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H. Salehi

*II. Institut für Theoretische Physik, Universität Hamburg*

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# On the Hawking Radiation Associated with an Oppenheimer–Snyder Collapsing Star

Hadi Salehi

II. Institute for Theoretical Physics, University of Hamburg, D-2000 Hamburg 50, West Germany

**Abstract**

We discuss on the basis of the geometrical optics approximation the development of the Hawking-radiation in the space-time of a black hole resulting from an Oppenheimer-Snyder collapsing star. To the extent that we can use this approximation it is shown that the asymptotic flow of radiation to future null infinity does not depend on the specific parameter describing the initial condition for the stellar collapse (in our model the initial radius) . Further a unique definition of the radial part of the radiation is given via the concept of normal ordering of the energy momentum tensor operator with respect to the time-like Killing vector field of the Schwarzschild-metric. The remaining ambiguity is related to the global features of states.



## 1 Introduction

It is still a debatable issue whether we can understand the thermodynamic parameters of black holes on the basis of the principles of quantum physics. This idea was put forward by Hawking. In his basic work [1] he opened the extremely attractive possibility of understanding the thermodynamic aspects of black holes as a manifestation of the instability of these objects at the quantum level. That is, by considering the quantum field theory in the space-time of a collapsing star he argued that after the formation of the black hole, this acts as a source of thermal radiation at a temperature that corresponds precisely to the assumed form of the information theoretic Beckenstein-entropy [2] ,viz.

$$T = \frac{c^3\hbar}{8\pi MG} \tag{1}$$

where $M$ denotes the mass of the black hole and G is the gravitational constant.
Subsequent papers were concentrated on the consideration of eternal static black holes. The early attempts along this line were based on the suggestion that the effect is more a consequence of the causal and topological structure of the space-time than a specific geometry inside the collapsing star. It was argued in [3] that the construction of an appropriate "vaccum state" on the maximally extended space-time of a spherical symmetric eternal static black hole (Kruskal space-time) will reproduce the effect of the gravitational collapse. In a different approach [4] attention was focused on the analogy of the axiomatic theory of quantum fields on an eternal static black hole background with that of fields in Minkowski space-time, the outside region of the black hole corresponding to the Rindler wedge and its Kruskal extension to the Minkowski space-time. Of central importance in this approach was the general theorem of Bisognano and Wichmann [5] which initially was stated without any motivations from black hole physics. Due to this theorem the vaccum state of the Minkowski space-time restricted to the observations in the Rindler wedge is a thermal state with respect to the Lorentz boosts and these boosts correspond to the time-translations for a uniformly accelerated observer in the wedge. The temperature agrees with (1) if the acceleration is replaced by the acceleration at the surface of the black hole. Unfortunately, there are however some differences from the Minkowski space-time theory. The most important one is the lack of a global time-like Killing vector field, whose ground state defines a global vaccum. Therefore, in the next stage a strict local point of view on the analysis of the Hawking radiation emerged. It was shown in [6] that the analysis of the local aspects of the states in the immediate vicinity of the horizon already singles out the value of the Hawking temperature for the theory. The subsequent stage of the development was connected with arguments based on global considerations. It was observed in [7] that the thermal properties of the states for the outside region of the black hole arise as a consequence of the more general concept of stationarity.
All of these results originate, of course, from the need of a deeper understanding of the Hawking radiation and it is certainly suggestive to base the analysis on the concept of an eternal static black hole. But, such a basis does not comprise the most important feature of the radiation, namely its space-time development due to the gravitational collapse. In a recent work a rigorous derivation of the Hawking radiation in the nonstatic case, i.e. in the gravitational field of a collapsing star, could be given by Fredenhagen and Haag [8]. They related for a linear quantum field living in the classical space-time of a spherical symmetric collapsing star

the asymptotic black hole radiation at large times (Hawking radiation) to the short distance behaviour of the states of the quantum field on the horizon. In their derivation, they used the concept of a point-like detector whose word line follows locally the time-like Killing-vector field of the Schwarzschild-metric and it was established that the asymptotic counting rate at large times of such a detector placed at large distance from the black hole corresponds to a steady flow of radiation at the above temperature. Although this result is rigorous it is not free from idealizations, e. g. the spatial extension and the internal structure of the detector is completely ignored. From another point of view an unsatisfactory aspect arises if we think of the detector employed as registering the deviation of the local states from a "ground state" of the Schwarzschild-metric. The point is that, locally, there are many stationary ground states, each of which corresponding to a different static extension of the Schwarzschild-metric beyond the region in which the detector is placed. Thus, concerning the global features of the underlying ground state an essential ambiguity arises which is closely related to the failure of a quantum version of the classical Birkhoff-theorem [9].

In the present work we use a different approach to the subject. We give a description of the asymptotic black hole radiation in terms of the asymptotic flow of information to future null infinity which is assumed to be controlled by the asymptotic form (at large times and at large distance from the black hole) of the retarded part of that particular state of the quantum field that coincides at early times (prior to the beginning of the collapse) with the stationary ground state of the global static metric. We will then establish that the mentioned ambiguity does not affect the radial part of the asymptotic radiation. At the same time we obtain direct information about the space-time development of the radiation within the geometrical optics approximation and relate the asymptotic radiation to the ultraviolet behaviour of the stationary ground state at initial time. Further it will be confirmed explicitly that the asymptotic flow of information to the future null infinity does not depend on the specific form of the initial condition associated with the stellar collapse.

Our arguments will be based in this paper on the explicit model for the spherical symmetric gravitational collapse of an incoherent star (dust cloud) of uniform energy density; i.e. the Oppenheimer-Snyder model [10]. The organization is as follows: Chapter 2 brings a discussion of our geometrical model. In chapter 3 we establish our quantum field theoretical notation by restricting to the case of a linear scalar quantum field. The linear dynamical laws are used to pass from the four dimensional smeared out field to a symplectic smearing of the covariant field on a given three dimensional Cauchy-surface which constitutes the basis of our calculation. Further the concept of the symplectic smeared out field will be used to replace the singular quantum dynamics by the problem of Cauchy-development of classical test-functions backwards in time. Chapter 4 brings a discussion of the covariant massles scalar field equation in the Oppenheimer-Snyder model and the associated geometrical optics Cauchy-development of the classical test-function backwards in time. In chapter 5 we give the closed form of the retarded part of the state in the geometrical optics approximation and derive its asymptotic form at large times. Finally we turn our attention to the radial part of the radiation and establish its universality in the sense above mentioned.



## 2 Geometrical Preliminaries

In this chapter we give a set of charts covering the space-time of a massive star which undergoes a spherical symmetric gravitational collapse to form a black hole. Outside the star we use as our first set of coordinates the Schwarzschild-coordinates. Due to the Birkhoff-theorem the metric will be the corresponding part of the asymptotically flat Schwarzschild-metric cut off by the surface of the star, that is, ( we shall use units in which $c = \hbar = G = 1$)

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{2}$$

Notice that if the radius $r = 2M$ (the Schwarzschild-radius) lies outside the star the Schwarzschild-metric will then hold only down to this radius.

The end of the quasi-static life-phase of the star, that is the beginning of the collapse, is denoted in the following by the Cauchy-surface $\Sigma_0$. We assume that the segment of $\Sigma_0$ lying outside the star corresponds to the value $t = 0$ of the Schwarzschild-time. The specific form of the metric in the past of $\Sigma_0$ inside the star does not concern us in this paper. In order to find the metric inside the star in the future of $\Sigma_0$ we have to make some assumption about the energy-momentum tensor and the state-equation of the stellar matter. We use a perfect-fluid description of the stellar matter and write the energy-momentum tensor in the form :

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{3}$$

where $U_\mu$ is the local value of the fluid four-velocity and $\rho$ (resp. $p$) is the local energy dencity (resp. pressure). Furthermore, we expect that the qualitative properties of a stellar collapse do not depend on the state-equation of the stellar matter. Therefore we shall take in this paper the simplest possible choice of the state-equation, namely $p = 0$ , describing incoherent stellar matter or simply dust cloud. To give the metric inside the star we make use of the Lagrangian (comoving) -coordinates $(\tau, R, \vartheta, \varphi)$ . The angular part of these coordinates is standard. $\tau$ is the proper time of the free falling dust particles whose geodesic trajectories are distinguished by the space-like coordinate $R$. We shall refer to $R$ as the Lagrangian-radius. We assume that the segment of $\Sigma_0$ lying inside the star corresponds to the value $\tau = 0$. In these coordinates the only non vanishing component of $T_{\mu\nu}$ is $T_{00} = \rho$ and the interior solution of the Einstein-equations takes the form [10]-[12]

$$ds^2 = -d\tau^2 + g_{11}(\tau, R)dR^2 + r^2(\tau, R)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{4}$$

$$g_{11}(\tau, R) = (\frac{\partial r}{\partial R})^2(1 + f(R))^{-1} \tag{5}$$

$$(\frac{\partial r}{\partial \tau})^2 = F(R)r^{-1} + f(R) \tag{6}$$

$$F'(R) = \frac{\partial r}{\partial R}\, r^2 8\pi\rho \tag{7}$$

where the functions $f(R)$ and $F(R)$ depend on the distribution and radial velocity of the stellar matter on the initial surface $\Sigma_0$ . The solution given by (4)-(7) is known as the general Tolman-solution. We note that the property of the Lagrangian-radius $R$ to be a space-like coordinate imposes the following positivity-condition on $f(R)$ :

$$1 + f(R) > 0. \tag{8}$$

We may also use the Lagrangian-coordinates in the outside region of the star as our second set of coordinates.
Let us now consider the general Tolman-solution in more specific terms. First, for simplicity we confine ourselves to a situation in which the radial velocity of the stellar matter on the initial surface $\Sigma_0$ vanishes. In this case it follows immediately from (6):

$$r(0,R)f(R) + F(R) = 0. \tag{9}$$

Without loss of generality we assume that the Lagrangian-radius $R$ coincides with the Schwarzschild-radius on the initial surface [1]. The equation (9) takes then the form:

$$F(R) = -Rf(R) \tag{10}$$

Now we relate the unknown function $F(R)$ to the initial-distribution of the stellar matter by projecting the equation (7) on the initial surface $\Sigma_0$. We obtain

$$F(R) = 8\pi \int_0^R dR' \, \rho(0,R')R'^2 = 2m(R) \tag{11}$$

where $m(R)$ denotes the mass within the sphere of radius $r = R$ . By (10) and (11) the functions $f(R)$ and $F(R)$ are determined. To find the unknown function $r(\tau, R)$ needed in (4) to fix the solution we extend the Lagrangian-coordinates beyond the star to the exterior region. The function $r(\tau,R)$ arises then in the future of $\Sigma_0$ in the whole space-time as the solution of the partial differential equation (6) together with the initial conditions:

$$r(0,R) = R \; , \; (\partial r(0,R)/\partial\tau) = 0. \tag{12}$$

The solution of this problem is given in the following parametric form:

$$r = \frac{R}{2}(1 + cos\ \eta) \; , \; 0 \leq \eta \leq \pi \tag{13}$$

$$\tau = \frac{R^{3/2}}{2\sqrt{F(R)}}(\eta + sin\ \eta) \; , \; 0 \leq \eta \leq \pi. \tag{14}$$

These equations are the analytical implementation of our qualitative knowledge about the stellar collapse. In fact it follows from (13) and (14) that the parameter-value $\eta = \pi$ corresponds to a space-time singularity and after the finite proper time $\pi R^{3/2}/2\sqrt{F(R)}$ each dust particle corresponding to the Lagrangian-radius $R$ reaches this singularity. In the interest of simplicity we confine further the analysis to initial matter distribution of uniform energy density and refer to the corresponding solution as the Oppenheimer-Snyder model [10]. In this model it follows from (10) and (11)

$$F(R) = \frac{8\pi}{3} R^3 \rho = 2M \frac{R^3}{a^3} \tag{15}$$

$$f(R) = -\frac{2M}{a}\frac{R^2}{a^2} \tag{16}$$

[1] If this is not the case we define $\bar{R} = r(0,R)$ as a new Lagrangian-radius.



where $a$ is the Lagrangian-radius corresponding to the surface of the star. Consequently, the equations (13) and (14) take in the interior region the form:

$$r = R(1 + cos\ \eta)/2 \tag{17}$$

$$\tau = \sqrt{\frac{a^3}{2M}}(\eta + sin\ \eta)/2. \tag{18}$$

We note here that the function $f(R)$ is monoton and under the typical astronomical conditions ($\frac{2M}{a} \approx 10^{-6}$) may be neglected together with all of its derivities in the interior region in comparison with one.
Finally we shall need the explicit form of the metric outside the star in the Lagrangian-coordinates. This may be found by the requirement of continuity of the function $F(R)$ on the surface of the star, leading to

$$F(R) = 2M \; , for\ R \geq a. \tag{19}$$

In contrast to the Schwarzschild-coordinates the Lagrangian-coordinates can be used to chart the black hole, i. e. the nonstatic region of the space-time below the Schwarzschild-radius . The transition between the both sets of charts in some overlap region is given by:

$$r = R(1 + cos\ \eta)/2 \tag{20}$$

$$\tau = \frac{R^{3/2}}{\sqrt{2M}}(\eta + sin\ \eta)/2 \tag{21}$$

$$t = 2M \ln\{\frac{\sqrt{\frac{R}{2M} - 1} + tan\ \eta}{\sqrt{\frac{R}{2M} - 1} - tan\ \eta}\} + R\sqrt{\frac{R}{2M} - 1}\ \frac{\eta + sin\ \eta}{2} + 2M\sqrt{\frac{R}{2M} - 1}\ \eta. \tag{22}$$

The first two equations may be obtained by replacing $F(R)$ in (13) and (14) by its expression $F(R) = 2M$ in the outside region of the star. The last equation is due to the property of the curves $R = const.$ to be geodesic curves in the Schwarzschild-metric.

## 3 Setting of the Quantum Field Theoretical Notation

Let $\phi$ be a quantum field living in the space-time considered in the preceding chapter, for simplicity, assumed to be scalar, neutral, massless and minimally coupled to gravity

$$\phi(\Box_g h) = 0, \; \Box_g = |g|^{-1/2}\partial_\mu g^{\mu\nu}|g|^{1/2}\partial_\nu, \; |g| = -det\ g_{\mu\nu} \tag{23}$$

where $h(x)$ is a four dimensional space-time test-function. A state is characterized by a hierarchy of expectation values

$$W^{(n)}(\phi(h_1)....\phi(h_n)) = <\phi(h_1)...\phi(h_n)> . \tag{24}$$

In the heuristic notation the field may be written as the four dimensional smeared out field

$$\phi(h) = \int d\mu(x)\phi(x)h(x) \; , d\mu(x) = |g|^{1/2}d^4x \tag{25}$$

Correspondingly we may write

$$<\phi(h_1)...\phi(h_n)>=\int d\mu(x_1)...d\mu(x_n)W^{(n)}(x_1...x_n)h_1(x_1)...h_n(x_n). \quad (26)$$

The function $W^{(n)}(x_1...x_n)$ is the n-point function of the state.
Let $\Sigma_H$ be the Cauchy-surface $\tau = \tau_0$ where $\tau_0$ indicates in the Lagrangian-chart the moment when the star radius crosses the Schwarzschild-radius and henceforth let $h(x)$ be a test-function having its support in the future of $\Sigma_H$ in the Schwarzschild-region far away from the black hole. To represent $\phi(h)$, it will be convenient for the purpose of this paper to pass to the concept of the three dimensional symplectic smeared out field. This may be done in the following way[2]: For each $t_0 \in [a,b]$ where

$$a = inf\{t : (t,r,\vartheta,\varphi) \in supp\ h\}, b = sup\{t : (t,r,\vartheta,\varphi) \in supp\ h\} \quad (27)$$

$h(x)$ induces a unique solution of the following Cauchy-problem

$$\Box_g f^{t_0}(x) = 0\ ,\ \partial_t f^{t_0}(t_0,r,\vartheta,\varphi) = h(t_0,r,\vartheta,\varphi), f^{t_0}(t_0,r,\vartheta,\varphi) = 0\ . \quad (28)$$

Using the one parameter family $\{f^{t_0}\}$ of classical solutions of (28) the four dimensional smeared out field may be written as

$$\phi(h) = \int dt_0 \int_{\Sigma(t=t_0)} d\Sigma^\mu \phi(x)\ \overleftrightarrow{\partial}_\mu\ f^{t_0}(x) \quad (29)$$

where $d\Sigma^\mu$ denotes the surface element. The essential point is now that by taking into account the dynamical equation (23) the right hand side of (29) becomes independent of the specific choice of the Cauchy-surface and hence we can write [3]

$$\phi(h) = \int d\Sigma^\mu \phi(x)\ \overleftrightarrow{\partial}_\mu\ f(x) \quad (30)$$

with

$$f(x) = \int dt^0 f^{t_0}(x). \quad (31)$$

Intuitively, the equation (30) expresses $\phi(h)$ in terms of the Cauchy-data of $\phi$ on an arbitrary chosen Cauchy surface. We shall refer to $f(x)$ as the symplectic test function associated with $h(x)$.
Inserting the representation (30) for each $\phi(h_i), i = 1...n$ into (24) and choosing the Cauchy-surface $\Sigma$ as $\Sigma_0$, we obtain the symplectic representation of a state in the future of $\Sigma_H$ at large distance from the black hole in the form

$$<\phi(h_1),.,\phi(h_n)>=\int_{\Sigma_0} d\Sigma^{\mu_1}..d\Sigma^{\mu_n}W^{(n)}(x_1,.,x_n)\ \overleftrightarrow{\partial}_{\mu_1}..\overleftrightarrow{\partial}_{\mu_n}\ f_1(x_1)..f_n(x_n). \quad (32)$$

We are concerned in this paper only with quasi-free states, i.e. that subclass of the states whose truncated n-point functions vanish for $n \neq 2$. Such a state may be characterized by its two-point function. Further we are interested in that particular quasi-free state that coincides

[2] The generalization to the case when we have not specified the space-time support of $h(x)$ is straightforward but not needed in this paper.
[3] The situation is analogous to that of the work [8] where the construction of $f(x)$ from $h(x)$ was firstly considered. The reader may also find further explanations in that work.



in the past of $\Sigma_0$ with the ground state of the static metric. Applying (32) we obtain the symplectic representation of this state in the future of $\Sigma_H$ at large distance from the black hole

$$<\phi(h_1),\phi(h_2)>=\int_{\Sigma_0} d\Sigma^{\mu_1}d\Sigma^{\mu_2}W_G^{(2)}(x_1,x_2)\ \overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\ f_1(x_1)f_2(x_2) \quad (33)$$

where $W_G^{(2)}(x_1,x_2)$ is the two-point function of the ground state of the static metric. For practical reasons we shall write this equation in the form

$$<\phi(\overline{h}_1),\phi(h_2)>=\int_{\Sigma_0} d\Sigma^{\mu_1}d\Sigma^{\mu_2}W_G^{(2)}(x_1,x_2)\ \overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\ \overline{f}_1(x_1)f_2(x_2). \quad (34)$$

Finally we give the characterization of the retarded part of the state given in the symplectic notation (34) by imposing the following radiation condition on the functions $f_i(x_i), i = 1,2$ in the future of $\Sigma_H$ at large distance from the black hole (we suppress the index i)

$$n^\mu \partial_\mu f = 0 \quad (35)$$

where $n^\mu$ is a tangent vector along the outgoing radial null rays of the Schwarzschild-metric. This condition means that $f(x)$ propagates by increasing the time towards the future null infinity. It must be noted that the compatibility of the condition (35) with the dynamical laws (28) is a consequence of the property of the space-time to be asymptotically flat.

## 4 Geometrical Optics Form of $f_i(x_i)$ , $i = 1,2$ in the past of $\Sigma_H$

Let us now consider the explicit construction of the symplectic test-functions $f_i(x_i), i = 1,2$ in the past of $\Sigma_H$ needed in (34) to represent the state in the future of $\Sigma_H$ at large distance from the black hole. The function $f(x)$ (we suppress the index $i$ ) is given by (31) and (28). Due to the spherical symmetry the angular part may be separated by expanding[4]

$$f(x) = \sum_{l,m} \Omega(x^0,x^1)\overline{f}_{l,m}(x^0,x^1)Y_{l,m}(\vartheta,\varphi)\ ,\ x = (x^0,x^1,\vartheta,\varphi) \quad (36)$$

where $\Omega(x_0,x_1)$ is some conformal factor which will be specified later. Correspondingly we expand $h(x)$

$$h(x) = \sum_{l,m} \Omega(x^0,x^1)\overline{h}_{l,m}(x^0,x^1)Y_{l,m}(\vartheta,\varphi). \quad (37)$$

In the Lagrangian-chart ($x^0 = \tau, x^1 = R$) the function $f_{l,m}$ has to satisfy the following equation in the gauge $\Omega(\tau,R) = 1$

$$[\sqrt{g_{11}^{-1}}\frac{\partial}{\partial\tau}(r^2\sqrt{g_{11}}\frac{\partial}{\partial\tau}) - \sqrt{g_{11}^{-1}}\frac{\partial}{\partial R}(r^2\sqrt{g_{11}^{-1}}\frac{\partial}{\partial R}) + l(l+1)]f_{l,m} = 0 \quad (38)$$

or more explicitly

$$\frac{\partial^2 f_{l,m}}{\partial\tau^2} - g_{11}^{-1}\frac{\partial^2 f_{l,m}}{\partial R^2} + \frac{\partial}{\partial\tau}(\ln(r^2\sqrt{g_{11}})\frac{\partial f_{l,m}}{\partial\tau} - g_{11}^{-1}\frac{\partial}{\partial R}(\ln(r^2\sqrt{g_{11}^{-1}})\frac{\partial f_{l,m}}{\partial R} + \frac{l(l+1)}{r^2}f_{l,m} = 0. \quad (39)$$

[4] The presence of $f_{l,m}(x^0,x^1)$ in the complex conjugated form is a matter of convenience.

In consequence of the assumed uniformity of the energy density inside the star the coefficients of this equation fail to be continuous on the surface of the star, denoted in the following by $\Gamma$. To remove this difficulty we impose the following transition conditions on $f_{l,m}$ along $\Gamma$

$$\{f_{l,m}\}^{\Gamma-0} = \{f_{l,m}\}^{\Gamma+0} \quad (40)$$

$$\{\frac{1}{\sqrt{g_{11}}}\frac{\partial f_{l,m}}{\partial R}\}^{\Gamma-0} = \{\frac{1}{\sqrt{g_{11}}}\frac{\partial f_{l,m}}{\partial R}\}^{\Gamma+0}. \quad (41)$$

The first condition is simply the requirement of continuity. Taking this into account the second condition may be derived by integrating both sides of the equation (38) on an interval $[a-\epsilon, a+\epsilon]$ and going to the limit $\epsilon \to 0$. Henceforth we neglect the small function $f(R)$ in the dominator of $g_{11}$ in (5). With this assumption it will be convenient to perform our calculations in the gauge $\Omega(\tau, R) = r^{-1}(\tau, R)$. In the region outside the star the function $f_{l,m}$ will then be a solution of the radial wave equation of the Schwarzschild-metric

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \overset{*}{r}^2} + v_l)f_{l,m} = 0, \quad (42)$$

$$v_l = (1 - \frac{2M}{r})(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}), \quad \overset{*}{r} = r + 2M\ln|\frac{r}{2M} - 1|. \quad (43)$$

Using the continuity of the function $r^{-1}(\tau, R)$ one can show that the transition conditions (40),(41) remain the same in the new gauge. Further the equation (39) in the region inside the star takes the simple form ($\alpha = \sqrt{\frac{2M}{a^3}}$)

$$\alpha^2 \frac{\partial^2 f_{l,m}}{\partial \eta^2} - \frac{\partial^2 f_{l,m}}{\partial R^2} + V_l(\eta, R) f_{l,m} = 0, \quad (44)$$

$$V_l(\eta, R) = \frac{l(l+1)}{R^2} + \alpha^2 \frac{\cos \eta}{1 + \cos \eta} \quad (45)$$

where the parameter $\eta$ is defined by (13) and (14).
Let $\Sigma_H(a, \infty)$ (resp. $\Sigma_H(0, a)$) denote the segment of $\Sigma_H$ lying outside (resp. inside) the star ($a$ corresponds to the Lagrangian-radius of the surface of the star). According to the fact that the space-time support of $f_{l,m}$ lies entirely outside the black hole the function $f_{l,m}$ is uniquely determined by a set of Cauchy-data on $\Sigma_H(a, \infty)$. In the geometrical optics approximation, that is by neglecting the potential terms in (42) and (44), the retarded part of the state characterized by the radiation condition (35) corresponds to an outgoing set of these Cauchy-data. This means that in the future of $\Sigma_H$ the function $f_{l,m}$ is only a function of the retarded time $u = t - \overset{*}{r}$; viz. $f_{l,m}(t, r) = f^0_{l,m}(u)$, where the function $f^0_{l,m}(u)$ is uniquely determined by $h_{l,m}$ in (37) and will not be specified at this moment. To determine the function $f_{l,m}$ in the past of $\Sigma_H$ we proceed as follows:
Let us imagine the interior solution, i. e. the form of $f_{l,m}$ in the interior region of the star, to be known. For the actual construction of $f_{l,m}$ in the past of $\Sigma_H$ in the exterior region of the star we need a boundary condition on $\Gamma$. For this we shall take

$$f_{l,m} = \{f_{l,m}\}^{\Gamma-0}, \quad on\ \Gamma. \quad (46)$$

With this choice the transition condition (40) is automatically satisfied. It should be noted that the boundary values $\{f_{l,m}\}^{\Gamma-0}$ depend on their side on the Cauchy-data on $\Sigma_H$ and these



are conletely determined by the function $f^0_{l,m}$. Now the linearity may be used to decompose the function $f_{l,m}$ in the past of $\Sigma_H$ in the exterior region into the two parts

$$f_{l,m} = f^{\Sigma}_{l,m} + f^{\Gamma}_{l,m} \quad (47)$$

where $f^{\Sigma}_{l,m}$ satisfies the Cauchy-data on $\Sigma_H$ (as prescribed by the function $f^0_{l,m}$) and vanishes on boundary $\Gamma$, $f^{\Gamma}_{l,m}$ takes zero Cauchy-data on $\Sigma_H$ and satisfies the boundary condition on $\Gamma$ (as prescribed by (46)). The function $f^{\Sigma}_{l,m}$ is given by

$$f^{\Sigma}_{l,m}(t, r) = f^0_{l,m}(u) - f^0_{l,m}(u_\Gamma(v)) \quad (48)$$

where $u_\Gamma(v)$ gives the retarded time as a function of the advanced time $v = t + \overset{*}{r}$ along the boundary $\Gamma$. The function $f^{\Gamma}_{l,m}$ must be only a function of the advanced time $v$ as it takes zero Cauchy-data on $\Sigma_H(a, \infty)$ and is related to the interior solution by

$$f^{\Gamma}_{l,m}(v) = f_{l,m}(a - 0, \tau_\Gamma(v)) \quad (49)$$

where $\tau_\Gamma(v)$ gives the proper time $\tau$ as a function of $v$ along $\Gamma$.
These observations show that for the actual construction of $f_{l,m}$ in the past of $\Sigma_H$ in the exterior region of the star one has only to express the interior solution in terms of the function $f^0_{l,m}$. Concerning this problem we note that the interior solution is uniquely determined by a set of zero Cauchy-data on $\Sigma_H(0, a)$, the regularity condition $f_{l,m}(0, \tau) = 0$ at $R = 0$ and a boundary condition on $\Gamma$. This unknown boundary condition may be determined by making use of the transition condition (41) in the following manner: Inserting (47) into the right hand side of (41) one obtains

$$\{\frac{\partial f_{l,m}}{\partial R}\}^{\Gamma-0} - \frac{\{\sqrt{g_{11}}\}^{\Gamma-0}}{\{\sqrt{g_{11}}\}^{\Gamma+0}}\{\frac{\partial f^{\Gamma}_{l,m}}{\partial R}\}^{\Gamma+0} = \frac{\{\sqrt{g_{11}}\}^{\Gamma-0}}{\{\sqrt{g_{11}}\}^{\Gamma+0}}\{\frac{\partial f^{\Sigma}_{l,m}}{\partial R}\}^{\Gamma+0}. \quad (50)$$

From (48) and (49) we have

$$\{\frac{\partial f^{\Sigma}_{l,m}}{\partial R}\}^{\Gamma+0} = \{f^{0'}_{l,m}(u)\}^{\Gamma+0}\{\frac{\partial u}{\partial R} - \frac{\partial u}{\partial \tau}(\frac{\partial v}{\partial \tau})^{-1}\frac{\partial v}{\partial R}\}^{\Gamma+0} \quad (51)$$

$$\{\frac{\partial f^{\Gamma}_{l,m}}{\partial R}\}^{\Gamma+0} = \{\frac{\partial f_{l,m}}{\partial \tau}\}^{\Gamma-0}\{(\frac{\partial v}{\partial \tau})^{-1}\frac{\partial v}{\partial R}\}^{\Gamma+0}. \quad (52)$$

This can be inserted into (50) to yield

$$\{\frac{\partial f_{l,m}}{\partial R}\}^{\Gamma-0} - \frac{\{\sqrt{g_{11}}\}^{\Gamma-0}}{\{\sqrt{g_{11}}\}^{\Gamma+0}}\{(\frac{\partial v}{\partial \tau})^{-1}\frac{\partial v}{\partial R}\}^{\Gamma+0}\{\frac{\partial f_{l,m}}{\partial \tau}\}^{\Gamma-0} = Q\{f^0_{l,m}\} \quad (53)$$

with

$$Q\{f^0_{l,m}\} = \frac{\{\sqrt{g_{11}}\}^{\Gamma-0}}{\{\sqrt{g_{11}}\}^{\Gamma+0}}\{f^{0'}_{l,m}(u)\}^{\Gamma+0}\{\frac{\partial u}{\partial R} - \frac{\partial u}{\partial \tau}(\frac{\partial v}{\partial \tau})^{-1}\frac{\partial v}{\partial R}\}^{\Gamma+0}. \quad (54)$$

This is the needed inhomogeneous boundary condition on $\Gamma$ to fix the interior solution. To simplify (53) we make use of the property of the curves $u = const.$ (resp. $v = const$) to be outgoing (resp. incoming) radial null rays in region outside the star. Along a radial null ray we have in the Lagrangian-coordinates

$$-d\tau^2 + g_{11}(\tau, R)dR^2 = 0. \quad (55)$$

It follows

$$(d\tau/dR)_{u=const.(resp.v=const.)} = +(resp.-)\sqrt{g_{11}} \tag{56}$$

and hence

$$\frac{\partial u(resp.\ v)/\partial R}{\partial u(resp.\ v)/\partial\tau} = -(resp.\ +)\sqrt{g_{11}}. \tag{57}$$

Inserting this into (53) and using $\{\sqrt{g_{11}}\}^{\Gamma\bar{+}0} = \{\partial r/\partial R\}^{\Gamma\bar{+}0}$ yields

$$\{\frac{\partial f_{l,m}}{\partial R}\}^{\Gamma-0} - \{\frac{\partial r}{\partial R}\}^{\Gamma-0}\{\frac{\partial f_{l,m}}{\partial\tau}\}^{\Gamma-0} = -2\{\frac{\partial r}{\partial R}\}^{\Gamma-0}\{\frac{\partial f^0_{l,m}}{\partial\tau}\}^{\Gamma+0}. \tag{58}$$

In order to make more transparent the last equation we shall write it in terms of the coordinates $(\eta, R)$. From (17) and (18) one can extract ( $\alpha = \sqrt{\frac{2M}{a^3}}$ )

$$\frac{\partial}{\partial\tau} = \sqrt{\frac{2M}{R}}\frac{1}{r}\frac{\partial}{\partial\eta}\ ,\ for\ R \geq a \tag{59}$$

$$\frac{\partial}{\partial\tau} = \alpha\frac{R}{r}\frac{\partial}{\partial\eta}\ ,\ for\ R\ \leq a \tag{60}$$

$$\{r\}^{\Gamma-0} = \{r\}^{\Gamma+0} = a\{\frac{\partial r}{\partial R}\}^{\Gamma-0}. \tag{61}$$

Using these results the boundary condition (58) takes the form

$$\{\frac{\partial f_{l,m}}{\partial R}\}^{\Gamma-0} - \alpha\{\frac{\partial f_{l,m}}{\partial\eta}\}^{\Gamma-0} = -2\alpha\{\frac{\partial f^0_{l,m}(u)}{\partial\eta}\}^{\Gamma+0}. \tag{62}$$

Notice here that the curves $\eta \pm \alpha R = const.$ are the radial null rays in the interior region of the star. Next we turn to the explicit construction of the function $f_{l,m}$. According to our geometrical optics approximation we neglect the potential term in (44) and write the radial wave equation in the form

$$\frac{\partial^2 f_{l,m}}{\partial R^2} - \alpha^2\frac{\partial^2 f_{l,m}}{\partial\eta^2} = 0. \tag{63}$$

By the Laplace-transformation

$$\tilde{f}_{l,m}(R,z) = \int_{-\infty}^{\eta_0} d\eta f_{l,m}(R,\eta)e^{z(\eta-\eta_0)} \tag{64}$$

the determination of $f_{l,m}(\eta, R)$ in the past of $\Sigma_H$ in the interior region of the star may be reduced to the solution of the ordinary differential equation

$$\frac{d^2\tilde{f}_{l,m}(R,z)}{dR^2} - \alpha^2 z^2\tilde{f}_{l,m}(R,z) = 0 \tag{65}$$

together with the boundary conditions

$$\frac{d\tilde{f}_{l,m}(a,z)}{dR} + \alpha z\tilde{f}_{l,m}(a,z) = 2\alpha z\tilde{q}(z)\ ,\ \tilde{f}_{l,m}(0,z) = 0 \tag{66}$$

where

$$\tilde{q}(z) = \int_{-\infty}^{\eta_0} d\eta f^0_{l,m}(u(a,\eta))e^{z(\eta-\eta_0)}. \tag{67}$$



The zero Cauchy-data on $\Sigma_H(0,a)$ are automatically satisfied. The solution of the above problem reads

$$\tilde{f}_{l,m}(R,z) = (e^{\alpha z(R-a)} - e^{-\alpha z(R+a)})\tilde{q}(z). \tag{68}$$

The inversion may be performed via the formula

$$e^{\lambda z}\tilde{q}(z) \longleftrightarrow f^0_{l,m}(u(a,\eta-\lambda)), \lambda \leq 0 \tag{69}$$

and the fact that for $\eta - \lambda \geq \eta_0$ we have $f^0_{l,m}(u(a,\eta-\lambda)) = 0$ . It follows

$$\begin{aligned} f_{l,m}(R,\eta) = {} & \theta(\eta_0 - \eta + \alpha R - \alpha a)f^0_{l,m}(u(a,\eta - \alpha R + \alpha a)) - \\ & \theta(\eta_0 - \eta - \alpha R - \alpha a)f^0_{l,m}(u(a,\eta + \alpha R + \alpha a)). \end{aligned} \tag{70}$$

Now we set in this equation $R = a$ to obtain

$$f_{l,m}(a,\eta) = f^0_{l,m}(u(a,\eta)) - \theta(\eta_0 - \eta - 2\alpha a)f^0_{l,m}(u(a,\eta + 2\alpha a)). \tag{71}$$

Notice here that in the interior region of the star the parameter $\eta$ is only a function of $\tau$ as one simply extracts from (18). Using (71) and (49) the function $f^{\Gamma}_{l,m}(v)$ may be written as

$$f^{\Gamma}_{l,m}(v) = f^0_{l,m}(u_\Gamma(v)) - \theta(\eta_0 - \eta_\Gamma(v) - 2\alpha a)f^0_{l,m}(u(a,\eta_\Gamma(v) + 2\alpha a)) \tag{72}$$

where $\eta_\Gamma(v)$ gives the parameter $\eta$ as a function of the advanced time $v$ along $\Gamma$. Inserting (72) and (48) into (47) we obtain the representation of $f_{l,m}$ in the past of $\Sigma_H$ in the exterior region of the star in the form

$$f_{l,m}(t,r) = f^0_{l,m}(u) - \theta(\eta_0 - \eta_\Gamma(v) - 2\alpha a)f^0_{l,m}(u(a,\eta_\Gamma(v) + 2\alpha a)). \tag{73}$$

Let us establish some properties of this solution. Due to the known results given by Dimock and Kay [13] the space-time support of $f^0_{l,m}(u)$ on $\Sigma_H$ tends towards the horizon $r \to 2M$ as the space-time support of $h(x)$ is shifted towards the time-like asymtotic future. This fact together with (73) implies that by shifting the space-time support of $h(x)$ sufficiently far towards the time-like asymptotic future the support of $f_{l,m}$ on $\Sigma_0$ will lie entirely outside the star. Under the typical astronomical conditions ($\frac{2M}{a} \approx 10^{-6}$) this will actually be the case if *supp* $h(x)$ lies in the future of the retarded time $u_0$ where $u_0$ is the value of the retarded time along that outgoing null ray which intersects the surface of the star at $r \approx 10M$. This class of test-functions is the most interesting class for our purpose and will be considered in this paper. As a consequence the first term in (73) vanishes on $\Sigma_0$, leading to

$$f_{l,m}(t,r) = -\theta(\eta_0 - \eta_\Gamma(v) - 2\alpha a)f^0_{l,m}(u(a,\eta_\Gamma(v) + 2\alpha a)). \tag{74}$$

# 5 Asymptotic Form of the Retarded Part of the State

We address ourselves now to the computation of the asymptotic form of the retarded part of the state given in the symplectic representation (34) . According to the property of the space-time support of the symplectic test functions $f_i(x_i), i = 1,2$ on $\Sigma_0$ mentioned at the end of the preceding chapter only the region outside the star contributes to the integral (34) and this contribution is given by

$$< \phi(\bar{h}_1)\phi(h_2) >= \int_{t_1=t_2=0} dr_1 r_1^2 d\Omega_1 dr_2 r_2^2 d\Omega_2 W_G^{(2)}(x_1,x_2)\ \overleftrightarrow{D}_1\overleftrightarrow{D}_2\ \bar{f}_1(x_1)f_2(x_2) \tag{75}$$

with

$$D_i = (1 - \frac{2M}{r})^{-1}\frac{\partial}{\partial t_i}\ ,\ i = 1, 2. \tag{76}$$

Using the Schwarzschild-coordinates in the static region the two-point function of the ground state may be expanded as

$$W_G^{(2)}(x_1, x_2) = \sum_{l,m} Y_{l,m}(\vartheta_1, \phi_1)\overline{Y}_{l,m}(\vartheta_2, \phi_2)(r_1 r_2)^{-1} \times$$

$$\int_0^\infty d\omega G_l(\omega, \overset{*}{r}_1)\overline{G}_l(\omega, \overset{*}{r}_2)e^{-i\omega(t_1 - t_2)} \tag{77}$$

where the function $G_l(\omega, \overset{*}{r})$ satisfies in the region outside the star the radial equation of the Schwarzschild-metric[5] and possesses the asymptotic representation

$$G_l(\omega, \overset{*}{r}) \overset{\overset{*}{r}\to\infty}{\longrightarrow} \frac{1}{2i\sqrt{\omega}}(e^{i\omega\overset{*}{r} + i\delta_l} - e^{-i\omega\overset{*}{r} - i\delta_l}). \tag{78}$$

In (78) the set of the scattering phases $\delta_l$ gives an asymptotic characterization of the global features of the ground state in space-like direction and depends on the interior geometry of the star (in our model only on its mass and radius). We note that different choices of $\delta_l$ correspond to different initial conditions for the stellar collapse. Inserting (77) into (75) and using the expansion (36) for each $f_i(x_i), i = 1, 2$ and performing the angular integration one obtains

$$< \phi(\bar{h}_1)\phi(h_2) > = \sum_{l,m}\int_0^\infty d\omega I(\omega, f_{1;l,m})\overline{I}(\omega, f_{2;l,m}) \tag{79}$$

where

$$I(\omega, f_{i;l,m}) = \int d\overset{*}{r}\ \{G_l(\omega, \overset{*}{r})e^{-i\omega t}\ \overset{\leftrightarrow}{\frac{\partial}{\partial t}}\ f_{i;l,m}(t, \overset{*}{r})\}_{t_1 = t_2 = 0}\ ,\ i = 1, 2 \tag{80}$$

To evalute this integral we write the functions $f_{i;l,m}$ given by (74) as the Fourier-integral

$$f_{i;l,m}(v) = \frac{1}{2\pi}\int d\bar{\omega}\tilde{f}_{i;l,m}(\bar{\omega})e^{i\bar{\omega}v} \tag{81}$$

$$\tilde{f}_{i;l,m}(\bar{\omega}) = -\int dv\theta(\eta_0 - \eta_\Gamma(v) - 2\alpha a)f^0_{i;l,m}(u(a, \eta_\Gamma(v) + 2\alpha a))\epsilon^{-i\bar{\omega}v}. \tag{82}$$

Further, due to the fact that the Space-time support of $f_{l,m}$ on $\Sigma_0$ lies entirely outside the star we approximate the function $G_l(\omega, \overset{*}{r})$ by its asymptotic form given by (78). Inserting (78) and (81) into (80) we get

$$I(\omega, f_{i;l,m}) = (-4\pi i\sqrt{\omega})^{-1}\int d\bar{\omega}d\overset{*}{r}\ \tilde{f}_{i;l,m}(\bar{\omega}) \quad \times$$

$$i[\epsilon^{+i\delta_l}(\omega + \bar{\omega})\epsilon^{i(\omega + \bar{\omega})\overset{*}{r}} - e^{-i\delta_l}(\omega + \bar{\omega})\epsilon^{-i(\omega - \bar{\omega})\overset{*}{r}}]. \tag{83}$$

Performing the integrations yields

$$I(\omega, f_{i;l,m}) \doteq -\epsilon^{-i\delta_l}\sqrt{\omega}\ \tilde{f}_{i;l,m}(\omega). \tag{84}$$

[5]The radial equation may be derived by separating the exponential factor $\epsilon^{-i\omega t}$ in the solutions of the equation (42.)



Now we use the Fourier-decomposition of $f^0_{i;l,m}(u)$

$$f^0_{i;l,m}(u) = \int d\omega'\tilde{f}^0_{i;l,m}(\omega')e^{-i\omega' u} \tag{85}$$

and obtain

$$I(\omega, f_{i;l,m}) = e^{-i\delta_l}\sqrt{\omega}\int d\omega'\ \tilde{f}^0_{i;l,m}(\omega')F(\omega, \omega') \tag{86}$$

where

$$F(\omega, \omega') = \int dv\ \theta(\eta_0 - \eta_\Gamma(v) - 2\alpha a)e^{-i\omega' s(v)}e^{-i\omega v}, \tag{87}$$

$$s(v) = u(a, \eta_\Gamma(v) + 2\alpha a). \tag{88}$$

Consequently the equation (79) takes the form

$$< \phi(\bar{h}_1)\phi(h_2) > = \sum_{l,m}\int d\omega'\ d\omega''\tilde{f}^0_{1;l,m}(\omega')\overline{\tilde{f}^0}_{2;l,m}(\omega'')K(\omega', \omega'') \tag{89}$$

with

$$K(\omega', \omega'') = \int_0^\infty d\omega\ \omega F(\omega, \omega')\overline{F}(\omega, \omega''). \tag{90}$$

Next we turn to the evaluation of the function $F(\omega, \omega')$. The explicit form of the function $s(v)$ is given in the appendix A. Under the typical astronomical conditions ($\frac{2M}{a} \approx 10^{-6}$) and for $u > u_0$ we have

$$s(v) = v_0 + \lambda_1(v - \bar{v}) - 4M\ln|\lambda_2(v - \bar{v})| \tag{91}$$

$$\lambda_1 \approx 1\ ,\ |\lambda_2| \approx \frac{1}{2M}\ ,\ \bar{v} = v(a, \eta_0 - 2\alpha a)\ ,\ v_0 = v(a, \eta_0) \tag{92}$$

where $\eta_0$ corresponds to $\tau = \tau_0$. To simplify the expression (91) we shall redefine the retarded time $u$ and the advanced time $v$ according to

$$u' = u - v_0\ \ ,\ \ v' = \lambda_1(v - \bar{v}). \tag{93}$$

Notice here that we have $\lambda_1 \neq 0$. By this redefinition we obtain for $u > u_0$ or equivalently $u' > u'_0 \equiv u_0 - v_0 \approx -2M$

$$u'(a, \eta_\Gamma(v') + 2\alpha a) = v' - 4M\ln|\lambda'_2 v'| \tag{94}$$

where $|\lambda'_2| = |\lambda_2(\lambda_1)^{-1}| \approx \frac{1}{2M}$. In the interest of simplicity in what follows the prime is suppressed. Therefore for $s(v)$ we shall write

$$s(v) = v - 4M\ln|\lambda_2 v|. \tag{95}$$

Now using the monotony of $\eta_\Gamma(v)$ (see appendix A) one gets

$$\theta(\eta_0 - \eta_\Gamma(v) - 2\alpha a) = \theta(-v). \tag{96}$$

Taking this into account we infer from (87)

$$F(\omega, \omega') = \int_{\xi_i}^0 dv\ \epsilon^{-i\omega' s(v)}\epsilon^{-i\omega v},\ \ \xi_i < 0 \tag{97}$$

where we have introduced a cut off parameter $\xi_i$ to express explicitly that only a finite region contributes effectively to the above integral as $f^0_{i;l,m}(u)$ on $\Sigma_H$ is a localized function near the horizon. There is some flexibility in the choice of $\xi_i$. The only condition that restricts the choice of $\xi_i$ is $\xi_i \leq s^{-1}(u_i^-)$ where $s^{-1}(u)$ is the inverse function of $s(v)$ and $u_i^-$ is the value of the retarded time along the earliest outgoing null ray that intersects the support of $h_i(x_i)$, i. e. for $u < u_i^-$ all outgoing null rays do not intersect the support of $h_i(x_i)$. Let us set at first $\xi_i = s^{-1}(u_i^-)$. From properties of the function $s(v)$ together with properties of the space-time support of $f^0_{i;l,m}(u)$ mentioned at the end of the preceding chapter one can extract that by shifting the space-time support of $h_i(x_i)$ towards the time-like asymptotic future $\xi_i$ tends to zero ($|\xi_i| \ll 2M$) and hence the contribution of the upper integration-limit to the integral (97) becomes significant. Let us give a some what different formulation of this fact. For $\omega' > 0$ the function $F(\omega,\omega')$ may be evaluated to (see appendix B)

$$F(\omega,\omega') = i\frac{1}{\omega'+\omega}\left(\frac{|\lambda_2|}{\omega'+\omega}\right)^{i4M\omega'} e^{-2M\pi\omega'}\gamma(1+i4M\omega', i\xi_i(\omega'+\omega)) \tag{98}$$

where $\gamma(,)$ is the incomplete gamma-function. The exponential damping factor in (98) determines the relevant energy scale; i.e. for $\omega' \gg \frac{1}{2M}$ the function $F(\omega,\omega')$ practically vanishes. This fact together with $\gamma(y,z) \stackrel{z\to 0}{\longrightarrow} 0$ implies that for $|\xi_i| \ll 2M$ only the ultraviolet part ($\omega \gg \frac{1}{2M}$) of the initial ground state contributes to $F(\omega,\omega')$. To evaluate this contribution we set $\xi_i = -2M$ and consider the asymptotic form of $F(\omega,\omega')$ for $\omega \gg \frac{1}{2M}$. Using the asymptotic expansion of $\gamma(y,z)$ in descending powers of $z$ [6]one obtains

$$F(\omega,\omega') = i\frac{1}{\omega'+\omega}\left(\frac{|\lambda_2|}{\omega'+\omega}\right)^{i4M\omega'} e^{-2M\pi\omega'}\Gamma(1+i4M\omega') - \\ i\frac{1}{\omega'+\omega} e^{i4M\omega' \ln|2M\lambda_2|} e^{i2M(\omega'+\omega)}[1 + O(1/|2M(\omega'+\omega)|)]. \tag{99}$$

The first two terms are the dominant terms for $\omega \gg \frac{1}{2M}$. Neglecting $4M\omega' \ln|2M\lambda_2|$ in the second term compared with $2M\omega$ and taking the dominant terms in (99) we get

$$F(\omega,\omega') = i\left[\frac{1}{\omega'+\omega}\left(\frac{|\lambda_2|}{\omega'+\omega}\right)^{i4M\omega'} e^{-2M\pi\omega'}\Gamma(1+i4M\omega') - \frac{e^{i2M(\omega'+\omega)}}{\omega'+\omega}\right]. \tag{100}$$

For $\omega' < 0$ we may analytically continue (anticlockwise) the above formula on the domain $\omega' < 0$, $-\omega < 0$, leading to the final result

$$F(\omega,\omega') = \theta(\omega')[A(\omega,\omega') + B(\omega,\omega')] + \theta(-\omega')[e^{4M\pi\omega'}A(-\omega,\omega') + B(-\omega,\omega')]. \tag{101}$$

Here $A(\omega,\omega')$ (resp. $B(\omega,\omega')$) denotes the first (resp. second) term in (100). Due to the logarithmic singularity of $A(\omega,\omega')$ at the origin ($\omega = 0, \omega' = 0$) we get by analytic continuation the factor $e^{4M\pi\omega'}$.

The formula (101) gives the effective part of the function $F(\omega,\omega')$ needed in (89) und (90) to represent the asymptotic form of the retarded part of the state.

Now let us evaluate the kernel $K(\omega',\omega'')$. Putting (101) and (90) together one infers

$$K(\omega',\omega'') = \int_0^\infty d\omega\, \omega\{\theta(\omega')\theta(\omega'')[A(\omega,\omega')\overline{A}(\omega,\omega'') + B(\omega,\omega')\overline{B}(\omega,\omega'')] + \\ \theta(-\omega')\theta(\omega'')[e^{4M\pi\omega'}A(-\omega,\omega')\overline{A}(\omega,\omega'') + B(-\omega,\omega')\overline{B}(\omega,\omega'')] + \\ \theta(\omega')\theta(-\omega'')[A(\omega,\omega')e^{4M\pi\omega''}\overline{A}(-\omega,\omega'') + B(\omega,\omega')\overline{B}(-\omega,\omega'')] + \\ \theta(-\omega')\theta(-\omega'')[e^{4M\pi(\omega'+\omega'')}A(-\omega,\omega')\overline{A}(-\omega,\omega'') + B(-\omega,\omega')\overline{B}(-\omega,\omega'')]\}. \tag{102}$$

[6]See[14]



Note that the cross terms between $A$ and $B$ do not contribute to $K(\omega',\omega'')$ as $B$ oscillates in the effective range of contribution of $\omega$ to the above integral ($\omega \gg (2M)^{-1}$) with great frequency. For the same reason the contribution of $B(-\omega,\omega')\overline{B}(\omega,\omega'')$ and $B(\omega,\omega')\overline{B}(-\omega,\omega'')$ may be neglegted. The contribution of the remaining terms inolving $B\overline{B}$ may be evaluated to

$$R(\omega',\omega'') \equiv e^{i2M(\omega'-\omega'')}\int_0^\infty d\omega\, \omega\left[\frac{\theta(\omega')\theta(\omega'')}{(\omega'+\omega)(\omega''+\omega)} + \frac{\theta(-\omega')\theta(-\omega'')}{(\omega'-\omega)(\omega''-\omega)}\right]. \tag{103}$$

Now due to the fact that the contribution of small $\omega$'s does not affect the asymptotic form of retarded part of the state we neglect in the expression for $A$ (in the dominator) $\omega'$ compared with $\omega$ and obtain after performing the integration

$$K(\omega',\omega'') = e^{-2M\pi(\omega'+\omega'')}|\lambda_2|^{i4M(\omega'-\omega'')}\Gamma(1+i4M\omega')\overline{\Gamma}(1+i4M\omega'') \\ 2\pi\delta(4M(\omega'-\omega''))\,[\theta(\omega')\theta(\omega'') + \theta(-\omega')\theta(\omega'')e^{4M\pi\omega'} \\ +\theta(\omega')\theta(-\omega'')e^{4M\pi\omega''} + \theta(-\omega')\theta(-\omega'')e^{4M\pi(\omega'+\omega'')}] + R(\omega',\omega''). \tag{104}$$

Inserting this expression into (89) and performing the integration over $\omega''$ and using the formula $|\Gamma(1+i4M\omega')|^2 = 4M\pi\omega'/sinh(4M\pi\omega')$ yields

$$<\phi(\overline{h}_1)\phi(h_2)> = \sum_{l,m}\int d\omega' \tilde{f}^0_{1;l,m}(\omega')\overline{\tilde{f}^0}_{2;l,m}(\omega')[ \\ \theta(\omega')\frac{4\pi^2\omega'}{e^{8M\pi\omega'}-1} + \theta(-\omega')\frac{4\pi^2\omega'}{1-e^{-8M\pi\omega'}}] + (R) \tag{105}$$

where

$$(R) = \sum_{l,m}\int d\omega'\, d\omega''\tilde{f}^0_{1;l,m}(\omega')\overline{\tilde{f}^0}_{2;l,m}(\omega'')R(\omega',\omega''). \tag{106}$$

The first two terms in (105) correspond to an outgoing radiation with temperature $(8M\pi)^{-1}$. To make more transparent the last term in (105) consider the static extension of the space-time and the corresponding extension of the stationary ground state of the static metric to the future of $\Sigma_0$. We show that in the time-like asymptotic future of the static extended space-time and far away from the star the last term in (105) may be identified as the retarded part of the stationary ground state of the static metric, $<\phi(\overline{h}_1)\phi(h_2)>_G$. To see this we use the symplectic representation of $<\phi(\overline{h}_1)\phi(h_2)>_G$ on a Cauchy-surface lying in the time-like asymptotic future. Repeating the same calculation given at the beginning of this chapter with respect to the retarded time (replacing in (81) $v$ by $u$) one obtains

$$<\phi(\overline{h}_1)\phi(h_2)>_G = \sum_{l,m}\int d\omega'\, d\omega''\tilde{f}^0_{1;l,m}(\omega')\overline{\tilde{f}^0}_{2;l,m}(\omega'')K_G(\omega',\omega'') \tag{107}$$

with

$$K_G(\omega',\omega'') = \int_0^\infty d\omega\, \omega F_G(\omega,\omega')\overline{F}_G(\omega,\omega''),\ F_G(\omega,\omega') = \int_{\zeta_i}^\infty du\, e^{-i(\omega'+\omega)u} \tag{108}$$

where the cut off parameter $\zeta_i$ indicates explicitly that only great values of $u$ contribute to the above integral. The effective contribution occurs by $\zeta_i \gg -2M$[7].The function $F_G(\omega,\omega')$ may be evaluated to

$$F_G(\omega,\omega') = -i\left[\theta(\omega')\frac{e^{-i(\omega'+\omega)\zeta_i}}{\omega'+\omega} + \theta(-\omega')\frac{e^{-i(\omega'-\omega)\zeta_i}}{\omega'-\omega}\right]. \tag{109}$$

[7]Note that we have $u_0' \approx -2M$.

Note that the cross terms between the two terms in (109) do not contribute to $K_G(\omega',\omega'')$ as these terms are rapidly oscillating ($\zeta_i \gg -2M$). Setting $\zeta_i = -2M$ one obtains

$$K_G(\omega',\omega'') = e^{i2M(\omega'-\omega'')} \int_0^\infty d\omega\omega\left[\frac{\theta(\omega')\theta(\omega'')}{(\omega'+\omega)(\omega''+\omega)} + \frac{\theta(-\omega')\theta(-\omega'')}{(\omega'-\omega)(\omega''-\omega)}\right]. \quad (110)$$

From this result one can extract that our statement holds.
The other interesting aspect is the universality of the radiation part of the state, i. e. the first two terms in (105), in the sense that it does not depend on the set of scattering phases $\delta_l$ (note that $\delta_l$ depends in our model on the radius and the mass of the star). This means that the asymptotic radiation transmitted to the future null infinity does not depend on the initial radius of the collapsing star.

# 6 Energy Momentum Consideration

The results of the preceding chapter show that the radiation part of the state, i.e. the first two terms in the expression (105), may be identified as a deviation of the retarded part of the state from that of the ground state of the static extended space-time which leads to a steady outgoing radiation continuing eternally. This violates the energy conservation of the total system. To satisfy the Bondy-Sachs energy balance [16],[17] the black hole must lose mass. The mechanism for this must be found in the back reaction of the quantum field to gravity which was neglegted up to this point in this paper. The fundamental quantity containing direct information about the mass decreasing is the expectation value of the radial component of the energy momentum tensor operator, i.e. $< T_{0r}(x) >$ with $T_{0r}(x) = \partial_0\phi(x)\partial_r\phi(x)$ . As a product of operator valued distribution the formal expression of $T_{0r}(x)$ is divergent and must be renormalized. As a first orientation let us consider in some observation domain in the Schwarzschild-region the most natural renormalization procedure which is assumed to consist in substracting from the operator $T_{0r}(x)$ its expectation value in a "local ground state "of the Schwarzschild-metric, viz.

$$: T_{or}(x) : = T_{or}(x) - < T_{or}(x) >_G \quad (111)$$

This corresponds locally to the normal ordering of $T_{0r}$ with respect to the time-like Killing vector field of the Schwarzschild-metric. At first sight the above definition of $T_{0r}(x)$ might appear as ambiguous, since, as already mentioned in the introduction, there exist many stationary states which are the ground states in a global static metric, the latter agreeing locally in the observation domain with the Schwarzschild-metric. All these stationary states correspond at large distance from the source to different sets of scattering phases in (78) . The essential point is now that this global ambiguity does not affect the radial part of the radiation given by (111). To demonstrate this fact let us consider at large distance from the source the difference between the expectation values of $T_{0r}$ in two different ground states; i.e.

$$\Delta \equiv < T_{0r} >_G - < T_{0r} >_{G'} \quad (112)$$

By introducing the retarded time $u$ and the advanced time $v$ as new coordinates (these are the Bondi-Sachs radiation coordinates) the operator $T_{0r}$ may be written as

$$T_{0r}(x) = \partial_v\phi(x)\partial_v\phi(x) - \partial_u\phi(x)\partial_u\phi(x) \quad (113)$$



Next we use the method of covariant point-separation [18] to obtain

$$\Delta = \frac{1}{2}\, lim_{x'\to x}\, D_{uv}[< \{\phi(x)\phi(x')\} >_G - < \{\phi(x)\phi(x')\} >_{G'}] \quad (114)$$

$$D_{uv} = g_v^{v'}\partial_v\partial_{v'} - g_u^{u'}\partial_u\partial_{u'} \quad (115)$$

where $g_u^{u'}$ (resp. $g_v^{v'}$ )is the retarded (resp. advanced) component of the bivector of parallel transport and the presence of the anticomutator goes back to the symmetrization of the arguments. Using the expansion (77) and the asymptotic formula (78) we obtain after a straightforward calculation

$$< \phi(x)\phi(x') >_G - < \phi(x)\phi(x') >_{G'} = \sum_{l,m} Y_{l,m}(\vartheta,\phi)\, \dot{Y}_{l,m}(\vartheta',\phi')(rr')^{-1} \times$$

$$\int_0^\infty \frac{d\omega}{2i\omega}[e^{-i\omega(v'-u)}(e^{i2\delta'_l} - e^{i2\delta_l}) + e^{-i\omega(u'-v)}(e^{-i2\delta'_l} - e^{-i2\delta_l})] \quad (116)$$

where $\delta_l$ (resp. $\delta'_l$) corresponds to $G$ (resp.$G'$).
This equation shows that only the cross terms between the retarded and the advanced part of $< \phi(x)\phi(x') >_G$ and $< \phi(x)\phi(x') >_{G'}$ contribute to $\Delta$ and this contribution vanishes in (114) by the differentiation. This surprising result makes the procedure of the normal ordering in (111) unambiguous. It must be stressed that the same can not be stated for the expectation values of other components of the energy momentum tensor operator, e. g. the expectation value of the energy density of the quantum field. This lack becomes crucial if we treat the back reaction problem via the Einstein-equations coupled to the renormalized expectation value of the energy momentum tensor operator. At the present time no renormalization procedure based on unambiguous principle seems to exist. The opposite interesting possibility is that perhaps no renormalization is needed. Conceptually the development of a future theory along this opposite line must be based on the specific knowledge about the local structure of states and the corresponding local dynamical laws. We shall address ourselves to this problem in an independent paper.

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### Appendix A

In this appendix we derive the explicit form of $u(a,\eta_\Gamma(v)+2\alpha a)$ given in (91). Let us consider, first, the function $u(a,\eta+2\alpha a)$. It may be written as

$$u(a,\eta+2\alpha a) = v(a,\eta+2\alpha a) - 2\,\dot{r}\,(a,\eta+2\alpha a) \quad (117)$$

Due to the presense of the $\theta$-function in (87) we are interessted only in the form of $u(a,\eta+2\alpha a)$ for $\eta \leq \bar{\eta} \equiv \eta_0 - 2\alpha a$. We take for $\frac{2M}{a}$ the typical value $10^{-6}$. For $\eta = \bar{\eta} - 0.01$ the radius of the star is $\approx 100M$. Therefor for $u \geq u_o$ all of the regular functions in (117) may be replaced by the linear terms in their corresponding Taylor-expansion in $\eta = \bar{\eta}$ . So doing one infers

$$u(a,\eta+2\alpha a) = -4M + v_0 + (\lambda'_1 - 2\lambda'_2)(\eta - \bar{\eta}) - 4M\ln\left|\frac{\lambda'_2}{2M}(\eta-\bar{\eta})\right| \quad (118)$$

$$v_0 = v(a,\eta_0)\ ,\ \lambda_1' = (\frac{\partial v(a,\eta+2\alpha a)}{\partial\eta})_{\eta=\bar\eta}\ ;\ \lambda_2' = (\frac{\partial r(a,\eta+2\alpha a)}{\partial\eta})_{\eta=\bar\eta}.$$

From the formulas (20)-(22) one can extract

$$v_o \approx 10^9 M\ ,\ \lambda_1' \approx 10^3 M\ ,\ \lambda_2' \approx -10^3 M. \tag{119}$$

Next we turn to the specification of the function $\eta_\Gamma(v)$. By considering the linear term in the Taylor-expansion of the function $v(a,\eta)$ in $\eta=\bar\eta$ one similarly obtains

$$v - \bar v = \lambda_3(\eta - \bar\eta) \tag{120}$$

$$\bar v = v(a,\bar\eta) \approx 10^9 M\ ,\ \lambda_3 = (\frac{\partial v(a,\eta)}{\partial\eta})_{\eta=\bar\eta} \approx 10^3 M.$$

This determines the function $\eta_\Gamma(v)$. Putting (118) and (120) together one obtains

$$u(a,\eta_\Gamma(v)+2\alpha a) = -4M + v_0 + \lambda_1(v-\bar v) - 4M\ln|\lambda_2(v-\bar v)| \tag{121}$$

$$\lambda_1 = (\lambda_1' - 2\lambda_2')\lambda_3^{-1} \approx 1, \lambda_2 = \frac{\lambda_2'}{2M\lambda_3} \approx -\frac{1}{2M}$$

Neglecting in the right hand side of (121) the first term compared with the second term yields the required result.

## Appendix B

Consider the integral

$$F(\omega,\omega') = \int_{\xi_i}^{0} dv e^{-i\omega' s(v)} e^{-i\omega v}, s(v) = v - 4M\ln|\lambda_2 v|,\ \xi_i < 0. \tag{122}$$

Substituation $y = \frac{v}{\xi_i}$ yields

$$F(\omega,\omega') = |\xi_i|^{1+i4M\omega'}|\lambda_2|^{i4M\omega'}\int_0^1 dy e^{-i\xi_i(\omega'+\omega)y} y^{i4M}. \tag{123}$$

By using the integral representation of the confluent hypergeometric function $\Phi(a,c,x)$ [19] one obtains

$$F(\omega,\omega') = |\xi_i|^{1+i4M\omega'}|\lambda_2|^{i4M\omega'}(1+4M\omega')^{-1}\Phi(1+i4M\omega', 2+i4M\omega', -i\xi_i(\omega'+\omega)). \tag{124}$$

Now let $\omega'$ be positiv so that $\omega'+\omega \geq 0$ . With this assumption the formula [8] $ax^{-a}\gamma(a,x) = \Phi(a,a+1,-x)$ , where $\gamma(a,x)$ is the incomplete gamma-function, may be applied to obtain

$$F(\omega,\omega') = i\frac{1}{\omega'+\omega}(\frac{|\lambda_2|}{\omega'+\omega})^{i4M\omega'} e^{-2M\pi\omega'}\gamma(1+i4M\omega', i\xi_i(\omega'+\omega)). \tag{125}$$

[8] See [14]



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