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BLOCKSPIN AND MULTIGRID FOR STAGGERED FERMIONS IN NON-ABELIAN GAUGE FIELDS * †

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ABSTRACT

We discuss blockspins for staggered fermions, i. e. averaging and interpolation procedures which are needed in a real space renormalization group approach to gauge theories with staggered fermions and in a multigrid approach to the computation of gauge covariant propagators. The discussion starts from the requirement that the symmetries of the free action should be preserved by the blocking procedure in the limit of a pure gauge. A definition of an averaging kernel as a solution of a gauge covariant eigenvalue equation is proposed, and the properties of a corresponding interpolation kernel are examined in the light of general criteria for good choices of blockspins. Some results of multigrid computations of bosonic propagators in an $SU(2)$ gauge field in 4 dimensions are also presented.

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1. Free Staggered Fermions

We begin with the discussion of block spins for free staggered fermions (without gauge fields) which respect the symmetries of free staggered fermion actions on a lattice. In sections 1.1. and 1.2. results of Joos and Schaefer [1] on the structure of these symmetries are briefly reviewed (see also [2]).

1.1. *Distribution of Spin and Flavor Over Sites of the Lattice*

A staggered fermion field on the d-dimensional lattice $\Lambda_{a/2} = (\frac{1}{2}a\mathbb{Z})^d$ of lattice spacing $a/2$ is a 1-component field χ. One should better interpret the theory as living on a lattice of period a which is composed of cells with 2^d sites each. Only translations by integer multiples $\vec{n}a$ of a are true space time translations. One cell consists of one site $x \in \Lambda_a = (a\mathbb{Z})^d$ in the hypercubic lattice of lattice spacing a and of all its translates in $\Lambda_{a/2}$ of the form

$$x_H \equiv x + \frac{1}{2}e_H\ , \quad e_H = \sum_{\mu\in H} e_\mu\ . \tag{1.1}$$

H is any set $H = \{\mu_1 \dots \mu_h,\ 0 \le h \le d,\ \mu_1 < \mu_2 \cdots < \mu_h\}$ of distinct indices μ_i, and e_μ is the lattice vector of length a in μ-direction in Λ_a. Spin and flavor are thought to be distributed over sites x_H in one cell as follows [1]:
Let γ^μ be the Euclidean Dirac matrices and

$$\gamma^H = \gamma^{\mu_1} \cdots \gamma^{\mu_h}\ ,\ \gamma^\emptyset \equiv \mathbb{1}\ . \tag{1.2}$$

One introduces a matrix $Z = (Z_a{}^b)$ of forms

$$Z = 2^{-[d/2]} \sum_H (\gamma^H)^T\ B\ dx^H \tag{1.3}$$

where $dx^H = dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_h}$ and

$$B dx^H = (-1)^{h(h-1)/2} dx^H\ . \tag{1.4}$$

One may expand

$$\sum_H \chi(x_H)\ dx^H = \sum_{a,b} \Phi_a{}^b(x)\ Z_a{}^b\ . \tag{1.5}$$

This exhibits the field $\chi(x_H)$ at site x_H in a cell as a linear combination of field components $\Phi_a{}^b$ with spinor index a and flavor index b. Each index takes values $1, \dots, 2^{[d/2]}$.
We will call H "pseudoflavor". Sites x_H in $\Lambda_{a/2}$ with the same pseudoflavor H form a sublattice $\Lambda_a{}^H$ of $\Lambda_{a/2}$ [figure 1],

$$\Lambda_a{}^H = \Lambda_a + \frac{1}{2}e_H\ . \tag{1.6}$$

1.2. *The Lattice Symmetry Group*

A staggered fermion field describes fermions with $2^{[d/2]}$ flavors. In the continuum limit, the action is invariant under elements g in the group $\mathcal{T}$ of all translations in d dimensions, under flavor transformations $f \in SU(2^{[d/2]})$ and under spinorial (Euclidean) Lorentz transformations.

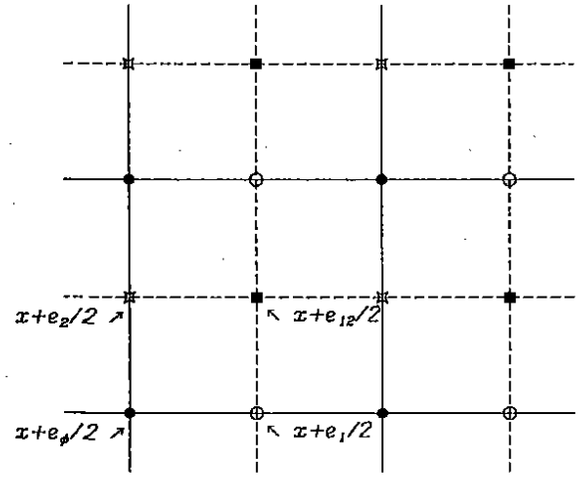

Fig. 1. Part of a pseudoflavour lattice Λ^H for $d = 2$, $H \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, embedded into $\Lambda = \mathbb{Z} \times \mathbb{Z}$.

On the lattice $\Lambda_{a/2}$, the symmetry under translations and flavor transformations is reduced to a subgroup

$$\mathcal{G}_a \subset T \otimes SU(2^{\lfloor \frac{d}{2} \rfloor}) \tag{1.7}$$

Translations by integer multiples $\tilde{n}a$ of a (*not* $a/2$) survive as true translation symmetries. Pure flavor transformations do not survive as symmetries at all. What survives are certain combinations ϵd^H of flavor transformations and translations by $\frac{1}{2}c^H$, where $\epsilon = \pm 1$ and H is a set of distinct indices as before. We call them pseudoflavor transformations. The square $(d^H)^2$ of any such transformation is a true lattice translation by $-c_H$. To keep track of the dependence on the lattice spacing, we write $T_a(*)$ for the action of an element $*$ of the symmetry group. ϵd^H acts on the staggered fermion field as

$$\begin{aligned} (T_a(\epsilon d^K)\chi)(y) &= \epsilon\, \hat{\rho}_{H,K}\, \chi(y + \frac{1}{2}c_K) \\ &\quad \text{if } y \in \Lambda_a^H \,. \end{aligned} \tag{1.8}$$

$\hat{\rho}_{H,K}$ is a sign factor which is the same as appears in the Clifford product of forms, given by $dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + \delta^{\mu\nu}$, and

$$dx^H \vee dx^K = \hat{\rho}_{H,K}\, dx^{H \triangle K} \tag{1.9}$$

where $H \triangle K$ is the symmetric difference $H \triangle K = (H \cup K) \backslash (H \cap K)$.
The action of translations is the obvious one. In particular, translations by the vectors c_H in Λ_a act as

$$T_a(c_H)\, \chi(y) = \chi(y - c_H) \,.$$

The spinorial Lorentz transformations in the continuum can be thought to be composed of "geometrical Lorentz transformations" and flavor transformations. A subgroup of the group of geometric Lorentz transformations survives on the lattice $\Lambda_{a/2}$. It is equal to the discrete group of rotations (and reflections) of a hypercubic lattice Λ_a.

1.3. *Blocking Consistent With the Symmetry*

We proceed to the consideration of the block lattice $\Lambda_{La/2}$ of lattice spacing $La/2$. We would like to map the staggered Fermi field χ on $\Lambda_{a/2}$ into a field Ξ, called block spin, on $\Lambda_{La/2}$ through application of a suitable averaging operation C

$$\Xi = C\chi \,. \tag{1.10}$$

C will be given by a kernel $C(x,z)$ with arguments $z \in \Lambda_{a/2}$ and $x \in \Lambda_{La/2}$ so that[a]

$$\Xi(x) = \int_{z \in \Lambda_{a/2}} C(x,z)\, \chi(z) \,. \tag{1.11}$$

We demand that the blocking is consistent with the symmetries. This means first of all that the symmetry group $\mathcal{G}_{La}$ on the coarser lattice should be a subgroup of the symmetry group $\mathcal{G}_a$ on the finer lattice,

$$\mathcal{G}_{La} \subset \mathcal{G}_a \,. \tag{1.12}$$

Moreover, the blocking map C should commute with the action of the surviving symmetry

$$T_{La}(g)\, C = C\, T_a(g) \quad \text{for } g \in \mathcal{G}_{La} \,. \tag{1.13}$$

On the right hand side, g should be regarded as an element of $\mathcal{G}_a$, defined through the embedding (1.12).
We write e'_μ for the lattice vectors of length La in Λ_{La}, and similarly for ϵ'_H. Thus,

$$\epsilon'_H = L e_H \,. \tag{1.14}$$

The elementary pseudoflavor transformations d'^K on $\Lambda_{La/2}$ must be linear combinations of pseudoflavor transformations d^K and translations e on $\Lambda_{a/2}$. We may temporarily think of the theory on $\Lambda_{a/2}$ as obtained from a continuum theory by a sequence of blockings, and regard again d^K as a combination of a true flavor transformation and a translation by $-\frac{1}{2}e_K$, and similarly for d'^K. It follows that we should have

$$-\frac{1}{2}e'_K = -\frac{1}{2}e_H + e \tag{1.15}$$

for some H, where e is a true translation. That is $e = \sum n^\mu e_\mu$ with integers n^μ. Since $e'_K = Le_K$, this is only possible if L is odd,

$$L = 3, 5, \ldots \tag{1.16}$$

and if $H = K$, $e = \frac{1}{2}(L-1)\, e_K$. In particular, $L = 2$ is not allowed.
The commutativity condition (1.13) specializes to

$$T_{La}(\epsilon'_K)\, C = C\, T_a(Le_K) \tag{1.17}$$

$$T_{La}(\epsilon d'^K)\, C = C\, T_a(\epsilon d^K)\, T_a(\frac{1}{2}(L-1)\, e_K) \tag{1.18}$$

[a] We use lattice notation $\int_{z \in \Lambda_{a/2}} \equiv (\frac{a}{2})^d \sum_{z \in \Lambda_{a/2}}$.

for translations and pseudoflavor transformations. This is easy to fulfill. We regard the block lattice $\Lambda_{La/2} = (\frac{1}{2}La\mathbb{Z})^d$ as a sublattice of $\Lambda_{a/2} = (\frac{1}{2}a\mathbb{Z})^d$ in the obvious way. If L is odd, then this identification is consistent with the flavor assignment. This means that a site $y \in \Lambda_{La/2}$ carries the same pseudoflavor whether we regard it as a site in $\Lambda_{La/2}$ or as a site in $\Lambda_{a/2}$. This is so because

$$\Lambda'^H_{La} = \Lambda_{La} + \epsilon'_H \subset \Lambda^H_a = \Lambda_a + \epsilon_H \ . \tag{1.19}$$

Commutativity conditions (1.17),(1.18) are fulfilled if C is translation invariant so that

$$C(x + \epsilon'_\mu, z + Le_\mu) = C(x, z) \tag{1.20}$$

and if in addition

$$C(x, z) = 0 \quad \text{unless } x \in \Lambda^H_{La/2},\ z \in \Lambda^H_{a/2} \tag{1.21}$$

for some H. The last condition means that if $y \in \Lambda^H_{La/2}$ then $\Xi(y)$ is a weighted sum of fields $\chi(z)$ at sites with the same pseudoflavor H. Blockings with such a property were considered for 2-dimensional staggered fermions in the presence of $U(1)$ gauge fields by Ben Av et al.[3] However they chose a scale factor $L = 2$.

1.4. *Block Averages*

In theories of scalar fields, block averages are a natural choice of block spin.[4] This means that $C(x, z)$ are step functions in z with supports on disjoint cubes $[x]$ of side L lattice spacings. The natural generalization of this choice of block spin to staggered fermions (without gauge fields) is as follows.

With each site x of the block lattice $\Lambda_{La/2}$ we associate a hypercube $[x] \subset \Lambda_{a/2}$ of side-length La, that is $2L$ lattice spacings $a/2$. When considered as a site in $\Lambda_{a/2}$, x will be the point in the center of this hypercube. These hypercubes overlap. If $x \in \Lambda^H_{La/2}$ then we call $z \in [x]$ an active site if $z \in \Lambda^H_{a/2}$, i. e. if it carries the same pseudoflavor as the central point x of the hypercube $[x]$. Every point z in $\Lambda_{a/2}$ is active site in exactly one block $[x]$. Every hypercube $[x]$ contains L^d active sites [figure 2].

It is necessary to admit overlapping blocks $[x]$ if one insists that $\chi(z)$ makes a contribution to some block spin $\Xi(x)$ for any z, i.e. in order that every z is active site of some block (no decimation).

If $x \in \Lambda^H_{La/2}$ then we set

$$\left(L\frac{a}{2}\right)^d C(x, z) = \begin{cases} 1 & \text{if } z \in [x] \cap \Lambda^H_{a/2} \\ 0 & \text{otherwise .} \end{cases} \tag{1.22}$$

$C(x, z)$ is nonvanishing only on active sites z of hypercube $[x]$, so that it has disjoint support in z for different x. Conditions (1.20) and (1.21) are fulfilled.

The natural dimension of mass of fermion fields is $(d-1)/2$; if we choose to consider the dimensionless fields which are obtained by multiplying with powers of lattice spacings, a factor $L^{(d-1)/2}$ should be inserted on the right hand side of eq. (1.22).

A variant of this blocking procedure was first introduced by G. Mai.[5] She considered blocking from the continuum to the lattice.

With a view to generalization to theories with gauge fields, we reformulate definition (1.22)

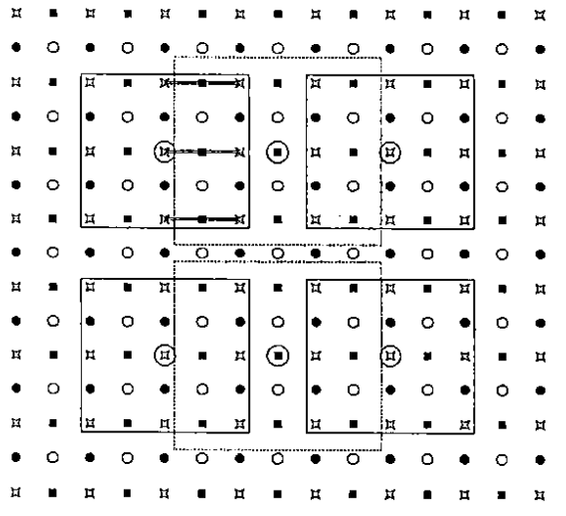

Fig. 2. Block averaging consistent with the symmetry on a 2-dimensional lattice with scale factor $L = 3$: only blocks of two pseudoflavors (stars and squares, the borders of a block indicated by solid and dotted lines, resp.) are shown. Central sites of blocks (sites of the blocklattice) are marked by a circle. Note the overlapping of blocks: The six double lines denote links between active sites of different blocks (different pseudoflavors). Each block contains L^2 active sites.

in a fancier way. The fundamental action for massless staggered fermions on a lattice $\Lambda_{a/2}$ is

$$S = \int_{z \in \Lambda_{a/2}} \bar{\chi}(z)\ \not{\nabla}\chi(z) \ . \tag{1.23}$$

The Dirac operator $\not{\nabla}$ is defined by

$$\not{\nabla}\chi(z) = \sum_{\mu=1}^{d} \eta_\mu(z)\ \nabla^\mu\chi(z) \tag{1.24}$$

$$\nabla^\mu\chi(z) = \frac{1}{a}\left[\chi(z + \frac{1}{2}e_\mu) - \chi(z - \frac{1}{2}e_\mu)\right] \tag{1.25}$$

$\eta_\mu(z) = \pm 1$ are a substitute for Dirac matrices. They are the special cases of the sign factors $\eta_K(z) = \hat{\rho}_{K,H}$ for $z \in \Lambda^H_{a/2}$, with $K = \{\mu\}$. In the absence of gauge fields,

$$\not{\nabla}^2 = \Delta = \text{Laplacian} \ . \tag{1.26}$$

Explicitly

$$\Delta\chi(z) = \frac{1}{a^2} \sum_{0 \neq \mu = -d}^{d} \left[\chi(z + e_\mu) - \chi(z)\right] \ . \tag{1.27}$$

If $z \in \Lambda_{a/2}^H$ then $\triangle\chi(z)$ depends only on $\chi(w)$ for $w \in \Lambda_{a/2}^H$ (i. e. z and w must have the same pseudoflavor).

We define the Laplacian with "Neumann boundary conditions" on the boundary of hypercubes $[x]$ as follows. If $x \in \Lambda_{La}^H$ then $\triangle_{N,x}$ is defined by the same formula (1.27) as $\triangle$ except that those terms in the sum are omitted where z and $z + \epsilon_\mu$ are both in Λ_a^H, but one is in the block $[x]$ and the other is outside $[x]$.

The choice (1.22) of C satisfies the condition (1.21) and the equation

$$-\triangle_{N,x} C^*(z,x) = \lambda_0(x)\ C^*(z,x)\ , \tag{1.28}$$

and $C(x,z)$ vanishes outside $[x]$. The Laplacian $\triangle_{N,x}$ acts on the argument z, and $\lambda_0(x)$ is its lowest eigenvalue, i. e. $\lambda_0(x) = 0$.
We use the abbreviation $C^*(z,x) \equiv \overline{C(x,z)}$; this is the kernel of the adjoint operator C^*.
C is determined up to a phase factor by these requirements together with the normalization condition

$$CC^* = 1 \quad \text{i.e.} \quad \int_z C(x,z)\ C^*(z,y) = \delta(x-y)\ . \tag{1.29}$$

So far we have imposed the requirement (1.21) that $C(z,x)$ vanishes for $z \notin \Lambda_a^H$ as a separate condition. However, this condition could also be regarded as a consequence of the eigenvalue equation (1.28). If we restrict attention to sublattices Λ_a^K with $K \neq H$ then the modification of $\triangle$ through "Neumann boundary conditions" as stated above become inoperative, and the only solution of the eigenvalue equation (1.28) which vanishes outside $[x]$ is $C(z,x) = 0$. [This way of specifying the solution amounts to supplementing the Neumann boundary conditions on Λ_a^H by Dirichlet boundary conditions on Λ_a^K, $K \neq H$.]

We remark finally that the solution and its uniqueness are preserved when one replaces $\triangle_{N,x}$ by another operator $-\triangle'_{N,x}$ which agrees with $-\triangle_{N,x}$ on Λ_a^H but not in general:

$$-\triangle'_{N,x} = \nabla^{\mu *}_{N,x} \nabla^\mu_{N,x}\ .$$

Here $\nabla^\mu_{N,x} f(z) = 0$ if $z \pm \frac{1}{2}e_\mu$ are both in Λ_a^H, but one of them is in $[x]$ and the other is not in $[x]$. For all other z, $\nabla^\mu_{N,x} f(z) = \nabla^\mu f(z)$.

2. Staggered Fermions in Non-Abelian Lattice Gauge Theories

2.1. *Considerations on Gauge Covariance*

We proceed to the consideration of staggered fermions in the presence of an external lattice gauge field U. For definiteness, the reader may suppose that the gauge group is $G = SU(2)$. The gauge field attached to the link $b = (z + \frac{1}{2}e_\mu, z)$ in $\Lambda_{a/2}$ will be denoted by $U(z,\mu)$ or by $U[z + \frac{1}{2}e_\mu, z] = U[b]$. In the first blocking step, U takes values in G. In subsequent blocking steps, dielectric gauge fields may be used.[6] For $G = SU(2)$ they are real multiples of $SU(2)$-matrices. In either case

$$U(z,\mu)^* = U(z + \frac{1}{2}e_\mu, -\mu)\ . \tag{2.1}$$

It will be convenient to introduce also the parallel transporters along two successive links

$$U(z,2\mu) = U(z + \frac{1}{2}e_\mu, \mu)\ U(z,\mu) \tag{2.2}$$

and the "forward-backward" parallel transport

$$U(z,\pm\mu) = U(z + \frac{1}{2}e_\mu, -\mu)\ U(z,\mu)\ . \tag{2.3}$$

The last quantity is $\mathbb{1}$ for ordinary gauge fields.
The staggered fermion field will be a 2-dimensional vector in color space.

We continue to regard the block lattice $\Lambda_{La/2}$ as a sublattice of the lattice $\Lambda_{a/2}$. In this way, a gauge transformation V on the fine lattice $\Lambda_{a/2}$ fixes a gauge transformation on the block lattice. We wish to specify a blocking procedure which is gauge covariant. A gauge transformation $\chi(z) \to V(z)\ \chi(z)$ of the original field should induce a gauge transformation $\Xi(x) \to V(x)\ \Xi(x)$ of the block spin.

The averaging kernel $C(x,z)$ will be a 2×2 matrix. The requirement of gauge covariance implies that it should satisfy

$$C(x,z)\ V(z) = V(x)\ C(x,z)\ . \tag{2.4}$$

The most general solution of this requirement is of the form

$$C(x,z) = \sum_{\mathcal{C}: z \to x} \rho(\mathcal{C})\ U(\mathcal{C})\ . \tag{2.5}$$

Summation is over paths $\mathcal{C}$ on $\Lambda_{a/2}$ from z to x. They consist of links $b_1 \dots b_n$, and $U(\mathcal{C})$ is the parallel transporter $U[b_n] \dots U[b_1]$ along this path. Finally, $\rho(\mathcal{C})$ are arbitrary complex factors. However we prefer not to define C by specifying the factors $\rho(\mathcal{C})$ explicitly, in contrast to ref. [3].

2.2. *Averaging Kernel as Solution of a Gauge Covariant Equation*

In a gauge theory of staggered fermions, the gauge covariant Dirac operator is defined by

$$\not{\nabla}\chi(z) = \frac{1}{a}\sum_{\mu=1}^{d} \eta_\mu(z)\ [U(z,\mu)^*\ \chi(z + \frac{1}{2}e_\mu) - U(z,-\mu)^*\ \chi(z - \frac{1}{2}e_\mu)]\ . \tag{2.6}$$

There exists also a gauge covariant Laplacian $\triangle$ which replaces the Laplace operator (1.27)

$$\triangle\chi(z) = \frac{1}{a^2}\sum_{0 \neq \mu = -d}^{d} [U(z,2\mu)^*\ \chi(z + e_\mu) - U(z,\pm\mu)^*\ \chi(z)]\ . \tag{2.7}$$

In the case of an ordinary gauge field, $U(z,\pm\mu) = 1$ by definition (2.3). We admit dielectric gauge fields and defined the Laplacian so that it is covariant under rescaling $U[b] \to \lambda\ U[b]$, $\lambda \in \mathbb{R}$. The square of the Dirac operator is not the Laplacian but

$$\not{\nabla}^2\chi(z) = \left([\triangle + \frac{1}{a^2}\sum_{\mu \neq \pm\nu} \sigma^{\mu\nu}(z) F^*_{\mu\nu}(z)]\ \chi\right)(z)\ . \tag{2.8}$$

The natural definition of $F_{\mu\nu}$ on the lattice [6] enters here

$$F_{\mu\nu}(z) = U(z + \frac{1}{2}e_\mu, \nu)\ U(z,\mu) - U(z + \frac{1}{2}e_\nu, \mu)\ U(z,\nu)\ . \tag{2.9}$$

Summation is over positive and negative μ, ν, and $\sigma_{\mu\nu}$ is a commutator of η's.

The Laplacian continues to enjoy the property that $\triangle\chi(z)$ depends only on $\chi(y)$ for $y \in \Lambda_{a/2}^H$ if $z \in \Lambda_{a/2}^H$. The same is *not* true of the square of the Dirac operator.

One possible generalization of the definition (1.28) of an averaging kernel C is now as follows. We define $C^*(z,x) = C(x,z)^\dagger$ where † denotes the adjoint of 2×2 matrices. We continue to demand property (1.21), i. e. C averages only over sites z of the same pseudoflavor. We define the covariant Laplacian $\triangle_{N,x}$ with Neumann boundary conditions on the boundary of block $[x] \subset \Lambda_{a/2}$ by eq. (2.7) for $z \in [x]$, but with the modification that only those terms in the sum over μ are retained for which $z + \epsilon_\mu \in [x]$. It is required that

$$-\triangle_{N,x}\, C^*(z,x) = \lambda_0(x)\, C^*(z,x)\ . \tag{2.10}$$

$\triangle_{N,x}$ acts on z and on the first index of the matrix $C^*(z,x)$. $\lambda_0(x)$ is the lowest eigenvalue of the positive semidefinite operator $-\triangle_{N,x}$ [considered as acting on functions which vanish on sites $z \notin \Lambda_{a/2}^H$ when $x \in \Lambda_{La/2}^H$]. In addition we may impose the normalization condition

$$CC^* = \mathbb{1}\ . \tag{2.11}$$

Eqs. (2.10),(2.11) are gauge covariant, and therefore the solution is also gauge covariant, modulo the arbitrariness in the solution. The arbitrariness consists in the possible substitution

$$C^*(z,x) \to C^*(z,x)\ \Omega(x) \tag{2.12}$$

where $\Omega(x)$ is a unitary 2×2 matrix. This amounts to a gauge transformation on the block lattice, plus multiplication of $C(x,z)$ by a constant phase factor. Such arbitrariness appears because we have not yet built in the information that we selected a distinguished site $x \in \Lambda_{a/2}$ (the block center) in every block $[x]$, and the covariance condition on C depends on this choice. If $G = SU(2)$ then we may eliminate the arbitrariness in a gauge covariant way by demanding that

$$C(x,x) = c(x) \cdot \mathbb{1} \ \text{ with } \ c(x) > 0\ .$$

This ensures that the unique solution is gauge covariant in the sense explained before.

There exists an efficient algorithm to compute the solution of the eigenvalue equation (2.10) numerically. It is true and proven and has been described by one of us in [7].

The adjoint spinor $\overline{\chi}(z)$ is an independent Grassmannian integration variable in lattice gauge theory. We may specify an averaging procedure for it in the obvious way

$$\overline{\Xi}(x) = \int_z \overline{\chi}(z)\ C^*(z,x)\ . \tag{2.13}$$

This does not contradict the statement of G. Mai in [5] that one needs different blocking procedures for adjoints. She considered blockings from the continuum to the lattice. The subtlety which she encountered is hidden in the fact that pseudoflavor H for χ and for $\overline{\chi}$ does not have the same physical meaning in terms of spin and flavor.

As an alternative to the above scheme one could think of using the square of the Dirac operator in place of the Laplacian in the eigenvalue equation (2.10). This may seem more natural, but it leads to a much greater computational complexity and storage space requirements. It is therefore appropriate to make some general remarks first.

We will discuss what is a good blockspin, i. e. a good averaging operator C, in some detail in section 4. Together with C one needs to consider an interpolation kernel $\mathcal{A}$. Given C, the optimal choice of $\mathcal{A}$ is such that $\phi = \mathcal{A}\Phi$ minimizes the action for arbitrary Φ, under the constraint that $\Phi = C\phi$ is prescribed. *If one adopts the Galerkin choice* $\mathcal{A} = C^*$ then the extremum principle which involves the action restricts the choice of C as well. The projective multigrid philosophy of Hulsebos, Smit and Vink [6], and of Brower, Rebbi and Vicari [9] has this as its central idea, i. e. the local action should define the blockspin[b]. Use of the square of the Dirac operator in the eigenvalue equation for C would be in the spirit of this philosophy.

In the rest of this section we will discuss the possibility of using the Dirac operator to define C, and why this leads to much greater computational complexity. In section 4. we present numerical evidence that the simpler ("Laplacian") choice of C should be good enough.

We may define $\not{\nabla}_{N,x}\, \chi(z)$ for $z \in [x]$ by eq. (2.6) with the modification that those terms in the sum over μ are omitted for which $z \pm \frac{1}{2}\epsilon_\mu$ are both in $\Lambda_{a/2}^H$ but one is in $[x]$, while the other is not in $[x]$. Setting

$$-\not{\nabla}^2_{N,x} = \not{\nabla}^*_{N,x}\ \not{\nabla}_{N,x} \tag{2.14}$$

one finds that $-\not{\nabla}^2_{N,x}\, \chi(z)$ agrees with $-\triangle_{N,x}\, \chi(z)$ for $z \in \Lambda_{a/2}^H$ when $F_{\mu\nu}(z) = 0$.

The lattice $\Lambda_{a/2}$ may be divided into sites of even and odd parity in chessboard fashion.[10] $\not{\nabla}\, \chi(z)$ at even z depends only on $\chi(w)$ at odd w. Because L is odd, the even/odd assignment is consistent with the identification of $\Lambda_{La/2}$ as a sublattice of $\Lambda_{a/2}$, i. e. odd blocks $[x]$ have block centers which are odd etc. We might thus require that the averaging kernel C be a solution of the eigenvalue equation (2.10) with $-\not{\nabla}^2_{N,x}$ substituted for $-\triangle_{N,x}$. But the square of the Dirac operator couples sites with different pseudoflavor. Therefore it is in general not consistent with the eigenvalue equation to require that $C(x,z) = 0$ when x and z carry different pseudoflavor. Instead we should supplement the boundary conditions - compare the discussion at the end of subsection 1.4. The requirement that $C(z,x)$ should vanish for sites outside $[x]$ but not in Λ_a^H can be built into the eigenvalue equation as an additional boundary condition by setting

$$\not{\nabla}^2_{ND,x} = \Theta_x\ \not{\nabla}^2_{N,x} \Theta_x\ . \tag{2.15}$$

where $\Theta_x \chi(z) = \Theta_x(z)\chi(z)$ and $\Theta_x(z) = 1$ for $z \in [x] \cup \Lambda_a^H$, and $= 0$ otherwise.

The eigenvalue equation becomes

$$-\not{\nabla}^2_{ND,x} C^*(z,x) = \lambda_0(x)\ C^*(z,x) \tag{2.16}$$

where $\lambda_0(x)$ is the lowest eigenvalue of the positive semidefinite operator $-\not{\nabla}^2_{ND,x}$ [considered as acting on functions which vanish on odd sites z if x is odd and on even sites otherwise].

We seek solutions which vanish outside $[x]$. The arbitrariness in the solution consists in the possibility of the substitution (2.12) where $\Omega(x)$ is an arbitrary 2×2 matrix.

[b] Let us note that this principle does not suffice to fix the blockspin uniquely. The local action is supposed to be the action restricted to one block. But to specify this restriction completely requires a choice of boundary conditions. This choice of boundary conditions can be a subtle issue.

The solution will in general not vanish when x and z carry different pseudoflavor. In the presence of a gauge field, the Dirac operator is no longer translation invariant. Because pseudoflavor transformations contain an admixture of translation, it is also not invariant under pseudoflavor transformations. For this reason we cannot rely on symmetry to rule out the possiblility that eq. (1.21) is violated.

If we call z an active site of block $[x]$ when $C(x,z) \neq 0$ then it is no longer true that any site z is an active site of exactly one block. In this sense, the blocks will overlap in a nontrivial way. In numerical applications to multigrid computations of nearly massless propagators for staggered fermions in a gauge field U this leads to a great increase in complexity and storage space requirement.

The overlap problem could be ameliorated, but not eliminated completely, by introducing nonoverlapping blocks $[x]_s$ and requiring that $C(z,x)$ vanishes outside $[x]_s$ when $z \notin \Lambda_a^H$. This is implemented by redefining Θ_x in eq.(2.15): $\Theta_x\chi(z) = \Theta_x(z)\chi(z)$ and $\Theta_x(z) = 1$ for $z \in [x]_s \cup \Lambda_a^H$, and $= 0$ otherwise. The redefinition amounts to imposing Neumann boundary on larger blocks $[x]$ in Λ_a^H and Dirichlet boundary conditions on smaller blocks $[x]_s$ in Λ_a^K, $K \neq H$. (Remember that H is fixed by x).

Proceeding in this way, the old solution for C is retained in the limit of a pure gauge, and therefore the blocking is consistent with the symmetries in the limit of a pure gauge.

The maximal nonoverlapping hypercubes $[x]_s$ are given by

$$z \in [x]_s \text{ if } \max_\mu |z^\mu - x^\mu| \leq \frac{1}{2}(L-1)\,\frac{a}{2}\,. \tag{2.17}$$

The overlap problem is eliminated if we choose $[x]_s = \emptyset$ instead. But this leads back to the Laplacian form of the eigenvalue equation which was considered before.

The analytical and numerical considerations which we present in section 4. gave no hint that the prize of overlap pays. Therefore we have so far only implemented the "Laplacian" choice (2.10) of the averaging kernel C on a computer.

3. Multigrid Approach to the Computation of Propagators for Staggered Fermions in an External Gauge Field

The multigrid method has been a successful method for the solution of discretizations of partial differential equations.[11] In lattice gauge theory, the problem arises to compute the solution $S(z)$ of the equation

$$(-\not{\nabla}^2 + m^2)\, S(z) = f(z) \tag{3.1}$$

where $\not{\nabla}$ is the gauge covariant Dirac operator, and the right hand side f is given. When $f(z) = \mathbb{1}\cdot\delta(z-w)$ [$\mathbb{1}$ = unit matrix in color space] then $(-\not{\nabla} + m)S$ is the Dirac propagator. In application to the quenched approximation of QCD, one needs smeared Dirac propagators so that f is actually not a δ-function. If one has an efficient multigrid routine to solve eq. (3.1) (for staggered fermions) it could substitute for the conjugate gradient subroutine in existing hybrid Monte Carlo programs for the simulation of the full (unquenched) QCD, see for instance [12]. There the right hand side f is determined by the instantaneous configuration of a pseudofermion field and changes in every Monte Carlo updating sweep.

Traditional methods for solving eq. (3.1) suffer from critial slowing down when m^2 is small. The hope is that this problem can be avoided in a multigrid approach. Several groups have reported work on 2-dimensional systems [3,9,13–15], including staggered fermions in a $U(1)$ gauge field.[3,14] Our own work concentrates on 4-dimensional staggered fermions in an $SU(2)$ gauge field. Generalization to $SU(3)$ would be tedious but not difficult.

Let us review very briefly the basic idea on which the multigrid approach is based. This will explain why its application to gauge theories is not trivial, especially for staggered fermions.

In the multigrid approach one introduces a sequence of lattices $\Lambda^0, \Lambda^1, \Lambda^2, \ldots$ of increasing lattice spacing a_i

$$a_{i+1} = L a_i \;\;,\;\; a_0 \equiv a\,. \tag{3.2}$$

For staggered fermions we prefer to call the lattice spacing $a_i/2$, and $\Lambda^i = \Lambda_{a_i/2}$. In applications one has (anti-) periodic boundary conditions rather than an infinite lattice.

Suppose one wants to solve an equation

$$D_0 S_0 = f_0 \tag{3.3}$$

on the fundamental lattice Λ^0, with D_0 equal to the negative Laplacian without gauge fields plus a small mass term, for instance. After some relaxation sweeps on gets an approximation $\tilde{S}_0$ to S_0 which differs from the exact solution by an error $e_0 = \tilde{S}_0 - S_0$. Often the error becomes small very slowly, because of critical slowing down, but it becomes *smooth* very fast. The error satisfies an equation

$$D_0 e_0 = r_0 \tag{3.4}$$

which involves the residual $r_0 = D_0\tilde{S}_0 - f_0$. If e_0 is smooth, it is determined to a very good accuracy by a function e_1 on the next coarser lattice Λ^1, and can be represented in the form

$$e_0 = \mathcal{A}\, e_1 \tag{3.5}$$

with an interpolation map $\mathcal{A}$ which should be so chosen that it maps functions on Λ^1 into smooth functions on Λ^0.
Conversely, e_1 can be obtained from e_0 with the help of an averaging map C which satisfies

$$C\mathcal{A} = \mathbb{1}\,. \tag{3.6}$$

It follows that $e_1 = Ce_0$. Inserting (3.5) into (3.4) and acting on the result with C we see that e_1 will satisfy the equation

$$D_1 e_1 = r_1 \tag{3.7}$$

with

$$D_1 = C\, D_0\, \mathcal{A} \;\;,\;\; r_1 = C r_0\,. \tag{3.8}$$

The problem has been reduced to an equation on the coarser lattice. If the treatment of this equation by relaxation or other methods suffers from critical slowing down again, one may repeat the procedure, going to coarser and coarser lattices, until one arrives at a lattice Λ^n of few points if necessary. On such a lattice there can be no critical slowing down.

Given C there exists an ideal choice of the interpolating kernel $\mathcal{A}$ such that the above equations are really true and there is complete decoupling between layers[c]. Its construction is described in [16] and is recalled in the next section. Given C, the ideal $\mathcal{A}$ is such that $\phi = \mathcal{A}\Phi$

[c] Complete decoupling between layers means that the propagator is a sum of contributions from layers, and these contributions from different layers satisfy effective difference equations which do not couple. As a result, the convergence speed is determined by the convergence speed on the individual layers.

minimizes the action, for arbitrary Φ, under the constraint that the blockspin $\Phi = C\phi$ is prescribed. D_1 is equal to $\mathcal{A}^* D_0 \mathcal{A}$ for this choice of $\mathcal{A}$, and therefore D_1 is selfadjoint when D_0 is. For a good choice of the block spin, i. e. of C, the kernel $\mathcal{A}(z,x)$ has exponential decay in $|z-x|$ with decay length of order a_1. We adopt this as the *definition of a "good block spin"*. In the bosonic case, with $-D_0$ equal to the ordinary Laplacian (with or without an added mass term), the block average is known to be a good block spin in this sense.[4] In numerical applications one approximates the ideal $\mathcal{A}$ by truncating its exponential tails.[17]

In these considerations, smoothness was crucial. It is therefore important to answer the question what is a smooth function in the presence of gauge fields. The standard definition of smoothness of ϵ would be that the derivatives are small,

$$(\nabla_\mu \epsilon, \nabla_\mu \epsilon) = (\epsilon, -\Delta\epsilon) \ll (\epsilon, \epsilon) \; . \tag{3.9}$$

When ∇_μ are ordinary derivatives then this definition of smoothness is not gauge invariant. We could take covariant derivatives instead. In the bosonic case this gives indeed a natural gauge invariant notion of smoothness. The lowest eigenvalue of the covariant Laplacian is a natural measure of disorder. When disorder is big, then there are no smooth functions in the sense of (3.9). But in this case one does not really need them either in realistic problems, because propagators $(-\Delta + m^2)^{-1}(z,w)$ for *nonnegative mass squared* $m^2 \geq 0$ fall off fast anyway and there will be no critical slowing down when one applies standard methods.

In the case of staggered fermions the situation is different. There are now two seemingly reasonable definitions. We may take for Δ in eq. (3.9) the covariant Laplacian as defined in eq. (2.7) or substitute the square of the Dirac operator for it. Unfortunately, both definitions are qualitatively different, except in the case of a pure gauge.

Laplacian smoothness of a function $\chi(z)$ implies only restrictions which relate the values of $\chi(z)$ at points $z \in \Lambda_{a/2}{}^H$ with the *same* pseudoflavor H. The same is *not* true of Diracian smoothness.

Moreover, if the gauge field is fairly disordered, then there will be no functions with Laplacian smoothness. But it is *not* true in general that a propagator $(- \not{\nabla}^2 + m^2)^{-1}(z,w)$ cannot have slow decay for $m \approx 0$.

If we adopt the Laplacian definition of smoothness, then the choice $\mathcal{A} = C^*$ together with the definition (2.10) of an averaging operator is natural. But the rational behind this construction breaks down when the gauge field is too disordered.

Data have already been collected by one of us (T. K.) for bosonic propagators $(-\Delta + m^2)^{-1}(z,w)$, where Δ is the usual covariant Laplacian. The case of a pure gauge was considered on a 9^4 and 18^4 lattice and also an equilibrated $SU(2)$ gauge field at $\beta = 2.7$ on a 9^4 lattice. Several adjustable parameters were built into the routine to govern the relaxation procedure. In particular, damping by multiplication of the residual r_1 in the equation (3.7) with a factor was allowed, and it was found to be important to optimize this parameter. Gauß Seidel or Jacobi relaxation on the fine lattice Λ^0 was used, while eq. (3.7) on the coarse lattice was solved exactly.

In the case of a pure gauge values of $m^2 = 10^{-10}, \ldots, 10^{-1}, 1$ were examined. Complete elimination of critical slowing down was observed [figure 3]. For the equilibrated $SU(2)$-gauge field, the lowest eigenvalue $-m_{cr}^2 \geq 0$ of $-\Delta$ was first determined to an accuracy of 10^{-6} or better, and m^2 was chosen negative,

$$m^2 = m_{cr}^2 + \Delta m^2 \quad \text{with } \Delta m^2 = 10^{-7} \ldots 10^{-1} \; .$$

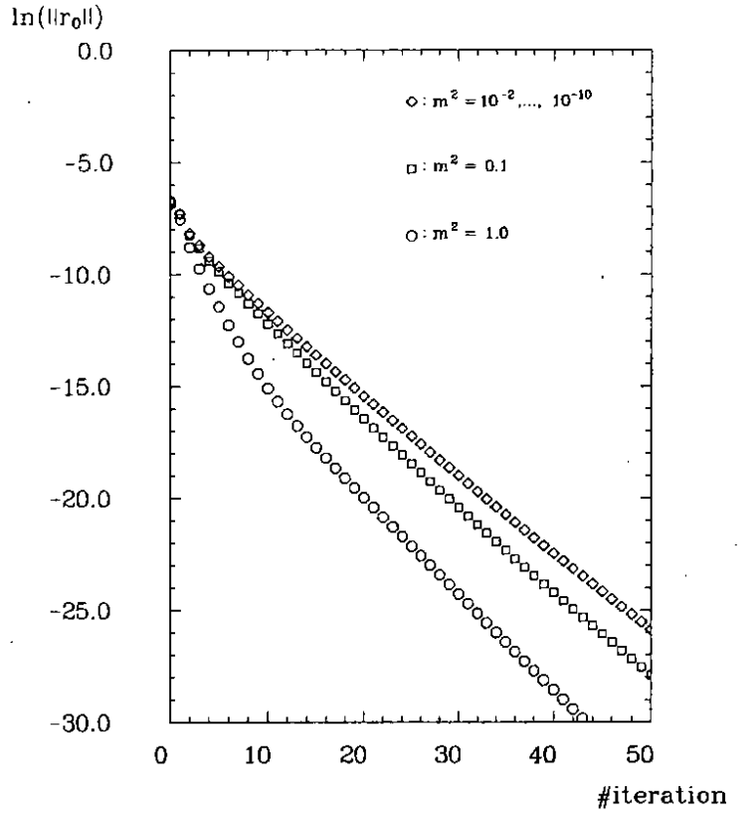

Fig. 3. Convergence of a multigrid computation of the bosonic propagator $(-\Delta + m^2)^{-1}$ in a pure gauge on an 18^4 lattice by a multigrid procedure. C is chosen as the lowest eigenmode of $-\Delta_{N,x}$, and $\mathcal{A} = C^*$, $L = 3$. The logarithm of the root-mean-square of the residual r_0 is plotted vs. number of iterations. One iteration involves one checkerboard Gauß Seidel (SOR) relaxation on the fundamental lattice, and exact solution of the residual equation on the 2nd layer by conjugate gradient. From left to right, the curves belong to $m^2 = 1$, $m^2 = 0.1$ and $10^{-2} \geq m^2 \geq 10^{-10}$. No mass dependence was observed for $m^2 \leq 10^{-2}$; the asymptotic relaxation time is $\tau = 2.9$ iterations in this range. For comparison: standard checkerboard Gauß Seidel (SOR) relaxation with optimal relaxation parameter on 1 layer gives $\tau = 0.188/m^2$ and undamped Jacobi iteration gives $\tau = 8/m^2$.

In this way a slow falloff of the propagator $(-\Delta + m^2)^{-1}$ was artificially enforced. The results were moderately encouraging [figure 4]. The multigrid procedure is much faster than

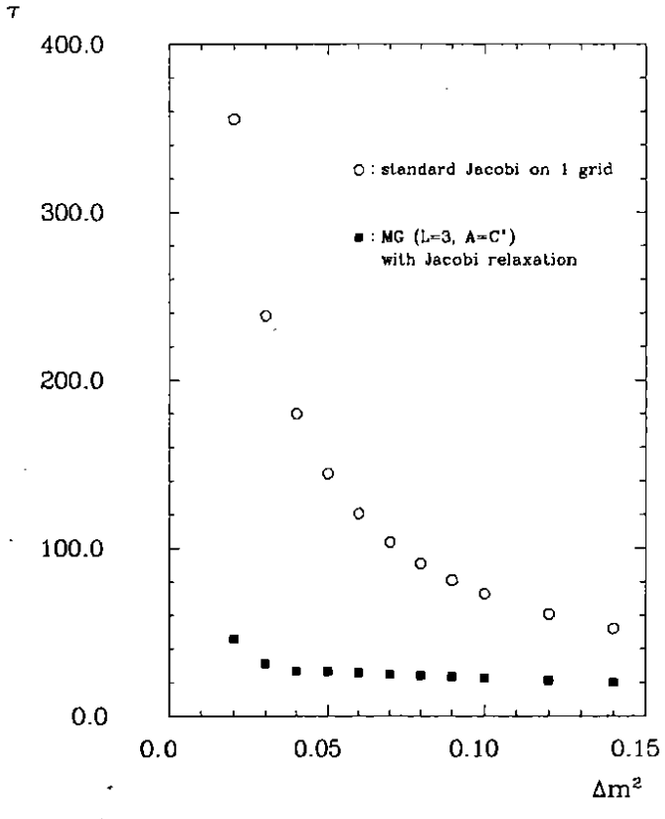

Fig. 4. Asymptotic relaxation time (in number of iterations) for computations of the bosonic propagator $(-\triangle + m^2)^{-1}$ in an $SU(2)$ gauge field equilibrated at $\beta = 2.7$ on a 9^4 lattice, with $m^2 = m_{cr}^2 + \Delta m^2$, $m_{cr}^2 = -0.8210607$.
Circles are for standard Jacobi relaxation on 1 layer.
Squares are for multigrid with $L = 3$, $\mathcal{A} = C^*$ and C the lowest eigenmode of $-\triangle_{N,x}$. One iteration involves one Jacobi relaxation sweep on the fundamental lattice, followed by exact solution of the residual equation on the 2nd layer by conjugate gradient. The mass dependence shows that critical slowing down is not completely eliminated.

ordinary relaxation on one layer, but critical slowing down was not completely eliminated and there was no gain compared to conjugate gradient on a 9^4 lattice. Details will be given elsewhere.[18]

This may be a warning sign that an $SU(2)$ gauge field at $\beta = 2.7$ is still too disordered to make the Laplacian definition of smoothness appropriate.

4. Criteria for Good Blockspins

The idea of the real space renormalization group approach to quantum field theory is to split a field ϕ on the original lattice into a low frequency part ψ and a high frequency part ζ, and to integrate out the high frequency field ζ in order to obtain an effective action. ψ should be smooth and determined by a function Φ on the block lattice. The split of the field becomes

$$\phi = \psi + \zeta = \mathcal{A}\Phi + \zeta \ . \tag{4.1}$$

Consider first a free field ϕ. The free field measure $d\mu_{S_0}(\phi)$ is determined by the free propagator S_0. We will mainly be interested in the case when S_0 depends on an external gauge field. In the case of a fermion field ϕ the normalized measure is

$$d\mu_{S_0}(\phi) = \left(det\ \frac{S_0}{2\pi}\right) e^{-\frac{1}{2}(\phi, S_0^{-1}\phi)} \prod_z d\phi(z)\ d\bar{\phi}(z) \ . \tag{4.2}$$

Given an averaging operator C with $CC^* = 1$ which associates a blockspin $\Phi = C\phi$ with every field ϕ, one can find $\mathcal{A}$ such that the measure $d\mu_{S_0}(\phi)$ factorizes into a measure for Φ and a measure for ζ

$$d\mu_{S_0}(\phi) = d\mu_{S_1}(\Phi)\ d\mu_\Gamma(\zeta) \ . \tag{4.3}$$

This is achieved by setting

$$\Gamma = (S_0^{-1} + \kappa C^* C)^{-1} \tag{4.4}$$

$$\mathcal{A} = \kappa\, \Gamma\, C^* \tag{4.5}$$

$$S_1 = C\, S_0\, C^* + \frac{1}{\kappa} \ . \tag{4.6}$$

κ is a free parameter which should be taken large enough.
For any κ the propagator S_0 splits into a propagator Γ for high frequency fields ζ and a propagator for the low frequency field ψ

$$S_0 = \mathcal{A}\, S_1\, \mathcal{A}^* + \Gamma \ .$$

S_1 is the propagator for the blockspin Φ. Factorization may be checked with the help of the formula for the characteristic function of a Gaussian measure, $\int d\mu_S(\phi) e^{i<J,\phi>} = e^{-\frac{1}{2}<J,SJ>}$. The limit $\kappa \to \infty$ exists and makes the split (4.1) unique [16], and eq. (3.6) holds[d]. In this case, an equivalent characterization of $\mathcal{A}$ is through an extremum principle. Given Φ, $\phi = \mathcal{A}\Phi$ extremises the action under the constraint that the blockspin $C\phi$ takes its prescribed value Φ.

ζ is a field with an infrared cutoff (i. e. a "mass") and therefore it is reasonable to demand that $\Gamma(z,w)$ should decay exponentially with distance $|z-w|$, with decaylength of order one block lattice spacing a_1. The same falloff properties should be shared by $\mathcal{A}(z,x)$. If $\kappa < \infty$ this is ensured by eq. (4.5). The effective action will have good locality properties if these requirements are fulfilled. This remains true when S_0 depends on a gauge field, and also in an interacting theory to all orders of perturbation theory. In special cases, this has also been shown nonperturbatively.[4,19] It will be said that C defines a good blockspin if the associated kernel $\mathcal{A}$ decays exponentially with decaylength a_1. It is plausible that a good block spin in

[d] In practice κ-independence is reached for $\kappa \gtrsim O(10^4)$.

this sense is essential in multigrid computations of gauge covariant propagators as well.

The problem is how C should be chosen for staggered fermions. For staggered fermions in 2 dimensions without gauge fields, G. Mai has shown that an averaging operator C which is related to the choice of section 1., but blocks from the continuum to the lattice, defines a good blockspin in this sense.[5] The method uses Fourier transformation and does not extend to gauge theories.

Balaban has presented a method to study falloff properties of gauge covariant propagators.[19] He considered the bosonic propagators

$$\Gamma = (-\triangle + \kappa C^* C)^{-1}$$

where $\triangle$ is the usual gauge covariant Laplacian, and showed exponential decay under general assumptions on C. It is instructive to see what difficulties arise when one tries to generalize this proof to staggered fermions. $\triangle$ gets replaced by the square of the Dirac operator.

In the bosonic case one starts from a lattice Λ_a of lattice spacing a, and one divides it into disjoint hypercubes $[x]$ of sidelength La. If L is odd, each block has a central point x; these block centers form a lattice Λ_{La}. Balaban proceeds by writing down a generalized random walk expansion for Γ. Its leading term is

$$(-\triangle_N + \kappa C^* C)^{-1} = \Gamma^0 \tag{4.7}$$

where $\triangle_N$ is the Laplacian with Neumann boundary conditions on all block boundaries. The full expansion is of the form

$$\Gamma = \Gamma^0 + \Gamma^0 Q \Gamma^0 + \Gamma^0 Q \Gamma^0 Q \Gamma^0 + \ldots \tag{4.8}$$

where Q is an operator which satisfies certain bounds on its norm, and whose kernel $Q(z,w)$ is only nonvanishing when z, w are in neighbouring blocks. (We oversimplify a little to bring out the main point).

Because the blocks are disjoint, the Neumann boundary conditions lead to a decoupling such that

$$\Gamma^0(z,w) = 0 \quad \text{unless } z, w \text{ are in the same block} . \tag{4.9}$$

It follows that the contributions to the series (4.8) up to n-th order make contributions to $\Gamma(z,w)$ only if z and w are no further than $n-1$ blocks away from each other, i. e. if

$$|z - w| < nLa . \tag{4.10}$$

Therefore the propagator Γ will have exponential decay if the Neumann series (4.8) converges.

Given the bounds on the norm of Q, convergence will hold if the lowest eigenvalue of

$$-\triangle_N + \kappa C^* C ,$$

restricted to one block, stays sufficiently far away from 0. $-\triangle_N$ has at most one eigenvalue close to zero, and if $C^* C$ is the projector on this eigenmode and κ is not too small, then the lowest eigenvalue of $-\triangle_N + \kappa C^* C$ is equal to the second lowest eigenvalue of $-\triangle_N$ (which is at least of order L^{-2}).

In conclusion, there are two essential ingredients. Neumann boundary conditions decouple

between different blocks, and the lowest eigenvalue of $-\triangle_N + \kappa C^* C$ is not too close to zero.

In the case of staggered fermions we want to replace the Laplacian $\triangle$ by the square of the Dirac operator $\not{\nabla}^2$ on $\Lambda_{a/2}$. The trouble is that blocks $[x]$ will overlap. This is innocuous in the case of a pure gauge. In this case $\not{\nabla}^2$ does not couple sites with different pseudoflavors H, and blocks of active sites $[x] \cap \Lambda^H_{a/2}$ do not overlap. As a result, one can proceed as in the bosonic case.

But this is no longer true when we have a nontrivial gauge field. One way out could be to retain $-\triangle_N + \kappa C^* C$ as 0-th order term, where $\triangle_N$ is still the Laplacian. But this means that the $F_{\mu\nu}$ term in $\not{\nabla}^2$ has to be absorbed into Q. If we use the Laplacian definition (2.10) of C, then the lowest eigenvalue of $-\triangle_N + \kappa C^* C$ is as big as we can hope to make it, and decoupling holds also, but a good enough bound on Q will require that $F_{\mu\nu}$ is small enough.

At this point we turned to numerical work. We considered equilibrated $SU(2)$ gauge field configurations on an 18^4 lattice whose probability distribution is given by the Wilson action for a pure gauge theory with β ranging from 2.7 to ∞ (pure gauge). To appreciate the decay properties, we compare with the case of a pure gauge, and with the decay properties of the $\mathcal{A}$-kernel in the bosonic case (i. e. without the $F_{\mu\nu}$-term). The results are encouraging [tables 1–6]. In the fermionic case, the decay properties of $\mathcal{A}$ at $\beta = 2.7$ are nearly as good as for a pure gauge.

We conclude that the kernel C of section 2, eq. (2.10), defines a good blockspin. Moreover, the effect of the $F_{\mu\nu}$-term is reasonably mild also in other respects. $\mathcal{A}(z,x)$ can be nonvanishing for sites z, x of different pseudoflavor when $F_{\mu\nu} \neq 0$. But we see that it stays reasonably small in this case, the RMS maximum is only 0.18 (0.12) compared to 1.0 for sites of equal pseudoflavor when $\beta = 2.7$ ($\beta = 3.0$).

This suggests to approximate $\mathcal{A}(z,x)$ by a kernel which is nonvanishing only when z, x have the same pseudoflavor. This saves much computational complexity and memory. We have implemented the scheme of section 3 for the computation of gauge covariant propagators with such choices of $\mathcal{A}$ (including $\mathcal{A} = C^*$) and with the Laplacian choice (2.10) of C. We expect to have numerical results soon.

Table 1. RMS of the kernel of the optimal interpolation operator $\mathcal{A}$ on blocks x for bosons in an $SU(2)$ gauge field on an 18^4 lattice: $\beta = \infty$ (pure gauge). A scale factor $L = 3$ is chosen. The kernel is centered at $y = 0$, i. e. the dimensionless quantity $(La)^d \sqrt{L^{-d} \sum_{z \in x} ||\mathcal{A}(z,0)||^2}$ is displayed, where $||\mathcal{A}(z,y)||^2 = \frac{1}{2}\mathrm{Tr}[\mathcal{A}(z,y)^\dagger \mathcal{A}(z,y)]$. The table shows the results in $x_1 - x_2$ planes — with x_1 (x_2) ranging from 0 to 5 from left (top) to right (bottom) — for fixed $x_3 = 0, \ldots, 3$ and $x_4 = 0$. Note the symmetry of the kernel for pure gauges.

KERNEL $\mathcal{A}$ FOR BOSONS ON AN 18^4 LATTICE, $L = 3$, $\beta = \infty$

$(x_3, x_4) = (0,0)$					
1.01351	0.13321	0.01688	0.00160	0.01688	0.13321
0.13321	0.02782	0.00398	0.00051	0.00398	0.02782
0.01688	0.00398	0.00016	0.00003	0.00016	0.00398
0.00160	0.00051	0.00003	0.00000	0.00003	0.00051
0.01688	0.00398	0.00016	0.00003	0.00016	0.00398
0.13321	0.02782	0.00398	0.00051	0.00398	0.02782

$(x_3, x_4) = (1,0)$					
0.13321	0.02782	0.00398	0.00051	0.00398	0.02782
0.02782	0.00751	0.00115	0.00019	0.00115	0.00751
0.00398	0.00115	0.00006	0.00001	0.00006	0.00115
0.00051	0.00019	0.00001	0.00000	0.00001	0.00019
0.00398	0.00115	0.00006	0.00001	0.00006	0.00115
0.02782	0.00751	0.00115	0.00019	0.00115	0.00751

$(x_3, x_4) = (2,0)$					
0.01688	0.00398	0.00016	0.00003	0.00016	0.00398
0.00398	0.00115	0.00006	0.00001	0.00006	0.00115
0.00016	0.00006	0.00001	0.00000	0.00001	0.00006
0.00003	0.00001	0.00000	0.00000	0.00000	0.00001
0.00016	0.00006	0.00001	0.00000	0.00001	0.00006
0.00398	0.00115	0.00006	0.00001	0.00006	0.00115

$(x_3, x_4) = (3,0)$					
0.00160	0.00051	0.00003	0.00000	0.00003	0.00051
0.00051	0.00019	0.00001	0.00000	0.00001	0.00019
0.00003	0.00001	0.00000	0.00000	0.00000	0.00001
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00003	0.00001	0.00000	0.00000	0.00000	0.00001
0.00051	0.00019	0.00001	0.00000	0.00001	0.00019

Table 2. RMS of the kernel of the optimal interpolation operator $\mathcal{A}$ on blocks x for bosons in an $SU(2)$ gauge field on an 18^4 lattice: $\beta = 3.0$. A scale factor $L = 3$ is chosen. The kernel is centered at $y = 0$, i. e. the dimensionless quantity $(La)^d \sqrt{L^{-d} \sum_{z \in x} ||\mathcal{A}(z,0)||^2}$ is displayed, where $||\mathcal{A}(z,y)||^2 = \frac{1}{2}\mathrm{Tr}[\mathcal{A}(z,y)^\dagger \mathcal{A}(z,y)]$. The table shows the results in $x_1 - x_2$ planes — with x_1 (x_2) ranging from 0 to 5 from left (top) to right (bottom) — for fixed $x_3 = 0, \ldots, 3$ and $x_4 = 0$. Note that the kernel is not symmetric for nontrivial gauge field and that it falls off a little more rapidly than in a pure gauge. This is an effect of the disorder in the gauge field.

KERNEL $\mathcal{A}$ FOR BOSONS ON AN 18^4 LATTICE, $L = 3$, $\beta = 3.0$

$(x_3, x_4) = (0,0)$					
1.01332	0.12022	0.01167	0.00157	0.01235	0.11862
0.11533	0.02132	0.00239	0.00039	0.00244	0.02188
0.01132	0.00225	0.00016	0.00004	0.00023	0.00234
0.00137	0.00037	0.00004	0.00001	0.00004	0.00035
0.01152	0.00249	0.00019	0.00004	0.00023	0.00260
0.12207	0.02220	0.00276	0.00040	0.00242	0.02381

$(x_3, x_4) = (1,0)$					
0.11704	0.02060	0.00258	0.00044	0.00250	0.02298
0.02134	0.00495	0.00065	0.00013	0.00065	0.00524
0.00256	0.00067	0.00006	0.00002	0.00008	0.00066
0.00040	0.00012	0.00001	0.00000	0.00001	0.00013
0.00252	0.00061	0.00006	0.00001	0.00007	0.00071
0.02308	0.00509	0.00071	0.00014	0.00065	0.00609

$(x_3, x_4) = (2,0)$					
0.01140	0.00221	0.00013	0.00004	0.00013	0.00237
0.00227	0.00061	0.00005	0.00001	0.00005	0.00061
0.00016	0.00006	0.00001	0.00000	0.00001	0.00005
0.00005	0.00001	0.00000	0.00000	0.00000	0.00002
0.00015	0.00005	0.00001	0.00000	0.00001	0.00005
0.00261	0.00067	0.00005	0.00002	0.00005	0.00066

$(x_3, x_4) = (3,0)$					
0.00155	0.00045	0.00003	0.00001	0.00004	0.00043
0.00041	0.00014	0.00002	0.00000	0.00001	0.00013
0.00004	0.00002	0.00000	0.00000	0.00000	0.00001
0.00001	0.00000	0.00000	0.00000	0.00000	0.00000
0.00005	0.00001	0.00000	0.00000	0.00000	0.00002
0.00045	0.00014	0.00001	0.00000	0.00002	0.00014

Table 3. RMS of the kernel of the optimal interpolation operator $\mathcal{A}$ on blocks x for bosons in an $SU(2)$ gauge field on an 18^4 lattice; $\beta = 2.7$. A scale factor $L = 3$ is chosen. The kernel is centered at $y = 0$, i. e. the dimensionless quantity $(La)^d \sqrt{L^{-d} \sum_{z \in x} ||\mathcal{A}(z,0)||^2}$ is displayed, where $||\mathcal{A}(z,y)||^2 = \frac{1}{2}\mathrm{Tr}[\mathcal{A}(z,y)^\dagger \mathcal{A}(z,y)]$. The table shows the results in $x_1 - x_2$ planes — with x_1 (x_2) ranging from 0 to 5 from left (top) to right (bottom) — for fixed $x_3 = 0, \ldots, 3$ and $x_4 = 0$. Note that the kernel is not symmetric for nontrivial gauge field and that it falls off a little more rapidly than for $\beta = \infty$. This is an effect of the disorder in the gauge field.

KERNEL $\mathcal{A}$ FOR BOSONS ON AN 18^4 LATTICE, $L = 3$, $\beta = 2.7$

$(x_3, x_4) = (0,0)$					
1.01291	0.11805	0.01033	0.00151	0.01047	0.12180
0.12336	0.02390	0.00238	0.00042	0.00228	0.02386
0.01100	0.00224	0.00023	0.00005	0.00015	0.00229
0.00144	0.00033	0.00004	0.00001	0.00003	0.00036
0.01034	0.00221	0.00027	0.00004	0.00024	0.00200
0.11389	0.01910	0.00201	0.00034	0.00268	0.02348
$(x_3, x_4) = (1,0)$					
0.11554	0.02237	0.00229	0.00037	0.00211	0.01961
0.02388	0.00606	0.00065	0.00011	0.00050	0.00525
0.00238	0.00062	0.00010	0.00001	0.00005	0.00062
0.00034	0.00010	0.00001	0.00000	0.00001	0.00011
0.00207	0.00058	0.00008	0.00002	0.00006	0.00054
0.01898	0.00448	0.00057	0.00012	0.00059	0.00475
$(x_3, x_4) = (2,0)$					
0.01028	0.00225	0.00016	0.00004	0.00013	0.00196
0.00207	0.00051	0.00006	0.00001	0.00005	0.00050
0.00018	0.00007	0.00001	0.00000	0.00001	0.00005
0.00004	0.00001	0.00000	0.00000	0.00000	0.00001
0.00018	0.00005	0.00001	0.00000	0.00001	0.00005
0.00211	0.00054	0.00005	0.00001	0.00004	0.00054
$(x_3, x_4) = (3,0)$					
0.00135	0.00034	0.00004	0.00001	0.00003	0.00038
0.00032	0.00010	0.00001	0.00000	0.00001	0.00011
0.00003	0.00001	0.00000	0.00000	0.00000	0.00001
0.00001	0.00000	0.00000	0.00000	0.00000	0.00000
0.00006	0.00002	0.00000	0.00000	0.00000	0.00002
0.00041	0.00011	0.00001	0.00000	0.00002	0.00013

Table 4. RMS of the kernel of the optimal interpolation operator $\mathcal{A}$ on blocks x for staggered fermions in an $SU(2)$ gauge field on an 18^4 lattice; $\beta = \infty$. A scale factor $L = 3$ is chosen. The kernel is centered at $y = 0$. For $x \in \Lambda^H_{La/2}$ the dimensionless average over sites z in $[x]$ with the same pseudoflavor as x, viz. $(L\frac{a}{2})^d \sqrt{L^{-d} \sum_{z \in [x] \cap \Lambda^H_{a/2}} ||\mathcal{A}(z,0)||^2}$, is displayed where $||\mathcal{A}(z,y)||^2 = \frac{1}{2}\mathrm{Tr}[\mathcal{A}(z,y)^\dagger \mathcal{A}(z,y)]$. The table shows the results in $x_1 - x_2$ planes — with x_1 (x_2) ranging from 0 to 5 from left (top) to right (bottom) — for fixed $x_3 = 0, \ldots, 3$ and $x_4 = 0$. Note the symmetry of the kernel for pure gauges and that the kernel is nonzero only for arguments (z, x) with $x = (x_1, x_2, x_3, x_4)$ where all x_i are even. [These blocks carry the same pseudoflavor as $y = 0$.] On this even sublattice the kernel $\mathcal{A}$ is the same as the one for bosons on a 9^4 lattice in a pure gauge.

KERNEL $\mathcal{A}$ FOR FERMIONS ON AN 18^4 LATTICE, $L = 3$, $\beta = \infty$

$(x_3, x_4) = (0,0)$					
1.0131	0.0	0.1469	0.0	0.1469	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.1469	0.0	0.0326	0.0	0.0326	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.1469	0.0	0.0326	0.0	0.0326	0.0
0.0	0.0	0.0	0.0	0.0	0.0
$(x_3, x_4) = (1,0)$					
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
$(x_3, x_4) = (2,0)$					
0.1469	0.0	0.0326	0.0	0.0326	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0326	0.0	0.0092	0.0	0.0092	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0326	0.0	0.0092	0.0	0.0092	0.0
0.0	0.0	0.0	0.0	0.0	0.0
$(x_3, x_4) = (3,0)$					
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0

Table 5. RMS of the kernel of the optimal interpolation operator $\mathcal{A}$ on blocks x for staggered fermions in an $SU(2)$ gauge field on an 18^4 lattice: $\beta = 3.0$. A scale factor $L = 3$ is chosen. The kernel is centered at $y = 0$. For $x \in \Lambda_{L a}^{H}{}_{/2}$ the dimensionless average over sites z in $[x]$ with the same pseudoflavor as x, viz. $(L\frac{a}{2})^d \sqrt{L^{-d} \sum_{z \in [x] \subset \Lambda_{a}^{H}{}_{/2}} \|\mathcal{A}(z,0)\|^2}$, is displayed where $\|\mathcal{A}(z,y)\|^2 = \frac{1}{2}\mathrm{Tr}_{\dagger}\mathcal{A}(z,y)^{\dagger}\mathcal{A}(z,y)$. The table shows the results in $x_1 - x_2$ planes — with x_1 (x_2) ranging from 0 to 5 from left (top) to right (bottom) — for fixed $x_3 = 0, \ldots, 3$ and $x_4 = 0$. Note that the kernel is not symmetric for nontrivial gauge field and that it has support on every other block, due to the fact that sites of different pseudoflavor are now coupled through the $F_{\mu\nu}$ term of $-\nabla^2$.

KERNEL $\mathcal{A}$ FOR FERMIONS ON AN 18^4 LATTICE, $L = 3$, $\beta = 3.0$

$(x_3, x_4) = (0,0)$					
1.0251	0.0	0.1859	0.0	0.1580	0.0
0.0	0.1211	0.0	0.0233	0.0	0.1219
0.1605	0.0	0.0467	0.0	0.0363	0.0
0.0	0.0248	0.0	0.0060	0.0	0.0245
0.1722	0.0	0.0415	0.0	0.0410	0.0
0.0	0.1300	0.0	0.0221	0.0	0.1408
$(x_3, x_4) = (1,0)$					
0.0	0.1194	0.0	0.0269	0.0	0.1313
0.1222	0.0	0.0331	0.0	0.0284	0.0
0.0	0.0290	0.0	0.0087	0.0	0.0255
0.0264	0.0	0.0070	0.0	0.0075	0.0
0.0	0.0337	0.0	0.0077	0.0	0.0337
0.1320	0.0	0.0301	0.0	0.0354	0.0
$(x_3, x_4) = (2,0)$					
0.1621	0.0	0.0472	0.0	0.0337	0.0
0.0	0.0273	0.0	0.0078	0.0	0.0256
0.0382	0.0	0.0137	0.0	0.0097	0.0
0.0	0.0087	0.0	0.0023	0.0	0.0081
0.0434	0.0	0.0169	0.0	0.0120	0.0
0.0	0.0364	0.0	0.0081	0.0	0.0280
$(x_3, x_4) = (3,0)$					
0.0	0.0201	0.0	0.0059	0.0	0.0256
0.0231	0.0	0.0066	0.0	0.0073	0.0
0.0	0.0066	0.0	0.0024	0.0	0.0068
0.0060	0.0	0.0023	0.0	0.0023	0.0
0.0	0.0067	0.0	0.0028	0.0	0.0095
0.0246	0.0	0.0074	0.0	0.0073	0.0

Table 6. RMS of the kernel of the optimal interpolation operator $\mathcal{A}$ on blocks x for staggered fermions in an $SU(2)$ gauge field on an 18^4 lattice: $\beta = 2.7$. A scale factor $L = 3$ is chosen. The kernel is centered at $y = 0$. For $x \in \Lambda_{L a}^{H}{}_{/2}$ the dimensionless average over sites z in $[x]$ with the same pseudoflavor as x, viz. $(L\frac{a}{2})^d \sqrt{L^{-d} \sum_{z \in [x] \subset \Lambda_{a}^{H}{}_{/2}} \|\mathcal{A}(z,0)\|^2}$, is displayed where $\|\mathcal{A}(z,y)\|^2 = \frac{1}{2}\mathrm{Tr}_{\dagger}\mathcal{A}(z,y)^{\dagger}\mathcal{A}(z,y)$. The table shows the results in $x_1 - x_2$ planes — with x_1 (x_2) ranging from 0 to 5 from left (top) to right (bottom) — for fixed $x_3 = 0, \ldots, 3$ and $x_4 = 0$. Note that the kernel is not symmetric for nontrivial gauge field and that it has support on every other block, due to the fact that sites of different pseudoflavor are now coupled through the $F_{\mu\nu}$ term of $-\nabla^2$.

KERNEL $\mathcal{A}$ FOR FERMIONS ON AN 18^4 LATTICE, $L = 3$, $\beta = 2.7$

$(x_3, x_4) = (0,0)$					
1.0358	0.0	0.1771	0.0	0.1352	0.0
0.0	0.1797	0.0	0.0290	0.0	0.1606
0.1665	0.0	0.0450	0.0	0.0378	0.0
0.0	0.0296	0.0	0.0071	0.0	0.0354
0.1892	0.0	0.0487	0.0	0.0382	0.0
0.0	0.1848	0.0	0.0304	0.0	0.1391
$(x_3, x_4) = (1,0)$					
0.0	0.1568	0.0	0.0250	0.0	0.1528
0.1825	0.0	0.0414	0.0	0.0309	0.0
0.0	0.0398	0.0	0.0107	0.0	0.0435
0.0311	0.0	0.0109	0.0	0.0083	0.0
0.0	0.0446	0.0	0.0079	0.0	0.0402
0.1683	0.0	0.0451	0.0	0.0368	0.0
$(x_3, x_4) = (2,0)$					
0.1719	0.0	0.0500	0.0	0.0317	0.0
0.0	0.0452	0.0	0.0092	0.0	0.0430
0.0473	0.0	0.0151	0.0	0.0110	0.0
0.0	0.0125	0.0	0.0031	0.0	0.0103
0.0486	0.0	0.0154	0.0	0.0128	0.0
0.0	0.0544	0.0	0.0110	0.0	0.0348
$(x_3, x_4) = (3,0)$					
0.0	0.0261	0.0	0.0052	0.0	0.0228
0.0261	0.0	0.0092	0.0	0.0073	0.0
0.0	0.0094	0.0	0.0027	0.0	0.0093
0.0070	0.0	0.0039	0.0	0.0027	0.0
0.0	0.0098	0.0	0.0030	0.0	0.0082
0.0225	0.0	0.0082	0.0	0.0075	0.0

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References

1. H. Joos and M. Schaefer, *The representation theory of the symmetry group of lattice fermions as a basis for kinematics in lattice QCD*, Z. Phys. **C34** (1987) 465

2. M.F.L. Golterman, *Irreducible representations of the staggered fermion symmetry group*, Nucl. Phys. **B278** (1986) 417

3. R. Ben-Av, A. Brandt, and S. Solomon, *The fermionic matrix, instantons, zero modes and multigrid*, Nucl. Phys. **B329** (1990) 193;

   R. Ben-Av, A. Brandt, M. Harmatz, E. Katznelson, P.G. Lauwers, S. Solomon, and K. Wolowesky, *Fermion Simulations Using Parallel Transported Multigrid*, Phys. Lett. **B253** (1991) 185;

   R. Ben-Av, A. Brandt, M. Harmatz, E. Katznelson, P.G. Lauwers, S. Solomon, and K. Wolowesky, *Parallel transported multigrid and its applications to the Schwinger model*, Proceedings LATTICE '90, Nucl. Phys. B (Proc. Suppl.) (to appear)

4. K. Gawędzki and A. Kupiainen, *A Rigorous Block Spin Approach to Massless Lattice Theories*, Commun. Math. Phys. **77** (1980) 31;

   K. Gawędzki and A. Kupiainen, *Massless Lattice ϕ^4_4 Theory: Rigorous Control of a Renormalizable Asymptotically Free Model*, Commun. Math. Phys. **99** (1985) 197

5. G. Mai, *Ein Blockspin für $2^{d/2}$ Fermionen*, diploma thesis, Hamburg, October 1989

6. G. Mack, *Dielectric lattice gauge theory*, Nucl. Phys. **B235** [FS11] (1984) 197

7. T. Kalkreuter, *Projective block spin transformations in lattice gauge theories*, DESY preprint 90-158, December 1990

8. A. Hulsebos, J. Smit and J.C. Vink, *Multigrid Monte Carlo in an external $U(1)$ gauge field*, Nucl. Phys. B (Proc.Suppl.) **9** (1989) 512;

   A. Hulsebos, J. Smit and J.C. Vink, *Multigrid Monte Carlo for a Bose field in an external gauge field*, Nucl. Phys. **B331** (1990) 531

9. R.C. Brower, C. Rebbi and E. Vicari, *Projective multigrid method for propagators in lattice gauge theory*, Phys. Rev. **D43** (1991) 1965

10. R. Edwards, D. Ritchie and D. Zwanziger, *Reduced Dirac Kähler lattice fermion action*, Nucl. Phys. **B296** (1988) 961

11. A. Brandt, "*Guide to multigrid development*", Proceedings of the 1st European MG conference, November 1981;

    W. Hackbusch, *Multi-Grid Methods and Applications*, Springer Series in Computational Mathematics 4, (Springer-Verlag, Berlin, 1985)

12. C.B. Chalmers, R.D. Kenway and D. Roweth, *Algorithms for calculating quark propagators on large lattices*, J. Comp. Phys. **70** (1987) 500

13. R.C. Brower, R.G. Edwards, C. Rebbi and E. Vicari, *Projective multigrid for Wilson fermions*, FSU-SCRI-91-54, April 1991

14. A. Hulsebos, J. Smit and J.C. Vink, *Multigrid inversion of the staggered fermion matrix*, ITFA-90-28, December 1990, Proceedings LATTICE '90, Nucl. Phys. B (Proc. Suppl.) (to appear)

15. V. Vyas, *A multigrid algorithm for calculating fermionic propagators using Migdal-Kadanoff transformation*, Wuppertal preprint WUB 91-10, February 1991

16. G. Mack, *Multigrid methods in quantum field theory*, Cargèse Lectures July 1987, *in* G. 't Hooft et al., Nonperturbative quantum field theory (Plenum Press, New York, 1989)

17. G. Mack and S. Meyer, *The effective action from multigrid Monte Carlo*, Nucl. Phys. B (Proc.Suppl.) **17** (1990) 293

18. T. Kalkreuter, *in preparation*

19. T. Balaban, *Regularity and decay of lattice Greens functions*, Comm. Math. Phys. **89** (1983) 571