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The Clique Density Theorem

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THE CLIQUE DENSITY THEOREM.

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ABSTRACT. Consider the following question: Fix an integer $r \geqslant 3$ as well as a real parameter $\gamma \in [0, 1/2)$ and let n be a large integer. How many cliques of size r must a graph on n vertices with at least γn^2 edges necessarily contain? As n tends to infinity, the answer turns out to be of the form

$$F_r(\gamma) \cdot n^r + O(n^{r-2})$$

for some constant $F_r(\gamma)$ the precise value of which has been conjectured by LOVÁSZ and SIMONOVITS in the 1970s. Despite of some impressive progress by BOLLOBÁS and these two authors themselves, their conjecture has remained widely open until very recently when RAZBOROV and NIKIFOROV managed to solve the cases $r = 3$ and $r = 4$, respectively. In the present paper we prove this Clique Density Conjecture inductively for all values of r. An interesting corollary to this result asserts that the graphs that are extremal with respect to this problem do not depend much on r – a fact that can be made precise by talking about graphs with weighted vertices.

1. INTRODUCTION.

Given an integer $r \geqslant 3$ and a graph $\mathcal{G}$ with n vertices and γn^2 edges, where $0 \leqslant \gamma < \frac{1}{2}$, how many cliques of size r must $\mathcal{G}$ necessarily have? This question presents itself almost naturally once one knows about TURÁN'S classical theorem ([11]), which informs us in particular of the mere existence of such cliques as soon as γ becomes larger than $\frac{r-2}{2(r-1)}$ and that this bound is indeed sharp. In awareness of the structure of the unique graphs which are extremal with respect to the absence of r–cliques, namely complete $(r-1)$–partite graphs the sizes of whose vertex classes are as close to one another as possible, it is quite tempting to propose the following construction for guessing the answer to the more general question we are just discussing: Form for some positive integer s a complete $(s+1)$–partite graph all of whose vertex classes except for one perhaps substantially smaller one are of equal size in such a way that the number of its edges is about γn^2 and count the number of r–cliques the graph thereby obtained possesses. Speaking rather

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roughly, ignoring for a moment the phenomenon that not all real numbers are integers and interpreting the word "about" in the foregoing sentence as meaning "precisely", this can be accomplished in exactly one way: Namely, one determines an integer $s \geqslant 1$ for which $\gamma \in \left[\frac{s-1}{2s}, \frac{s}{2(s+1)}\right]$, takes $\alpha \in \left[0, \frac{1}{s}\right]$ to be a solution of $\gamma = \frac{s}{2(s+1)}(1-\alpha^2)$ and arranges the s larger vertex classes of the multipartite graph one intends to exhibit to consist of exactly $\frac{n(1+\alpha)}{s+1}$ vertices, whereas the remaining one comprises just $\frac{n(1-s\alpha)}{s+1}$ vertices. The resulting graph is easily computed to contain

$$\frac{1}{(s+1)^r}\binom{s+1}{r}(1+\alpha)^{r-1}\left(1-(r-1)\alpha\right)\cdot n^r$$

cliques of size r. This calculation motivates the clique Clique Density Conjecture proposed by Lovász and Simonovits (see [6]), which is the main result of this paper, an that reads as follows:

Conjecture 1.1. *If $r \geqslant 3$ and $\gamma \in \left[0, \frac{1}{2}\right)$, then every graph on n vertices with at least γn^2 edges contains at least*

$$\frac{1}{(s+1)^r}\binom{s+1}{r}(1+\alpha)^{r-1}\left(1-(r-1)\alpha\right)\cdot n^r$$

cliques of size r, where $s \geqslant 1$ is an integer for which $\gamma \in [\frac{s-1}{2s}, \frac{s}{2(s+1)}]$ and $\alpha \in [0, \frac{1}{s}]$ is implicitly defined by $\gamma = \frac{s}{2(s+1)}(1-\alpha^2)$.

Before we start to prove this, a few historical and mathematical remarks are in order and to make the former ones more perspicuous we start with the latter ones.

First one observes that if $s \leqslant r-2$, then the binomial coefficient $\binom{s+1}{r}$ vanishes, which means that in accordance with Turán's result no r–cliques are guaranteed to exist whenever $\gamma \leqslant \frac{r-2}{2(r-1)}$. As soon as γ passes this threshold value by an absolute positive amount however tiny, the minimal number of r–cliques starts to become proportional to n^r. Second, if γ happens to be of the form $\frac{t}{2(t+1)}$ for some positive integer t, then there are two legitimate choices for the pair (s, α), namely $(t, 0)$ and $(t+1, \frac{1}{t+1})$. Yet it is not hard to verify that both of these give rise to the same lower bound of

$$\frac{1}{(t+1)^r}\binom{t+1}{r}\cdot n^r$$

on the number of r–cliques. Incidentally, as we shall see below, these boundary cases are the easiest to deal with. Third, it should be clear in the light of the considerations sketched above that this claim, if true, is asymptotically best possible when we think of r and γ as being fixed and of n as tending to infinity. In other words, the bizarre expression appearing in the above formula before the multiplication dot is insofar "correct" as it is the best possible one not depending on n. On the much more ambitious question where one also regards n as being fixed and asks for the minimal number of r–cliques in an absolute sense, we will not say anything intelligent in this paper going beyond Conjecture 1.1. It nevertheless appears worth while to mention at this point an observation apparently due to NIKIFOROV ([7]) without giving further details here: If one repeats the above calculation more carefully, i.e. using real integers instead of their approximations, one may see that for fixed values of γ and r the error inherent in the above formula is of a lower order than one would perhaps guess at first, i.e. it is not only $O(n^{r-1})$, but even $O(n^{r-2})$.

We now turn to some historical comments: The apart from TURÁN'S work first attempt to bound the number $k_r(\mathcal{G})$ of r–Cliques contained in a graph $\mathcal{G}$ from below was undertaken by GOODMAN ([5]) who proved $k_3(\mathcal{G}) \geqslant \frac{1}{3}\gamma(4\gamma - 1)n^3$ whenever $\mathcal{G}$ has n vertices and at least γn^2 edges. This was extended by LOVÁSZ and SIMONOVITS ([6]) to

$$k_r(\mathcal{G}) \geqslant \frac{1}{r!} \cdot 2\gamma(4\gamma-1)(6\gamma-2) \cdot \ldots \cdot (2(r-1)\gamma - (r-2)) \cdot n^r$$

for all $r \geqslant 3$ and $\gamma \geqslant \frac{r-2}{2(r-1)}$. It might be helpful to observe that this lower bound agrees with Conjecture 1.1 whenever $\gamma = \frac{t}{2(t+1)}$ where $t \geqslant r-2$ denotes an integer, and that it is piecewise convex as a function of γ between these values. The first and in this generality as far as we know hitherto never defeated improvement over this result was obtained by BOLLOBÁS ([1], see also section 1 of chapter VI from [2]), who devised a miraculous argument demonstrating that the piecewise linear function interpolating between these boundary values also serves as a lower bound of $k_r(\mathcal{G})$. The first person who almost achieved a whole continuum of cases of Conjecture 1.1 was FISHER ([3]) who attacked the case $r = 3$ and $s = 2$ by means of ideas belonging to the spectral theory of graphs and relying on a then unproved hypothesis regarding the zeros of clique polynomials he managed to solve this case; the conjecture on which he relied has later been established by GOLDWURM and SANTINI ([4]). An altogether different proof of this case has later been given by RAZBOROV in the fifth section of [9]. The approach described there is of a highly infinitary nature and is based upon what one might call the "differential calculus of flag algebra homomorphisms", which in turn constitutes an important part of RAZBOROV'S flag algebraic investigations that are designed to provide us with a framework for thinking about problems from extremal combinatorics in general in a much more systematic fashion than it has ever been accomplished before. Shortly afterwards, Razborov dedicated a whole paper to the application of his calculus to the minimal triangle density of graphs with given edge density in which he entirely proved the case $r = 3$ of Conjecture 1.1. His argument proceeds by induction on s. Upon seeing it, Nikiforov realized that one could get finitary analogues of some of Razborov's equations in a setting similar to the one utilized by Motzkin and Strauss in their alternative proof of Turán's theorem ([8]) by exploiting Lagrange multipliers and this insight allowed him to give another proof of this result in a more structural spirit and dealing with all values of s at the same time. With some modifications, this proof applies to the case $r = 4$ as well ([7]). Given these quite rapid recent developments, we may think of ourselves as "standing on the shoulders of giants", as Sir Isaac Newton would express it. Slightly less informally, we shall try to explain below how to emulate still more substantial fragments of Razborov's flag algebraic differential calculus in the finitary analytical setting used by Nikiforov and then perform a technically more elaborate version of Razborov's original argument on triangles.

It should be pointed out at this occasion, however, that the preference we have given to a rather concrete presentation over entirely abstract manipulations exclusively serves expository purposes and is in no way enforced by the structure of our argument itself. On the contrary, the reader who has seriously studied [9] and [10] will have no difficulty whatsoever in recasting the relevant passages of this paper in terms of flag algebra homomorphisms.

2. Weighted Graphs.

Given a set X and a positive integer r, we use $X^{(r)}$ to denote the collection of all r–element subsets of X. Also, if n refers to a positive integer, then $[n]$ is, by definition, shorthand for $\{1, 2, \ldots, n\}$. By a *weighted graph of order* n, we mean a pair consisting of a sequence $(x_1, x_2, \ldots, x_n)$ of non–negative real numbers the sum of which is equal to 1 and a function $\mathcal{A} : [n]^{(2)} \longrightarrow [0, 1]$. Whenever $\mathcal{G}$ is such a weighted graph of order n and r is a positive integer, we set

$$\mathcal{G}(K_r) = \sum_{M \in [n]^{(r)}} \prod_{E \in M^{(2)}} \mathcal{A}(E) \prod_{i \in M} x_i.$$

Notice for instance that

$$\mathcal{G}(K_1) = x_1 + x_2 + \ldots + x_n = 1.$$

The following variant of Conjecture 1.1 is, as we are soon going to see, equivalent to it and has to the best of our knowledge first been formulated explicitly by NIKIFOROV.

Claim 2.1. *Let $r \geqslant 3$ denote an integer and $\mathcal{G}$ a weighted graph. Suppose that a positive integer s and a real number $\alpha \in \left[0, \frac{1}{s}\right]$ are chosen in such a way that $\mathcal{G}(K_2) = \frac{s}{2(s+1)}(1-\alpha^2)$. Then*

$$\mathcal{G}(K_r) \geqslant \frac{1}{(s+1)^r}\binom{s+1}{r}(1+\alpha)^{r-1}\left(1-(r-1)\alpha\right).$$

To see that this indeed entails Conjecture 1.1, take a graph $\mathcal{G}$ and an integer $r \geqslant 3$ about which you want to know the latter statement, label the vertices of $\mathcal{G}$ arbitrarily as $\{v_1, v_2, \ldots, v_n\}$ and construct a weighted graph $\mathcal{G}'$ of order n by the stipulations $x_1 = x_2 = \ldots = x_n = \frac{1}{n}$ and

$$\mathcal{A}(\{i,j\}) = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are joined in } \mathcal{G} \text{ by an edge,} \\ 0 & \text{otherwise} \end{cases}$$

for all $\{i,j\} \in [n]^{(2)}$. Plainly $\mathcal{G}$ has exactly $\mathcal{G}'(K_2) \cdot n^2$ edges and $\mathcal{G}'(K_r) \cdot n^r$ cliques of size r, which proves the desired implication.

As we shall not need the converse direction, we only give a sketch of its proof. Given a weighted graph of order n specified by the sequence $(x_1, x_2, \ldots, x_n)$ of real numbers as well as a function $\mathcal{A} : [n]^{(2)} \longrightarrow [0,1]$, take a very large integer k and form a graph $\mathcal{H}$ whose vertices fall into n independent classes $V_1, V_2, \ldots, V_n$ whose sizes are approximately $kx_1, kx_2, \ldots, kx_n$, respectively and in which for each $\{i,j\} \in [n]^{(2)}$ roughly a proportion of $\mathcal{A}(\{i,j\})$ among all possible edges from V_i to V_j is present in a sufficiently quasirandom way. Such a graph $\mathcal{H}$ can in particular be arranged to have k vertices, $\mathcal{G}(K_2) \cdot k^2 \pm O(k)$ edges and $\mathcal{G}(K_r) \cdot k^r \pm O(k^{r-1})$ cliques of size r, so letting k tend to infinity we might in fact derive Claim 2.1 from Conjecture 1.1.

Throughout the rest of this paper, we follow NIKIFOROV'S suggestion to think about the clique density problem in terms of weighted graphs, as this allows us to apply our knowledge concerning continuous and piecewise differentiable or convex functions rather directly without necessitating the usage of flag algebraic circumlocutions. In other words, among all possible limits of graphs one could refer to in terms of flag algebra homomorphisms we only exploit a few somewhat special ones that are rather easily visualized.

3. The bound obtained by Lovász and Simonovits.

We are now going to provide an analogue of the inequality due to Lovász and Simonovits formulated in the language of weighted graphs. Although it is clearly weaker then Claim 2.1 and thus soon going to be superseded, we are nevertheless needing it for two purposes. First, it gives an optimal result if $\gamma = \frac{t-1}{2t}$ for some positive integer t and our later arguments do not apply to that case – basically because the function by means of which $\mathcal{G}(K_r)$ is going to be estimated in terms of γ is not differentiable at those values. Second, it will tell us that if we choose a counterexample to Claim 2.1 that is extremal in a sense on which we shall elaborate below, then it still cannot be wicked to such an extent that the analytical estimates we intend to apply later on are precluded from leading to the envisaged contradiction – despite the fact that they are in some sense only "locally true".

Clearly the easiest way to obtain the inequality stated as Proposition 3.1 below would have been to accept the result of Lovász and Simonovits as a matter of fact and then to argue that the proposition itself follows from it by using the same construction we have indicated at the end of the previous section. Yet there is a reason which makes us thinking that it might be useful to include the argument presented below into this paper. Namely in section 5 the reader is going to encounter a very similar but technically somewhat more demanding computation. So what we are doing here might be regarded as a helpful preparation for other things that follow.

Proposition 3.1. *Given a weighted graph $\mathcal{G}$ and an integer $r \geqslant 2$ such that the quantity $\gamma = \mathcal{G}(K_2)$ is not smaller than $\frac{r-2}{2(r-1)}$, we have*

$$\mathcal{G}(K_r) \geqslant \frac{1}{r!} \cdot 2\gamma(4\gamma-1)(6\gamma-2)\cdot\ldots\cdot(2(r-1)\gamma-(r-2)).$$

Proof. Notice that this follows by means of an easy induction on r from the following statement:

(∗) *If a weighted graph $\mathcal{G}$ satisfies $\gamma = \mathcal{G}(K_2) \geqslant \frac{r-2}{2(r-1)}$ for some $r \geqslant 2$, then*

$$\mathcal{G}(K_r) \geqslant \frac{2(r-1)\gamma-(r-2)}{r} \cdot \mathcal{G}(K_{r-1}) \qquad \textit{and} \qquad \mathcal{G}(K_{r-1}) > 0.$$

.

For this reason, it suffices to verify (∗) instead, which will again be done by induction on r. The case $r = 2$ of (∗) is obvious in view of $\mathcal{G}(K_1) = 1$. So suppose now that $\mathcal{G}$ is a weighted graph satisfying $\gamma = \mathcal{G}(K_2) \geqslant \frac{r-1}{2r} > \frac{r-2}{2(r-1)}$ for some $r \geqslant 2$ and

$$r \cdot \mathcal{G}(K_r) \geqslant (2(r-1)\gamma - (r-2)) \cdot \mathcal{G}(K_{r-1}) \qquad \text{as well as} \qquad \mathcal{G}(K_{r-1}) > 0.$$

For the induction step, we cursorily remark that these assumptions trivially entail $\mathcal{G}(K_r) > 0$ so that it only remains to estimate $\mathcal{G}(K_{r+1})$ from below in terms of $\mathcal{G}(K_r)$. Let n denote the order of $\mathcal{G}$ and suppose that $\mathcal{G}$ is given by the sequence $(x_1, x_2, \ldots, x_n)$ of reals as well as the function $\mathcal{A} : [n]^{(2)} \longrightarrow [0,1]$.

For each $M \subseteq [n]$ we write

$$A_M = \prod_{E \in M^{(2)}} \mathcal{A}(E) \qquad \text{and} \qquad X_M = \prod_{i \in M} x_i.$$

Now consider any $M \in [n]^{(r+1)}$ and define

$$B_M = \sum_{E \in M^{(2)}} \prod_{F \in M^{(2)} - \{E\}} \mathcal{A}(F),$$

as well as

$$C_M = \sum_{N \in M^{(r)}} A_N.$$

We claim that these expressions satisfy

$$(\circledast) \qquad 2B_M - C_M \leqslant (r^2-1)A_M.$$

To see this, we note that our inequality is linear in each of its variables $\mathcal{A}(E)$, where $E \in M^{(2)}$, which entails that we only need to look at the case where $\mathcal{A}(E) \in \{0,1\}$ for all $E \in M^{(2)}$. Now if additionally $K = \#\{E \in M^{(2)} \mid \mathcal{A}(E) = 0\}$ is at least 2, then $A_M = B_M = 0$ and $C_M \geqslant 0$; if $K = 1$ then $A_M = 0$, $B_M = 1$, $C_M = 2$, and finally if $K = 0$ then $A_M = 1$, $B_M = \frac{1}{2}r(r+1)$ and $C_M = r+1$. This completes the proof of ($\circledast$).

Multiplying our estimate by X_M and summing over all possibilities for M, we infer

$$(\circledast) \qquad \sum_{M \in [n]^{(r+1)}} (2B_M - C_M)X_M \leqslant (r^2-1)\mathcal{G}(K_{r+1}).$$

Setting

$$\eta_L = \sum_{i \in [n]-L} x_i \prod_{\ell \in L} \mathcal{A}(\{i, \ell\})$$

for all $L \in [n]^{(r-1)}$, we now investigate the sum

$$\sum_{L \in [n]^{(r-1)}} A_L X_L \eta_L^2.$$

Expanding the squares, we get several purely quadratic terms whose sum may in view of the inequality $\mathcal{A}(E)^2 \leqslant \mathcal{A}(E)$, that is valid for all $E \in [n]^{(2)}$, be bounded from above by

$$\sum_{Q\in[n]^{(r)}} A_Q X_Q \Big(1 - \sum_{i\in[n]-Q} x_i\Big) = \mathcal{G}(K_r) - \sum_{M\in[n]^{(r+1)}} C_M X_M .$$

Moreover, we get several mixed terms that sum up to

$$2 \cdot \sum_{M\in[n]^{(r+1)}} B_M X_M .$$

Combining this with (⊛), we get

$$\sum_{L\in[n]^{(r-1)}} A_L X_L \eta_L^2 \leqslant \mathcal{G}(K_r) + \sum_{M\in[n]^{(r+1)}} (2B_M - C_M) X_M \leqslant \mathcal{G}(K_r) + (r^2-1)\mathcal{G}(K_{r+1}).$$

Substituting this together with

$$\sum_{L\in[n]^{(r-1)}} A_L X_L \eta_L = r \cdot \sum_{Q\in[n]^{(r)}} A_Q X_Q = r \cdot \mathcal{G}(K_r)$$

and

$$\sum_{L\in[n]^{(r-1)}} A_L X_L = \mathcal{G}(K_{r-1})$$

into the inequality

$$\Bigg(\sum_{L\in[n]^{(r-1)}} A_L X_L \eta_L\Bigg)^2 \leqslant \sum_{L\in[n]^{(r-1)}} A_L X_L \cdot \sum_{L\in[n]^{(r-1)}} A_L X_L \eta_L^2,$$

which is an instance of the Cauchy–Schwarz–Inequality, we deduce

$$r^2 \mathcal{G}(K_r)^2 \leqslant \mathcal{G}(K_{r-1}) \left(\mathcal{G}(K_r) + (r^2-1)\mathcal{G}(K_{r+1})\right).$$

Invoking now the induction hypothesis, we obtain, after a permissable cancelation of $\mathcal{G}(K_{r-1})$,

$$(2(r-1)r\gamma - r(r-2))\, \mathcal{G}(K_r) \leqslant \mathcal{G}(K_r) + (r^2-1)\mathcal{G}(K_{r+1})$$

and hence indeed

$$(2r\gamma - (r-1))\, \mathcal{G}(K_r) \leqslant (r+1)\mathcal{G}(K_{r+1}),$$

which completes the induction step. This finally proves (∗) and thus the proposition. □

Remark 3.2. The reader may find it both instructive and amusing to rewrite this argument in terms of flag algebra homomorphisms.

We would now like to make those consequences of Proposition 3.1 explicit that we shall really utilize in the sequel.

Corollary 3.3. *Suppose that r and s are integers satisfying $r \geqslant 2$ and $s \geqslant r-1$. Then for every weighted graph $\mathcal{G}$ satisfying $\gamma = \mathcal{G}(K_2) > \frac{s-1}{2s}$ one has*

$$\mathcal{G}(K_r) > \frac{1}{s} \cdot \binom{s}{r} \cdot \left(\frac{2\gamma}{s-1}\right)^{r-1}.$$

Proof. Clearly $\frac{s-1}{2s} \geqslant \frac{r-2}{2(r-1)}$, wherefore Proposition 3.1 tells us

$$\mathcal{G}(K_r) \geqslant \frac{1}{r!} \cdot 2\gamma(4\gamma-1)(6\gamma-2) \cdot \ldots \cdot (2(r-1)\gamma-(r-2)).$$

Now for each $i \in \{1, 2, \ldots, r-1\}$ we have

$$2i\gamma - (i-1) > 2\gamma\left(i - \frac{s(i-1)}{s-1}\right) = \frac{2\gamma(s-i)}{s-1} \geqslant 0$$

and hence

$$\mathcal{G}(K_r) > \left(\frac{2\gamma}{s-1}\right)^{r-1} \cdot \frac{(s-1)\cdot \ldots \cdot (s-r+1)}{r!} = \frac{1}{s} \cdot \binom{s}{r} \cdot \left(\frac{2\gamma}{s-1}\right)^{r-1}.$$

□

Corollary 3.4. *Under the additional assumption $\alpha \in \left\{0, \frac{1}{s}\right\}$, Claim 2.1 holds.*

Proof. As we have already seen in the introduction, we have in these cases $\gamma = \frac{t}{2(t+1)}$ for some non–negative integer t. If $t \leqslant r-2$ our claim is obvious, which means that we may suppose $t \geqslant r-1$ from now on. Thus γ is large enough for Proposition 3.1 to be applicable and the desired result follows. □

4. Some analytical preparations.

Now we intend to provide a thorough analysis of the function occurring in Claim 2.1. All of the observations we are going to make are rather easily established, but the amount of calculation on which some of them rely is not negligible. So in order to help the main proof of this paper to appear a bit shorter than it actually is, it seems to be advantageous to gather these rather routine computations in a separate section, the present one. The lazy and trustful reader who just wants to know how the clique density problem may be solved without digesting every mundane detail can safely take all the statements he finds here on belief and ignore their occasionally laborious verifications. We would also like to precaution the reader that we did not even attempt to motivate what is going to happen in the following pages. It might therefore be a sensible idea to interrupt the reading of this section once the function H makes its first appearance. Thereafter one may directly jump to the next and hopefully more interesting section large portions of which should be fairly understandable then and whose remaining places should offer an intrinsic explanation of why certain properties of F_r and related entities to be found here are worth our while to know about.

Throughout this section, we fix integers $r \geqslant 3$ and $s \geqslant r-1$ as well as a real number $M \geqslant 1$ satisfying

$$\left(\frac{s-1}{s}\right)^{r-2} > \frac{s-r+1}{s-1} \cdot M^{r-2}.$$

Define the function $F_r : \left[0, \frac{1}{2}\right) \longrightarrow \left[0, \frac{1}{r!}\right)$ as follows: Given $\gamma \in \left[0, \frac{1}{2}\right)$, choose the unique positive integer t for which $\gamma \in \left[\frac{t-1}{2t}, \frac{t}{2(t+1)}\right)$, determine the real number $\alpha \in \left(0, \frac{1}{t}\right]$ solving the equation $\gamma = \frac{t}{2(t+1)}(1-\alpha^2)$ and set

$$F_r(\gamma) = \frac{1}{(t+1)^r}\binom{t+1}{r}(1+\alpha)^{r-1}\left(1-(r-1)\alpha\right).$$

In terms of this function, the statement of Claim 2.1 can be shortened to the inequality $\mathcal{G}(K_r) \geqslant F_r(\mathcal{G}(K_2))$, that is allegedly valid for all weighted graphs $\mathcal{G}$. We have more or less already seen earlier that F_r is continuous and clearly it is piecewise differentiable as well. Moreover, F_r is identically vanishing on the interval $\left[0, \frac{r-2}{2(r-1)}\right]$ and if $\gamma \in \left(\frac{t-1}{2t}, \frac{t}{2(t+1)}\right)$ for some integer $t \geqslant r-1$, then differentiating the equation locally defining $F_r(\gamma)$ with respect to α, we infer

$$-\frac{t\alpha}{t+1} \cdot F_r'(\gamma) = -\frac{(r-1)r}{(t+1)^r}\binom{t+1}{r}\alpha(1+\alpha)^{r-2}.$$

As $\alpha > 0$, it follows that

$$F_r'(\gamma) = \frac{(r-1)r}{t(t+1)^{r-1}}\binom{t+1}{r}(1+\alpha)^{r-2} > 0.$$

Thus F_r is strictly increasing on the interval $\left[\frac{r-2}{2(r-1)}, \frac{1}{2}\right)$ and as

$$\lim_{\gamma \longrightarrow 1/2} F_r(\gamma) = \lim_{t \longrightarrow \infty} \frac{1}{(t+1)^r}\binom{t+1}{r} = \frac{1}{r!}$$

it possesses an inverse $F_r^{-1} : \left[0, \frac{1}{r!}\right) \longrightarrow \left[\frac{r-2}{2(r-1)}, \frac{1}{2}\right)$. Moreover, the above expression for $F_r'(\gamma)$ decreases as α decreases, whence F_r is in an obvious sense piecewise concave. Note that the identity function F_2 on $\left[0, \frac{1}{2}\right)$ has essentially the same properties as F_r. This concludes our discussion of the most elementary properties of these functions.

Next we propose to look at the function $H : \left[\frac{r-2}{r-1} \cdot M, M\right] \longrightarrow \mathbb{R}_0^+$ given by

$$\eta \longmapsto \frac{1}{s^{r-1}} \binom{s}{r-1} \cdot \frac{(r-1)\eta - (r-2)M}{\eta^{r-1}}.$$

Claim 4.1. *The function H is strictly increasing and satisfies*

$$F_{r-1}\left(\frac{s-2}{2(s-1)}\right) < H(M) \leqslant F_{r-1}\left(\frac{s-1}{2s}\right).$$

Proof. If $\eta \in \left[\frac{r-2}{r-1} \cdot M, M\right)$, then

$$H'(\eta) = \frac{(r-2)(r-1)}{s^{r-1}} \binom{s}{r-1} \cdot \frac{M-\eta}{\eta^r} > 0,$$

which entails the first part of our claim. Furthermore, by our smallness condition imposed on M, we have

$$H(M) = \frac{1}{s^{r-1}} \binom{s}{r-1} \cdot \frac{1}{M^{r-2}} > \frac{s-r+1}{s(s-1)^{r-1}} \binom{s}{r-1} = \frac{1}{(s-1)^{r-1}} \binom{s-1}{r-1},$$

i.e.

$$H(M) > F_{r-1}\left(\frac{s-2}{2(s-1)}\right).$$

Finally, using $M \geqslant 1$, we get

$$H(M) = \frac{1}{s^{r-1}} \binom{s}{r-1} \cdot \frac{1}{M^{r-2}} \leqslant \frac{1}{s^{r-1}} \binom{s}{r-1} = F_{r-1}\left(\frac{s-1}{2s}\right).$$

□

Notice in particular that the composed function $F_{r-1}^{-1} \circ H$ is defined everywhere on the interval $\left[\frac{r-2}{r-1} \cdot M, M\right]$ and that its range is contained in $\left[\frac{r-3}{2(r-2)}, \frac{s-1}{2s}\right]$, which in turn lies in $\left[0, \frac{1}{2}\right)$. For $t \in \{r-2, r-1, \dots, s-1\}$ there exists a unique real number $\vartheta_t \in \left[\frac{r-2}{r-1} \cdot M, M\right]$ satisfying $H(\vartheta_t) = F_{r-1}\left(\frac{t-1}{2t}\right)$. The number ϑ_{s-1} will play a special role later and sometimes it will just be denoted by ϑ. Evidently one has

$$\tfrac{r-2}{r-1} \cdot M = \vartheta_{r-2} < \vartheta_{r-1} < \ldots < \vartheta_{s-1} = \vartheta < M.$$

Later on we shall need some rudimentary knowledge concerning the magnitude of these numbers.

Claim 4.2. *If the integer t belongs to the interval $[r-2, s-2]$, then $\vartheta_t \leqslant \frac{t}{t+1} \cdot M$. In addition, we have $\vartheta \geqslant \frac{s-1}{s} \cdot M$.*

Proof. Whenever t belongs to the specified interval, we have $H(\vartheta_{t+1}) \leqslant H(M)$, i.e.

$$\frac{1}{(t+1)^{r-1}}\binom{t+1}{r-1} \leqslant \frac{1}{s^{r-1}}\binom{s}{r-1}\frac{1}{M^{r-2}}.$$

Multiplying this by $(t-r+2)(t+1)^{r-2}/t^{r-1}$ we infer $H(\vartheta_t) \leqslant H(\frac{t}{t+1}\cdot M)$, thus proving the first part of our claim. Similarly but slightly easier we infer from $M \geqslant 1$ that $H(\frac{s-1}{s}\cdot M) \leqslant H(\vartheta)$, which leads to the second part of the claim. □

Claim 4.3. *If $\eta \in \left[\frac{r-2}{r-1}\cdot M, M\right]$ and $\nu = (F_{r-1}^{-1}\circ H)(\eta)$, then the function $Q : [0,\nu] \longrightarrow \mathbb{R}$ defined by*

$$\delta \longmapsto (r-1)\binom{s}{r-1}\delta - s^{r-1}\eta^{r-2}F_r(\delta)$$

attains its global maximum at $\delta = \nu$.

Proof. Choose an integer $t \in [r-2, s-1]$ as well as a real number $\beta \in [0, \frac{1}{t}]$ such that $\nu = \frac{t}{2(t+1)}(1-\beta^2)$. Since Q is piecewise convex and as convex functions attain their global maxima at boundary values, it suffices to establish the following two statements:

(A) The function Q is increasing on $\left[\frac{t-1}{2t}, \nu\right]$.
(B) If $d \in [t-1]$, then $Q(\frac{d-1}{2d}) \leqslant Q(\frac{t-1}{2t})$.

For the proofs of both of these subclaims, we use

(C) $M^{r-2}\cdot\frac{1}{t^{r-1}}\binom{t}{r-1} \leqslant \frac{1}{s^{r-1}}\binom{s}{r-1}$,

which is an obvious consequence of $H(\vartheta_t) \leqslant H(M)$. Now, to verify ($A$), take any $\delta \in \left(\frac{t-1}{2t}, \nu\right)$ and write $\delta = \frac{t}{2(t+1)}(1-\alpha^2)$, where $\alpha \in \left(\beta, \frac{1}{t}\right)$. Multiplying ($C$) by $(r-1)s^{r-1}$ one obtains

$$s^{r-1}M^{r-2}\frac{(r-1)r}{t(t+1)^{r-1}}\binom{t+1}{r}\left(\frac{t+1}{t}\right)^{r-2} \leqslant (r-1)\binom{s}{r-1}.$$

In view of $\eta \leqslant M$ and $\alpha \leqslant \frac{1}{t}$ this implies

$$s^{r-1}\eta^{r-2}F_r'(\delta) \leqslant (r-1)\binom{s}{r-1},$$

whence $Q'(\delta) \geqslant 0$. Thereby we have proved assertion (A).

Let us now turn our attention to (B). If $t \leqslant r-1$, then F_r vanishes at all relevant numbers and our claim is obvious. So henceforth we may suppose $t \geqslant r$ and for similar reasons $d \geqslant r-1$ as well. The function $\Phi : \left[\frac{1}{t}, \frac{1}{r-1}\right] \longrightarrow [0,1]$ defined by

$$x \longmapsto (1-x)(1-2x)\cdot\ldots\cdot(1-(r-1)x)$$

is obviously convex, wherefore

$$\frac{\Phi\left(\frac{1}{t}\right)-\Phi\left(\frac{1}{d}\right)}{\frac{1}{d}-\frac{1}{t}} \leqslant -\Phi'\left(\tfrac{1}{t}\right).$$

Since

$$-\Phi'\left(\tfrac{1}{t}\right)=\left\{\frac{t}{t-1}+\frac{2t}{t-2}+\ldots+\frac{(r-1)t}{t-r+1}\right\}\Phi\left(\tfrac{1}{t}\right)\leqslant\frac{(r-1)rt}{2(t-r+1)}\Phi\left(\tfrac{1}{t}\right),$$

it follows that

$$\begin{aligned}\frac{1}{t^r}\binom{t}{r}-\frac{1}{d^r}\binom{d}{r} &\leqslant \left(\frac{1}{d}-\frac{1}{t}\right)\frac{(r-1)r}{2(t-r+1)}\cdot\frac{1}{t^{r-1}}\binom{t}{r}\\ &=(r-1)\left(\frac{t-1}{2t}-\frac{d-1}{2d}\right)\cdot\frac{1}{t^{r-1}}\binom{t}{r-1}.\end{aligned}$$

Multiplying this by

$$s^{r-1}\eta^{r-2}\cdot\frac{1}{t^{r-1}}\binom{t}{r-1}\leqslant\binom{s}{r-1},$$

which is a consequence of (C) as $\eta \leqslant M$, we deduce

$$s^{r-1}\eta^{r-2}\left\{\frac{1}{t^r}\binom{t}{r}-\frac{1}{d^r}\binom{d}{r}\right\}\leqslant(r-1)\binom{s}{r-1}\left(\frac{t-1}{2t}-\frac{d-1}{2d}\right),$$

which is easily seen to be equivalent to (B). □

The following will only be used for $k = 2$ and $k = r$, but the general case is not really harder.

Claim 4.4. *For each integer $k \geqslant 2$ the function $J : [\frac{r-2}{r-1}\cdot M, M] \longrightarrow \mathbb{R}$ defined by*

$$\eta \longmapsto \eta^k(F_k \circ F_{r-1}^{-1} \circ H)(\eta)$$

is concave or convex on the intervals $[\vartheta_t, \vartheta_{t+1}]$, where $t = r-2, r-1, \ldots, s-2$, as well as $[\vartheta, M]$ depending on whether $k \geqslant r-1$ or $k \leqslant r-1$.

Proof. Treating both cases for k at the same time, we select any $t \in \{r-2, r-1, \ldots, s-1\}$ and intend to verify that the second derivative of J has the expected sign on $(\vartheta_t, \vartheta_{t+1})$, where for convenience $\vartheta_s = M$. Utilizing that $x \longmapsto \frac{2x-1}{x^2}$ is strictly increasing on $(0, 1)$ we may define a function $S : (\vartheta_t, \vartheta_{t+1}) \longrightarrow \left(\frac{t}{t+1}, 1\right)$ such that

$$(F_{r-1}^{-1}\circ H)(\eta)=\frac{t}{2(t+1)}\cdot\frac{2S(\eta)-1}{S(\eta)^2}$$

holds for all $\eta \in (\vartheta_t, \vartheta_{t+1})$. Since the right hand side may be rewritten as

$$\frac{t}{2(t+1)}\times\left\{1-\left(\frac{1}{S(\eta)}-1\right)^2\right\},$$

we have

$$\frac{1}{s^{r-1}}\binom{s}{r-1}\cdot\frac{(r-1)\eta-(r-2)M}{\eta^{r-1}}=\frac{1}{(t+1)^{r-1}}\binom{t+1}{r-1}\cdot\frac{(r-1)S(\eta)-(r-2)}{S(\eta)^{r-1}}.$$

Differentiating with respect to η and dividing by $(r-2)(r-1)$, we find

$$\frac{1}{s^{r-1}}\binom{s}{r-1}\cdot\frac{M-\eta}{\eta^{r}}=\frac{1}{(t+1)^{r-1}}\binom{t+1}{r-1}\cdot\frac{1-S(\eta)}{S(\eta)^{r}}\cdot S'(\eta),$$

and the combination of both equations yields

$$S'(\eta)=\frac{S(\eta)(M-\eta)\left[(r-1)S(\eta)-(r-2)\right]}{\eta(1-S(\eta))\left[(r-1)\eta-(r-2)M\right]}.$$

Furthermore

$$J(\eta)=\frac{1}{(t+1)^{k}}\binom{t+1}{k}\cdot\frac{kS(\eta)-(k-1)}{S(\eta)^{k}}\cdot\eta^{k}.$$

Differentiating and using the above formula for $S'(\eta)$, we get

$$J'(\eta)=\frac{1}{(t+1)^{k}}\binom{t+1}{k}\cdot\frac{k\eta^{k-1}\left[(r-1)S(\eta)\eta-(k-1)\eta+(k-r+1)S(\eta)M\right]}{S(\eta)^{k}\left[(r-1)\eta-(r-2)M\right]}.$$

A repetition of that argument leads to

$$J''(\eta)=\frac{1}{(t+1)^{k}}\binom{t+1}{k}\cdot\frac{k(k-1)(r-k-1)\eta^{k-2}\left[S(\eta)M-\eta\right]^{2}}{S(\eta)^{k}\left(1-S(\eta)\right)\left[(r-1)\eta-(r-2)M\right]^{2}},$$

which entails the desired conclusion in view of the presence of the factor $r-k-1$ in the numerator. □

Claim 4.5. *For each $\eta\in\left[\frac{r-2}{r-1}\cdot M, M\right]$ the difference*

$$(r-1)\binom{s}{r-1}\eta^{2}\nu-s^{r-1}\eta^{r}F_{r}(\nu)$$

is at most

$$\frac{r-2}{(s-1)(s+1)}\binom{s+1}{r}\cdot\left(\tfrac{1}{2}(r-1)s\vartheta^{2}-(r-1)s\vartheta M+r(s-1)M\eta\right),$$

where $\nu=(F_{r-1}^{-1}\circ H)(\eta)$.

Proof. The difference under consideration, which we shall denote by $T(\eta)$ in the sequel, is piecewise convex by Claim 4.4. Using the definition of ϑ it is not hard to see that for $\eta=\vartheta$ one has equality in the inequality we seek to establish and for these reasons it suffices to prove the statements

$$(A)\qquad \text{If } r-2\leqslant t\leqslant s-2,\text{ then } \lim_{\eta\longrightarrow\vartheta_t^{+}}T'(\eta)\geqslant(r-2)\tbinom{s}{r-1}M$$

and

$$(B)\qquad \lim_{\eta\longrightarrow M^{-}}T'(\eta)\leqslant(r-2)\tbinom{s}{r-1}M,$$

where the superscripted plus or minus signs below the limit are intended to signify that η is supposed to approach the boundary value in question from the right or from the left, respectively. Throughout the computations that follow we use the function S as well as the formulae for $J'(\eta)$ corresponding to $k = 2$ and $k = r$ from the foregoing proof.

To verify (A), we divide into two cases.

First Case: $t = r - 2$

Note that the hypothesis of (A) yields $s \geqslant r$, wherefore $H(\vartheta_{r-1}) \leqslant H(M)$, i.e.

$$\frac{1}{(r-1)^{r-1}} \leqslant \frac{1}{s^{r-1}}\binom{s}{r-1} \cdot \frac{1}{M^{r-2}}.$$

Now let $\eta \in (\vartheta_{r-2}, \vartheta_{r-1})$ be arbitrary. Then

$$\frac{1}{(r-1)^{r-1}} \cdot \frac{(r-1)S(\eta) - (r-2)}{S(\eta)^{r-1}} \leqslant \frac{1}{s^{r-1}}\binom{s}{r-1} \cdot \frac{(r-1)S(\eta) - (r-2)}{S(\eta)^{r-1}} \cdot \frac{1}{M^{r-2}},$$

which by the definition of S and H yields $H(\eta) \leqslant H(S(\eta)M)$, and thus $\eta \leqslant S(\eta)M$. Also, $\eta \in \left(\frac{r-2}{r-1} \cdot M, M\right)$ gives

$$\eta \geqslant (r-1)\eta - (r-2)M > 0,$$

which by $S(\eta) \leqslant 1$ may be weakened to

$$\eta \geqslant S(\eta)\left((r-1)\eta - (r-2)M\right).$$

Consequently

$$\eta\left(S(\eta)M - \eta\right) \geqslant S(\eta)\left(S(\eta)M - \eta\right)\left((r-1)\eta - (r-2)M\right),$$

i.e.

$$\eta\left[(r-1)S(\eta)\eta - \eta - (r-3)S(\eta)M\right] \geqslant MS(\eta)^2\left[(r-1)\eta - (r-2)M\right].$$

Multiplying by $(r-2)\binom{s}{r-1}S(\eta)^{-2}\left[(r-1)\eta - (r-2)M\right]^{-1}$ we infer

$$T'(\eta) \geqslant (r-2)\binom{s}{r-1}M,$$

as required.

Second Case: $r - 1 \leqslant t \leqslant s - 2.$

Notice that Claim 4.2 entails

$$(r-2)(M - \vartheta_t)(M - \tfrac{t+1}{t}\vartheta_t) \geqslant 0,$$

whence

$$\vartheta_t\left[(r-2-\tfrac{1}{t})\vartheta_t-(r-3)M\right] \geqslant (M-\tfrac{1}{t}\vartheta_t)\left((r-1)\vartheta_t-(r-2)M\right).$$

By $t>r-2$ the second factor of the right hand side is positive, wherefore

$$\vartheta_t\cdot\frac{(r-2-\frac{1}{t})\vartheta_t-(r-3)M}{(r-1)\vartheta_t-(r-2)M} \geqslant M-\tfrac{1}{t}\vartheta_t.$$

Multiplying by $(r-1)$ and subtracting $M-\frac{r-1}{t}\vartheta_t$ we obtain

$$(r-1)\vartheta_t\cdot\frac{(r-2-\frac{1}{t})\vartheta_t-(r-3)M}{(r-1)\vartheta_t-(r-2)M}-(M-\tfrac{r-1}{t}\vartheta_t) \geqslant (r-2)M.$$

If we now multiply by $\binom{s}{r-1}$ and use

$$\lim_{\eta\longrightarrow\vartheta_t^+} S(\eta)=\frac{t}{t+1}$$

as well as the definition of ϑ_t, we get indeed

$$\lim_{\eta\longrightarrow\vartheta_t^+} T'(\eta) \geqslant (r-2)\binom{s}{r-1}M.$$

This completes the verification of (A).

So let us now continue with (B). From $H(M)>F_{r-1}\left(\frac{s-2}{2(s-1)}\right)$ one deduces easily

$$S(M)>\frac{s-1}{s}\geqslant\frac{r-2}{r-1},$$

where we have made the obvious definition

$$S(M)=\lim_{\eta\longrightarrow M^-} S(\eta).$$

This entails

$$\frac{(s-r+1)S(M)}{(r-1)S(M)-(r-2)}\leqslant s-1,$$

which in turn implies

$$(r-1)\left\{\frac{(s-r+1)S(M)}{(r-1)S(M)-(r-2)}-(s-1)\right\}\left(\frac{1-S(M)}{S(M)}\right)^2\leqslant 0.$$

Adding $(r-2)s$ and rearranging our terms, we infer

$$(r-1)(s-1)\frac{2S(M)-1}{S(M)^2}-\frac{(s-r+1)\left(rS(M)-(r-1)\right)}{S(M)\left((r-1)S(M)-(r-2)\right)}\leqslant (r-2)s.$$

Multiplying this by $\frac{M}{s}\cdot\binom{s}{r-1}$ and exploiting the equation

$$\frac{1}{M^{r-2}}=\frac{(r-1)S(M)-(r-2)}{S(M)^{r-1}},$$

that follows easily from the definition of S, one gets assertion (B), whereby Claim 4.5 has finally been proved. □

5. Clique densities.

We now come to the unique central section of this paper in which we are going to provide a proof of Claim 2.1, thus solving the clique density problem.

Theorem 5.1. *Whenever $\mathcal{G}$ denotes a weighted graph and $r \geqslant 2$ an integer, we have $\mathcal{G}(K_r) \geqslant F_r(\mathcal{G}(K_2))$. To say the same thing more verbosely, the estimate*

$$\mathcal{G}(K_r) \geqslant \frac{1}{(s+1)^r} \cdot \binom{s+1}{r} \cdot (1+\alpha)^{r-1} (1-(r-1)\alpha)$$

holds, where s refers to a positive integer for which $\gamma = \mathcal{G}(K_2)$ belongs to the interval $\left[\frac{s-1}{2s}, \frac{s}{2(s+1)}\right]$ and $\alpha \in \left[0, \frac{1}{s}\right]$ is required to satisfy $\gamma = \frac{s}{2(s+1)}(1-\alpha^2)$.

Proof. Since F_2 is the identity function confined to $\left[0, \frac{1}{2}\right)$, this is clear for $r = 2$. Arguing indirectly, let $r \geqslant 3$ denote the least integer for which this can fail* and take n to be the least order that counterexamples can possibly have. As we have already mentioned earlier, the function $F_r : \left[0, \frac{1}{2}\right) \longrightarrow \left[0, \frac{1}{r!}\right)$ is continuous. Now the totality of all weighted graphs of order n may in an obvious fashion be regarded as a compact topological space, which implies that the continuous function defined on it by

$$\mathcal{G} \longmapsto \mathcal{G}(K_r) - F_r(\mathcal{G}(K_2))$$

attains an absolute minimum. Now fix once and for all a weighted graph $\mathcal{G}$ of order n for which this minimal value occurs and choose an integer $s \geqslant 1$ as well as a real number $\alpha \in \left[0, \frac{1}{s}\right]$ such that the number $\gamma = \mathcal{G}(K_2)$ can be written as $\gamma = \frac{s}{2(s+1)}(1-\alpha^2)$. By the hypothesized failure of our theorem, we have

$$\mathcal{G}(K_r) < \frac{1}{(s+1)^r} \cdot \binom{s+1}{r} \cdot (1+\alpha)^{r-1} (1-(r-1)\alpha)\,,$$

which clearly can only happen if $s \geqslant r-1$. Also, Corollary 3.4 tells us that $\alpha \in \left(0, \frac{1}{s}\right)$, wherefore the function F_r is in particular differentiable at γ. As we have seen in Section 4, its derivative $\lambda = F_r'(\gamma)$ is given by

$$\lambda = \frac{(r-1)r}{s(s+1)^{r-1}} \binom{s+1}{r} (1+\alpha)^{r-2}.$$

*By the results of Razborov and Nikiforov ([10], [7]) we could assume $r \geqslant 5$ here, but actually there is no need for doing so.

Let $\mathcal{G}$ as usual be presented by the sequence $(x_1, x_2, \ldots, x_n)$ of non–negative reals summing up to 1 as well as the function $\mathcal{A} : [n]^{(2)} \longrightarrow [0, 1]$. Having thereby explained how to select that potential counterexample towards whose investigation we will direct our whole efforts, we hope to render the still rather lengthy remainder of the proof more intelligible by dividing it into five steps.

First Step: Exploiting extremality.

Clearly each of the numbers $x_1, x_2, \ldots, x_n$ has to be positive, for if one among them vanished there existed another counterexample of lower order than n. For $\{i, j\} \in [n]^{(2)}$ it is sometimes convenient to write $\mathcal{A}_{ij}$ instead of $\mathcal{A}(\{i, j\})$. Given a sequence $i_1, i_2, \ldots, i_m$ of distinct integers from $[n]$ as well as another integer $\rho \geqslant 1$, we set

$$\mathcal{G}_{i_1,i_2,\ldots,i_m}(K_\rho) = \sum_{M \in I^{(\rho)}} \prod_{(k,j)\in[m]\times M} \mathcal{A}_{i_k,j} \prod_{E\in M^{(2)}} \mathcal{A}(E) \prod_{j\in M} x_j,$$

where for typographical reasons we have temporarily written I instead of $[n] - \{i_1, i_2, \ldots, i_m\}$. Note that for $m = 0$ this coincides with our earlier notation, so no confusion can arise. Since $(x_1, x_2, \ldots, x_n)$ is an interior point of the simplex

$$\{(\xi_1, \xi_2, \ldots, \xi_n) \in [0, 1]^n \mid \xi_1 + \xi_2 + \ldots + \xi_n = 1\},$$

Lagrange's Theorem concerning the extremal values of multivariate functions reveals the existence of a certain real constant μ such that

$$\mathcal{G}_i(K_{r-1}) = \lambda \mathcal{G}_i(K_1) - \mu$$

holds for all $i \in [n]$. The impact that extremality has on $\mathcal{A}$ is studied in

(∗) *For each $\{i, j\} \in [n]^{(2)}$ one has $\mathcal{A}_{ij}(\lambda - \mathcal{G}_{ij}(K_{r-2})) \geqslant 0$.*

This is obvious whenever $\mathcal{A}_{ij}$ vanishes, so let us suppose now that this number is positive. If η denotes any sufficiently small positive real number, we may construct a weighted graph $\mathcal{G}^\eta$ agreeing entirely with $\mathcal{G}$ except for the stipulation $\mathcal{A}^\eta_{ij} = \mathcal{A}_{ij} - \eta$. Clearly one has $\mathcal{G}^\eta(K_2) = \gamma - \eta x_i x_j$ and $\mathcal{G}^\eta(K_r) = \mathcal{G}(K_r) - \eta x_i x_j \mathcal{G}_{ij}(K_{r-2})$, wherefore

$$\mathcal{G}^\eta(K_r) - F_r(\mathcal{G}^\eta(K_2)) = \mathcal{G}(K_r) - F_r(\gamma) + \eta x_i x_j (\lambda - \mathcal{G}_{ij}(K_{r-2})) \pm O(\eta^2),$$

which in view of the assumed extremality of $\mathcal{G}$ entails $\lambda \geqslant \mathcal{G}_{ij}(K_{r-2})$ and hence (∗).

Second Step: A morally flag algebraic consideration.

As in the proof of Proposition 3.1, we set

$$A_M = \prod_{E \in M^{(2)}} \mathcal{A}(E) \qquad \text{and} \qquad X_M = \prod_{i \in M} x_i$$

for all $M \subseteq [n]$. This time, however, we need the further stipulations

$$B_M = \sum_{i \in M} \Big(\sum_{j \in M - \{i\}} \mathcal{A}_{ij} \Big) A_{M-\{i\}},$$

$$C_M = \sum_{Q \in M^{(r)}} A_Q$$

and

$$D_M = \sum_{i \in M} \sum_{\{j,k\} \in (M-\{i\})^{(2)}} (1 - \mathcal{A}_{ij})(1 - \mathcal{A}_{ik}) A_{M-\{i\}}$$

for all $M \in [n]^{(r+1)}$. What we shall need to know about these expressions is:

($\circledast$) *If $M \in [n]^{(r+1)}$, then $B_M - (r-1)C_M + D_M \geqslant (r+1)A_M$.*

To see this, we note again that our inequality is linear in each of its variables, for which reason we may suppose $\mathcal{A}(E) \in \{0, 1\}$ for all $E \in M^{(2)}$. Form a graph $\mathcal{H}$ with vertex set M by putting an edge between $i, j \in M$ exactly if $\mathcal{A}_{ij} = 1$. If $\mathcal{H}$ is free from cliques of size r, then $A_M = B_M = C_M = D_M = 0$. If $\mathcal{H}$ contains a unique such clique and i further edges, where $0 \leqslant i \leqslant r-2$, then $A_M = 0$, $B_M = i$, $C_M = 1$ and $D_M = \binom{r-i}{2}$, wherefore indeed $B_M - (r-1)C_M + D_M = \binom{r-i-1}{2} \geqslant A_M$. If the graph $\mathcal{H}$ possesses exactly two cliques of size r, then it misses precisely one edge and $A_M = 0$, $B_M = 2(r-1)$, $C_M = 2$ as well as $D_M = 0$. Finally, if $\mathcal{H}$ happens to be a clique, then $A_M = 1$, $B_M = r(r+1)$, $C_M = r+1$ and $D_M = 0$. This analysis proves ($\circledast$) in all possible cases.

Multiplying the inequality just obtained by X_M and summing over M, we deduce

$$(\circledast) \qquad \sum_{M \in [n]^{(r+1)}} (B_M - (r-1)C_M + D_M) X_M \geqslant (r+1)\mathcal{G}(K_{r+1}).$$

Next, we propose to ponder the sum

$$\sum_{i \in [n]} x_i \mathcal{G}_i(K_1) \mathcal{G}_i(K_{r-1}).$$

Expanding the product, we get a plethora of terms involving each of the variables $x_1, x_2, \ldots, x_n$ at most linearly and their sum is easily seen to equal

$$\sum_{M \in [n]^{(r+1)}} B_M X_M.$$

The remaining "quadratic" terms sum up to

$$\sum_{\{i,j\}\in[n]^{(2)}} (x_i^2 x_j + x_i x_j^2)\mathcal{A}_{ij}^2 \mathcal{G}_{ij}(K_{r-2}).$$

Utilizing the inequality $(*)$ from our first step, this sum is estimated to be

$$\geqslant \sum_{\{i,j\}\in[n]^{(2)}} (x_i^2 x_j + x_i x_j^2)\mathcal{A}_{ij} \mathcal{G}_{ij}(K_{r-2}) - \lambda\Phi,$$

where for brevity

$$\Phi = \sum_{\{i,j\}\in[n]^{(2)}} (x_i^2 x_j + x_i x_j^2)(\mathcal{A}_{ij} - \mathcal{A}_{ij}^2).$$

The first of these sums may also be written as

$$(r-1)\sum_{Q\in[n]^{(r)}} A_Q X_Q \Big(1 - \sum_{q\in[n]-Q} x_q\Big) = (r-1)\mathcal{G}(K_r) - (r-1)\sum_{M\in[n]^{(r+1)}} C_M X_M$$

Thus we have altogether

$$\sum_{i\in[n]} x_i \mathcal{G}_i(K_1)\mathcal{G}_i(K_{r-1}) \geqslant (r-1)\mathcal{G}(K_r) + \sum_{M\in[n]^{(r+1)}} (B_M - (r-1)C_M)X_M - \lambda\Phi.$$

Exploiting $(\circledast)$ the right hand side can further be bounded from below by

$$(r-1)\mathcal{G}(K_r) + (r+1)\mathcal{G}(K_{r+1}) - \lambda\Phi - \sum_{M\in[n]^{(r+1)}} D_M X_M.$$

Now notice that trivially

$$\sum_{M\in[n]^{(r+1)}} D_M X_M \leqslant \sum_{i\in[n]} \sum_{\{j,k\}\in([n]-\{i\})^{(2)}} \mathcal{A}_{jk}(1-\mathcal{A}_{ij})(1-\mathcal{A}_{ik}) x_i x_j x_k \mathcal{G}_{jk}(K_{r-2}).$$

Hence writing

$$V_{\{i,j,k\}} = \mathcal{A}_{jk}(1-\mathcal{A}_{ij})(1-\mathcal{A}_{ik}) + \mathcal{A}_{ik}(1-\mathcal{A}_{ij})(1-\mathcal{A}_{jk}) + \mathcal{A}_{ij}(1-\mathcal{A}_{ik})(1-\mathcal{A}_{jk})$$

for all $\{i,j,k\}\in[n]^{(3)}$ and applying $(*)$ again, we obtain

$$(r-1)\mathcal{G}(K_r) + (r+1)\mathcal{G}(K_{r+1}) - \lambda\Phi - \lambda\sum_{T\in[n]^{(3)}} V_T X_T \leqslant \sum_{i\in[n]} x_i \mathcal{G}_i(K_1)\mathcal{G}_i(K_{r-1}).$$

The Lagrange equations entail the right hand side to equal

$$\lambda\sum_{i\in[n]} x_i \mathcal{G}_i(K_1)^2 - 2\gamma\mu$$

and in view of the by now straightforward calculation

$$\begin{aligned}\sum_{i\in[n]} x_i\mathcal{G}_i(K_1)^2 &= \sum_{\{i,j\}\in[n]^{(2)}} (x_i^2x_j + x_ix_j^2)\mathcal{A}_{ij}^2 \\ &\quad + 2\sum_{\{i,j,k\}\in[n]^{(3)}} (\mathcal{A}_{ij}\mathcal{A}_{jk} + \mathcal{A}_{jk}\mathcal{A}_{ki} + \mathcal{A}_{ki}\mathcal{A}_{ij})X_{\{i,j,k\}} \\ &= -\Phi + \sum_{\{i,j\}\in[n]^{(2)}} x_ix_j\mathcal{A}_{ij}\Big(1 - \sum_{k\in[n]-\{i,j\}} x_k\Big) \\ &\quad + \sum_{\{i,j,k\}\in[n]^{(3)}} (\mathcal{A}_{ij} + \mathcal{A}_{jk} + \mathcal{A}_{ki})X_{\{i,j,k\}} \\ &\quad - \sum_{T\in[n]^{(3)}} V_TX_T + 3\mathcal{G}(K_3) \\ &= -\Phi - \sum_{T\in[n]^{(3)}} V_TX_T + \gamma + 3\mathcal{G}(K_3)\end{aligned}$$

we finally arrive at the main estimate of this step, namely

$$(r-1)\mathcal{G}(K_r) + (r+1)\mathcal{G}(K_{r+1}) \leqslant \lambda(\gamma + 3\mathcal{G}(K_3)) - 2\gamma\mu.$$

Third Step: Introducing and estimating M.

Let us now define a real number M such that

$$\mu = \frac{(r-2)r}{(s+1)^r}\binom{s+1}{r}(1+\alpha)^{r-1}M.$$

Since

$$r\cdot\mathcal{G}(K_r) = \sum_{i\in[n]} x_i\mathcal{G}_i(K_{r-1}) = \sum_{i\in[n]} x_i(\lambda\mathcal{G}_i(K_1) - \mu) = 2\gamma\lambda - \mu,$$

we have

$$\mathcal{G}(K_r) = \frac{1}{(s+1)^r}\binom{s+1}{r}(1+\alpha)^{r-1}\left[1 - (r-1)\alpha - (r-2)(M-1)\right],$$

which in view of the presumed smallness of the left hand side gives $M > 1$. Eventually we shall prove $M \leqslant 1$ as well, thereby reaching a final contradiction, but before we can realistically hope to do so we first need to provide a much weaker upper bound on M, so that the results from our fourth section become available. This is our next immediate task. To achieve it, we find it convenient to introduce the abbreviations

$$A = \frac{r}{s(s+1)^{r-1}}\binom{s+1}{r},$$

$$B = \frac{(r-2)r}{(s+1)^r}\binom{s+1}{r}$$

and

$$C = \sqrt[r-2]{\frac{(2\gamma A)^{r-1}}{r \cdot \mathcal{G}(K_r)}}.$$

Notice that the last of these stipulations is permissible, as Proposition 3.1 entails $\mathcal{G}(K_r) > 0$ in view of $\gamma > \frac{s-1}{2s} \geqslant \frac{r-2}{2(r-1)}$. Applying the inequality between the arithmetic and geometric mean of $r-1$ positive real numbers, one of which is equal to $r \cdot \mathcal{G}(K_r)$ while each of the remaining $r-2$ numbers equals $C(1+\alpha)^{r-1}$, we get

$$r \cdot \mathcal{G}(K_r) + (r-2)C(1+\alpha)^{r-1} \geqslant 2(r-1)A\gamma(1+\alpha)^{r-2} = 2\gamma\lambda = r \cdot \mathcal{G}(K_r) + \mu$$

and hence $(r-2)C \geqslant BM$. Rising both sides to their $(r-2)^{\text{nd}}$ powers we infer after some easy simplifications

$$\left(\frac{2\gamma}{s}\right)^{r-1} \cdot \binom{s+1}{r} \geqslant (s+1)\mathcal{G}(K_r)M^{r-2}.$$

Using now Corollary 3.3, we obtain

$$\left(\frac{s-1}{s}\right)^{r-2} > \frac{s-r+1}{s-1} \cdot M^{r-2},$$

as desired.

Fourth Step: Induction on n.

By $(*)$ we have for each $i \in [n]$ the inequality

$$(r-1)\mathcal{G}_i(K_{r-1}) = \sum_{j \in [n]-\{i\}} x_j \mathcal{A}_{ij} \mathcal{G}_{ij}(K_{r-2}) \leqslant \sum_{j \in [n]-\{i\}} \lambda x_j \mathcal{A}_{ij} = \lambda \mathcal{G}_i(K_1)$$

and hence by the Lagrange equation

$$(r-1)\left(\lambda \mathcal{G}_i(K_1) - \mu\right) \leqslant \lambda \mathcal{G}_i(K_1),$$

i.e.

$$\mathcal{G}_i(K_1) \leqslant \tfrac{(r-1)\mu}{(r-2)\lambda} = \tfrac{s}{s+1}(1+\alpha)M.$$

Thus if we define the real number η_i to obey $\mathcal{G}_i(K_1) = \frac{s}{s+1}(1+\alpha)\eta_i$, then $\eta_i \leqslant M$. Similarly but easier we have

$$0 \leqslant \mathcal{G}_i(K_{r-1}) = \lambda \mathcal{G}_i(K_1) - \mu,$$

whence $\mathcal{G}_i(K_1) \geqslant \frac{\mu}{\lambda}$, so that altogether we get $\eta_i \in \left[\frac{r-2}{r-1} \cdot M, M\right]$.
The main objective of this step is the verification of

($\boxplus$) *For each $i \in [n]$ one has*

$$\lambda \mathcal{G}_i(K_2) - \mathcal{G}_i(K_r) \leqslant \frac{(r-2)s(1+\alpha)^r}{(s-1)(s+1)^{r+1}} \binom{s+1}{r} \times \left(\tfrac{1}{2}(r-1)s\theta^2 - (r-1)s\theta M + r(s-1)\eta_i M\right).$$

Plainly it suffices to show this for $i = n$ and for brevity we are henceforth going to write η instead of η_n. As $\mathcal{G}_n(K_1)$ is positive, we may construct a weighted graph $\mathcal{G}^*$ of order $n-1$ specified by the numbers $x_i^* = \frac{\mathcal{A}_{in} x_i}{\mathcal{G}_n(K_1)}$ for $i = 1, 2, \ldots, n-1$ as well as the restriction of $\mathcal{A}$ to $[n-1]^{(2)}$. Our minimal choices of r and n entail $\mathcal{G}^*(K_{r-1}) \geqslant F_{r-1}(\delta)$ and $\mathcal{G}^*(K_r) \geqslant F_r(\delta)$, where $\delta = \mathcal{G}^*(K_2)$. By our construction of $\mathcal{G}^*$ and the LAGRANGE equation for $i = n$, we have

$$\mathcal{G}^*(K_{r-1}) = \frac{\mathcal{G}_n(K_{r-1})}{\mathcal{G}_n(K_1)^{r-1}} = \frac{\lambda \mathcal{G}_n(K_1) - \mu}{\mathcal{G}_n(K_1)^{r-1}}.$$

Expressing the right hand side in terms of r, s, M and η, this simplifies to $\mathcal{G}^*(K_{r-1}) = H(\eta)$, where H refers to the function defined just before Claim 4.1. Stipulating therefore $\nu = F_{r-1}^{-1}(H(\eta))$, as in the hypothesis of Claim 4.3, we have $\delta \leqslant \nu$. Now in view of

$$\mathcal{G}^*(K_2) = \frac{\mathcal{G}_n(K_2)}{\mathcal{G}_n(K_1)^2} \qquad \text{and} \qquad \mathcal{G}^*(K_r) = \frac{\mathcal{G}_n(K_r)}{\mathcal{G}_n(K_1)^r},$$

we get

$$\lambda \mathcal{G}_n(K_2) - \mathcal{G}_n(K_r) = \frac{s(1+\alpha)^r \eta^2}{(s+1)^r} \times \left\{(r-1)\binom{s}{r-1}\delta - s^{r-1}\eta^{r-2}\mathcal{G}^*(K_r)\right\}.$$

By our bound on $\mathcal{G}^*(K_r)$, the difference in curly braces is at most $Q(\delta)$, where Q signifies the function introduced in Claim 4.3, and by that claim itself this is in turn at most $Q(\nu)$. Estimating now the product $\eta^2 Q(\nu)$ by means of Claim 4.5 we finish proving ($\boxplus$).

Fifth Step: An excellent finish.

Notice that

$$\frac{s}{s+1}(1+\alpha) \sum_{i \in [n]} x_i \eta_i = \sum_{i \in [n]} x_i \mathcal{G}_i(K_1) = 2\gamma = \frac{s}{s+1}(1-\alpha^2),$$

wherefore

$$\sum_{i \in [n]} x_i \eta_i = 1 - \alpha.$$

Thus multiplying (⊞) by x_i and adding up the n resulting inequalities we conclude

$$3\lambda\mathcal{G}(K_3) - (r+1)\mathcal{G}(K_{r+1}) \leqslant \frac{(r-2)s(1+\alpha)^r}{(s-1)(s+1)^{r+1}}\binom{s+1}{r}\times$$
$$\left(\tfrac{1}{2}(r-1)s\theta^2 - (r-1)s\theta M + r(s-1)(1-\alpha)M\right).$$

Combining this with the main result from our second step and plugging in the formulae expressing γ, λ, μ and $\mathcal{G}(K_r)$ in terms of r, s and M from the first and third step we get an estimate that on first sight looks rather lengthy. After massive cancelations, however, it just reads

$$(1-\alpha) - 2M \leqslant \frac{(1+\alpha)s^2}{s^2-1}(\vartheta^2 - 2M\vartheta).$$

Since $\vartheta \in \left[\frac{s-1}{s}M, M\right]$ by Claim 4.2, we have

$$\vartheta^2 - 2M\vartheta = (M-\vartheta)^2 - M^2 \leqslant -\tfrac{s^2-1}{s^2}\cdot M^2,$$

whence

$$(1-\alpha) - 2M \leqslant -(1+\alpha)M^2,$$

i.e.

$$(1-M)((1-\alpha)-(1+\alpha)M) \leqslant 0.$$

But if M really was greater than 1, as suggested by our third step, then both factors of the left hand side were negative. This contradiction finally proves Theorem 5.1. □

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