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A Semiclassical Approach to Quantum Gravity

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Abstract

The interaction of a scalar quantum field with gravity is investigated in the semiclassical context where the space-time is treated classically. It is essentially understood as a self-interaction of the quantum field , mediated by its own states. The relevant states here are not arbitrary but are selected by the principle of equivalence which is incorporated in form of specific nonlinear constraint equations. The quantum field is then subjected to a state dependent (nonlinear) field equation. Concluding, we comment on some problems concerninig the consistency of the scheme employed.

1 Introduction

Since Hawking's orginal discovery of black hole radiation a great deal of work has been done on the foundation of "the semiclassical model of selfconsistent dynamics" describing the interaction of linear quantum fields with gravity. The general framework adopted in this model may be indicated as follows: One starts by considering a quantum field obeying a linear covariant dynamical equation and the standard commutation relations on a fixed global space-time, the latter understood classically in the sense of the general theory of relativity. The central assumption is that the back reaction of the quantum field to gravity can be described in a selfconsistent manner via the Einstein-equations coupled to the renormalized expectation value of the energy momentum tensor operator of the quantum field in some appropriately chosen state, viz.

$$G_{\mu\nu} = -\kappa < T_{\mu\nu} >_{ren.} . \qquad (1)$$

In its underlying structure this model originates, of course, from striving for a semiclassical approach to quantum gravity. But, for this purpose its basic assumptions have turned out to be very restrictive. Looking, for example, at the technical side there is a complete lack of success in dealing with the problem of how to define the right hand side of (1). Indeed, despite several attemps, e.g. [1]-[10], no truly satisfactory procedure for renormalization of $< T_{\mu\nu} >$ has been developed.

At the present time there is a feeling around that the conventional approach based on this model is not even consistent to serve as a basis for a semiclassical quantum gravity. But outside that model no attempts at a formulation of a selfconsistent semiclassical scheme have been made.

It may be, of course, that the incorporation of gravity into the quantum field theory could be accomplished only at the level of the fully quantized theory of gravity. At present one radical school of thought shares this conviction and maintains that the principles of the semiclassical quantum gravity ultimately will not define a theory. There are, however, other aspects. Nobody knows today the principles of the fully quantized theory of gravity. Granted this ignorance the semiclassical approach remains the natural one towards the incorporation of gravity into the quantum field theory.

In any case, there is the desire to understand the inherent objective weaknesses of the conventional semiclassical approach. Concerning this task we have to take seriously the many conceptual difficulties surrounding the nature of its underlying assumptions. The history of science teaches us that such an investigation may help to establish the guiding line along which the future theory should be formulated. In this context it is important to realize, first, that the conventional framework indicated is based on the inadmissible notion of a rigid global background metric.This introduces, of course, necessarily a nonlocal element in the theory and degenerates the characteristic feature of general theory of relativity, in which the space-time becomes a dynamical object and all physical laws are strict local. Conceptually this feature of general theory of relativity must be preserved in any theory incorporating the gravitational interaction. In order to have an example of the kind of the difficulty one encounters consider the problem of the general covariance. It is obvious that the notion of a rigid global background metric implies the existence of a priori causal relations between observables of different space-time regions. On the other hand, since the group of all local diffeomorphisms does not leave the causal relations unchanged, so the latter should not be a

priori given if the former is regarded as the symmetry group.
Another unsatisfactory aspect of the conventional frame concerns the nature of the dynamical laws. It is by no means clear that a model based on linear covariant dynamical equations for the quantum field could fit into the essentially nonlinear gravitational interaction. On the contrary, we expect that the incorporation of gravity into the quantum field theory can only be provided by a nonlinear theory.
Our main goal in the present paper is to investigate how we can improve our understanding of the conceptual frame of the semiclassical approach to quantum gravity. We shall study in particular how the semiclassical theory can be formulated without referring to any rigid global background metric. In arriving at dynamical laws our guiding principle will be the principle of equivalence. We demonstrate a possibility of incorporating that principle into the quantum field theory.
Our discussion will be mainly based on the algebraic approach to generally covariant quantum field theory, presented by Fredenhagen and Haag [11], in which the principle of locality is advanced in its most stringent form as employed by Einstein in the formulation of the general theory of relativity. Their work seems to clarify considerably the question of how the general covariance and the strict locality can be incorporated into the quantum field theory.
To begin with let us quickly sketch the algebraic framework which we shall adopt.
At the most basic level we consider a four dimensional manifold M, not yet equipped with a metric, and associate to each open set $\mathcal{O} \subset \mathcal{M}$ an involutive algebra $\mathcal{A}(\mathcal{O})$. The selfadjoint elements of $\mathcal{A}(\mathcal{O})$ are interpreted as observation procedures which are pure descriptions of laboratory measurements in $\mathcal{O}$. There should not be any a priori relations between procedures associated with different space-time regions, with other words the algebra $\mathcal{A} = \bigcup \mathcal{A}(\mathcal{O})$ has to be flexible.
This interpretation allows us to implement the principle of the general covariance by considering the group of all local diffeomorphisms of the manifold as acting by automorphisms on $\mathcal{A}$, i.e. each local diffeomorphism χ is represented by an automorphism α_χ of $\mathcal{A}$ such that

$$\alpha_\chi(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\chi(\mathcal{O})). \tag{2}$$

In the construction of the algebra of observables from the algebra of procedures the concept of "physical state" emerges. Any physical state ω corresponds to some positive linear functional on $\mathcal{A}$ and generates via the GNS-construction a representation π_ω of $\mathcal{A}$ by an operator algebra in a Hilbert space $\mathcal{H}_\omega$. Once the representation π_ω is given, one can select a family of related states on $\mathcal{A}$ (the so called folium of ω), namely those represented by vectors and density matrices of $\mathcal{H}_\omega$.
Having specified a physical state ω on $\mathcal{A}$, one can consider in each subalgebra $\mathcal{A}(\mathcal{O})$ the equivalence relation

$$A \sim B, \iff \omega'(A - B) = 0 \ \forall \omega' \in \mathcal{F}_\omega. \tag{3}$$

Here $\mathcal{F}_\omega$ denotes the folium of the state ω. The set of such equivalence relations generates a two sided ideal $\mathcal{J}(\mathcal{O})$ in $\mathcal{A}(\mathcal{O})$. The construction of the algebra of observables $\mathcal{A}_{obs}(\mathcal{O})$ from the algebra of procedures is then accomplished by taking the quotient

$$\mathcal{A}_{obs}(\mathcal{O}) = \mathcal{A}(\mathcal{O})/\mathcal{J}(\mathcal{O}). \tag{4}$$

This standpoint in the treatment of local observables is essential for our approach to semicalssical quantum gravity. Clearly, in this setting the emphasis in the specification of the physical laws, i.e. the relations between local observables, is placed on the characterization of the admissible folia of physical states.[1] If there are superselection rules there exist several folia (sectors) of physical states on $\mathcal{A}$ which correspond to different unitary inequivalent representations of $\mathcal{A}$.

To approach the problem of specification of the admissible folia of physical states we shall make the basic assumption that the relevant states (and the associated folia) are everywhere primary (the von Neumann algebras resulting from the GNS-representation of such states have only trivial center for a sufficiently small neighbourhood of a point). Each primary folium of local physical states provides us with a realization of the principle of local definiteness in the sense of the work [12], where a fixed gravitational background was assumed. The characteristic change here is that, unlike the situation in that work, for each sufficiently small neighbourhood of a point there will be now different primary folia of local states. This fact can be understood on the basis of our interpretation of the local algebras as the algebras of procedures.

Our main objective is, first, the question of how to specify the primary folia of local physical states.

We can formulate now one general criterion selecting the primary folia of physical interest. Let us comment first on the physical background. The axioms of quantum field theory in Minkowski space exclude the existence of observables at a single point. In that theory, due to the exact Lorentz-invariance, the observables in space like complement of a single point generates the total algebra. It is not hard to see that this statement ignores the existence of the Planck length, $l_p = (\hbar\kappa/c)^{1/2} \approx 10^{-33}$ cm (κ is the gravitational constant), as the smallst possible length scale that can even in principle be measured by experiments. In reality the above statement need not hold in the gravitational case. The best we can do is to require the validity of that statement in the Minkowskian limit $\kappa \to 0$ where the Planck length tends to zero. Therefore in the limit $\kappa \to 0$ the algebra $\mathcal{A}_{obs}(\mathcal{O})$ has to move into the commutant of the total algebra as $\mathcal{O}$ contracts to a single point. Thinking in terms of states this requires that, if we ignore the Planck regime, two states in the same primary folium should become indistinguishable in a sufficiently small neighbourhood of a point. Clearly, this statement converts the ignorance of the Planck regime into the requirement of a common leading short distance singularity (ultraviolet tail) of different states in the same primary folium. The full significance of the primary folia exhibiting this property will become evident in the light of our considerations in this work.

The required features of the local algebras are incorporated in a simple model, the so called tensor algebra over the space of scalar test functions on the space-time manifold. The monomials of the local algebra $\mathcal{A}(\mathcal{O})$ in this model are smooth functions $f^{(n)} : M \times .. \times M \to C$ with support in $\mathcal{O}$. The algebraic product is the tensor product of functions:

$$f^{(n)}.g^{(m)} = h^{(n+m)}\ ,\ h^{(n+m)}(p_1, .., p_{n+m}) = f^{(n)}(p_1, .., p_n) g^{(m)}(p_{n+1}, ... p_{n+m}). \tag{5}$$

The involution is the complex conjugation together with the inversion of the sequence of arguments. A diffeomorphism sending the point p to χp acts as the automorphism α_χ on $\mathcal{A}$

[1] One should note, however, that the whole information about the physical laws contained in the algebra of observables can be expressed by direct specification of the two sided ideals in the algebra of procedures as well. This second alternative is widely used in the traditional treatments of quantum field theory. But for the treatment of gravity in quantum field theory it appears as inevitable to convert the physical laws into appropriate mathematical constraints on states rather than observables.

according to

$$(\alpha_\chi f^{(n)})(p_1,..,p_n) = f^{(n)}(\chi^{-1}p_1,..,\chi^{-1}p_n). \tag{6}$$

A state ω on $\mathcal{A}(\mathcal{O})$ is given by a hierarchy of distributions (the n-point functions) $\omega^{(n)} \in \mathcal{D}'(\mathcal{O} \times .. \times \mathcal{O})$. $\omega^{(n)}(f^{(n)})$ is the expectation value of the monomials $f^{(n)}$ in the state ω. In the present work we shall take this model as the kinematical model for the local algebras of a scalar field. It must be emphasized that this intepretation departs from the similar Borchers interpretation of the ordinary Wightman field theory, [13], in an essential feature. We do not admit, namely, any a priori relations between observables. In order to work with the more familiar notion of a covariant "quantum field ϕ" we shall write for the degree 1 elements of the algebra $\phi(f^{(1)})$ instead of $f^{(1)}$. Heuristically we may pass from $\phi(f^{(1)})$ in each chart $x = \varphi(p)$ to $\phi(x)$ according to

$$\phi(f^{(1)}) = \int d^4x \phi(x) f^{(1)}(x). \tag{7}$$

Correspondingly we may pass from $\omega^{(n)}(f^{(n)})$ to $W^{(n)}(x_1,..,x_n)$, where $W^{(n)}(x_1,..,x_n)$ is referred to as the n-point function of the state.
Depending on the specific theory in mind we also need concern ourselves in the following with the hierarchy of truncated n-point functions, $W_T^{(n)}$, in terms of which the hierarchy $W^{(n)}$ is obtained by standard formulas.
The formalism described so far does not initially include any notion of space-time geometry. Therefore the central problem is how one can transform it in a semiclassical theory. We address ourselves now to this problem.

2 The local structure of physical states

The concept of space-time metric is naturally tied to the subjective ignorance of the Planck regime. On the other hand, as was already indicated, that ignorance requires a common leading short distance singularity of different states in the same primary folium. This raises the question of whether we can in some sense combine these two aspects.
In this section we want to exhibit the precise correspondence between the space-time metric and the local structure of states in one primary folium. So we shall, first, ignore the Planck regime and consider its effect later.
On general grounds we expect that the two point function plays the dominant role in the theory. Specifically, the space-time metric should be encoded basically in the local structure of that function. Therefore, in this work our attention will be focused on the specification of the local structure of the two point function, leaving the specification of higher functions to future work.
Let us now consider a "sufficiently small" contractible neighbourhood $\mathcal{O}_p$ of a point $p \in M$ and a primary folium, denoted by $\mathcal{F}_{\mathcal{O}_p}$, of local states on $A(\mathcal{O}_p)$. We set in some chart $x = \{x^\mu\} = \varphi(p)$

$$d_{\mathcal{O}_p} = sup_{x' \in \mathcal{O}_p} |x^\mu - x'^\mu| \; .$$

For a given state $\omega \in \mathcal{F}_{\mathcal{O}_p}$ we shall assume that there exists at least one smooth scalar function $F^{(2)} : \mathcal{O}_p \times \mathcal{O}_p \to R$, so that $\tau_x^{(2)}(x') \equiv F^{(2)}(x,x') W_T^{(2)}(x,x')$ is bounded as a function of x' in $\mathcal{O}_p$ and the limit

$$\|\tau_x^{(2)}\| := \lim_{d_{\mathcal{O}_p} \to 0} sup_{x' \in \mathcal{O}_p} |F^{(2)}(x,x') W_T^{(2)}(x,x')|, \tag{8}$$

exists and is nonvanishing. Here $W_T^{(2)}$ is the truncated two point function of the state ω. For practical reasons the quantity arising from the above limit is assumed to be dimensionless[2]. One might think of the function $F^{(2)}$ as describing the structure of the leading short distance singularity of the two point function of the state involved. Since the structure of this singularity should be common for all states in the same primary folium[3] in what follows the function $F^{(2)}$ is taken to be universal, i.e. independent on the individual states. Concerning the specification of that function we shall assume that the limit

$$\lim_{d_{\mathcal{O}_p}\to 0} d_{\mathcal{O}_p}^{-2} sup_{x'\in\mathcal{O}_p} |F^{(2)}(x,x')|$$

exists and is nonvanishing. Expanding now the function $F^{(2)}(x,x')$ in the coordinate differences $\xi^\mu = x'^\mu - x^\mu$, the above condition asserts that the leading term in this expansion must be of second order, viz.

$$F^{(2)}(x,x') = \tilde{g}_{\mu\nu}(x)\xi^\mu\xi^\nu + ... \tag{9}$$

The dimensionless quantity $\tilde{g}_{\mu\nu}(x)$ that arises from this expansion transforms like a tensor and is determined by the above assumption up to a conformal factor (note that ξ^μ does not transform in general like a vector). In view of this fact one is led to conclude that the macroscopic metric $g_{\mu\nu}(x)$ is obtained from $\tilde{g}_{\mu\nu}(x)$ by a conformal transformation, viz.

$$g_{\mu\nu}(x) = \Omega^{-1}(x)\tilde{g}_{\mu\nu}(x). \tag{10}$$

This important observation may be regarded as the quantum version of the classical result that the knowledge of the null cone at each point of the space-time enables one to measure the metric at this point up to a conformal factor, see [14]. We can use this analogy further to give the function $F^{(2)}(x,x')$ an intrinsic geometrical meaning by requiring that the equation $F^{(2)}(x,x') = 0$ define the null cone at point x. Therefore by this requirement $F^{(2)}(x,x')$ can be identified up to the conformal factor $\Omega^{-1}(x)$ with the square of the geodesic distance $\sigma(x,x')$ between the points x and x', viz.

$$\sigma(x,x') = \Omega^{-1}(x)F^{(2)}(x,x'). \tag{11}$$

We may determine the conformal factor in the last equation by normalizing $\|\tau_x^{(2)}\|$ in (8) to one which results in $F^{(2)}(x,x')$ coinciding with $\sigma(x,x')$.

Having introduced the notion of local macroscopic metric, we now take on the problem of writing down an expansion determining the local structure of the two point function of the state considered. At this point there are several ways to proceed. The most convenient way consists in applying the techniques of covariant Taylor expansion, developed in [15] and [2]. We shall base our analysis on an expansion for the symmetric part of the truncated two point function $W_{T,S}^{(2)}$ of the form

$$W_{T,S}^{(2)}(x,x') = \sigma^{-1}(1 + a_\mu\sigma^{;\mu} + a_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu} + ...). \tag{12}$$

Here $a_\mu, a_{\mu\nu}, ..$ are (smooth) tensors at point x and the semicolon denotes covariant derivatives with respect to the symmetric affine connection defined by the metric.

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[2]We shall adopt in our discussion the natural units in which $c = \hbar = 1$. Accordingly the field φ will have the dimension of an inverse length.

[3]In the notation of the work [11] this statement corresponds to the well established fact that the scaling limit coincides for all states in one primary folium.

It should be noted that this is not to say that such an expansion could not include additional singular terms which respect the norm condition (8). For example we could allow $W^{(2)}_{T,S}(x,x')$ to involve an additional logarithmic singularity, such as in the case of Hadamard expansion. But, in that expansion the logarithmic singularity occurs because the equations governing the dynamics of the quantum field are supposed to be linear. As already mentioned in the introduction we are not satisfied with this idea. Generally there is no real justification for regarding such additional singularities as fundamental. We therefore adopt the view that additional singularities are not present. It is quite likely that at some future time we may have the occasion for improving the expansion (12), e.g. by a return to an additional singularity. But at this stage we must adhere to the principle of simplicity. In this sense the expansion is the simplest thing that one can write.

Another point is that in general there would be states in one primary folium whose behaviour do differ from that given by (12). We assert to have in (12) only a condition singling out the subclass of "smooth states". These are such that the amount of the energy momentum dencity produced by them is finit. This point will be illustrated in section 5.

Now, for a reason which is apparant from mathematics we shall refer to the expansion (12), when terminated at some order, as the jet class associated with this order. For example the jet class of order two is determined by the tensors a_μ and $a_{\mu\nu}$. This terminology will help us to avoid confusion.

One important point should be noted about the expansion (12). In reality we must always confine ourselves in (12) to a separation of the points x and x' of scales greater than Planck length, as we are dealing with semiclassical quantum gravity. Further, we must always avoid the possibility that the separation of the points x and x' becomes too large, as we have in (12) a local expansion. In actual situations there would be always a domain of many order of magnitude on which the expansion (12) can be valid.

Thus, if we want to develop the theory with the expansion (12) that part of the (symmetric) truncated two point function which corresponds to a separation of the points x and x' of scales comparable with the Planck length remains unspecified. Basically, one is dealing here with a lack of determinacy. There are, however, important indications that the theory should become finite at scales below the Planck length. Once this assumption is made the Planck length would act as natural cut-off in the semiclassical theory and hence wherever we use the expansion (12) to make some calculations the end results must be replaced by their average value over the Planck regime as $x' \to x$. In this way one gets a theory in which no singularity occurs.

We shall adopt this point of view in our discussion. It will be used in the form that the average of σ^{-1} over the Planck regime gets replaced by the inverse value of the gravitational constant, κ^{-1}. We then have in the theory a sort of a general principle which asserts that the effect of gravity should always be included in the local structure of states. We shall refer to this principle as Planck structure hypothesis. This hypothesis reduces the occurence of singularities to a peculiarity of the Minkowskian limit $\kappa \to 0$.

The discussion so far has led to a semiclassical interpretation of the theory, i.e. disregarding the Planck-regime, the local macroscopic geometry arises as a common intrinsic property of a primary folium of local physical states. The next central question concerns the physical significance of jet classes and the problem of their specification. At this stage we need the notion of dynamical laws in order to proceed.

3 The local laws

The problem of specifying the jet classes in the present context is closely related to what one calls in the conventional approach the problem of renormalization of the energy momentum tensor operator. First we note that if we wish to have a theory based on differential equations the actual construction of the jet classes must be subjected to a certain "maximal set" of differential equations relating them to the macroscopic geometry defined by the primary folium employed. There is an objective criterion telling us what kind of equations one should incorporate in a semiclassical theory. Indeed, following the intuitive idea that the admissible physical states should carry a finite inertial and gravitational mass the equations employed have to provide us with a realization of the principle of equivalence (equality of inertial and gravitational mass). Thus the problem becomes one of how to convert this idea into appropriate mathematical constraint equations on states.

As a first concrete step towards this goal let us assume that among all local observables of a bounded region $\mathcal{O}$ there is a specified observable, called Q, whose expectation value vanishes in each "smooth" state belonging to a primary folium $\mathcal{F}_{\mathcal{O}}$ of local physical states, viz.

$$< Q >_{\omega} = 0, \ \forall \omega \in \mathcal{F}'_{\mathcal{O}} \ . \tag{13}$$

where $\mathcal{F}'_{\mathcal{O}}$ denotes the class of smooth states as a subset of $\mathcal{F}_{\mathcal{O}}$. One may think of Q as being for each state sensitive to a deviation of the inertial mass from the gravitational mass. Viewed in this way the condition (13) is an essential constraint to which the relevant states must be subjected. Therefore we shall try to present the theory directly in terms of some postulates about Q.

In the present work we are primarely concerned with only one feature of Q, its scaling behaviour at a point $p \in M$. On the heuristic level we shall assume here that as $\mathcal{O}$ contracts to a single point p the scaling behaviour of Q is controlled in each chart $x = \varphi(p)$ by a symmetric tensor operator $Q_{\mu\nu}(x)$. Heuristically we may then replace the equation (13) by the following equations at a single point

$$< Q_{\mu\nu} >_{\omega} = 0, \ \forall \omega \in \mathcal{F}'_{\mathcal{O}} \ . \tag{14}$$

Now, as we are dealing with the principle of equivalence we would expect that the operator $Q_{\mu\nu}$ involves the field operator ϕ in a nonlinear manner.

It should be emphasized that the condition (14) need not hold for all states in one primary folium. Rather, we would expect that there are local states for which the right hand side of (14) differes from zero, leading to a state dependent residual quantity. But, it is quite reasonable to think of such states as either describing irreversible processes, the residual quantity corresponding to the local entropy production, or not being at all "well behaved". In this sense we shall interprete (14) as a condition characterizing "the local equilibrium states"[4].

It should be clearly understood that the behaviour of the ϕ-field established by (14) does not happen in the ordinary Minkowski-theories. It is an entirely new feature emerging in theories including the gravitational interaction. Therefore we are led to formulate a correspondence principle. According to this principle the physical effect of the equations (14) should disappear

[4]To explain the extent to which the equations (14) are the defining characteristic of local equilibrium, and to esablish their structural connection with some local stability group remains to be explored.

in the nongravitational limit $\kappa \to 0$ where the space-time metric should become globally the Minkowski metric. We may establish this fact by requiring that the expectation value $< Q_{\mu\nu} >_\omega$ in every state of one primary folium should satisfies the asymptotic condition

$$< Q_{\mu\nu} >_\omega \stackrel{\kappa \to 0}{\longrightarrow} \kappa^{-1} G_{\mu\nu}. \tag{15}$$

where $G_{\mu\nu}$ is the Einstein tensor corresponding to the macroscopic metric defined by the folium of local states considered. This ensures, indeed, that in the limit $\kappa \to 0$ the rquirement (14) is no longer a constraint on the states but is reduced to the identity $G_{\mu\nu} = 0$, as already satisfied in the Minkowski-theories.

We may also expect here a close relationship between the equations (14) and the semiclassical Einstein equations. In the next section we shall establish this relationship in more specific terms. Notice now that by (15) the whole of $Q_{\mu\nu}$ must have the dimension of a length to the power -4.

To construct $Q_{\mu\nu}$ in terms of the field operator ϕ we may start from the statement that the equations (14), as local equilibrium condition, need not hold for an arbitrary field cofiguration but only for fields which satisfy the dynamical laws. Thus the question arises of how to supplement them by a field equation. Here we are, of course, greatly hampered by the absence of a natural approach. But, tentatively, we may write the field equation in the form

$$\Box\phi + \kappa\phi\, Q_\alpha^{\ \alpha} = 0. \tag{16}$$

where $\Box$ is the invariant d'Alemberian depending on the local primary folium employed. Notice now that as a consequence of (15) and (16) in the nongravitational limit $\kappa \to 0$ the theory becomes one of a scalar massless field propagating in Minkowski space-time.

In view of (16) we would expect now that the operator $Q_{\mu\nu}$ involves the derivatives of the field ϕ up to first order (otherwise we would obtain certain pathologies). Further, because of complete homogeneity of space-time under equilibrium condition we would expect that $Q_{\mu\nu}$ can not explicitly contain the field operator ϕ and hence must be expressible only in terms of derivatives of ϕ.

The simplest candidat for $Q_{\mu\nu}$ incorporating all the expected features will be

$$Q_{\mu\nu} = \phi_{;\mu}\phi_{;\nu}. \tag{17}$$

The hypothesis that we want to advance is that the nescessary dynamical informations for the semiclassical quantum gravity situation are always contained in the equations (14), (16) and (17).

4 The Einstein equations

In this section we study more closely the kind of restrictions which the constraint equations (14) impose upon the structure of jet classes. Before entering into the discussion we want to collect some technical facts. First, in the standard notations of the point separation method, see [2], the equation (14) may be expressed as[5]

$$< Q_{\mu\nu} >_\omega = 2 \lim_{x' \to x} g_\nu^{\ \nu'} W^{(2)}_{S;\mu;\nu'}(x, x') = 0. \tag{18}$$

[5]In the following expression a symmetrization with respect to the indices μ and ν must be done so that $Q_{\mu\nu}$ becomes symmetric. For simplicity we shall make the symmetrization only at the end.

Here $W_S^{(2)}$, is the symmetric part of two point function of ω, and $g_\nu^{\nu'}$ is the bivector of the parallel transport (here and in what follows the unprimed indices refer to tensors in tangent space at x while the primed indices refer to tensor in tangent space at x').
An important feature of this equation is that it restricts only the structure of the jet class of order four. We refer for the discussion of the analogous situation in the frame of the conventional approach to the publications [2],[3], where attention was directed to the problem of renormalizing the energy momentum tensor operator and singularities arising from the Hadamard expansion of the two point function.
Now, let us write down explicitly the expansion that would determine the local stucture of the symmetric part of the truncated two point function, $W_{T,S}^{(2)}$, corresponding to the jet class of order four

$$W_{T,S}^{(2)}(x,x') = -6\ \sigma^{-1}(1 + a_\mu \sigma^{;\mu} + a_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu} + a_{\mu\nu\delta}\sigma^{;\mu}\sigma^{;\nu}\sigma^{\delta} + a_{\mu\nu\delta\gamma}\sigma^{;\mu}\sigma^{;\nu}\sigma^{;\delta}\sigma^{;\gamma}) \tag{19}$$

which is similar to (12) [6]. The requirement of symmetry determines the tensors a_μ and $a_{\mu\nu\delta}$ to

$$a_\mu = 0 \tag{20}$$
$$a_{\mu\nu\delta} = -\frac{1}{2} a_{\mu\nu;\delta}. \tag{21}$$

The simple proof may be found by looking at the symmetric covariant Taylor series, see [10].
We are now prepared to give the calculational results concerning the local behaviour of the expectation value $< Q_{\mu\nu} >_\omega$. Using the expansion (19) and the formula (18) we find after collecting terms in like powers of $\sigma^{;\mu}$

$$< Q_{\mu\nu} >_\omega = < Q_{\mu\nu} >_\omega^{quartic} + < Q_{\mu\nu} >_\omega^{quadratic} + < Q_{\mu\nu} >_\omega^{0} \tag{22}$$

where

$$< Q_{\mu\nu} >_\omega^{quartic} = -12 \lim_{x'\to x} \sigma^{-2}(-2\sigma^{-1}\sigma_{;\mu}\sigma_{;\nu} + g_{\mu\nu}) \tag{23}$$

$$< Q_{\mu\nu} >_\omega^{quadratic} = -12 \lim_{x'\to x} \{\sigma^{-3}(-2a_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta}\sigma_{;\mu}\sigma_{;\nu}) + \sigma^{-2}[(\frac{1}{6}R_{\mu\alpha\nu\beta} + g_{\mu\nu}a_{\alpha\beta})\sigma^{;\alpha;}\sigma^{;\beta} + 2a_{\mu\alpha}\sigma^{;\alpha}\sigma_{;\nu} + 2a_{\nu\alpha}\sigma^{;\alpha}\sigma_{;\mu}] - 2\sigma^{-1}a_{\mu\nu}\} \tag{24}$$

$$\begin{aligned} < Q_{\mu\nu} >_\omega^{0} = {} & W^{(1)}_{;\mu}W^{(1)}_{;\nu} - 12 \lim_{x'\to x}\{\sigma^{-3}(-2a_{\alpha\beta\delta\gamma}\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\delta}\sigma^{;\gamma}\sigma_{;\mu}\sigma_{;\nu}) + \\ & \sigma^{-2}[(\frac{1}{40}R_{\mu\alpha\nu\beta;\delta\gamma} + \frac{7}{360}R^{\tau}_{\alpha\mu\beta}R_{\tau\delta\nu\gamma} + g_{\mu\nu}a_{\alpha\beta\delta\gamma} + \frac{1}{6}R_{\mu\alpha\nu\beta}a_{\delta\gamma})\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\delta}\sigma^{;\gamma} + \\ & (-\frac{2}{3}a_{\alpha\lambda}R^{\lambda}{}_{\beta\mu\delta} + a_{\alpha\beta\delta;\mu} + 4a_{\mu\alpha\beta\delta})\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\delta}\sigma_{;\nu} + \\ & (\frac{1}{3}a_{\lambda\alpha}R^{\lambda}{}_{\beta\nu\delta} + 4a_{\nu\alpha\beta\delta})\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\delta}\sigma_{;\mu}] + \\ & \sigma^{-1}(-24a_{\mu\nu\alpha\beta} - 3a_{\mu\nu\alpha;\beta} - 3a_{\mu\nu\beta;\alpha})\sigma^{;\alpha}\sigma^{;\beta}\}. \end{aligned} \tag{25}$$

[6] For technical reasons we have separated off the factor -6.

Here is $W^{(1)}$ the one point function. In writing the above expressions we have supressed direction-dependent terms involving odd powers of $\sigma^{;\alpha}$, since such terms may be eliminated by averaging over a separation of the point x in the $\sigma^{;\alpha}$ direction and one in the $-\sigma^{;\alpha}$ direction. There remains still a difficulty concerning direction-dependent terms involving even powers of $\sigma^{;\mu}$. To get rid of direction-dependence of such terms, that is, in order for $< Q_{\mu\nu} >_\omega$ to be a true tensor at point x, one has to average over all directions using a suitable measure. Following the work of Adler,Lieberman,Ng [3] in what follows we use the elementary averaging procedure which consists in making the replacements

$$\sigma^{;\alpha}\sigma^{;\beta} \rightarrow \frac{1}{2}\sigma g^{\alpha\beta}$$

$$\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\delta}\sigma^{;\gamma} \rightarrow \frac{1}{6}\sigma^2(g^{\alpha\beta}g^{\delta\gamma} + g^{\alpha\delta}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\delta})$$

$$\begin{aligned}\sigma^{;\mu}\sigma^{;\nu}\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\delta}\sigma^{;\gamma} \rightarrow \frac{1}{24}\sigma^3\{ & g^{\mu\nu}(g^{\alpha\beta}g^{\delta\gamma} + g^{\alpha\delta}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\delta}) + \\ & g^{\mu\alpha}(g^{\nu\beta}g^{\delta\gamma} + g^{\nu\delta}g^{\beta\gamma} + g^{\nu\gamma}g^{\beta\delta}) + \\ & g^{\mu\beta}(g^{\nu\alpha}g^{\delta\gamma} + g^{\nu\delta}g^{\alpha\gamma} + g^{\nu\gamma}g^{\alpha\delta}) + \\ & g^{\mu\delta}(g^{\nu\alpha}g^{\beta\gamma} + g^{\nu\beta}g^{\alpha\gamma} + g^{\nu\gamma}g^{\beta\alpha}) + \\ & g^{\mu\gamma}(g^{\nu\alpha}g^{\beta\delta} + g^{\nu\beta}g^{\alpha\delta} + g^{\nu\delta}g^{\alpha\beta})\}.\end{aligned} \tag{26}$$

In consequence of this averaging the term $< Q_{\mu\nu} >_\omega^{quartic}$ vanishes identically. For the second term in (22) we find

$$< Q_{\mu\nu} >_\omega^{quadratic} = -\lim_{x'\rightarrow x} \sigma^{-1}(R_{\mu\nu} - 8a_{\mu\nu} + 2g_{\mu\nu}a_\alpha^{\ \alpha}). \tag{27}$$

Notice here that $< Q_{\mu\nu} >_\omega^{quadratic}$ involves the tensor $a_{\mu\nu}$ through the trace less expression $-8a_{\mu\nu} + 2g_{\mu\nu}a_\lambda^{\ \lambda}$.

Now, according to our Planck structure hypothesis, stated in section 3, we have to replace the expression (27) by

$$< Q_{\mu\nu} >_\omega^{quadratic} = -\kappa^{-1}(R_{\mu\nu} - 8a_{\mu\nu} + 2g_{\mu\nu}a_\alpha^{\ \alpha}). \tag{28}$$

Turning now to the evaluation of the last term in (22) we find after averaging

$$< Q_{\mu\nu} >_\omega^0 = W^{(1)}_{;\mu}W^{(1)}_{;\nu} + \tau_{\mu\nu} + H_{\mu\nu} \tag{29}$$

where

$$\tau_{\mu\nu} = 156\ a_{\mu\nu\lambda}^{\ \ \ \ \lambda} - 36 g_{\mu\nu}a_\alpha^{\ \alpha}{}_\lambda^{\ \lambda} \tag{30}$$

and

$$\begin{aligned}H_{\mu\nu} = -&\frac{1}{20}(\Box R_{\mu\nu} + R_\mu^{\ \lambda}{}_\nu^{\ \xi}{}_{;\lambda;\xi} + R_\mu^{\ \lambda}{}_\nu^{\ \xi}{}_{;\xi;\lambda}) - \\ &\frac{7}{180}(R^\tau{}_{\lambda\mu}{}^\lambda R_{\tau\xi\nu}{}^\xi + R^\tau{}_{\lambda\mu\xi}R_\tau{}^\lambda{}_\nu{}^\xi + R^\tau{}_{\lambda\mu\xi}R_\tau{}^\xi{}_\nu{}^\lambda) - \\ &\qquad\qquad \frac{1}{3}(a_\alpha^{\ \alpha}R_{\mu\nu} + 2R_{\mu\lambda\nu\alpha}a^{\lambda\alpha}) +\end{aligned}$$

$$\frac{4}{3}(a_{\alpha\lambda}R^{\lambda\alpha}{}_{\mu\nu} + a_{\alpha\lambda}R^{\lambda}{}_{\nu\mu}{}^{\alpha} + a_{\nu\lambda}R^{\lambda}{}_{\mu}) -$$
$$\frac{2}{3}(a_{\lambda\alpha}R^{\lambda\alpha}{}_{\nu\mu} + a_{\lambda\alpha}R^{\lambda}{}_{\mu\nu}{}^{\alpha} + a_{\lambda\mu}R^{\lambda}{}_{\nu}) +$$
$$+2a_{\mu\lambda}R^{\lambda}{}_{\alpha\nu}{}^{\alpha} - 4(a_{\nu\lambda}R^{\lambda}{}_{\alpha\mu}{}^{\alpha} + a_{\alpha\lambda}R^{\lambda}{}_{\nu\mu}{}^{\alpha} + a_{\alpha\lambda}R^{\lambda\alpha}{}_{\mu\nu}) -$$
$$2(a^{\lambda}{}_{\lambda\nu;\mu} + 2a_{\mu\lambda}{}^{\lambda}{}_{;\nu}) + 18a_{\nu\lambda}{}^{\lambda}{}_{;\mu}. \tag{31}$$

Now putting all these results together and looking back at (18) we find

$$< Q_{\mu\nu} >_{\omega} = -\kappa^{-1}(R_{\mu\nu} - 8a_{\mu\nu} + 2g_{\mu\nu}a_{\alpha}{}^{\alpha}) + W^{(1)}_{;\mu}W^{(1)}_{;\nu} +$$
$$+H_{\mu\nu} + \tau_{\mu\nu} = 0. \tag{32}$$

We may use at this point the correspondence principle (15) to obtain

$$a_{\mu\nu} = \frac{1}{4}R_{\mu\nu}. \tag{33}$$

This determines the jet class of order two. Consequently, the equations (32) take the form [7]

$$G_{\mu\nu} = -\kappa S_{(\mu\nu)}\{\omega\} \tag{34}$$

where

$$S_{\mu\nu}\{\omega\} = W^{(1)}_{;\mu}W^{(1)}_{;\nu} + H_{\mu\nu} + \tau_{\mu\nu}. \tag{35}$$

In (34) we have a set of 10 equations to which the actual construction of the allowed jet class of order four has to be subjected. These equations relate the one- and the two point function and correspond to the standard form of the semiclassical Einstein equations, the quantum source of the gravity being $S_{\mu\nu}\{\omega\}$.
Let us now look at the tensor $\tau_{\mu\nu}$. We immediately see a connection between that tensor and the amount of energy momentum contained in local part of the two point function. Actually, the tensor $\tau_{\mu\nu}$ is the basic dynamical variable occuring in the theory and one should always imagine different states in one primary folium to differ in the behaviour of $\tau_{\mu\nu}$. Only in this way we get a theory which is basically in accord with the standard ideas of general relativity. Now, from the standpoint of Cauchy-problem the equations (34) alone do not provide a determinate mathematical problem. We need, namely, a equation by which the quantum source of gravity can be computed independently. This gap is now filled by taking into account the field equation (16). Indeed, using the point separation method we may derive from (16) the following equations that would determine the one point function and the symmetric part of the two point function

$$\Box W^{(1)}(x) = -2\ \kappa \lim_{x'\to x}\lim_{x''\to x'} g^{\alpha''}{}_{\alpha'}W^{(3);\alpha'}_{;\alpha''}(x,x',x'') \tag{36}$$

$$\Box W^{(2)}_{S}(x,x') = -4\ \kappa \lim_{x''\to x}\lim_{x'''\to x''} g^{\alpha''}{}_{\alpha'''}W^{(4)\ ;\alpha'''}_{;\alpha''}(x,,x'',x''',x'). \tag{37}$$

where $W^{(3)}_{S}$ respectively $W^{(4)}$ is the three point function (symmetrized in x' and x'') respectively the four point function (symmetrized in each of the pair of points x,x' and x'',x''').
There is just a technical problem if we try to treat the Cauchy problem, because of the term

[7] in the following the round around the indices denotes the symmetric part of a tensor

$H_{\mu\nu}$ in (35). That term involves, namely, the fourth order derivatives of the metric and terms which are quadratic in curvature.
Now, if we adhere to the idea that the constraint equations (19) correspond to local statistical equilibrium, the effect of $H_{\mu\nu}$ might appear as small in comparison with other terms in (34) and hence one could put the theory in a more sensible form by neglecting that tensor. But, one can not get a reasonable interpretation of equations by adopting this picture. It is, namely, quite likely that the tensor $H_{\mu\nu}$, even if it is small, would lead to inappropriate stability properties of solutions.
Fortunately, there is one further possibility. It may be, of course, that $H_{\mu\nu}$ could be compensated entirely by a corresponding counter term in the expression of $\tau_{\mu\nu}$. But before one moves to this topic much care is needed to the local structure of higher functions of the states.

5 Conclusion ant Outlook

We hope to have demonstrated a new possibility of thinking about semiclassical quantum gravity. Let us summarize once again the basic steps.
Starting from the principle of equivalence we have attributed the corresponding nonlinear constraint equations (14) to quantum gravitation. The basic input here was the assumption that the relevant local states belong to one primary folium exhibiting a specific universal short distance structure. The latter property was essential in introducing the notion of macroscopic space-time metric. This acts as a superselection quantity seperating different folia of local states. To answer the question which folium of local states is actually realized we have to solve the nonlinear field equation (16) together with the constraint equations (14), (17) subject to appropriate boundary conditions. In this sense different folia of local states are connected by dynamical laws.
The nature of the dynamics in this scenario is, however, at this stage of development obscure, e. g. it is still not clear whether the Cauchy development respects the local structure of the truncated two point function assumed in (19), on which the results of this work are based. But, to this problem some understanding of the local behaviour of the higher functions seems to be an essential prerequisite. We feel confidence that a rigorous justification of this scenario can be given .
Concerning the thermodynamic aspects of the theory there is the problem of a deeper underestanding of constraint equations which we have called condition of local equilibrium. There must also be some change introduced into these equations in order to include the effect of local entropy production.
The other important question remains to be answered concerns the relation of our approach to a "Lagrangian" and its corresponding energy momentum tensor. From the conceptual point of view it is, of course, entirely open whether investigations in quantum gravity should follow the orthodox picture of Lagrangian formalism. Here we merely note that it is perhaps possible that the basic nature of the macroscopic metric to be essentially a state dependent quantity limits the effectiveness of such a picture.
In conclusion, let us point out that we have concentrated in this paper on the broed line of the development of a "possible theory", rather than on any attemps at a rigorous justification of our assumptions. It is our belief that a rigorous formulation of a theory along the line suggested will have a beneficial effect upon our understanding of quantum gravity.

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