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GENERALIZED NUCLEARITY CONDITIONS AND THE SPLIT PROPERTY IN QUANTUM FIELD THEORY

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ABSTRACT

Generalized nuclearity conditions that are applicable in arbitrary superselection sectors of a quantum field theory and to theories with a maximal temperature are discussed. They are shown to imply the (distal) split property and to impose specific restrictions on the spectral properties of modular operators associated with local algebras and vectors of compact energy support.

1. INTRODUCTION

The idea that a quantum field theory with a reasonable physical interpretation should satisfy some restrictions on the number of its local degrees of freedom [1,2] can be expressed mathematically in different ways. In [3] a heuristic argument based on thermodynamical considerations was given for the following formulation:

Suppose the Hilbert space $\mathcal{H}$ of the theory contains a vacuum vector Ω. Let $\mathcal{A}(\mathcal{O})$ denote the C^*-algebra of operators on $\mathcal{H}$ representing the observables associated with the region $\mathcal{O}$ of Minkowski space, and let H denote the Hamiltonian (assumed to be positive). Then the linear map $\Theta_{\Omega,\beta}$: $\mathcal{A}(\mathcal{O}) \to \mathcal{H}$, given by

$$\Theta_{\Omega,\beta}(A) = e^{-\beta H} A\Omega, \tag{1.1}$$

should, for any bounded $\mathcal{O}$, be a compact map of arbitrarily small order if $\beta > 0$ is sufficiently large.

We recall from [3] that the order q of a continuous linear map Θ of a Banach space $\mathcal{E}$ into another Banach space $\mathcal{F}$ is defined as the nonnegative number (if it exists)

$$q = \limsup_{\varepsilon \searrow 0} \frac{\ln \ln N(\varepsilon)}{\ln 1/\varepsilon}, \tag{1.2}$$

where $N(\varepsilon)$, the ε-content of Θ, is the maximal number of elements E_ι in the unit ball of $\mathcal{E}$ such that $\|\Theta(E_\iota - E_\kappa)\| > \varepsilon$ if $\iota \neq \kappa$.

The heuristic basis of this condition, referred to as Condition N in [4], is the same as for the nuclearity condition proposed earlier in [2]. There it was required that the maps $\Theta_{\Omega,\beta}$ are nuclear for all $\beta > 0$ and that their nuclear norms $\nu(\cdot)$ satisfy a bound

$$\nu(\Theta_{\Omega,\beta}) \leq e^{c\beta^{-n}} \tag{1.3}$$

as $\beta \searrow 0$ with some constants c and n. The formulation in terms of the order of maps (or equivalently, in terms of "p-nuclearity" [3]) turns out to be mathematically somewhat more convenient, however. Furthermore, Condition N involves only the behaviour of the maps $\Theta_{\Omega,\beta}$ for large β and thus applies to a wider class of theories of potential physical interest, e.g. theories with a maximal temperature. Yet both of these conditions refer to the vacuum vector Ω and therefore cannot be applied in charged superselection sectors as such.

In [4] Condition N was compared with some alternative compactness and nuclearity criteria that have been suggested. This analysis led to a somewhat more stringent condition, $N^\sharp$, that is in a sense dual to Condition N. It requires that the maps Ξ_β of a local algebra $\mathcal{A}(\mathcal{O})$ into the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, defined by

$$\Xi_\beta(A) = e^{-\beta H} A e^{-\beta H}, \tag{1.4}$$

are of small order for large β. In the vacuum sector Condition $N^\sharp$ implies Condition N, but since the former does not involve the vacuum state it makes sense in charged sectors as well.

In a recent article [5] Borchers and Schumann propose a generalization of Condition (1.3) that can likewise be applied in charged sectors. Moreover, they were able to derive from it an important consequence of all previous nuclearity conditions [2,3,6], the split property.

We recall that an inclusion $\mathcal{A} \subset \mathcal{B}$ of two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ is called split if there is a type-I-factor $\mathcal{M}$ such that $\mathcal{A} \subset \mathcal{M} \subset \mathcal{B}$. A quantum field theory, specified by its local net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of C^*-algebras, has the split property if the inclusion $\mathcal{A}(\mathcal{O})'' \subset \mathcal{A}(\widehat{\mathcal{O}})''$ is split for all bounded regions $\mathcal{O} \subset\subset \widehat{\mathcal{O}}$, where the symbol $\subset\subset$ indicates that the closure of $\mathcal{O}$ is contained in the interior of $\widehat{\mathcal{O}}$. More generally, the theory is said to have the distal split property if the above inclusions are split for pairs of regions $\mathcal{O}$ and $\widehat{\mathcal{O}}$ with a sufficiently large inner distance $\delta := \sup\{\tau \mid \mathcal{O} + (t, 0) \subset \widehat{\mathcal{O}} \text{ for all } |t| < \tau\}$.

The Borchers-Schumann condition, in the following referred to as BS, requires that for each bounded region $\mathcal{O}$ there is a vector Φ of compact energy support such that the map

$$A \mapsto e^{-\beta H} A e^{\beta H} \Phi, \qquad A \in \mathcal{A}(\mathcal{O}) \tag{1.5}$$

is nuclear for all $0 < \beta < \beta_0$ and the nuclear norm satisfies bounds of the type (1.3) as $\beta \searrow 0$. This condition is weaker than the analogous condition for the maps (1.4). On the other hand, since BS requires nuclearity for arbitrarily small β it is not applicable to theories with a maximal temperature, where the distal split property is the best one can expect.

In this letter we consider instead of (1.5) the maps $\Theta_{\Phi,\beta}$: $\mathcal{A}(\mathcal{O}) \to \mathcal{H}$, defined by

$$\Theta_{\Phi,\beta}(A) = e^{-\beta H} A \Phi \tag{1.6}$$

for suitable $\Phi \in \mathcal{H}$ and $\beta > 0$, the map (1.1) being a particular example. These maps have the following property which is not evidently shared by the maps (1.5): If (1.6) satisfies some nuclearity condition for a given β, it satisfies the same condition for all larger β. Since we are primarily interested in the behavior of the maps for large β we regard this as an advantage.

As a straightforward generalization of Condition N that can be used in arbitrary sectors (with positive energy) and that is less stringent than Condition $N^\natural$ we require that for each bounded region $\mathcal{O}$ and each sufficiently large β there is a Φ in the domain of $e^{\beta H}$ such that the order q_β of the map (1.6) is sufficiently small. We show in Sect. 2 that this condition is sufficient to derive the distal split property. In fact, our method of proof gives a fairly accurate connection between the minimal inner distance for which the inclusion of the local algebras is split and the dependence of q_β on β. In particular, the theory has the split property if q_β tends to zero faster than $1/\beta$ as $\beta \to \infty$. We recall in this context that in theories with a maximal temperature the order q_β is expected to behave like $1/\beta$ for large β [3].

Our proof of the (distal) split property is based on a simple estimate for analytic functions and does not involve modular theory. However, as discussed in [3] and [7], there is an intimate connection between the split property and nuclearity properties of modular operators. In particular, it was shown in [3] that Condition N is equivalent to an analogous condition for the modular operators associated with local algebras and the vacuum vector Ω. Moreover, it was observed in [7] that this modular nuclearity condition implies that the same condition is satisfied by the modular operators associated with a dense set of cyclic and separating vectors for the local algebras.

As further evidence for a tight connection between the spectral properties of the Hamiltonian and the modular operators we show in Sect. 3 that Condition $N^\natural$ implies that a corresponding nuclearity condition holds uniformly for all modular operators associated with vectors of compact energy support.

In the last section we discuss briefly the following variant of our condition, akin to that of [2] and the BS condition: The maps (1.6) are supposed to be nuclear for all $\beta > 0$ and to satisfy an estimate of the type (1.3). We show that also this form of the condition implies the split property. The proof we give makes use of the fact that the concepts of strong and weak analyticity coincide for functions of a complex variable

with values in a Banach space. This fact could also be used to simplify the proof of the split property given in [5].

2. FROM GENERALIZED NUCLEARITY TO THE SPLIT PROPERTY

Our main result on the (distal) split property is the following theorem.

Theorem 2.1. *Suppose for a given bounded region $\mathcal{O}_0$ in Minkowski space there exists a $\beta > 0$ and a vector $\Phi \neq 0$ in the domain of $e^{\beta H}$ such that the map $\Theta_{\Phi,\beta}$: $\mathcal{A}(\mathcal{O}_0) \to \mathcal{H}$, defined by (1.6), is of order $q_\beta < 1/3$. Then the inclusion $\mathcal{A}(\mathcal{O})'' \subset \mathcal{A}(\widehat{\mathcal{O}})''$ is split for all regions $\mathcal{O}$, $\widehat{\mathcal{O}}$ with $\mathcal{O} \subset\subset \widehat{\mathcal{O}} \cap \mathcal{O}_0$, provided the inner distance δ between $\mathcal{O}$ and $\widehat{\mathcal{O}}$ satisfies the inequality*

$$\delta > \frac{2}{\pi}\beta \Big| \ln \tan\Big(\frac{\pi}{4}(1 - 3q_\beta)\Big)\Big|. \tag{2.1}$$

If in particular there exists a family of vectors Φ such that $q_\beta \cdot \beta \to 0$ for $\beta \to \infty$, then the inclusion is split for all $\delta > 0$.

To simplify notation we put $\mathcal{A} := \mathcal{A}(\mathcal{O})''$ and $\mathcal{B} := \mathcal{A}(\widehat{\mathcal{O}})''$. In order to show that the inclusion $\mathcal{A} \subset \mathcal{B}$ is split it suffices to verify as in Prop. 4.1 of [3] that the map $\Xi_{\Phi,*}$ of $\mathcal{A}$ into the predual $\mathcal{B}'_*$ of $\mathcal{B}'$, given by

$$\Xi_{\Phi,*}(A)(\cdot) = (\Phi, \cdot A\Phi), \qquad A \in \mathcal{A} \tag{2.2}$$

is of order $q_* < 1/3$. Theorem 2.1 will thus follow as an immediate corollary once the following result has been proved.

Proposition 2.2. *Suppose Φ is in the domain of $e^{\beta H}$ for some $\beta > 0$. If $\Theta_{\Phi,\beta}$ is of finite order q_β, then $\Xi_{\Phi,*}$ is of finite order $q_* \leq q_\beta \cdot (1-(2\alpha/\pi))^{-1}$ with $\alpha = 2\arctan e^{-\pi\delta/(2\beta)}$.*

For the proof of Proposition 2.2 we need two lemmas. The first is implicitly contained in Lemmas 2.2 and 2.3 in [3] and is stated here for completeness only:

Lemma 2.3. (a) *Let Θ_i, $i = 1,2$ be continuous linear mappings of a (real or complex) Banach space $\mathcal{E}$ into Banach spaces $\mathcal{F}_i$, $i = 1,2$. If*

$$\|\Theta_1(A)\| \leq \text{const} \cdot \|\Theta_2(A)\|^k \tag{2.2}$$

for all $A \in \mathcal{E}$, $\|A\| \leq 1$, with some $k > 0$, and if Θ_2 has finite order q_2, then Θ_1 has finite order $q_1 \leq q_2/k$.

(b) *Let Θ_i, $i = 1,2$ be (real or complex) continuous linear mappings of a Banach space $\mathcal{E}$ into a Banach space $\mathcal{F}$. If Θ_i has finite order q_i, $i = 1,2$, then $\Theta = \Theta_1 + \Theta_2$ has finite order $q \leq \max\{q_1, q_2\}$.*[1]

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[1] The proof of the corresponding Lemma 2.2 in [3] contains a misprint: The correct estimate for the ε-content of the map Θ is $N_\Theta(\varepsilon) \leq N_{\Theta_1}(\varepsilon/2) \cdot N_{\Theta_2}(\varepsilon/2)$.

The second lemma concerns analytic functions of a complex variable.

Lemma 2.4. *Let $G \subset \mathbf{C}$ be a simply connected domain containing 0 whose boundary Γ is the union of Jordan arcs Γ_i, $i = 1, \ldots n$. Suppose F is a bounded analytic function on G, piecewise continuous at the boundary, and put*

$$M_i = \sup_{z \in \Gamma_i} |F(z)|.$$

Then

$$|F(0)| \leq \prod_{i=1}^{n} M_i^{\mu_i},$$

where μ_i denotes the normalized harmonic measure of $\Gamma_i \subset \Gamma$ with respect to 0.

Proof. The lemma is a consequence of the fact that $\ln |F(z)|$ is subharmonic on G, cf. [8], Ch. III ("Zweikonstantensatz").

Proof of Proposition 2.2. For fixed $A \in \mathcal{A}$ and $B' \in \mathcal{B}'$ we consider the functions

$$F_+(z) = \langle \Phi, A e^{izH} B' e^{-izH} \Phi \rangle$$

if $0 \leq \operatorname{Im} z \leq \beta$, and

$$F_-(z) = \langle \Phi, e^{izH} B' e^{-izH} A \Phi \rangle$$

if $-\beta \leq \operatorname{Im} z \leq 0$. Both functions are bounded and analytic in the interior of their domains and continuous at the boundary. Moreover, since A and $e^{itH} B' e^{-itH}$ commute for $|t| < \delta$, we have $F_+(z) = F_-(z)$ for $z = t \in \mathbf{R}$, $|t| < \delta$. Hence there is a common analytic continuation F of $F_\pm$ to the doubly cut strip

$$G_{\beta,\delta} = \{z \mid |\operatorname{Im} z| < \beta\} \setminus \{z \mid \operatorname{Im} z = 0, |\operatorname{Re} z| \geq \delta\}.$$

The unit disk $\{w \mid |w| < 1\}$ is mapped conformally onto $G_{\beta,\delta}$ by

$$w \mapsto z = \frac{\beta}{\pi} \left\{ \ln \frac{1 + w e^{i\alpha}}{1 - w e^{i\alpha}} - \ln \frac{1 - w e^{-i\alpha}}{1 + w e^{-i\alpha}} \right\} \tag{2.3}$$

with $\alpha = 2 \arctan e^{-\pi\delta/(2\beta)}$. For the boundary values of this map, $w \nearrow e^{i\phi}$ with $-\pi < \phi \leq \pi$, we have

$$\operatorname{Re} z = \frac{\beta}{\pi} \ln \left| \cot \frac{\phi + \alpha}{2} \cot \frac{\phi - \alpha}{2} \right| \tag{2.4}$$

and

$$\operatorname{Im} z = \begin{cases} 0 & \text{if } |\phi| < \alpha \text{ or } \pi - \phi < \alpha \text{ or } \pi + \phi < \alpha \\ \beta & \text{if } \alpha < \phi < \pi - \alpha \\ -\beta & \text{if } \alpha < -\phi < \pi - \alpha. \end{cases} \tag{2.5}$$

Moreover, $z = 0$ if $w = 0$. From Lemma 2.4 we thus obtain the estimate

$$|F(0)| \leq \sup_{|t| \geq \delta} |F(t + i0)|^{\frac{\alpha}{\pi}} \cdot \sup_{|t| \geq \delta} |F(t - i0)|^{\frac{\alpha}{\pi}} \cdot \sup_t |F(t + i\beta)|^{\frac{1}{2} - \frac{\alpha}{\pi}} \cdot \sup_t |F(t - i\beta)|^{\frac{1}{2} - \frac{\alpha}{\pi}} \tag{2.6}$$

and hence

$$|\langle \Phi, A B' \Phi \rangle| \leq \left(\|\Phi\|^2 \cdot \|A\| \cdot \|B'\| \right)^{\frac{2\alpha}{\pi}} \cdot \left(\|e^{\beta H} \Phi\|^2 \cdot \|B'\|^2 \cdot \|e^{-\beta H} A \Phi\| \cdot \|e^{-\beta H} A^* \Phi\| \right)^{\frac{1}{2} - \frac{\alpha}{\pi}}. \tag{2.7}$$

Taking the supremum over the unit ball in $\mathcal{B}'$ we obtain for $\|A\| \leq 1$

$$\|\Xi_{\Phi,*}(A)\| \leq \text{const} \cdot \|e^{-\beta H} A \Phi\|^{\frac{1}{2} - \frac{\alpha}{\pi}} \cdot \|e^{-\beta H} A^* \Phi\|^{\frac{1}{2} - \frac{\alpha}{\pi}}, \tag{2.8}$$

and thus

$$\|\Xi_{\Phi,*}(A \pm A^*)\| \leq \text{const} \cdot \|e^{-\beta H}(A \pm A^*)\Phi\|^{1 - \frac{2\alpha}{\pi}}. \tag{2.9}$$

Since the real linear maps $A \mapsto A \pm A^*$ are bounded, the proposition now follows from Lemma 2.3. As noted above, Theorem 2.1 follows as a corollary.

3. CONNECTIONS WITH MODULAR NUCLEARITY

A vector $\Phi \in \mathcal{H}$ of compact energy support is in the domain of $e^{\beta H}$ for all β and thus a potential candidate for nuclearity conditions as considered in Theorem 2.1. Moreover, it is an entire analytic vector for H. By the Reeh-Schlieder property it is therefore cyclic and separating for the local von Neumann algebras $\mathcal{A}(\mathcal{O})''$ and one may consider the corresponding modular operators $\Delta_{\Phi,\mathcal{O}}$. If $\mathcal{O} \subset \widehat{\mathcal{O}}$ are two regions with inner distance $\delta > 0$ we define in analogy to [3] linear mappings $\Xi_\Phi^\natural \colon \mathcal{A}(\mathcal{O})'' \to \mathcal{H}$ by

$$\Xi_\Phi^\natural(A) = \Delta_{\Phi,\widehat{\mathcal{O}}}^{1/4} A \Phi. \tag{3.1}$$

For the case where Φ is the vacuum vector Ω it was shown in [3] that the order of the maps (3.1) can be estimated by the order of the maps (1.6). Proposition 2.4 above combined with Lemma 3.1 in [3] shows that this relation holds in fact true for all vectors of compact energy support.

Theorem 3.1. *Let $\Phi \neq 0$ be a vector of compact energy support and suppose that for a given $\mathcal{O}$ the map $\Theta_{\Phi,\beta} \colon \mathcal{A}(\mathcal{O}) \to \mathcal{H}$ has order q_β. Then the map $\Xi_\Phi^\natural$, defined by (3.1), has order*

$$q^\natural \leq 2 q_\beta \cdot (1 - (2\alpha/\pi))^{-1} \tag{3.2}$$

with $\alpha = 2 \arctan e^{-\pi\delta/(2\beta)}$.

If the maps $\Xi_\beta \colon \mathcal{A}(\mathcal{O}) \to \mathcal{B}(\mathcal{H})$, defined in (1.4), have finite order $\hat{q}_\beta$, then the preceding conclusion holds uniformly for all vectors of compact energy support, i.e.

$$q^\natural \leq 2 \hat{q}_\beta \cdot (1 - (2\alpha/\pi))^{-1} \tag{3.3}$$

for all such vectors Φ. In particular $q^\natural = 0$ if $\hat{q}_\beta \cdot \beta \to 0$ for $\beta \to \infty$.

Proof. The first statement follows immediately from Lemma 3.1 in [3] and Proposition 2.4. The second statement, concerning the maps Ξ_β, is a simple corollary, for $\Theta_{\Phi,\beta}(A) = \Xi_\beta(A)e^{\beta H}\Phi$, and this implies that $q_\beta \leq \hat{q}_\beta$.

In [3] it was shown that conversely nuclearity properties of $\Xi^{\natural}_\Omega$ imply similar properties of $\Theta_{\Omega,\beta}$. The proof makes use of the Lorentz invariance of the vacuum vector Ω and it is therefore not clear at the moment how this part of the statement carries over to more general vectors Φ.

4. A VARIANT OF THE GENERALIZED NUCLEARITY CONDITION

In this last section we derive the split property from a condition for the maps (1.6) that is similar to the original nuclearity condition of [2] and Condition BS. We recall that a map Θ between Banach spaces $\mathcal{E}$ and $\mathcal{F}$ is nuclear if it has a representation

$$\Theta(\cdot) = \sum_{k=1}^{\infty} \ell_k(\cdot)\xi_k \tag{4.1}$$

with $\xi_k \in \mathcal{F}$ and ℓ_k in the dual space $\mathcal{E}'$, such that

$$\sum_{k=1}^{\infty} \|\ell_k\|\,\|\xi_k\| < \infty. \tag{4.2}$$

The infimum over all sums (4.2) for which (4.1) holds is the nuclear norm of Θ, denoted by $\nu(\Theta)$.

Theorem 4.1. *Suppose for each bounded region $\mathcal{O}$ of Minkowski space there exists a vector Φ in the domain of $e^{\beta_0 H}$ for some $\beta_0 > 0$ such that the map $\Theta_{\Phi,\beta}$: $\mathcal{A}(\mathcal{O}) \to \mathcal{H}$ is nuclear for all $\beta > 0$ and the nuclear norm satisfies the bound*

$$\nu(\Theta_{\Phi,\beta}) \leq e^{c\beta^{-n}} \tag{4.3}$$

as $\beta \searrow 0$ with some constants c and n. Then the theory has the split property.

Proof. Consider a fixed pair $\mathcal{O} \subset \widehat{\mathcal{O}}$ of bounded regions with inner distance $\delta > 0$ and put as before $\mathcal{A} := \mathcal{A}(\mathcal{O})''$ and $\mathcal{B} := \mathcal{A}(\widehat{\mathcal{O}})''$. We have to show that the map $\Xi_{\Phi,*}$: $\mathcal{A} \to \mathcal{B}'_*$, defined in (2.2), is nuclear. To do this we consider again the functions $F^\pm$ used in the proof of Proposition 2.4, but this time regarded as linear maps $\mathcal{A} \to \mathcal{B}'_*$ for each z, i.e. we define for $A \in \mathcal{A}$

$$\Xi^+_z(A)(\cdot) := (\Phi, Ae^{izH} \cdot e^{-izH}\Phi) \tag{4.4}$$

if $0 < \text{Im } z < \beta_0$, and

$$\Xi^-_z(A)(\cdot) := (\Phi, e^{izH} \cdot e^{-izH}A\Phi) \tag{4.5}$$

if $0 < -\text{Im } z < \beta_0$. For $0 < \pm\text{Im } z < \beta_0$ the maps $\Xi^\pm_z$ are nuclear and we have

$$\nu(\Xi^\pm_z) \leq \|e^{\beta_0 H}\Phi\| \cdot \nu(\Theta_{\Phi,|\text{Im} z|}) \leq \text{ const} \cdot e^{c|\text{Im} z|^{-n}}. \tag{4.6}$$

Moreover, for each finite set of elements $A_i \in \mathcal{A}$ and $B'_i \in \mathcal{B}'$, $i = 1, \dots m$, the functions

$$z \mapsto \sum_{i=1}^{m} \Xi^\pm_z(A_i)(B'_i) \tag{4.7}$$

are analytic for $0 < \pm\text{Im } z < \beta_0$. We are thus in the following situation: The functions $z \mapsto \Xi^\pm_z$ take values in the Banach space $\mathcal{N}(\mathcal{A}, \mathcal{B}'_*)$ of nuclear maps $\mathcal{A} \to \mathcal{B}'_*$ and are uniformly bounded on compact subsets of their domains. Moreover, the functions (4.7) are analytic for each choice of $\sum_{i=1}^m A_i \otimes B'_i$ in the algebraic tensor product $\mathcal{A} \odot \mathcal{B}'$. Now $\mathcal{A} \odot \mathcal{B}'$ is a dense subspace of the dual space of $\mathcal{N}(\mathcal{A}, \mathcal{B}'_*)$ (it separates points) and by a standard result about analytic functions with values in a Banach space (see e.g. Theorem 1.37 and Remark 1.38 in Ch.III of [9]) it follows that the functions $z \mapsto \Xi^\pm_z$ are strongly analytic on their domains of definition.

As in [2] and [5] we can choose an integration contour Γ around the origin and an auxiliary function $a(z)$ that is analytic in the doubly cut strip $G_{\beta_0,\delta}$, satisfies $a(0) = 1$, and tends to zero at the end points of the interval $[-\delta, \delta]$ in such a way that $|a(z)| \cdot \nu(\Xi^\pm_z)$ is bounded on Γ. If $\Gamma^\pm$ denotes the parts of Γ that lie above and below the real axis it follows that

$$\Xi := \int_{\Gamma^+} \frac{dz}{2\pi i z}\, a(z) \cdot \Xi^+_z + \int_{\Gamma^-} \frac{dz}{2\pi i z}\, a(z) \cdot \Xi^-_z \tag{4.8}$$

is well defined as a Bochner integral in the Banach space $\mathcal{N}(\mathcal{A}, \mathcal{B}'_*)$. On the other hand, by locality we have $\lim_{\beta \searrow 0} \Xi^\pm_{t \pm i\beta}(A)(B') = \Xi_{\Phi,*}(A)(e^{itH}B'e^{-itH})$ for fixed $A \in \mathcal{A}$, $B' \in \mathcal{B}'$ and $|t| < \delta$. Hence the functions $z \mapsto \Xi^\pm_z(A)(B')$ analytically continue each other across the real axis, and the Cauchy formula gives $\Xi(A)(B') = \lim_{\beta \searrow 0} \Xi^\pm_{\pm i\beta}(A)(B') = \Xi_{\Phi,*}(A)(B')$. Hence $\Xi_{\Phi,*} \in \mathcal{N}(\mathcal{A}, \mathcal{B}'_*)$, and the proof is complete.

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