end{aligned} \tag{2.11}$$

with f_k vanishing rapidely for $|x-y| \gg k^{-1}$. Different 'average schemes' correspond to different choices of f_k. We consider in particular a two parameter family of schemes parametrized by β and a

$$f_k(q) = \exp\left\{-a\left(\frac{q^2}{k^2}\right)^\beta\right\} \tag{2.12}$$

It may be reduced to a one parameter family with an implicite definition of $a(\beta)$ through

$$\beta = \frac{\exp 2a - 1}{2a} \tag{2.13}$$

The average action Γ_k is the effective action for averages of fields over a volume with size k^{-d} and describes the the physics at distance scales $\gtrsim k^{-1}$. (For more details on the properties of Γ_k compare ref.[3], with the identification $\nu = q^2(\frac{1}{2}q^2)$ for complex (real) scalar fields.) It can be obtained as the logarithm of the partition function with a constrained action S_k

$$\begin{aligned}\exp(-\Gamma_k[\varphi]) &= \int \mathcal{D}\chi \exp(-S_k[\varphi,\chi]) \\ S_k[\varphi,\chi] &= -\ln P_k[\varphi,\chi] + S[\chi]\end{aligned} \tag{2.14}$$

This makes the use of saddle point approximations and similar methods straightforward. Since Γ_k has the same symmetries as S we can expand in the number of derivatives and obtain in analogy to (2.1)

$$\Gamma_k = \int d^dx\Big\{U_k(\rho) + \frac{1}{2}Z_k(\rho)\partial_\mu\varphi^a\partial^\mu\varphi_a + \frac{1}{4}Y_k(\rho)\partial_\mu\rho\partial^\mu\rho + \dots\Big\} \tag{2.15}$$

$$\rho = \frac{1}{2}\varphi^a\varphi_a \tag{2.16}$$

(and similar for complex $\varphi_a(x)$ with $\rho = \varphi^\dagger\varphi$).

We want to understand the behaviour of the functions $U_k(\rho)$, $Z_k(\rho)$ and $Y_k(\rho)$ for a given action, i.e. given functions $V(\rho)$, $\bar{Z}(\rho)$ and $\bar{Y}(\rho)$. Even if we start with the 'classical action'

(2.7) for the φ^4-theory (regularized by a momentum cutoff Λ), the average potential $U_k(\rho)$ becomes a complicated function of ρ rather than a simple quartic polynomial. The wave function renormalization in the Goldstone direction, $Z_k(\rho)$, also depends on ρ and differs from the wave function renormalization for the radial mode which is given by

$$\tilde{Z}_k(\rho) = Z_k(\rho) + Y_k(\rho)\rho \tag{2.17}$$

Instead of dealing with general functions $V(\rho)$, $\bar{Z}(\rho)$, $\bar{Y}(\rho)$ we may also use a set of 'bare couplings' $\bar{\alpha}_i$

$$V = V(\bar{\alpha}_i;\xi),\ \ \bar{Z} = \bar{Z}(\bar{\alpha}_i;\xi),\ \ \bar{Y} = \bar{Y}(\bar{\alpha}_i;\xi) \tag{2.18}$$

to parametrize these functions. (They comprise mass term, quartic coupling etc.). Similarly, U_k, Z_k and Y_k can be described by an (infinite) set of couplings $\alpha_i(k)$ [1]

$$U_k = U_k(\alpha_i(k);\rho) \tag{2.19}$$

and similar for Z_k and Y_k. Our task consists in a computation of the 'average' couplings $\alpha_i(k)$ for given $\bar{\alpha}_i$.

We use the method of a renormalization group improved saddle point expansion. The scale dependence of the average action can be written in the form of evolution equations ($t = \ln(k/\Lambda)$)

$$\begin{aligned}\frac{\partial}{\partial t}U_k(\rho) &= \zeta_k(\rho) \\ \frac{\partial}{\partial t}Z_k(\rho) &= -\eta_k(\rho)Z_k(\rho) \\ \frac{\partial}{\partial t}\tilde{Z}_k(\rho) &= -\tilde{\eta}_k(\rho)\tilde{Z}_k(\rho)\end{aligned} \tag{2.20}$$

We first perform a one loop computation of U_k (sect. 4) and Z_k, Y_k (sect. 5). Taking the logarithmic derivative with respect to k gives ζ_k, η_k and $\tilde{\eta}_k$ as functions of k and the bare couplings $\bar{\alpha}_i$. The 'renormalization group improvement' replaces in ζ_k, η_k and $\tilde{\eta}_k$ the bare couplings $\bar{\alpha}_i$ by the couplings $\alpha_i(k)$. This transforms (2.20) into a coupled system of differential equations for $\alpha_i(k)$. It can be solved with 'initial values' $\alpha_i(k=\Lambda) = \bar{\alpha}_i$.

This procedure for the renormalization group improvement is commonly used in standard perturbation theory in four dimensions. For $d = 4$ and weak couplings the running of all dimensionless couplings is slow. The difference between $\alpha_i(k)$ and $\bar{\alpha}_i$ appears only in higher order in perturbation theory and is of no worry for a one loop calculation. For $d < 4$, however, we will see that some of the couplings run very fast such that some $\alpha_i(k)$ differ strongly from $\bar{\alpha}_i$. The renormalization group improvement is nevertheless justified by the following argument: The one loop contributions $U_k^{(1)}$, $Z_k^{(1)}$, $Y_k^{(1)}$ are given by momentum integrals which can be expressed in terms of the vertices and propagators of the theory. The 'average scale' k enters as an effective infrared cutoff through the propagators. A partial resummation of higher loop effects results in the replacement of the bare vertices (parametrized by $\bar{\alpha}_i$) by n-point functions evaluated at an appropriate momentum scale q_o^2. We argue that $q_o^2 = k^2$ is the most reasonable choice. Indeed, the contribution of the ultraviolet region $q^2 \gg k^2$ to the momentum integrals for ζ_k, η_k and $\tilde{\eta}_k$ turns out to be exponentially suppressed. For

[1] As a possible choice for α_i one could take the coefficients of an expansion of U_k, Z_k and Y_k in powers of ρ. We will use later a more convenient set of couplings.

most of the quantities studied in this paper the infrared cutoff implied by the averaging is strong enough to suppress also the contributions from $q^2 \ll k^2$. Then ζ_k etc. are entirely dominated by a small momentum range $q^2 \approx k^2$ implying the choice $q_o^2 = k^2$. Finally, the n-point functions at $q_o^2 = k^2$ are related to $\alpha_i(k)$ in essentially the same way as the 'bare' vertices are related to $\bar{\alpha}_i$. The renormalization group improvement $\bar{\alpha}_i \to \alpha_i(k)$ therefore corresponds to a partial resummation of higher orders in the loop expansion. We emphasize that this argument is only applicable to the derivatives ζ_k, η_k, $\bar{\eta}_k$ and not to $U_k^{(1)}$, $Z_k^{(1)}$, $Y_k^{(1)}$ for which all modes with $q^2 \gtrsim k^2$ can contribute. Furthermore, we will encounter examples where the contribution from the infrared region $q^2 \ll k^2$ is not suppressed effectively enough. For this case we will give in appendix E a more detailed discussion of the renormalization group improvement which goes beyond the replacement $\bar{\alpha}_i \to \alpha_i(k)$.

3 The classical average action

The classical field equations are derived from the constrained action S_k (2.14). They simplify considerably [2, 3] for all solutions for which $\xi = \frac{1}{2}\chi^a\chi_a$ and $\tau = \frac{1}{2}\partial_\mu\chi^a\partial^\mu\chi_a$ are independent of x^μ. For real scalar fields and with $V' = \partial V/\partial\xi$ etc. they read in this case

$$\chi^a(q) = f_k(q)\left\{1 + (1 - f_k^2(q))\left(\bar{Z} - 1 + \frac{V' + \bar{Z}'\tau}{q^2}\right)\right\}^{-1} \varphi^a(q) \tag{3.1}$$

Consider first the configuration $\varphi(x) = \varphi = $ const. and choose $\varphi^a = \varphi\delta_1^a$. There is always a solution $\chi_0^a(x) = $ const. For $\beta > 1$ it is simply

$$\chi_0^a(x) = \varphi\delta_1^a; \quad \frac{1}{2}\chi_0^a\chi_{0a} = \rho = \frac{1}{2}\varphi^2 \tag{3.2}$$

We expand S_k up to terms quadratic in small fluctuations $\delta\chi(q)$ around the solution (3.2),

$$S_k = \Omega V(\rho) + S_k^{(2)} \tag{3.3}$$

$$\begin{aligned} S_k^{(2)} = \frac{1}{2}\Omega\sum_q\Big\{&\left(\frac{q^2 f_k^2(q)}{1 - f_k^2(q)} + \bar{Z}(\rho)q^2 + V'(\rho)\right)\delta\chi_a^*(q)\delta\chi^a(q) \\ &+\rho\left(\bar{Y}(\rho)q^2 + 2V''(\rho)\right)\delta\chi_1^*(q)\delta\chi_1(q)\Big\} \end{aligned} \tag{3.4}$$

The constraint modifies the propagator leading to an effective infrared cutoff at a scale around k. We denote

$$\begin{aligned} P_z(q) &= \frac{q^2}{1 - f_k^2(q)} + \left(\bar{Z}(\rho) - 1\right)q^2 \\ P_y(q) &= \frac{q^2}{1 - f_k^2(q)} + \left(\rho\bar{Y}(\rho) + \bar{Z}(\rho) - 1\right)q^2 \end{aligned} \tag{3.5}$$

$$\bar{k}_z^2 = \min P_z(q), \quad \bar{k}_y^2 = \min P_y(q) \tag{3.6}$$

The constant solution is a local minimum of S_k if

$$\begin{aligned} \bar{k}_z^2 &+ V'(\rho) > 0 \\ \bar{k}_y^2 &+ V'(\rho) + 2V''(\rho)\rho > 0 \end{aligned} \tag{3.7}$$

Only if these two conditions are satisfied a saddlepoint approximation around the constant solution can be used. In this case the classical potential $U_k^{(0)}$ is simply given by V

$$U_k^{(0)}(\rho) = V(\rho) \tag{3.8}$$

If (3.7) is violated, spin waves [2] ($N \geq 2$) or kinks ($N = 1$) will minimize the constraint action, leading to a different classical potential $U_k^{(0)}$.

In order to compute the classical approximation to Z_k and Y_k we consider configurations with a small space dependent piece added to a constant field which read in momentum space

$$\begin{aligned} \varphi_1(0) &= \varphi \\ \varphi_a(Q) &= \delta\varphi_a, \quad \varphi_a(-Q) = \delta\varphi_a^* \end{aligned} \tag{3.9}$$

Exact solutions of the field equations are known for finite $\delta\varphi$ if $N \geq 3$, $\delta\varphi_3 = i\delta\varphi_2$, $\delta\varphi_a = 0$ otherwise [3]. Here we can restrict our discussion to sufficiently small $\delta\varphi$ such that only terms quadratic in the fluctuations around the constant solution have to be kept

$$\begin{aligned} S_k \approx \Omega V(\rho) + S_k^{(2)} + \frac{1}{2}\Omega \sum_{q=\pm Q} \Big\{ & \frac{q^2}{1-f_k^2(q)} \delta\varphi_a^*(q)\delta\varphi^a(q) \\ & - \frac{q^2 f_k(q)}{1-f_k^2(q)} \Big(\delta\varphi_a^*(q)\delta\chi^a(q) + \delta\chi_a^*(q)\delta\varphi^a(q)\Big)\Big\} \end{aligned} \tag{3.10}$$

This yields the solution

$$\begin{aligned} \chi_0^a(Q) &= A_z(Q)\delta\varphi^a(Q) \quad \text{for} \quad a \neq 1 \\ \chi_0^1(Q) &= A_y(Q)\delta\varphi^1(Q) \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} A_z(Q) &= f_k(Q)\left\{1 + \left(1 - f_k^2(Q)\right)\left(\bar{Z} - 1 + \frac{V'}{Q^2}\right)\right\}^{-1} \\ A_y(Q) &= f_k(Q)\left\{1 + \left(1 - f_k^2(Q)\right)\left(\bar{Z} - 1 + \bar{Y}\rho + \frac{V' + 2V''\rho}{Q^2}\right)\right\}^{-1} \end{aligned} \tag{3.12}$$

Inserting this solution into S_k gives the classical average action

$$\begin{aligned} \Gamma_k^{(0)} &= \Omega V(\rho) + \Omega \sum_a \delta\varphi_a^*(Q)\delta\varphi_a(Q)\Big\{A_a^2(Q)\Big(V'(\rho) + 2V''(\rho)\rho\delta_{a1} + \\ &+ Q^2\bar{Z}(\rho) + Q^2\bar{Y}(\rho)\rho\delta_{a1}\Big) + Q^2 B_a(Q)\Big\} \end{aligned} \tag{3.13}$$

with

$$B_a(Q) = \frac{(1 - f_k(Q)A_a(Q))^2}{1 - f_k^2(Q)} \tag{3.14}$$

and $A_1 \equiv A_y$, $A_a = A_z$, for $a \neq 1$. For small $Q^2 \ll k^2$ one finds with (2.12)

$$A_y(Q) = 1 - 2a\frac{V' + 2V''\rho}{k^2}\left(\frac{Q^2}{k^2}\right)^{\beta-1} + O\Big(\left(\frac{Q^2}{k^2}\right)^{\beta}\Big) \tag{3.15}$$

$$B_y(Q) = 2a\left(\frac{V' + 2V''\rho}{k^2}\right)^2\left(\frac{Q^2}{k^2}\right)^{\beta-2} + O\Big(\left(\frac{Q^2}{k^2}\right)^{\beta-1}\Big) \tag{3.16}$$

and similar for A_z, B_z. We restrict ourself to $\beta > 2$ where there is no classical wave function renormalization in lowest order in Q^2.

$$\begin{aligned} Z_k^{(0)}(\rho) &= \bar{Z}(\rho) \\ Y_k^{(0)}(\rho) &= \bar{Y}(\rho) \\ \tilde{Z}_k^{(0)}(\rho) &= \bar{Z}(\rho) + \rho\bar{Y}(\rho) \end{aligned} \tag{3.17}$$

The classical average action is local for $Q^2 \ll k^2$ whereas the strong nonlocality at $Q^2 \approx k^2$ reflects the effective cutoff at this scale. Also, for infinitesimal $\delta\varphi_a$ the condition (3.7) remains unchanged. The modifications of the classical wave function renormalization for $\beta = 2$ is briefly outlined in appendix A.

4 Evolution equation for the average potential

The one loop average potential obtains from (3.4) by Gaussian integration

$$\begin{aligned} U_k^{(1)}(\rho) &= \frac{1}{2}(2\pi)^{-d}\int d^dq\Big\{\ln\left(P_y(q)+V'(\rho)+2V''(\rho)\rho\right) \\ &+ (N-1)\ln\left(P_z(q)+V'(\rho)\right)\Big\}+const. \end{aligned} \tag{4.1}$$

Differentiation with respect to $\ln k$ yields the one loop evolution equation

$$\begin{aligned} \zeta_k^{(1)} &= \frac{\partial}{\partial t}U_k^{(1)} = \frac{1}{2}(2\pi)^{-d}\int d^dq\frac{\partial}{\partial t}P(q) \\ &\Big\{\left(P_y(q)+V'+2V''\rho\right)^{-1}+(N-1)\left(P_z(q)+V'\right)^{-1}\Big\} \end{aligned} \tag{4.2}$$

with (2.12)

$$\frac{\partial}{\partial t}P(q)\equiv\frac{\partial}{\partial t}P_z(q)\equiv\frac{\partial}{\partial t}P_y(q)=k\frac{\partial}{\partial k}\left(\frac{q^2}{1-f_k^2(q)}\right)=4a\beta\left(\frac{q^2}{k^2}\right)^{\beta}\frac{q^2f_k^2(q)}{\left(1-f_k^2(q)\right)^2} \tag{4.3}$$

The integral (4.2) is ultraviolet finite due to the exponential decay of $\frac{\partial}{\partial t}P(q)$ for large $q^2\gg k^2$. It is also infrared finite since $\frac{\partial}{\partial t}P(q)\sim 2\beta P(q)$ for $q^2\ll k^2$. The integral is therefore dominated by q^2 in the vicinity of k^2. As discussed in sect. 2 we improve the evolution equation by substituting in (4.2) U_k, Z_k and Y_k instead of the bare quantities V, $\bar{Z}$ and $\bar{Y}$. Using $x=q^2$ we obtain for $\zeta_k(\rho)$ a one dimensional integral which only depends on $U_k'(\rho)$, $U_k''(\rho)$, $Z_k(\rho)$ and $Y_k(\rho)$

$$\zeta_k=v_d\int_0^\infty dx\, x^{\frac{d}{2}-1}\left(\frac{\frac{\partial}{\partial t}P_y}{P_y+U_k'+2U_k''\rho}+\frac{(N-1)\frac{\partial}{\partial t}P_z}{P_z+U_k'}\right) \tag{4.4}$$

with

$$v_d^{-1}=2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right) \tag{4.5}$$

In the spontaneously broken regime the average potential U_k has its minimum at $\rho_0\neq 0$, determined by

$$U_k'(\rho_0)=0 \tag{4.6}$$

We are interested in the scale dependence of the minimum at $\rho_0(k)$ which obtains by taking the total t-derivative of eq. (4.6)

$$U_k''(\rho_0)\frac{d\rho_0}{dt}=-\frac{\partial U_k'}{\partial t}(\rho_0) \tag{4.7}$$

Here the partial derivative $\partial U_k'/\partial t$ should be taken at fixed ρ_0. We also define

$$\bar\lambda=U_k''(\rho_0(k)) \tag{4.8}$$

The evolution equations for the scale dependence of ρ_0 and $\bar\lambda$ are easily derived by differentiation of (4.4) with respect to ρ.

$$\begin{aligned} \bar\delta &= \frac{d\rho_0}{dt}=-\frac{1}{\bar\lambda}\frac{\partial\zeta_k}{\partial\rho}(\rho_0) \\ &= v_d\int dx\, x^{\frac{d}{2}-1}\frac{\partial P}{\partial t}\Big\{\frac{3+2U'''\rho_0/\bar\lambda+(Y+Y'\rho_0+Z')x/\bar\lambda}{\left(P_y+2\bar\lambda\rho_0\right)^2} \\ &+ (N-1)\frac{1+Z'x/\bar\lambda}{P_z^2}\Big\} \end{aligned} \tag{4.9}$$

$$\bar\beta_\lambda=\frac{d\bar\lambda}{dt} = \frac{\partial^2\zeta_k}{\partial\rho^2}(\rho_0)-U'''(\rho_0)\bar\delta \tag{4.10}$$

$$\begin{aligned} &= v_d\int dx x^{\frac{d}{2}-1}\frac{\partial P}{\partial t}\Big\{2\frac{(3\bar\lambda+2U'''\rho_0+(Y+Y'\rho_0+Z')x)^2}{(P_y+2\bar\lambda\rho_0)^3} \\ &+ 2(N-1)\frac{(\bar\lambda+Z'x)^2}{P_z^3}-\frac{5U'''+2U^{(4)}\rho_0+(2Y'+Y''\rho_0+Z'')x}{\left(P_y+2\bar\lambda\rho_0\right)^2} \\ &- (N-1)\frac{U'''+Z''x}{P_z^2}\Big\}+U'''\bar\delta \end{aligned} \tag{4.11}$$

Here we use the shorthands Y, Z', U''' etc. instead of $Y_k(\rho_0)$, $Z_k'(\rho_0)$, $U_k'''(\rho_0)$. It is convenient to introduce the dimensionless integrals

$$\begin{aligned} L_n^d(w) &= k^{2n-d}\int dx x^{\frac{d}{2}-n-1}\frac{\partial}{\partial t}\left(\frac{P_z(x)+w}{x}\right)^{-n} \\ &= -nk^{2n-d}\int dx x^{\frac{d}{2}-1}\frac{\partial P}{\partial t}\left(P_z+w\right)^{-(n+1)} \end{aligned} \tag{4.12}$$

and similar for $\bar L_n^d$ by replacing $P_z\to P_y$. In terms of these integrals the evolution equations read

$$\begin{aligned} \bar\delta &= -v_dk^{d-2}\Big\{\left(3+2U'''\rho_0/\bar\lambda\right)\bar L_1^d\left(2\bar\lambda\rho_0\right) \\ &+ (N-1)L_1^d(0)\Big\} \\ &- v_dk^d\Big\{\frac{(Y+Y'\rho_0+Z')}{\bar\lambda}\bar L_1^{d+2}\left(2\bar\lambda\rho_0\right) \\ &+ \frac{(N-1)Z'}{\bar\lambda}L_1^{d+2}(0)\Big\} \\ \bar\beta_\lambda &= -v_dk^{d-4}\Big\{\left(3\bar\lambda+2U'''\rho_0\right)^2\bar L_2^d\left(2\bar\lambda\rho_0\right) \\ &+ k^2\left(Y+Y'\rho_0+Z'\right)\left(6\bar\lambda+4U'''\rho_0\right)\bar L_2^{d+2}\left(2\bar\lambda\rho_0\right) \\ &+ k^4\left(Y+Y'\rho_0+Z'\right)^2\bar L_2^{d+4}\left(2\bar\lambda\rho_0\right) \\ &+ (N-1)\bar\lambda^2L_2^d(0)+2(N-1)k^2Z'\bar\lambda L_2^{d+2}(0) \\ &+ (N-1)k^4Z'^2L_2^{d+4}(0)\Big\} \\ &+ v_dk^{d-2}\Big\{\left(5U'''+2U^{(4)}\rho_0\right)\bar L_1^d\left(2\bar\lambda\rho_0\right) \end{aligned} \tag{4.13}$$

$$
\begin{aligned}
&+\; k^2\left(2Y'+Y''\rho_0+Z''\right)\tilde{L}_1^{d+2}\left(2\bar{\lambda}\rho_0\right)\\
&+\; (N-1)U'''L_1^d(0)+(N-1)k^2Z''L_1^{d+2}(0)\Big\}\\
&+\; U'''\tilde{\delta}
\end{aligned}
\tag{4.14}
$$

The integrals L_n^d and $\tilde{L}_n^d$ are evaluated in appendix B similarly as in ref. [4]. We define (for $n>0$)

$$
\begin{aligned}
L_n^d(0) &= -2Z^{-n}l_n^d\\
\tilde{L}_n^d\left(2\bar{\lambda}\rho_0\right)/L_n^d(0) &= s_n^d\left(2\bar{\lambda}\rho_0\right)
\end{aligned}
\tag{4.15}
$$

and find that the β dependent quantities l_n^d are positive and of order one. They depend only weakly on Z. For the special case $d=2n$ the integrand is a total derivative and one finds the exact result

$$
l_n^{2n}=1 \tag{4.16}
$$

The function s_n^d has the qualitative properties of a step function with

$$
s_n^d\left(2\bar{\lambda}\rho_0\right)\approx\begin{cases}\left(Z/\tilde{Z}\right)^n & \text{for}\quad 2\bar{\lambda}\rho_0\ll\tilde{Z}k^2\\ 0 & \text{for}\quad 2\bar{\lambda}\rho_0\gg\tilde{Z}k^2\end{cases}
\tag{4.17}
$$

In particular one finds for large β

$$
\lim_{\beta\to\infty} l_n^d = 1 \tag{4.18}
$$

$$
\lim_{\beta\to\infty} s_n^d\left(2\bar{\lambda}\rho_0\right) = \left(\frac{Zk^2}{\tilde{Z}k^2+2\bar{\lambda}\rho_0}\right)^n \tag{4.19}
$$

We observe that $\tilde{\delta}$ is expressed in terms of $\bar{\lambda}$, U''', Z, Z', Y and Y' whereas $\tilde{\beta}_\lambda$ involves one further ρ-derivative of these quantities. One can derive the evolution equations for all of these quantities. They will involve even higher ρ-derivatives on the r.h.s. . For example, $\frac{\partial}{\partial t}U'''$ involves $U^{(5)}$. In consequence, the evolution equations constitute a complicated system of coupled nonlinear differential equations for infinitely many functions. The only accessible analytic way of dealing with such a system is a truncation to a finite number of functions. We will concentrate in this paper on a standard kinetic term, i.e. $Z(\rho)=\tilde{Z}(\rho)=$ const. . This corresponds to the truncation

$$
\begin{aligned}
Z' &= Z''=\cdots Z^{(n)}=0\\
Y &= Y'=\cdots Y^{(n)}=0
\end{aligned}
\tag{4.20}
$$

on the r.h.s. of (4.13) and (4.14). This approximation is valid as long as

$$
\begin{aligned}
|\,Z'k^2\,| &\ll \bar{\lambda},\quad |\,\tilde{Z}'k^2s\,|\ll\bar{\lambda}\\
|\,Z''k^4\,| &\ll \bar{\lambda}^2,\quad |\,\tilde{Z}''k^4s\,|\ll\bar{\lambda}^2
\end{aligned}
\tag{4.21}
$$

(Here s is used symbolically to denote an additional suppression factor for $Zk^2\ll 2\bar{\lambda}\rho_0$ as arising, for example, from $s_n^d\left(2\bar{\lambda}\rho_0\right)$.) We can then take $\tilde{L}_n^d\left(2\bar{\lambda}\rho_0\right)=L_n^d\left(2\bar{\lambda}\rho_0\right)$ and (4.13), (4.14) simplify considerably

$$
\tilde{\delta} = 2v_dl_1^dk^{d-2}Z^{-1}\left\{\left(3+2U'''\rho_0/\bar{\lambda}\right)s_1^d\left(2\bar{\lambda}\rho_0\right)+N-1\right\} \tag{4.22}
$$

$$
\begin{aligned}
\tilde{\beta}_\lambda &= 2v_dl_2^dk^{d-4}Z^{-2}\left\{\left(9\bar{\lambda}^2+12\bar{\lambda}U'''\rho_0+4(U''')^2\rho_0^2\right)s_2^d\left(2\bar{\lambda}\rho_0\right)+(N-1)\bar{\lambda}^2\right\}\\
&\quad -4v_dl_1^ds_1^d\left(2\bar{\lambda}\rho_0\right)k^{d-2}Z^{-1}\left\{U'''+U^{(4)}\rho_0-\frac{(U''')^2}{\bar{\lambda}}\rho_0\right\}
\end{aligned}
\tag{4.23}
$$

In addition, we consider in this paper only the potential in the vicinity of ρ_0 which we approximate by a φ^4-potential. This corresponds to the truncation

$$
U'''(\rho_0)=\cdots U^{(n)}(\rho_0)=0 \tag{4.24}
$$

which is valid for

$$
|\,U'''\rho_0s\,|\ll\bar{\lambda},\quad |\,Zk^2U'''s\,|\ll\bar{\lambda}^2,\quad |\,Zk^2U^{(4)}\rho_0s\,|\ll\bar{\lambda}^2 \tag{4.25}
$$

We observe that this truncation automatically holds if the terms $\sim s_n^d\left(2\bar{\lambda}\rho_0\right)$ can be neglected. With (4.22) we recover the result of ref. [3] up to effects from the wave function renormalization Z

$$
\tilde{\delta} = 2v_dl_1^dk^{d-2}Z^{-1}\left\{N-1+3s_1^d\left(2\bar{\lambda}\rho_0\right)\right\} \tag{4.26}
$$

$$
\tilde{\beta}_\lambda = 2v_dl_2^dk^{d-4}Z^{-2}\bar{\lambda}^2\left\{N-1+9s_2^d\left(2\bar{\lambda}\rho_0\right)\right\} \tag{4.27}
$$

The explicit dependence of the evolution equations on the scale k and the wave function renormalization Z can be removed[2] by a redefinition of variables. We introduce the dimensionless normalized field κ which corresponds to the potential minimum

$$
\kappa = k^{2-d}Z\rho_0 \tag{4.28}
$$

and the dimensionless quartic scalar coupling

$$
\lambda = k^{d-4}Z^{-2}\bar{\lambda} \tag{4.29}
$$

The corresponding evolution equations read

$$
\frac{\partial\kappa}{\partial t} = \beta_\kappa = (2-d-\eta)\kappa+2v_dl_1^d\left(N-1+3s_1^d\left(2\bar{\lambda}\rho_0\right)\right) \tag{4.30}
$$

$$
\frac{\partial\lambda}{\partial t} = \beta_\lambda = (d-4+2\eta)\lambda+2v_dl_2^d\lambda^2\left(N-1+9s_2^d\left(2\bar{\lambda}\rho_0\right)\right) \tag{4.31}
$$

The anomalous dimension $\eta=-\partial\ln Z/\partial t$ will be calculated in the next sections. We will discuss approximation schemes where the contribution of the anomalous dimension to β_λ can be neglected in a first approximation whereas we have to keep it for β_κ in some cases. In four dimensions and for $N\geq 2$ the theory is infrared free ($\lim_{k\to 0}\lambda=0$) in the spontanously broken phase even for a finite momentum cutoff Λ . In contrast, the logarithmic running of λ is finally stopped by the scalar mass in the symmetric phase. In two and three dimensions one finds an infrared fixpoint λ whose approximate value depends on the ratio

$$
2\bar{\lambda}\rho_0/Zk^2 = 2\lambda\kappa \tag{4.32}
$$

[2] This holds in the limit where l_n^d is independent of Z and s_n^d depends only on the ratio $\bar{\lambda}\rho_0/Zk^2=\lambda\kappa$.

We distinguish between two regimes
Linear regime : $\lambda\kappa \ll \frac{1}{2}$

$$\lambda_* = \frac{4-d}{2v_d l_2^d (N+8)} \tag{4.33}$$

Goldstone regime : $\lambda\kappa \gg \frac{1}{2}$

$$\lambda_* = \frac{4-d}{2v_d l_2^d (N-1)} \tag{4.34}$$

(The last equation applies only for $N \geq 2$.) The fixpoint is approached rather rapidly. In leading order $\tilde{\lambda}$ scales according to its canonical dimension

$$\tilde{\lambda}(k) \sim k^{4-d} \tag{4.35}$$

We observe for large N

$$v_d \lambda_* \sim N^{-1} \tag{4.36}$$

For $d < 4$ a perturbative expansion in powers of $v_d\lambda$ is questionable for moderate values of N.

5 Wave function renormalization

For a one loop computation of the wave function renormalization effects we need an expansion of S_k around the solution (3.11) up to quadratic fluctuations. We are only interested in lowest order in Q^2 and therefore simplify eq. (3.11) by taking $A(Q) = 1$. One obtains

$$\begin{aligned} S_k^{(2)} &= \int d^d x \Big[\frac{1}{2} \delta\chi^a \delta\chi^b \Big\{ V'(\rho+\Delta)\,\delta_{ab} \\ &+ V''(\rho+\Delta)\,\varphi_a\varphi_b + \frac{1}{2}\bar{Z}'(\rho+\Delta)\,\delta_{ab}\partial_\mu\delta\varphi^c\partial^\mu\delta\varphi_c \\ &+ \frac{1}{2}\bar{Z}''(\rho+\Delta)\,\varphi_a\varphi_b\partial_\mu\delta\varphi^c\partial^\mu\delta\varphi_c + \frac{1}{2}\bar{Y}(\rho+\Delta)\,\partial_\mu\delta\varphi_a\partial^\mu\delta\varphi_b \\ &+ \bar{Y}'(\rho+\Delta)\,\varphi_a\varphi_c\partial_\mu\delta\varphi^c\partial^\mu\delta\varphi_b \\ &+ \frac{1}{4}\Big(\bar{Y}'(\rho+\Delta)\,\delta_{ab} + \bar{Y}''(\rho+\Delta)\,\varphi_a\varphi_b\Big)\,\varphi^c\varphi^d\partial_\mu\delta\varphi_c\partial^\mu\delta\varphi_d \Big\} \\ &+ (\partial_\mu\delta\chi^a)\,\delta\chi^b \Big\{ \bar{Z}'(\rho+\Delta)\,\varphi_b\partial^\mu\delta\varphi_a \\ &+ \frac{1}{2}\bar{Y}(\rho+\Delta)\,(\varphi_a\partial^\mu\delta\varphi_b + \varphi^c\partial^\mu\delta\varphi_c\delta_{ab}) \\ &+ \frac{1}{2}\bar{Y}'(\rho+\Delta)\,\varphi_a\varphi_b\varphi_c\partial^\mu\delta\varphi^c \Big\} \\ &+ \frac{1}{2}\partial_\mu\delta\chi^a\partial^\mu\delta\chi^b \Big\{ \bar{Z}(\rho+\Delta)\,\delta_{ab} + \frac{1}{2}\bar{Y}(\rho+\Delta)\,\varphi_a\varphi_b \Big\} \Big] \\ &+ \frac{1}{2}\Omega\sum_q \frac{q^2 f_k^2(q)}{1-f_k^2(q)}\delta\chi_a^*(q)\delta\chi^a(q) \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} \Delta &= \frac{1}{2}\delta\varphi_a\delta\varphi^a + \varphi\delta\varphi_1 \quad , \rho = \frac{1}{2}\varphi^2 \\ \varphi_a &= \varphi\delta_{a1} + \delta\varphi_a \end{aligned} \tag{5.2}$$

Writing (with $S_{ab}(q,q') = S_{ba}^*(q',q)$)

$$S_k^{(2)} = \frac{1}{2}\Omega\sum_{q,q'}\delta\chi^{a*}(q)S_{ab}(q,q')\delta\chi^b(q') \tag{5.3}$$

the one loop contribution to the average action reads

$$\Gamma_k^{(1)} = \frac{1}{2}\ln \mathrm{Det}\; S_{ab}(q,q') + const \tag{5.4}$$

We first compute Z_k for $N \geq 2$ by choosing the configuration (in momentum space)

$$\begin{aligned} \varphi_1(0) &= \varphi \\ \delta\varphi_2(Q) &= \delta\varphi \end{aligned} \tag{5.5}$$

The last term in (2.11) vanishes in quadratic order in $\delta\varphi$ and we have

$$Z_k(\rho) = \frac{1}{\Omega} \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \frac{\partial \Gamma_k}{\partial(\delta\varphi^* \delta\varphi)} (\varphi, \delta\varphi = 0) \tag{5.6}$$

We expand $S_{ab}(q, q')$ in terms containing zero, one or two powers of $\delta\varphi$

$$S(q,q') = S_0(q,q') + S_1(q,q') + S_2(q,q') + \cdots \tag{5.7}$$

The lowest order term is diagonal

$$\begin{aligned} S_0(q,q') &= \hat{S}_0(q)\delta(q-q') \\ \left(\hat{S}_0\right)_{ab} &= \left(\frac{q^2 f_k^2(q)}{1-f_k^2(q)} + \bar{Z}(\rho)q^2 + V'(\rho)\right)\delta_{ab} \\ &+ \left(\bar{Y}(\rho)\rho q^2 + 2V''(\rho)\rho\right)\delta_{a1}\delta_{b1} \end{aligned} \tag{5.8}$$

such that

$$\left(\hat{S}_0\right)_{aa} = \begin{cases} P_z + V' & \text{for } a \neq 1 \\ P_y + V' + 2V''\rho & \text{for } a = 1 \end{cases} \tag{5.9}$$

The first order term S_1 has only an off diagonal contribution

$$\begin{aligned} S_1(q,q') &= S_1^\dagger(q',q) \\ (S_1)_{11} &= (S_1)_{22} = 0,\ (S_1)_{ab} = 0 \text{ for } a,b > 2 \\ (S_1)_{12} &= \hat{S}_1(q,Q)\delta(q-q'-Q) + \hat{S}_1^*(q,-Q)\delta(q-q'+Q) \end{aligned} \tag{5.10}$$

with

$$\hat{S}_1 = V''\varphi\delta\varphi + \frac{1}{2}\bar{Y}\varphi\delta\varphi q^2 - \bar{Z}'\varphi\delta\varphi(q_\mu - Q_\mu)Q^\mu \tag{5.11}$$

We will only need the diagonal elements of S_2, namely

$$\begin{aligned} (S_2)_{aa}(q,q) &= \delta\varphi^*\delta\varphi\,(V'' + 2V''\delta_{a2} + 2V'''\rho\delta_{a1}) \\ &+ q^2\delta\varphi^*\delta\varphi\left(\bar{Z}' + \bar{Y}\delta_{a2} + 2\bar{Y}'\rho\delta_{a1}\right) \\ &+ Q^2\delta\varphi^*\delta\varphi\left(\bar{Z}' + \bar{Y}\delta_{a2} + 2\bar{Z}''\rho\delta_{a1}\right) \end{aligned} \tag{5.12}$$

The determinant in (5.4.) can be expanded in the small quantity $\delta\varphi$ (for $\Omega \to \infty$)

$$\begin{aligned} \Omega^{-1}\Gamma_k^{(1)} &= \frac{1}{2}(2\pi)^{-d}\int d^dq\Big\{\ln\det\hat{S}_0(q) \\ &+ tr\left[\hat{S}_0^{-1}(q)S_2(q,q)\right] \\ &- \left(\hat{S}_0^{-1}(q)\right)_{11}\left(\hat{S}_0^{-1}(q-Q)\right)_{22}\hat{S}_1(q,Q)\hat{S}_1^*(q,Q) \\ &- \left(\hat{S}_0^{-1}(q)\right)_{11}\left(\hat{S}_0^{-1}(q+Q)\right)_{22}\hat{S}_1(q,-Q)\hat{S}_1^*(q,-Q)\Big\} \end{aligned} \tag{5.13}$$

The first term accounts for the one loop contribution to the average potential whereas the other terms correspond to the graphs in fig. 1, evaluated in a constant background field φ

with modified inverse propagators P_z, P_y. They contribute

$$\begin{aligned} Z_k^{(a)} &= \frac{1}{2}(2\pi)^{-d}\int d^dq\,(P_z + V')^{-1}\left[(N-1)\bar{Z}' + \bar{Y}\right] \\ &+ \frac{1}{2}(2\pi)^{-d}\int d^dq\,(P_y + V' + 2V''\rho)^{-1}\left[\bar{Z}' + 2\bar{Z}''\rho\right] \end{aligned} \tag{5.14}$$

$$\begin{aligned} Z_k^{(b)} &= -2\rho\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\Big\{\frac{1}{2}(2\pi)^{-d}\int d^dq\,(P_y(q) + V' + 2V''\rho)^{-1} \\ &\quad (P_z(q-Q) + V')^{-1}\left[V''^2 + V''\bar{Y}q^2 + \frac{1}{4}\bar{Y}^2q^4\right. \\ &- \left.\bar{Z}'\left(2V'' + \bar{Y}q^2\right)\left(qQ - Q^2\right) + \bar{Z}'^2(qQ)^2\right] + (Q \leftrightarrow -Q)\Big\} \end{aligned} \tag{5.15}$$

and we obtain the one loop expression[3]

$$Z_k^{(1)} = \bar{Z} + Z_k^{(a)} + Z_k^{(b)} \tag{5.16}$$

We also want to compute the quantity Y_k or the wave function renormalization of the radial mode $\tilde{Z}_k = Z_k + Y_k\rho$ ($\tilde{Z}_k \equiv Z_k$ for $N = 1$). Instead of the Goldstone fluctuation (5.5) we now consider a purely radial configuration

$$\begin{aligned} \varphi_1(0) &= \varphi \\ \delta\varphi_1(Q) &= \delta\varphi \end{aligned} \tag{5.17}$$

such that

$$\tilde{Z}_k = \frac{1}{\Omega}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\frac{\partial\Gamma_k}{\partial(\delta\varphi^*\delta\varphi)}(\varphi,\ \delta\varphi = 0) \tag{5.18}$$

Using again the expansion (5.7) for S_{ab} the lowest order term S_0 coincides with (5.9). The first and second order term are now diagonal in internal space

$$\begin{aligned} S_1(q,q') &= \hat{S}_1(q,Q)\delta(q-q'-Q) + \hat{S}_1^*(q,-Q)\delta(q-q'+Q) \\ \hat{S}_1(q,Q) &= \varphi\delta\varphi\Big\{\left[V'' + \left(q^2 - qQ\right)\bar{Z}' + \frac{1}{2}Q^2\bar{Y}\right]\delta_{ab} \\ &+ \left[2V'' + 2V'''\rho + \left(q^2 - qQ\right)\left(\bar{Y} + \bar{Y}'\rho\right) + Q^2\left(\bar{Z}' + \frac{1}{2}\bar{Y} + \bar{Y}'\rho\right)\right]\delta_{a1}\delta_{b1}\Big\} \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} (S_2)_{aa}(q,q) &= \delta\varphi^*\delta\varphi\Big\{(V'' + 2V'''\rho)\,\delta_{ab} + \left(2V'' + 10V'''\rho + 4V^{(4)}\rho^2\right)\delta_{a1}\delta_{b1} \\ &+ \left(\bar{Z}' + \bar{Y} + \bar{Y}'\rho\right)Q^2\delta_{ab} + \left(2\bar{Z}''\rho + 4\bar{Y}'\rho + 2\bar{Y}''\rho^2\right)Q^2\delta_{a1}\delta_{b1} \\ &+ \left(\bar{Z}' + 2\bar{Z}''\rho\right)q^2\delta_{ab} + \left(\bar{Y} + 5\bar{Y}'\rho + 2\bar{Y}''\rho^2\right)q^2\delta_{a1}\delta_{b1}\Big\} \end{aligned} \tag{5.20}$$

[3] The careful reader may have noted that (5.5) is not exactly a Goldstone configuration since ρ is changed in order $\delta\varphi^*\delta\varphi$. The appropriate modification $\varphi_1(0) = \varphi - \delta\varphi^*\delta\varphi/\varphi$, $\varphi_1(2Q) = -\delta\varphi^2/2\varphi$ does not introduce a Q^2 dependent term in order $\delta\varphi^*\delta\varphi$. It can therefore be neglected for a computation of Z_k. Its net effect is a cancellation of the Q^2 independent terms in (5.12) which are replaced by $2\delta\varphi^*\delta\varphi V''(\delta_{a2} - \delta_{a1}) + q^2\delta\varphi^*\delta\varphi\bar{Y}(\delta_{a2} - \delta_{a1})$.

We use again the expansion of the determinant

$$\begin{aligned}
\Omega^{-1}\Gamma_k^{(1)} &= \frac{1}{2}(2\pi)^{-d}\int d^dq\Big\{\ln\det\hat S_0 + \sum_a\left(\hat S_0^{-1}\right)_{aa}(q)(S_2)_{aa}(q,q)\\
&- \frac{1}{2}\sum_a\Big[\left(\hat S_0^{-1}\right)_{aa}(q)\left(\hat S_0^{-1}\right)_{aa}(q-Q)\left(\hat S_1\right)_{aa}(q,Q)\left(\hat S_1^*\right)_{aa}(q,Q)\\
&+ \left(\hat S_0^{-1}\right)_{aa}(q)\left(\hat S_0^{-1}\right)_{aa}(q+Q)\left(\hat S_1\right)_{aa}(q,-Q)\left(\hat S_1^*\right)_{aa}(q,-Q)\Big]\Big\} \qquad (5.21)
\end{aligned}$$

and obtain for the wave function renormalization

$$\bar Z_k = \bar Z + \bar Y\rho + \bar Z_k^{(a)} + \bar Z_k^{(b)} \qquad (5.22)$$

$$\begin{aligned}
\bar Z_k^{(a)} &= \frac{1}{2}(2\pi)^{-d}\int d^dq\Big\{(N-1)(P_z+V')^{-1}\left(\bar Z' + \bar Y + \bar Y'\rho\right)\\
&+ (P_y+V'+2V''\rho)^{-1}\left(\bar Z' + \bar Y + 2\bar Z''\rho + 5\bar Y'\rho + 2\bar Y''\rho^2\right)\Big\} \qquad (5.23)
\end{aligned}$$

$$\begin{aligned}
\bar Z_k^{(b)} &= -\frac{1}{2}(2\pi)^{-d}\rho\int d^dq\Big\{(P_y(q)+V'+2V''\rho)^{-1}\\
&\cdot \frac{\partial}{\partial Q^2}\left[(P_y(q-Q)+V'+2V''\rho)^{-1} + (P_y(q+Q)+V'+2V''\rho)^{-1}\right]\Big|_{Q^2=0}\\
&\cdot \left[3V''+2V'''\rho + q^2\left(\bar Z' + \bar Y + \bar Y'\rho\right)\right]^2\\
&+ (N-1)(P_z(q)+V')^{-1}\left(V''+q^2\bar Z'\right)^2\\
&\cdot \frac{\partial}{\partial Q^2}\left[(P_z(q-Q)+V')^{-1} + (P_z(q+Q)+V')^{-1}\right]\Big|_{Q^2=0}\\
&+ \frac{8}{d}q^2(P_y(q)+V'+2V''\rho)^{-1}\frac{\partial}{\partial q^2}(P_y(q)+V'+2V''\rho)^{-1}\\
&\cdot \left[3V''+2V'''\rho + q^2\left(\bar Z' + \bar Y + \bar Y'\rho\right)\right]\left(\bar Z' + \bar Y + \bar Y'\rho\right)\\
&+ \frac{8(N-1)}{d}q^2\left(V''+q^2\bar Z'\right)\bar Z'(P_z(q)+V')^{-1}\frac{\partial}{\partial q^2}(P_z(q)+V')^{-1}\\
&+ 2(P_y+V'+2V''\rho)^{-2}\left(\bar Z' + \bar Y + \bar Y'\rho\right)\\
&\cdot \left[6V''+4V'''\rho + \left(2+\frac{1}{d}\right)q^2\left(\bar Z' + \bar Y + \bar Y'\rho\right)\right]\\
&+ 2(N-1)(P_z+V')^{-2}\left(V''\bar Y + q^2\bar Z'\bar Y + \frac{1}{d}q^2\bar Z'^2\right)\Big\} \qquad (5.24)
\end{aligned}$$

These expressions correspond to the graphs in fig. 2.

6 Anomalous dimensions with standard kinetic term

The anomalous dimensions obtain as the t-derivatives of $\ln Z_k$ and $\ln\bar Z_k$ at the minimum of the potential

$$\eta = -\frac{d}{dt}\ln Z_k(\rho_0) = -Z_k^{-1}(\rho_0)\left(\frac{\partial}{\partial t}Z_k(\rho_0) + Z_k'(\rho_0)\bar\delta\right) = \eta_k(\rho_0) - Z^{-1}Z'\bar\delta \qquad (6.1)$$

$$\begin{aligned}
\bar\eta &= -\frac{d}{dt}\ln\bar Z_k(\rho_0) = -\bar Z_k^{-1}(\rho_0)\left(\frac{\partial}{\partial t}\bar Z_k(\rho_0) + \bar Z_k'(\rho_0)\bar\delta\right)\\
&= \bar\eta_k(\rho_0) - \bar Z^{-1}(Z' + Y + Y'\rho_0)\bar\delta \qquad (6.2)
\end{aligned}$$

The partial derivatives $\frac{\partial}{\partial t}Z_k(\rho_0)$, $\frac{\partial}{\partial t}\bar Z_k(\rho_0)$ are taken for fixed ρ_0 and can be computed from (5.14),(5.15),(5.23) and (5.24). We use the renormalization group improvement to express these derivatives in terms of U_k, Z_k and Y_k instead of V, $\bar Z$ and $\bar Y$. We conclude that η depends on λ, Z, Y, Z' and Z'' whereas $\bar\eta$ involves in addition U''', Y' and Y''. The evolution equations for Z', Y' etc. will in turn need even higher ρ-derivatives.

We compute the anomalous dimension η for a standard kinetic term, i.e. we use the truncation $Y = Z' = Z'' = 0$ on the r.h.s. of (6.1). A set of sufficient conditions for this approximation to yield the leading contribution depends on the value of $\lambda\kappa$:
Linear regime $\lambda\kappa \ll \frac{1}{2}$:

$$\left|\frac{Yk^2}{\lambda}\right| \ll \lambda\kappa,\quad \left|\frac{Z'k^2}{\lambda}\right| \ll \lambda\kappa,\quad \left|\frac{Y\rho_0}{Z}\right| \ll 1,\quad \left|\frac{Z'\rho_0}{Z}\right| \ll 1,\quad \left|\frac{Z''\rho_0}{Z'}\right| \ll 1 \qquad (6.3)$$

Goldstone regime $\lambda\kappa \gg \frac{1}{2}$:

$$|Z'\rho_0 s| \ll Z,\quad \left|Z''\rho_0^2 s\right| \ll Z,\quad |Y\rho_0 s| \ll Z \qquad (6.4)$$

With this truncation $Z_k^{(a)}$ (5.14) vanishes and $Z_k^{(b)}$ (5.15) simplifies considerably

$$\begin{aligned}
Z_k^{(b)} &= -4V''^2\rho\frac{\partial}{\partial Q^2}I\left(Q^2=0\right)\\
I(Q) &= \frac{1}{2}(2\pi)^{-d}\int d^dq\,(P_z(q)+V'+2V''\rho)^{-1}\\
&\cdot (P_z(q+Q)+V')^{-1} \qquad (6.5)
\end{aligned}$$

We expand in powers of Q and use $P = P_z$, $x = q^2$

$$\begin{aligned}
\frac{\partial}{\partial Q^2}I\left(Q^2=0\right) &= -v_d\int_0^{\Lambda^2}dx\,x^{\frac{d}{2}-1}(P+V'+2V''\rho)^{-1}\\
&(P+V')^{-2}\left(\dot P + \frac{2}{d}\ddot Px - \frac{4}{d}\frac{\dot P^2x}{P+V'}\right) \qquad (6.6)
\end{aligned}$$

where

$$\dot{P} = \frac{\partial P}{\partial x}, \quad \ddot{P} = \frac{\partial^2 P}{\partial x^2} \tag{6.7}$$

We write

$$\begin{aligned} Z_k^{(b)} &= \frac{8v_d}{d} V''^2 \rho (J_d + B_d) \\ J_d &= \int dx x^{\frac{d}{2}} \dot{P}^2 (P+V')^{-2} (P + V' + 2V''\rho)^{-2} \\ B_d &= \int dx \frac{d}{dx} \left[x^{\frac{d}{2}} \dot{P} (P+V')^{-2} (P + V' + 2V''\rho)^{-1} \right] \end{aligned} \tag{6.8}$$

and find for $d < 6$ that Z_k is ultraviolet finite and $B_d = 0$ for $\Lambda \to \infty$. For $V' = 0$ the integral J_d reads

$$J_d = \int \frac{dx}{x} x^{\frac{d-2}{2}} \dot{P}^2 \left(\frac{x}{P}\right)^2 (P + 2V''\rho)^{-2} \tag{6.9}$$

and we note that the factor $\dot{P}^2 \left(\frac{x}{P}\right)^2$ becomes constant for both $x \gg k^2$ and $x \ll k^2$. For $d > 2$ the integral is dominated by values of x of the order $k^2 + 2V''\rho/Z$ and we roughly estimate

$$Z_k^{(b)} \sim \begin{cases} V''^2 \rho k^{d-6} & \text{for} \quad Zk^2 \gg 2V''\rho \\ \rho^{-1} k^{d-2} & \text{for} \quad Zk^2 \ll 2V''\rho \end{cases} \tag{6.10}$$

For $d = 2$ and $Zk^2 \ll 2V''\rho$, however, we observe logarithmic contributions from two regions with $Zk^2 \ll P \lesssim 2V''\rho$, namely for $x \ll k^2$ and $x \gg k^2$. The contribution from $x > k^2$ is the expected wave function renormalization from Goldstone modes with momenta $q^2 > k^2$

$$Z_{k(0)}^{(b)} = v_2 \rho^{-1} \ln \frac{2V''\rho}{k^2} \tag{6.11}$$

The contribution from the region $x < k^2$ is more problematic. The averaging procedure is not sufficient to eliminate effectively all contributions from modes with momenta $q^2 \ll k^2$ for this special case. One obtains for $\beta > 1$ the contribution

$$Z_{k(IR)}^{(b)} = v_2 \rho^{-1} (\beta - 1)^2 \ln \frac{k^2}{x_c} = v_2 \rho^{-1} (\beta - 1) \ln \frac{2V''\rho}{k^2} \tag{6.12}$$

with

$$P(x_c) = 2V''\rho, \quad x_c \approx k^2 \left(\frac{k^2}{2V''\rho}\right)^{\frac{1}{\beta-1}} \tag{6.13}$$

For a computation of $\eta_k(\rho) = -\frac{\partial}{\partial t} \ln Z_k(\rho)$ we only need the t-derivative of (6.8). The dependence of η_k on the momentum cutoff is exponentially suppressed and we can take the limit $\Lambda \to \infty$ for all d. There is no scale dependence from the boundary term B_d

$$\frac{\partial}{\partial t} B_d = 0 \tag{6.14}$$

and we only need to compute

$$\frac{\partial}{\partial t} J_d = M_{2,2}^d (U', U' + 2U''\rho) k^{d-6} \tag{6.15}$$

The integrals

$$M_{n_1,n_2}^d (w_1, w_2) = k^{2(n_1+n_2-1)-d} \int dx x^{\frac{d}{2}} \frac{\partial}{\partial t} \left[\dot{P}_z^2 (P_z + w_1)^{-n_1} (P_y + w_2)^{-n_2} \right] \tag{6.16}$$

are discussed in appendix C. For the approximation $Y = Z' = Z'' = 0$ we summarize

$$\eta_k(\rho) = -\frac{8v_d}{d} Z^{-1} U''^2 \rho k^{d-6} M_{2,2}^d (U', U' + 2U''\rho) \tag{6.17}$$

In the linear regime we can neglect the mass terms in the propagator in a first approximation and use

$$M_{2,2}^d \left(0, 2\bar{\lambda}\rho_0\right) \approx M_{4,0}^d(0) = -2Z^{-2} m_4^d \tag{6.18}$$

with constants m_4^d given by (C.18)-(C.20). We quote our final result for the anomalous dimension in terms of the dimensionless quantities κ and λ (4.28)(4.29)

$$\eta = \frac{16v_d}{d} m_4^d \lambda^2 \kappa \tag{6.19}$$

We conclude that the anomalous dimension in one loop order is suppressed by $\kappa \ll 1$. Its contribution to the evolution equations (4.30), (4.31) can be neglected in leading order.

In the Goldstone regime the integral $M_{2,2}$ is proportional to $\left(\bar{\lambda}\rho_0\right)^{-2}$ and we approximate

$$M_{2,2}^d \left(0, 2\bar{\lambda}\rho_0\right) \approx -2 \left(\frac{k^2}{2\bar{\lambda}\rho_0}\right)^2 m_{2,2}^d \tag{6.20}$$

with $m_{2,2}^d$ displayed in (C.51), (C.58). We obtain for the anomalous dimension

$$\eta = \frac{4v_d}{d} m_{2,2}^d \kappa^{-1} \tag{6.21}$$

As discussed previously the evaluation of η in two dimensions receives a questionable contribution from the momentum range $q^2 \ll k^2$. The anomalous dimension in the Goldstone regime for $d = 2$ is discussed in detail in appendix D, including contributions from nonvanishing Y. We propose an improved treatment resulting in

$$\begin{aligned} m_{2,2}^2 &= 1 \\ \eta &= 2v_2\kappa^{-1} = \frac{1}{4\pi\kappa} \quad \text{for } d = 2 \end{aligned} \tag{6.22}$$

We conclude that the contribution $\sim \eta\lambda$ in β_λ (4.31) is suppressed by a factor $(\lambda\kappa)^{-1}$ compared to the contribution $\sim \lambda^2$. It can be neglected in a first approximation. On the other hand, the contribution $\sim \eta\kappa$ in β_κ (4.30) is of the same order as the contribution independent of κ and has to be taken into account.

We finally turn to the computation of the anomalous dimension for the radial mode $\bar{\eta}$ (6.2). We make the additional truncations $Y' = Y'' = 0$. The approximation (4.20) gives the leading contribution to $\bar{\eta}$ if (6.3) holds for the linear regime, whereas for the Goldstone regime we have in addition to (4.21) the condition

$$\left|\frac{Yk^2}{\lambda}\right| \ll 1 \tag{6.23}$$

(We also assume (6.4) and similar conditions for $Y'\rho_0$ and $Y''\rho_0^2$.) In this limit $\bar{Z}_k^{(a)}$ vanishes whereas $\bar{Z}_k^{(b)}$ can be estimated to be of the order

$$\bar{Z}_k^{(b)} \sim V''^2 \rho k^{d-6} \tag{6.24}$$

Due to the Goldstone fluctuations this holds also for $Zk^2 \ll 2V''\rho$, in contrast to $Z_k^{(b)}$ (6.10). For a more precise evaluation of $\bar{\eta}$ we start from (5.24) and perform all steps in analogy to the computation η, resulting in

$$\begin{aligned} \bar{\eta}_k(\rho) &= -\frac{\partial}{\partial t}\ln \bar{Z}_k(\rho) = -\frac{4v_d}{d}k^{d-6}\bar{Z}^{-1}\rho\ \Big\{(3U''+2U'''\rho)^2 \bar{M}_{4,0}^d(U'+2U''\rho) \\ &- (N-1)U''^2 M_{4,0}^d(U')\Big\} \end{aligned} \tag{6.25}$$

(with $\bar{M} = M\ \left(Z \leftrightarrow \bar{Z}\right)$). We also neglect in this paper the terms $\sim U'''$ and define

$$t(w) = \bar{M}_{4,0}^d(w)/M_{4,0}^d(0) \tag{6.26}$$

This yields in terms of λ and κ

$$\bar{\eta} = \frac{8v_d}{d}m_4^d Z\bar{Z}^{-1}\lambda^2\kappa\left(N-1+9t\left(2\bar{\lambda}\rho_0\right)\right) \tag{6.27}$$

In the linear regime we approximate $t\left(2\bar{\lambda}\rho_0\right) \approx 1$. Comparing $\bar{\eta}$ with η (6.20) we find that the anomalous dimensions for the radial and Goldstone modes are actually different but both are small. In leading order the difference between Z and $\bar{Z}$ can be neglected since it evolves only slowly

$$\begin{aligned} \frac{\partial}{\partial t}\ \frac{\bar{Z}-Z}{Z} &= -\frac{\bar{Z}}{Z}(\bar{\eta}-\eta) \\ &= -\frac{8v_d}{d}m_4^d(N+6)\lambda^2\kappa \end{aligned} \tag{6.28}$$

This justifies the approximation $|\ Y\rho_0\ | \ll Z$ at least for a certain range of the evolution for which $Z,\ \bar{Z} \approx \bar{Z}$. For a justification of the approximation $|\ Z'\rho_0\ | \ll Z$ we evaluate

$$\frac{d}{dt}\left(\frac{Z'(\rho_0)\rho_0}{Z(\rho_0)}\right) = \rho_0\frac{\partial}{\partial t}\left(\frac{Z'}{Z}\right)_{|\rho_0} + \delta\left(\frac{Z'}{Z} + \frac{Z''\rho_0}{Z} - \frac{Z'^2\rho_0}{Z^2}\right) \tag{6.29}$$

Neglecting the second term we can invert the $\partial/\partial t$ and $\partial/\partial\rho$ differentiations such that

$$\frac{\partial}{\partial t}\left(\frac{Z'\rho_0}{Z}\right)_{|\rho_0} = \rho\frac{\partial}{\partial t}\frac{\partial}{\partial\rho}\ln Z_{|\rho_0} = -\rho\frac{\partial}{\partial\rho}\eta_{k|\rho_0} \approx -\eta \tag{6.30}$$

Since η is small in the linear regime the ratio evolves only slowly. A similar argument justifies $|\ Z''\rho_0/Z'\ | \ll 1$. On the other hand, an inspection of (6.10.) and (6.24) naively suggests

$$Z' \sim Y \sim \lambda^2 k^{2-d} \tag{6.31}$$

and contradicts the first line of (6.3).

In the Goldstone regime we neglect the contribution $\sim t$ in (6.27). Due to the Goldstone fluctuations the anomalous dimension $\bar{\eta}$ becomes very large ($\bar{\eta} \sim \lambda^2\kappa$) and we expect $Y\rho_0$ to grow fast. Indeed, eq. (6.24) suggests

$$Y \sim \lambda^2 k^{2-d} \tag{6.32}$$

and (6.23) becomes questionable. We conclude that the truncation with a standard kinetic term (4.20) is at best a crude approximation. We will treat with a general kinetic term in a forthcoming paper, where we will solve the evolution equations for Y, Z' etc... Although the qualitative conclusions reached in the next section will not be affected, we will see that the quantitative evaluation of fixpoints etc. becomes indeed modified.

7 Phase transitions in two, three and four dimensions

a) d = 4

In four dimensions the theory is infrared free at the phase transition. We can restrict the discussion to small values of λ without loss of generality. In the linear regime we can omit in leading order the anomalous dimension in the evolution equation for κ (4.30)

$$\begin{aligned} \frac{\partial \kappa}{\partial t} &= -2\kappa + 2v_4(N+2)l_1^4 \\ v_4 &= 1/32\pi^2 \end{aligned} \tag{7.1}$$

This equation has an ultraviolet fixpoint within the linear regime

$$\kappa_* = v_4(N+2)l_1^4 \tag{7.2}$$

For $\kappa < \kappa_*$ the expectation value $\kappa(k)$ decreases with decreasing k and reaches zero for some value k_0. Then $\varphi_0 = 0$ and we have to continue the evolution in the symmetric regime ($U'(0) \geq 0$). The theory is in the symmetric phase. For $\kappa > \kappa_*$ the value of κ increases until the approximation $2\kappa\lambda \ll 1$ breaks down at k_0. For $k < k_0$ one has to continue the evolution in the Goldstone regime. The theory is in the spontaneously broken phase.

The fixpoint κ_* corresponds to the phase transition between the symmetric and spontaneously broken phase. At the phase transition the renormalized minimum value $Z\rho_0$ scales $\sim k^2$. As long as κ is very near κ^* it evolves only very slowly. We call this the scaling region. In the scaling region we insert (7.2) into (6.19) and obtain for the anomalous dimension

$$\begin{aligned} \eta_* &= 4v_4 m_4^4 \lambda^2 \kappa_* \\ &= 4v_4^2(N+2)l_1^4 m_4^4 \lambda^2 \end{aligned} \tag{7.3}$$

It is interesting to observe within our truncated one loop calculation a value $\eta \sim \lambda^2$ similar to the two loop result in standard perturbation theory

$$\eta_{PT} = 2v_4^2(N+2)\lambda^2 \tag{7.4}$$

We should mention, however, that the ratio

$$\frac{\eta_*}{\eta_{PT}} = 2l_1^4 m_4^4 \tag{7.5}$$

depends on the average scheme (β). It varies between 1.3 and 1.75 for $2.5 \leq \beta \leq 3.5$. Also, the contributions from the omitted terms $\sim Z', Y$ are estimated to be of the same order as (7.3).

We may also evaluate the contribution to $\beta_\lambda \sim \lambda^3$ in our approximation using

$$s_2^d\left(2\bar{\lambda}\rho_0\right) = 1 - 4\frac{l_3^d}{l_2^d}\lambda\kappa + 0\left((\lambda\kappa)^2\right) \tag{7.6}$$

One finds

$$\beta_\lambda = 2v_4(N+8)\lambda^2 - 8v_4^2(N+2)l_1^4\left(9l_3^4 - m_4^4\right)\lambda^3 \tag{7.7}$$

which should be compared with the two loop result in perturbation theory

$$\beta_{\lambda PT} = 2v_4(N+8)\lambda^2 - 4v_4^2(9N+42)\lambda^3 \tag{7.8}$$

Similarly, we compute the first correction $\sim \lambda$ to β_κ

$$\begin{aligned} \beta_\kappa &= -2\kappa + 2v_4(N+2)l_1^4 + 6v_4 l_1^4\left(s_1^4 - 1\right) \\ &= -(2 + 12v_4\lambda)\kappa + 2v_4(N+2)l_1^4 \end{aligned} \tag{7.9}$$

The normalized mass term for the radial mode at the potential minimum is defined by

$$m^2(k) = 2\bar{\lambda}(k)\rho_0(k)Z_k^{-1} = 2\lambda\kappa k^2 \tag{7.10}$$

It obeys the evolution equation (in order λ)

$$\begin{aligned} \frac{\partial}{\partial t}m^2 &= \left(2 + \frac{\beta_\lambda}{\lambda} + \frac{\beta_\kappa}{\kappa}\right)m^2 \\ &= 4v_4(N+2)l_1^4\lambda k^2 + 2v_4(N+2)\lambda m^2 \end{aligned} \tag{7.11}$$

At the phase transition $m^2(k)$ is proportional k^2.

$$m_*^2(k) = 2\lambda(k)\kappa_* k^2 \tag{7.12}$$

The deviation from the phase transition is measured by

$$\delta m^2 = m^2 - m_*^2 \tag{7.13}$$

Here a positive (negative) δm^2 corresponds to the symmetric (spontaneously broken) phase. The evolution of δm^2 (for a given $\lambda(k)$) is determined by the anomalous mass dimension ω

$$\frac{\partial}{\partial t}\delta m^2 = \omega\delta m^2 \tag{7.14}$$

$$\omega = 2v_4(N+2)\lambda \tag{7.15}$$

The value (7.15) reproduces the result of standard one loop perturbation theory. The next term $\sim \lambda^2$ requires a computation of β_κ up to terms $\sim \lambda^2$.

It is remarkable that the one loop expansion of the average action not only reproduces β_λ, η and ω correctly in leading order in λ ($\eta = 0$). It also accounts at least partially for higher orders in λ which require, in standard perturbation theory, an expansion beyond one loop. It remains an interesting question to see to what extent a systematic procedure keeping all next to leading terms by relaxing the truncations (4.20) (4.24) accounts for the two loop perturbative result.

b) d = 3

The leading order evolution equation for κ in the linear regime

$$\beta_\kappa = \frac{\partial \kappa}{\partial t} = -\kappa + 2v_3(N+2)l_1^3 \tag{7.16}$$

implies $\beta_\kappa > 0$ for $\kappa \ll v_3 N$ and suggests an ultraviolet fixpoint at

$$\kappa_*^{(L)} = 2v_3(N+2)l_1^3 \tag{7.17}$$

At this fixpoint one obtains with (4.33)

$$\lambda_*^{(L)}\kappa_*^{(L)} = \frac{N+2}{N+8}\frac{l_1^3}{l_2^3} \tag{7.18}$$

Since this value is not much smaller than $\frac{1}{2}$, the fixpoint corresponds at best to the boundary region for the validity of the linear regime. In the Goldstone regime we infer from (6.21)

$$\beta_\kappa = -\kappa + 2v_3(N-1)l_1^3 - \frac{4v_3}{3}m_{2,2}^3 \tag{7.19}$$

Thus β_κ becomes negative for $\kappa \gg v_3 N$. Taken together with the result from the linear regime for $\kappa \ll v_3 N$ this leads to the conclusion that there must exist an ultraviolet fixpoint for κ of the order $v_3 N$. In leading order in the Goldstone regime it is evaluated at

$$\kappa_*^{(G)} = 2v_3\left\{(N-1)l_1^3 - \frac{2}{3}m_{2,2}^3\right\} \tag{7.20}$$

with

$$\lambda_*^{(G)}\kappa_*^{(G)} = \frac{l_1^3 - \frac{2}{3(N-1)}m_{2,2}^3}{l_2^3} \tag{7.21}$$

The transition between the symmetric and spontaneously broken phase occurs on the boundary between the linear regime and the Goldstone regime with $\lambda_*\kappa_* = 0(1)$. At the phase transition neither the expansion in κ nor in κ^{-1} is expected to converge rapidely. In particular, the radial or 'longitudinal' mode cannot be neglected near the phase transition of the three dimensional nonlinear σ-model! The anomalous dimension in the linear regime (6.19) reads, for $\lambda = \lambda_*$ (4.33) and $\kappa = \kappa_*$ (7.17)

$$\eta_*^{(L)} = \frac{8}{3}\frac{(N+2)}{(N+8)^2}\frac{m_4^3 l_1^3}{(l_2^3)^2} \tag{7.22}$$

whereas we obtain from (6.21) (7.20) in the Goldstone regime

$$\eta_*^{(G)} = \frac{2}{3}\frac{m_{2,2}^3}{(N-1)l_1^3 - \frac{2}{3}m_{2,2}^3} \tag{7.23}$$

For a quantitative evaluation we observe that l_n^3, m_4^3 and $m_{2,2}^3$ all depend on β (see also table 1)

$$\begin{aligned}
l_1^3 &= (2a)^{-\frac{1}{2\beta}}\Gamma\left(1+\frac{1}{2\beta}\right)\\
l_2^3 &= (2a)^{\frac{1}{2\beta}}\Gamma\left(1-\frac{1}{2\beta}\right)\left(2-2^{\frac{1}{2\beta}}\right)\\
l_3^3 &= (2a)^{\frac{3}{2\beta}}\Gamma\left(1-\frac{3}{2\beta}\right)\left(3-3\cdot 2^{\frac{3}{2\beta}}+3^{\frac{3}{2\beta}}\right)\\
m_{2,2}^3 &= -\frac{1}{2}l_1^3\left\{\left(\beta-\frac{3}{2}\right)\zeta\left(1+\frac{1}{2\beta}\right)-\left(\beta+\frac{1}{2}\right)\zeta\left(2+\frac{1}{2\beta}\right)\right\}\\
m_4^3 &= (2a)^{\frac{3}{2\beta}}\Gamma\left(1-\frac{3}{2\beta}\right)\left\{(6\beta-1)2^{\frac{3}{2\beta}-4}-1\right\}
\end{aligned} \tag{7.24}$$

For $m_{2,2}^3 = -\frac{1}{2}ql_1^3$ we find

$$\kappa_*^{(G)}/\kappa_*^{(L)} = \frac{N-1+\frac{1}{3}q}{N+2} \tag{7.25}$$

with $q = (1.1, 4.6, 9.1)$ for $\beta = (2.5, 3, 3.5)$. We observe that the contribution from the anomalous dimension partly compensates for the fewer degrees of freedom in the Goldstone regime. It depends, however, strongly on β. Nevertheless, the ratio (7.25) is close to one, especially for large N. We display in table 2 the values $\lambda_*^{(L)}$, $\lambda^{*(G)}$, $\kappa_*^{(L)}$, $\kappa_*^{(G)}$, $\eta_*^{(L)}$ and $\eta_*^{(G)}$ for three values of β and N = 4, using and $v_3 = 1/8\pi^2$. The fixpoint value η_* equals the critical exponent η and should be independent of β. In the Goldstone regime η_* depends strongly on β and has the wrong sign. On the other hand, the negative sign of $\eta_*^{(G)}$ suggests that the true value of η_* is below $\eta_*^{(L)}$. The difference between $\lambda_*^{(L)}\kappa_*^{(L)}$ and $\kappa_*^{(G)}\lambda_*^{(G)}$ makes a value $\kappa_*\lambda_* \approx \frac{1}{2}$ plausible. It is possible to determine κ_* λ_* and η_* by solving the integrals $\tilde{L}_n^d$ numerically as a function of $\lambda\kappa$. We will turn to this in a subsequent publication where we also include the effects of the deviation of U_k from a quartic polynomial.

Table 1: The coefficients l_n^3, m_4^3, $m_{2,2}^3$

β	l_1^3	l_2^3	l_3^3	m_4^3	$m_{2,2}^3$
2.5	0.83	1.09	1.14	0.97	-0,47
3.0	0.83	1.10	1.20	1.23	-1.91
3.5	0.84	1.10	1.22	1.47	-3.80

Table 2: Fixpoints and anomalous dimensions for $d = 3$, $N = 4$

β	$\kappa_*^{(L)}$	$\kappa_*^{(G)}$	$\lambda_*^{(L)}$	$\lambda_*^{(G)}$	$\kappa_*^{(L)}\lambda_*^{(L)}$	$\kappa_*^{(G)}\lambda_*^{(G)}$	$\eta_*^{(L)}$	$\eta_*^{(G)}$
2.5	0.13	0.07	3.01	12.0	0.38	0.86	0.08	-0.11
3.0	0.13	0.10	2.98	11.9	0.38	1.14	0.09	-0.34
3.5	0.13	0.13	2.98	11.9	0.38	1.52	0.11	-0.50

c) d = 2

In two dimensions the evolution equation for κ in the linear regime

$$\frac{\partial\kappa}{\partial t} = 2v_2(N+2) \tag{7.26}$$

drives κ quickly to zero. Whenever the couplings enter the linear regime of the evolution equations the theory is in the symmetric phase. The evolution in the Goldstone regime depends critically on the value of N. For $N \geq 2$ we use (6.22) for the anomalous dimension and obtain in leading order

$$\frac{\partial\kappa}{\partial t} = \beta_\kappa = 2v_2(N-2) = \frac{N-2}{4\pi} \tag{7.27}$$

It is striking that β_κ is average scheme independent and that the contribution $\sim \eta\kappa$ cancels exactly the contribution of one of the Goldstone modes in (4.30)! We also observe that the renormalized coupling of the nonabelian ($N > 2$!) nonlinear σ model is given by (2.6)

$$g^2 = \frac{1}{2}\kappa^{-1} \tag{7.28}$$

The evolution equation (7.27) correctly reproduces the one loop renormalization group equation in the nonlinear σ-model[5]

$$\frac{\partial}{\partial t} g^2 = -\frac{(N-2)}{2\pi} g^4 \tag{7.29}$$

The expansion in powers of κ^{-1} in the Goldstone regime corresponds to an expansion in powers of the coupling in the nonlinear σ-model. The average action is a convenient tool to describe simultaneously the linear scalar theory <u>and</u> the nonlinear σ-model which corresponds to the Goldstone regime!

For $N > 2$ the evolution equation (with $\lambda = \lambda_*$)

$$\begin{aligned} \frac{\partial}{\partial t}(\lambda\kappa) &= 2v_2(N-2)\lambda. \\ &= \frac{2}{l_2^2}\frac{(N-2)}{(N-1)} \end{aligned} \tag{7.30}$$

implies that $\lambda\kappa$ decreases until it reaches $\lambda\kappa(\Lambda_\sigma) = \frac{1}{2}$ at some scale $k = \Lambda_\sigma$. For $k < \Lambda_\sigma$ the evolution equation has to be continued in the linear regime and finally, for $\kappa = 0$, in the symmetric regime. There is no phase transition and the theory is always in the symmetric phase, in accordance to the Mermin-Wagner theorem [6] for continuous symmetries in two dimensions. Nevertheless, the evolution (7.29) is only logarithmic. The scale Λ_σ therefore depends exponentially on the short distance value of κ which we identify for simplicity with the minimum of the 'bare' potential $\bar{V}$

$$\begin{aligned} \bar{V}'(\sigma) &= 0 \\ \kappa(\Lambda) &= \bar{Z}\sigma \end{aligned} \tag{7.31}$$

We may define Λ_σ by

$$\kappa(\Lambda_\sigma) = \frac{1}{2}\lambda_*^{-1} = \frac{(N-1)l_2^2}{16\pi} \tag{7.32}$$

and observe the exponentially small ratio Λ_σ/Λ for large enough $\kappa(\Lambda)$

$$\Lambda_\sigma = \Lambda \exp - \left\{ \frac{4\pi\kappa(\Lambda)}{N-2} - \frac{l_2^2}{4}\frac{(N-1)}{(N-2)} \right\} \tag{7.33}$$

The scale Λ_σ indicates the momentum scale for which the effective coupling among Goldstone modes $g(\Lambda_\sigma)$ becomes strong. The strongly interacting regime in the nonabelian nonlinear σ-model turns out to correspond to the linear scalar theory in the symmetric phase! The masses of the N scalar excitations are of the order Λ_σ. Their quartic interaction is given by (4.33) with a similar fixpoint formula for the φ_6 interaction etc.

For observables involving only momenta $q^2 \gg \Lambda_\sigma^2$, however, the theory looks effectively as an interacting theory for the massless Goldstone bosons – in complete analogy to asymptotic

freedom in four dimensional nonabelian gauge theories like QCD. For practical purposes in statistical mechanics the finite volume $\sim l^d$ always constitutes an effective infrared momentum cutoff $\sim l^{-1}$. The physics of the modes with $q^2 \gtrsim l^{-2}$ shows effectively all features of a spontaneously broken phase if Λ_σ is much smaller than l^{-1}!

For $N = 2$ the contribution of the Goldstone mode to the evolution of κ vanishes in leading order. We expect this to remain true for the pure Goldstone boson contribution in arbitrary order in an expansion in κ^{-1}. The corresponding nonlinear σ-model is abelian and has no self interaction. There remain the contributions from the radial mode and from the interaction between the Goldstone mode and the radial mode. They are suppressed by some power of κ^{-1} (or perhaps even exponentially)

$$\frac{\partial\kappa}{\partial t} = c\kappa^{-b} \tag{7.34}$$

The exact phase structure depends on the sign of c according to

$$\kappa(k) = \kappa(\Lambda)\left(1 - \frac{c}{1+b}\kappa(\Lambda)^{-(1+b)}\ln\frac{\Lambda}{k}\right)^{\frac{1}{1+b}} \tag{7.35}$$

For positive c κ decreases for decreasing k and finally enters the linear regime. No phase transition would be expected in this case. For $c < 0$, however, κ increases in the Goldstone regime, $\beta_\kappa < 0$. Combining this with $\beta_\kappa > 0$ in the linear regime (7.26) we conclude that there must be a phase transition corresponding to some ultraviolet fixpoint κ_*. Such a fixpoint should lie within the validity of the Goldstone regime. We identify this possible phase transition with the 'Kosterlitz-Thouless' phase transition [7]. A computation of c needs a next to leading order computation in the Goldstone regime.

In any case, the evolution of κ is very slow for large $\kappa(\Lambda)$. The exact phase structure becomes irrelevant in presence of an effective infrared cutoff (like finite volume in statistical mechanics). We may use the lowest order result $\beta_\kappa = 0$ for κ larger than some critical value κ_c. (This corresponds to an 'effective Kosterlitz-Thouless phase' even for $c > 0$). This approximation is analogous to the neglection of the running of a small enough quartic coupling λ in four dimensions. Then $\kappa = const$ is a free parameter (given in statistical mechanics as a function of $T - T_C$). At first sight, the appearance of an 'order parameter' $\kappa \neq 0$ in the Kosterlitz-Thouless phase seems to contradict the Mermin-Wagner theorem [6]. This theorem only implies, however, that the minimum value $\rho_0(k)$ must vanish for $k \to 0$. (We remember that the kinetic term for φ is normalized at some fixed scale μ whereas κ corresponds to a normalization at the running scale k.) With

$$Z(k) = \bar{Z}\left(\frac{k}{\Lambda}\right)^{-\eta} = \bar{Z}\left(\frac{k}{\Lambda}\right)^{-\frac{1}{4\pi\kappa}} \tag{7.36}$$

we find that the Goldstone boson contribution in (4.26) indeed assures $\bar{\delta} > 0$ and drives ρ_0 to zero for $k \to 0$

$$\rho_0(k) = \sigma\left(\frac{k}{\Lambda}\right)^{\frac{1}{4\pi\bar{Z}\sigma}} \tag{7.37}$$

This holds for arbitrary values of σ. Only in this sense the Kosterlitz-Thouless phase has no symmetry breaking.

We observe that the anomalous dimension in the Goldstone regime $\eta = (4\pi\kappa)^{-1}$ is positive and depends on κ. This leads to a power law in the momentum dependence of the inverse propagator of the Goldstone boson in the Kosterlitz-Thouless phase

$$P_G \sim Z\left(p^2\right) p^2 \sim \left(p^2\right)^{1-\frac{1}{8\pi\kappa}} \tag{7.38}$$

This anomalous propagator avoids Coleman's no go theorem [8] for 'free' massless scalar fields in two dimensions.

8 Conclusions

The average action in continuous spacetime has been applied earlier [2, 3] to the N component scalar theory in four dimensions. In lowest order the results of perturbation theory for the behaviour near the minimum of the potential were reproduced. In this paper we have studied the scalar theory in two and three dimensions. These dimensions are not directly accessible to perturbation theory due to the strong infrared divergences originating from the Goldstone bosons. The perturbative expansions known so far employ an expansion in d - 4, the deviation of the number of dimensions from four (ε - expansion)[9]. The computation of the average action provides an effective infrared cutoff through the averaging procedure. This makes it possible to explore the long distance physics by means of evolution equations, varying the average scale k. Expansion methods directly formulated in two or three dimensions become available. Our results demonstrate that the average action is not only a nice conceptual tool but also permits the computation of quantities not easily accessible otherwise.

We have found that the wave function renormalization plays an important role for the qualitative behaviour of the scalar theory in less than four dimensions (especially for $d = 2$). The wave function renormalization is much more sensitive to the averaging procedure than the potential. We demonstrate this by considering a family of definitions of averages parametrized by a parameter β. If the momentum cutoff implied by the averaging procedure is too sharp ($\beta \to \infty$) the wave function renormalization diverges in one loop order. On the other hand, a very smooth cutoff ($\beta < 2$, $\beta \neq 1$) leads to a divergence of the classical contribution to the wave function renormalization. We conclude that the averaging procedure has to be chosen with care (we take β near three) in order to obtain a reliable loop expansion.

We have performed a one loop computation of the evolution equation for the minimum of the scalar potential $\varphi_0^2(k)$ and the quartic coupling $\bar{\lambda}(k)$. This yields the correct phase structure for two, three and four dimensions, and discrete, abelian and nonabelian continuous symmetry. In view of the qualitatively different behaviour for different dimensions and symmetries we find this very encouraging. We have developed two expansion methods for small and large quartic scalar coupling. Similar as on the lattice [10] they can be applied simultaneously. These two expansion methods correspond to an expansion as a linear or nonlinear σ-model (linear regime and Goldstone regime). (Compare also ref. 11 for the Goldstone regime.) It is interesting that the phase transition in three dimensions occurs on the boundary of validity between the linear and nonlinear treatment.

The investigation of the scale dependence of the average action provides a unified picture for the N-component scalar theory for different dimensions and different symmetries. Consider the model with finite momentum cutoff Λ (for example by a lattice in statistical mechanics) and in a finite volume $\sim l^d$, with $\Lambda l \gg 1$. We are only interested in observables involving length scales smaller than l and much larger than Λ^{-1} ($l^{-2} \ll q^2 \ll \Lambda^2$) such that boundary effects from the finite volume and cutoff can be neglected. We want to know the relevant excitations, their propagation and their couplings at a given length scale $\sim k^{-1}$. We find that the scalar theories for d = 2, 3, 4 with discrete ($N = 1$), abelian (N = 2) or nonabelian ($N > 2$) symmetry behave all qualitatively similar. There always exist two regions in parameter space, corresponding to the symmetric and spontaneously broken regime. In the symmetric regime the $SO(N)$ symmetry acts linearly on N degenerate excitations ($\varphi_0(k) = 0$). In the spontaneously broken regime the symmetry is realized nonlinearly on $N-1$ massless Goldstone excitations. The radial excitation has a nonvanishing mass $m(k)$. If this mass is much smaller than k it can be treated as a small correction. In this 'linear regime' the spontaneously broken regime is very similar to the symmetric regime with a small mass term. In contrast, for $m \gg k$ one can expand in inverse powers of m. This 'Goldstone regime' is dominated by the Goldstone modes. Within the spontaneously broken regime we always find a 'scaling regime' where $m(k)$ is proportional to k. All other dimensionful quantities also scale with appropriate powers of k. The scaling regime corresponds to an (approximate) ultraviolet fixpoint in parameter space. Three features in the scaling regime are common to all models:

i) The dimensionless normalized quartic scalar coupling λ and the dimensionless normalized minimum of the potential κ run very slowly (or not at all).

ii) The radial excitation can never be neglected since $m(k)$ decreases with k.

iii) There are nonvanishing anomalous dimensions (corresponding to critical exponents) which can be expressed in dependence on λ and κ.

A second order phase transition exists if the scaling region extends to $\Lambda l \to \infty$, $k \to 0$. For discrete symmetry (N = 1) a phase transition occurs for d = 2, 3 and 4. For continous symmetry ($N \geq 2$) the differences between various dimensions appear only in the limit $\Lambda l \to \infty$, $k \to 0$: In four dimensions there is a phase transition. The theory is infrared free since $\lambda(k \to 0) \to 0$. (This holds actually in the whole spontaneously broken phase and not only at the phase transition.) For small λ the running of λ is very slow ($\beta_\lambda \sim \lambda^2$). We can treat λ as a free parameter for all practical pruposes when a physical infrared cutoff exists. The second free parameter corresponds to the deviation from the phase transition or the physical scalar mass. The three dimensional models also exhibit a phase transition. Here λ approaches fast an infrared stable fixpoint λ_*. No free parameter remains near the phase transition except the mass term. In two dimensions and for $N > 2$ the potential minimum κ always runs to zero as $k \to 0$. The scaling regime breaks down at some critical scale $k = \Lambda_\sigma$ and there is no phase transition. Nevertheless, the running of κ is only logarithmic. For large κ we can effectively regard κ^{-1} as a free parameter. It corresponds to the coupling in the nonabelian nonlinear σ-model. For $N = 2$ the running of κ is even slower than logarithmic for large κ. There is presumably a phase transition which we associate with the Kosterlitz-Thouless phase transition.

In this paper we have mainly been concerned with the qualitative aspects of a unified description of scalar field theories. Major quantitative improvements for the critical exponents seem possible even within the one loop calculation. The threshold between the linear and nonlinear regime (relevant for d = 3) can be treated with a numerical evaluation of the integrals $\tilde{L}$ as a function of $\lambda\kappa$. One can also account for the effects of the φ^6 coupling and the different wave function renormalizations for the radial mode and the Goldstone modes. The corresponding couplings can also be computed as a function of k. We will turn to these improvements in a subsequent publication. It is an interesting question whether a one loop computation with a less severe truncation of the average action is sufficient for a quantitative computation of the critical exponents.

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Appendix A: Classical wave function renormalization

For $1 < \beta \leq 2$ the difference between $\Gamma_k^{(0)}$ and S_k becomes important even for small Q^2. We define the classical kinetic terms from (3.13) by

$$\tilde{K}_z^{(0)}(Q^2) = \Omega^{-1} \frac{\partial \Gamma_k^{(0)}}{\partial(\delta\varphi_a^*(Q)\delta\varphi_a(Q))} \quad , a \neq 1$$

$$\tilde{K}_y^{(0)}(Q^2) = \Omega^{-1} \frac{\partial \Gamma_k^{(0)}}{\partial(\delta\varphi_1^*(Q)\delta\varphi_1(Q))} \tag{A.1}$$

$$K_{z,y}^{(0)}(Q^2) = \tilde{K}_{z,y}^{(0)}(Q^2) - \tilde{K}_{z,y}^{(0)}(0) \tag{A.2}$$

In the leading approximation for A_a, B_a (3.15) (3.16) one obtains

$$K_z^{(0)}(Q^2) = Q^2\left\{\bar{Z}(\rho) - 2a\left(\frac{V'(\rho)}{k^2}\right)^2\left(\frac{Q^2}{k^2}\right)^{\beta-2}\right\}$$

$$K_y^{(0)}(Q^2) = Q^2\left\{\bar{Z}(\rho) + \bar{Y}(\rho)\rho - 2a\left(\frac{V'(\rho)+2V''(\rho)\rho}{k^2}\right)^2\left(\frac{Q^2}{k^2}\right)^{\beta-2}\right\} \tag{A.3}$$

The wave function renormalizations are defined by

$$Z_k(\rho; Q^2) = \frac{\partial}{\partial Q^2} K_z(Q^2)$$

$$\tilde{Z}_k(\rho; Q^2) = \frac{\partial}{\partial Q^2} K_y(Q^2) \tag{A.4}$$

We find that Z_k and $\tilde{Z}_k$ diverge for $Q^2 \to 0$ if $\beta < 2$. In this case the wave function renormalizations are only defined for $Q^2 > 0$.

For $\beta = 2$ the classical wave function renormalizations at $Q^2 = 0$ are finite

$$Z_k^{(0)}(\rho) \equiv Z_k^{(0)}(\rho; Q^2=0) = \bar{Z} - 2a\left(\frac{V'(\rho)}{k^2}\right)^2$$

$$\tilde{Z}_k^{(0)}(\rho) \equiv \tilde{Z}_k^{(0)}(\rho; Q^2=0) = \bar{Z} + \bar{Y}\rho - 2a\left(\frac{V'(\rho)+2V''(\rho)\rho}{k^2}\right)^2 \tag{A.5}$$

The corresponding classical evolution equations read

$$\eta_k^{(0)}(\rho) = -\frac{8a}{Z_k(\rho)}\left(\frac{U_k'(\rho)}{k^2}\right)^2$$

$$\tilde{\eta}_k^{(0)}(\rho) = -\frac{8a}{\tilde{Z}_k(\rho)}\left(\frac{U_k'(\rho)+2U_k''(\rho)\rho}{k^2}\right)^2 \tag{A.6}$$

Choosing $\rho=\rho_0$ at the minimum of the average potential ($U'(\rho_0)=0$) we find a nonvanishing classical anomalous dimension only for the radial excitation ($U''(\rho_0) = \bar{\lambda}$, $\tilde{Z} = \tilde{Z}_k(\rho_0)$)

$$\eta^{(o)} = \eta_k^{(o)}(\rho_o) = 0$$

$$\tilde{\eta}^{(o)} = \tilde{\eta}_k^{(o)}(\rho_o) = -32a\,\tilde{Z}^{-1}\,\bar{\lambda}^{2}\rho_o^{2}\,k^{-4} \tag{A.7}$$

Since $\tilde{\eta}^{(o)}$ vanishes for $\beta > 2$ one suspects that a computation of the wave function renormalization at $Q^2 = 0$ becomes very sensitive to the precise value of β if β is near two. The convergence of the loop expansion is presumably poor. We will choose β sufficiently larger than two in order to avoid complications from the classical wave function renormalization.

<u>Appendix B:</u> The integrals L_n^d, $\tilde{L}_n^d$

In this appendix we evaluate the dimensionless integrals (4.11)

$$L_n^d(w) = k^{2n-d}\frac{\partial}{\partial t}\int dx\, x^{\frac{d}{2}-1}\left(P_z + w\right)^{-n}$$

$$\tilde{L}_n^d(w) = k^{2n-d}\frac{\partial}{\partial t}\int dx\, x^{\frac{d}{2}-1}\left(P_z + Y\rho x + w\right)^{-n} \tag{B.1}$$

We start with

$$L_n^d \equiv L_n^d(0) = k^{2n-d}\int\frac{dx}{x}\, x^{\frac{d}{2}-n}\frac{\partial}{\partial t}T^n \tag{B.2}$$

where

$$T = x/P_z = \frac{1-f_k^2}{Z - f_k^2(Z-1)} \tag{B.3}$$

With

$$\frac{\partial}{\partial t}T = -2x\frac{\partial}{\partial x}T \tag{B.4}$$

we obtain an exact result for $n = d/2$

$$L_n^{2n}(o) = -2Z^{-n} \tag{B.5}$$

For $n \neq \frac{d}{2}$ we first note the behaviour of T for large and small x

$$\lim_{x\to\infty} T = Z^{-1}$$

$$\lim_{x\to 0} T = 2a\left(\frac{x}{k^2}\right)^{\beta} \tag{B.6}$$

and find that $L_n^d(o)$ is well defined for

$$n > -\frac{d}{2(\beta-1)} \tag{B.7}$$

Since $x\frac{\partial}{\partial x}T$ decreases exponentially for large x the integrals are always dominated by the integration region $x \approx k^2$. We observe the relation

$$x\frac{\partial}{\partial x}f_k^2 = \beta a\frac{\partial}{\partial a}f_k^2$$

$$x\frac{\partial}{\partial x}T = \beta a\frac{\partial}{\partial a}T \tag{B.8}$$

which yields, after partial integration, a power dependence of L on a according to

$$\left(n - \frac{d}{2} - \beta a \frac{\partial}{\partial a}\right) L_n^d = 0$$

$$L_n^d \sim a^{\frac{2n-d}{2\beta}} \tag{B.9}$$

Another general relation follows from the observation that T and $\frac{\partial}{\partial t}$ T only depend on the variable $z = 2a(x/k^2)^\beta$ since $f^2 = e^{-z}$. With

$$L_n^d = -2\,(2a)^{\frac{2n-d}{2\beta}} \int_0^\infty dz\, z^{\frac{d-2n}{2\beta}} \frac{\partial}{\partial z} T^n \tag{B.10}$$

we see that $L_n^d(o)$ depends on β and d only through the combination $(d-2n)/2\beta$ (for given n).

The dependence of L on Z becomes complicated due to the term $\sim f^2(Z-1)$ in the denominator of (B.3). Using

$$\frac{\partial P_z}{\partial Z} = x \tag{B.11}$$

we obtain the exact relation

$$\frac{\partial}{\partial Z} L_n^d = -n L_{n+1}^{d+2} \tag{B.12}$$

We first consider the approximation

$$P_z = \frac{Zx}{1-f_k^2}\ , \quad T = Z^{-1}(1-f_k^2) \tag{B.13}$$

which is justified if the integral is dominated by a region where $f^2(Z-1)/Z \ll 1$. In this limit the quantity (4.14)

$$1_n^d = -\frac{1}{2} Z^n L_n^d(o) \tag{B.14}$$

becomes independent of Z. Deviations from this approximation can be measured by the deviation of the ratio $1_n^d/1_{n+1}^{d+2}$ from one and result in the general relation

$$Z \frac{\partial}{\partial Z} 1_n^d = n(1_n^d - 1_{n+1}^{d+2}) \tag{B.15}$$

The approximation (B.13) becomes exact[1] for Z = 1.

With (B.13) and $y = x/k^2$ one has [ ]

$$\ell_n^d = -\frac{1}{2}(d-2n)\int_0^\infty dy\, y^{\frac{d}{2}-n-1}\left\{\left(1-f_k^2(y)\right)^n - \Theta_n\right\} \tag{B.16}$$

with $\Theta_n = 0$ for $n > d/2$, $\Theta_n = 1$ for $n < \frac{d}{2}$ and $f(y) = \exp(-a\, y^\beta)$. We conclude that all 1_n^d are positive. The integral (B.16) can be solved for $n < \frac{d}{2}$ in terms of Γ functions and we find in particular

$$\ell_1^d = (2a)^{-\frac{d-2}{2\beta}}\, \Gamma\left(1 + \frac{d-2}{2\beta}\right) \tag{B.17}$$

For $n > \frac{d}{2}$ we start directly from (B.10) with $p = (d-2n)/2\beta$

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1) In a different approach, we could also change the definition of the average action by considering suitable averages of normalized fields, such that (B.13) is valid exactly. The relations (B.19) (B.12) and (B.15) do not hold in this case and many formulae of this paper should be changed correspondingly.

$$\ell_n^d = (2a)^{-\frac{d-2n}{2\beta}} \hat{\ell}_{np}$$

$$\hat{\ell}_{np} = \int_0^\infty dz\, z^p \frac{\partial}{\partial z} (1 - e^{-z})^n$$

$$= n \int_0^\infty dz\, z^p e^{-z} (1 - e^{-z})^{n-1} \tag{B.18}$$

For $p > -1$ we can again express the result in terms of Γ-functions, e.g.

$$\hat{l}_{2p} = (2 - 2^{-p})\, \Gamma(1+p)$$

$$\hat{l}_{3p} = (3 - 3 \cdot 2^{-p} + 3^{-p})\, \Gamma(1 + p) \tag{B.19}$$

and we obtain in particular for $d = 2,3$

$$\ell_2^d = (2a)^{-\frac{d-4}{2\beta}} (2 - 2^{-\frac{d-4}{2\beta}})\, \Gamma(1 + \frac{d-4}{2\beta}) \tag{B.20}$$

For $p < -1$ one has to perform further partial integrations until the power of z exceeds -1.

We also discuss briefly the case $Z \gg 1$ where we can write formally

$$T = Z^{-1} - Z^{-2} \frac{f_k^2}{1-f_k^2} + O(Z^{-3}) \tag{B.21}$$

An expansion in powers of Z^{-1} is only meaningful, however, if f^2 is not near one. The conclusion $L_n^d(o) \sim Z^{-(n+1)}$ would be premature. By partial integration of (B.10) one obtains

$$L_n^d = -2(2a)^{\frac{2n-d}{2\beta}} \frac{2n-d}{\beta} \int_0^\infty dz\, z^{\frac{d-2n-2\beta}{2\beta}} (T^n - Z^{-n}) \tag{B.22}$$

and, in particular, for $n = 1$

$$L_1^d = -2Z^{-2}(2a)^{\frac{2-d}{2\beta}} \frac{d-2}{2\beta} \int_0^\infty dz\, z^{\frac{d-2-2\beta}{2\beta}} (e^z - 1 + Z^{-1})^{-1} \tag{B.23}$$

For large Z this integral is dominated by small z where $e^z - 1 \approx z$ and, for $\beta > \frac{d}{2} - 1$

$$\ell_1^d \approx \frac{d-2}{2\beta} (2a)^{\frac{2-d}{2\beta}} Z^{\frac{2-d}{2\beta}} \int_0^\infty dz\, z^{\frac{d-2-2\beta}{2\beta}} (z+1)^{-1} \tag{B.24}$$

For $d = 3,4$ and $\beta = 3$ (or larger) the dependence of l_1^d on Z is weak even for $Z \gg 1$. In addition, this Z-dependence can be absorbed by a Z-dependent rescaling of k which results in a rescaling of a. With this understanding we will neglect in the following the Z-dependence of l_n^d and use the approximations (B.17), (B.20).

For an evaluation of $\tilde{L}_n^d(w)$ in the <u>linear regime</u> $w \ll Zk^2$ we replace Z by $\tilde{Z}$ and expand $L_n^d(2\bar{\lambda}\rho_0)$ in powers of $w = 2\bar{\lambda}\rho_0$

$$L_n^d(2\bar\lambda\rho_0) = L_n^d(0) - 2n k^{-2}\bar\lambda\rho_0 L_{n+1}^d(0) + \dots \tag{B.25}$$

With $w/Zk^2 = 2\lambda\varkappa$ this results in the expansion

$$\tilde L_n^d(2\bar\lambda\rho_0) = -2\tilde Z^{-n}\left\{ \ell_n^d - 2n\frac{Z}{\tilde Z}\lambda\varkappa\,\ell_{n+1}^d + \dots\right\} \tag{B.26}$$

We finally need $\tilde L_n^d(w)$ for the <u>Goldstone regime</u> $Zk^2 \ll w$. We first consider

$$\begin{aligned}\tilde L_1^d(w) &= k^{2-d}\int dx\, x^{\frac{d}{2}-1}\frac{\partial}{\partial t}\left(P_y+w\right)^{-1}\\ &= 2k^{2-d}\int\frac{dx}{x}\,x^{\frac{d}{2}+1}\left(P_y+w\right)^{-2} x\frac{\partial}{\partial x}\left(\frac{P_y}{x}\right)\end{aligned} \tag{B.27}$$

The contribution from $x \gg k^2$ is exponentially suppressed and the integration region $x \approx k^2$ gives a contribution $\sim k^4/w^2$. We concentrate first on the integration region $0 < x < x_o$, $x_o \ll k^2$, $P_z(x_o) \ll w$, where we approximate $P_y \approx P_z$ and

$$\frac{P_z}{x} \approx \frac{1}{2a}\left(\frac{x}{k^2}\right)^{-\beta} \quad , \quad x \approx k^2\left(\frac{2a P_z}{k^2}\right)^{-\frac{1}{\beta-1}}$$

$$\begin{aligned} x\frac{\partial}{\partial x}\left(\frac{P_z}{x}\right) &\approx -\beta\frac{P_z}{x}\\ \frac{dx}{x} &\approx -\frac{1}{\beta-1}\,\frac{dP_z}{P_z}\end{aligned} \tag{B.28}$$

One obtains with $p_o = P_z(x_o)$

$$\tilde L_1^{d\,(i)}(w) = -\frac{2\beta}{\beta-1}k^2\int_{p_o}^{\infty} dp\,(p+w)^{-2}\left(\frac{2ap}{k^2}\right)^{-\frac{d}{2(\beta-1)}} \tag{B.29}$$

For $\beta > \frac{d}{2}+1$ we can extend the integration region to $p_o = 0$ such that

$$\begin{aligned}\tilde L_1^d(w) &= -2\frac{k^2}{w}\left(\frac{k^2}{2aw}\right)^{\frac{d}{2(\beta-1)}}\hat L_\beta^d\\ \hat L_\beta^d &= \frac{\beta}{\beta-1}\int_0^\infty d\tilde p\,(1+\tilde p)^{-2}\,\tilde p^{-\frac{d}{2(\beta-1)}}\end{aligned} \tag{B.30}$$

For $\beta\to\infty$ one approaches the limit [ ] $\hat L_\infty^d = 1$

$$\lim_{\beta\to\infty}\tilde L_1^d(w) \approx -2\frac{k^2}{w} \tag{B.31}$$

For finite values of β, however, $\tilde L_1^d(w)$ is suppressed by an additional power of $(k^2/2aw)$.

It remains the region $x \approx k^2$ ($x_0 < x < x_1$, $k^2 \ll x_1 \ll w$) which dominates the integral for $\beta < \frac{d}{2} + 1$. Here we can expand in powers of the small parameter $P_z/(w + Y_{\rho_0} x)$

$$\tilde{L}_1^{d(ii)}(w) = k^{2-d} \int_{x_0}^{x_1} dx\, x^{\frac{d}{2}-1} \frac{\partial}{\partial t} \left\{ (w+Y_{\rho_0}x)^{-1} - P_z\,(w+Y_{\rho_0}x)^{-2} + \ldots \right\} \tag{B.32}$$

The first term gives no contribution whereas the second reads in terms of the dimensionless variable $y = x/k^2$

$$\tilde{L}_1^{d(ii)}(w) = -2\left(\frac{k^2}{w}\right)^2 \hat{L}_d^{\beta}$$

$$\hat{L}_d^{\beta} = \int_{y_0}^{y_1} dy\, y^{\frac{d}{2}} \left(1 + \frac{Y_{\rho_0} k^2}{w} y\right)^{-2} T^{-2}\, y \frac{\partial}{\partial y} T \tag{B.33}$$

It can be evaluated using the approximation (B.13) but we will not need its explicit form. For $\beta > \frac{d}{2}+1$ the integral diverges for $y_0 \to 0$ and the breakdown of the expansion in powers of $P_z/(w + Y_{\rho_0} x)$ results in the power behaviour (B.30).

This phenomenon is closely related to a general breakdown of an expansion of $\tilde{L}_1^d(w)$ in powers of w^{-1} around $w^{-1} = 0$. The integral (B.27) vanishes to first non-leading order w^{-1}. The expansion involves, nevertheless, higher and higher powers of P_z. In consequence the expansion coefficients always become infrared divergent integrals for a sufficiently high order of the expansion (in our case in order w^{-2}). This is a general feature of all

the integrals L and M. It makes the expansion in $\varkappa^{-1}$ in the Goldstone regime cumbersome beyond a certain order. More generally, the effects of heavy particles with mass M on the average action at much lower energy scales $k^2 \ll M^2$ are not expected to be analytic in k^2/M^2. Typically only a few orders in the expansion in k^2/M^2 exist.

The integrals $\tilde{L}_n^d(w)$ for $n > 1$ can finally be obtained from $\tilde{L}_1^d(w)$ by repeated differentiation with respect to w, using

$$\tilde{L}_{n+1}^{d}(w) = -\frac{k^2}{n} \frac{\partial}{\partial w} L_n^d(w) \tag{B.34}$$

This implies

$$\tilde{L}_n^d(w) \sim \left(\frac{k^2}{w}\right)^{n-1} \tilde{L}_1^d(w) \tag{B.35}$$

Appendix C: The integrals $M^d_{n\ ,n}\ (w_1,w_2)$

In this appendix we evaluate the dimensionless integrals $M^d_{n_1,n_2}$ introduced in sect. 6 for $n_1, n_2 \geqslant 0$ and $n_1 + n_2 \geqslant 2$

$$M^d_{n_1,n_2}(w_1,w_2) = k^{2(n_1+n_2-1)-d} \frac{\partial}{\partial t} \int dx\, x^{\frac{d}{2}} \dot{P}_z^{\,2} \left(P_z + w_1\right)^{-n_1} \left(P_y + w_2\right)^{-n_2} \tag{C.1}$$

We start with

$$M_n^d = M_{n,0}^d(0) = k^{2n-d-2} \int dx\, x^{\frac{d}{2}-n} \frac{\partial}{\partial t}\left[\dot{P}_z^2 \left(\frac{x}{P_z}\right)^n \right]$$

$$= -2k^{2n-d-2} \int dx\, x^{\frac{d}{2}-n+1} \frac{\partial}{\partial x}\left[\dot{P}_z^2 \left(\frac{x}{P_z}\right)^n \right] \tag{C.2}$$

For d = 2(n-1) the integral can be evaluated exactly

$$M_n^{2n-2} = -2\, z^{2-n} \text{ for } d > 2$$

$$M_2^2 = -2 + 2(\beta - 1)^2 \tag{C.3}$$

For general $d \neq 2n - 2$ the integral (C.2) depends on β and becomes rather complicated. Using (3.4)

$$T = \frac{x}{P_z} = (1 - f_k^2)(1 + (z-1)(1-f_k^2))^{-1}$$

$$S = \frac{x\dot{P}_z}{P_z} = 1 - \frac{x}{T}\frac{\partial T}{\partial x} = \frac{1 - f_k^2 + x\frac{\partial}{\partial x} f_k^2 + (z-1)(1-f_k^2)^2}{(1-f_k^2)(1+(z-1)(1-f_k^2))} \tag{C.4}$$

we write

$$M_n^d = -2\, k^{2n-d-2} \int dx\, x^{\frac{d}{2}-(n-1)} \frac{\partial}{\partial x}\left(S^2\, T^{n-2} \right) \tag{C.5}$$

and observe the limits for small and large x

$$\lim_{x\to\infty} S = 1$$

$$\lim_{x\to\infty} \frac{\partial}{\partial x}\left(S^2 T^{n-2}\right) \sim \exp\left(-2a\left(\frac{x}{k^2}\right)^\beta\right)$$

$$\lim_{x\to 0} S = 1 - \beta \tag{C.6}$$

We conclude that M_n^d is both ultraviolet and infrared finite for

$$n > 2 - \frac{d-2}{2(\beta-1)} \tag{C.7}$$

With this condition the integral M_n^d is dominated by values of x near k^2. It is convenient to use

$$x\frac{\partial}{\partial x} f_k^2 = \beta a \frac{\partial}{\partial a} f_k^2 = -2\beta a \left(\frac{x}{k^2}\right)^\beta f_k^2 \tag{C.8}$$

and the dimensionless variable $y = x/k^2$ such that

$$M_n^d = -2\beta a \frac{\partial}{\partial a} \tilde{M}_n^d \tag{C.9}$$

$$\tilde{M}_n^d = \int_0^\infty dy\, y^{\frac{d}{2}-n} \{ S^2(y) T^{n-2}(y) - \tilde{c}_d \} \tag{C.10}$$

Here we choose the constant $\tilde{c}_d$ such that M_n^d is finite

$$\tilde{c}_d = \begin{cases} 0 & \text{for } d < 2n-2 \\ z^{2-n} & \text{for } d > 2n-2 \end{cases} \tag{C.11}$$

Let us concentrate (as in appendix B) on the case where we can approximate

$$T = z^{-1}(1-f_k^2)$$

$$S = T - 2\beta a\, y^\beta f_k^2(y)\, Z^{-1} \tag{C.12}$$

One obtains, with m_n^d independent of z,

$$M_n^d = -\,2z^{2-n}\, m_n^d \tag{C.13}$$

$$m_n^d = 2a\beta \int_0^\infty dy\, y^{\frac{d}{2}-n+\beta} f_k^2 \cdot$$

$$\{ (2-2\beta+4a\beta y^\beta)(1-f_k^2)^{n-4}(1-f_k^2-2a\beta y^\beta f_k^2)$$

$$+ (n-4)(1-f_k^2)^{n-5}(1-f_k^2-2a\beta y^\beta f_k^2)^2 \} \tag{C.14}$$

For $n \geq 4$ and $\beta > n - \frac{d}{2} - 1$ this integral can be solved explicitly with the help of

$$F_q^m(a,\beta) = \int_0^\infty dy\, y^q f_k^{2m}(y) = \frac{1}{\beta}\,\Gamma\left(\frac{1+q}{\beta}\right)(2ma)^{-\frac{1+q}{\beta}}$$

$$m > 0;\ q > -1 \tag{C.15}$$

In particular, one finds for $n = 4$

$$m_4^d = 4a\beta \{ -(\beta-1) F^1_{\frac{d}{2}-4+\beta} + 2a\beta\, F^1_{\frac{d}{2}-4+2\beta}$$

$$+ (\beta-1) F^2_{\frac{d}{2}-4+\beta} + 2a\beta(\beta-2) F^2_{\frac{d}{2}-4+2\beta} - 4a^2\beta^2 F^2_{\frac{d}{2}-4+3\beta} \}$$

$$= 2\,\Gamma\left(1+\frac{d-6}{2\beta}\right)(2a)^{-\frac{d-6}{2\beta}} \cdot$$

$$\left\{ \frac{d}{2} - 2 - 2^{-\frac{d-6}{2\beta}}\left(\frac{d^2}{32} - \frac{d}{8} + \frac{1}{8} + \frac{d-6}{16}\beta\right) \right\} \tag{C.16}$$

We note that m_4^d diverges for a sharp momentum cutoff ($\beta \to \infty$) except for $d = 6$, $m_4^6 = 1$. The origin of this behaviour is discussed in more detail in appendix E. For finite β and $\beta > 2$ the constant m_4^d is always positive for $2 \leq d \leq 6$. For example, one finds for $\beta = 3$

$$m_4^d = 2\,\Gamma\left(\frac{d}{6}\right)(2a)^{1-\frac{d}{6}}\left\{\frac{d}{2} - 2 + 2^{1-\frac{d}{6}}\left(1 - \frac{d}{16} - \frac{d^2}{32}\right)\right\} \tag{C.17}$$

For easy reference, we quote the values for $d = 2, 3, 4$ explicitly

$$m_4^2 = 2\,\Gamma\left(1-\frac{2}{\beta}\right)(2a)^{\frac{2}{\beta}}\left(\frac{\beta}{4}\,2^{\frac{2}{\beta}} - 1\right) \tag{C.18}$$

$$m_4^3 = \Gamma\left(1-\frac{3}{2\beta}\right)(2a)^{\frac{3}{2\beta}}\left\{(6\beta-1)\,2^{\left(\frac{3}{2\beta}-4\right)} - 1\right\} \tag{C.19}$$

$$m_4^4 = \frac{1}{4}\,\Gamma\left(1-\frac{1}{\beta}\right)(4a)^{\frac{1}{\beta}}\,(\beta - 1) \tag{C.20}$$

For $n < 4$ we use partial integration of (C.5)

$$m_n^d = \frac{2n-d-2}{2\beta}\,(2a)^{\frac{2n-d-2}{2\beta}}\,\tilde{m}_n^d \tag{C.21}$$

$$\tilde{m}_n^d = \int_0^\infty dz\, z^{-\left(1+\frac{2n-d-2}{2\beta}\right)}\,.$$

$$\left\{\left(1-e^{-z}+\beta z e^{-z}\right)^2\left(1-e^{-z}\right)^{n-4} - c_d\right\} \tag{C.22}$$

with $c_d = 1$ for $d > 2n - 2$ and $c_d = 0$ otherwise. For $n = 3$ and $d < 4$ one obtains, with $q = (d-4)/2\beta$ and $\beta > 1$

$$\tilde{m}_3^d = \left(2\beta - \frac{1}{q}\right)\int dz\, z^q e^{-z}$$

$$-\beta^2 \int dz\, z^{1+q} e^{-z}$$

$$+\beta^2 \int dz\, z^{1+q}\left(e^z - 1\right)^{-1} \tag{C.23}$$

This can be expressed in terms of Γ- functions and Riemann ζ-functions

$$\tilde{m}_3^d = \left\{\beta^2(1+q)\left[\zeta(2+q) - 1\right] + 2\beta - \frac{1}{q}\right\}\Gamma(1+q) \tag{C.24}$$

such that

$$\tilde{m}_3^2 = \left\{3\beta + \beta(\beta-1)\left[\zeta\left(2-\frac{1}{\beta}\right) - 1\right]\right\}\Gamma\left(1-\frac{1}{\beta}\right) \tag{C.25}$$

$$\tilde{m}_3^3 = \left\{4\beta + \frac{1}{2}\beta(2\beta-1)\left[\zeta\left(2-\frac{1}{2\beta}\right) - 1\right]\right\}\Gamma\left(1-\frac{1}{2\beta}\right) \tag{C.26}$$

For $n = 2$ and $d > 2$ we have $q = (d-2)/2\beta$ and find

$$\tilde{m}_2^d = \left\{\left(2\beta + \beta^2(1+q)\right)\zeta(1+q) - \beta^2(1+q)\,\zeta(2+q)\right\}\Gamma(1+q) \tag{C.27}$$

and in particular

$$\tilde{m}_2^3 = \{\beta(\beta+\tfrac{5}{2})\,\varphi(1+\tfrac{1}{2\beta}) - \beta(\beta+\tfrac{1}{2})\,\varphi(2+\tfrac{1}{2\beta})\}\,\Gamma(1+\tfrac{1}{2\beta}) \tag{C.28}$$

$$\tilde{m}_2^4 = \{\beta(\beta+3)\,\varphi(1+\tfrac{1}{\beta}) - \beta(\beta+1)\,\varphi(2+\tfrac{1}{\beta})\}\,\Gamma(1+\tfrac{1}{\beta}) \tag{C.29}$$

We next need the integrals $M^d_{n_1,n_2}(0,w)$ for $w > 0$. We observe the general relation

$$M^d_{n_1,n_2}(w_1,w_2) + \frac{Y\rho k^2}{w_2 - w_1} M^{d+2}_{n_1,n_2}(w_1,w_2)$$

$$= \frac{k^2}{w_2 - w_1}\left(M^d_{n_1,n_2-1}(w_1,w_2) - M^d_{n_1-1,n_2}(w_1,w_2)\right) \tag{C.30}$$

which implies, in particular, a useful identity for $n_2 = 2$

$$M^d_{n,2}(0,w) + \frac{Yk^2}{\bar{\lambda}} M^{d+2}_{n,2}(0,w) + \frac{1}{4}\left(\frac{Yk^2}{\bar{\lambda}}\right)^2 M^{d+4}_{n,2}(0,w)$$

$$= \frac{k^4}{w^2}\left(M^d_{n,0}(0) + M^d_{n-2,2}(0,w) - 2M^d_{n-1,1}(0,w)\right) \tag{C.31}$$

In the <u>linear regime</u> $w \ll Zk^2$ we expand

$$(P_y + w)^{-n} = P_z^{-n} - n(Y\rho x + w)\,P_z^{-(n+1)}$$

$$+ \frac{1}{2} n(n+1)(Y\rho x + w)^2\, P_z^{-(n+2)} - \dots \tag{C.32}$$

and therefore

$$M^d_{n_1,n_2}(0,w) = M^d_{n_1+n_2} - n_2\left(Y\rho M^{d+2}_{n_1+n_2+1} + \frac{w}{k^2} M^d_{n_1+n_2+1}\right)$$

$$+ \frac{1}{2} n_2(n_2+1)\left(Y^2\rho^2 M^{d+4}_{n_1+n_2+2} + \frac{2Y\rho w}{k^2} M^{d+2}_{n_1+n_2+2}\right.$$

$$\left. + \frac{w^2}{k^4} M^d_{n_1+n_2+2}\right) - \dots \tag{C.33}$$

In the <u>Goldstone regime</u> $w \gg Zk^2$ we restrict our discussion to $Y = 0$. This is a valid approximation as long as $Y\rho k^2 \ll w$. We use partial integration and evaluate the integral

$$M^{d}_{n_1,n_2}(0,w) = \frac{k^{2(n_1+n_2-1)-d}}{(n_1-1)(n_2-1)} \frac{\partial}{\partial t} \int dx\, x^{\frac{d}{2}} \frac{\partial}{\partial x} P_z^{-n_1+1} \frac{\partial}{\partial x} (P_z+w)^{-n_2+1}$$

$$= - \frac{k^{2(n_1+n_2-1)-d}}{(n_1-1)(n_2-1)} \frac{\partial}{\partial t} \int dx\, (P_z+w)^{-n_2+1}$$

$$\left\{ \frac{d}{2} x^{\frac{d}{2}-1} \frac{\partial}{\partial x} P_z^{-n_1+1} + x^{\frac{d}{2}} \frac{\partial^2}{\partial x^2} P_z^{-n_1+1} \right\} \tag{C.34}$$

The contribution for large $x \gg k^2$ is exponentially suppressed and the small x region is power suppressed according to (B.6) (C.7) once $P_z(x)$ becomes larger than w. In consequence, the integral is dominated by values of x for which $P_z(x) \lesssim w$.

For $n_1 > 2 - \frac{d-2}{2(\beta-1)}$ we can restrict the integration range $x_0 < x < x_1$ with $k^2 \ll x_1 \ll w$ and $x_0 < k^2$, $P_z(x_0) = \alpha w$, $k^2/w \ll \alpha \ll 1$. We consider first this integration region where $P_z + w$ is dominated by w. We expand in powers of w^{-1}

$$M^{d}_{n_1,n_2}(0,w) \approx \frac{2k^{2n_1-d}}{(n_1-1)(n_2-1)} \left(\frac{k^2}{w}\right)^{n_2-1} \left\{ j_0 - \frac{n_2-1}{w} j_1 \right.$$

$$\left. + \frac{1}{2} \frac{n_2(n_2-1)}{w^2} j_2 + \dots \right\} \tag{C.35}$$

The lowest order term

$$j_0 = \int_{x_0}^{\infty} dx\, x^{\frac{d}{2}-n_1-1} \left\{ \left(x\frac{\partial}{\partial x}\right)^3 T^{n_1-1} \right.$$

$$\left. + \left(\frac{d}{2} - 2n_1 + 1\right)\left(x\frac{\partial}{\partial x}\right)^2 T^{n_1-1} + \left(n_1 - \frac{d}{2}\right)(n_1-1)\, x\frac{\partial}{\partial x} T^{n_1-1} \right\} \tag{C.36}$$

remains finite in the limit $x_0 \to 0$ for $n_1 > 1 - \frac{d-2}{2(\beta-1)}$. We also evaluate all integrals j_m for $x_1 \to \infty$ since the UV-contribution is exponentially suppressed. We write

$$j_0 = \bar{j}_0 - \int_0^{x_0} dx\, x^{\frac{d}{2}-n_1-1} \left(x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + 1 - n_1\right)\left(x\frac{\partial}{\partial x} + \frac{d}{2} - n_1\right) T^{n_1-1} \tag{C.37}$$

and establish that the integral $\bar{j}_0$ over the range $0 \le x \le \infty$ vanishes

$$\bar{j}_0 = -\frac{1}{4}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\partial}{\partial t} + n_1 - 1\right)\left(\frac{\partial}{\partial t} + 2n_1 - d\right)\int_0^{\infty} dx\, x^{\frac{d}{2}-n_1-1} T^{n_1-1} = 0 \tag{C.38}$$

by use of the identity

$$0 = \frac{\partial}{\partial t} L_n^d(0) = k^{2n-d}\left(\frac{\partial}{\partial t} + 2n - d\right)\int_0^\infty dx\, x^{\frac{d}{2}-n-1} \frac{\partial}{\partial t} T^n \tag{C.39}$$

Since x_o is much smaller than k^2 we can approximate for x around x_o or smaller

$$T = 2a\left(\frac{x}{k^2}\right)^\beta$$

$$x_o^{\beta-1} = \frac{1}{2a\alpha}\; k^{2(\beta-1)}\; \frac{k^2}{w} \tag{C.40}$$

and therefore

$$j_o \sim x_o^{\frac{d}{2}-1}\; w^{-(n_1-1)} \tag{C.41}$$

As expected, this is negligible compared to the contribution from j_1 as long as $n_1 > 2 - \frac{d-2}{2(\beta-1)}$. For the special values $n_1 = 2$, $d = 2$ we obtain

$$j_o = -\frac{\beta(\beta-1)}{\alpha w}\left(1 + O\left(\left(\frac{k^2}{\alpha w}\right)^{\frac{\beta}{\beta-1}}\right)\right) \tag{C.42}$$

It remains to evaluate the contributions from j_m, $m \geqslant 1$,

$$j_m = -\frac{1}{2}\frac{\partial}{\partial t}\int_{x_o}^\infty dx\, x^{\frac{d}{2}-1-n_1+m}\, T^{-m}$$

$$\cdot\left\{\left(x\frac{\partial}{\partial x}\right)^2 + \left(\frac{d}{2} - 2n_1 + 1\right)x\frac{\partial}{\partial x} + (n_1-1)\left(n_1 - \frac{d}{2}\right)\right\} T^{n_1-1} \tag{C.43}$$

which are naively of order $w^{-(n_2+m-1)}$. For $m < n_1 - 1 + \frac{d-2}{2(\beta-1)}$ the integral j_m is infrared finite and we take $x_o \to 0$. In this case one has

$$j_m \sim k^{d+2m-2n_1} \tag{C.44}$$

and the naive estimate holds. In the opposite case j_m depends strongly on x_o. This signals a breakdown of the expansion in powers of w^{-1} as discussed in appendix B. The integral j_1 has an exact solution for $d = 2n_1 - 2$

$$j_1 = (n_1-1)\, Z^{-n_1+2} - T^{-1}(x_o)\left\{\left(x\frac{\partial}{\partial x}\right)^2 - n_1 x\frac{\partial}{\partial x} + (n_1-1)\right\} T^{n_1-1}\Big|_{x_o} \tag{C.45}$$

(and similar for other j_m if $d - 2n_1 + 2m = 0$). We define

$$M^d_{n_1,n_2}(0,w) = -2\left(\frac{k^2}{w}\right)^{n_2} Z^{-(n_1-2)}\, m^d_{n_1,n_2} \tag{C.46}$$

and obtain the leading order contribution for $n_1 > 2 - \frac{d-2}{2(\beta-1)}$

$$m^{d}_{n_1,n_2} = \frac{1}{n_1-1} Z^{n_1-2} k^{2n_1-d-2} f_1 \tag{C.47}$$

and, in particular, for $d > 2$ $(n_1 > 2)$

$$m^{2n_1-2}_{n_1,n_2} = 1 \tag{C.48}$$

We are particularly interested in the case $n_1 = n_2 = 2$. For $\underline{d > 2}$ we can take $x_0 \to 0$ and obtain for the leading contribution $(y = x/k^2)$

$$m^{d}_{2,2} = k^{2-d} f_1$$

$$= -\frac{1}{2}(d-2)\int_0^\infty dy\, y^{\frac{d}{2}-2} T^{-1}(y)\left\{\left(y\frac{\partial}{\partial y}\right)^2 + \left(\frac{d}{2}-3\right) y \frac{\partial}{\partial y}\right\} T(y) \tag{C.49}$$

We approximate again $T \sim (1 - f_k^2)$ and use the integration variable $z = 2ay^\beta$

$$m^{d}_{2,2} = -\frac{1}{2}(d-2)(2a)^{\frac{2-d}{2\beta}} \cdot$$

$$\left\{\left(\frac{d}{2}+\beta-3\right)\int_0^\infty dz\, z^{\frac{d-2}{2\beta}} \frac{e^{-z}}{1-e^{-z}}\right.$$

$$\left. -\beta\int_0^\infty dz\, z^{\frac{d-2}{2\beta}+1} \frac{e^{-z}}{1-e^{-z}}\right\} \tag{C.50}$$

The integrals can be expressed in terms of Riemann's ζ function

$$m^{d}_{2,2} = -\frac{1}{2}(d-2)(2a)^{\frac{2-d}{2\beta}} \Gamma\left(1+\frac{d-2}{2\beta}\right)$$

$$\cdot\left\{\left(\frac{d}{2}+\beta-3\right)\zeta\left(1+\frac{d-2}{2\beta}\right) - \left(\frac{d}{2}+\beta-1\right)\zeta\left(2+\frac{d-2}{2\beta}\right)\right\} \tag{C.51}$$

The case $\underline{d=2}$ is special since it is at the edge where the expansion in powers of w^{-1} breaks down in leading order. One finds (C.45)

$$f_1 = 2\, T^{-1} x \frac{\partial}{\partial x} T_{|x_0} - T^{-1}\left(x\frac{\partial}{\partial x}\right)^2 T_{|x_0} \tag{C.52}$$

and therefore, with (C.40),

$$f_1 = -\beta(\beta-2)\left(1 + O\left(\left(\frac{k^2}{\alpha w}\right)^{\frac{\beta}{\beta-1}}\right)\right) \tag{C.53}$$

Similarly, we obtain

$$j_2 = -\alpha\beta(\beta-1)w\left(1 + O\left(\left(\frac{k^2}{\alpha w}\right)^{\frac{\beta}{\beta-1}}\right)\right) \tag{C.54}$$

whereas contributions from higher j_m are suppressed by higher powers of α.

For $n_1 \leq 2 - \frac{d-2}{2(\beta-1)}$ we finally have to add the contribution from the integration region $0 \leq x \leq x_o$ in (C.34) where we can use the approximation (C.40)[2]

$$\Delta M^d_{n_1,n_2}(0,w) = -\frac{k^{2(n_1+n_2-1)-d}}{(n_1-1)(n_2-1)}\,\tilde{j} \tag{C.55}$$

$$\tilde{j} = (\beta-1)(n_1-1)\left[\frac{d}{2} - 1 + (\beta-1)(n_1-1)\right] \cdot \frac{\partial}{\partial t}\int_0^{x_o} dx\, x^{\frac{d}{2}-2}\, P_2^{-n_1+1}\,(P_2+w)^{-n_2+1}$$

$$= (n_1-1)\left[\frac{d}{2} - 1 + (\beta-1)(n_1-1)\right] \cdot \frac{\partial}{\partial t}\int_{P_2(x_0)}^{\infty} dP\, P^{-n_1}(P+w)^{-n_2+1}\left(2aPk^{-2\beta}\right)^{-\frac{d-2}{2(\beta-1)}} \tag{C.56}$$

This integral is easily solved for d = 2 since the only k dependence arises from the integration boundary

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2) The derivative $\frac{\partial}{\partial t}$ has to be taken at fixed x_o.

$$\tilde{j} = -(n_1-1)^2(\beta-1)\frac{\partial P_2(x_0)}{\partial t}\, P_2(x_0)^{-n_1}\left(P_2(x_0)+w\right)^{-n_2+1}$$

$$= -2\beta(\beta-1)(n_1-1)^2\, w^{-(n_1+n_2-2)}\, \alpha^{-n_1+1}(1+\alpha)^{-n_2+1} \tag{C.57}$$

Combining (C.35) and (C.54) we obtain, with (C.42), (C.53), (C.54) and (C.56),

$$M^2_{2,2}(0,w) = -2\beta\,\frac{k^4}{w^2}$$

$$m^2_{2,2} = \beta \tag{C.58}$$

We finally turn to the case $n_1 < 2 - \frac{d-2}{2(\beta-1)}$. (This is relevant for $n_1 = 0, 1$.) Here $M^d_{n_1,n_2}(0,w)$ is dominated by small values of x, $x \ll x_o$. It is most convenient to start from (C.1) and we write

$$M^d_{n_1,n_2}(0,w) = \left(\frac{k^2}{w}\right)^{n_1+n_2-1-\frac{d}{2}} \frac{\partial}{\partial t}\,\tilde{M}(w) \tag{C.59}$$

$$\tilde{M}(w) = \int\frac{dx}{x}\, x^{\frac{d}{2}+1}\, \dot{P}_2^{\,2}\, P_2^{-n_1}(P_2+w)^{-n_2}\, w^{n_1+n_2-1-\frac{d}{2}}$$

$$= \int\frac{dx}{x}\, S^2\, T^{-n_1-2}\left(\frac{x}{w}\right)^{\frac{d}{2}+1-n_1}\left(\frac{w}{P_2+w}\right)^{n_2} \tag{C.60}$$

The integral $\tilde{M}(w)$ exists for $\Lambda \to \infty$ if

$$n_1 + n_2 > \frac{d}{2} + 1$$

$$n_1 + n_2 > 2 - \frac{d-2}{2(\beta-1)} \tag{C.61}$$

Then $\tilde{M}$ is a function of (k^2/w) and

$$\frac{\partial}{\partial t} \tilde{M}(w) = -2w \frac{\partial}{\partial w} \tilde{M}(w) \tag{C.62}$$

For an evaluation of $\tilde{M}$ we concentrate on the region $0 < x < x_o$, $x_o \ll k^2$, $P_z(x_o) \ll w$. We use (B.28) with $\tilde{p} = P_z(x)/w$, $P_z(x_o) = \alpha_o w$, $\alpha_o \ll 1$. We differ here slightly from the treatment above and choose for convenience α_o independent of k and w

$$\tilde{M}^{(i)}(w) = (\beta-1)(2a)^{\frac{2-d}{2(\beta-1)}} \left(\frac{k^2}{w}\right)^{\frac{(d-2)\beta}{2(\beta-1)}} \hat{m}^d_\beta$$

$$\hat{m}^d_\beta = \int_{\alpha_o}^{\infty} d\tilde{p}\, \tilde{p}^{\left(1-n_1-\frac{d-2}{2(\beta-1)}\right)} (\tilde{p}+1)^{-n_2} \tag{C.63}$$

For $n_1 < 2 - \frac{d-2}{2(\beta-1)}$ the contribution from $\tilde{p} \ll 1$ can be neglected and we evaluate $\hat{m}^d_\beta$ for $\alpha_o = 0$

$$M^d_{n_1,n_2}(o,w) = (d-2)\beta(2a)^{\frac{2-d}{2(\beta-1)}} \hat{m}^d_\beta \left(\frac{k^2}{w}\right)^{n_1+n_2-2+\frac{d-2}{2(\beta-1)}} \tag{C.64}$$

For $d > 2$ we find that $M^d_{n_1,n_2}(o,w)$ is suppressed by a power of w^{-1} which differs from the naive estimate $M \sim w^{-n_2}$.

We may also discuss the boundary case $d = 2$, $n_1 = 2$ in this formulation

$$\frac{\partial}{\partial t} \tilde{M}(w) = -2n_2 \int \frac{dx}{x} \frac{P_z}{w} \left(\frac{w}{P_z + w}\right)^{n_2+1} S^2 \tag{C.65}$$

This integral is dominated by the two regions for x where $P_z(x) \approx w$. For $x < x_o$ we use

$$\frac{dx}{x} S^2 = -(\beta-1) \frac{dP_z}{P_z} \tag{C.66}$$

and find

$$\frac{\partial}{\partial t} \tilde{M}^{(i)}(w) = -2n_2(\beta-1) \int_0^\infty d\tilde{p} (1+\tilde{p})^{-(n_2+1)}$$
$$= -2(\beta-1) \tag{C.67}$$

Similarly we obtain for $x > x_1 \gg k^2$

$$\frac{dx}{x}\ S^2 - \frac{dP_z}{P_z} \tag{C.68}$$

$$\frac{\partial}{\partial t}\ \tilde{M}^{(iii)}(w) = -2 \tag{C.69}$$

and generalize (C.58) to arbitrary n_2

$$M^2_{2,n}(0,w) = -2\beta\left(\frac{k^2}{w}\right)^{n_2}$$

$$m^2_{2,n} = \beta \tag{C.70}$$

<u>Appendix D:</u> Anomalous dimension in the Goldstone regime for two dimensions

We evaluate here the wave function renormalization Z_k (5.14) (5.15) and the corresponding anomalous dimension η for d=2 and $Zk^2 \ll 2\bar{\lambda}\rho_0 = w$. We account for the difference between Z and $\tilde{Z}$ but omit the terms proportional to the derivatives of Z, i.e. we put Z' = Z" = 0. From (5.14) we find

$$Z_k^{(a)} = v_2 Y \int_0^{\Lambda^2} dx\ P_z^{-1} \tag{D.1}$$

$$\eta^{(a)} = -v_2 Y Z^{-1} L_1^2(0) = 2 v_2 Y Z^{-2} \tag{D.2}$$

Similarly, we obtain from (5.15)

$$Z_k^{(b)} = 4\rho_0 \bar{\lambda}^2 v_2 \int_0^{\Lambda^2} dx\,(P_y + w)^{-1}\left(1 + \frac{Y\rho_0 x}{w}\right)^2 \cdot$$
$$(\dot{P}_z + x\ddot{P}_z - 2x\dot{P}_z^2 P_z^{-1})\ P_z^{-2}$$
$$= v_2 \rho_0^{-1} \mathcal{Z} \tag{D.3}$$

with

$$\mathcal{Z} = -\int_0^{\Lambda^2} \frac{dx}{x}\,(P_z + w + Y\rho_0 x)^{-1}(w + Y\rho_0 x)^2 \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1}$$
$$= \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 \tag{D.4}$$

$$\mathcal{Z}_1 = -w \int_0^{\Lambda^2} \frac{dx}{x}\left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \tag{D.5}$$

$$\mathcal{Z}_2 = -Y\rho_0 \int_0^{\Lambda^2} dx \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \tag{D.6}$$

$$Z_3 = \int_\varepsilon^{\Lambda^2} \frac{dx}{x}\, P_z \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1}$$

$$- \int_\varepsilon^{\Lambda^2} \frac{dx}{x}\, (P_z + w + Y\beta_0 x)^{-1} P_z^2 \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \tag{D.7}$$

The infrared divergences of the two terms in Z_3 cancel and we can take the limit $\varepsilon \to 0$ for the sum without a problem.

One finds that $Z_1 \sim w/(Z\Lambda^2)$ vanishes for $\Lambda \to \infty$ and gives no contribution to the anomalous dimension, $\eta_1^{(b)} = 0$, whereas Z_2 has a simple k dependence since the integrand is dimensionless. Its contribution to the anomalous dimension

$$\eta_2^{(b)} = v_2 Y Z^{-1} \frac{\partial}{\partial t} \int_0^{\Lambda^2} \frac{dx}{x} \left[x \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \right]$$

$$= -2 v_2 Y Z^{-1} \int_0^{\Lambda^2} dx \frac{d}{dx} \left[x \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \right]$$

$$= -2 v_2 Y Z^{-2} \tag{D.8}$$

exactly cancels the contribution $\eta^{(a)}$(D.2). The contribution of the first term in Z_3 to the anomalous dimension is also easily evaluated

$$\eta_3^{(b)} = -v_2 \kappa^{-1} \frac{\partial}{\partial t} \int_\varepsilon^{\Lambda^2} \frac{dx}{x}\, P_z \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1}$$

$$= 2 v_2 \kappa^{-1} \int_\varepsilon^{\Lambda^2} dx \frac{d}{dx} \left[P_z \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \right]$$

$$= 2 v_2 \kappa^{-1} \left(1 - (\beta - 1)^2 \right) \tag{D.9}$$

Only the second term in (D.7) needs some closer consideration

$$\eta_{3'}^{(b)} = -v_2 \kappa^{-1} \frac{\partial}{\partial t} Z_{3'}$$

$$Z_{3'} = -\int_\varepsilon^{\Lambda^2} \frac{dx}{x} (P_z + w + Y\beta_0 x)^{-1} P_z^2 \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \tag{D.10}$$

Since $Z_{3'}$ is dimensionless it can only depend on the ratios Λ^2/w, w/k^2, and k^2/ε. We may therefore evaluate the anomalous dimension using

$$\eta_{3'}^{(b)} = 2 v_2 \kappa^{-1} \left(\Lambda^2 \frac{\partial}{\partial \Lambda^2} + w \frac{\partial}{\partial w} + \varepsilon \frac{\partial}{\partial \varepsilon} \right) Z_{3'}$$

$$= -2 v_2 \kappa^{-1} \left\{ \frac{Z}{Z + Y\beta_0} - w I_w - (\beta - 1)^2 \right\} \tag{D.11}$$

with

$$I_w = \int_0^\infty \frac{dx}{x}\,(P_z + w + Y_{\rho_0} x)^{-2}\, P_z^2 \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \tag{D.12}$$

We split the integration into three regions ($I_w = I^{(i)} + I^{(ii)} + I^{(iii)}$) where the following approximations hold:

(i) $\quad 0 < x \le x_0 \quad , \; x_0 \ll k^2$

$$P_z \approx (2a)^{-1}\, k^{2/\beta}\, x^{1-\beta}$$

$$\left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \approx (\beta-1)^2 P_z^{-1} \quad , \quad \frac{dx}{x} \approx -\frac{1}{\beta-1}\,\frac{dP_z}{P_z}$$

$$Y_{\rho_0} x \ll w + P_z$$

(ii) $\quad x_0 < x < x_1 \quad , \; k^2 \ll x_1 \ll w$

$$P_z \ll w + Y_{\rho_0} x$$

(iii) $\quad x \geqslant x_1$

$$P_z \approx Z x \quad , \quad \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \approx P_z^{-1} \tag{D.13}$$

Only the approximation $P_z \ll w + Y_{\rho_0} x$ in region ii) relies on the assumption $Zk^2 \ll w$. The contribution of the first region reads (with $p_0 = P_z(x_0)$)

$$I^{(i)} = (\beta-1)\int_{p_0}^\infty dP\left\{(P+w)^{-2} - 2Y_{\rho_0}\, x(P)(P+w)^{-3} + \cdots\right\} \tag{D.14}$$

where $x(P_z)$ is given by (B.28). One finds in analogy with (B.29) (for $\beta > 2$)

$$I^{(i)} = \frac{\beta-1}{w+p_0} + \frac{(\beta-1)^2}{2\beta}\,\frac{Y_{\rho_0}}{k^2}\,\tilde{L}_2^2(w) + \ldots \tag{D.15}$$

The contribution of the third region

$$I^{(iii)} = Z\int_{x_1}^\infty dx\,\left[w + (Y_{\rho_0}+Z)x\right]^{-2}$$

$$= \frac{Z}{Y_{\rho_0}+Z}\left[w + (Y_{\rho_0}+Z)x_1\right]^{-1} \tag{D.16}$$

cancels in leading order the first term in (D.11). It remains the central region

$$I^{(ii)} = \int_{x_0}^{x_1} \frac{dx}{x}\,(w + Y_{\rho_0} x)^{-2}\, P_z^2 \left(x\frac{\partial}{\partial x}\right)^2 P_z^{-1} \tag{D.17}$$

which is obviously suppressed and gives at most contributions

$\sim x_1/(w + Y\rho_o x_1)^2$ or p_o/w^2. We neglect the difference between this integral and the extrapolation of (D.14) (D.16) where we take $p_o = 0$ and $x_1 = 0$ respectively. Combining the different contributions we arrive at the result

$$\eta = 2 v_2 \varkappa^{-1} \left(1 + (\beta - 1) + \frac{(\beta-1)^2}{2\beta} \frac{Y\rho_o w}{k^2} \tilde{L}_2^2(w) \right) \tag{D.18}$$

The last term is suppressed by a factor $\frac{Y\rho_o k^2}{w}\left(\frac{k^2}{w}\right)^{\frac{1}{\beta-1}}$ and will be neglected in the following.

In summary, the anomalous dimension depends on β

$$\eta = 2 v_2 \varkappa^{-1} \beta \tag{D.19}$$

and diverges for $\beta \to \infty$ as suggested by the discussion of appendix E. This finding seems to contradict, however, the scheme independence of the phase structure as discussed in sect. 7. A closer inspection of the integral (D.4) shows indeed a peculiar behaviour in two dimensions. Only $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are dominated by $x \gtrsim k^2$, whereas $\mathcal{Z}_3$ also receives an important contribution from $x \ll k^2$ (which is absent for $d > 2$). This contribution is responsible for the term $\sim (\beta - 1)$ in (D.18) (compare the discussion in sect. 6). If we consider an expansion of Z in powers of $\varkappa^{-1}$ we observe that only the lowest order terms $Z^{(a)} + v_2 \rho_o^{-1}(\mathcal{Z}_1 + \mathcal{Z}_2)$ obey the criterion for the validity of the renormalization group improvement discussed in sect. 2, namely that the momentum integrals are dominated by $x \gtrsim k^2$. For $d > 2$ this property also holds for the term $\sim \varkappa^{-1}$ and breaks only down in higher order in $\varkappa^{-1}$. In two dimensions, however, our method of simply replacing the bare couplings by renormalized couplings at the scale k becomes insufficient already in order $\varkappa^{-1}$. Contributions from higher loops (and from terms in the average action containing more than two derivatives) are expected to become important already in this order.

In the present case we attempt to account for the higher loop effects more appropriately and push the renormalization group improvement further by replacing in $\mathcal{Z}_3$ the effective coupling $\bar{\lambda}$ by a momentum dependent vertex in analogy to (4.27)

$$\bar{\lambda} \rightarrow \bar{\lambda}(x) = \lambda Z^2 x \tag{D.20}$$

$$w \rightarrow 2\lambda Z^2 \rho_o x \tag{D.21}$$

This procedure modifies the region $x \gtrsim k^2$ of the integral

$$\mathcal{Z}_3 = \int \frac{dx}{x} \frac{(2\bar{\lambda} + Yx)\varkappa}{(2\bar{\lambda} + Yx)\varkappa + Z P_z} P_z \left(x \frac{\partial}{\partial x} \right)^2 P_z^{-1} \tag{D.22}$$

only in next to leading order. A behaviour of $\bar{\lambda}(x)$ different from (D.20) for $x \gtrsim k^2$ only influences corrections $\sim \varkappa^{-2}$ in η. On the other hand, the region $x \ll k^2$ is substantially altered. In this approach the computation of $\eta_3^{(b)}$ simplifies considerably and one obtains with (D.20)

$$\eta_3^{(b)} = - v_2 \varkappa^{-1} \frac{\partial}{\partial t} \mathcal{Z}_3$$
$$= 2 v_2 \varkappa^{-1} \frac{Y + 2\lambda Z^2}{Y + 2\lambda Z^2 + Z^2 \varkappa^{-1}} \tag{D.23}$$

The β-dependence of the anomalous dimension has disappeared and we find, up to corrections of the order $\varkappa^{-2}$,

$$\eta = 2 v_2 \varkappa^{-1} \tag{D.24}$$

Although we did not attempt to develop this further improvement systematically we believe that (D.24) is actually more reliable than (D.19), especially for the scaling region which occurs within the validity of the Goldstone regime (see sect. 7).

In the scaling region the evaluation of η simplifies considerably since k is the only scale present. All other dimensionful quantities are proportional to appropriate powers of k. (The only other scale dependence arises implicitly from the slow running of Z and can be neglected on the r.h.s. of (D.1) and (D.3) where we take Z and Y as scale independent constants.) The appropriate effective coupling $\bar{\lambda}(x)$ must have the generic form

$$\bar{\lambda}(x) = \lambda Z^2 k^2 \tilde{\lambda}\left(\frac{x}{k^2}\right) \tag{D.25}$$

Since Z_k is dimensionless it can only depend on the ratio Λ/k. We can therefore obtain the anomalous dimension in the scaling region by variation of Λ

$$\eta = 2Z^{-1}\Lambda^2 \frac{\partial}{\partial\Lambda^2}\left(Z_k^{(a)} + Z_k^{(b)}\right) \tag{D.26}$$

We only assume that $\bar{\lambda}(x)$ grows somewhat slower than x such that w/x can be neglected at very large x. One finds

$$\eta = 2v_2\kappa^{-1}\left(1 + Z^2 Y^{-1}\kappa^{-1}\right)^{-1}$$

$$= 2v_2\kappa^{-1}\left(1 - \frac{Z}{\tilde{Z}}\right) \tag{D.27}$$

This result coincides with (D.24) in leading order.[3] We assume that (D.27) is appropriate not only in the scaling region but within the whole range of validity of the Goldstone regime.

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3) We assume here $Y \neq 0$ in the Goldstone regime such that $Z \ll \tilde{Z}$ as suggested by (6.24).

Appendix E: Problems with sharp cutoff

In the limit of a sharp momentum cutoff, $\beta \to \infty$, the one loop contribution to η diverges (except for d = 6). The origin of this problem can be understood by evaluating the Q^2 dependence of $\partial\Gamma_k^{(1)}/\partial(\delta\varphi\,\delta\varphi^*)$ directly in the limit $\beta\to\infty$. The integral I(Q) (6.5) reads, after a shift in the integration variable and using

$$\lim_{\beta\to\infty} P^{-1}(q) = \frac{1}{q^2}\,\Theta(q^2 - k^2) \tag{E.1}$$

$$I_\infty(Q) = \lim_{\beta\to\infty} I(Q) = \frac{1}{2}(2\pi)^{-d}\int d^d q \left\{\left(q+\frac{Q}{2}\right)^2 + V'\right\}^{-1} \cdot \left\{\left(q-\frac{Q}{2}\right)^2 + V' + 2V''\varphi^2\right\}^{-1} \Theta\left(\left(q+\frac{Q}{2}\right)^2 - k^2\right) \cdot \Theta\left(\left(q-\frac{Q}{2}\right)^2 - k^2\right) \tag{E.2}$$

(for $Z = 1$).
With $Q_\mu = Q\,\delta_{\mu 1}$ and $Q > 0$ we introduce the variables

$$x = \sum_{\mu=2}^{d} q_\mu q^\mu$$

$$y = q_1$$

$$\left(q \pm \frac{Q}{2}\right)^2 = x + \left(y \pm \frac{Q}{2}\right)^2 \tag{E.3}$$

and observe that the shape of the integration region differs for k^2 smaller or bigger than $\frac{1}{4}Q^2$ (compare fig. 3)

$$I_\infty = I_0 - I_I - I_{II} + \theta\left(k^2 - \frac{Q^2}{4}\right)\left(I_A + I_B\right) \tag{E.4}$$

$$I_R = \frac{1}{2\pi}\, v_{d-1} \int_R dx\, dy\, \tilde{k}(x,y;Q) \tag{E.5}$$

$$\tilde{k}(x,y;Q) = x^{\frac{d-3}{2}} \left\{ x + y^2 + Qy + \frac{1}{4}Q^2 + V' \right\}^{-1} \cdot \left\{ x + y^2 - Qy + \frac{1}{4}Q^2 + V' + 2V''\varphi^2 \right\}^{-1} \tag{E.6}$$

The integral I_0 extends over $(x,y) \in \mathbb{R}^2$ and is obviously independent of k. The only k dependence arises from the k dependence of the boundary of the other integration regions

(I) : $-\frac{Q}{2} - k < y < -\frac{Q}{2} + k$
$0 < x < k^2 - (y + \frac{Q}{2})^2$

(II): $\frac{Q}{2} - k < y < \frac{Q}{2} + k$
$0 < x < k^2 - (y - \frac{Q}{2})^2$

(A) : $\frac{Q}{2} - k < y < 0$
$0 < x < k^2 - (y - \frac{Q}{2})^2$

(B) : $0 < y < -\frac{Q}{2} + k$
$0 < x < k^2 - (y + \frac{Q}{2})^2$ (E.7)

We are interested in small values $Q^2 \ll 4k^2$ where

$$\frac{\partial}{\partial t} I_\infty = -\frac{1}{2\pi} v_{d-1}\, k \frac{\partial}{\partial k} \int_0^{\frac{Q}{2}+k} dy \int_0^{k^2-(y-\frac{Q}{2})^2} dx \left(\tilde{k}(x,y;Q) + \tilde{k}(x,y;-Q) \right)$$

$$= -\frac{v_{d-1}}{\pi} \left\{ \frac{k^2}{k^2+V'+2V''\varphi^2} \int_{-\frac{Q}{2}}^{k} dz\, \frac{(k^2-z^2)^{\frac{d-3}{2}}}{2Qz+Q^2+k^2+V'} + \frac{k^2}{k^2+V'} \int_{-\frac{Q}{2}}^{k} dz\, \frac{(k^2-z^2)^{\frac{d-3}{2}}}{2Qz+Q^2+k^2+V'+2V''\varphi^2} \right. \tag{E.8}$$

We note that (E.8) is not symmetric under $Q \to -Q$ (the integration boundaries involve $|Q|$ rather than Q). One finds in particular

$$\xi = \frac{\partial}{\partial Q} \frac{\partial}{\partial t} I_\infty \Big|_{Q=0} = -\frac{v_{d-1}}{\pi} \frac{k^{d-1}}{(k^2+V')(k^2+V'+2V''\varphi^2)} \cdot \left\{ 1 - \frac{2}{d-1}\left(\frac{k^2}{k^2+V'} + \frac{k^2}{k^2+V'+2V''\varphi^2} \right) \right\} \tag{E.9}$$

For a sharp momentum cutoff we conclude that the Q dependence of $\frac{\partial}{\partial t} I$ for small Q has the form

$$\frac{\partial}{\partial t} I_\infty = c + \xi \sqrt{Q^2} + \xi' Q^2 + \ldots \tag{E.10}$$

The derivative $\frac{\partial}{\partial Q^2}\frac{\partial}{\partial t} I_\infty$ does not exist at $Q^2 = 0$ and the one loop contribution to the kinetic term has not the standard form. We therefore have to choose a smooth definition of the average field with a finite value of β, e.g. $\beta = 3$.

We finally observe that ξ vanishes for $V' = V''\varphi^2 = 0$ if $d = 5$ and not for $d = 6$ as naively suggested by the results of appendix C. This indicates

$$\frac{\partial}{\partial t} \lim_{\beta\to\infty} I(Q) \neq \lim_{\beta\to\infty} \frac{\partial}{\partial t} I(Q) \qquad \text{(E.11)}$$

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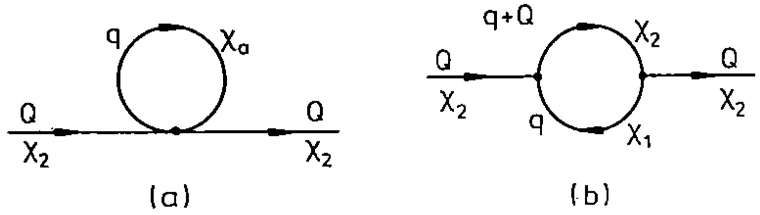

Fig. 1

Fig. 1: Contributions to the wave function renormalization of the Goldstone field

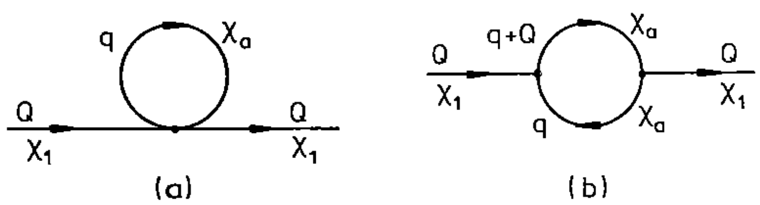

Fig. 2

Fig. 2: Contributions to the wave function renormalization of the radial field

$k < \frac{Q}{2}$

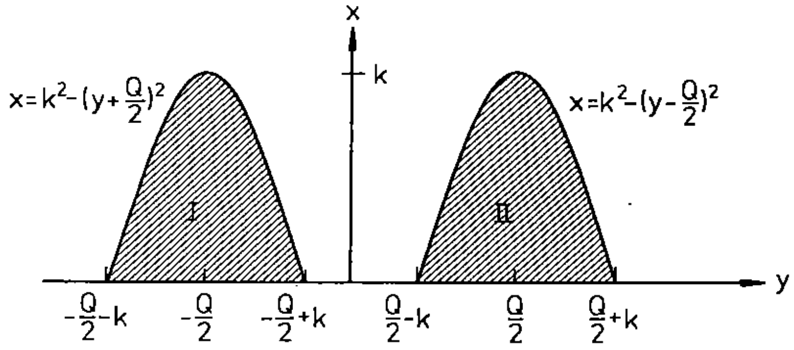

Fig. 3a

$k > \frac{Q}{2}$

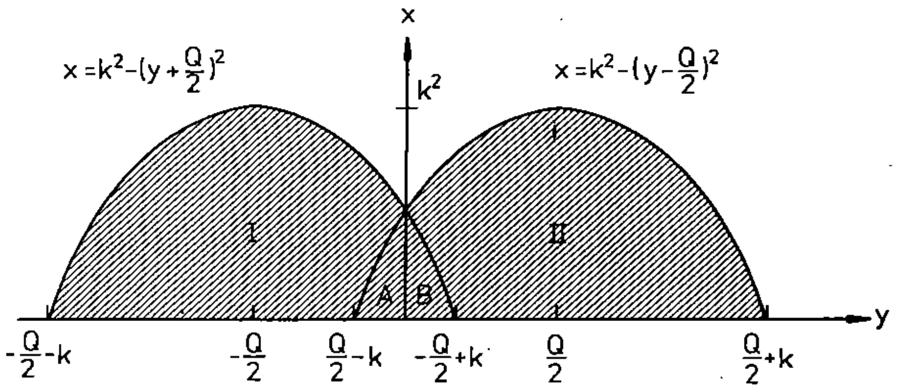

Fig. 3b

Fig. 3: Integration region for the wave function renormalization with sharp cutoff (see text)