DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 91-009
February 1991

受入
91-4-8
高工研図書室

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ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

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DESY 91-009
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ISSN 0418-9833

Local behaviour of exactly solvable potentials

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abstract.- *It is found that the local behaviour of exactly solvable potentials corresponds either to a Pöschl–Teller potential or to a harmonic oscillator potential.*

The use of continuous groups of transformations to solve differential equations introduced by Lie has proved through the years to be very effective. This method has actually developed to a widely used tool in theoretical physics. By relating the Hamiltonian of a physical system to some invariant differential operator of a Lie algebra, the calculation of the physical relevant quantities such as energies, masses, transition probabilities, or scattering matrices[1–7] can be achieved by algebraic manipulation. The differential equations describing the physical systems need not be explicitly solved, or are considerably simplified. The representation space together with the generators of the Lie algebra determine the properties of the systems. The range of potentials that can be treated this way, cor-

responds to the factorizable potentials[8] or the exactly solvable potentials as discussed in the supersymmetric quantum mechanics[9].
This powerful technique has however the following inconvenience. In physics, the system and not the symmetry is given, and there is no general theory to determine which symmetry, if any, is relevant to the problem. In this paper we provide a result to help coping with this awkwardness of the algebraic technique for systems whose description can be reduced to the study of an ordinary differential equation. We show that the physical systems which can be treated with the algebraic techniques behave locally like an oscillator or a Pöschl–Teller potential problem. Path integral methods do also lead to a similar conclusion on the relevance of these two potentials[10].

Even if the applications to physics are restricted to Lie structures, we want to keep our arguments as general as possible, and hence consider Riemannian manifolds, to become specific later on. After introducing the geodesic polar coordinates on a Riemannian manifold, and linking its differential structure with a Schrödinger equation, some examples are presented. The central result of the paper on the local behaviour of the exactly solvable potentials is then stated, and some applications are indicated.

Let M be a Riemannian manifold, x be some point of M. There exists a neighbourhood N of x in M on which every point can be parametrized by the shortest distance r of a geodesic reaching the point, and by the direction of the geodesic. i.e. the coordinates of the intersection between the tangent to the geodesic and the unit sphere on the tangential space M_x. The mapping $x \mapsto (r, \theta_1, \ldots, \theta_n)$ is called a system of geodesic polar coordinates. For a proof of the existence of such a coordinate system see for instance Ref. 11.

For M of dimension two the metric takes in these coordinates the following form,

$$ds^2 = dr^2 + h^2(r, \theta) d\theta^2.$$

In this paper we consider the case where h depends only on the radial variable r. However, for sufficiently small values of r this is no constraint, since

$$h(r) = r - \frac{1}{6}\kappa r^3 + R(r, \theta), \tag{1}$$

where $\lim_{r\to 0} (R(r, \theta)/r^3) = 0$, and κ is the Gaussian curvature at the point x.

For such a function $h(r)$, the Laplace Beltrami operator Δ_M takes the form

$$\Delta_M = \frac{1}{h(r)}\frac{\partial}{\partial r}h(r)\frac{\partial}{\partial r} + \frac{1}{h^2(r)}\frac{\partial}{\partial \theta^2}.$$

From the above equation it follows that the eigenfunctions of Δ_M are given by,

$$\Delta_M F_{\lambda,m} = \lambda F_{\lambda,m},$$

where $F_{\lambda,m} = \Phi_\lambda(r)e^{im\theta}$, with Φ_λ obeying the following differential equation:

$$\left[\frac{1}{h(r)}\frac{\partial}{\partial r}h(r)\frac{\partial}{\partial r} - \frac{m^2}{h^2(r)}\right]\Phi_\lambda(r) = \lambda\Phi_\lambda(r) \tag{2}$$

Equation (2) is of the Sturm–Liouville type and can be transformed into the form of a Schrödinger equation with a potential in two ways:

a- After multiplying (2) by $h^2(r)$ and using the new variable $\xi = \int \frac{1}{h(r)}dr$ we obtain the equation

$$\left[-\frac{d^2}{d\xi^2} + \lambda h^2(\xi)\right]\Phi_\lambda(\xi) = m^2\Phi_\lambda(\xi). \tag{3}$$

b- The normalization condition

$$\int \Phi_\lambda(u)^2 h(u) du = 1$$

suggests the introduction of the function Ψ_λ,

$$\Psi_\lambda(r) = \Phi_\lambda(r)h^{1/2}(r).$$

Equation (2) yields

$$\left[-\frac{d^2}{dr^2} + \frac{h''(r)}{2h(r)} - \left(\frac{h'(r)}{2h(r)}\right)^2 + \frac{m^2}{h^2(r)}\right]\Psi_\lambda(r) = -\lambda\Psi_\lambda(r). \tag{4}$$

If we take for the manifold M a globally symmetric space X=G/K of dimension two; i.e. G is a connected semisimple Lie group and K is a maximal compact subgroup, and use some representation for K, then the transformations **a-** and **b-** provide us with a potential group[12] or with a dynamical group[13] of a physical system respectively. The following table displays some examples.

Examples	
SO(3)/SO(2)	
$h(r)=\sin(r), \quad 0<r<\pi/2, \quad -\infty<\xi<\infty$	
Trans. **a-**	$\left[-\frac{d}{d\xi^2}+\frac{\lambda}{(\cosh(\xi)^2}\right]\Phi_\lambda(\xi)=m^2\Phi_\lambda(\xi)$
Trans. **b-**	$\left[-\frac{d^2}{dr^2}+\frac{m^2-1/4}{\sin^2(r)}\right]\Psi_\lambda(r)=(-\lambda+1/4)\Psi_\lambda(r)$
SO(2,1)/SO(2)	
$h(r)=\sinh(r), \quad -\infty<r<\infty, \quad -\infty<\xi<\infty$	
Trans. **a-**	$\left[-\frac{d}{d\xi^2}+\frac{\lambda}{(\sinh(\xi)^2}\right]\Phi_\lambda(\xi)=m^2\Phi_\lambda(\xi)$
Trans. **b-**	$\left[-\frac{d^2}{dr^2}+\frac{m^2-1/4}{\sinh^2(r)}\right]\Psi_\lambda(r)=-(\lambda+1/4)\Psi_\lambda(r)$
E(2)/SO(2)	
$h(r)=r, \quad 0<r<\infty, \quad -\infty<\xi<\infty$	
Trans. **a-**	$\left[-\frac{d}{d\xi^2}+\lambda e^{2\xi}\right]\Phi_\lambda(\xi)=m^2\Phi_\lambda(\xi)$
Trans. **b-**	$\left[-\frac{d^2}{dr^2}+\frac{m^2+1/4}{r^2}\right]\Psi_\lambda(r)=-\lambda\Psi_\lambda(r)$

As the table shows, there is a variety of systems which can be studied with the algebraic methods. Actually, as already mentioned, all exactly solvable potentials fall into this group. However, in spite of this great range of systems, the local behaviour of all of them is characterized by only two potentials: the harmonic oscillator and the Pöschl–Teller. To prove this result, consider for sufficiently small values of r the function h of any manifold of dimension two. This function has according to equation (1) the following form

$$h(r)=r-\frac{1}{6}\kappa r^3.$$

Transformation **a-** yields then for small r i.e. $\xi\to\infty$

$$\left[-\frac{d^2}{d\xi^2}+\frac{\lambda e^{-2\mu}}{4}\left(\frac{1}{\cosh^2(\xi-\mu)}\right)\right]\Phi_\lambda(\xi)=m^2\Phi_\lambda(\xi), \qquad (5)$$

here is $e^{-2\mu}=6/\kappa$.

And transformation **b-** gives for small r ,

$$\left[-\frac{d}{dr^2}+\frac{M^2-1/4}{r^2}-\frac{\kappa}{4}+\frac{\kappa^2}{48}r^2\right]\Psi_\lambda(r)=-\lambda\Psi_\lambda(r). \qquad (6)$$

To conclude, we point out that in a sense the Pöschl–Teller potential and the harmonic oscillator potential occupy a central position within the exactly solvable potentials. Not only because they have being exhaustively treated, but mainly because they actually describe the local behaviour of all such potentials. This result can be used to decide on the solvability of a given potential. Also if a perturbative treatment applies, they naturally provide the starting input for the perturbation series.

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The author would like to thank Dr. Ian Cooper from the Newcastle University UK for the valuable discussions.

References

1. G. Gell-Mann and Ne'eman, *The Eightfold Way*, Benjamin, New York, 1964.

2. A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1974); **111**, 201 (1978); **123**, 468 (1979).

3. F. Iachello, Chem. Phys. Lett. **78**, 581 (1981);
F. Iachello and R.D. Levine, J. Chem. Phys. **77**, 3046 (1982).

4. Y. Alhassid, F. Gürsey and F. Iachello, Phys.Rev.Lett **50**, 873 (1983); Ann.Phys. (N.Y.) **148**, 346 (1983); **167**, 181 (1986).

5. A. Frank and K.B. Wolf, Phys.Rev.Lett. **52**, 1737 (1984);
A. Frank, Y. Alhassid and F. Iachello, Phys.Rev. **A34**, 677 (1986).

6. Y. Alhassid, J. Engel and J. Wu, Phys.Rev.Lett. **53**, 17 (1984);
Y.Alhassid, F. Iachello and R.D. Levine, Phys.Rev.Lett. **54**, 1746 (1985);
J.Wu, F. Iachello and Y. Alhassid, Ann.Phys. (N.Y.) **173**, 68 (1987);
J.Wu, Y. Alhassid, and F. Gürsey, Ann.Phys. (N.Y) **196**, 163 (1989).

7. A. Perelomov, *Generalized Coherent States and Their Applications*, Texts and Monographs in Physics (Springer Verlag, 1985).
R. F. Wehrhahn Phys. Rev. Lett. **65**, 1294 (1990).
R. F. Wehrhahn and R. D. Levine *Symmetry Scattering for Systems with Internal Structure*. (in press).

8. E. Schrödinger, Proc. Roy. Irish Acad. **A46**, 9, 183 (1940).
L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1951).
A. A. Adrianov, N. V. Borisov and m. v. Ioffe, Phys. Lett. **105A**, 19 (1984).
C. V. Sukumar, J. Phys. **A18**, L57 (1985).

9. E. Witten. Nucl. Phys. **B 185**. 513 (1981);
F. Cooper and B. Freedman. Ann. Phys. (NY) **146** (1983);
C.V. Sukumar. J. Phys. **A 18**. 2917 (1985);
W. Kwong and L. Rosner. Prog. Theor. Phys. Supp. **86**, 366 (1985);
F. Cooper, J.N. Ginocchio and A. Khare, Phys. Rev. **D 36**, 2458 (1987);
R. Duff, A. Khare and U.P. Sukhatme, Am. J. Phys. **56**, 163 (1988).

10. C. Grosche and F. Steiner, *Feynman Path Integrals* (in preparation).

11. S. Helgason, *Differential Geometry, Lie Groups and Symmetric spaces*, Academic Press. New York 1978.

12. F. Gürsey, *Group Theoretical Methods in Physics*, Lecture Notes in Physics No. 180 (Springer Verlag, 1983);
Y. Alhassid, *Boson in Nuclei*, (World Scientific, Singapore 1983).

13. Y. Alhassid, F. Iachello, and J. Wu Phys. Rev. Lett. **56** 271 (1986).