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FROM NON-SEMISIMPLE HOPF ALGEBRAS TO CORRELATION FUNCTIONS FOR LOGARITHMIC CFT

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Abstract

We use factorizable finite tensor categories, and specifically the representation categories of factorizable ribbon Hopf algebras H, as a laboratory for exploring bulk correlation functions in local logarithmic conformal field theories. For any ribbon Hopf algebra automorphism ω of H we present a candidate for the space of bulk fields and endow it with a natural structure of a commutative symmetric Frobenius algebra. We derive an expression for the corresponding bulk partition functions as bilinear combinations of irreducible characters; as a crucial ingredient this involves the Cartan matrix of the category. We also show how for any candidate bulk state space of the type we consider, correlation functions of bulk fields for closed oriented world sheets of any genus can be constructed that are invariant under the natural action of the relevant mapping class group.

1 Introduction

Understanding a quantum field theory includes in particular having a full grasp of its correlators on various space-time manifolds, including the relation between correlation functions on different space-times. This ambitious goal has been reached for different types of theories to a variable extent. Next to free field theories and to topological ones, primarily in two and three dimensions, two-dimensional rational conformal field theories are, arguably, best under control.

This has its origin not only in the (chiral) symmetry structures that are present in conformal field theory, but also in the fact that for rational CFT these symmetry structures have particularly strong representation theoretic properties: they give rise to modular tensor categories, and are thus in particular finitely semisimple. In many applications semisimplicity is, however, not a physical requirement. Indeed there are physically relevant models, like those describing percolation problems, which are not semisimple, but still enjoy certain finiteness properties.

It is thus natural to weaken the requirement that the representation category of the chiral symmetries should be a modular tensor category. A natural generalization is to consider *factorizable finite ribbon categories* (see Remark 3.6(i) for a definition of this class of categories). By Kazhdan-Lusztig type dualities such categories are closely related to categories of finite-dimensional modules over finite-dimensional complex Hopf algebras [FGST1]. For this reason, we study in this paper structures in representation categories of finite-dimensional factorizable ribbon Hopf algebras.

Let us summarize the main results of this contribution. We concentrate on bulk fields. In the semisimple case, the structure of bulk partition functions has been clarified long ago; in particular, partition functions of automorphism type have been identified as a significant subclass [MS2]. Here we deal with the analogue of such partition functions without imposing semisimplicity. Specifically, we assume that the representation category of the chiral symmetries has been realized as the category of finite-dimensional left modules over a finite-dimensional factorizable ribbon Hopf algebra. For any ribbon Hopf algebra automorphism ω we then obtain a description of the space of bulk fields for the corresponding automorphism invariant and show that it has a natural structure of a commutative symmetric Frobenius algebra (Theorem 2.5). The proof that the space of bulk fields is a commutative algebra also works for arbitrary factorizable finite ribbon categories (Proposition 2.3).

We are able to express the resulting bulk partition functions as bilinear combinations of irreducible characters – such a decomposition can still exist because characters behave additively under short exact sequences. We find (Theorem 4.4) that the crucial ingredient (apart from ω) is the Cartan matrix of the underlying category, This result is most gratifying, as the Cartan matrix has a natural categorical meaning and is stable under Morita equivalence and under Kazhdan-Lusztig correspondences of abelian categories. The Cartan matrix enters in particular in the analogue of what in the semisimple case is the charge-conjugation partition function. As the latter is, for theories with compatible boundary conditions, often called the Cardy case, we refer to its generalization in the non-semisimple case as the *Cardy-Cartan modular invariant*.

Finally we describe how for any bulk state space of the type we consider, correlation functions of bulk fields for closed oriented world sheets of any genus can be found that are invariant under the natural action of the mapping class group on the relevant space of chiral blocks. The construction of these correlators is algebraically natural, once one has realized (see Proposition 3.4) that the monodromy derived from the braiding furnishes a natural action of a canonical Hopf algebra object in the category of chiral data on any representation of the chiral algebra.

2 The bulk state space

2.1 Holomorphic factorization

The central ingredient of chiral conformal field theory is a chiral symmetry algebra. Different mathematical notions formalizing this physical idea are available. Any such concept of a chiral algebra $\mathcal{V}$ must provide a suitable notion of representation category $\mathcal{R}ep(\mathcal{V})$, which should have, at least, the structure of a $\mathbb{C}$-linear abelian category.

For concreteness, we think about $\mathcal{V}$ as a vertex algebra with a conformal structure. A vertex algebra $\mathcal{V}$ and its representation category $\mathcal{R}ep(\mathcal{V})$ allow one to build a system of sheaves on moduli spaces of curves with marked points, called conformal blocks, or chiral blocks. These sheaves are endowed with a (projectively flat) connection. Their monodromies thus lead to (projective) representations of the fundamental groups of the moduli spaces, i.e. of the mapping class groups of surfaces. This endows the category $\mathcal{R}ep(\mathcal{V})$ with much additional structure. In particular, from the chiral blocks on the three-punctured sphere one extracts a monoidal structure on $\mathcal{R}ep(\mathcal{V})$, which formalizes the physical idea of operator product of (chiral) fields. Furthermore, from the monodromies one obtains natural transformations which encode a braiding as well as a twist. In this way one keeps enough information to be able to recover the representations of the mapping class groups from the category $\mathcal{R}ep(\mathcal{V})$. Moreover, if $\mathcal{R}ep(\mathcal{V})$ is also endowed with left and right dualities, the braiding and twist relate the two dualities to each other, and in particular they can fit together to the structure of a ribbon category. In this paper we assume that $\mathcal{R}ep(\mathcal{V})$ indeed is a ribbon category; there exist classes of vertex algebras which do have such a representation category and which are relevant to families of logarithmic conformal field theories.

The category $\mathcal{R}ep(\mathcal{V})$ – or any ribbon category $\mathcal{C}$ that is ribbon equivalent to it – is called the category of chiral data, or of Moore-Seiberg [MS1, BK1] data. For sufficiently nice chiral algebras $\mathcal{V}$ the number of irreducible representations is finite and $\mathcal{R}ep(\mathcal{V})$ carries the structure of a factorizable finite ribbon category. For the purposes of this paper we restrict our attention to the case that $\mathcal{R}ep(\mathcal{V})$ has this structure.

While chiral CFT is of much mathematical interest and also plays a role in modeling certain physical systems, like in the description of universality classes of quantum Hall fluids, the vast majority of physical applications of CFT involves full, local CFT. It is generally expected that a full CFT can be obtained from an underlying chiral theory by suitably "combining" holomorphic and anti-holomorphic chiral degrees of freedom or, in free field terminology, left- and right movers. Evidence for such a *holomorphic factorization* comes from CFTs that possess a Lagrangian description (see e.g. [Wi]). Conversely, the *postulate* of holomorphic factorization can be phrased in an elegant geometric way as the requirement that correlation functions on a surface Σ are specific sections in the chiral blocks associated to a double cover $\widehat{\Sigma}$ of Σ. These particular sections are demanded to be invariant under the action of the mapping class group of Σ, and to be compatible with sewing of surfaces.

These conditions only involve properties of the representations of mapping class groups that are remembered by the additional structure of the category $\mathcal{R}ep(\mathcal{V})$ of chiral data. Accordingly, to find and characterize solutions to these constraints it suffices (and is even appropriate) to work at the level of $\mathcal{R}ep(\mathcal{V})$ as an abstract factorizable ribbon category. We refer to [BK2] for details on how the vector bundles of chiral blocks, which form a complex analytic modular functor (in the terminology of [BK2]) can be recovered from representation theoretic data that correspond to a topological modular functor, and to [FRS3, Sects. 5 & 6.1] for a more detailed discussion of this relationship for the chiral blocks that appear in the simplest correlation functions, involving few points on a sphere.

In *rational* conformal field theories, for which the category $\mathcal{R}ep(\mathcal{V})$ has the structure of a (semisimple) modular tensor category, the problem of finding and classifying solutions to all these constraints has a very satisfactory solution (see e.g. [ScFR] for a review). In *logarithmic* CFTs, on the other hand, no evidence for holomorphic factorization is available from a Lagrangian formulation. Instead, in this paper we take holomorphic factorization as a starting point. Until recently, only few model-independent results for logarithmic CFTs were available. Here we will present a whole class of solutions for correlators of bulk fields, on orientable surfaces of any genus with any number of insertions. It is remarkable that, once relevant expressions, like e.g. sums over isomorphism classes of simple objects, that are suggestive in the semisimple case have been substituted with the right categorical constructions, we find a whole class of solutions which work very much in the same spirit as for semisimple theories.

In this contribution we first focus on the *bulk state space* F of the theory. Invoking the state-field correspondence, F is also called the space of bulk fields. A bulk field carries both holomorphic and anti-holomorphic degrees of freedom. When the former are expressed in terms of $\mathcal{C} \simeq \mathcal{R}ep(\mathcal{V})$, then for the latter one has to use the *reverse category* $\mathcal{C}^{\mathrm{rev}}$, which is obtained from $\mathcal{C}$ by inverting the braiding and the twist isomorphisms. As a consequence, a bulk field, and in particular the bulk state space F, is an object in the *enveloping category* $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, the Deligne tensor product of $\mathcal{C}^{\mathrm{rev}}$ and $\mathcal{C}$.

It should be appreciated that owing to the opposite braiding and twist in its two factors, the enveloping category is in many respects simpler than the category $\mathcal{C}$ of chiral data. This is a prerequisite for the possibility of having correlation functions that are local and invariant under the mapping class group of the world sheet. If $\mathcal{C}$ is semisimple and factorizable, then a mathematical manifestation of this simplicity of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ is the fact that its class in the Witt group [DMNO] of non-degenerate fusion categories vanishes.

The most direct way of joining objects of $\mathcal{C}^{\mathrm{rev}}$ and $\mathcal{C}$ to form bulk fields that comes to mind is to combine the 'same' objects in each factor, which in view of the distinction between $\mathcal{C}^{\mathrm{rev}}$ and $\mathcal{C}$ means that any object U of $\mathcal{C}$ is to be combined with its (right, say) dual $U^\vee$ in $\mathcal{C}^{\mathrm{rev}}$. When restricting to *simple* objects only, this idea results in the familiar expression

$$F = F_C := \bigoplus_{i \in \mathcal{I}} S_i^\vee \boxtimes S_i \tag{2.1}$$

for the bulk state space, where $(S_i)_{i \in \mathcal{I}}$ is a collection of representatives for the isomorphism classes of simple objects of $\mathcal{C}$.

In rational CFT, where the index set $\mathcal{I}$ is finite, the object (2.1) is known as the charge conjugation bulk state space, and its character

$$\chi_{F_C} = \sum_{i \in \mathcal{I}} \chi_i^* \, \chi_i \tag{2.2}$$

as the *charge conjugation modular invariant.* Moreover, it can be shown [FFFS2] that for any modular tensor category $\mathcal{C}$ the function (2.2) is not only invariant under the action of the modular group, as befits the torus partition function of a CFT, but that it actually appears as part of a consistent full CFT, and hence F_C as given by (2.1) is indeed a valid bulk state space.

2.2 The bulk state space as a coend

It is, however, not obvious how the formula (2.1), which involves only simple objects, relates to the original idea of combining *every* object of $\mathcal{C}$ with its conjugate. Fortunately there is a purely categorical construction by which that idea can be made precise, namely via the notion of a *coend*. Basically, the coend provides the proper concept of summing over *all objects* of a category, namely doing so in such a manner that at the same time *all relations* that exist between objects are accounted for, meaning that all morphisms between objects are suitably divided out.

The notion of a coend can be considered for any functor G from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to any other category $\mathcal{D}$. That it embraces also the morphisms of $\mathcal{D}$ manifests itself in that the coend of G is not just an object D of $\mathcal{D}$, but it also comes with a *dinatural family* of morphisms from $G(U,U)$ to D (but still one commonly refers also to the object D itself as the coend of G). For details about coends and dinatural families we refer to Appendix A.1. In the case at hand, $\mathcal{D}$ is the enveloping category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, and the coend of our interest is

$$\int^U U^\vee \boxtimes U \,\in\, \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \,. \tag{2.3}$$

Whether this coend indeed exists as an object of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ depends on the category $\mathcal{C}$, but if it exists, then it is unique. If $\mathcal{C}$ is cocomplete, then the coend exists. In particular, the coend does indeed exist for all finite tensor categories (to be defined at the beginning of Subsection 2.4). This includes all modular tensor categories, and thus all categories of chiral data that appear in rational CFT. Moreover, a modular tensor category $\mathcal{C}$ is semisimple, so that accounting for all morphisms precisely amounts to disregarding any non-trivial direct sums, and thus to restricting the summation to simple objects. This yields

$$\int^U U^\vee \boxtimes U = F_{\mathcal{C}} \tag{2.4}$$

with $F_{\mathcal{C}}$ as given by (2.1). In short, once we realize the physical idea of summing over all states in the proper way that is suggested by elementary categorical considerations, it does explain the ansatz (2.1) for the bulk state space.

This result directly extends to all bulk state spaces that are of *automorphism type*. Namely, for any ribbon automorphism of $\mathcal{C}$, i.e. any autoequivalence ω of $\mathcal{C}$ that is compatible with its ribbon structure, we can consider the coend

$$F_\omega := \int^U U^\vee \boxtimes \omega(U) \,. \tag{2.5}$$

In the rational case this gives

$$F_\omega = \bigoplus_{i\in\mathcal{I}} S_i^\vee \boxtimes \omega(S_i) \,, \tag{2.6}$$

with associated torus partition function

$$\chi_{F_\omega} = \sum_{i\in\mathcal{I}} \chi_i^* \, \chi_{\bar\omega(i)} \tag{2.7}$$

where $\bar\omega$ is the permutation of the index set $\mathcal{I}$ for which $S_{\bar\omega(i)}$ is isomorphic to $\omega(S_i)$.

2.3 The bulk state space as a center

In rational CFT, the object (2.1) of $\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$ can also be obtained by another purely categorical construction from the category $\mathcal{C}$, and one may hope that this again extends beyond the rational case. To explain this construction, we need the notions of the *monoidal center* (or Drinfeld center) $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$ and of the *full center* of an algebra. $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category; its objects are pairs (U,z) consisting of objects and of so-called half-braidings of $\mathcal{C}$. The full center $Z(A)$ of an algebra $A\in\mathcal{C}$ is a uniquely determined commutative algebra in $\mathcal{Z}(\mathcal{C})$ whose half-braiding is in a suitable manner compatible with its multiplication. For details about the monoidal center of a category and the full center of an algebra see Appendix A.2. If $\mathcal{C}$ is modular, then [Mü, Thm. 7.10] the monoidal center is monoidally equivalent to the enveloping category,

$$\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C} \,\simeq\, \mathcal{Z}(\mathcal{C})\,, \tag{2.8}$$

so that in particular $Z(A)$ is an object in $\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$.

Now any monoidal category $\mathcal{C}$ contains a distinguished algebra object, namely the tensor unit $\mathbf{1}$, which is even a symmetric Frobenius algebra (with all structural morphisms being identity morphisms). We thus know that $Z(\mathbf{1})$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$. If $\mathcal{C}$ is modular, then this algebra is an object in $\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$, and gives us the bulk state space (2.1),

$$Z(\mathbf{1}) = F_{C}\,. \tag{2.9}$$

Moreover, let us assume for the moment that a rational CFT can be consistently formulated on any world sheet, including world sheets with boundary, with non-degenerate two-point functions of bulk fields on the sphere and of boundary fields on the disk. It is known [FFRS2] that *any* bulk state space of such a CFT is necessarily of the form

$$F = Z(A) \tag{2.10}$$

for some simple symmetric special Frobenius algebra A in $\mathcal{C}$. Further, the object $Z(A)$ decomposes into a direct sum of simple objects of $\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$ according to [FRS2]

$$F = Z(A) = \bigoplus_{i,j\in\mathcal{I}} \big(S_i^\vee\boxtimes S_j\big)^{\oplus Z_{ij}(A)} \tag{2.11}$$

with the multiplicities $Z_{ij}(A)$ given by the dimensions

$$Z_{ij}(A) := \dim_{\Bbbk}\!\big(\mathrm{Hom}_{A|A}(S_i^\vee\otimes^+ A\otimes^- S_j, A)\big) \tag{2.12}$$

of morphisms of A-bimodules. Here the symbols $\otimes^\pm$ indicate the two natural ways of constructing induced A-bimodules with the help of the braiding of $\mathcal{C}$. (In case A is an Azumaya algebra, this yields the automorphism type bulk state spaces (2.6), for which $Z_{ij}(A)=\delta_{\bar{j},\bar\omega(i)}$, see formula (2.7)).

The result (2.10) implies further that F is not just an object of $\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$, but in addition carries natural algebraic structure:

Proposition 2.1. [RFFS, Prop. 3]
For A a symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$, the full center $Z(A)$ is a commutative symmetric Frobenius algebra in $\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$.

In field theoretic terms, the multiplication on F describes the operator product of bulk fields, while the non-degenerate pairing which supplies the Frobenius property reflects the non-degeneracy of the two-point functions of bulk fields on the sphere.

2.4 Logarithmic CFT and finite ribbon categories

It is natural to ask whether the statements about rational CFT collected above, and specifically Proposition 2.1, have a counterpart beyond the rational case.

Of much interest, and particularly tractable, is the class of non-semisimple theories that have been termed *logarithmic* CFTs. It appears that the categories of chiral data of such CFTs, while not being semisimple, still share crucial finiteness properties with the rational case. A relevant concept is the one of a *finite tensor category*; this is [EO] an abelian rigid monoidal category with finite-dimensional morphism spaces and finite set $\mathcal{I}$ of isomorphism classes of simple objects, such that each simple object has a projective cover and the Jordan-Hölder series of every object has finite length.

Unless specified otherwise, in the sequel $\mathcal{C}$ will be assumed to be a (strict) finite tensor category with a ribbon structure or, in short, a *finite ribbon category.* For all such categories the coend $F_{\mathcal{C}} = \int^U U^\vee \boxtimes U$ exists [FSS1].

Remark 2.2. Finiteness of $\mathcal{I}$ and existence of projective covers are, for instance, manifestly assumed in the conjecture [QS, GR] that the bulk state space of charge conjugation type decomposes as a left module over a single copy of the chiral algebra $\mathcal{V}$ as

$$F_{\mathcal{C}} \rightsquigarrow \bigoplus_{i\in\mathcal{I}} P_i^\vee \otimes_{\mathbb{C}} S_i \,, \tag{2.13}$$

where P_i the projective cover of the simple $\mathcal{V}$-module S_i.
Note, however, that the existence of such a decomposition does by no means allow one to deduce the structure of $F_{\mathcal{C}}$ as an object of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$. In particular there is no reason to expect that simple or projective objects of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ appear as direct summands of $F_{\mathcal{C}}$, nor that $F_{\mathcal{C}}$ is a direct sum of '$\boxtimes$-factorizable' objects $U \boxtimes V$ of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$. We do *not* make any such assumption; our working hypothesis is solely that the bulk state space $F_{\mathcal{C}}$ for a logarithmic CFT can still be described as the coend (2.3).

We are now going to establish an algebra structure on the coend (2.3), without assuming semisimplicity. As already pointed out, the coend is not just an object, but an object together with a dinatural family. For the bulk state space $F_{\mathcal{C}} = \int^U U^\vee \boxtimes U$ we denote this family of morphisms by $\imath^\circ$ and its members by

$$\imath^\circ_U : \quad U^\vee \boxtimes U \to F_{\mathcal{C}} \tag{2.14}$$

for $U \in \mathcal{C}$. The braiding of $\mathcal{C}$ is denoted by $c = (c_{U,V})$. We also need the canonical isomorphims that identify, for $U, V \in \mathcal{C}$, the tensor product of the duals of U and V with the dual of $V \otimes U$; we denote them by

$$\gamma_{U,V} : \quad U^\vee \otimes V^\vee \xrightarrow{\cong} (V \otimes U)^\vee \,. \tag{2.15}$$

Now we introduce a morphism $m_{F_{\mathcal{C}}}$ from $F_{\mathcal{C}} \otimes F_{\mathcal{C}}$ to $F_{\mathcal{C}}$ by setting

$$m_{F_{\mathcal{C}}} \circ (\imath^\circ_U \otimes \imath^\circ_V) := \imath^\circ_{V\otimes U} \circ (\gamma_{U,V} \boxtimes c_{U,V}) \tag{2.16}$$

for all $U, V \in \mathcal{C}$. This family of morphisms from $(U^\vee \boxtimes U) \otimes (V^\vee \boxtimes V) = (U^\vee \otimes V^\vee) \boxtimes (U \otimes V)$ to $F_{\mathcal{C}}$ is dinatural both in U and in V and thereby determines $m_{F_{\mathcal{C}}}$ completely, owing to the universal property of coends.

Proposition 2.3.

(i) *The morphism* (2.16) *endows the object* F_C *with the structure of an* (*associative, unital*) *algebra in* $\mathcal{C}^{\rm rev}\boxtimes\mathcal{C}$.

(ii) *The multiplication* m_{F_C} *of the algebra* F_C *is commutative.*

Proof. (i) For $U,V,W\in\mathcal{C}$ we have

$$\begin{aligned} & m_{F_C}\circ(id_{F_C}\otimes m_{F_C})\circ(\imath_U^\circ\otimes\imath_V^\circ\otimes\imath_W^\circ) \\ &\qquad\qquad = \imath^\circ_{W\otimes V\otimes U}\circ(\gamma_{V\otimes U,W}\boxtimes c_{V\otimes U,W})\circ[(\gamma_{U,V}\boxtimes c_{U,V})\otimes id_{W^\vee\boxtimes W}] \qquad \text{and} \\ & m_{F_C}\circ(m_{F_C}\otimes id_{F_C})\circ(\imath_U^\circ\otimes\imath_V^\circ\otimes\imath_W^\circ) \\ &\qquad\qquad = \imath^\circ_{W\otimes V\otimes U}\circ(\gamma_{U,W\otimes V}\boxtimes c_{U,W\otimes V})\circ[id_{U^\vee\boxtimes U}\otimes(\gamma_{V,W}\boxtimes c_{V,W})]\,. \end{aligned} \tag{2.17}$$

Using the braid relation

$$c_{V\otimes U,W}\circ(c_{U,V}\otimes id_W) = c_{U,W\otimes V}\circ(id_U\otimes c_{V,W}) \tag{2.18}$$

and the obvious identity

$$\gamma_{V\otimes U,W}\circ(\gamma_{U,V}\otimes id_{W^\vee}) = \gamma_{U,W\otimes V}\circ(id_{U^\vee}\otimes\gamma_{V,W}) \tag{2.19}$$

in $\mathrm{Hom}_{\mathcal{C}}(U^\vee\otimes V^\vee\otimes W^\vee,(W\otimes V\otimes U)^\vee)$, it follows that for any triple U,V,W the two morphisms in (2.17) coincide, and thus

$$m_{F_C}\circ(id_{F_C}\otimes m_{F_C}) = m_{F_C}\circ(m_{F_C}\otimes id_{F_C})\,. \tag{2.20}$$

This shows associativity. Unitality is easy; the unit morphism is given by

$$\eta_{F_C} = \imath_{\mathbf{1}}^\circ \ \in \mathrm{Hom}_{\mathcal{C}^{\rm rev}\boxtimes\mathcal{C}}(\mathbf{1}\boxtimes\mathbf{1},F_C)\,. \tag{2.21}$$

(ii) Denote by $c^{\mathcal{C}^{\rm rev}\boxtimes\mathcal{C}}$ the braiding in $\mathcal{C}^{\rm rev}\boxtimes\mathcal{C}$. We have

$$\begin{aligned} m_{F_C}\circ c^{\mathcal{C}^{\rm rev}\boxtimes\mathcal{C}}_{F_C,F_C}\circ(\imath_U^\circ\otimes\imath_V^\circ) &= m_{F_C}\circ(\imath_V^\circ\otimes\imath_U^\circ)\circ(c^{-1}_{V^\vee,U^\vee}\boxtimes c_{U,V}) \\ &= \imath^\circ_{U\otimes V}\circ(\gamma_{V,U}\boxtimes c_{V,U})\circ(c^{-1}_{V^\vee,U^\vee}\boxtimes c_{U,V}) \\ &= \imath^\circ_{U\otimes V}\circ[(\gamma_{V,U}\circ c^{-1}_{V^\vee,U^\vee})\boxtimes(c_{V,U}\circ c_{U,V})] \\ &= \imath^\circ_{V\otimes U}\circ[\gamma_{U,V}\boxtimes(c^{-1}_{V,U}\circ c_{V,U}\circ c_{U,V})] \ = m_{F_C}\circ(\imath_U^\circ\otimes\imath_V^\circ)\,. \end{aligned} \tag{2.22}$$

Here the crucial step is the fourth equality, in which the dinaturalness property of $\imath^\circ$ (together with $(c_{U,V})^\vee = c_{U^\vee,V^\vee}$) is used. We conclude that $m_{F_C}\circ c^{\mathcal{C}^{\rm rev}\boxtimes\mathcal{C}}_{F_C,F_C} = m_{F_C}$, i.e. the algebra (F_C,m_{F_C},η_{F_C}) is commutative.

Let us also point out that working with a non-strict monoidal structure would make the formulas appearing here more lengthy, but the proof would carry over easily. □

2.5 The coend as a bimodule of a factorizable Hopf algebra

Proposition 2.3 is about as far as we can get, for now, for general finite ribbon categories. To obtain a stronger result we specialize to a particular subclass, consisting of categories $\mathcal{C}$ that are equivalent to the category H-Mod of (finite-dimensional left) modules over a *factorizable Hopf algebra* H. Such an algebra is, in short, a finite-dimensional Hopf algebra $(H, m, \eta, \Delta, \varepsilon, \mathrm{s})$ (over an algebraically closed field $\Bbbk$ of characteristic zero) that is endowed with an R-matrix $R \in H \otimes_{\Bbbk} H$ and a ribbon element $v \in H$ and for which the monodromy matrix $Q = R_{21} \cdot R$ is non-degenerate. Some more details about this class of algebras are supplied in Appendix A.4.

It is worth pointing out that categories belonging to this subclass which are relevant to CFT models are are well known, namely [DPR, CGR, FFSS] the semisimple representation categories of Drinfeld doubles of finite groups. Logarithmic CFTs for which the category of chiral data is believed to be at least very close to the type of category considered here are the $(1, p)$ triplet models, see e.g. [FGST1, FGST2, NT, TW, RGW].

For H a finite-dimensional ribbon Hopf algebra, the category H-Mod carries a natural structure of finite ribbon category. The monoidal structure (which again we tacitly take to be strictified) and dualities precisely require the algebra A to be Hopf: the tensor product is obtained by pull-back of the H-action along the coproduct Δ, the tensor unit is $\mathbf{1} = (\Bbbk, \varepsilon)$, and left and right dualities are obtained from the duality for $\mathcal{V}ect_{\Bbbk}$ with the help of the antipode. The braiding c on H-Mod is given by the action of the R-matrix composed with the flip map τ, while the twist θ is provided by acting with the inverse v^{-1} of the ribbon element.

In a fully analogous manner one can equip the category H-Bimod of finite-dimensional H-bimodules with the structure of a finite ribbon category as well: Pulling back both the left and right H-actions along Δ gives again a tensor product. Explicitly, the tensor product of H-bimodules (X, ρ_X, ϱ_X) and (Y, ρ_Y, ϱ_Y) is the tensor product over $\Bbbk$ of the underlying $\Bbbk$-vector spaces X and Y together with left and right actions of H given by

$$\begin{aligned} \rho_{X\otimes Y} &:= (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (\Delta \otimes id_X \otimes id_Y) \qquad \text{and} \\ \varrho_{X\otimes Y} &:= (\varrho_X \otimes \varrho_Y) \circ (id_X \otimes \tau_{Y,H} \otimes id_H) \circ (id_X \otimes id_Y \otimes \Delta)\,. \end{aligned} \tag{2.23}$$

The tensor unit is the one-dimensional vector space $\Bbbk$ with both left and right H-action given by the counit, $\mathbf{1}_{H\text{-Bimod}} = (\Bbbk, \varepsilon, \varepsilon)$. A braiding c on the so obtained monoidal category is obtained by composing the flip map with the action of the R-matrix R from the right and the action of its inverse R^{-1} from the left, and a twist θ is provided by

$$\theta_X = \rho \circ (id_H \otimes \varrho) \circ (v \otimes id_X \otimes v^{-1})\,, \tag{2.24}$$

i.e. by acting with the ribbon element v from the left and with its inverse from the right. (For a visualization of the braiding and the twist isomorphisms in terms of the graphical calculus for the symmetric monoidal category $\mathcal{V}ect_{\Bbbk}$ see formulas (3.3) and (4.19), respectively, of [FSS1].)

The category H-Bimod with this structure of ribbon category is of interest to us because it can be shown [FSS1, App. A.2] to be braided equivalent to $(H \otimes_{\Bbbk} H^{\mathrm{op}})$-Mod and thus to the enveloping category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} = H\text{-Mod}^{\mathrm{rev}} \boxtimes H\text{-Mod}$. We will henceforth identify the enveloping category with H-Bimod and present our results in the language of H-bimodules. In particular we think of the coend $F_{\mathcal{C}} = \int^U U^\vee \boxtimes U$ as an H-bimodule; we find

Theorem 2.4. [FSS1, Prop. A.3]
The coend $\int^U U^\vee \boxtimes U$ *in the category* H-Bimod *is the* coregular bimodule*, that is, the vector space* $H^* = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ *dual to* H *endowed with the duals of the left and right regular* H*-actions, i.e.*

$$\begin{aligned} \rho_{F_C} &= (d_H^{\Bbbk} \otimes id_{H^*}) \circ (id_{H^*} \otimes m \otimes id_{H^*}) \circ (id_{H^*} \otimes \mathrm{s} \otimes b_H^{\Bbbk}) \circ \tau_{H,H^*} \qquad \text{and} \\ \reflectbox{ρ}_{F_C} &= (d_H^{\Bbbk} \otimes id_{H^*}) \circ (id_{H^*} \otimes m \otimes id_{H^*}) \circ (id_{H^*} \otimes id_H \otimes \tau_{H^*,H}) \circ (id_{H^*} \otimes b_H^{\Bbbk} \otimes \mathrm{s}^{-1}) \,. \end{aligned} \tag{2.25}$$

together with the dinatural family of morphisms given by

$$\imath_U^\circ := \big[(d_U^{\Bbbk} \circ (id_{U^*} \otimes \rho_U)) \otimes id_{H^*} \big] \circ \big[id_{U^*} \otimes ((\tau_{U,H^*} \otimes id_{H^*}) \circ (id_U \otimes b_{H^*}^{\Bbbk})) \big] \tag{2.26}$$

for any H*-module* (U, ρ_U)*.*

Here $d^{\Bbbk}$ and $b^{\Bbbk}$ are the evaluation and coevaluation maps for the duality in $\mathcal{V}ect_{\Bbbk}$, respectively. Thus in particular (2.26) describes $\imath_U^\circ$ in the first place only as a linear map from $U^* \otimes_{\Bbbk} U$ to H^*. But it can be checked [FSS1, Lemma A.2] that if $H^* = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ is given the structure of the coregular H-bimodule and $U^* \otimes_{\Bbbk} U$ the H-bimodule structure

$$(U^* \otimes_{\Bbbk} U, \rho, \reflectbox{ρ}) := \big(U^* \otimes_{\Bbbk} U \,,\, \rho_{U^\vee} \otimes id_U \,,\, id_{U^*} \otimes (\rho_U \circ \tau_{U,H} \circ (id_U \otimes \mathrm{s}^{-1})) \big) \tag{2.27}$$

that is implied by the equivalence between H-Mod$^{\mathrm{rev}} \boxtimes H$-Mod and H-Bimod, then $\imath_U^\circ$ is actually a morphism in H-Bimod.

Now due to our finiteness assumptions, H has an integral and cointegral. Denote by $\Lambda \in H$ and $\lambda \in H^*$ the integral and cointegral, respectively, normalized according to the convention (A.31). We henceforth denote the coend $\int^U U^\vee \boxtimes U$ again by F_C and set

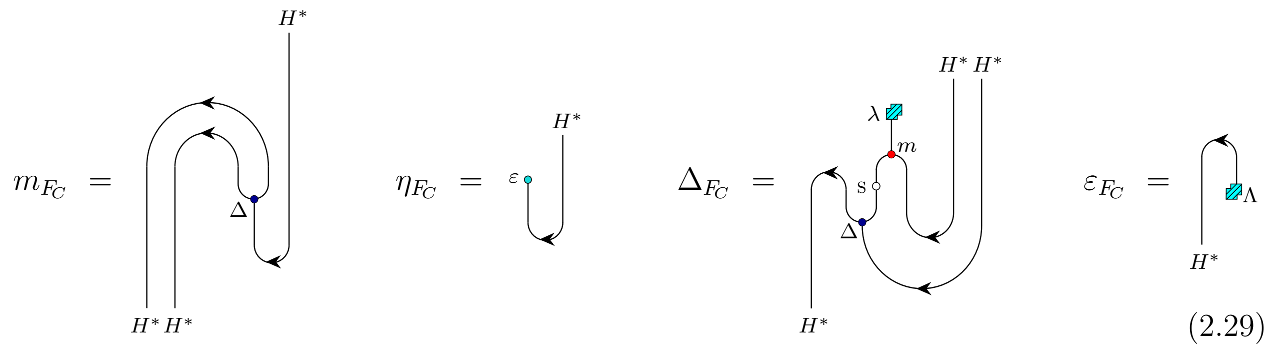
$$\begin{aligned} & m_{F_C} := \Delta^* : \quad F_C \otimes F_C \to F_C \,, \qquad\qquad \eta_{F_C} := \varepsilon^* : \quad \mathbf{1} \to F_C \,, \\ & \Delta_{F_C} := [(id_H \otimes (\lambda \circ m)) \circ (id_H \otimes \mathrm{s} \otimes id_H) \circ (\Delta \otimes id_H)]^* : \quad F_C \to F_C \otimes F_C \qquad \text{and} \\ & \varepsilon_{F_C} := \Lambda^* : \quad F_C \to \mathbf{1} \,. \end{aligned} \tag{2.28}$$

Again these are introduced as linear maps between the respective underlying vector spaces, but are actually morphisms of H-bimodules, as indicated. This way F_C is endowed with a Frobenius algebra structure, as befits the bulk state space of a conformal field theory:

Theorem 2.5. [FSS2, Thm. 2]
For H *a factorizable Hopf algebra, the bimodule morphisms* (2.28) *endow the coend* F_C *with a natural structure of a commutative symmetric Frobenius algebra with trivial twist in the ribbon category* H-Bimod.

We note that this assertion is in full agreement with the result of Proposition 2.3 which holds for general finite ribbon categories. Indeed, by implementing the explicit form (2.26) of the dinaturality morphisms, the product m_{F_C} that we defined in (2.16) for any finite ribbon category $\mathcal{C}$ reproduces the expression for m_{F_C} in (2.28), and likewise for η_{F_C}.

It will be convenient to work with the pictorial expressions for the maps (2.28) that are furnished by the graphical calculus for tensor categories. They are



(2.29)

(Such pictures are to be read from bottom to top.)

The result just described generalizes easily from F_C to the automorphism-twisted versions F_ω as defined in (2.5). Namely [FSS1, Prop. 6.1], for any Hopf algebra automorphism ω of H the H-bimodule that is obtained from the coregular bimodule by twisting the right H-action by ω, i.e.

$$F_\omega = \big(H^*, \rho_{F_C}, \varrho_{F_C} \circ (id_{H^*} \otimes \omega)\big)\,, \tag{2.30}$$

carries the structure of a Frobenius algebra, which is commutative, symmetric and has trivial twist. The structural morphisms for the Frobenius structure are again given by (2.28), i.e. as linear maps they are identical with those for F_C.

We also note if H is semisimple, then F_ω carries the structure of a Lagrangian algebra in the sense of [DMNO, Def. 4.6].

3 Handle algebras

Besides in the description of the bulk state space, there is another issue in CFT in which one needs to perform a sum over all states for the full local theory, namely when one wants to specify the relation between correlators on world sheets that are obtained from each other by [So] *sewing* (respectively, looking at the process from the other end, by *factorization*) as 'summing over all intermediate states'.

Let us first consider this relationship for rational CFT, and at the level of spaces of chiral blocks. A rational CFT furnishes a modular functor, and this functor is representable [BK2, Lemma 5.3.1]. Accordingly the space $V(E)$ of chiral blocks for a Riemann surface E is isomorphic to the morphism space $\mathrm{Hom}_{\mathcal{C}}(U_E, \mathbf{1})$ for a suitable object $U_E \in \mathcal{C}$; in particular, if E has genus zero and n ingoing (say) chiral insertions $U_1, U_2, ..., U_n$, then $U_E \cong U_1 \otimes U_2 \otimes \cdots \otimes U_n$. Let now the Riemann surface E^1 be obtained from the connected Riemann surface E° by removing two disjoint open disks $D_\pm$ and gluing the resulting boundary circles to each other, whereby the genus increases by 1. Then there is an isomorphism

$$\bigoplus_{i\in\mathcal{I}} V(E^\circ_{i\bar\imath}) \xrightarrow{\cong} V(E^1) \tag{3.1}$$

between the space $V(E^1)$ and the direct sum of all spaces $V(E^\circ_{i\bar\imath})$, where the surface $E^\circ_{i\bar\imath}$ is obtained from E° by introducing chiral insertions S_i and $S_{\bar\imath} \cong S_i^\vee$, respectively, in the disks $D_\pm$

(see e.g. [BK2, Def. 5.1.13(iv)]). In terms of morphism spaces of $\mathcal{C}$ this amounts to

$$V(E^1) \;\cong\; \bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(S_i^\vee \otimes S_i \otimes U_{E^\circ}, \mathbf{1}) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(L \otimes U_{E^\circ}, \mathbf{1})\,, \tag{3.2}$$

where on the right hand side we introduced the object

$$L := \bigoplus_{i\in\mathcal{I}} S_i^\vee \otimes S_i \;\in \mathcal{C}\,. \tag{3.3}$$

By induction, the space of chiral blocks for a genus-g surface with ingoing field insertions U_1, $U_2, , ... , U_n$ is then

$$V(E) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(L^{\otimes g} \otimes U_1 \otimes U_2 \otimes \cdots \otimes U_n, \mathbf{1})\,, \tag{3.4}$$

i.e. the object L appears to a tensor power given by the number of handles. Also, as we will point out soon, the object L of $\mathcal{C}$ carries a natural structure of a Hopf algebra internal to $\mathcal{C}$; it is therefore called the (chiral) *handle Hopf algebra*.

Invoking holomorphic factorization, the correlation function $\mathrm{Cor}(\Sigma)$ for a world sheet Σ is an element in the space of chiral blocks for the *complex double* $\widehat{\Sigma}$ of the world sheet. Taking Σ to be orientable and with empty boundary and all field insertions on Σ to be the whole bulk state space F, one has

$$\mathrm{Cor}(\Sigma) \,\in\, V(\widehat{\Sigma}) \;\cong\; \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes\,\mathcal{C}}(K^{\otimes g} \otimes F^{\otimes m}, \mathbf{1}) \tag{3.5}$$

with

$$K := L \boxtimes L = \bigoplus_{i,j\in\mathcal{I}} (S_i^\vee \boxtimes S_j^\vee) \otimes (S_i \boxtimes S_j) \;\in \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}\,. \tag{3.6}$$

K is called the *bulk handle Hopf algebra.* Note that each of the objects $S_i \boxtimes S_j$ with $i,j \in \mathcal{I}$ is simple and together they exhaust the set of simple objects of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, up to isomorphism.

It is tempting to think of the gluing of a handle to a Riemann surface as a means for inserting a complete set of intermediate states. In view of the isomorphism (3.4) then immediately the question arises why it is precisely the object L that does this job. And again the categorical notion of a coend turns out to provide the proper answer. Indeed, just like the bulk state space F_C, the objects $L \in \mathcal{C}$ and $K \in \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ can be recognized as the coends of suitable functors, namely as

$$L = \int^{U\in\mathcal{C}} U^\vee \otimes U \qquad\text{and}\qquad K = \int^{X\in\mathcal{C}^{\mathrm{rev}}\boxtimes\,\mathcal{C}} X^\vee \otimes X\,, \tag{3.7}$$

respectively (together with corresponding families of dinatural transformations, whose explicit form we do not need at this point).

Moreover, just like in the discussion of F_C this description remains valid beyond the semisimple setting: These coends exist not only when $\mathcal{C}$ is a modular tensor category, i.e. for rational CFT, but also for more general categories, and in particular for any finite tensor category $\mathcal{C}$. Note that here the statement for K is redundant, as it is just obtained from the from the one for L by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, and $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ inherits all relevant structure and properties from $\mathcal{C}$. This applies likewise to other issues, like e.g. the Hopf algebra structure on these objects, and accordingly we will usually refrain from spelling them out for L and K separately.

As already announced, we have

Theorem 3.1. [Ly1, Ke1]
For $\mathcal{C}$ a finite ribbon category, the coend $L = \int^{U\in\mathcal{C}} U^\vee \otimes U$ carries a natural structure of a Hopf algebra in $\mathcal{C}$. It has an integral $\Lambda_L \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, L)$ and a Hopf pairing $\varpi_L \in \mathrm{Hom}_{\mathcal{C}}(L\otimes L, \mathbf{1})$.

The structural morphisms of L as a Hopf algebra are given by

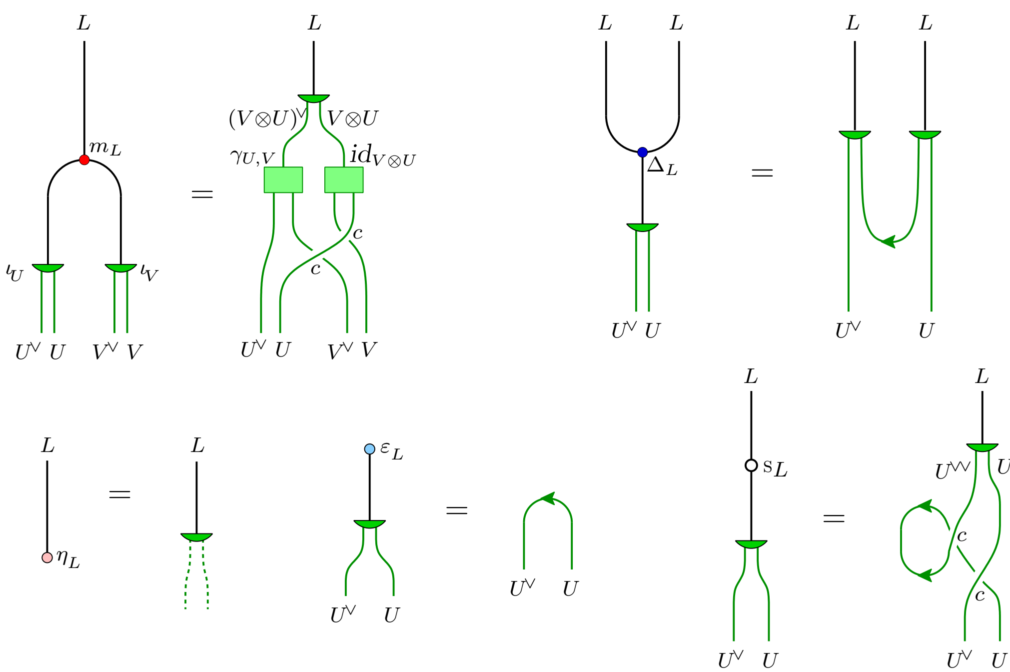
$$\begin{aligned}
& m_L \circ (\iota_U \otimes \iota_V) := \iota_{V\otimes U} \circ (\gamma_{U,V} \otimes id_{V\otimes U}) \circ (id_{U^\vee} \otimes c_{U,V^\vee\otimes V}) \,, \\
& \eta_L := \iota_{\mathbf{1}} \,, \\
& \Delta_L \circ \iota_U := (\iota_U \otimes \iota_U) \circ (id_{U^\vee} \otimes b_U \otimes id_U) \,, \\
& \varepsilon_L \circ \iota_U := d_U \,, \\
& \mathrm{s}_L \circ \iota_U := (d_U \otimes \iota_{U^\vee}) \circ (id_{U^\vee} \otimes c_{U^{\vee\vee},U} \otimes id_{U^\vee}) \circ (b_{U^\vee} \otimes c_{U^\vee,U}) \,,
\end{aligned} \tag{3.8}$$

and the Hopf pairing is

$$\varpi_L \circ (\iota_U \otimes \iota_V) := (d_U \otimes d_V) \circ \big[id_{U^\vee} \otimes (c_{V^\vee,U} \circ c_{U,V^\vee} \otimes id_V) \big] \,. \tag{3.9}$$

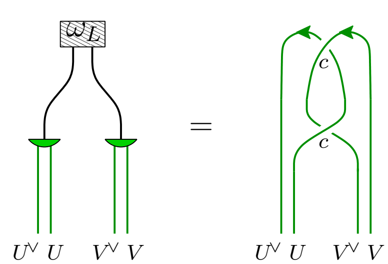
Here d and b are the evaluation and coevaluation morphisms for the (right) duality in H-Modand c is the braiding of H-Mod, while ι is the dinatural family of the coend L, and the isomorphisms $\gamma_{U,V}$ are the ones defined in (2.15).

In terms of graphical calculus in $\mathcal{C}$,



(3.10)

and



(3.11)

Remark 3.2.

(i) That the object L that is associated with the creation of handles carries a Hopf algebra structure is by no means a coincidence. Indeed, Hopf algebras are ubiquitous in constructions with three-dimensional cobordisms, and specifically the *handle*, i.e. a torus with an open disk removed, is a Hopf algebra 1-morphism in the bicategory $\mathcal{C}ob$ of three-dimensional cobordisms with corners [CrY, Ye]. Moreover, there exists a surjective functor from the braided monoidal category freely generated by a Hopf algebra object to $\mathcal{C}ob$ [Ke2].

(ii) The Hopf algebra $L = L(\mathcal{C})$ is directly associated with the category $\mathcal{C}$, and thereby with the CFT having $\mathcal{C}$ as its category of chiral data, and analogously for K: for given $\mathcal{C}$ there is a uniquely (up to isomorphism) determined chiral handle Hopf algebra, and likewise a unique bulk handle Hopf algebra. This is in contrast to the bulk state space: for given $\mathcal{C}$ there is typically more than one possibility. Specifically, in rational CFT the different bulk state spaces are in bijection with Morita classes of simple symmetric special Frobenius algebras in $\mathcal{C}$, see formula (2.10) above.

(iii) Via its integrals and Hopf pairing, the Hopf algebra L gives rise to three-manifold invariants as well as to representations of mapping class groups (see [Ly1, Ly2, Ke1, Vi], and [FS, Sects. 4.4 & 4.5] for an elementary introduction). Even though for non-semisimple $\mathcal{C}$ these cannot be normalized in such a way that they fit together to furnish a three-dimensional topological field theory, one may still hope that this hints at a close relationship with three-dimensional topology even in the non-semisimple case.

(iv) Obviously, the object L of $\mathcal{C}$ is obtained from the object $F_\mathcal{C}$ of $\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}$ by applying the functor that on $\boxtimes$-factorizable objects of $\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}$ acts as $U \boxtimes V \mapsto U \otimes V$. This functor is called the *diagonal restriction functor* in [Ly4].

It will be relevant to us that the algebra K acts (and coacts) on the object $F_\mathcal{C}$ of $\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}$, and in fact on any object of $\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}$. We formulate the relevant statements directly for objects in $\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}$; with K replaced by L, they apply analogously in $\mathcal{C}$.

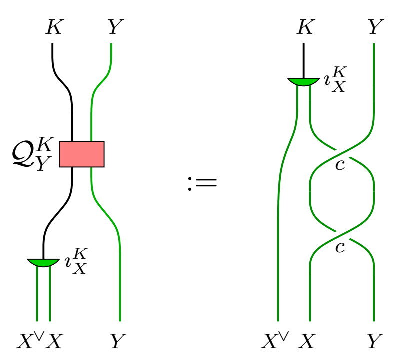
For $Y \in \mathcal{C}^{\rm rev} \boxtimes \mathcal{C}$ set

$$\delta_Y^K := (id_Y \otimes \imath_Y^K) \circ (b_Y \otimes id_Y) \;\in \mathrm{Hom}_{\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}}(Y, Y \otimes K) \tag{3.12}$$

as well as

$$\varkappa_Y^K := (\varepsilon_K \otimes id_Y) \circ \mathcal{Q}_Y^K \;\in \mathrm{Hom}_{\mathcal{C}^{\rm rev} \boxtimes \mathcal{C}}(K \otimes Y, Y)\,, \tag{3.13}$$

where in the latter formula the morphism $\mathcal{Q}_Y^K$ is the *partial monodromy* between K and Y, defined as



(3.14)

Proposition 3.3. [Ly2, Fig. 7]

For any object Y of a finite ribbon category $\mathcal{C}$, the morphism (3.12) *endows Y with the structure of a right K-comodule.*

Proposition 3.4. [FSS3, Rem. 2.3]

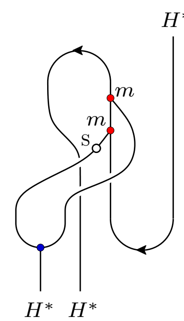
(i) *For any object Y of a finite ribbon category $\mathcal{C}$, the morphism* (3.13) *endows Y with the structure of a left K-module.*

(ii) *The module and comodule structures* (3.13) *and* (3.12) *fit together to the one of a left-right Yetter-Drinfeld module over K. This affords a fully faithful embedding of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ into the category of left-right Yetter-Drinfeld modules over K internal to $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$.*

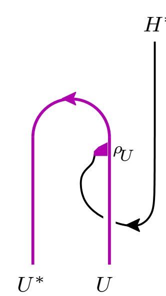
Since the crucial ingredient of $\varkappa_Y^K$ is a double braiding, we refer to $\varkappa_Y^K$ as the *partial monodromy action* of K on Y.

Remark 3.5. (i) The second part of Proposition 3.4 fits nicely with the result [Ye, Thm. 3.9] that every 1-morphism of the cobordism bicategory $\mathcal{C}ob$ carries a structure of left-right Yetter-Drinfeld module over the one-holed torus (compare Remark 3.2(i)).
(ii) The category $\mathcal{C}_L$ of L-modules in $\mathcal{C}$ is braided equivalent to the monoidal center $\mathcal{Z}(\mathcal{C})$, see Theorem 8.13 [BV].
(iii) The full subcategory $\mathcal{C}_L^{\mathcal{Q}}$ of $\mathcal{C}_L$ consisting of the modules $(U, \varkappa_U^L)$ for $U \in \mathcal{C}$, with $\varkappa_U^L$ defined analogously as in (3.13), is a monoidal subcategory: by the definition of Δ_L and the functoriality of the braiding, one has $\varkappa_{U\otimes V}^L = (\varkappa_U^L \otimes \varkappa_V^L) \circ \Delta_L$.

Next let us specialize to the situation considered in Section 2.5, i.e. that the finite ribbon category $\mathcal{C}$ is equivalent to the category H-Mod for H a factorizable Hopf algebra. Then the chiral handle Hopf algebra $L \in H$-Mod is the vector space H^* dual to H endowed with the coadjoint H-action

$\rho_\triangleright :=$  (3.15)

and the members of the dinatural family $\imath^L$ are the linear maps (see [Ke1, Lemma 3] and [Vi, Sect. 4.5])

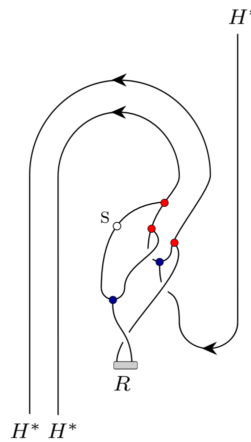
$\imath_U^L =$  (3.16)

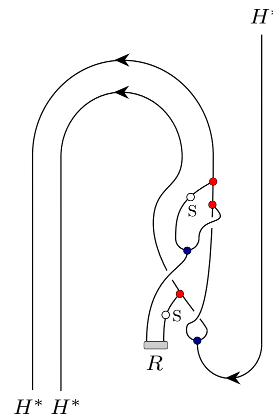
When expressed in terms of vector space elements, these morphisms are nothing but the matrix elements of left multiplication in H.

The unit, counit and coproduct of the Hopf algebra L, as given by (3.10) for the case of general finite ribbon categories, now read

$$\begin{aligned} \eta_L &= (\varepsilon_H)^* \equiv (\varepsilon_H \otimes id_{H^*}) \circ b_H^{\Bbbk} \,, \\ \varepsilon_L &= (\eta_H)^* \equiv d_H^{\Bbbk} \circ (id_{H^*} \otimes \eta_H) \qquad \text{and} \qquad \Delta_L = (m_H^{\mathrm{op}})^* \,, \end{aligned} \tag{3.17}$$

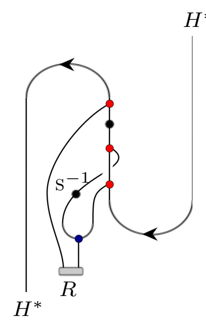
while two equivalent descriptions of the product are

$m_L \; =$ 

$=$ 

(3.18)

and the antipode is

$\mathrm{S}_{\triangleright\triangleleft} \; =$ 

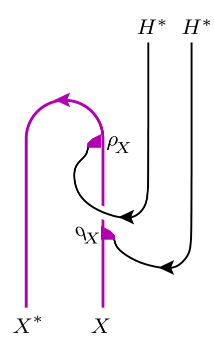
(3.19)

Further, our finiteness assumptions imply that L now comes with an integral and a cointegral, given by

$$\Lambda_L = \lambda^* \qquad \text{and} \qquad \lambda_L = \Lambda^*, \tag{3.20}$$

respectively. Both of them are two-sided, even though the cointegral λ of H in general is only a right cointegral.

Similarly, identifying, as in Section 2.5, the enveloping category $H\text{-Mod}^{\mathrm{rev}} \boxtimes H\text{-Mod}$ with the category H-Bimod of bimodules (with the ribbon structure presented there), the coend $K \in H\text{-Bimod}$ is the *coadjoint bimodule*, that is, the tensor product $H^* \otimes_{\Bbbk} H^*$ of two copies of the dual space H^* endowed with the coadjoint left H-action (3.15) on the first tensor factor and with the coadjoint right H-action on the second factor, with dinatural family

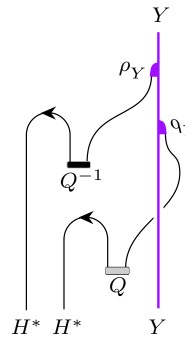
$\imath_X^K \; :=$ 

(3.21)

for any H-bimodule $X = (X, \rho_X, \varrho_X)$. The structural morphisms of K as a Hopf algebra and its integral and cointegral are straightforward analogues of the expression given for L above; for explicit formulas we refer to (A.32) – (A.36) of [FSS1].

The partial monodromy action (3.13) of K on an H-bimodule $(Y, \rho_Y, \text{ϙ}_Y)$ is given in terms of the monodromy matrix Q and its inverse by

$$\varkappa_Y^K = \qquad (3.22)$$



i.e. the natural K-action is nothing but the H-bimodule action composed with variants of the Drinfeld map (A.26).

Remark 3.6.

(i) For H a ribbon Hopf algebra, the Hopf pairing (3.11) of the handle Hopf algebra L is non-degenerate iff H is factorizable. It is thus natural to call more generally a finite ribbon category $\mathcal{C}$ *factorizable* iff the Hopf pairing (3.11) of $L(\mathcal{C})$ is non-degenerate.

(ii) Factorizability implies e.g. that the integral of $L(\mathcal{C})$ is two-sided and that $L(\mathcal{C})$ also has a two-sided cointegral (Prop. 5.2.10 and Cor. 5.2.11 of [KL]). If $\mathcal{C}$ is semisimple, then being factorizable is equivalent to being modular. Thus factorizability may be seen as a generalization of modularity to non-semisimple categories (the authors of [KL] even use the qualification 'modular' in place of 'factorizable').

(iii) A quasitriangular Hopf algebra H is factorizable iff its Drinfeld double $D(H)$ is isomorphic, in a particular manner, to a two-cocycle twist of the tensor product Hopf algebra $H \otimes H$ [Sc, Thm. 4.3], and thus [ENO, Rem. 4.3] iff the functor that acts on objects $U \boxtimes V$ of the enveloping category of H-Mod as

$$U \boxtimes V \longmapsto (U \otimes V, z_{U\otimes V}) \qquad \text{with} \qquad z_{U\otimes V}(W) := (c_{U,W} \otimes id_V) \circ (id_U \otimes c_{W,V}^{-1}) \qquad (3.23)$$

furnishes a monoidal equivalence

$$H\text{-Mod}^{\text{rev}} \boxtimes H\text{-Mod} \xrightarrow{\simeq} \mathcal{Z}(H\text{-Mod}) \qquad (3.24)$$

between the enveloping category and the monoidal center of H-Mod.

(iv) Now the bulk state space in conformal field theory is an object in $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$. Thus if we want to be able to describe the bulk state space, in line with the semisimple case (2.11), as a full center, we should better be allowed to regard the full center $Z(A)$ of an algebra A in a factorizable finite ribbon category $\mathcal{C}$, which by definition is an object in $\mathcal{Z}(\mathcal{C})$, also as an object in $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$, and thus want $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$ to be monoidally equivalent. [1]

We do not know whether this requirement is satisfied for *all* factorizable finite ribbon categories. On the other hand, for the condition to be satisfied it is certainly not required that $\mathcal{C}$ is ribbon equivalent to H-Mod for a ribbon Hopf algebra H. Specifically, the notion of factorizability can

[1] In fact, in [ENO] this property is used to *define* factorizability for braided monoidal categories that are not necessarily ribbon.

be extended from Hopf algebras to weak Hopf algebras [NTV, Def. 5.11], and again a weak Hopf algebra H is factorizable iff the functor (3.23) is a monoidal equivalence [ENO, Rem. 4.3]. This covers in particular the case of all semisimple $\mathcal{C}$, because every semisimple finite tensor category is equivalent to the representation category of some semisimple finite-dimensional weak Hopf algebra [Os, Thm. 4.1 & Rem. 4.1(iv)].

4 The torus partition function

4.1 The partition function as a character

By definition, the torus partition function Z of a CFT, whether rational or not, is the character of the bulk state space F. Here the term character refers to F as a module over the tensor product of the left and right copies of the chiral algebra $\mathcal{V}$. That is, the character is a real-analytic function of the modulus τ of the torus, which takes values in the complex upper half plane, and it is the generating function for dimensions of homogeneous subspaces of $\mathcal{V}\otimes_{\mathbb{C}}\mathcal{V}$-modules. As such, Z is a sum of characters of simple $\mathcal{V}\otimes_{\mathbb{C}}\mathcal{V}$-modules, even though F is, in general, not fully reducible.

Referring to the chiral algebra $\mathcal{V}$ is not necessary, though. Rather, as for our purposes we are allowed to work at the level of $\mathcal{R}ep(\mathcal{V})$ as an abstract factorizable ribbon category, we can regard F just as an object of $\mathcal{C}^{\text{rev}}\boxtimes\mathcal{C}\simeq\mathcal{R}ep(\mathcal{V}\otimes_{\mathbb{C}}\mathcal{V})$. Indeed, we know from (3.5) that the torus partition function – the zero-point correlator on the torus T – is an element of the space

$$V(\mathrm{T}\sqcup-\mathrm{T}) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(L,\mathbf{1})\otimes_{\mathbb{C}}\mathrm{Hom}_{\mathcal{C}}(L,\mathbf{1}) \;\cong\; \mathrm{Hom}_{\mathcal{C}^{\text{rev}}\boxtimes\,\mathcal{C}}(K,\mathbf{1}) \tag{4.1}$$

of chiral blocks. Now the morphism space $\mathrm{Hom}_{\mathcal{C}^{\text{rev}}\boxtimes\,\mathcal{C}}(K,\mathbf{1})$ contains in particular the *characters* of the algebra K. An immediate conjecture for the torus partition function is thus the character χ_F^K of the bulk state space as a module (with action $\varkappa_F^K$ as defined in (3.13)) over the bulk handle Hopf algebra K. As we will see in Section 5 below, χ_F^K is in fact just the particular member $(g,n)=(1,0)$ of a family of morphisms that are natural candidates for correlation functions at any genus g and with any number n of bulk insertions.

In this description the term character now refers to F as a K-module. The notion of the character of a module over an associative $\Bbbk$-algebra is standard and is explained in detail in Appendix A.3. For an algebra A in a monoidal category $\mathcal{C}$ one can set up representation theory in much the same way as for a $\Bbbk$-algebra, i.e. for an algebra in $\mathcal{V}ect_{\Bbbk}$. The notion of character then still makes sense provided that $\mathcal{C}$ is sovereign, which for the categories of our interest is the case. [2] Concretely, the formula (A.14) for the character of a module M over an algebra A in $\mathcal{V}ect_{\Bbbk}$ gets modified to

$$\chi_M^A = \mathrm{tr}_M(\rho) = \tilde{d}_M\circ(\rho\otimes\pi_M)\circ(id_A\otimes b_M) \;\in \mathrm{Hom}_{\mathcal{C}}(A,\mathbf{1})\,, \tag{4.2}$$

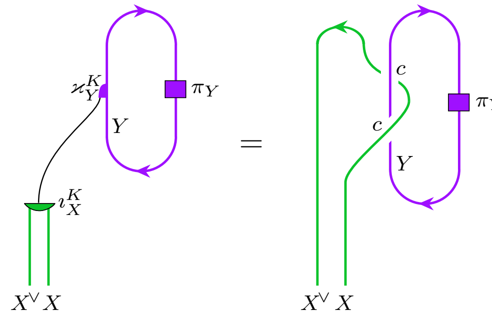
with $\pi_M\colon M^\vee\to{}^\vee\!M$ the sovereignty isomorphism between the right and left duals of M.

In the case at hand the relevant algebra is the bulk handle Hopf algebra K, and its action

[2] Any ribbon category is sovereign, i.e. (see e.g. Def. 2.7 of [Dr]) the left and right dualities are connected by a monoidal natural isomorphism.

is given by (3.13). We thus have

$$\chi_Y^K \circ \imath_X^K \;=\; \qquad\qquad (4.3)$$



for any K-module $(Y, \varkappa_Y^K)$ in $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, and analogously for the character χ_U^L of an L-module $(U, \varkappa_U^L)$ in $\mathcal{C}$.

Remark 4.1. Since L and K are Hopf algebras, there are natural notions of left and right dual modules. The character of the L-module $U^\vee$ right dual to $U = (U, \varkappa_U^L)$ is given by the same morphism as the one for χ_U^L, except that the braidings in (4.3) get replaced by inverse braidings.

It is worth being aware that so far the coend $F = F_C$ (2.4) (respectively, $F = F_\omega$ (2.5)) is only conjecturally the bulk state space of a conformal field theory, and similarly the morphism χ_F^K is merely a candidate for the torus partition function Z of that CFT. But just like we could verify that, for the case $\mathcal{C} \simeq H$-Mod (and $F = F_\omega$ for any ribbon automorphism ω of H) the coend has the desired properties of being a commutative symmetric Frobenius algebra, we will see below that in this case χ_F^K has the desired property of being a bilinear combination of suitable chiral characters with non-negative integral coefficients. Moreover, these coefficients turn out to be quantities naturally associated with the category $\mathcal{C}$.

The status of χ_F^K can be corroborated further by establishing modular invariance. Indeed, this follows as a corollary from the mapping class group invariance of general correlation functions that we will present in Section 5 below. The partition function should in addition be compatible with sewing. At this point we have no handle on this property yet. Thus, while we can prove modular invariance at any genus, as far as sewing is concerned the state of affairs bears some similarity with the situation in rational CFT prior to the development of the TFT construction [FRS1] of correlators: While modular invariance is a crucial property of the partition function, it is only necessary, but in general not sufficient, and indeed there are plenty of modular invariants which are incompatible with sewing. On the other hand, for all rational CFTs the charge conjugation modular invariant *is* compatible with sewing [FFFS1], and accordingly we do expect that, for any factorizable finite ribbon category $\mathcal{C}$, at least for $F = F_C$ the character χ_F^K does provide the torus partition function of a CFT with $\mathcal{C} \simeq \mathcal{R}ep(\mathcal{V})$.

4.2 Chiral decomposition

The simple modules of $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{V}$, i.e. the simple objects of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \simeq \mathcal{R}ep(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{V})$, are of the form $S_i \boxtimes S_j$ with S_i, for $i \in \mathcal{I}$, the simple $\mathcal{V}$-modules. For rational CFT, i.e. for semisimple $\mathcal{C}$, the category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ is semisimple, too, so that in particular the bulk state space F decomposes as in formula (2.11) into a direct sum of simple objects $S_i \boxtimes S_j$ for appropriate $i, j \in \mathcal{I}$. When $\mathcal{C}$ is non-semisimple, this is no longer the case. Moreover, for non-semisimple $\mathcal{C}$ one even cannot, in general, write F as a direct sum of $\boxtimes$-factorizable objects, i.e. of objects of the form $U \boxtimes V$.

Nevertheless, since characters split over exact sequences, a chiral decomposition analogous to the one in rational CFT does exist for the torus partition function. Specifically, if $\mathcal{C}$ is a finite tensor category, for which the index set $\mathcal{I}$ is finite, the torus partition function can be written as a finite sum

$$Z = \sum_{i,j\in\mathcal{I}} Z_{ij}\, \chi_i^{\mathcal{V}} \otimes_{\mathbb{C}} \chi_j^{\mathcal{V}} \tag{4.4}$$

with $Z_{ij} \in \mathbb{Z}_{\geq 0}$.

For non-rational CFT the space of zero-point chiral blocks for the torus is *not* exhausted by the characters – that is, the characters of $\mathcal{V}$-modules in the vertex algebra description, respectively by the characters χ_U^L, for $U \in \mathcal{C}$, of the L-modules $(U, \varkappa_U^L)$. Rather, this space also includes linear combinations of so-called *pseudo-characters* [FGST1, FG, GT, AN]. Specifically, for any C_2-cofinite vertex algebra these functions can be constructed with the help of symmetric linear functions on the endomorphism spaces of suitable decomposable projective modules [Mi, Ar]. The existence of an expression of the form (4.4) thus means in particular that the pseudo-characters do not contribute to the torus partition function. This certainly fits nicely with the physical idea of counting states; mathematically it is a non-trivial statement that a decomposition of the form (4.4) exists, even without requiring integrality of the coefficients.

In a purely categorical setting, the analogue of the space of zero-point blocks for the torus is the space

$$C\!\ell(L) := \{ f \in \mathrm{Hom}_{\mathcal{C}}(L,\mathbf{1}) \,|\, f\circ m_L = f\circ m_L \circ c_{L,L} \} \tag{4.5}$$

of central forms, or class functions, on L. The characters of simple L-modules form a subspace of $C\!\ell(L)$, and this is a proper subspace unless $\mathcal{C}$ is semisimple. One should expect that in analogy with (4.4) the character χ_F^K satisfies

$$\chi_F^K \;\in\; C\!\ell(L) \otimes_{\Bbbk} C\!\ell(L) \;\subseteq\; \mathrm{Hom}_{\mathcal{C}}(L,\mathbf{1}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(L,\mathbf{1}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}}(K,\mathbf{1}) \tag{4.6}$$

and thus decomposes into products of simple L-characters χ_k^L as

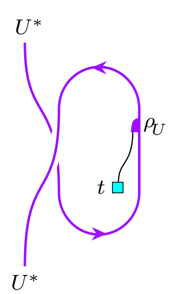
$$\chi_F^K = \sum_{k,l} x_{kl}(F)\, \chi_k^L \otimes_{\Bbbk} \chi_l^L \tag{4.7}$$

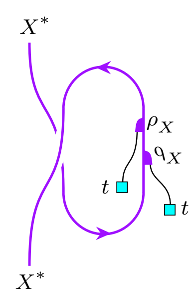
with $x_{kl} \in \mathbb{Z}_{\geq 0}$. We will now establish that this is indeed true in the case that $\mathcal{C} = H$-Mod and $F = F_\omega$.

4.3 The Cardy-Cartan modular invariant and its relatives

Let us thus specialize again to the case that $\mathcal{C} = H$-Mod for some factorizable Hopf algebra H.

The sovereign structure for the ribbon categories H-Mod and H-Bimod is given by

$\pi_U^{H\text{-Mod}} =$ 

and $\pi_X^{H\text{-Bimod}} =$ 

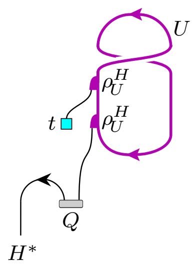
(4.8)

respectively, with t an invertible group-like element of H obtained as the product of the Drinfeld element u (A.30) and the inverse of the ribbon element of H,

$$t = u\, v^{-1}. \tag{4.9}$$

Using the formulas (3.16) and (3.21) for the dinatural families $\imath^L$ and $\imath^K$ of the coends L and K, the characters of L-modules $(U, \varkappa_U^L)$ with $U = (U, \rho_U^H) \in H$-Mod and those of K-modules $(X, \varkappa_X^K)$ with $X = (X, \rho_X^H, \reflectbox{ρ}_X^H) \in H$-Bimod – as described, for the case of K, in (4.3) – can then be written as

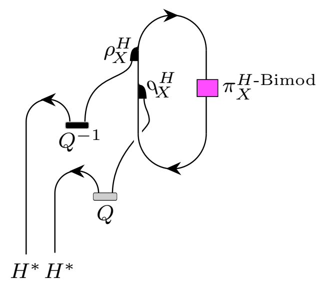
$$\chi_U^L \;=$$



$$= \; \chi_U^H \circ m \circ (t \otimes f_Q) \;=\; \chi_U^H \circ m \circ (f_Q \otimes t) \tag{4.10}$$

and as

$$\chi_X^K \;=$$



$$= \; \chi_X^{H\otimes H^{\rm op}} \circ (m \otimes m) \circ (t \otimes f_{Q^{-1}} \otimes f_Q \otimes t)\,, \tag{4.11}$$

respectively, with f_Q the Drinfeld map (A.26) and $f_{Q^{-1}}$ the analogous morphism in which the monodromy matrix Q is replaced by its inverse. (In (4.11), each of the two occurrences of the element t in $\pi_X^{H\text{-Bimod}}$ can be treated analogously as the single t in (4.10); for details see Lemmas 6 and 8 of [FSS2].)

The result (4.11) is actually a rather direct corollary of (4.10): the categories of $H \otimes H$-modules and of H-bimodules are ribbon equivalent (an equivalence functor is given in explicitly in [FSS1, Eq. (A.22)]), and this equivalence maps the $H \otimes H$-module $L \otimes_{\Bbbk} L$ and the H-bimodule K are mapped to one another.

Remark 4.2. Since H is by assumption factorizable, the Drinfeld map f_Q is invertible. The group-like element t is invertible as well. As a consequence the result (4.10) implies that the set $\mathcal{X} = \{\chi_{S_i}^L \,|\, i \in \mathcal{I}\}$ of characters is linearly independent and that the character of any L-module of the form $(U, \varkappa_U^L)$ is an integral linear combination of the characters in $\mathcal{X}$. It follows that the simple objects, up to isomorphisms, of the full monoidal subcategory $H\text{-Mod}_L^{\mathcal{Q}}$ of $H\text{-Mod}_L$ that consists of the modules $(U, \varkappa_U^L)$ are precisely the modules $(S_i, \varkappa_{S_i}^L)$ with $\{S_i \,|\, i \in \mathcal{I}\}$ the simple H-modules. As a consequence, in the chiral decomposition (4.7) the simple L-characters are $\chi_k^L = \chi_{S_k}^L$ and the summation extends over the same index set $\mathcal{I}$ as the summation in e.g. (4.4).

Next we note that a finite-dimensional Hopf algebra H in $\mathcal{V}ect_{\Bbbk}$ carries a natural structure of a Frobenius algebra and thus is in particular self-injective. According to (A.22) the character of H as the *regular bimodule* (i.e., with regular left and right actions) over itself can thus be written as

$$\chi_H^{H\otimes H^{\rm op}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\, \chi_i^H \otimes \chi_j^H \tag{4.12}$$

with $c_{i,j}$ the entries (A.21) of the Cartan matrix of the category H-Mod. If H is factorizable, then the *coregular* bimodule F_C (see Theorem 2.4) is isomorphic to the regular bimodule, with an intertwiner given by the *Frobenius map*

$$\Phi := ((\lambda\circ m)\otimes id_{H^*})\circ(\mathrm{s}\otimes b_H^{\Bbbk})\,, \tag{4.13}$$

and hence the character of F_C decomposes like in (4.12),

$$\chi_F^{H\otimes H^{\rm op}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\, \chi_i^H \otimes \chi_j^H\,. \tag{4.14}$$

Now compose the equality (4.14) with $(m\otimes m)\circ(t\otimes f_{Q^{-1}}\otimes f_Q\otimes t)$. Then by comparison with (4.11) we learn that

$$\chi_X^K = \sum_{i,j\in\mathcal{I}} c_{i,j}\, \big[\chi_i^H\circ m\circ(t\otimes f_{Q^{-1}})\big]\otimes\big[\chi_j^H\circ m\circ(f_Q\otimes t)\big]\,. \tag{4.15}$$

Here the second tensor factor equals χ_j^L as given in (4.10). For the first factor, the presence of $f_{Q^{-1}}$ instead of f_Q amounts to replacing the braiding in $\varkappa_{S_i}^L$ by its inverse, and thus according to Remark 4.1 we deal with the L-character the dual module. We conclude that [FSS2, Thm. 3]

$$\chi_F^K = \sum_{i,j\in\mathcal{I}} c_{i,j}\, \chi_{\bar\imath}^L\otimes\chi_j^L = \sum_{i,j\in\mathcal{I}} c_{\bar\imath,j}\, \chi_i^L\otimes\chi_j^L\,, \tag{4.16}$$

where $\chi_i^L = \chi_{S_i}^L$ is the character of the simple L-module $(S_i,\varkappa_{S_i}^L)$. This is the desired chiral decomposition, of the form (4.4).

Remark 4.3. (i) By definition (see (A.21)) the numbers $c_{i,j}$ are non-negative integers. And they are naturally associated with the category $\mathcal{C}\simeq H$-Mod – they depend only on $\mathcal{C}$ as an abelian category.
(ii) Among the simple objects of $\mathcal{C}$ is in particular the tensor unit $\mathbf{1}\cong S_0$. In general, the corresponding diagonal coefficient $c_{0,0}$ in (4.16) is larger than 1. This is *not* in conflict with the uniqueness of the vacuum – it just accounts for the fact that for non-semisimple $\mathcal{C}$ the tensor unit has non-trivial extensions and is in particular not projective.
(iii) The result (4.16) fits well with predictions for the bulk state space of concrete classes of logarithmic CFTs, namely [GR] the $(1,p)$ triplet models and [QS] WZW models with supergroup target spaces, compare Remark 2.2.

We refer to the character (4.14) as the *Cardy-Cartan modular invariant*, because in the semisimple case, for which $c_{i,j}=\delta_{i,j}$, the expression (4.14) reduces to the charge conjugation modular invariant, which in the context of studying compatible conformally invariant boundary conditions of the CFT is also known as the Cardy case.

Next we generalize the Cardy-Cartan modular invariant to the situation that we perform a twist by a ribbon Hopf algebra automorphism of H. This is achieved as follows. First note that an automorphism ω of H induces an endofunctor $G_\omega\colon H\text{-Mod} \to H\text{-Mod}$. If ω is a *Hopf algebra automorphism*, i.e. both an algebra and a coalgebra automorphism and commuting with the antipode, then the functor G_ω is rigid monoidal, and if ω is a *ribbon Hopf algebra automorphism*, i.e. in addition satisfies

$$(\omega \otimes \omega)(R) = R \qquad \text{and} \qquad \omega(v) = v\,, \tag{4.17}$$

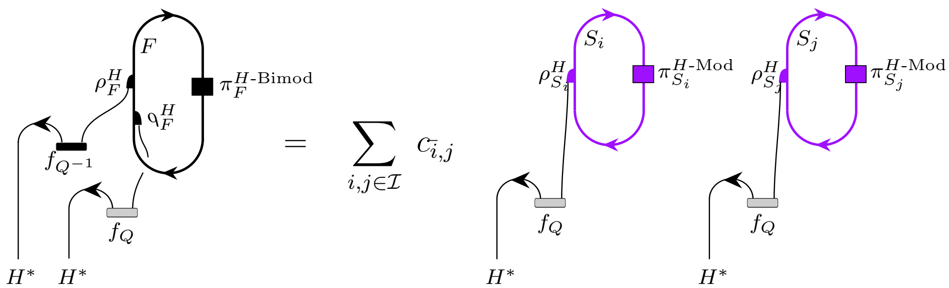
then G_ω is is even a ribbon functor. Given two automorphisms ω and ω', one has $G_\omega \circ G_{\omega'} = G_{\omega\omega'}$, as a strict equality of functors. It follows that G_ω has $G_{\omega^{-1}}$ as an inverse and is thus an equivalence of categories. In particular, ω induces a bijection $\overline{\omega}$ from the index set $\mathcal{I}$ to itself, in such a way that $\{S_{\overline{\omega}(i)} \,|\, i \in \mathcal{I}\}$ is again a full set of representatives of the isomorphism classes of simple H-modules.

With this information we are in a position to establish

Theorem 4.4. *For ω a ribbon Hopf algebra automorphism of a factorizable Hopf algebra H, the character of the automorphism-twisted coregular bimodule $F_\omega = (H^*, \rho_{F_C}, \text{\textsf{o}}_{F_C} \circ (id_{H^*} \otimes \omega))$ (see* (2.30)*) has the chiral decomposition*

$$\chi^K_{F_\omega} = \sum_{i,j\in\mathcal{I}} c_{\bar\imath,\overline{\omega}(j)}\, \chi^L_i \otimes \chi^L_j\,. \tag{4.18}$$

Proof. Pictorially, (4.15) reads



(4.19)

Now compose this equality with $id_{H^*} \otimes (\omega^{-1})^*$ and use that, by the first equality in (4.17), $(\omega\otimes\omega)(Q) = Q$, so that the automorphism ω^{-1} can be pushed through the Drinfeld map on both sides of the equality. This yields

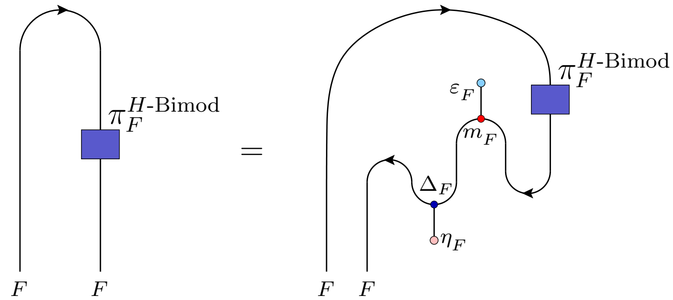
$$\chi^K_{F_\omega} = \sum_{i,j\in\mathcal{I}} c_{\bar\imath,j}\, \chi^L_i \otimes \chi^L_{S_j^{\omega^{-1}}} = \sum_{i,j\in\mathcal{I}} c_{\bar\imath,j}\, \chi^L_i \otimes \chi^L_{S_{\overline{\omega}(j)^{-1}}}\,. \tag{4.20}$$

A relabeling of the summation index j then gives (4.18). ☐

5 Correlation functions

As already pointed out, the conjecture that the character χ^K_F gives the torus partition function of a full CFT with bulk state space $F = F_C$ constitutes a special case of a proposal for general correlation functions $\mathrm{Cor}_{g;n}$ of bulk fields, for orientable world sheets of arbitrary genus g and with an arbitrary number n of insertions of the bulk state space. This proposal [FSS1, FSS3] is based on the idea that it should be possible to express correlators entirely and very directly through the basic structures of their ingredients – that is, the topology of the world sheet and the structure of the bulk state space as a symmetric Frobenius algebra and as a module $(F, \varkappa^K_F)$ over the bulk handle Hopf algebra.

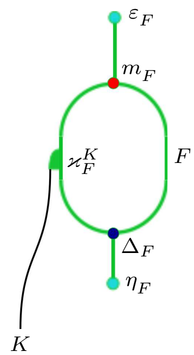
Let us first see how this works in the case of the torus partition function. To this end we note the equalities



(5.1)

where we first use the Frobenius property and then the symmetry of F. The equality of the left and right hand sides of (5.1) allows us to rewrite the expression (4.3) for χ_F^K as

$\chi_F^K =$

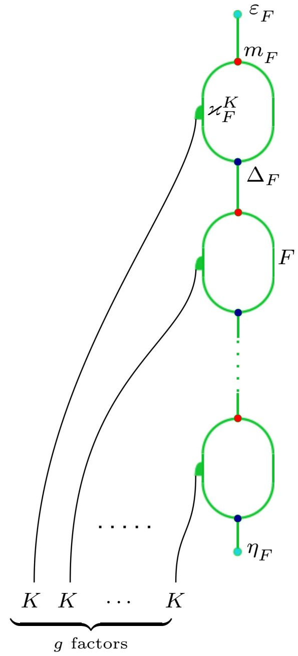


(5.2)

Thus, basically the morphism χ_F^K consists of an F-'loop' combined with the action of the handle Hopf algebra K. This can be seen as a manifestation of the fact that we deal with a world sheet having one handle. In a similar vein, for the correlator $\mathrm{Cor}_{g;0}$, i.e. the partition function of an orientable world sheet Σ of arbitrary genus g, we are lead to the following construction:

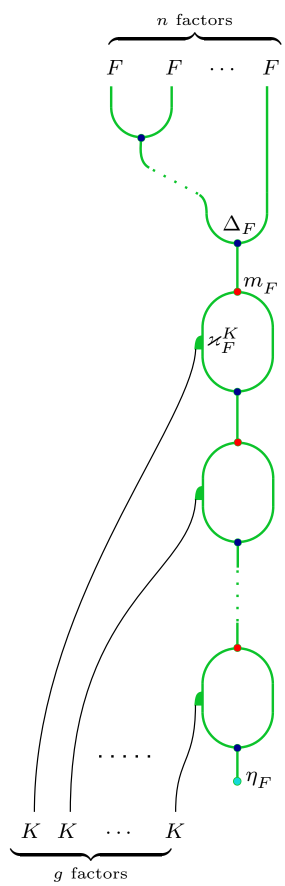
- Select a skeleton Γ for Σ and label each edge of the skeleton by the Frobenius algebra F.
- Orient the edges of Γ in such a manner that each vertex of Γ has either one incoming and two outgoing edges or vice versa. Label each of these three-valent vertices either with the coproduct Δ_F of F or with the product m_F, depending on whether one or two of its three incident edges are incoming.
- To avoid having to introduce any duality morphisms (analogously as in the description (4.3) of $\mathrm{Cor}_{0;0} = \chi_F^K$), when implementing the previous part of the construction allow for adding further edges that connect one three-valent and one uni-valent vertex, the latter being labeled by the unit η_F or counit ε_F of F.
- For each handle of Σ attach one further edge, labeled by the handle Hopf algebra K, to the corresponding loop of the skeleton, and label the resulting new trivalent vertex by the representation morphism $\varkappa_F^K$.
- The so obtained graph defines a morphism in $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}}(K^{\otimes g}, \mathbf{1})$.

At genus $g = 1$ this prescription precisely reproduces the morphisms (5.2) in $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}}(K, \mathbf{1})$. At higher genus several different choices for the skeleton Γ are possible, but with the help of the symmetry and Frobenius property of F one sees that they all yield one and the same morphism in $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, namely

$\mathrm{Cor}_{g;0} =$ 

(5.3)

Our ansatz generalizes easily to world sheets with bulk field insertions: For n outgoing (say) insertions of the bulk state space, just replace the counit $\varepsilon_F \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}}(F, \mathbf{1})$ in (5.3) with an n-fold coproduct $\Delta_F^{(n)} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}}(F, F^{\otimes n})$. When doing so, the order of taking coproducts is immaterial owing to coassociativity of Δ_F, and the order of factors in $F^{\otimes n}$ does not matter due to cocommutativity of Δ_F; with any choice of ordering, the resulting morphism in $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}}(K^{\otimes g}, F^{\otimes n})$ equals

$\mathrm{Cor}_{g;n}(F) \;=$ 

(5.4)

for $n>0$.

Likewise one can generalize the ansatz to correlators $\mathrm{Cor}_{g;p,q}$ with any numbers q of incoming and p of outgoing insertions. The incoming insertions are incorporated by replacing the unit $\eta_F \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}}(\mathbf{1},F)$ in (5.3) with a p-fold product $m_F^{(q)} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}}(F^{\otimes q},F)$. Furthermore, the case of genus zero is included by just omitting the F-'loop'. Altogether the prescription can be summarized as follows:

$$
\begin{aligned}
&\mathrm{Cor}_{0;1,1} := id_F\,,\\
&\mathrm{Cor}_{1;1,1} := m_F \circ (\rho_F^K \otimes id_F) \circ (id_K \otimes \Delta_F)\,,\\
&\mathrm{Cor}_{g;1,1} := \mathrm{Cor}_{1;1,1} \circ (id_K \otimes \mathrm{Cor}_{g-1;1,1}) \quad \text{for} \quad g>1\,,\\
&\mathrm{Cor}_{g;p,q} := \Delta_F^{(p)} \circ \mathrm{Cor}_{g;1,1} \circ \big(id_{K^{\otimes g}} \otimes m_F^{(q)}\big)\,.
\end{aligned}
\tag{5.5}
$$

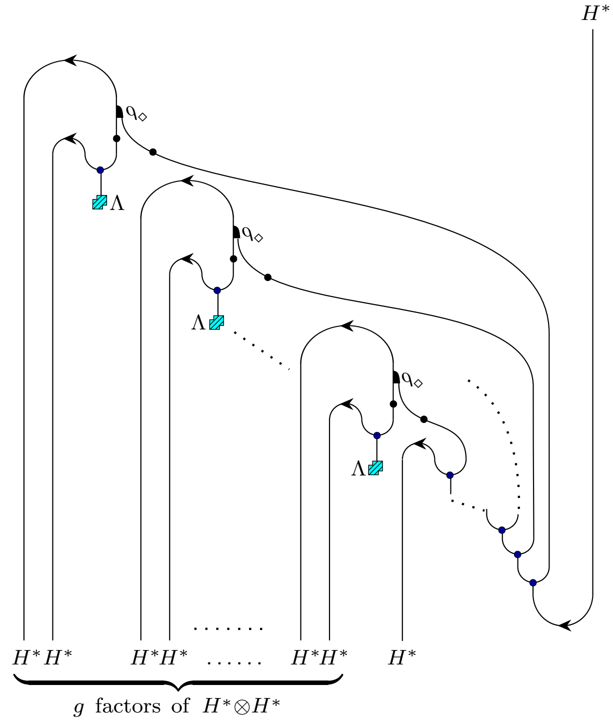
Remark 5.1.

(i) One may be tempted to work with ribbons instead of with edges. But since F has trivial twist, $\theta_F = id_F$, the framing does not matter and can be neglected in our discussion.

(ii) Our ansatz results from the description (5.2) of the torus partition function $\mathrm{Cor}_{1;0,0}$ and the knowledge that $\mathrm{Cor}_{g;p,q}$ must be an element of the morphism space $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}}(K^{\otimes g}\otimes F^{\otimes q}, F^{\otimes p})$. It would be much more elegant to derive the prescription from a three-dimensional approach, which in the case of rational CFT should be related by a kind of folding trick to the TFT construction of [FRS1].

What enters in the expressions for correlation functions above is only the structure of $\mathcal{C}$ as a factorizable finite tensor category and of F as a bulk state space, carrying the structure of a Frobenius algebra that is commutative and symmetric and has trivial twist. Again we can be more explicit for the case that $\mathcal{C}$ is equivalent to the category H-Mod of finite-dimensional modules over some factorizable Hopf algebra H and that $F = F_\omega$ for any ribbon automorphism ω of H. Let us present the correlator $\mathrm{Cor}_{g;p,q}$ for the case that $p=q=1$, the extension to $p,q>1$ being easy, and first take F to be the coregular H-bimodule F_C. Then by inserting the expressions 2.29 for the structural morphisms of the Frobenius algebra F_C and writing out the braiding of the category $H\text{-Mod}^{\mathrm{rev}} \boxtimes H\text{-Mod} \simeq H\text{-Bimod}$ (which appears in the representation morphism $\varkappa_F^K$), after a few rearrangements one obtains

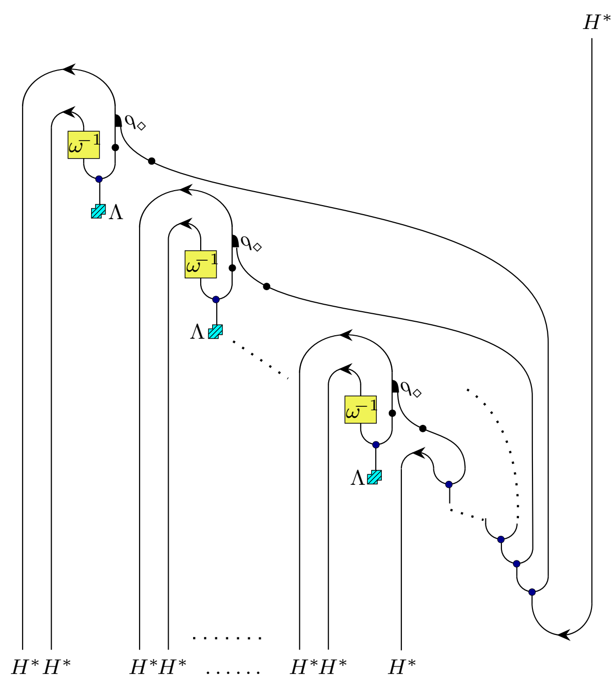
$$\mathrm{Cor}_{g;1,1}(F_C) \;=\; \tag{5.6}$$



with $\varrho_\diamond$ the right-adjoint action of H on itself.

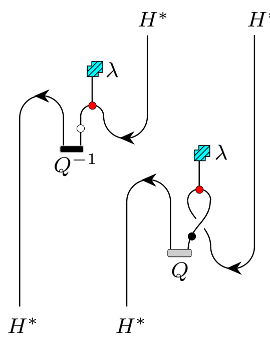
For general F_ω the result differs from (5.6) only by a few occurences of the automorphism ω (recall formula (2.30) and that the structural morphisms of the Frobenius algebra F_ω coincide with those of F_C as linear maps):

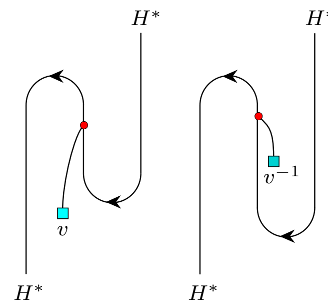
$$\mathrm{Cor}_{g;1,1}(F_\omega) \;=\; \tag{5.7}$$



The correlators of a rational conformal field theory must be invariant under an action of the mapping class group $\mathrm{Map}_{g:n}$ of closed oriented surfaces of genus g with n boundary components, where g is the genus of the world sheet and $n = p + q$ is the number of (incoming plus outgoing) field insertions. One expects that this can still be consistently imposed for logarithmic CFTs. And indeed we are able to establish mapping class group invariance of the ansatz for correlators that we presented above, i.e. of the morphisms (5.5) for the case that $\mathcal{C} \simeq H$-Mod and $F = F_\omega$.

Recall that the morphism $\mathrm{Cor}_{g;p,q}$ is an element of the space $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}}(K^{\otimes g} \otimes F^{\otimes q}, F^{\otimes p})$. A natural action $\pi^{K;F}_{g:p+q}$ of $\mathrm{Map}_{g:p+q}$ on this morphism space has been found in [Ly2, Ly3]. $\pi^{K;F}_{g:p+q}$ is described in some detail in Appendix A.5; here we just note that $\mathrm{Map}_{g:n}$ is generated by suitable Dehn twists, that most of them are represented by pre-composing with an endomorphim of $K^{\otimes g}$ or $F^{\otimes q}$ or post-composing with an endomorphim of $F^{\otimes p}$, and that at genus 1 the relevant endomorphims of $K = H^* \otimes_{\Bbbk} H^*$ are

$S^K \;=$ 

and $T^K \;=$ 

(5.8)

which amount to an S- and T-transformation, respectively.

Remark 5.2. In general, the mapping class group action considered in [Ly2, Ly3] is only projective. But owing to the fact that the category relevant to us is an enveloping category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, with a ribbon structure in which the two factors are treated in an opposite fashion, in the situation at hand the action is in fact a genuine linear representation (compare Remark 5.5 of [FSS1]).

Denote by $\mathrm{Map}_{g:p,q}$ the subgroup of $\mathrm{Map}_{g:p+q}$ that leaves the subsets of incoming and outgoing insertions separately invariant, and by $\pi^{K;F}_{g:p,q}$ the representation of $\mathrm{Map}_{g:p,q}$ that is obtained (compare (A.40)) from $\pi^{K;F}_{g:p+q}$. The following is the main result of [FSS3] (Theorem 3.2, Remark 3.3 and Theorem 6.7):

Theorem 5.3. *For H a factorizable ribbon Hopf algebra and ω a ribbon automorphism of H, and for any triple of integers $g, p, q \geq 0$, the morphism $\mathrm{Cor}_{g;p,q}(F_\omega)$ is invariant under the action $\pi^{K;F}_{g:p,q}$ of the group Map$_{g:p,q}$.*

Remark 5.4. Besides invariance under the action of $\pi^{K;F}_{g:p,q}$, the other decisive property of correlation functions is compatibility with sewing. That is, there are sewing relations at the level of chiral blocks, and the correlators must be such that the image of a correlator, as a specific vector in a space of chiral blocks, under these given chiral relations, is again a correlator. Compatibility with sewing allows one to construct all correlation functions by starting from a small set of fundamental corrleators and thereby amounts to a kind of locality property.
Like mapping class group invariance, compatibility with sewing is a requirement in rational CFT, and again one expects that one can consistently demand it for logarithmic CFTs as well. For now, checking compatibility of our ansatz with sewing is still open. It is in fact fair to say that already for rational CFT the study of sewing [FFRS1, FFSt] still involves some brute-force arguments. Understanding sewing in a way suitable for logarithmic CFT may require (or, amount to) deeper insight into the nature of sewing.

A Appendix

A.1 Coends

For a category $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\mathrm{op}}$ is the one with the same objects, but reversed morphisms, i.e. a morphism $f\colon U \to V$ in $\mathcal{C}$ is taken to be a morphism $V \to U$ in $\mathcal{C}^{\mathrm{op}}$. Given $\Bbbk$-linear abelian categories $\mathcal{C}$ and $\mathcal{D}$ and a functor G from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathcal{D}$, a *dinatural transformation* from F to an object $B \in \mathcal{D}$ is a family $\varphi = \{\varphi_U\colon G(U,U) \to B\}_{U\in\mathcal{C}}$ of morphisms with the property that the square

$$\begin{tikzcd}
 & G(V,U) \arrow[dl, "G(f{,}\mathrm{id}_U)"'] \arrow[dr, "G(\mathrm{id}_V{,}f)"] & \\
G(U,U) \arrow[dr, "\varphi_U"'] & & G(V,V) \arrow[dl, "\varphi_V"] \\
 & B &
\end{tikzcd} \tag{A.1}$$

of morphisms commutes for all $f \in \mathrm{Hom}(U,V)$.

A *coend* (D,ι) for the functor G is an initial object among all such dinatural transformations, that is, it is an object $D \in \mathcal{D}$ together with a dinatural transformation ι such that for any dinatural transformation φ from G to any $B \in \mathcal{D}$ there exists a unique morphism $\kappa \in \mathrm{Hom}_{\mathcal{D}}(D,B)$ such that $\varphi_U = \kappa \circ \iota_U$ for every object U of $\mathcal{C}$. In other words, given a diagram

$$\begin{tikzcd}
 & G(V,U) \arrow[dl, "G(f{,}\mathrm{id}_U)"'] \arrow[dr, "G(\mathrm{id}_V{,}f)"] & \\
G(U,U) \arrow[dr, "\iota_U"'] \arrow[ddr, bend right, "\varphi_U"'] & & G(V,V) \arrow[dl, "\iota_V"] \arrow[ddl, bend left, "\varphi_V"] \\
 & D \arrow[d, dashed, "\kappa"] & \\
 & B &
\end{tikzcd} \tag{A.2}$$

with commuting inner and outer squares for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(U,V)$, there exists a unique morphism κ such that also the triangles in the diagram commute for all $U,V \in \mathcal{C}$.

If the coend exists, then it is unique up to unique isomorphism. The underlying object, which by abuse of terminology is referred to as the coend of G as well, is denoted by an integral sign,

$$D = \int^{U\in\mathcal{C}} G(U,U) \tag{A.3}$$

The finiteness properties of the categories we are working with in this paper guarantee the existence of all coends we need. Specifically, the bulk state space $F_{\mathcal{C}}$ is the coend (2.3) of the functor $G^{\mathcal{C}}_{\boxtimes}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ that acts on objects as $(U,V) \mapsto U^{\vee} \boxtimes V$, while the chiral and full handle Hopf algebras L and K are the coends (3.7) of the functors $G^{\mathcal{C}}_{\otimes}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ and

$G_{\otimes}^{\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}}\colon (\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C})^{\mathrm{op}}\times(\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C})\to\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$ that act on objects as $(U,V)\mapsto U^{\vee}\otimes V\in\mathcal{C}$ and as $(X,Y)\mapsto X^{\vee}\otimes Y\in\mathcal{C}^{\mathrm{rev}}\boxtimes\mathcal{C}$, respectively.

If the category $\mathcal{C}$ is cocomplete, then an equivalent description of the coend of G (see e.g. section V.1 of [May]) is as the coequalizer of the morphisms

$$\coprod_{f\colon V\to W} G(V,W) \;\underset{t}{\overset{s}{\rightrightarrows}}\; \coprod_{U\in\mathcal{C}} G(U,U) \tag{A.4}$$

whose restrictions to the 'fth summand' are $s_f = F(f,id)$ and $t_f = F(id,f)$, respectively. Thus, morally, the coend of G is the universal quotient of $\coprod_U G(U,U)$ that enforces the two possible actions of G on any morphism f in $\mathcal{C}$ to coincide.

A.2 The full center of an algebra

Given an object U of a monoidal category $\mathcal{C}$, a *half-braiding* $z=z(U)$ on U is a natural family of isomorphisms $z_V\colon U\otimes V\to V\otimes U$, for all $V\in\mathcal{C}$, such that (assuming $\mathcal{C}$ to be strict) $z_{\mathbf{1}}=id_U$ and

$$(id_V\otimes z_W)\circ(z_V\otimes id_W) = z_{V\otimes W} \tag{A.5}$$

for all $V,W\in\mathcal{C}$. The *monoidal center* $\mathcal{Z}(\mathcal{C})$ is the category which has as objects pairs (U,z) consisting of objects of $\mathcal{C}$ and of half-braidings, while its morphisms are morphisms f of $\mathcal{C}$ that are naturally compatible with the half-braidings of the source and target of f. The category $\mathcal{Z}(\mathcal{C})$ is again monoidal, with tensor product

$$(U,z)\otimes(U',z') := (U\otimes U', (z_V\otimes id_{U'})\circ(id_U\otimes z'_V)) \tag{A.6}$$

and tensor unit $(\mathbf{1},id)$, and with respect to this tensor product it is braided, with braiding isomorphisms $c_{(U,z),(U',z')} := z_{U'}$. The forgetful functor $\mathrm{F}_{\mathcal{C}}^{\mathcal{Z}}$ from $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$, acting on objects as

$$\mathrm{F}_{\mathcal{C}}^{\mathcal{Z}}:\quad (U,z)\longmapsto U\,, \tag{A.7}$$

is faithful (but in general neither full nor essentially surjective) and monoidal.

For $A=(A,m,\eta)$ a (unital, associative) algebra in $\mathcal{C}$, we say that an object (U,z) of $\mathcal{Z}(\mathcal{C})$ together with a morphism $r\in\mathrm{Hom}_{\mathcal{C}}(U,A)$ is *compatible* with the product of A iff

$$m\circ(id_A\otimes r)\circ z_A = m\circ(r\otimes id_A) \tag{A.8}$$

in $\mathrm{Hom}_{\mathcal{C}}(U\otimes A,A)$. Given the algebra A in $\mathcal{C}$, the *full center* $Z(A)$ of A is [Da] a pair consisting of an object in $\mathcal{Z}(\mathcal{C})$ – by abuse of notation denoted by $Z(A)$ as well – and a morphism $\zeta_A\in\mathrm{Hom}_{\mathcal{C}}(\mathrm{F}_{\mathcal{C}}^{\mathcal{Z}}(Z(A)),A)$ that is terminal among all pairs $((U,z),r)$ in $\mathcal{Z}(\mathcal{C})$ that are compatible with the product of A. That $Z(A)$ is terminal among compatible pairs means that for any such pair $((U,z),r)$ there exists a unique morphism $\kappa\in\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((U,z),Z(A))$ such that the equality

$$\zeta_A\circ\mathrm{F}_{\mathcal{C}}^{\mathcal{Z}}(\kappa) = r \tag{A.9}$$

holds in $\mathrm{Hom}_{\mathcal{C}}(U,A)$.

For the categories $\mathcal{C}$ relevant to us in this paper, the full center of any algebra in $\mathcal{C}$ exists. Being defined by a universal property, $Z(A)$ unique up to unique isomorphism. Further, $Z(A)$ has a unique structure of a (unital, associative) algebra in $\mathcal{Z}(\mathcal{C})$ such that ζ_A is an algebra morphism in $\mathcal{C}$, and this algebra structure is commutative [Da, Prop. 4.1]. Furthermore, if A and B are Morita equivalent algebras in $\mathcal{C}$, then the algebras $Z(A)$ and $Z(B)$ in $\mathcal{Z}(\mathcal{C})$ are isomorphic [Da, Cor. 6.3].

If the category $\mathcal{C}$ is braided, then there is also a more familiar notion of center of an algebra inside $\mathcal{C}$ itself, albeit there are two variants (unless the braiding is symmetric), the left center and the right center. The *left center* of an algebra A in $\mathcal{C}$ is obtained with the help of an analogue of the compatibility condition (A.8) in which the half-braiding is replaced by the braiding c of $\mathcal{C}$, according to

$$m \circ (id_A \otimes q) \circ c_{U,A} = m \circ (q \otimes id_A) . \tag{A.10}$$

Again one considers pairs of objects U in $\mathcal{C}$ together with morphisms $q \in \mathrm{Hom}_{\mathcal{C}}(U, A)$ obeying (A.10), and defines the left center $C_l(A) \equiv (C_l(A), \zeta_A^l)$ to be terminal among such compatible pairs. The right center $C_r(A)$ is defined analogously. $C_l(A)$ has a unique structure of an algebra in $\mathcal{C}$ such that the morphism $\zeta_A^l \in \mathrm{Hom}_{\mathcal{C}}(C_l(A), A)$ is an algebra morphism. This algebra structure is commutative (Prop. 2.37(i) of [FrFRS] and Prop. 5.1 of [Da]); clearly, if already A is commutative, then $C_l(A) = A = C_r(A)$. If $\mathcal{C}$ is ribbon and A is Frobenius, then $C_l(A)$ has trivial twist [FrFRS, Lemma 2.33].

We mention the left center here because it was instrumental for the construction by which the full center $Z(A)$ was introduced originally, for modular tensor categories [RFFS, Eq. (A.1)]. For the more general categories of our interest, there is the following variant, which also makes use of the functor $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}} : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ that is right adjoint to the forgetful functor $\mathrm{F}_{\mathcal{C}}^{\mathcal{Z}}$. Let us assume that $\mathcal{C}$ is a factorizable finite ribbon category for which $\mathcal{Z}(\mathcal{C})$ is monoidally equivalent to the enveloping category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ (compare Remark 3.6(iv)). Then the right adjoint functor $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}$ exists [RGW, Thm. 3.20]). Moreover, $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}$ is lax monoidal, implying that for any algebra A in $\mathcal{C}$, the object $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(A) \in \mathcal{Z}(\mathcal{C})$ is again an algebra. Also, the natural transformations $\varepsilon \colon \mathrm{F}_{\mathcal{C}}^{\mathcal{Z}} \circ \mathrm{R}_{\mathcal{C}}^{\mathcal{Z}} \Rightarrow Id_{\mathcal{C}}$ and $\eta \colon Id_{\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}} \Rightarrow \mathrm{R}_{\mathcal{C}}^{\mathcal{Z}} \circ \mathrm{F}_{\mathcal{C}}^{\mathcal{Z}}$ of the adjunction are monoidal [Da, Lemma. 5.3]. And further, provided that the natural transformation ε is epi, the full center of an algebra $A \in \mathcal{C}$ can be expressed as ([Da, Thm. 5.4] and [RGW, Thm. 3.24])

$$Z(A) = C_l(\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(A)) \qquad \text{with} \qquad \zeta_A = \varepsilon_A \circ \zeta^l_{\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(A)} . \tag{A.11}$$

The functor $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}$ can be given explicitly; once we identify $\mathcal{Z}(\mathcal{C})$ with $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$, it acts on objects $U \in \mathcal{C}$ as [RGW, Eq. (3.43)]

$$\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(U) = (U \boxtimes \mathbf{1}) \otimes \mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(\mathbf{1}) . \tag{A.12}$$

Moreover, the algebra $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(\mathbf{1})$ in $\mathcal{Z}(\mathcal{C})$ is commutative [RGW, Lemma 3.25] and hence equals $Z(\mathbf{1})$. Together with the formulas (2.9) and (2.4) for the bulk state space $F_{\mathcal{C}}$ this shows that $\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(\mathbf{1})$ can be obtained as a coend,

$$\mathrm{R}_{\mathcal{C}}^{\mathcal{Z}}(\mathbf{1}) \cong \int^U U^\vee \boxtimes U . \tag{A.13}$$

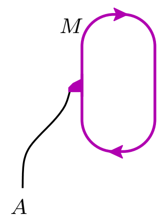
Hereby for modular tensor categories, for which $F_{\mathcal{C}}$ is given by the finite direct sum (2.1), the formula (A.11) for the full center reduces to (A.1) of [RFFS].

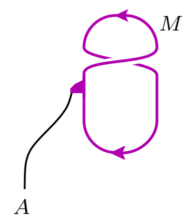
A.3 Algebras and characters

Let $A = (A, m, \eta)$ be a (unital, associative) finite-dimensional algebra over a field $\Bbbk$, and let $M = (M, \rho)$ be a finite-dimensional left A-module. The *character* χ_M^A of the module M is defined to be the partial trace of the representation morphism ρ, with the trace taken in the sense of linear maps. This means

$$\chi_M^A = \mathrm{tr}_M(\rho) = \tilde{d}_M^{\Bbbk} \circ (\rho \otimes id_{M^\vee}) \circ (id_A \otimes b_M^{\Bbbk}) \ \in \mathrm{Hom}(A, \Bbbk)\,, \tag{A.14}$$

with $b_M^{\Bbbk}$ the (right) coevaluation and $\tilde{d}_M^{\Bbbk}$ the (left) evaluation map of $\mathcal{V}ect_{\Bbbk}$. Now the map $\tilde{d}_M^{\Bbbk} \in \mathrm{Hom}_{\Bbbk}(M \otimes_{\Bbbk} M^*, \Bbbk)$ can be expressed through the right evaluation map $d_M^{\Bbbk} \in \mathrm{Hom}_{\Bbbk}(M^* \otimes_{\Bbbk} M, \Bbbk)$ as $\tilde{d}_M^{\Bbbk} = d_M^{\Bbbk} \circ \tau_{M,M^*}$ with τ the flip map (and similarly for the left and right coevaluations). Thus two equivalent descriptions of the character are, pictorially,

χ_M^A = 

= 

(A.15)

Characters are class functions, i.e. satisfy $\chi_M^A \circ m = \chi_M^A \circ m \circ \tau_{A,A}$. A is semisimple iff the space of class functions is already exhausted by linear combinations of characters of A-modules. Furthermore, characters behave additively under short exact sequences. As a consequence, taking $\{S_i \,|\, i \in \mathcal{I}\}$ to be a full set of representatives of the isomorphism classes of simple A-modules, with characters $\chi_i^A \equiv \chi_{S_i}^A$, and writing $[\, M : S_i \,]$ for the multiplicity of S_i in the Jordan-Hölder series of M, one has

$$\chi_M^A = \sum_{i \in \mathcal{I}} [\, M : S_i \,]\, \chi_i^A\,. \tag{A.16}$$

The simple modules S_i have projective covers P_i, from which they can be recovered as the quotients $S_i = P_i / J(A)\, P_i$ with $J(A)$ the Jacobson radical of A. The modules $\{P_i \,|\, i \in \mathcal{I}\}$ constitute a full set of representatives of the isomorphism classes of indecomposable projective left A-modules. There is a (non-unique) collection $\{e_i \in A \,|\, i \in \mathcal{I}\}$ of primitive orthogonal idempotents such that $P_j = A\, e_j$ for all $j \in \mathcal{I}$, as well as $Q_j = e_j\, A$ for a full set of representatives of the isomorphism classes of indecomposable projective right A-modules. The algebra A decomposes as a left module over itself (with the *regular* action, given by the product m) as

$$_AA \,\cong\, \bigoplus_{i \in \mathcal{I}} P_i \otimes_{\Bbbk} \Bbbk^{\dim(S_i)}. \tag{A.17}$$

Of particular interest to us is A regarded as a *bi*module over itself, with regular left and right actions. The decomposition of this bimodule into indecomposables is considerably more involved than the decomposition as a module and cannot be expressed in a 'model-independent' manner analogous to (A.17). But we can use that the structure of an A-bimodule is equivalent to the one of a left $A \otimes A^{\mathrm{op}}$-module. Accordingly, by the *character of A as an A-bimodule* we mean its character as an $A \otimes A^{\mathrm{op}}$-module.

Now if $\Bbbk$ has characteristic zero, then for any two finite-dimensional $\Bbbk$-algebras A and B complete sets of simple modules over the tensor product algebra $A \otimes B$ are [CuR, Thm. (10.38)] given by $\{S_i^A \otimes_{\Bbbk} S_j^B \,|\, i \in \mathcal{I}_A\,,\ j \in \mathcal{I}_B\}$. In view of (A.16), the character of any $A \otimes B$-module X can therefore be written as the bilinear combination

$$\chi_X^{A \otimes B} = \sum_{i \in \mathcal{I}_A,\, j \in \mathcal{I}_B} [\, X : S_i^A \otimes_{\Bbbk} S_j^B \,]\, \chi_i^A \otimes_{\Bbbk} \chi_j^B\,. \tag{A.18}$$

For the case of our interest, i.e. $B = A^{\rm op}$ and $X = A$, this decomposition reads

$$\chi_A^{A\otimes A^{\rm op}} = \sum_{i,j\in\mathcal{I}} [\, A : S_i \otimes_\Bbbk T_j \,]\, \chi_{S_i\otimes_\Bbbk T_j}^{A\otimes A^{\rm op}} \tag{A.19}$$

with $T_k = Q_k / J(A)\, Q_k$ the simple quotients of the projective right A-modules Q_k.

Next we use that (for details see [FSS2, App. A])

$$[\, A : S_i \otimes_\Bbbk T_j \,] = \dim_\Bbbk\big(\mathrm{Hom}_A(P_i, P_j)\big) = c_{i,j} \,, \tag{A.20}$$

where the non-negative integers $c_{i,j}$ are defined by

$$c_{i,j} := [\, P_i : S_j \,] \,. \tag{A.21}$$

The matrix $C = \big(c_{i,j} \big)$ is called the *Cartan matrix* of the algebra A, or of the category A-Mod. It obviously depends only on A-Mod as an abelian category.

Assume now that A is *self-injective*, i.e. injective as a left module over itself. Then $T_k \cong S_k^*$ as right A-modules, so that in view of (A.20) we can rewrite the character (A.19) as

$$\chi_A^{A\otimes A^{\rm op}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\, \chi_i^A \otimes \chi_j^A \,. \tag{A.22}$$

A.4 Factorizable Hopf algebras

In this paper we deal with finite-dimensional Hopf algebras over a field $\Bbbk$ that is algebraically closed and has characteristic zero. In the application to logarithmic CFT, $\Bbbk$ is the field $\mathbb{C}$ of complex numbers. We denote by $m \in \mathrm{Hom}_\Bbbk(H \otimes_\Bbbk H, H)$ the product, by $\eta \in \mathrm{Hom}_\Bbbk(\Bbbk, H)$ the unit, by $\Delta \in \mathrm{Hom}_\Bbbk(H, H \otimes_\Bbbk H)$ the coproduct, by $\varepsilon \in \mathrm{Hom}_\Bbbk(H, \Bbbk)$ the counit, and by $\mathrm{s} \in \mathrm{Hom}_\Bbbk(H, H)$ the antipode of H.

An *R-matrix* for H is an invertible element R of $H \otimes_\Bbbk H$ which intertwines the coproduct and opposite coproduct in the sense that

$$R\, \Delta\, R^{-1} = \tau_{H,H} \circ \Delta \equiv \Delta^{\rm op} \tag{A.23}$$

and which satisfies the equalities

$$(\Delta \otimes id_H) \circ R = R_{13} \cdot R_{23} \qquad \text{and} \qquad (id_H \otimes \Delta) \circ R = R_{13} \cdot R_{12} \tag{A.24}$$

in $H \otimes_\Bbbk H \otimes_\Bbbk H$. (The notation R_{13} means that R is to be considered as an element in the tensor product of the first and third factors of $H \otimes_\Bbbk H \otimes_\Bbbk H$, and similarly for R_{23} etc.) A Hopf algebra $(H, m, \eta, \Delta, \varepsilon, \mathrm{s})$ together with an R-matrix R is called a *quasitriangular Hopf algebra*. For more information about quasitriangular Hopf algebras see e.g. Chapters 1 and 2 of [Maj].

For a quasitriangular Hopf algebra, the invertible element

$$Q := R_{21} \cdot R \tag{A.25}$$

of $H \otimes_\Bbbk H$ is called the *monodromy matrix*. A quasitriangular Hopf algebra for which Q is non-degenerate, meaning that it can be expressed as $\sum_\ell h_\ell \otimes k_\ell$, in terms of two vector space bases $\{h_\ell\}$ and $\{k_\ell\}$ of H, is called *factorizable*. Equivalently, factorizability means that the *Drinfeld map*

$$f_Q := (d_H \otimes id_H) \circ (id_{H^*} \otimes Q) \ \in \mathrm{Hom}(H^*, H) \tag{A.26}$$

is invertible. With a view towards the non-quasitriangular Hopf algebras considered for logarithmic conformal field theories in e.g. [FGST1], one should note that for the notion of factorizability we only need the existence of a monodromy matrix Q, but not of an R-matrix; moreover, the properties of Q can be formulated without any reference to R [Br, Sect. 2].

A factorizable Hopf algebra is minimal in the sense that it does not contain a proper quasitriangular Hopf subalgebra [Ra2, Prop. 3b], and is thus [Ra1] a quotient of the Drinfeld double $D(B)$ of some Hopf algebra B.

A *ribbon element* for a quasitriangular Hopf algebra H is an invertible element v of the center of H that obeys

$$\mathrm{s} \circ v = v\,, \qquad \varepsilon \circ v = 1 \qquad \text{and} \qquad \Delta \circ v = (v \otimes v) \cdot Q^{-1}. \tag{A.27}$$

A quasitriangular Hopf algebra together with a ribbon element is called a *ribbon Hopf algebra*.

By a slight abuse of terminology, for brevity we refer in this paper to a finite-dimensional factorizable ribbon Hopf algebra over $\Bbbk$ just as a *factorizable Hopf algebra*. There exist plenty of such algebras. For example, the Drinfeld double of a finite-dimensional Hopf algebra B is factorizable provided that the square of the antipode of B obeys a certain condition [KaR, Thm. 3]. This includes e.g. the Drinfeld doubles of finite groups (for which explicit formulas for the morphisms $\mathrm{Cor}_{g;p,q}$ (5.4) can be found in [FFSS]). Another large class $\{U_{(N,\nu,\omega)}\}$ (with $N > 1$ an odd integer, ω a primitive Nth root of unity, and $\nu < N$ a positive integer such that N does not divide ν^2) of factorizable Hopf algebras is described in [Ra2, Sect. 5.2]. $U_{(N,\nu,\omega)}$ has dimension $N^3/(N,\nu^2)^2$ [Ra2, Prop. 10b)], and this family comprises the small quantum group that is a finite-dimensional quotient of $U_q(\mathrm{sl}(2))$ as the special case $\nu = 2$ with $q = \omega^{-2}$, compare [Ra2, p. 260)] and [LyM, Prop. 4.6].

We also need the notions of integrals and cointegrals for Hopf algebras. A *left integral* of H is an element $\Lambda \in H$ obeying

$$m \circ (id_H \otimes \Lambda) = \Lambda \circ \varepsilon\,, \tag{A.28}$$

or, in other words, a morphism of left H-modules from the trivial H-module $(\Bbbk, \varepsilon)$ to the regular H-module (H, m). Dually, a *right cointegral* of H is an element $\lambda \in H^*$ that satisfies

$$(\lambda \otimes id_H) \circ \Delta = \eta \circ \lambda\,. \tag{A.29}$$

Right integrals and left cointegrals are defined analogously.

For a finite-dimensional Hopf algebra there is, up to normalization, a unique non-zero left integral Λ and a unique non-zero right cointegral λ, and the number $\lambda \circ \Lambda \in \Bbbk$ is invertible. Also, the antipode s of H is invertible. If H is quasitriangular, then the square of the antipode is an inner automorphism, acting as $h \mapsto u^{-1}\, h\, u$ with $u \in H$ the *Drinfeld element*

$$u := m \circ (\mathrm{s} \otimes id_H) \circ R_{21}\,. \tag{A.30}$$

And if H is factorizable, then it is *unimodular*, i.e. the left integral Λ is also a right integral, which implies that $\mathrm{s} \circ \Lambda = \Lambda$. Moreover, $f_Q(\lambda)$ is an integral, too, and thus is a non-zero multiple of Λ. One may then fix the normalizations of the integral and cointegral in such a way that

$$\lambda \circ \Lambda = 1 \qquad \text{and} \qquad f_Q(\lambda) = \Lambda\,. \tag{A.31}$$

This convention is adopted throughout this paper; it determines Λ and λ uniquely except for a common sign factor.

A.5 Representations of mapping class groups

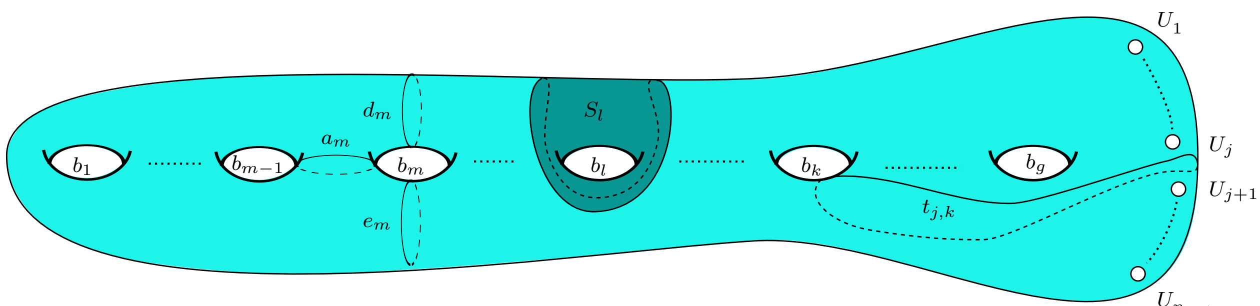
As has been established in [Ly3], the spaces (3.4) of chiral blocks come with natural representations of mapping class groups for surfaces with holes (that is, with open disks excised). To describe these or, rather, the representations on spaces of blocks with outgoing instead of incoming field insertions (see formula (A.34) below), it is convenient to present these groups through generators (and relations, but these are irrelevant for us, as we are interested in invariants). A suitable set of generators of $\mathrm{Map}_{g:n}$, the mapping class group of genus-g surfaces with n holes, is obtained by noticing the exact sequence

$$1 \longrightarrow B_{g,n} \longrightarrow \mathrm{Map}_{g:n} \longrightarrow \mathrm{Map}_{g;0} \longrightarrow 1 \tag{A.32}$$

of groups, where $B_{g,n}$ is a central extension of the surface braid group by $\mathbb{Z}^n$ (compare Theorem 9.1 of [FM]). As a set of generators one may thus take the union of those for some known presentations of $\mathrm{Map}_{g;0}$ [Wa] and of $B_{g,n}$ [Sco]. This amounts to the following (non-minimal) system of generators [Ly1, Ly3]:

1. Braidings which interchange neighboring boundary circles.
2. Dehn twists about boundary circles.
3. Homeomorphisms S_l, for $l = 1, 2, \dots, g$, which act as the identity outside a certain region $\mathcal{T}_l$ and as a modular S-transformation in a slightly smaller region $\mathcal{T}_l' \subset \mathcal{T}_l$ that has the topology of a one-holed torus.
   (For the relevant regions $\mathcal{T}_l$ and $\mathcal{T}_l'$, as well as the cycles appearing in the subsequent entries of the list, see the picture below.)
4. Dehn twists in tubular neighborhoods of certain cycles a_m and e_m, for $m = 2, 3, \dots, g$.
5. Dehn twists in tubular neighborhoods of certain cycles b_m and d_m, for $m = 1, 2, \dots, g$.
6. Dehn twists in tubular neighborhoods of certain cycles $t_{j,m}$, for $j = 1, 2, \dots, n-1$ and $m = 1, 2, \dots, g$.

In particular, for the torus without holes ($g = 1$ and $n = 0$), the generators $S = S_1$ and $T = d_1$ furnish the familiar S- and T-transformations which generate the modular group $\mathrm{SL}(2, \mathbb{Z})$. The regions $\mathcal{T}_l$ and $\mathcal{T}_l'$ and cycles a_m, b_m, e_m, d_m, and $t_{j,m}$ are exhibited in the following picture:



(A.33)

$\mathcal{T}_l$ is the shaded region in (A.33); it is a one-holed torus forming a neighborhood of the lth handle, while $\mathcal{T}_l'$ is the smaller region indicated by the dotted line inside $\mathcal{T}_l$.

Also shown in (A.33) are decorations of the boundary circles by objects $U_1, U_2, \dots, U_n$ of a factorizable finite tensor category $\mathcal{C}$. The representation of $\mathrm{Map}_{g:n}$ constructed in [Ly3] acts on the space

$$V_{g:n}^U := \mathrm{Hom}_{\mathcal{C}}(L^{\otimes g}, U) \tag{A.34}$$

of morphisms of $\mathcal{C}$, where

$$U := \bigoplus_{\sigma\in\mathfrak{N}} U_{\sigma(1)} \otimes U_{\sigma(2)} \otimes \cdots \otimes U_{\sigma(n)} \,, \tag{A.35}$$

with L the handle Hopf algebra (3.7) and with $\mathfrak{N} = \mathfrak{N}(U_1,...,U_n)$ the subgroup of the symmetric group $\mathfrak{S}_n$ generated by those permutations σ for which for at least one value of i the objects U_i and $U_{\sigma(i)}$ are non-isomorphic. In this representation $\pi_{g,n}^U$ of $\mathrm{Map}_{g:n}$ the different types of generators described in the list above act on the space (A.34) as follows (see [FSS3, Prop. 2.4]):

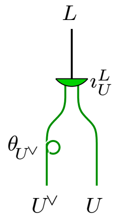
1. Post-composition with a braiding morphism which interchanges the objects that label neighboring field insertions.
2. Post-composition with a twist isomorphism of the object labeling a field insertion.
3. Pre-composition with an isomorphism $id_{L^{\otimes g-l}} \otimes S^L \otimes id_{L^{\otimes l-1}} \in \mathrm{End}_{\mathcal{C}}(L^{\otimes g})$, for $l = 1, 2, \dots, g$.
4. Pre-composition with an isomorphism $id_{L^{\otimes g-m}} \otimes [\mathcal{O}^L \circ (T^L \otimes T^L)] \otimes id_{L^{\otimes m-2}}$, respectively $id_{L^{\otimes g-m}} \otimes \big[(T^L \otimes \theta_{L^{\otimes m-1}}) \circ \mathcal{Q}^L_{L^{\otimes m-1}}\big]$, for $m = 2, 3, \dots, g$.
5. Pre-composition with an isomorphism $id_{L^{\otimes g-m}} \otimes (S^{L^{-1}} \circ T^L \circ S^L) \otimes id_{L^{\otimes m-1}}$, respectively $id_{L^{\otimes g-m}} \otimes T^L \otimes id_{L^{\otimes m-1}}$, for $m = 1, 2, \dots, g$.
6. The map that sends $f \in \mathrm{Hom}_{\mathcal{C}}(L^{\otimes g}, U_1 \otimes \cdots \otimes U_n) \subseteq V_{g:n}^U$ to

$$\begin{aligned} \Big(\Big[(id_{U_1\otimes\cdots\otimes U_j} \otimes \tilde{d}_{U_{j+1}\otimes\cdots\otimes U_n}) \circ (f \otimes id_{{}^\vee U_n\otimes\cdots\otimes{}^\vee U_{j+1}}) \\ \circ \{ id_{L^{\otimes g-m}} \otimes [\mathcal{Q}^L_{L^{\otimes m-1}\otimes{}^\vee U_n\otimes\cdots\otimes{}^\vee U_{j+1}} \circ (T^L \otimes \theta_{L^{\otimes m-1}\otimes{}^\vee U_n\otimes\cdots\otimes{}^\vee U_{j+1}})]\} \Big] \\ \otimes id_{U_{j+1}\otimes\cdots\otimes U_n} \Big) \circ \big(id_{LL^{\otimes g}} \otimes \tilde{b}_{U_{j+1}\otimes\cdots\otimes U_n} \big) \end{aligned} \tag{A.36}$$

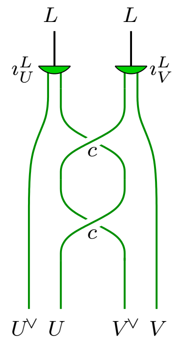
   and acts analogously on the other direct summands $\mathrm{Hom}_{\mathcal{C}}(L^{\otimes g}, U_{\sigma(1)} \otimes U_{\sigma(2)} \otimes \cdots \otimes U_{\sigma(n)})$ of $V_{g:n}^U$, with $\sigma \in \mathfrak{N}$, for $j = 1, 2, \dots, n-1$ and $m = 1, 2, \dots, g$.

Here we have introduced the abbreviations S^L, T^L, $\mathcal{O}^L$, and $\mathcal{Q}^L_W$ for $W \in \mathcal{C}$, for specific morphisms of $\mathcal{C}$ involving tensor powers of L. These morphisms are defined, with the help of dinatural families, by

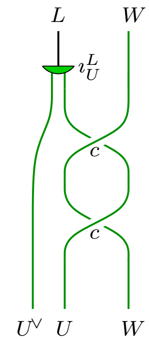
$$T^L \circ \imath_U^L \; :=$$



$$\mathcal{O}^L \circ (\imath_U^L \otimes \imath_V^L) \; :=$$



$$\mathcal{Q}_W^L \circ (\imath_U^L \otimes id_W) \; :=$$



(A.37)

while

$$S^L := (\varepsilon_L \otimes id_L) \circ \mathcal{O}^L \circ (id_L \otimes \Lambda_L)\,. \tag{A.38}$$

For applications in CFT, we also need to generalize the prescriptions above to the situation that there are both outgoing and incoming field insertions. To treat this case, we must partition the set of boundary circles into two subsets having, say, p and q and elements. Denoting the objects labeling the corresponding insertions by $U_1, U_2, \dots, U_p$ and by $W_1, W_2, \dots, W_q$, respectively, we can define objects U and W analogously as in (A.35) and consider the linear isomorphism

$$\varphi: \quad \mathrm{Hom}_{\mathcal{C}}(L^{\otimes g} \otimes W, U) \xrightarrow{\;\cong\;} \mathrm{Hom}_{\mathcal{C}}(L^{\otimes g}, U \otimes W^\vee) \tag{A.39}$$

that is supplied by the right duality of $\mathcal{C}$. Then by setting

$$\pi_{g,p,q}^{W,U}(\gamma) := \varphi^{-1} \circ \pi_{g,p+q}^{U\otimes W^\vee}(\gamma) \circ \varphi \tag{A.40}$$

for $\gamma \in \mathrm{Map}_{g:p+q}$ we obtain a representation of the subgroup $\mathrm{Map}_{g:p,q}$ of the mapping class group $\mathrm{Map}_{g:p+q}$ that leaves each subset of circles separately invariant, on the space $\mathrm{Hom}_{\mathcal{C}}(L^{\otimes g} \otimes W, U)$.

Also, in the application to correlation functions of bulk fields in full CFT, we deal with the category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ instead of $\mathcal{C}$, and accordingly with the *bulk* handle Hopf algebra K instead of L. Then in particular for $\mathcal{C} \simeq H$-Mod the S- and T-transformations result in the pictures (5.8) for S^K and T^K presented in the main text. For further details we refer to [FSS3, St], e.g. $\mathcal{O}^K$ for $\mathcal{C} \simeq H$-Mod is given by formula (4.2) of [FSS3].

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