THE ODD ORIGIN OF GERSTENHABER, BV AND THE MASTER EQUATION

RALPH M. KAUFMANN, BENJAMIN C. WARD, AND J. JAVIER ZÚÑIGA

Abstract. In this paper we show that Gerstenhaber brackets, BV operators and related master equations arise in a very natural way when considering odd operads and their generalizations.

We show that many known examples such as BV operators in the Calabi–Yau setting, brackets in string field theory, the master equation in that setting, the master equation for Feynman transforms come from this type of setup.

We give a systematic and comprehensive treatment of all the usual setups involving (cyclic/modular) operads and PROP(erad)s including new results. Further generalizations and categorical constructions will be presented in a sequel.

Introduction

In recent years there have been many algebraic constructions which in their background have some operadic origin. Perhaps the most prominent are Lie brackets, Gerstenhaber brackets and master equations. The Lie algebras of Kontsevich [K, CV] as well as Deligne's conjecture [KS, McCS, V, BF, T, K2], its cyclic generalization [K3] and its A_∞ version which was studied in [TZ, KSch, Wa, K6], and notably string topology [CS] are of this type, especially when considered in the algebraic framework [TZ, K2, K3]. Among master equations the relevant constructions go back to Sen and Zwiebach [SZ, KSV] and newer ones include [ASZK,Schw,HVZ,Bar,MMS,S1–S3]. There is a plethora of further incidences which would fill pages.

This paper is a systematic study of these algebraic operations, i.e. brackets and BV operators Δ their occurrence in master equations and the origin of these equations. Without being too specific in this introduction there are several incarnations of the master equation going by various names.

$$\{S \bullet S\} = 0, \quad dS + \frac{1}{2}\{S \bullet S\} = 0, \quad dS + \Delta(S) + \frac{1}{2}\{S \bullet S\} = 0. \qquad (0.1)$$

The first is a type of classical master equation, with the differential the equation is sometimes called Maurer-Cartan equation and with Δ is called the quantum master equation. Of course, one has to —and we will— specify where S lies and what the definition of $\{\,\cdot\,\}$, Δ is.

The mantra we provide for all these constructions is that

Metatheorem 1. *Odd non–self–gluings give rise to Gerstenhaber brackets.*

Metatheorem 2. *Odd self–gluings give rise to BV operators.*

Metatheorem 3. *Algebraically, the master equation classifies dg algebras over the relevant (co)–bar construction.*

Metatheorem 4. *Topologically, the master equation drives the compactification.*

The first two statements are scattered throughout many places e.g. [G, GV, KM, KSV, Schw, MMS, Bar]. Usually the emphasis is not on the odd, but on even structures which differ by a suspension in the operdic/PROP(eradic) case. That this reverses the logic is one of the main points we argue. In fact, as we show, even classically the odd point of view explains why the signs and degrees in Gerstenhaber's original construction [G] of the bracket on the Hochschild complex appear naturally when considered in the odd operadic framework. The third statement has an incarnation in [Bar]. The last statement is actively researched [HVZ, S1–S3] among others and goes back to [SZ].

We will gather these type of constructions in a systematic fashion. In order to reach a broader audience, we present the most important cases here, gradually increasing the technical level. The text is aimed to be fairly self–contained and balanced between exposition and technical details. There are basically three stages, those of operads, cyclic operads and modular operads. The cases of PROPs, properads, wheeled PROPs and properads [MMS] fit in between. We will make the metatheorems above concrete in each of these setups. The operad case is classical albeit spread out through the literature, while our analysis of the cyclic operad case is to our knowledge new. The modular operad case was studied in [Bar]. Even in the known cases, there is a slight subtlety here about which multiplicative structure is meant when using the terminology Gerstenhaber or BV. Without a multiplicative structure one only has an odd Lie algebra and respectively a differential.

Taking the cue from [SZ, Schw, HVZ], we show that there is a natural way to add a multiplication by going to the "non connected" version of the operadic type construction, which are new operadic gadgets that we define.

Another possibility is that there already exists an internal multiplication. This situation is *a priori* very different and the resulting equations are up to homotopy. We comment on this in §6.3.

We also would like to mention several points that have been somewhat confounded in the literature. The existence of the bracket and BV operators relies solely on the odd gluing structures. *Examples* come from generalized symplectic structures [K, CV, Gi], but a symplectic structure is not necessary. Another type of *example* of these odd versions arises via the relevant (co)bar constructions or Feynman transform. In these examples the structures are free if one disregards the dg structure. Again, this need not be the case for the bracket and BV operator to exist.

A further desideratum is to put all these constructions under one roof. That is to provide a language in which the metatheorems become actual theorems. As we are dealing with several operadic structures of different natures such as cyclic operads, modular operads, twisted modular operads, PROPs, etc. at the same time a great deal of abstraction is required for this. It is furnished by our new categorical constructions revolving around our new perspective on operad type structures called Feynman categories. The plus side is that then all the various theorems have a common root and more applications, such as the open/closed case [HVZ] etc. are covered as well. The drawback is that since the setting is quite general it is equally technical.

So for the exposition we are faced with the problem, that either the statements or their context becomes rather complicated. To cut the Gordian knot, we separate the results into two papers. This first part contains the examples and theorems most relevant for the "practicing mathematician or physicist" in an accessible form, and the categorical constructions which provide an underlying layer in [Fey] for the experts and interested. This also has the benefit of letting one specify which bar construction is the relevant one for the considerations at hand.

In this paper, we will show that there is an odd generalization of the classical structures of operads, (anti)–cyclic operads and modular operads. Going through this list we show that odd non–self gluing will lead to an odd Poisson bracket and odd self–gluing gives rise to a differential operator Δ. These results then readily extend to other odd settings such as odd (wheeled) PROP(erad)s or nc odd modular operads, which we define. When disconnected graphs are allowed, the operator Δ becomes a BV operator.

The modular operad case is in this respect the most difficult as the input/output distinction is lifted — as basically is already done in the cyclic case — and gluing procedures along general graphs are allowed. In order to obtain the odd gluing, one has to go up one layer of abstraction and introduce twisted operads. With hindsight all the constructions presented before can be better understood in this generality. Before passing to the modular situation, the odd versions can be obtained by certain shifts and to make the exposition more accessible and transparent we introduce them in this way in §§1–3.

Although one can still define modular operads via basic gluing operations it is not possible to describe their odd counterparts in this fashion. The correct formalism is furnished by $\mathfrak{K}$–modular operads of [GeK2]. The even version, which we call anti-modular is then actually still twisted.

In order to define the twisted versions of modular operads one needs to first pass to another equivalent definition of operads involving triples of functors. One can then twist these triples with what are called co–boundaries. The idea of triples nicely ties in with the fact that given an operad there are unique compositions along rooted trees and composing them corresponds to gluing rooted trees at their leaves, resulting again in a rooted tree. For each of the types of operads we have encountered there is a corresponding set of graphs. We do not wish to go through the tedious set theoretical definition of graphs and their morphisms in the main text and thus refer to the appendix or [BoM, MSS, Fey] for even more details.

The main reason to go to triples is that the twisted versions are necessary for the odd self–gluings of Metatheorem 2, so although the ideas and constructions are basically of the same order of complexity as in the previous section, we need to use more technical or less widely known language. Being aware of this, we review the salient features of these constructions in §4.

As we show below the cyclic bracket comes from odd gluings ${}_s\circ_t$. These are non–self gluings as they involve two different elements. These type of gluings (brackets and nested brackets) in graph language only involve trees. For trees, one can describe the twists in a simpler fashion, and we will, in terms of anti–cyclic and odd operads.

If we are able to self–glue elements, then we can again ask if there is an operation if these self–gluings are odd. Indeed this is the case and the origin of the BV operator, which will be treated subsequently in §5, see also 3.4.1 for the wheeled PROP(erad) case. The most commonly known example of self–gluings occurs in modular operads. This is however not odd. To get an odd gluing, as we explained one has to consider $\mathfrak{K}$–modular operads. These arise naturally as the Feynman transform of a modular operad; viz. the relevant (co)bar construction which in the case of modular operads is called Feynman transform. We review this construction in §7.

This type of BV operator also occurs in the hom spaces between operads that differ by a twist by $\mathfrak{K}$. This is what allows us to put the result on BV algebras in [MMS, Theorem 3.4.3] into a broader framework.

With all the technical assumptions in place, Metatheorem 1 and 2 transform to:

Theorem A. *The direct sum of coinvariants of (nc) (cyclic/modular) operads or (wheeled) PROP(erad)s or chain level EMOs, the sum over all non–self gluings gives an odd Lie bracket, which is Gerstenhaber in the presence of a horizontal composition.*[1]

Theorem B. *On the direct sum of coinvariants of (nc) (modular) operads, wheeled PROP(erad)s or linear EMOs, the sum over all self gluings gives a differential Δ, which is BV in the presence of a horizontal composition.*

In these theorems, which coinvariants to take is subtle and depends on the type of operadic gadget. Also whether or not the Lie bracket comes from a pre–Lie structure depends on the particular situation. We give precise theorems in the text. The cyclic bracket of an odd cyclic operad is new to our knowledge.

Metatheorem 3 takes the concrete form

Theorem C. *Metatheorem 3 is true for (cyclic/modular) operads and (wheeled) Properads. It furthermore holds for pairs of a $\mathfrak{D}$ twisted Feynman transform and a $\mathfrak{D}\mathfrak{K}^{-1}$ twisted structure.*

For the non–connected versions the statement means that the horizontal composition is not resolved by the relevant (co)bar construction/Feynman transform. An example of this is the wheeled PROP classifying solutions of the master equation found in [MSS].

[1]For PROPs the non–self gluings refer to the dioperadic gluings.

Finally, we analyze the geometric situation in §8. Here Metatheorem 4 turns into a definition for the Feynman transform. We give several examples of this.

Theorem D. *The blowups $\overline{M}^{KSV}_{g,n}$ of [KSV], $\overline{\mathcal{M}}^{HVZ\,b,\vec{m}}_{g,n}$ of [HVZ] as well as the Stasheff associahedra and the complex K^1 for the little discs [KSch] are compactifications satisfying the master equation*

In the sequel [Fey] we will give a general setup where all of the above type of operadic structures, for which there is not yet even a common name, as functors from so–called Feynman categories. This lets us abstractly treat all the cases at once including the twists. Making these concrete one is again led to the constructions we present here.

The Appendix contains more detailed definition of graphs and the definition of the algebras used in the text.

Acknowledgments

We would like to thank Bill Browder, Maxim Kontsevich, Yuri I. Manin, Jim McClure, Albert Schwarz, Jim Stasheff, Dennis Sullivan, and Sasha Voronov for enlightening discussions. We also benefited from the interest and discussions with Clemens Berger, Dennis Borisov, and Benoit Fresse.

RK thankfully acknowledges support from NSF DMS-0805881 and the Humboldt Foundation. He also thanks the Institut des Hautes Etudes Scientifiques, the Max–Planck–Institute for Mathematics in Bonn and the Institute for Advanced Study for their support and the University of Hamburg for its hospitality.

This project was worked on substantially while RK was visiting the IAS. While at the IAS RKs work was supported by the NSF under agreement DMS–0635607.

BW would like to thank the Purdue Research Foundation for its support.

Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science foundation.

Conventions

For convenience, we usually work in the in the category $g\mathcal{V}ect$ of graded vector spaces over a fixed ground field k of characteristic 0. Some constructions lie in $dg\mathcal{V}ect$.

For most constructions, this is not necessary and one can generalize to any additive category (or better a category enriched over graded Abelian groups) which is cocomplete. Or even less, where the particular colimits we use exist.

Sometimes we however use the isomorphism between $\mathbb{S}_n$ invariants and $\mathbb{S}_n$ co–invariants for all n. In this case, we need characteristic 0. Usually this step is again convenient but not strictly necessary and it can be omitted at the price of less succinct statements.

We will also use the notion of disjoint union of sets. Here one has to be a bit careful what one means. Either the usual definition, which is neither symmetric nor associative, or its strictification. By this we mean the $\amalg$ gives a symmetric monoidal structure on the category of finite sets with set maps and according to MacLane's coherence theorem [McL2] we can replace the category by a strict monoidal one. We usually choose the latter or alternatively, we implicitly assume the use of associators and commutators. Apart from this remark we do not wish to burden the reader with these details.

Finally to make the analogies more clear, we will use common notation for all the animals in the bestiary. That is $\mathcal{O}$ will be an operad, cyclic operad, modular operad, PROP, wheeled PROP, properad, wheeled properad and their twisted versions. Likewise we will use $\mathbb{T}$ for triple again regardless of the specific details. This fits well with [Fey] where $\mathcal{O}$ is just a monoidal functor in each case and $\mathbb{T}$ is the standard triple from a natural forgetful and the adjoint free functor.

1. Classical theory: Operads and Gerstenhaber's bracket

In this section, we start by collecting together the facts about operads and brackets. The main example is furnished by the Hochschild complex and the Gerstenhaber bracket. At the end, we take a slightly different point of view in accordance with our mantra by switching from operads to odd operads —which we define. The benefit is that this gives agreement of the signs and grading from the operadic and the Hochschild point of view. Another thing which is special in the case of operads is that the bracket has a pre–Lie structure. This traces back to the fact that for operads one is dealing with rooted trees.

1.1. Basic Background.

1.1.1. **Canonical Example.** For a finite dimensional vector space V set $\mathcal{E}nd(V)(n) = Hom(V^{\otimes n}, V)$. Notice that these spaces are again vector spaces. Another way to say this is that there is an internal hom in the category. These spaces have an obvious $\mathbb{S}_n$ action by permuting the variables (factors of V) of the multilinear functions. There are composition maps $\circ_i$: $\mathcal{E}nd(V)(n) \otimes \mathcal{E}nd(V)(m) \to \mathcal{E}nd(V)(n+m-1); f \otimes g \mapsto f \circ_i g$ which are given by substituting g in the i–th place of the function f. There is a unit for these compositions which is the identity function $id : V \to V$. These compositions are associative and equivariant under the action of the relevant symmetric groups in a natural universal manner. That is for every pair of permutations $(\sigma, \sigma') \in \mathbb{S}_n \times \mathbb{S}_m$ there is a unique permutation $\sigma \circ_i \sigma' \in \mathbb{S}_{n+m-1}$ s.t. $\sigma f \circ_i \sigma' g = (\sigma \circ_i \sigma') f \circ_{\sigma^{-1}(i)} g$.

1.1.2. **Operads.** We will briefly recall the salient features of the definition of an operad, which is an abstraction of the example above. A full definition can be found in [MSS]. Technically we will be dealing with pseudo operads, but with the exception of this subsection, we will not mention the "pseudo" any more. A unital (pseudo) operad is given by a collection $\{\mathcal{O}(n)\}$ in $g\mathcal{V}ect$ or more generally in a symmetric monoidal category $\mathcal{C}$ together with:

(1) operadic compositions or gluing maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m + n - 1) : 1 \leq i \leq n$$

(2) an $\mathbb{S}_n$ action for each $\mathcal{O}(n)$.
(3) and a unit $id \in \mathcal{O}(1)$

Such that the gluing maps satisfy the associativity relations

$$(\mathcal{O}(n) \circ_i \mathcal{O}(m)) \circ_j \mathcal{O}(l) =$$
$$\begin{cases} C_{\mathcal{O}(m),\mathcal{O}(l)}(\mathcal{O}(n) \circ_j \mathcal{O}(l)) \circ_{i+l-1} \mathcal{O}(m) & \text{if } 1 \leq j < i \\ \mathcal{O}(n) \circ_i (\mathcal{O}(m) \circ_{j-i+1} \mathcal{O}(l)) & \text{if } i \leq j \leq i+m-1 \\ C_{\mathcal{O}(m),\mathcal{O}(l)}(\mathcal{O}(n) \circ_{j-m+1} \mathcal{O}(l)) \circ_i \mathcal{O}(m) & \text{if } i+m \leq j \end{cases} \tag{1.1}$$

where C is the commutator map in the symmetric monoidal category. In the category $g\mathcal{V}ect$ C is given by $C(a \otimes b) = (-1)^{deg(a)deg(b)} b \otimes a$, where deg is the degree.

The unit satisfies

$$\forall a \in \mathcal{O}(n), 1 \leq i \leq n : \quad id \circ_1 a = a; \quad a \circ_i id = a$$

and the gluing maps are required to be $\mathbb{S}_n$ equivariant. We omit the rather lengthy formal definition of the equivariance in favor of the canonical example above from which it can be easily abstracted; see also [MSS] for a definition.

The collections of $\mathbb{S}_n$ modules $\mathcal{O}(n)$ is called an $\mathbb{S}$–module.

1.1.3. **Rooted trees.** The associativity means that any planar rooted tree τ with leaves labeled by $1, \dots, n$ determines a unique composition by using it as a flow chart. Here associativity says that the order of the compositions is irrelevant. If we add the $\mathbb{S}$ equivariance, then any rooted tree gives an operation. More in §4.

1.1.4. **Algebras over operads.** The operad $\mathcal{E}nd(V)$ plays a special role. An algebra V over an operad is an operadic morphism from $\mathcal{O}$ to $\mathcal{E}nd(V)$ (of degree 0). Here operadic morphism is the straightforward notion obtained by requiring that all the compositions and $\mathbb{S}_n$ actions are respected.

The operad $\mathcal{E}nd(V)$ can also be generalized to any closed symmetric monoidal category $\mathcal{C}$ where now V is an object.

1.1.5. **Weaker structures.** Dropping the unit from the data and axioms yield the notion of a non–unital pseudo–operad. The distinction between pseudo or not is irrelevant in the unital case as these notions are equivalent; see [MSS]. Dropping the $\mathbb{S}_n$ action and the $\mathbb{S}_n$ equivariance, we arrive at the definition of a *non*–Σ operad.

Notation 1.1. Given an operad $\mathcal{O} = \{\mathcal{O}(n)\}$ we set $\mathcal{O}^{\oplus} = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(n)$.

If $a \in \mathcal{O}(n)$ with (internal) degree $deg(a)$, we set $ar(a) = n$ and $|a| = deg(a) + ar(a)$.

We will also consider the co–invariants $\mathcal{O}(n)_{\mathbb{S}_n}$ and set $\mathcal{O}^{\oplus}_{\mathbb{S}} := \bigoplus_{n \in \mathbb{N}} \mathcal{O}(n)_{\mathbb{S}_n}$.

There is a natural map $\mathcal{O}(n) \to \mathcal{O}(n)_{\mathbb{S}_n}$ and we denote it as follows $a \mapsto [a]$. This induces a map $\mathcal{O}^{\oplus} \to \mathcal{O}^{\oplus}_{\mathbb{S}}$ which we denote by the same symbol.

1.2. **Lie bracket.** There is a natural Lie bracket on $\mathcal{O}^{\oplus}$ [GV, K2] and on its coinvariants [KM] $\mathcal{O}^{\oplus}_{\mathbb{S}}$.

Theorem 1.2. *Given an operad $\{\mathcal{O}(n)\}$ in $g\mathcal{V}ect$. Set*

$$a \circ b := \sum_{i=1}^{ar(a)} a \circ_i b \tag{1.2}$$

then $\circ$ is a pre–Lie multiplication and hence

$$[a \circ b] := a \circ b - (-1)^{deg(a)deg(b)} b \circ a \tag{1.3}$$

defines a Lie bracket on $\mathcal{O}^{\oplus}$. This Lie bracket descends to a Lie bracket on $\mathcal{O}^{\oplus}_{\mathbb{S}}$ — for the grading deg.

Furthermore the pre–Lie and Lie structures already exist for non–Σ operads.

Proof. The proof is a straight-forward calculation checking the pre–Lie property of the operation $\circ$, that is the 2–3 symmetry of the associator. The fact that the bracket descends to coinvariants is checked by setting $[a] \circ [b] = [a \circ b]$ and remarking that this is well defined (viz. independent of choices). This due to the $\mathbb{S}$–equivariance of the operadic compositions $\circ_i$. The last claim follows from the fact that neither the formula nor the verification of the conditions use the $\mathbb{S}_n$ action. That check is basically the proof found in [G]. □

It is the Lie algebra on the coinvariants that Kapranov and Manin [KM] identified as the Lie algebra of derivations of the respective tensor functor. This provides a point of contact with the Maurer-Cartan formalism.

1.3. **Odd Lie bracket.** In Gerstenhaber's original work [G] the bracket is not Lie but odd Lie. This is because he introduces certain signs in the summation. We will show that these signs can be understood in terms of suspensions and shifts. Although they are defined in a bit of an *ad hoc* fashion, they are indeed the natural deeper structure as one can view from the bigger picture provided by the metatheorems.

In particular, doing an operadic suspension one almost gets the signs. That is after one more shift, the signs are the ones of the Hochschild complex. What seems *prima vista* unfortunate, namely that a naïve shift of an operad ceases to be an operad, is actually completely natural, as according to the mantra the bracket should come from an odd gluing. Let us formalize this.

1.3.1. **Shifts and odd Lie brackets.** Given a graded vector space $V = \bigoplus_i V^i$, we set $\Sigma V := V[-1]$ this means that $(\Sigma V)^i = V^{i-1}$ and call it the suspension of V. The inverse operation of suspension is called desuspension. We set $(\Sigma^{-1}V)^i = V^{i+1}$

If $|\,.\,|$ is the grading of V, we set $s(a) := |a|+1$ then $s(a)$ is the natural degree of a thought of as an element in ΣV.

Definition 1.3. A bilinear map $\{\bullet\}$ on graded vector space V with grading $|\,.\,|$ is an odd Lie bracket if

(1) odd anti–symmetry

$$\{a \bullet b\} = -(-1)^{s(a)s(b)}\{b \bullet a\}$$

(2) odd Jacobi

$$0 = \{a \bullet \{b \bullet c\}\} + (-1)^{s(c)(s(a)+s(b))}\{c \bullet \{a \bullet b\}\} + (-1)^{s(a)(s(b)+s(c))}\{b \bullet \{c \bullet a\}\}$$

Alternatively, a direct calculation yields the following useful characterization.

Lemma 1.4. $\{\ \bullet\ \}$ *is an odd Lie bracket on* V *if and only if it is a Lie bracket on* ΣV. □

1.3.2. **Shifted compositions and Gerstenhaber's bracket.** Following Gerstenhaber, given $\mathcal{O}$ in $g\mathcal{V}ect$ we define new composition maps $\bullet_i$ as follows.

$$a \bullet_i b := (-1)^{(i-1)s(b)} a \circ_i b \tag{1.4}$$

Set

$$a \bullet b = \sum_{i=1}^{ar(a)} a \bullet_i b \tag{1.5}$$

Remark 1.5. Notice that even if all the $\circ_i$ are even then $\circ : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ does not preserve the total degree $|\,.\,|$. However in the same situation $\bullet$ does preserve degree for the shifted grading $s(a) := |a| - 1$.

Analogously to the Lie situation, set

$$\{a \bullet b\} := a \bullet b - (-1)^{s(a)s(b)} b \bullet a \tag{1.6}$$

With this definition one readily verifies that:

Proposition 1.6. *[G, GV, KM, K2] The bi–linear operation* $\{\ \bullet\ \}$ *is an odd Lie bracket and it descends to co–invariants* $\mathcal{O}_{\mathbb{S}}^{\oplus}$. □

Remark 1.7. Since we are dealing with signs only, the shift in degree can be made to be $+1$ or -1.

1.4. **Signs an essential Remark.** There are two ways in which to view the signs

(1) Simply as the shifted signs which may seem rather odd.
(2) By setting $deg(a) = |a|$, $deg(\bullet) = 1$ and using the Koszul rule of sign for when permuting symbols.[2] Here the symbols "{" and "}" are assigned degree 0. That is as a $\mathbb{Z}/2\mathbb{Z}$ graded operation $\bullet$ is odd.

The relevant calculation equating both sign formalisms for the odd Lie property is that:

$$s(a)s(b) + 1 = (|a|-1)(|b|-1) + 1 \equiv |a||b| + |a| + |b| \mod 2 \tag{1.7}$$

This is essentially, why we can shift instead of using a triple, see §4.

[2] In the geometric considerations the $\bullet$ indeed often comes from an S^1 action, which one can consider the $\bullet$ to represent.

Remark 1.8. In the operad or anti–cyclic operad case (see the next section) the first version is viable, while in the modular (see section 4) or more general case the second version is preferable and in a sense necessary. Thus with hindsight, we will see that the second version is actually natural also in the non–modular context.

1.5. **Suspensions and Shifts for Operads.** Let sgn_n be the one–dimensional sign representation of $\mathbb{S}_n$.

Definition 1.9. Given an operad $\mathcal{O}$ we define $s\mathcal{O}$, the operadic suspension of $\mathcal{O}$, to be the graded $\mathbb{S}$–module $s\mathcal{O}(n) = \Sigma^{n-1}(\mathcal{O}(n) \otimes sgn_n)$ with the natural induced operad structure. We will use the standard isomorphism identifying $\mathcal{O}(n) \otimes k \simeq \mathcal{O}(n)$

Denote the induced operadic compositions for $s\mathcal{O}$ by $\bullet_i$. Explicitly: set $\bullet_1$

$$\Sigma^{ar(a)-1} a \bullet_1 \Sigma^{ar(b)-1} b := \Sigma^{ar(a)+ar(b)-2}(a \circ_1 b) \tag{1.8}$$

then by $\mathbb{S}$–equivariance the $\bullet_i$ are necessarily given by

$$\Sigma^{ar(a)-1} a \bullet_i \Sigma^{ar(b)-1} b = (-1)^{(i-1)(ar(b)-1)} \Sigma^{ar(a)+ar(b)-2}(a \circ_i b) \tag{1.9}$$

Notice that in this operad the operations $\bullet_i$ are of degree 0. The operations $\bullet_i$ satisfy the following associativity relations.

$$(a \bullet_i b) \bullet_j c = \begin{cases} (-1)^{(|b|-1)(|c|-1)}(a \bullet_j c) \bullet_{i+l-1} b & \text{if } 1 \leq j < i \\ a \bullet_i (b \bullet_{j-i+1} c) & \text{if } i \leq j \leq i+m-1 \\ (-1)^{(|b|-1)(|c|-1)}(a \bullet_{j-m+1} c) \bullet_i b & \text{if } i+m \leq j \end{cases} \tag{1.10}$$

These relations are the correct graded associativity equations for the grading by $s(a)$ — it is off from the grading $|\,.\,|$ by one though. Hence we obtain:

Proposition 1.10. *The operadic suspension $s\mathcal{O}$ of an operad $\mathcal{O}$ together with the compositions $\bullet_i$ is an operad in $g\mathcal{V}ect$ for the grading s.* □

Proposition 1.11. *Identifying elements of $\mathcal{O}$ with their counterparts in $s\mathcal{O}$, the Gerstenhaber's bracket $\{\ \bullet\ \}$ is the natural Lie bracket $[\ \circ\ ]$ for the shifted operad $s\mathcal{O}$.*

Proof. Indeed the Gerstenhaber's bracket is odd Lie for the grading $|\,.\,|$ and hence using Lemma 1.4 is Lie for the natural grading s of $s\mathcal{O}$ which is obtained from $|\,.\,|$ by one naïve shift. □

1.5.1. **Motivational example for $s\mathcal{O}$.** Consider the endomorphism operad $\mathcal{E}nd(V)$ with $\mathcal{E}nd(V)(n) = Hom(V^{\otimes n}, V)$ having degree 0.

The operadic shift then comes about if one considers $V[1]$ instead of V. A map of degree 0 from $V^{\otimes n} \to V$ gives a map of degree $n-1$ from $(V[1])^{\otimes n} = V^{\otimes n}[n] \to V[1]$. One has that $\mathcal{E}nd(V[1]) \simeq s\mathcal{E}nd(V)$ (see e.g. [MSS]). And in general:

Proposition 1.12. *[MSS] V is an $\mathcal{O}$–algebra if and only if $V[1]$ is an $s\mathcal{O}$ algebra.*

$a \in$	natural degree of a
$\mathcal{O}(n)$	$deg(a)$
$s\mathcal{O}(n)$	$s(a) = deg(a) + n - 1$
$\Sigma s\mathcal{O}(n)$	$\lvert a\rvert = deg(a) + n$

TABLE 1. Natural degrees in suspensions and shifts

1.5.2. **Degrees in the Hochschild complex.** If A is an associative algebra $\mathcal{E}nd(A)$ actually is a complex, the Hochschild cochain complex $CH^*(A,A)$. It is given by $CH^n(A,A) = Hom(A^{\otimes n}, A)$ with the Hochschild differential, which is immaterial at the moment. As vector spaces $CH^n(A,A) = \mathcal{E}nd(A)(n)$, but it is put in degree n, however. Thus an element $a \in CH^*(A,A)$ has natural degree $|a|$. This is not the natural operadic grading however which is either $deg(a)$ in $\mathcal{E}nd(A)$ or $s(a)$ in $s\mathcal{E}nd(A) = \mathcal{E}nd(A[1])$.

So although the operadic shift $s\mathcal{E}nd(V)$ of $\mathcal{E}nd(V)$ is a graded operad and it provides Gerstenhaber's signs as the signs of the natural Lie bracket, as a graded vector space it is still one shift short from the Hochschild complex.

Adding one more naïve shift Σ, we obtain the right grading, so that $CH^*(A,A)$ is a graded algebra with respect to the cup product and the bracket has Gerstenhaber's signs, that is $CH^*(A,A) = \Sigma s\mathcal{E}nd(A)$; formal definitions can be found below.

1.5.3. **Naïve shifts and odd operads.** One thing that is somewhat dramatically altered is that when we do a naïve shift we are not dealing with an operad any more, but an odd operad which will formalize now.

Definition 1.13. For an $\mathbb{S}$–module $\mathcal{O}$ its suspension $\Sigma\mathcal{O}$ is the $\mathbb{S}$–module $\{\Sigma\mathcal{O}(n)\}$. Likewise we define $\Sigma^{-1}\mathcal{O}$.

Definition 1.14. An odd operad $\mathcal{O}$ in $g\mathcal{V}ect$ is an $\mathbb{S}$–module with operations $\bullet_i$ such that $\Sigma^{-1}\mathcal{O}$ together with the $\bullet_i$ is an operad.

Notice this means that in $\mathcal{O}$ the operations satisfy the equations 1.10 where $|\,.\,|$ is now just the degree in $\mathcal{O}$.

Proposition 1.15. *Given an odd operad $\mathcal{O}$, the vector space $\mathcal{O}^{\oplus}$ carries an odd bracket $\{\ \bullet\ \}$.*

Proof. This follows directly from Lemma 1.4. □

Corollary 1.16. *Given an operad $\mathcal{O}$ the odd operad $\Sigma s\mathcal{O}$ naturally carries an odd Lie bracket, which is the shift of the natural Lie bracket on $s\mathcal{O}$.*

1.5.4. **The Hochschild complex as an odd operad.** To sum up this section, the most natural way to think about the Hochschild complex is as an odd operad $CH^*(A,A) = \Sigma s\mathcal{E}nd(A)$. This provides all the correct signs and degrees. Furthermore in this fashion one can generalize the bracket to the cyclic and modular cases.

We briefly collect together the relevant degrees in Table 1.

1.6. **Monoidal structure and tree picture and twisted Operads.** One may be tempted to introduce a new monoidal structure on $g\mathcal{V}ect$ where $\otimes$ is of degree 1. That is an element $a \otimes b$ has degree $deg(a) + deg(b) + 1$. Then the natural commutativity constraint would be $C(a \otimes b) = -(-1)^{(deg(a)-1)(deg(b)-1)}$. It turns out that this constraint however does not satisfy the Hexagon Axiom for the usual associator. In the standard way of defining tensors there is no way to remedy the situation without violating the Pentagon Axiom.

One way to think about odd operads is that in the normal picture of operads the trees have been replaced by rooted trees whose internal edges and root edge each have weight one. This is the same as giving the vertices weight one as in a rooted tree every vertex has a unique outgoing edge.

If we allow tensor products on rooted trees, then the associators can be fixed by just enumerating the symbols $\otimes$ according to the vertex they correspond to in the tree picture for the bracketing. We get a sign according to the permutation of the respective vertices.

An even better picture is to place the symbols $\otimes$ on the edges in the tree picture for operations, see §4. To fast–forward a bit, this is one reason to introduce more general operads. In the case of modular operads this is simply captured by the notion of $\mathfrak{K}$–modular operad. In the present case, we could introduce twisted operads. Then indeed, such a change of sign is described by a twisting cocycle for a triple as we show in §4. This point of view would give an alternative definition; see Theorem 4.13.

These twists are also naturally incorporated in the setup of Feynman categories [Fey].

2. Cyclic, Anti–cyclic operads and a cyclic bracket

The first generalization we will give is for the cyclic case. We briefly recall the definitions in terms of operads with extra structure and in terms of arbitrary finite sets.

2.1. **The $\mathbb{S}_{n+}$ definition of cyclic operads.** In an operad one can think of $\mathcal{O}(n)$ as having n inputs and one output. The $\mathbb{S}_n$ action then permutes the inputs. The idea of a cyclic operad is that the output is also treated democratically, i.e. there is an action of $\mathbb{S}_{n+1}$ on $\mathcal{O}(n)$ which also permutes the output. Usually one labels the inputs by $\{1, \dots, n\}$ and the output by 0. In order to formalize this we follow [GeK2] and define $\mathbb{S}_{n+}$ to be the bijections of the set $\{0, 1, \dots, n\}$. Then $\mathbb{S}_n$ is naturally included into $\mathbb{S}_{n+}$ as the bijections that keep 0 fixed. As a group $\mathbb{S}_{n+} \simeq \mathbb{S}_{n+1}$ and it is generated by $\mathbb{S}_n$ and the long cycle $\tau = (01234 \cdots n)$. Let $C_{n+} \subset \mathbb{S}_{n+}$ be the cyclic group generated by τ.

Given an $\mathbb{S}_{n+}$ module (M, ρ) we denote the action of τ by T, i.e. for $m \in M$. $T(m) = \rho(\tau)(m)$. We also define the operator $N = 1 + T + \cdots + T^n$ on $\mathcal{O}(n)$.

Definition 2.1. [GeK1] A unital operad $\mathcal{O}$ is a *cyclic operad* if there is a $\mathbb{S}_{n+}$ action on each $\mathcal{O}(n)$ which extends the action of $\mathbb{S}_n$ such that the following conditions are met

(1) $T(id) = id$ where $id \in \mathcal{O}(1)$ is the operadic unit.
(2) $T(a \circ_1 b) = (-1)^{|a||b|} T(b) \circ_{ar(b)} T(a)$

The collection of objects $\mathcal{O}(n)$ together with their $\mathbb{S}_{n+}$ action is called a cyclic $\mathbb{S}$–module. In order to get the same indexing for the symmetric groups and the operad one sets $\mathcal{O}((n)) := \mathcal{O}(n-1)$. Here, morally, n is the number of inputs and outputs.

Example 2.2. The standard example of a cyclic operad is $\mathcal{E}nd(V)$ where V is a (graded) vector space of finite type with a (graded) non–degenerate even bilinear form $\langle\ ,\ \rangle$. The operation T on $f \in \mathcal{E}nd(n)$ is then defined via $\langle\ ,\ \rangle$ by

$$\langle v_0, Tf(v_1 \otimes \cdots \otimes v_n)\rangle = \pm\langle v_n, f(v_0 \otimes \cdots \otimes v_{n-1})\rangle \tag{2.1}$$

where in the graded case $\pm$ is the sign given by the Koszul sign rules. Another way to phrase this is as follows. $\langle\ ,\ \rangle$ gives an isomorphism between V and its dual space $\check{V}$. Thus

$$\mathcal{E}nd(V)(n) = Hom(V^{\otimes n}, V) \simeq \check{V}^{\otimes n} \otimes V \xrightarrow{\langle\ ,\ \rangle} \check{V}^{\otimes n+1} \tag{2.2}$$

Now on the last term there is an obvious $\mathbb{S}_{n+}$ action permuting the factors and this action can be transferred to $\mathcal{E}nd(V)(n)$ via the isomorphism.

Definition 2.3. [GeK1] A unital operad $\mathcal{O}$ is an *anti–cyclic operad* if there is a $\mathbb{S}_{n+}$ action on each $\mathcal{O}(n)$ which extends the action of $\mathbb{S}_n$ such that the following conditions are met

(1) $T(id) = -id$ where $id \in \mathcal{O}(1)$ is the operadic unit.
(2) $T(a \circ_1 b) = -(-1)^{|a||b|} T(b) \circ_{ar(b)} T(a)$

Example 2.4. The standard example of an anti–cyclic operad is furnished by the endomorphism operad of a symplectic vector space. That is $\mathcal{E}nd(V)$ where now V has a symplectic form ω. The action is then given as in the last example. The extra minus sign comes from the fact that the symplectic form is skew symmetric.

Remark 2.5. The last two examples can be unified using the notion of operadic correlation functions from [K5]. Here the correlation functions are given on $\check{V}^{\otimes n}$ and the propagators by the Casimir elements of $\langle\ ,\ \rangle$, where now these elements encode the signs. This fits well with the tree picture and Feynman diagrams since the propagators are associated to the *edges* and not the vertices.

2.1.1. **Algebras over (anti)–cyclic operads.** An algebra over a cyclic respectively anti-cyclic operad $\mathcal{O}$ is a vector space V together with a non–degenerate even symmetric form or respectively a non–degenerate even skew symmetric form and a morphism of cyclic, respectively anti–cyclic operads from $\mathcal{O}$ to $\mathcal{E}nd(V)$.

2.2. **Forgetful functor.** By simply forgetting the $\mathbb{S}_{n+}$ structure and only retaining the $\mathbb{S}_n$ structure on $\mathcal{O}(n)$, we get back an operad.

2.3. **Products.** For operads there are several products. We will be concerned with the naïve product defined as follows. Let $\{\mathcal{O}(n)\}$ and $\{\mathcal{P}(n)\}$ be operads then set $(\mathcal{O}\otimes\mathcal{P})(n) := \mathcal{O}(n)\otimes\mathcal{P}(n)$ with the diagonal $\mathbb{S}_n$ action.

For (anti–)cyclic operads, we use the diagonal $\mathbb{S}_{n+}$ action.

The product of two cyclic operads or two anti–cyclic operads is a cyclic operad while the product of a cyclic and an anti–cyclic operad is anti–cyclic.

Example 2.6. Given a cyclic operad $\mathcal{O}$ and a symplectic vector space V the operad $\mathcal{O}\otimes\mathcal{E}nd(V)$ is anti–cyclic.

Examples of cyclic operads are given by the cyclic extension of the operads *Comm*, *Lie* and *Assoc*. These are the operads whose algebras are precisely associative and commutative, Lie and associative algebras [CV, K].

Example 2.7. Given an anti–cyclic operad $\mathcal{O}$ and a vector space V with a symmetric non–degenerate pairing the operad $\mathcal{O}\otimes\mathcal{E}nd(V)$ is still anti–cyclic. Here a natural candidate is *pLie* the operad for pre–Lie algebras. The fact that this and several other operads are anti–cyclic is found in [Ch].

2.4. **Suspension for (anti)–cyclic operads.**

Definition 2.8. The operadic suspension $s\mathcal{O}$ of an (anti)–cyclic operad is given by the operad $s\mathcal{O}$ with the $\mathbb{S}_{n+}$ module structure on $s\mathcal{O}(n)$ given by the diagonal $\mathbb{S}_{n+}$ action on $\mathcal{O}(n)\otimes sgn_{n+1}$. Here we used that both sgn_n and sgn_{n+1} are both isomorphic to k as k–modules

An easy computation shows that

Lemma 2.9. *The operadic suspension of a cyclic operad is an anti–cyclic operad and vice–versa.*

Example 2.10. In the case of $\mathcal{E}nd(V)$ for a pair $(V,\langle\ ,\ \rangle)$, we have the isomorphism $\mathcal{E}nd(V[1])\simeq s\mathcal{E}nd(V)$. Now $\langle\ ,\ \rangle$ gives a pairing between $V[1]$ and $V[-1]$ so that we get an isomorphism $\mathcal{E}nd(V[1])(n)\simeq(\check{V}[-1])^{\otimes n}\otimes V[1]$. This space has natural degree $n-1$ and has a natural $\mathbb{S}_{n+}$ action. Since all the degrees are shifted by one, we see that if $\langle\ ,\ \rangle$ is symmetric, $s\mathcal{E}nd$ is anti–cyclic and if it is skew $s\mathcal{E}nd$ is cyclic.

2.5. **Naïve suspension and odd versions.** We can again use a naïve shift like in 1.5.3. Just like in that section we define an odd cyclic operad to be the result of the naïve shift of an anti–cyclic operad. This terminology ensures that $\Sigma s\mathcal{O}$ is odd cyclic.

2.6. **(Cyclic) Coinvariants.** Given a (anti)–cyclic operad $\mathcal{O}$ we define its space of coinvariants to be $\mathcal{O}^{\oplus}_{\mathbb{S}_+} := \bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+}}$.

We will also consider just the cyclic coinvariants $\mathcal{O}^{\oplus}_{C} := \bigoplus \mathcal{O}(n)_{C_{n+}}$ where C_{n+} is the cyclic subgroup generated by T in $\mathbb{S}_{n+}$.

2.6.1. **Non–Σ cyclic operad.** A weaker structure than that of cyclic operad is that of a non–Σ cyclic operad. Here one only requires an action of C_{n+}, the cyclic subgroup of $\mathbb{S}_{n+}$, on $\mathcal{O}(n)$.

2.7. **Cyclic operads via arbitrary indexing sets.** A nice way to think about cyclic operads is to look at operads in arbitrary sets. We think of the inputs and the output labeled by a set S. That is we get objects $\mathcal{O}(S)$ for any finite set S together with isomorphisms $\phi_* : \mathcal{O}(S) \to \mathcal{O}(S')$ for each bijection $\phi : S \to S'$. As well as structure maps

$$ {}_s\circ_t : \mathcal{O}(S) \otimes \mathcal{O}(T) \to \mathcal{O}((S \setminus \{s\}) \amalg (T \setminus \{t\})) \tag{2.3} $$

these maps are equivariant with respect to bijections and associative in the appropriate sense.

The cyclic or anti–cyclic condition then translate to

$$ a\, {}_s\circ_t b = \pm(-1)^{deg(a)deg(b)} b_t\circ_s a \tag{2.4} $$

where the extra minus sign is present in the anti–cyclic case.

2.8. **Moving between the two pictures.** We do not want to go into all these details about the correspondence between the two pictures and refer to [MSS] for full details.

Given a cyclic operad $\mathcal{O}$, one sets

$$ \mathcal{O}(S) = \left(\bigoplus_{\text{bijections } S \leftrightarrow \{0,1,\dots,|S|-1\}} \mathcal{O}(|S|-1) \right)_{\mathbb{S}_{n+}} \tag{2.5} $$

Where $\mathbb{S}_{n+}$ acts diagonally on both the sum, by acting on the bijections, and the summands. Given the full finite set version, the version using the natural numbers is basically given by inclusion.

For operads switching from $\mathcal{O}(n)$ to $\mathcal{O}(X)$ corresponds to switching from the category of finite sets with bijections to its skeleton, the category with objects the natural numbers and only automorphisms, where n represents the set $\{1, \dots, n\}$ and $Aut(n) = \mathbb{S}_n$. For cyclic operads n actually represents the set $\{0, 1, \dots, n\}$ and $Aut(n) = \mathbb{S}_{n+}$.

Following Markl, we will call the skeletal version involving only the natural numbers the biased version. The finite set version is then the un–biased one.

Caveat 2.11. Here there is one serious caveat. When composing, for operads one can identify the set $n \setminus \{i\} \amalg m$ with $n+m-1$ by first enumerating the first n elements until i is reached then enumerating the m elements of the second set and the rest of the elements of the first set. That is the set above has a natural linear order.

On the other hand, in the cyclic case, the set $n \setminus \{i\} \amalg m \setminus \{j\}$ does not have a canonical linear order, but only a cyclic one. If $j = 0$ and $i \neq 0$, then we are in the case above and we do have such an order. Likewise if $i = 0$ and $j \neq 0$, we again can make a linear order by switching the factors. This is essentially equivalent to the condition in the definition of a cyclic operad.

Notice that things are completely unclear where both $i = 0$ and $j = 0$. More on this below; see §2.11.

2.8.1. **Categorical formulation for $\mathbb{S}$–modules.** Consider the category Fin of finite sets with bijections. Then an $\mathbb{S}$–module is just a functor from that category to $g\mathcal{V}ect$ or the fixed category $\mathcal{C}$. Now Fin has as a skeleton the natural numbers in either the form $\{1,\dots,n\}$ or the form $\{0,\dots,n\}$. The former is used for operads and the latter for cyclic operads. That is for an operad $\mathcal{O}(n)=\mathcal{O}(\{1,\dots,n\})$ while for a cyclic operad $\mathcal{O}(n)=: O((n+1))=\mathcal{O}(\{0,\dots,n\})$. The equivalence on this level is then obvious.

2.8.2. **Tree picture.** One way to consider the relationship is that operads correspond to rooted trees whereas cyclic operads correspond to trees. There is an obvious forgetful functor from rooted trees to trees, which gives the inclusion of the operations corresponding to a rooted tree into those of a cyclic operad. The conditions on a cyclic operad vice–versa guarantee that the operation of a rooted tree is equivariant under changes of the root.

On the other hand given just a tree, to make it rooted, there is a choice of a root and there is no canonical choice. The only thing to do is to sum over all of these choices. In the $\mathbb{S}$–module operad picture this corresponds to using the operator N. All these considerations appear naturally in the realm of Feynman categories where these operations are realized by pull–backs and push–forwards given by Kan–extensions.

2.8.3. **Coinvariants.** Things become nicer on the level of coinvariants. Here it suffices to take $\mathcal{O}^{\oplus}_{\mathbb{S}_+}$. The categorical proof is that this represents the colimit over the category of finite sets with bijections of $\mathcal{O}$ viewed as the functor that assigns $\mathcal{O}(S)$ to a set S.

A pedestrian way to say this is that taking coinvariants, we can first identify sets which are in bijection with each other and then only have to mod out by automorphisms. For each finite set S we can choose $\{0,\dots,|S|-1\}$ as such a representative.

2.9. **The bracket in the anti–cyclic case.**

Definition 2.12. Let $\mathcal{O}$ be an anti–cyclic operad For $a\in\mathcal{O}(S)$ and $b\in\mathcal{O}(T)$ we define

$$[a\odot b]:=\sum_{s\in S,t\in T} a\,{}_s\circ_t b \tag{2.6}$$

Proposition 2.13. $[\ \odot\ ]$ *is anti–symmetric and satisfies the Jacobi identity for any three elements in the sense that for $a\in\mathcal{O}(S), b\in\mathcal{O}(T), c\in\mathcal{O}(U)$*

$$[a\odot b]=-(-1)^{deg(a)deg(b)}[b\odot a]\in\bigoplus_{s\in S\, t\in T}\mathcal{O}((S\setminus s)\amalg(T\setminus t))$$

$$(-1)^{deg(a)deg(c)}[a\odot[b\odot c]]+(-1)^{deg(a)deg(b)}[b\odot[c\odot a]]+(-1)^{deg(c)deg(b)}[c\odot[a\odot b]]=0$$
$$\in\bigoplus_{s\in S,t\in T,u\in U}\mathcal{O}((S\setminus s)\amalg(T\setminus t)\amalg(U\setminus u)) \tag{2.7}$$

Proof. The proof is a straightforward calculation. The first equation directly follows from the antisymmetry of the operations ${}_s\circ_t$ for an anti–cyclic operad.

Checking the Jacobi identity is straight forward: $(-1)^{deg(a)deg(c)}[a\odot[b\odot c]] =$

$$
\begin{aligned}
& (-1)^{deg(a)deg(c)} \sum_{\substack{t'\in T\coprod_{s\in S} U\setminus\{t,u\}}} \sum_{\substack{t\in T\\ u\in U}} a_s\circ_{t'}(b_t\circ_u c) \\
= \; & (-1)^{deg(a)deg(c)} \sum_{\substack{t'\in T\setminus\{t\}\\ s\in S}} \sum_{\substack{t\in T\\ u\in U}} a_s\circ_{t'}(b_t\circ_u c) + (-1)^{deg(a)deg(c)} \sum_{\substack{t'\in U\setminus\{u\}\\ s\in S}} \sum_{\substack{t\in T\\ u\in U}} a_s\circ_{t'}(b_t\circ_u c) \\
= \; & (-1)^{deg(a)deg(c)} \sum_{\substack{t'\in T\setminus\{t\}\\ s\in S}} \sum_{\substack{t\in T\\ u\in U}} (a_s\circ_{t'} b)_t\circ_u c - (-1)^{deg(a)deg(b)} \sum_{\substack{t'\in U\setminus\{u\}\\ s\in S}} \sum_{\substack{t\in T\\ u\in U}} (b_t\circ_u c)_{t'}\circ_s a \\
= \; & -(-1)^{deg(b)deg(c)} \sum_{\substack{t'\in T\setminus\{t\}\\ s\in S}} \sum_{\substack{t\in T\\ u\in U}} c_u\circ_t(a_s\circ_{t'} b) - (-1)^{deg(a)deg(b)} \sum_{\substack{t'\in U\setminus\{u\}\\ s\in S}} \sum_{\substack{t\in T\\ u\in U}} b_t\circ_u(c_{t'}\circ_s a) \\
= \; & -(-1)^{deg(b)deg(c)}[c\odot[a\odot b]] - (-1)^{deg(a)deg(b)}[b\odot[c\odot a]]
\end{aligned}
$$

□

Notice that in this statement, we use the conventions stated in the beginning.

In view of §2.8.3 the following theorem is now straightforward.

Theorem 2.14. *If $\mathcal{O}$ is an anti–cyclic operad then $[\ \odot\ ]$ induces a Lie bracket on $\mathcal{O}^{\oplus}_{\mathbb{S}_+}$.*

□

We will denote this Lie bracket by the same symbol.

Remark 2.15. Notice that unlike in the operad case, this bracket is *not the anti–symmetrization of a pre–Lie structure*. It is actually the choice of the root that gives this extra structure in the operad case through the linear orders on the compositions. Here no such consistent choice for linear orders exists. See also 2.11 and §2.11.

Example 2.16. [K, CV] Fixing a sequence of vector spaces of dimension $2n$ with a symplectic form on them, we immediately get three sequences of Lie algebras from the anti–cyclic operads $Comm\otimes\mathcal{E}nd(V^n), Lie\otimes\mathcal{E}nd(V^n)$ and $Assoc\otimes\mathcal{E}nd(V^n)$. These are exactly the three sequences considered by Kontsevich in his seminal paper [K] and further studied by [CV]. There is also the generalization of this construction to cyclic quadratic Koszul operads [Gi].

Example 2.17. Likewise we can fix a sequence of dimension n vectors spaces V^n with a symmetric non–degenerate bilinear form and consider the sequence of Lie algebras obtained $pLie\otimes\mathcal{E}nd(V^n)$.

This begs the

Question 2.18. What is the underlying geometry in the $pLie$ case? Or in the other cases of [Ch]?

Example 2.19. Of course by §2.3 and §2.4 any suspension of a cyclic operad will yield an anti–cyclic one and hence a Lie algebra and any tensor product of a cyclic operad with an anti–cyclic one will give and anti–cyclic operad and hence a Lie algebra.

2.10. **Lift to the cyclic co–invariants, non–Σ version.** As mentioned before, the set $n \setminus \{i\} \amalg m \setminus \{j\}$ has no canonical linear order, but it does have a cyclic order. Hence we can identify it with $n+m-1$ up to the action of C_{n+m-1+}. Using this identification, we can restrict to the C_{n+} coinvariants of the sets n to obtain a bracket on the cyclic co–invariants and since we are only taking C_{n+} coinvariants it actually suffice to take a non–Σ cyclic operad.

Theorem 2.20. *If $\mathcal{O}$ is an anti–cyclic operad then $[\ \odot\ ]$ induces a Lie bracket on the cyclic co–invariants $\mathcal{O}_C^{\oplus} := \bigoplus \mathcal{O}(n)_{C_{n+}}$.*

The same result holds true for $\mathcal{O}$ a non–Σ anti–cyclic operad.

Example 2.21. The necklace Lie algebra of Bocklandt and Le Bruyn [BLB,Sch] is an example of such a Lie algebra structure. Here the cyclic operad structure is on the oriented cycles and the necklace words are the cyclic invariants.

2.11. **The bracket in the biased setting and compatibilities.** Using the above description, we can relate the original brackets to those arising in the operad setting. The obstruction is that the two brackets lift to different spaces, but we can use the operator N which maps $\mathcal{O}(n)$ to $\mathcal{O}(n)^{C_{n+}}$ to make the connection.

We first introduce the operations

$$a_i \bar{\circ}_j b = T^{1-i} a \circ_1 T^{-j} b \tag{2.8}$$

Notice that $a \circ_i b = a_i \bar{\circ}_0 b$ and $b \circ_j a = a_j \bar{\circ}_0 b$

Proposition 2.22. *$[N(a) \circ N(b)]$ is in the image of N. Moreover, if we set $c = \sum_{i,j} a_i \bar{\circ}_j b$ then $[N(a) \circ N(b)] = N(c)$.*

The map N induces a map of Lie algebras from $\mathcal{O}_C^{\oplus}$ with bracket $[\ \odot\ ]$ to $\mathcal{O}^{\oplus}$ with bracket $[\ \circ\ ]$ via $[a] \mapsto N(a)$.

Proof.

$$[N(a)\circ N(b)] = \left[\sum_{i=0}^{n} T^i(a), \sum_{j=0}^{m} T^j(b)\right]$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} T^i(a)\circ \sum_{j=0}^{m} T^j(b) - \sum_{j=0}^{m} T^j(b)\circ \sum_{i=0}^{n} T^i(a)\\
&= \sum_{x=1}^{n}\left(\sum_{i=0}^{n} T^i(a)\circ_x \sum_{j=0}^{m} T^j(b)\right) - \sum_{y=1}^{n}\left(\sum_{j=0}^{m} T^j(b)\circ_y \sum_{i=0}^{n} T^i(a)\right)\\
&= \sum_{x=1}^{n}\sum_{i=0}^{n}\sum_{j=0}^{m}\big(T^i(a)\circ_x T^j(b)\big) - \sum_{y=1}^{n}\sum_{j=0}^{m}\sum_{i=0}^{n}\big(T^j(b)\circ_y T^i(a)\big)\\
&= \sum_{x=1}^{n} T^{x-1}\sum_{i=0}^{n}\sum_{j=0}^{m}\big(T^{i-x+1}(a)\circ_1 T^j(b)\big) - \sum_{y=1}^{n} T^{y-m}\sum_{j=0}^{m}\sum_{i=0}^{n}\big(T^{j+m-y}(b)\circ_m T^i(a)\big)\\
&= \sum_{x=1}^{n} T^{x-1}\sum_{i=0}^{n}\sum_{j=0}^{m}\big(T^{i-x+1}(a)\circ_1 T^j(b)\big) + \sum_{y=1}^{m} T^{y-m-1}\sum_{j=0}^{m}\sum_{i=0}^{n}\big(T^i(a)\circ_1 T^{j+m-y+1}(b)\big)\\
&= \sum_{x=0}^{n-1} T^{x}\sum_{i=0}^{n}\sum_{j=0}^{m}\big(T^{i-x}(a)\circ_1 T^j(b)\big) + \sum_{x=-m}^{-1} T^{x}\sum_{j=0}^{m}\sum_{i=0}^{n}\big(T^i(a)\circ_1 T^{j-x}(b)\big) \qquad (2.9)
\end{aligned}
$$

$$(2.10)$$

$$
\begin{aligned}
&= \sum_{x=1}^{n-1} T^{x}\sum_{i'=0}^{n}\sum_{j'=0}^{m}\Big(T^{-i'+1}(a)\circ_1 T^{-j'}(b)\Big) + \sum_{x=-m}^{-1} T^{x}\sum_{j''=0}^{m}\sum_{i''=0}^{n}\Big(T^{-i''+1}(a)\circ_1 T^{-j''}(b)\Big)\\
&= \sum_{x=-m}^{n-1} T^{x}\sum_{k=0}^{n}\sum_{l=0}^{m}\Big(T^{-k+1}(a)\circ_1 T^{-l}(b)\Big)\\
&= \sum_{x=-m}^{n-1} T^{x}(a_l\bar{\circ}_k b) = \sum_{x=-m}^{n-1} T^{x}c = N(c)
\end{aligned}
$$

To go from line (2.9) to the next line one needs to re–index, which is possible since for a fixed x the interior double sum takes all combinations of $[n]\times[m]$.

To get the second statement we remark that the image of c in the co–invariants satisfies $[c] = [[a]\odot[b]]$. □

To compare the brackets on the cyclic invariants we will use the standard isomorphism between invariants and coinvariants, so let the characteristic we are working in be 0. Consider the usual sequence of invariants and coinvariants.

$$0 \longrightarrow \mathcal{O}(n)^{C_{n+}} \underset{\frac{1}{n+1}N}{\overset{i}{\rightleftarrows}} \mathcal{O}(n) \underset{s}{\overset{p}{\rightleftarrows}} \mathcal{O}(n)_{C_{n+}} \longrightarrow 0 \qquad (2.11)$$

where i is the inclusion, p is the projection and $s([a]) = \frac{1}{n+1}Na$.

Plugging in, we get

Corollary 2.23. *For $a \in \mathcal{O}(n)$ and $b \in \mathcal{O}(m)$:*

$$p[s([a] \circ s([b])] = \frac{n+m-2}{(n+1)(m+1)}[[a] \odot [b]] \tag{2.12}$$

That is on the cyclic coinvariants the two brackets coincide up to a coboundary 2–cocycle.

2.12. **The odd Lie bracket and odd cyclic operads.** We can now adapt Gerstenhaber's construction to the cyclic operad setting. We first note the following.

Proposition 2.24. *If $\mathcal{O}$ is a cyclic operad then $s\mathcal{O}$ is an anti–cyclic operad with a Lie bracket $[\ \odot\ ]$. This Lie bracket yields an odd Lie bracket $\{\ \odot\ \}$ on $\mathcal{O}^{\oplus}_{\mathbb{S}_+}$ when using the degree $|\,.\,|$. More precisely it is an odd Lie bracket on the odd cyclic operad $\Sigma s\mathcal{O}$.*

Proof. The only thing to check is that the signs are correct. This follows from the fact that the degree of a in $s\mathcal{O}$ is indeed $s(a) = |a| - 1$. In particular $[\ \odot\ ]$ is a Lie bracket for the grading s and hence after applying a shift, again by Lemma 1.4, it is odd Lie for the grading $|\,.\,|$ which is given by an additional naïve shift. □

3. (Wheeled) PROPs and Properads.

There are several further generalizations of operad structures. For an operad $\mathcal{O}$ it is natural to consider $\mathcal{O}(n)$ as having n–inputs and one output. The first generalization is to include multiple inputs and outputs. The next generalization is to allow non–connected graphs. Using both of them one arrives as PROPs, which where actually first historically [McL1,BV]. Restricting back to the connected graphs, one arrives at the notion of properads [Va].

The next step, which will take us to the realm of Metatheorem 2 is to allow self–gluings. This leads to the notions of wheeled PROPs and wheeled properads [MMS]. Here it will become apparent that the odd gluing is essential. For the wheeled cases there is still a shift, which will allow to make the gluings odd. This is intimately related to the fact that PROPs just like operads have distinct inputs and outputs.

Finally, wheeled PROPs as they deal with non–connected graphs are the first instance, where a multiplication for the BV operator and the Gerstenhaber bracket naturally appears.

3.1. **PROPs.** A unital PROP in the biased definition has an underlying sequence of objects $\mathcal{O}(m,n)$ of $\mathcal{C}$ or say $dg\mathcal{V}ect$ which carry an $\mathbb{S}_n \times \mathbb{S}_m$ action. For this collection of bimodules to be a PROP, it has to have the following additional structures.

(1) Vertical compositions $\boxminus : \mathcal{O}(n,m) \otimes \mathcal{O}(m,k) \to \mathcal{O}(n,k)$ which are equivariant

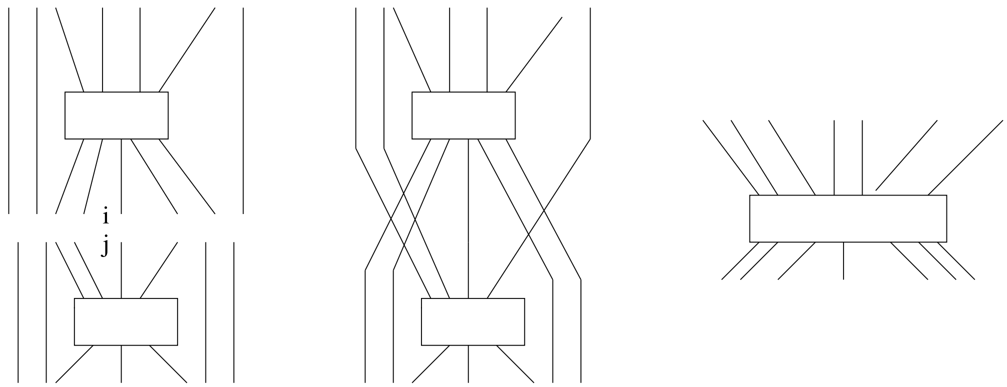

FIGURE 1. The dioperadic compositions

(2) Horizontal compositions $\boxminus : \mathcal{O}(n,m)\otimes\mathcal{O}(k,l)\to\mathcal{O}(n+k,m+l)$ which are compatible in the sense that $(a \boxbar b)\boxminus(c\boxbar d) = (a\boxminus c)\boxbar(b\boxminus d)$
(3) Unit. $\mathbb{1}\in\mathcal{O}(1,1)$, s.t. $(\mathbb{1}\boxminus\cdots\boxminus\mathbb{1})\boxbar a = a\boxbar(\mathbb{1}\boxminus\cdots\boxminus\mathbb{1}) = a$

The collection of objects $\mathcal{O}(n,m)$ together with the $\mathbb{S}_n\times\mathbb{S}_m$ action is called an $\mathbb{S}$–bimodule.

The graphs that one can compose along are not necessarily connected oriented graphs without oriented loops.

We define the compositions $a\,{}_i\circ_j b$ by adding identities in all slots other than into the input slot i of a and the output slot j of b and gluing i and j together. These operations, gluing one input to one output, are called dioperadic operations ${}_i\circ_j : \mathcal{O}(n,m)\otimes\mathcal{O}(k,l)\to\mathcal{O}(n+k-1,m+l-1)$

In the unbiased version one has a functor $\mathcal{O}$ from $Fin\times Fin$ to $\mathcal{C}$. Using the unit, one obtains compositions ${}_s\circ_t : \mathcal{O}(U,S)\otimes\mathcal{O}(T,V)\to\mathcal{O}(U\amalg T\setminus\{t\}, V\amalg (S\setminus\{s\})$.

Example 3.1 (Endomorphism PROP)**.** The canonical example is the endomorphism PROP $\mathcal{E}nd(V)(n,m) = Hom(V^{\otimes n}, V^{\otimes m})$ with the obvious $\mathbb{S}_n\times\mathbb{S}_m$ action permuting the variables and functions together with the obvious compositions.

Remark 3.2. Every PROP contains an operad given by the $\mathcal{O}(n,1)$ and the dioperadic operations ${}_i\circ_1 =: \circ_i$.

Example 3.3 (PROP generated by an operad)**.** An operad can be thought of as giving a sequence $\mathcal{O}(n,1)$. Setting

$$\mathcal{O}(n,m) := \bigoplus_{(n_1,\dots,n_m):\sum n_i = n} \mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_m)$$

we obtain a PROP by defining $\boxminus$ to be essentially the identity, i.e. just tensoring together the two factors followed by the inclusion of the summand. The $\mathbb{S}_m$ action permutes the factors and the $\mathbb{S}_n$ action acts via the identification of the disjoint union of the sets $\{1,\dots,n_i\}$ with the set $\{1,\dots,n\}$ by first enumerating them one after another in the order given by i. This is a good example of a non–connected generalization treated in §6.1.

3.1.1. **Properads.** Looking at the definition of a PROP one can see that the associativity implies that there are compositions defined for any oriented graph Γ.

Restricting to the situation where compositions are defined for non all *connected* oriented graphs one obtains the notion of a properad [Va]. For instance the horizontal composition $\boxminus$ is dropped.

3.1.2. **Algebras.** An algebra over a PROP(erad) $\mathcal{O}$ is then a vector space V together with a morphism of PROP(erad)s $\mathcal{O} \to \mathcal{E}nd(V)$

3.1.3. **Coinvariants.** We let $\mathcal{O}_S^{\oplus}$ be the sum over the coinvariants $\mathcal{O}(n,m)_{\mathbb{S}_n \times \mathbb{S}_m}$.

3.2. **Poisson–Lie bracket.** Analogously to the structure of the Lie bracket for operads, we can define for $a \in \mathcal{O}(n,m)$ and $b \in \mathcal{O}(k,l)$

$$a \circ b := \sum_{i,j} a\, {}_i\circ_j b, \quad [a \circ b] := a \circ b - (-1)^{deg(a)deg(b)} b \circ a \tag{3.1}$$

As before we let $\mathcal{O}^{\oplus} = \bigoplus_{n,m} \mathcal{O}(n,m)$ and $\mathcal{O}_{\mathbb{S}}^{\oplus} := \bigoplus_{n,m} \mathcal{O}(n,m)_{\mathbb{S}_n \times \mathbb{S}_m}$. In the case of a PROP, we also have a natural multiplicative structure given by $\boxminus$.

Theorem 3.4. *For a PROP(erad) $\mathcal{O}$, the product above is pre–Lie on $\mathcal{O}^{\oplus}$ and hence induces a Lie bracket $[\ \circ\ ]$. This Lie bracket descends to $\mathcal{O}_{\mathbb{S}}^{\oplus}$.*

For a PROP $\mathcal{O}$ The induced Lie bracket on $\mathcal{O}_{\mathbb{S}}^{\oplus}$ is Poisson w.r.t $\boxminus$.

The Lie bracket for operad induces a Poisson bracket on the PROP generated by that operad coinciding with the natural Poisson bracket above.

Proof. The proof of the pre–Lie and hence Jacobi–identity can be adapted from the operad case. To show the Poisson property, we see that $a \circ (b \boxminus c) = (a \circ b) \boxminus c + (-1)^{deg(a)deg(b)} b \boxminus (a \circ c)$ up to symmetric group actions depending if an output of a is glued to b or c, where the sign comes from the commutativity constraint in $g\mathcal{V}ect$. The last statement follows by the definition of the Poisson property and Example 3.2. □

Adding a vertical composition formally to Properads, by using not necessarily connected graphs, we end up back with PROPs. For cyclic operads things are a bit more complicated, and we have to first introduce the notion of non–connected cyclic operads. This is done in §6.1.

3.3. **Odd versions.** The odd versions of the concepts above can again be defined by using shifts and suspensions.

3.3.1. **Suspension.** The *suspension* of a PROP $\mathcal{O}$ is the PROP $s\mathcal{O}$ whose $\mathbb{S}$–bimodule is

$$\Sigma^{n-m}\mathcal{O}(n,m)\otimes(sgn_n\otimes sgn_m) \tag{3.2}$$

Just like for operads we have the following version of Proposition 1.12:

Proposition 3.5. *[MMS] $s\mathcal{O}$ is indeed a PROP and V is an $\mathcal{O}$–algebra if and only if $V[1]$ is an $s\mathcal{O}$ algebra.*

This explains both the shift and the sign representations.

3.3.2. **Naïve/output shift.** Now the naïve shift is a bit more complicated than before. We can again take $\mathcal{E}nd$ as a guide. Naively shifting it as an operad and then taking the PROP it generates we see that we are led to the following definition.

Given an $\mathbb{S}$–bimodule $\mathcal{O}$, we let $s_{\mathrm{out}}\mathcal{O}$ be the bimodule

$$s_{\mathrm{out}}\mathcal{O}(n,m)=\Sigma^{m}\mathcal{O}(n,m)\otimes sgn_m \tag{3.3}$$

Just like in the case of operads (which is a subcase), one obtains slightly different signs in the associativity equations than one would expect for the induced operations.

Definition 3.6. An odd PROP(erad) is the naïve shift of a PROP. That is $\mathcal{O}$ is an odd PROP(erad) if and only if $s_{\mathrm{out}}^{-1}\mathcal{O}$ is a PROP.

Example 3.7. An example of such an odd PROP(erad) is given by

$$\mathcal{P}(n,m)=\check{V}^{\otimes n}\otimes\Sigma^{m}(V^{\otimes m}\otimes sgn_m) \tag{3.4}$$

with the natural $\mathbb{S}_n\times\mathbb{S}_m$ action. The vertical composition given by the natural pairing are given by the natural pairing $\check{V}\otimes V\to k$ and the horizontal composition is induced by tensoring together the factors..

We will also consider the suspension given by $s_{\mathrm{in}}\mathcal{O}(n,m)=\Sigma^{n}sgn_n\otimes\mathcal{O}(n,m)$. With this notation, we see that $s=s_{\mathrm{in}}s_{\mathrm{out}}^{-1}$.

With these notions in place, we can extend Metatheorem 1 and due to the existence of $\boxminus$ the resulting bracket is even Gerstenhaber.

Theorem 3.8. *An odd PROP(erad) $\mathcal{O}$ carries an odd (pre)–Lie bracket on $\mathcal{O}^{\oplus}$ and $\mathcal{O}^{\oplus}_{\mathbb{S}}$. The odd Lie bracket is Gerstenhaber w.r.t. $\boxminus$ for an odd PROP on $\mathcal{O}^{\oplus}_{\mathbb{S}}$. The odd Lie bracket on an odd operad induces an odd Lie bracket on the odd PROP generated by that operad and it is a Gerstenhaber bracket there.*

Proof. The only thing to check is that the effective shift for the dioperadic operations is indeed one. This is the case, since before the dioperadic operation, the total shift is $n+m$ and after the shift it is $n+m-1$. $\square$

3.4. **Wheeled versions.** The dioperadic operations and $\boxminus$ are not quite enough to recover the PROP structure. After one such operation, to get to the operation $\boxdot$ one would have to do self–gluings of one input to an output. This is precisely what is allowed in the wheeled version.

That is in the unbiased version a wheeled PROP has the operations $\boxminus$, ${}_s\circ_t$ and self–gluing operations $\circ_{st} : \mathcal{O}(S,T) \to \mathcal{O}(S\setminus\{s\}, T\setminus\{t\})$ which again satisfy natural equivariance, associativity and compatibility.

The compositions are defined for not necessarily connected oriented graphs with wheels.

Dropping the horizontal composition $\boxminus$ one obtains the notion of a wheeled properad. The compositions are defined for *non–connected* oriented graphs with wheels.

Example 3.9. The PROP(erad) $\mathcal{E}nd_V(n,m) \simeq \check{V}^{\otimes n} \otimes V^{\otimes m}$ has such a natural wheeling by simply contracting tensors for the self–gluings.

3.4.1. **Wheeled odd PROP(erad)s.** The odd versions are described just as above. These are by definition the images under the suspension s_{out}. Again, we denote the image of the compositions ${}_i\circ_j$ and $\circ_{ij}$ by ${}_i\bullet_j$ and $\bullet_{ij}$.

Lemma 3.10. *In on odd wheeled PROP(erad), we have $\bullet_{ij} \bullet_{kl} a = - \bullet_{k'l'} \bullet_{i'j'} a$, where i', j', k', l' are the names of the appropriately renumbered flags.*

Proof. This is due to the shift. Now if we interchange the order, we interchange outputs j and l resulting in a minus sign. Since the inputs are unaltered, switching i and k gives no sign. $\square$

This is the first time we encounter odd–self gluings, and we indeed find the first occurrence of Metatheorem 2.

Theorem 3.11. *For an odd wheeled PROP(erad) $\mathcal{O}$, the operator Δ defined on each $\mathcal{O}(n,m)$ by*

$$\Delta(a) := \sum_{ij} \bullet_{ij}(a) \tag{3.5}$$

satisfies $\Delta^2 = 0$.

Moreover on the cyclic coinvariants for a PROP the operator Δ is a BV operator on $\mathcal{O}_{\mathbb{S}}^{\oplus}$ for the multiplication $\boxminus$ and its associated bracket (see Appendix) is the Gerstenhaber bracket induce by $\{\ \bullet\ \}$.

Proof. The reason for the vanishing of Δ is Lemma 3.10. For the BV bracket we notice that $\Delta(a \boxminus b)$ splits into four sums depending on the gluing. The inputs of a glued to the outputs of a, this gives $\Delta(a)b$, the inputs of b to the outputs of b, the term $a\Delta(b)$, the outputs of a to the inputs of b and vice–versa, which gives $a \bullet b$ and $b \bullet a$ respectively — all up to permutations.

The only thing that remains to be checked is that the signs that work out which they do by a straightforward computation. Δ has degree 1 since each $\bullet_{ij}$ has degree 1 after the shift. Finally, the structures descend as we sum over all possible gluings. $\square$

Remark 3.12. Notice that there is no BV in the unshifted case. We need the odd composition to get a differential. This also shows that the Gerstenhaber bracket is actually the deeper one and the regular Lie bracket is actually a shift of the odd one rather than vice–versa.

4. Modular operads, triples and twisting

We will now turn to self–gluings for operads. This leads to the notion of modular operads. This is the first notion, where the odd version is not given by a simple shift. It is rather a twist, namely what is know as a $\mathfrak{K}$–modular operad. For this we will need to introduce triples. With hindsight, we will see that all the other odd versions also arise from twisted triples. We will deal with the triples for twisted modular operads quite explicitly and against this background will be more casual for the other triples.

4.1. **Modular operads.** We will introduce modular operads in the unbiased setting.

A modular operad is a collection $\mathcal{O}(g,S)$ bi–indexed by finite sets and the natural numbers, usually taken with the condition that $2g+2-|S|>0$ together with gluing maps

$$ {}_s\circ_t : \mathcal{O}(g,S)\otimes\mathcal{O}(g',T)\to\mathcal{O}(g+g',S\setminus\{s\}\amalg T\setminus\{t\}) \quad \forall s\in S, t\in T \tag{4.1}$$

and self gluing maps

$$ \circ_{ss'} : \mathcal{O}(g,S)\to\mathcal{O}(S\setminus\{s,s'\},g+1) \quad \text{for all distinct } s,s'\in S \tag{4.2}$$

which are compatible associative and equivariant with respect to bijections. The details of these conditions are straightforward, but tedious and we refer to [GeK2, MSS]. An alternative definition utilizing of triples is below; see §4.

Example 4.1. The motivating example are the Deligne–Mumford compactifications $\bar{M}_{g,S}$ of curves of genus g with $|S|$ punctures labeled by the set S.

A linear example is then given by the $H_*(\bar{M}_{g,n})$.

For the biased version, just like in the cyclic case, one uses the sets $\{0,1,\dots,|S|-1\}$ and the notation $\mathcal{O}((g,n)) := \mathcal{O}(g,n-1)$.

4.2. **Triples.** Before delving into the categorical depth of triples, we will consider a relevant example in the case of operads. The main idea connecting the definition via triples to the previous ones is that the associativity of the gluing operations $\circ_i$ (or $\circ_s$ in the unbiased case) guarantees that each S–labeled rooted tree τ gives a unique composition $\circ_\tau$ from $\mathcal{O}(\tau)$ (defined by equation (4.3) below) to $\mathcal{O}(s)$.

4.2.1. **Forgetful and Free Functor.** Given an operad $\mathcal{O}$ we can forget the gluing maps and only retain the $\mathbb{S}$–module. This gives a functor G between the respective categories. The functor G has a left adjoint functor F which is the

type	graphs for triple	local sets at a vertex v
operad	rooted trees	incoming flags
non–Σ operad	planar rooted trees	incoming flags
cyclic operads	trees	flags
non–Σ cyclic operad	planar trees	flags
modular operad	stable graphs	(flags, $g(v)$)
PROP	not necessarily connected directed graphs	(incoming flags, outgoing flags)
properad	connected directed graphs	(incoming flags, outgoing flags)
wheeled PROP	not necessarily connected directed graphs with wheels	(incoming flags, outgoing flags)
wheeled properad	connected directed graphs with wheels	(incoming flags, outgoing flags)

TABLE 2. Types of operads and the graphs underlying their triples

free functor. Explicitly, given an $\mathbb{S}$–module $\mathcal{V}$, the free operad $F(\mathcal{V})$ on $\mathcal{V}$ is constructed as follows. For a rooted tree τ one sets

$$\mathcal{V}(\tau) = \bigotimes_{v \text{ vertex of } \tau} \mathcal{V}(In(v)) \tag{4.3}$$

where $In(v)$ is the set of flags or half edges at incoming at v. Recall that in a rooted tree there is a natural orientation towards the root and this defines the outgoing edge or flag at each vertex. All other flags are incoming.

The composition $\circ_\tau$ is obtained by contracting all edges, that is for each edge we perform $\circ_i$ operation where i is the input flag of the edge.

Rooted trees whose tails are labeled by a set S form a category $\mathcal{I}so\mathcal{R}T(S)$, by allowing isomorphism of labeled rooted trees as the only morphisms. The free operad is the given by the $\mathbb{S}$–module

$$F(\mathcal{V}) = colim_{\mathcal{I}so\mathcal{R}T(n)}\mathcal{V} = \bigoplus_{\tau\in\mathcal{R}T(n)} \mathcal{V}(\tau)/\sim \; = \bigoplus_{[\tau] \text{ iso classes}} \mathcal{V}(\tau) \tag{4.4}$$

where $\sim$ is the equivalence under push–forward with respect to isomorphism. The operad structure on the $F(\mathcal{V})(S)$ is given summand by summand. If there are two summands indexed by τ in $F(\mathcal{V})(S)$ and τ' in $F(\mathcal{V})(T)$ under the composition $\circ_s$ their tensor product maps to the summand $\tau \circ_s \tau'$ which is the tree where τ' is glued onto τ at the leaf indexed by s.

4.3. **Operads as triples.** Let $\mathbb{T} = GF$ which is an endo–functor from $\mathbb{S}$–modules to $\mathbb{S}$–modules. Since F and G are an adjoint pair, there are natural transformations $\epsilon : FG \to id$ and $\eta : id \to GF$. In our particular case, the first is given by sending the summand of τ to its image under the composition $\circ_\tau$. This is well defined up to isomorphism because of the equivariance of the gluings. The second is just inclusion of the summand given by the S labeled tree with one vertex. Actually, one can prove that they indeed form an adjoint pair using these natural transformations; see e.g. [GM].

4.3.1. **Triples.** Using these on $\mathbb{T}$ one gets the following natural transformations $\mu : \mathbb{T}\mathbb{T} \to \mathbb{T}$ via $G(FG)F \stackrel{\epsilon}{\to} GF$ and $\eta : id \to \mathbb{T}$. These natural transformation satisfy the equations of an associative unital monoid. In general *a triple* is an endo–functor $\mathbb{T}$ together with μ and η which satisfies just these equations. Our triple was constructed using an adjoint pair and it is a fact that all triples actually arise this way [EM, Kl].

4.3.2. **Operads.** Now if $\mathcal{O}$ is an operad, we also get a map $\alpha : \mathbb{T}\mathcal{O} \to \mathcal{O}$ by sending each summand $\mathcal{O}(\tau)$ indexed by an S–labeled tree τ to $\mathcal{O}(S)$ using $\circ_\tau$. Due to the associativity these maps satisfy the module equations when considering the two possible ways to map $\mathbb{T}\mathbb{T}\mathcal{O}$ to $\mathcal{O}$.

Vice–versa, given an $\mathbb{S}$–module $\mathcal{V}$ if we are given a morphism $\alpha : \mathbb{T}\mathcal{V} \to \mathcal{V}$, we have equivariant maps $\circ_\tau$ and moreover if they satisfy the module equations, then these $\circ_\tau$ decompose into elementary maps $\circ_s$, where the $\circ_s$ come from rooted trees with exactly one internal edge. It is straightforward to check that the $\circ_s$ define an operad structure on the $\mathcal{V}(S)$.

The natural transformation μ also has a nice tree interpretation. Let τ_0 be the tree index of the first application of $\mathbb{T}$, then in the next application one picks up a collection of indices τ_v, one for each vertex v of τ_0. In order to show the associativity, one can see that the corresponding summand of $\mathbb{T}\mathbb{T}\mathcal{V}$ is the same as $\mathcal{V}(\tau_1)$ where τ_1 is obtained from τ_0 by blowing up each vertex v into the tree τ_v. Vice–versa, τ_0 is obtained from τ_1 by contracting the subtrees τ_v to a vertex. One sometimes writes $\tau_1 \to \tau_0$ since this is a morphism in the naïve category of graphs.

4.3.3. **Algebras over triples.** In general an algebra over a triple $\mathbb{T}$ is an object $\mathcal{V}$ of the underlying category together with a map $\alpha : \mathbb{T}\mathcal{V} \to \mathcal{V}$ such that α, μ and η satisfy the axioms of a module over an algebra with a unit; see [MSS] for the precise technical details.

From the above, we obtain:

Proposition 4.2. *Operads are precisely algebras over the triple $\mathbb{T}$ of rooted trees.* □

4.4. **Other cases.** The method is now set to define all the other cases as algebras over a triple. We only have to specify the triple. Taking the cue from above, we have to (1) fix the type of graph and the category of isomorphisms (2) fix the value of $\mathcal{V}$ on each graph, i.e. the analogue of equation (4.3); in all common examples this is local in the vertices. And (3) set $F(\mathcal{V}) = colim_{\mathcal{I}so\mathcal{G}raph}\mathcal{V}$ where the colimit is taken over the category of isomorphisms of S–labeled graphs of the given type. (4) give μ via gluing the graphs together by inserting the graphs indexed by a vertex into that vertex. Think of this as the blow–up which is inverse to the operation of contracting the subgraph.

For (1) we use Table 2 where we take the S–labeled version of the respective graphs. For (2) we use the general formula

$$\mathcal{O}(\Gamma) = \bigotimes_{v \text{ vertex of } \Gamma} \mathcal{O}(loc(v)) \tag{4.5}$$

where $loc(v)$ is the local set at v given in Table 2, and for (4) we use the gluing together of flags; see appendix.

Notice that in each of the examples the underlying objects are graphs of some sort. These form a naïve category of graphs, by allowing isomorphisms and contractions of edges, with the respective change of data. For modular operads for instance, when contracting a loop edge, one also has to increase the genus by one.

Proposition 4.3. *[MSS, Mar, MMS, Va, GeK2] The types of operads listed in Table 2 are precisely algebras over the respective triple defined above.*

We will make this explicit for modular operads. Here the graphs are stable S–labeled graphs, which means that they are arbitrary graphs together with a labeling by S of the tails and a genus function g from the vertices of the given graph Γ to $\mathbb{N}$, such that $2g(\Gamma) - 2 - |S| > 2$ where $g(\Gamma) = \sum_{\text{vertices } v} g(v) + dimH^1(\Gamma)$ is the total genus of the graph. The basic gluings ${}_s\circ_t$ come from trees with one edge where s and t are the flags of the unique edge and the gluings $\circ_{ss'}$ come from the one vertex graph with one loop whose flags are indexed by s and s'.

For various gradings the following formula is useful for an S–labeled Γ

$$\sum_v (|Flags(v)| - 2 + g(v)) = 2g(\Gamma) - 2 + |S| \tag{4.6}$$

4.5. **Twisted modular operads.** The idea is to get new notions of operads by twisting the triple $\mathbb{T}$. In order to do this one alters the definition of F by using $\mathcal{O}(\Gamma) \otimes \mathfrak{D}(\Gamma)$ and then again takes the colimit.

$$\begin{aligned} \mathcal{V}_{\mathfrak{D}}(\Gamma) &= \mathcal{V}(\Gamma) \otimes \mathfrak{D}(\Gamma) \\ \mathbb{T}_{\mathfrak{D}}\mathcal{V}(g,S) &= colim_{\Gamma \in \mathcal{I}so\mathcal{G}raph_{mod}(S)} \mathcal{V} \\ &\simeq \bigoplus_{\Gamma \in \mathcal{I}so\mathcal{G}raph_{mod}(S)} \mathcal{V}(\Gamma) \otimes \mathfrak{D}(\Gamma)/\sim \\ &\simeq \bigoplus_{[\Gamma]} (\mathcal{V}(\Gamma) \otimes \mathfrak{D}(\Gamma))_{Aut(\Gamma)} \end{aligned} \tag{4.7}$$

where here $\mathcal{G}raph_{mod}(S)$ are S–labeled stable graphs with a genus function and the last sum is over isomorphism classes of such graphs. Taking coinvariants with respect to the automorphism group is new, since the automorphism groups of rooted S–labeled trees are trivial.

In order for this to work $\mathfrak{D}$ has to be what is called a hyper–operad in [GeK2]. The relevant problem being that if we do the inverse of contracting edges along subgraphs —so as to build the composition along a graph— we have to know how $\mathfrak{D}$ behaves. So let Γ_1 be a stable graph and Γ_0 a graph obtained from Γ by contracting subtrees Γ_v, where v runs through the vertices of Γ_0 and Γ_v is the preimage of v under the contraction. This is also what is needed to define the transformation $\mathbb{T}_{\mathfrak{D}}\mathbb{T}_{\mathfrak{D}} \to \mathbb{T}_{\mathfrak{D}}$.

The datum of $\mathfrak{D}$ is given by specifying all the $\mathfrak{D}(\Gamma)$ and maps

$$\mathfrak{D}(\Gamma_0) \otimes \bigotimes_{v \text{ vertices of } \mathcal{G}raph} \mathfrak{D}(\Gamma_v) \to \mathfrak{D}(\Gamma_1) \tag{4.8}$$

for each morphism $\Gamma_1 \to \Gamma_0$ which again have to satisfy some natural associativity, see [MSS, GeK2]. One also fixes that $\mathfrak{D}(*_{g,S}) = k$, where $*_{g,S}$ is the graph with one vertex of genus g and S tails. These are necessary to show that the twisted objects are again triples with unit. Notice that there might be no contractions of edges in $\Gamma_1 \to \Gamma_0$. For this subcase we have that $\mathfrak{D}$ is compatible with the $\mathbb{S}_n$ action.

4.5.1. **Compositions in twisted modular operads.** A good way to understand twisted modular operads is as follows. For a modular operad the algebra over a triple picture says that for each S–labeled graph Γ with total genus g there is a unique operation $\circ_\Gamma$ from $\mathcal{O}(\Gamma) \to \mathcal{O}(g, S)$. Now for a twisted modular operad this ceases to be the case. One actually has to specify more information on the graph. One way to phrase this is that $\mathfrak{D}(\Gamma)$ is a vector space of operations for each graph Γ and we get a well defined operation when we specify an element of that vector space. Of course basis elements suffice. To make this precise, we use adjointness of $\otimes$ or in other words the fact that the category is closed monoidal.

Lemma 4.4. *Being an algebra over a $\mathfrak{D}$ twisted triple in a closed monoidal category is equivalent to having equivariant, compatible composition maps*

$$\circ_{ord,\Gamma} : \mathcal{O}(\Gamma) \to \mathcal{O}(g, S) \tag{4.9}$$

for each S–labeled Γ of total genus g and each element $ord \in \mathfrak{D}(\Gamma)$.

Proof. The triple gives compatible compositions maps $\phi : \mathfrak{D}(\Gamma) \otimes \mathcal{O}(\Gamma) \to \mathcal{O}(g, S)$ that is

$$\phi \in Hom(\mathfrak{D}(\Gamma) \otimes \mathcal{O}(\Gamma), \mathcal{O}(g, S)) \simeq Hom(\mathfrak{D}(\Gamma), Hom(\mathcal{O}(\Gamma), \mathcal{O}(g, S)))$$

In other words if $ord \in \mathfrak{D}(\Gamma)$ then we get a composition $\circ_{ord,\Gamma} : \mathcal{O}(\Gamma) \to \mathcal{O}(g, S)$ and the collection of these compositions is equivalent to ϕ. □

4.6. **Standard Twists.** Table 3 lists some of the standard twists and the operads they correspond to.

Here given a (graded) finite dimensional vector space V or an edge e composed of two flags s and t:

$$\mathrm{Det}(V) = \Sigma^{-dim(V)}\Lambda^{dim(V)}V \tag{4.10}$$

$$Or(e) = \Sigma^2 Det(\{s, t\}) = span(\Sigma^2(\Sigma^{-1}(s) \wedge \Sigma^{-1}t)) \tag{4.11}$$

Name	Value on Γ	appears in
Det	$\mathrm{Det}(H_1(\Gamma))$	
$\mathfrak{K}$	$\mathrm{Det}(Edge(\Gamma))$	Feynman transform
$\mathfrak{T}$	$\mathrm{Det}(\bigoplus_{e\in E_\Gamma} Or(e))$	anti-symmetric of degree 1
$\mathfrak{L}$	$\mathrm{Det}(Flag(\Gamma))Det^{-1}(Tail(\Gamma))$	

Table 3. Standard twists for operads

The most important feature about Det is:

$$Det(\bigoplus V_i) = \bigotimes_i Det(V_i) \tag{4.12}$$

Our main interest here are $\mathfrak{K}$–modular operads, which are the correct odd version of operads. They turn up naturally in two situations. The first is as the Feynman transform of a modular operad, (see §7) and the second is on the chain and homology level of modular operads with twist gluing or a degree one gluing; see §8.

The twists $\mathfrak{L}$ and $\mathfrak{T}$ occur for endomorphism operads with (anti)–symmetric bilinear forms. The twist Det is trivial on trees and is the main difference between the tree or oriented case and the higher genus modular case as we will explain below.

4.6.1. **Odd edge interpretation of $\mathfrak{K}$.** The interpretation which explains why $\mathfrak{K}$–modular operads are the odd version of modular operads is that in a $\mathfrak{K}$–modular operad each edge gets weight -1 and so permutations of the edges give rise to signs. Also permuting the vertices of an edge, gives the shifted sign. These are exactly the Gerstenhaber signs as we discuss below §4.7.2.

4.6.2. **Twisted endomorphism operads and algebras over twisted modular operads.** Some of these twists appear when one considers the extension of the operad $\mathcal{E}nd(V)$ to the modular case. Much as in the cyclic case one has to add a non–degenerate form. As an $\mathbb{S}$–module, one sets $\mathcal{E}((g,n)) = V^{\otimes n}$.

The composition is then given by contracting with the form as in the cyclic case. If the form is of degree l and symmetric or anti–symmetric the resulting operad structure is a twisted modular operad where the twists are; (see e.g. [Bar]):

$$\begin{array}{rl} \mathfrak{K}^{\otimes l} & \text{if the form is symmetric of degree } l \\ \mathfrak{K}^{\otimes l-2}\mathfrak{L} & \text{if the form is anti-symmetric of degree } l \end{array} \tag{4.13}$$

These operads are then the natural receptacle in the formulation of an algebra over an operad. That is an algebra over a $\mathfrak{K}^{\otimes l}$ or $\mathfrak{K}^{\otimes l-2}\mathfrak{L}$ twisted modular operad $\mathcal{O}$ is a map of twisted modular operads from $\mathcal{O}$ to $\mathcal{E}(V)$ (of degree 0), where V is a vector space with a non degenerate symmetric respectively anti–symmetric form of degree l.

In the cyclic operad case one uses the isomorphism $Hom(V^{\otimes n}, V) \simeq V^{\otimes n+1}$.

Name	Value on $*_{((g,n))}$	appears in
$\mathfrak{s}$	$\Sigma^{-2(g-1)-n}sgn_n$	operadic suspension
$\tilde{\mathfrak{s}}$	$\Sigma^{-n}sgn_n$	shifts of $\mathcal{E}$
Σ	Σk	naïve shift

Table 4. List of coboundary twists and their natural habitats Here n refers to the standard notation $\mathcal{O}((n)) = \mathcal{O}(n-1)$ with $\mathbb{S}_{(n-1)+} \simeq \mathbb{S}_n$ action in the cyclic/modular case.

4.6.3. **Coboundaries.** A special type of twist is given by a functor from the one vertex graphs to invertible elements in the target category. In the main application, this means a one–dimensional vector space in some degree. That is a collection of $\mathfrak{l}(*_v)$ for each possible vertex type functorial under automorphisms; in the modular case the vertex types are given by (g, S) and the automorphisms are $\mathbb{S}_{|S|}$.

If Γ has total genus g and tails S, then

$$\mathfrak{D}_{\mathfrak{l}}(\Gamma) = \mathfrak{l}(g, S) \otimes \bigotimes_{v\in\Gamma} \mathfrak{l}((g(v), Flag(v))^{-1}$$

The most common coboundaries are listed in Table 4.

These coboundaries behave nicely with respect to conjugation: if $\mathfrak{l}$ is the functor of tensoring with $\mathfrak{l}$ then

$$\mathfrak{l} \circ \mathbb{T}_{\mathfrak{D}} \circ \mathfrak{l}^{-1} \simeq \mathbb{T}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}} \tag{4.14}$$

This equation also proves

Proposition 4.5. *The categories of algebras over the triple $\mathbb{T}_{\mathfrak{D}}$ and algebras over the triple $\mathbb{T}_{\mathfrak{D}\mathfrak{D}_{\mathfrak{l}}}$ are equivalent, with the equivalence given by tensoring with $\mathfrak{l}$.* □

This is the underlying reason for the form of our definition of odd operads and PROP(erad)s; see §4.7.2.

Remark 4.6. It is important to notice that although $\mathfrak{l}$ determines $\mathfrak{D}_{\mathfrak{l}}$, it can happen that different $\mathfrak{l}$ give rise to the same twist $\mathfrak{D}$. For instance $\mathfrak{D}_{\mathfrak{s}^2} \simeq \mathbb{1}$ [GeK2].

4.6.4. **Relations.** The standard twists are not independent, but rather they satisfy the relations:

$$\begin{aligned}
\mathfrak{K} &\simeq \mathfrak{T} \otimes \mathfrak{D}_{\mathfrak{s}} && (4.15)\\
Det &\simeq \mathfrak{T} \otimes \mathfrak{D}_{\Sigma}^{-1} \simeq \mathfrak{K} \otimes \mathfrak{D}_s{}^{-1} \otimes \mathfrak{D}_{\Sigma}^{-1} && (4.16)\\
\mathfrak{D}_s &\simeq \mathfrak{L}^{-1} \otimes \mathfrak{K}^{\otimes 2} && (4.17)\\
\mathfrak{D}_{\tilde{s}} &\simeq \mathfrak{L}^{-1} && (4.18)\\
& && (4.19)
\end{aligned}$$

Remark 4.7. Notice that *in case the graph Γ is a tree*, we see that Det is trivial and hence $\mathfrak{K} \simeq \mathfrak{D}_s \otimes \mathfrak{D}_{\Sigma}$ and $\mathfrak{T} \simeq \mathfrak{D}_{\Sigma}$.

Remark 4.8. We also see that $\mathfrak{K}^{\otimes 2} = \mathfrak{D}_{\mathfrak{s}}\mathfrak{D}_{\tilde{\mathfrak{s}}}^{-1}$ and hence twists by $\mathfrak{K}$ and $\mathfrak{K}^{-1}$ are equivalent.

Also $\mathfrak{K}^{\otimes 2}(\Gamma) = \Sigma^{-2|E(\Gamma)|}$. This also means that if we are only looking at the $\mathbb{Z}/2\mathbb{Z}$ degree then $\mathfrak{K} = \mathfrak{K}^{-1}$

4.6.5. **Suspension: Shifting V.** If V has a symmetric/anti-symmetric form B of deg l then $V[1]$ carries a anti-symmetric/symmetric induced form $\tilde{B}$ of degree $l-2$ where $\tilde{B}(x[1], y[1]) = (-1)^{|x|}B(x, y)$.

As a twisted modular operad $\mathcal{E}(V[1])((g,n)) := V[1]^{\otimes n}$ is additionally $\tilde{s}$ shifted.

We wish to point out that the difference between $\mathcal{E}$ and $\mathcal{E}nd$ gives different answers to what suspension is natural. Before we had s as the usual suspension; now it is $\tilde{\mathfrak{s}}$. Likewise, the operadic suspension $\mathfrak{s}$ is actually s^{-1} in the case of a cyclic operad. The same for operads and PROP(erad)s. All these are natural, depending on the definition of the endomorphism operad, and it is a matter of choice which ones to use, see in particular Proposition 4.6 and Remark 4.8

4.6.6. **Tensor products.**

Lemma 4.9. *If $\mathcal{O}$ is a $\mathfrak{D}$ twisted modular operad and $\mathcal{O}'$ is a $\mathfrak{D}'$ twisted modular operad then $(\mathcal{O}\otimes\mathcal{O}')((g,n)) := \mathcal{O}((g,n))\otimes\mathcal{O}'((g,n))$ is a $\mathfrak{D}\mathfrak{D}'$ twisted modular operad.* □

4.7. **Generalization of twists.** The theory of twisted triples works equally well for the other triples in Table 2. In all these cases one has to specify the following things. First, what the category of graphs is. This is given by contractions of edges and in the non–connected case also by so called mergers, where two vertices are fused together keeping all inputs and outputs; see Appendix. Furthermore one has to specify a vertex type $*_\Gamma$ for each graph, such that the component $[\Gamma]$ of the morphism $\mathbb{T}\mathcal{O} \to \mathcal{O}$ yields $\circ_\Gamma : \mathcal{O}(\Gamma) \to \mathcal{O}(*_\Gamma)$. Equivalently the morphism $\mathbb{T}\mathbb{T} \to \mathbb{T}$ expands a vertex $*_\Gamma$ to all graphs with that vertex type.1 In all the cases there is a canonical choice given by the result of a total contraction of all edges followed by a total merger [BoM].

Again as in Lemma 4.9, tensoring together twisted versions tensors the twists.

4.7.1. **Odd and anti– as coboundaries.** Notice that the twists $\mathfrak{K}$ always make sense and $\mathfrak{s}$ for the cyclic situation. If we restrict $\mathfrak{K}$ to trees, we find that the twist by $\mathfrak{K}$ is precisely the twist by $\mathfrak{D}_\Sigma\mathfrak{D}_s$. But the shift Σs was exactly what we associated to the grading of the Hochschild complex. Hence with hindsight, we could have worked with $\mathfrak{K}$ twisted operads and $\mathfrak{K}$ twisted cyclic operads.

More precisely we have the list of operad–like types given in Table 5, which could equivalently be defined as algebras over twisted triples.

For (cyclic) operads, we have already clandestinely encountered these twists. Namely, the odd (cyclic) operads are nothing but algebras of the triple of rooted trees, (respectively trees), twisted by $\mathfrak{D}_\Sigma$. And one can check that indeed anti–cyclic operads are equivalent to algebras over the triple twisted by $\mathfrak{D}_s$ which by the previous considerations agrees with the twist by $\mathfrak{L}^{-1}\otimes\mathfrak{K}^{\otimes 2}$. See Lemma 4.10 for the proofs.

Type	defining twist	value (of $\mathfrak{l}$ if coboundary)	on	isomorphic twist
odd operad	$\mathfrak{D}_\Sigma$	Σk	$*_n$	$\mathfrak{D}_{\Sigma s} \simeq \mathfrak{K}$
operad	$1\!\!1$	k	$*_n$	$\mathfrak{K}\mathfrak{D}_\Sigma^{-1}$
odd cyclic operads	$\mathfrak{D}_{\Sigma s}$	$\Sigma^{n-1} sgn_n$	$*_{((n))}$	$\mathfrak{K}$
anti–cyclic operads	$\mathfrak{D}_s = \mathfrak{D}_{\mathfrak{s}^{-1}}$	$\Sigma^{n-2} sgn_n$	$*_{((n))}$	$\mathfrak{D}_\mathfrak{s} \simeq \mathfrak{K}\mathfrak{D}_\Sigma^{-1}$
odd PROP(erad)	$\mathfrak{D}_{s_{out}}$	$\Sigma^m sgn_m$	$*_{n,m}$	$\mathfrak{K}$
PROP(erad)	$1\!\!1$	k	$*_{n,m}$	$\mathfrak{K}\mathfrak{D}_{\Sigma_{out}}^{-1}$
$\mathfrak{K}$–modular	$\mathfrak{K}$	$Det(Edge)$	Γ	$\mathfrak{K}$
anti–modular	$Det\ \mathfrak{D}_\mathfrak{s}$	$\Sigma^{-1} Det(Edge)$	Γ	$\mathfrak{K}\mathfrak{D}_\Sigma^{-1}$

Table 5. Types of operads defined by certain twisted triples via Proposition 4.5

Lemma 4.10. *We have the following isomorphisms: For operads $\mathfrak{D}_s \simeq 1\!\!1$ and all the isomorphisms listed in Table 5*

Proof. D_s is concentrated in degree 0 and the $\mathbb{S}_n$ action is trivial. Indeed for an n–tree the shift is $n-1+\sum_v(1-ar(v)) = n-1+|V|-|E_{int}|+n = 0$.

For $D_{\Sigma s}$ the value on an S labeled rooted tree is $\mathfrak{D}_{\Sigma s}(T) = Det^{-1}(S) \otimes \bigotimes_v Det(In(v)) \simeq Det(Edge) = \mathfrak{K}(T)$.

For the cyclic operad case, we have $\mathfrak{K} \simeq \mathfrak{D}_\mathfrak{s}\mathfrak{D}_\Sigma$ by Remark 4.7

Finally, for the PROP(erad)s for Γ of type (n,m) that is n inputs and m outputs $D_{s_{out}}(\Gamma) = Det^{-1}(Tail_{out}(\Gamma)) \otimes \bigoplus_v Det(Flag_{out}(v)) \simeq Det(Edge) \simeq \mathfrak{K}$ where we used that the set of non–tail flags is in bijection with the edges. $\square$

Lemma 4.11. *Notice that for PROP(erads) by an analogous argument $\mathfrak{D}_{s_{out}} \simeq \mathfrak{D}_{s_{in}} \simeq \mathfrak{K}$ so that $\mathfrak{D}_s \simeq \mathfrak{D}_{s_{in}}\mathfrak{D}_{s_{out}}^{-1} \simeq 1\!\!1$. Thus a suspended PROP(erad) is a PROP(erad).* $\square$

Remark 4.12. In [MMS] the following cocycles are also used: $\mathfrak{s} = s^{-1}$, $w = \mathfrak{K}^{-1}\mathfrak{s}$. It seems although stated differently, that in [MMS] they use $\mathfrak{D}_{s_{out}^{-1}} \simeq \mathfrak{K}^{-1}$ to twist, which is equivalent since the categories of the twisted PROP(erad)s are equivalent by Proposition 4.5

4.7.2. **Odd operads and anti–cyclic operads as twisted operads and their relation to $\mathfrak{K}$.** Now we can make the Metatheorem 1 precise by using $\mathfrak{K}$ twisted instead of odd.

Theorem 4.13. *All $\mathfrak{K}$ twisted versions in Table 5 carry a natural odd Lie bracket on the direct sum of their coinvariants. Their Σ_{out}^{-1} shifts accordingly carry a Lie bracket.*

Proof. The first statement is just a rephrasing of our previous results, using Proposition 4.5 and Lemma 4.10 except for the case of $\mathfrak{K}$–modular operads which for the bracket reduces to the case of odd cyclic, since the gluing is only along trees. $\square$

Notice that we included anti–modular in the list. This is the natural candidate to carry the even bracket and we see that this is as twisted as the $\mathfrak{K}$–modular operad. The main point is that the cocycle Det is not a coboundary in the modular version.

5. Odd self–gluing and the BV differential

In this paragraph, we deal with Metatheorem 2. For this we need odd–self gluings. We have already treated odd wheeled PROP(erads). We now turn to $\mathfrak{K}$–modular operads.

The most important fact that we need is that $\mathfrak{K}$–modular operads have an odd self–gluing structure that is the operations $\bullet_{ss'} : \mathcal{O}(S) \to \mathcal{O}(S \setminus \{s, s'\})$ such that for four element subsets $\{s, s', t, t'\} \subset S$ and $a \in \mathcal{O}(S)$

$$\bullet_{ss'} \bullet_{tt'}(a) = - \bullet_{tt'} \bullet_{ss'}(a) \in \mathcal{O}(S \setminus \{s, s', t, t'\}) \tag{5.1}$$

Using the language of graphs, the two different operations correspond to a graph with one vertex flags indexed by S and two pairs of flags $\{s, s'\}$ and $\{t, t'\}$ joined together as edges e_1 and e_2, the two compositions however correspond to $\circ_{e_1 \wedge e_2, \Gamma}$ and $\circ_{e_2 \wedge e_1, \Gamma}$ in the notation of Lemma 4.4, which differ by a minus sign.

Proposition 5.1. *The operator Δ defined on each $\mathcal{O}(g, S)$ defined by*

$$\Delta(a) = \sum_{\{s,s'\} \in S, s \neq s'} \bullet_{ss'}(a) \in \bigoplus_{\{s,s'\} \in S, s \neq s', g+1} \mathcal{O}(S \setminus \{s, s'\}) \tag{5.2}$$

satisfies $\Delta^2 a = 0$ for any $a \in \mathcal{O}(g, S)$.

Proof. We consider the component $S \setminus \{s, s', t, t'\}$ for fixed s, s', t, t'. It will get six contributions which appear pairwise. Each pair corresponds to an ordered partition $\{a, b\} \amalg \{c, d\}$ of $\{s, s, t, t'\}$ and the two terms appear with opposite sign. These are the compositions for the $S \setminus \{s, s', t, t'\}$–labeled graph with one vertex and two edges in both orders of the two edges. □

Remark 5.2. Here we chose to index by two element subsets of S. If we index by tuples (s, s') and we are in characteristic different from two then we obtain the more familiar form:

$$\Delta(a) = \frac{1}{2} \sum_{(s,s') \in S, s \neq s'} \bullet_{ss'}(a) \in \bigoplus_{\{s,s'\} \in S \times S, s \neq s'} \mathcal{O}(S \setminus \{s, s'\})$$

Passing to coinvariants, we obtain an instance of Metatheorem 2

Proposition 5.3. Δ *induces a differential on $\mathcal{O}_{\mathbb{S}}^{\oplus}$ that is $\Delta^2 = 0$. This differential lifts to the cyclic invariants and to the biased setting.*

Proof. On $\mathcal{O}_{\mathbb{S}}^{\oplus}$ the equality follows directly from (5.1). For the lifts, we remark that $\{0, \dots \hat{i}, \dots \hat{j}, \dots n\}$ has a natural cyclic and linear order. □

Remark 5.4. In the biased setting as shown in [SZ, Schw] it is sufficient to lift Δ to $\bullet_{n-1n}$ on $\mathcal{O}(n)$. The compatibility follows from the standard sequence (2.11).

Now we have Metatheorem 2 in the form:

Theorem 5.5. *The $\mathfrak{K}$ twisted version of modular operads, wheeled PROP(erad)s and the chain level Schwarz EMOs carry a differential Δ on their coinvariants.*

Where the EMOs are discussed in 8.1.3.

6. Multiplication, Gerstenhaber and BV

So far for cyclic and modular operads, we have only constructed (odd) Lie brackets and differentials. In order to upgrade them to Gerstenhaber respectively Poisson algebras and BV operators, we need an additional multiplicative structure.

Following [SZ, Schw, HVZ] we show that there is a natural external multiplication one can introduce by going to disconnected graphs. It is the external multiplication that is natural to consider in the master equation as that equation is a linearization of an equation involving an exponential.

There is a second type of multiplicative structure that is possible. This is an internal product; that is an element $\mu \in \mathcal{O}(2)$ which is associative. Although a little bit outside the main focus of the paper, we deal with the second type of multiplication in order to contrast it with the one above. This second type of structure appears in Deligne's conjecture [KS, McCS, V, BF, T, K2], its cyclic generalization [K3]. A last possibility is an A_∞ version which was studied in [TZ, KSch, Wa, K6], but that goes beyond the scope of this paper.

6.1. **Non–connected versions.** *A priori* an operad of the above kinds has no multiplication. We can however add a generic one, by passing from connected graphs to non–connected ones. The cue to use this type of multiplication comes from [SZ, Schw, HVZ].

6.1.1. **Non–connected (odd) operads.** For operads the notion of the non–connected (nc) generalization exists and is called the PROP generated by the operad. To get the triple one should look at the appropriate subcategory for PROPs. Namely,the non–connected version is given by forests of rooted trees. For an algebra over such a triple we need a new composition $\boxminus : \mathcal{O}(S) \times \mathcal{O}(T) \to \mathcal{O}(S \amalg T)$ called a horizontal composition. This needs to be compatible in the obvious way with the other compositions. In particular we take the category $\mathcal{RF}$ to be the category of rooted forests where now we have the additional morphism coming from the disjoint union $\amalg$. In categorical terms this is nothing but a free monoidal category based on $\mathcal{RT}$, the category of rooted trees. The elements are collections of rooted trees, with the obvious isomorphisms and other morphisms given by combining collections and contracting edges.

In order to achieve the correct odd notion, we again have to twist the triple. The twist is now by $\mathfrak{K}$ as previously. We call an algebra over such a triple a non–connected odd operad. Notice that $\{\ \bullet\ \}$ is well defined as the sum over the non–self gluings.

Theorem 6.1. *Given a non–connected odd operad, the odd Lie bracket* $\{ \bullet \}$ *is Gerstenhaber with respect to* $\boxminus$.

Proof. This just boils down to the fact that before anti–symmetrizing on the left hand side of (A-1), we have a summand corresponding to connecting the inputs/output of a to any element of the set $S \amalg T$ if $b \in \mathcal{O}(S)$ and $c \in \mathcal{O}(T)$ say. The ones connecting the root to S are the first term, while the ones connecting the root to T are the second term of the rhs. □

6.1.2. **NC–cyclic.** For cyclic operads and modular operad the non–connected notions have not appeared in the literature yet — as far as we are aware. The relevant triples are those of forests (collections of trees). We will call the algebras over these triples nc–cyclic operads. Again the relevant morphisms are given by isomorphisms, contracting edges and combining collections. The combining of two one vertex graphs gives a horizontal composition $\boxminus : \mathcal{O}(S) \otimes \mathcal{O}(T) \to \mathcal{O}(S \amalg T)$. The twist by $\mathfrak{K}$ makes sense and we obtain the notion of odd–nc–cyclic operad.

Theorem 6.2. *Given an odd nc–cyclic operad, the odd Lie bracket* $\{ \odot \}$ *is Gerstenhaber with respect to* $\boxminus$ *on the coinvariants* $\mathcal{O}_{\mathbb{S}}^{\oplus}$.

Proof. This just boils down to the fact that on the left hand side of (A-1), we have a summand corresponding to connecting the root of c to any element of the set $S \amalg T$. The ones connecting to S are the first term, while the ones connecting to T are the second term of the rhs. □

6.1.3. **NC–modular operads.** For NC modular operads the basic underlying triple will be non–connected graphs. We must however deal with the genus labeling. Since the graphs are not connected one should replace g by χ where χ is the Euler characteristic. For any graph, its Euler characteristic is given by the Euler characteristic of its realization. Viewing it as a 1–dimensional CW complex and contracting any tails, we get that

$$\chi(\Gamma) = b_0(|\Gamma|) - b_1(|\Gamma|) = |\text{vertices of } G| - |\text{internal edges of } \Gamma|;$$

If Γ is connected then $1 - \chi(\Gamma) = g$.

We replace the genus labeling by the labeling by γ. That is a function $\gamma :$ vertices of $\Gamma \to \mathbb{N}$.

The total γ is now

$$\gamma(\Gamma) = 1 - \chi(\Gamma) + \sum_{v \text{ vertex of } \Gamma} \gamma(v)$$

This means we get non–self gluing ${}_s\circ_t$ for which γ is again additive in γ and self–gluings $\circ_{ss'}$ increase γ by one. There is also the collecting together which gives a horizontal map $\boxminus : \mathcal{O}(\gamma, S) \otimes \mathcal{O}(\gamma', T) \to \mathcal{O}(\gamma + \gamma', S \amalg T)$.

The triple is now given as usual. Just as in the modular case, the multiplication in the triple expands the vertices into graphs of the corresponding type $(Flags(v), \gamma(v))$.

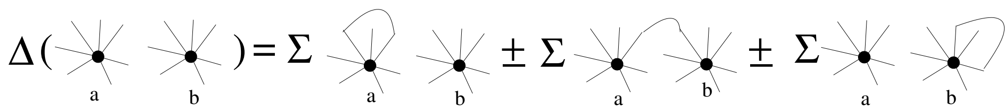

FIGURE 2. The tree terms for checking the BV property

The twist by $\mathfrak{K}$ makes sense and we obtain the notion of an nc–$\mathfrak{K}$–modular operad.

Again Δ is well defined as the sum over all self–gluings.

Theorem 6.3. *For an nc–$\mathfrak{K}$–modular operad $\mathcal{O}$, the sum over non–self gluings gives an odd Lie bracket $\{\ \odot\ \}$ on the coinvariants (both cyclic and full) which is Gerstenhaber for the horizontal multiplication on $\mathcal{O}^{\oplus}_{\mathbb{S}}$.*

The differential Δ given by summing over the self–gluings is a BV operator for the horizontal multiplication on $\mathcal{O}^{\oplus}_{\mathbb{S}}$ and its Gerstenhaber bracket is the bracket induced by $\{\ \odot\ \}$.

Proof. The proof can either be done by direct calculation or by the following argument which is essentially an adaption of that of [HVZ]. If we look at the equation (A-2) then taking $\Delta(a \boxminus b)$ decomposes into three terms. All self–gluings of a, all self–gluings of b and all non–self gluings between a and b, which if one is careful with the signs give all the gluings. A pictorial representation is given in Figure 2. Again one has to be careful that one uses coinvariants, which is where $\{\ \odot\ \}$ satisfies the Jacobi identity. □

6.1.4. **NC–extension.** Just like there is the PROP generated by an operad, a cyclic or (twisted) modular operad generates an nc version.

Here the operation $\boxminus$ is just taken to be $\otimes$ and one sets

$$\mathcal{O}^{nc}((\gamma,n)) = \bigoplus_k \bigotimes_{\substack{(n_1,\dots,n_k):\sum_i n_i = n \\ (g_1,\dots,g_k):\sum 1-g_i=\gamma}} \mathcal{O}((n_i,g_i))$$

6.2. **$\mathfrak{K}$ –twisted Realization of the Mantra.** We can now formulate Metatheorem 2 in this context.

Theorem 6.4. *For the nc–versions of odd cyclic operads and $\mathfrak{K}$ twisted modular operads as well as for $\mathfrak{K}$–twisted version of wheeled PROPs the operator Δ is a BV operator on the coinvariants which induces the previously constructed Gerstenhaber bracket.*

6.3. **Operads with multiplication.** Let $\mu \in \mathcal{O}(2)$, s.t. $\mu \circ_1 \mu = \mu \circ_2 \mu$. An operad together with such an element is called an operad with multiplication. Indeed on $\mathcal{O}^{\oplus}$, μ defines a graded associative multiplication via $a \otimes b \mapsto (\mu \circ_2 b) \circ_1 a$.

Such an element also gives rise to a differential $da := \{a \bullet \mu\}$

Theorem 6.5. *[G] For an operad with multiplication the odd bracket $\{\bullet\}$ is odd Poisson, aka Gerstenhaber up to homotopy, that is the equations hold up to $im(d)$.*

In the cyclic situation for an operad with a unit $1 \in \mathcal{O}(0)$ for μ, one can define degeneracy maps via $s_i(a) := a \circ_i 1$

Then one can define the operator $B = s(1-t)N$ on the complex $\mathcal{O}^{\oplus}$ with the differential d (or the sum of the internal differential and d). On the reduced complex is just sN. The calculation in [K3] shows that

Theorem 6.6. *For a cyclic operad, B is a differential on the reduced complex and descends to a BV operator for μ on the cohomology. Moreover the induced bracket agrees with the one coming from the Gerstenhaber structure.*

This type of BV operator is internal and has *a priori* nothing to do with the external Δ we considered above. They also yield different Gerstenhaber brackets, namely $\{\bullet\}$ and $\{\odot\}$.

Thus taking coinvariants, they are related *a posteriori.* Moreover if μ is cyclic, then the gluing can be thought of as composing both elements with μ and putting in a co–unit. The precise relationship and interplay between the two BV formalisms is an interesting open problem.

7. (Co)bar constructions and Feynman transform and the master equation

For an algebra there is a standard resolution given by the co–bar bar construction. Here one takes a (dg) algebra and makes it into a dg–coalgebra and then into a dg–algebra which is a free resolution for the first algebra. For operads there is a similar story, although one usually takes duals so that the operation goes operad to dg–operad to dg–operad. In the first step the operad is actually free as an operad, but not as a dg–operad. This is sometimes called quasi–free. The difference is exactly measured by the master equation in its various guises. For modular operads this was first proved in [Bar].

Since in the end one always ends up with a differential on the structure one should allow to start with one. This is why in this section we work in the category $dg\mathcal{V}ect$.

7.1. **(Co)bar construction aka Feynman transform.** The underlying operad of the dualizing complex $D(\mathcal{O})$ of a (cyclic) operad $\mathcal{O}$ is $F\mathfrak{D}_{\Sigma}\mathfrak{D}_s(G\mathcal{O})^*$ [GiK, MSS] that is the free operad on the $\Sigma \otimes s$ twisted $\mathbb{S}$–module which is the dual of the underlying $\mathbb{S}$–module $(G\mathcal{O})^*(S) = Hom(\mathcal{O}(S), k)$ of $\mathcal{O}$.[3] By our previous results on twists we could regard this operad not as an operad, but as a free $\mathfrak{K}$ twisted or odd operad.

This is exactly the way one proceeds in the case of modular operads. For a modular operad the underlying $\mathfrak{K}$–modular operad of the Feynman transform is $\mathcal{F}_{\mathfrak{K}}(G\mathcal{O})^*$ where $\mathcal{F}_{\mathfrak{K}}$ is the free functor for $\mathfrak{K}$–modular operads and is called the

[3]As before F and G are the free and forgetful functors.

Feynman transform. The underlying cyclic operad is $Cyl(\mathcal{F}\mathcal{O}) = \Sigma\mathfrak{s}BCyl\mathcal{O} = D(Cyl(\mathcal{O}))^4$.

We will now consider only the modular version. The cyclic case is just a subcase and the original operad case is simply given by an analogous construction. All of them can be found in [MSS].

More generally $\mathcal{F}_{\mathfrak{D}}\mathcal{O}$ of a $\mathfrak{D}$ modular operad, where $\mathfrak{D}$ is invertible, is the free $\check{\mathfrak{D}} = \mathfrak{K} \otimes \mathfrak{D}^{-1}$ modular operad in the dual of the underlying $\mathbb{S}$–module.

The fact that we took duals gives a differential in all cases this is dual to the composition given by contracting an edge. This is actually only the external part of the differential. The total differential on $\mathcal{F}_{\mathfrak{D}}O$ is the sum $d\mathcal{F} = \partial_{\mathcal{O}^*} + \partial$ where $\partial_{\mathcal{O}^*}$ is the internal differential induced from the differential on the $\mathcal{O}(S)$ by dualizing and taking tensors, and ∂ is a new external differential whose value on the term $(\mathfrak{D}^{-1}(\Gamma) \otimes \mathfrak{K}(\Gamma) \otimes \mathcal{O}^*(\Gamma))_{Aut(\Gamma)}$ is given as follows. Consider $\hat{\Gamma}$ together with an edge e such that $\hat{\Gamma}/e \simeq \mathcal{G}raph$. Then there is a map $\circ_e : \mathfrak{D}(\hat{\Gamma}) \otimes \mathcal{O}(\hat{\Gamma}) \to \mathfrak{D}(\Gamma) \otimes \mathcal{O}(\Gamma)$ which composes along e. Since $\mathcal{O}$ is an algebra over $\mathbb{T}$ for such a pair there is a map

$$\partial_{\hat{G},e} : \mathfrak{K}(\Gamma) \otimes \mathfrak{D}^{-1}(\Gamma) \otimes \mathcal{O}^*(\Gamma) \xrightarrow{\epsilon_e \otimes \circ_e^*} \mathfrak{K}(\hat{\Gamma}) \otimes \mathfrak{D}^{-1}(\hat{\Gamma}) \otimes \mathcal{O}^*(\hat{\Gamma}) \tag{7.1}$$

where ϵ_e is the multiplication by the basis element $[e]$ of $Det(\{e\})$. Now the matrix element ∂ between $(\mathfrak{D}^{-1}(\hat{\Gamma}) \otimes \mathfrak{K}(\hat{\Gamma}) \otimes \mathcal{O}^*(\hat{\Gamma}))_{Aut(\hat{\Gamma})}$ and $(\mathfrak{D}^{-1}(\Gamma) \otimes \mathfrak{K}(\Gamma) \otimes \mathcal{O}^*(\Gamma))_{Aut(\Gamma)}$ is the sum over all $\partial_{\hat{\mathcal{G}raph},e}$ for which $\hat{\mathcal{G}raph}/e \simeq \mathcal{G}raph$. If there is no such edge, then the matrix element is zero.

The reason to introduce the twist by $\mathfrak{K}$ into the picture is to make ∂ into a differential. Indeed applying it twice inserts two edges in all possible ways and each term appears twice: once with each possible ordering of the two edges. Due to the presence of the tensor factor $Det(Edges)$ these terms differ by a minus sign and cancel.

Remark 7.1. The Feynman transform actually does give a resolution if applied twice; see [GeK2]. We will not need this important fact however.

7.2. **Algebras over the bar construction, the Feynman transform and the Master Equation.** One has to distinguish: As an operad the transform is free as a dg operad it is not.

Since $\mathcal{F}(\mathcal{O})$ is free and the free functor is adjoint to the forgetful functor, a mere algebra V over $F(\mathcal{O})$, ignoring the dg–structure, is fixed by the maps of $\mathbb{S}$–modules $\mathcal{O}^*((g,n)) \to \mathcal{E}nd(V)((g,n))$. This means that the algebra structure is given by $\mathbb{S}_n$–equivariant maps $\hat{m}_{g,n}$

$$\hat{m}_{n,g} \in Hom_{\mathbb{S}_n}(\mathcal{O}^*((g,n)), \mathcal{E}(V)((g,n))) \simeq (V^{\otimes n} \otimes \mathcal{O}((g,n)))^{\mathbb{S}_n} \ni m_{g,n}$$

or isomorphically $\mathbb{S}_n$ invariant elements $m_{g,n}$ where the action is the diagonal one. Summing up these elements to a formal series

$$S := \sum_{g,n} m_{g,n} \tag{7.2}$$

[4] see Remark 7.6 below

it determines the structure of an algebra over the operad $\mathcal{F}_{\mathfrak{D}}\mathcal{O}$, ignoring the dg–structures. Also since the morphisms are degree preserving, S has degree 0 and vice–versa any such degree 0 series gives rise to a morphism. Now $\mathcal{O}((g,n)) \otimes V^{\otimes n} = \mathcal{O} \otimes \mathcal{E}(V)((g,n))$ is a $\mathfrak{K}$ twisted operad, since $\mathcal{E}(V)$ is $\mathfrak{K}$ twisted by definition. Thus its co–invariants carry $\{\ \odot\ \}$ and the operator Δ using the standard isomorphism between invariants and co-invariants.

Theorem 7.2. *[Bar] The series S defines a dg–algebra over $\mathcal{F}_{\mathfrak{D}}\mathcal{O}$ if and only S satisfies the quantum master equation*

$$dS + \Delta S + \frac{1}{2}\{S \odot S\} = 0. \tag{7.3}$$

where $dm = ((-1)^{|m|+1} d_{\mathcal{F}} + d_V)m$, the sum of the differential on the Feynman transform, suitably dualized and the internal differential of V. This gives a bijective correspondence between degree 0 solutions to the quantum master equation and dg–algebra structures of V over $\mathcal{F}_{\mathfrak{D}}\mathcal{O}$.

The theorem is basically an unraveling of definitions. The fact that the two terms Δ and $\{\ \odot\ \}$ appear is because the differential is the sum over inserting edges. Namely, each such edge corresponds to a self or a non–self gluing.

Remark 7.3. If one wishes, one can keep track of the genus, since $\{\ \odot\ \}$ leaves it invariant and Δ increases it by one, for $S(\lambda) := \sum_{g,n} \lambda^g m_{g,n}$ we get

$$dS + \lambda\Delta S + \frac{1}{2}\{S \odot S\} = 0. \tag{7.4}$$

7.2.1. **NC–generalization.** In the nc extension of the above situation the solutions to the master equations are also exactly the solutions of

$$(d + \lambda\Delta S)e^S = 0 \tag{7.5}$$

Here the exponential is formal for the product given by $\boxminus$. This is in accordance with quantum field theory, where the exponential gives the sum over all not–necessarily connected Feynman graphs.

7.2.2. **Extension to other targets.** There was actually nothing special about the target operad $\mathcal{E}$ we used, except that the tensor product of the original operad and the target was $\mathfrak{K}$–modular. That is fix $\mathcal{O}$ and $\mathcal{P}$ to be two twisted operads in $dg\mathcal{V}ect$, such that $\mathcal{O} \otimes \mathcal{P}$ is $\mathfrak{K}$–modular.

Theorem 7.4. *The dg morphisms $Hom_{dg}(\mathcal{F}\mathcal{O}, \mathcal{P})$, $\mathcal{P}$ considered with its internal differential, are given by solutions S of (7.3) for $S \in (\mathcal{O} \otimes \mathcal{P})^{\mathbb{S}}_{\oplus}$, of degree 0.*

7.3. **(Cyclic) operad version.** For the case of operads and cyclic operads we have the analogous statements. Here $S \in \bigoplus_n (\mathcal{O}(n) \otimes V^{\otimes n+1})^{\mathbb{S}_n}$ or respectively $S \in \bigoplus_n (\mathcal{O}((n)) \otimes \mathcal{E}nd_V(n))^{\mathbb{S}_n}$

Theorem 7.5. *An algebra over the (cyclic) operad $D(\mathcal{O})$ given by S (of degree 0) is a dg algebra if and only if it S satisfies:*

$$dS + \frac{1}{2}\{S \bullet S\} = 0 \tag{7.6}$$

$$dS + \frac{1}{2}\{S \odot S\} = 0 \tag{7.7}$$

□

Notice that $\mathcal{O}((n)) \otimes V^{\otimes n}$ is again a cyclic operad namely just the operad product of $\mathcal{O}$ and $\mathcal{E}nd(v)$ and as such there its direct sum is a Lie algebra. In other words, the possible algebra structures are in 1–1 correspondence with Maurer–Cartan elements in that Lie algebra. The analogous statement holds true for operads.

Remark 7.6. Strictly speaking the original definition of the dualizing complex of an operad $D(\mathcal{O})$ yields that $D(\mathcal{O}) = F\mathfrak{s}\Sigma G\mathcal{O}$ [MMS], where as before G and F are the forgetful and free functors. This is up to the final twist $(\mathfrak{s}\Sigma)^{-1}$ a $\mathfrak{K}^{-1}$ twisted operad. But since $\mathfrak{K}^{\otimes 2}(T) = \Sigma^{-2|E(T)|} = \Sigma^{2|V(T)|-2} = \mathfrak{D}_{\Sigma^{-2}}(T)$ we see that these structures only differ by a twist on trees and hence all categories of triples are equivalent.

7.4. **Feynman transform in the (wheeled)PROP(erad) case.** Here we deal with the other structures, we have encountered. Although the paper [MMS] is very thorough it seems to have missed Theorem 7.9, which we now furnish. In [MMS] with the use of the cocycle w^{-1} a (co)–bar construction was given. Dually we give the Feynman transform here. This is what allows us to put the result on BV algebras in [MSS][Theorem 3.4.3] into a broader framework.

As for modular operads the Feynman transform for invertible twists $\mathfrak{D}$ produces a $\mathfrak{K}\mathfrak{D}^{-1}$ twisted PROP(erad). In particular the Feynman transform of a wheeled PROP(erad) is a $\mathfrak{K}$-wheeled PROP(erad). In general, just like for modular operads if $\mathfrak{D}$ is an invertible twist, then the Feynman transform $\mathcal{F}_{\mathfrak{D}}$ turns $\mathfrak{D}$ twisted PROP(erads) into $\mathfrak{K}\mathfrak{D}^{-1}$ twisted PROP(erads).

Definition 7.7. Given a $\mathfrak{D}$ twisted (wheeled) PROP(erad) $\mathcal{O}$, we let $\mathcal{F}_{\mathfrak{D}}(\mathcal{O})$ be the free $\mathfrak{K}\mathfrak{D}^{-1}$ twisted (wheeled) PROP(erad) with the differential that is the sum of the differential induced by the one on $\mathcal{O}$ and the external differential which is defined by "insertion of all possible edges of weight -1" which is made precise by matrix elements as defined in §7.

Remark 7.8. There are a several versions of the Feynman transform for PROP-(erad)s. The first —and this is the one we use here— is essentially the dioperadic resolution [Ga]. This corresponds to the bracket we have introduced.

The second version is more complicated uses the resolution of [Va] and an accordingly changed bracket. We defer that discussion to [Fey], where we introduce transforms depending on a fixed set of generators.

For the quantum master equation, we never want to resolve the horizontal composition. This operation yields the multiplication for the Gerstenhaber/BV structure and is inherent in the definition of e^S which is the physically relevant exponentiated action [SZ, ASZK]; see e.g. equation (7.5).

Theorem 7.9. *Let $\mathcal{O}$ be a (wheeled) PROP(erad) and let $\mathcal{P}$ be a $\mathfrak{K}$- twisted (wheeled) PROP(erad). Then there are one to one correspondences*

$$Hom_{dg}(\mathcal{F}(\mathcal{O}), \mathcal{P}) \overset{1-1}{\leftrightarrow} (Q)ME((\mathcal{O} \otimes \mathcal{P})^{\mathbb{S}}_{\oplus})$$

Where for the non–wheeled case ME are the degree 0 solutions to the master equation (7.6), and for the wheeled case QME is the is the set of degree 0 solutions to the quantum master equation (7.3).

More generally using $\mathcal{F}_{\mathfrak{D}}$ the same holds true for a pair of a $\mathfrak{D}$ twisted $\mathcal{O}$ and $\mathfrak{K}\mathfrak{D}^{-1}$ twisted $\mathcal{P}$. In particular this works for $\mathcal{O}$ being $\mathfrak{D} = \mathfrak{K}$–twisted and $\mathcal{P}$ being untwisted.

Proof. The proof is completely analogous to the modular case. Since the $\mathcal{F}(\mathcal{O})$ is free when forgetting the differential, we get the series S. Now looking at the differential part, we have the two internal differentials and the external differential packed into d. Without the external one the equation for a dg–morphism would just be $dS = 0$. The external part, suitably dualized, just adds edges. If these are self–gluings they appear in the term with Δ if not they appear in the term with $\{ \bullet \}$. $\square$

It should be noted that one important example of this theorem has been proved in [MMS]. In particular they define a wheeled PROP $PolyV^{\circlearrowright} := F\mathfrak{K}^{-1}(Com) = \mathfrak{K}^{-1}\mathcal{F}(Com)$ where Com is the $\mathbb{S}$ bimodule with the trivial representation in each bi–degree, and also its obvious extension to a wheeled PROP. Here $\mathfrak{K} = \mathfrak{s}_{out}$ is used in its form as a coboundary and F is the free PROP. In our language this is equivalent to being in the image of the Feynman transform. The differential defined in [MMS] in this interpretation is exactly the one induced by the Feynman transform.

Using the theorem above one can recover.

Proposition 7.10 ([MMS])**.** *The $PolyV^{\circlearrowright}$ algebra structures on a finite dimensional complex (M_0, d_0) are in bijective correspondence with the symmetric ME solutions in the dgBV algebra $\wedge^{\bullet}T_M$, where $M = \Pi\Omega^1 M_0$ (Π is the parity reversal see e.g. [MMS] or [ASZK]). The first fact is immediate from the definitions.*

Proof. Since $PolyV^{\circlearrowright} = \mathfrak{K}^{-1}\mathcal{F}(Com)$, we see that its morphisms to $\mathcal{E}nd_M = \mathfrak{K}\mathfrak{K}^{-1}\mathcal{E}nd_M$ are equivalent to those of $\mathcal{F}(Com) \to \mathfrak{K}^{-1}\mathcal{E}nd(M_0))$. By the theorem above These are given by solutions in $(Com \otimes End(M_0))^{\mathbb{S}}_{\oplus} \cong (\mathfrak{K}^{-1}End_M)^{\mathbb{S}}_{\oplus} = (\wedge^{\bullet}T_M)^{\mathbb{S}}$. $\square$

Actually, in [MMS] the symmetry of the tensor is implicit in the representation and they work in the category of $\mathbb{Z}/2\mathbb{Z}$–graded spaces which goes through in the same way. Notice that in this case actually $\mathfrak{K} = \mathfrak{K}^{-1}$ and S has to be even.

8. Geometric examples

In this section we give some geometric examples which lead to occurrences of Metatheorem 4. There are basically two kinds: open and closed. These are motivated by the constructions of [HVZ] and [KSV], and ultimately by [SZ]. Informally speaking the common feature of the following closed examples is an S^1–action on the outputs, which can be transferred to a twist gluing. Such a twist gluing will be an S^1 family. Passing to homology or chains this 1–parameter family gives degree 1 to the gluing making the gluing odd.

The other type of gluing is a gluing at boundary punctures. In order for it to be odd one must consider orientations and for it to get degree one, one has to pick a grading by codimension as we explain below. The paradigm for this is contained in [HVZ], but was previously also inherently present in Stasheff's associahedra and more recently in [KSch] for the Gerstenhaber operad.

8.1. **Topological $\mathbb{S}_n \wr S^1$ modular operads.** Suppose we have a topological modular operad $\mathcal{O}$. We also assume that $\mathcal{O}((g,n))$ has an $(S^1)^{\times n}$ action which together with the $\mathbb{S}_n$ action gives an action of $\mathbb{S}_n \wr S^1$. For $\phi \in S^1 = \mathbb{R}/\mathbb{Z}$ let $\rho_i(\phi)a = (0,\dots,0,\phi,0,\dots)(a)$ where the non–zero entry is in the i–th place.

Definition 8.1. A topological S^1–modular operad is a modular operad $\mathcal{O}$ with an $\mathbb{S} \wr S^1$ action that is *balanced* which means that

$$\rho_i(\phi)(a)\,{}_i\circ_j b = a\,{}_i\circ_j \rho_j(-\phi)(b) \text{ and } \circ_i^j\,\rho_i(\phi(a)) = \circ_i^j(\rho_j(-\phi(a)) \tag{8.1}$$

Likewise we define the S^1–twisted versions of (cyclic) (twisted) operads and (wheeled) (twisted) PROP(erads) or also di–operads, etc.

Notation 8.2. To shorten the statements, we will call any $\mathcal{O}$ belonging to any of the categories in the previous sentence of *composition type*.

Definition 8.3. The twist gluing ${}_i\circ_j^{S^1}$ of a and b is the S^1 family given by $\rho_i(S^1)a\,{}_i\circ_j b$

This type of twist gluing does not give a nice operad type structure on the topological level, unless as suggested by Voronov, one uses the category of suitable spaces with correspondences as morphisms. It does however give nice operations on singular chains and hence on homology.

Namely, given two chains $\alpha \in S_k(\mathcal{O}(n))$ and $\beta \in S_l(\mathcal{O}(m))$ we define the chains

$$\alpha\,{}_i\bullet_j\beta := S_*({}_i\circ_j)EZ\,S_*(id \times \rho_j)(\alpha \times \rho_i \times \beta) \tag{8.2}$$

as chains parameterized over $\Delta^k \times \Delta^1 \times \Delta^l$ pushed forward with ρ_j and the Eilenberg Zilber map to give a chain in $S_{k+l+1}(\mathcal{O}(n) \times S^1 \times \mathcal{O}(m))$. Here Δ^1 maps to the fundamental class $[S^1]$. Likewise we define

$$\bullet_{ij}\,\alpha := S_*(\circ_{ij})S_*(\rho_i)([S^1] \times \alpha) \tag{8.3}$$

This type of operation of course generalizes and restricts to all $\mathcal{O}$ of composition type.

Theorem 8.4. *The chain and homology of any S^1–twisted $\mathcal{O}$ of composition type are $\mathfrak{K}$–twisted versions of that type.*

Proof. We see that the compositions are along the graphs of the triple, where the edges are now decorated by the fundamental class of S^1. This lives in degree 1 and hence the compositions get degree $+1$. If we now shift the source of the morphisms by -1 we get operations of degree 0 and hence we get composition morphisms for the $\mathfrak{K}$ twist of $\mathcal{O}(\Gamma)$. □

8.1.1. **Examples: $\mathcal{A}rc$, framed little discs and string topology.** One example is given by the $\mathcal{A}rc$ operad of [KLP], which has such a balanced S^1 action. The twist gluing and BV operator are discussed in [K7]. The $\mathcal{A}rc$ operad contains the well known operad of framed little discs [K1] which is a cyclic S^1 operad.

A rigorous topological version of the Sullivan PROP was given in [K4] this structure is actually a quasi–PROP which is only associative up to homotopy, but it has a cellular PROP chain model. Just like in the $\mathcal{A}rc$ operad there is an action of S^1 on the inputs, as these are fixed to have arcs incident to them. Thus we can twist glue by gluing in the S^1 families.

8.1.2. **Co–invariants.** Given an S^1–twisted $\mathcal{O}$ of composition type, we can consider its S^1–coinvariants. For concreteness we will treat modular operads, the other types work analogously. Here $\mathcal{O}_{S^1}((g,n)) := \mathcal{O}((g,n))_{(S^1)^{\times n}}$. Let $[\,] : \mathcal{O} \to \mathcal{O}_{S^1}$ denote the projection.

Then the twist gluings provide a natural family of gluings on the coinvariants: Namely if $[\alpha]$ and $[\beta]$ are two classes in the coinvariants, we can set

$$[\alpha]\, {}_i\bullet_j [\beta] := [\alpha\, {}_i\circ_j^{S^1} \beta] \quad \bullet_{ij}\,([\alpha]) := [\circ_{ij}^{S_1} \alpha] \tag{8.4}$$

Proposition 8.5. *These operations are well defined and furnish a $\mathfrak{K}$ twisted composition structure on the chain and homology level.*

Proof. The fact that this is well defined follows from the fact that the action is balanced. The second part is as above. □

Remark 8.6. The co–invariants of the Sullivan PROP are also what gives rise to an L_∞ structure [CS], which seems to be true in general.

8.1.3. **MO and EMO.** There are other early examples like the Schwarz–modular operads MO [Schw] where there are only self–gluings and a horizontal composition. In order to get an odd operation on the chain level Schwarz considers so called EMOs (extended modular operads), these carry just as above an S^1 action which gives an $\mathbb{S}_n \wr S^1$ action on each $\mathcal{O}((n))$.

8.2. **The paradigm: Real blow–ups and the Master equation.**

8.2.1. **Closed version [KSV].** A particularly interesting type of situation occurs if one augments an operad with an S^1 action. The prototype for this is the collection $\overline{M}_{g,n}^{KSV}$ of real blow ups of the Deligne–Mumford spaces along their compactification divisors as defined in [KSV].

Here, before the blow–up, the spaces $\overline{M}_{g,n}$ form a modular operad — even the archetypical one. The gluing of two curves is given by identifying the marked points and producing a node. One feature of the compactification is that the compactification divisor is composed of operadic compositions. More precisely for each genus labeled graph Γ of type $((g,n))$ there is a map $\overline{M}(\Gamma) \to \overline{M}_{g,n}$ where $\overline{M}(\Gamma) = \times_{v\in V(\Gamma)} M_{(g(v),Flag(v))}$ and in particular the one–edge trees define a normal crossing divisor.

Now after blowing up, the spaces $\overline{M}_{g,n}^{KSV}$ do not form a modular operad anymore, since one has to specify a vector over the new node. This is the origin of the twist gluing. One could have also added tangent vectors at each marked point and the nodes. This would give a modular operad. The KSV–construction is then just the twist gluing on the co–invariants.

The master equation now plays the following role. Let $S = \sum_{g,n}[\overline{M}_{g,n}^{KSV}/\mathbb{S}_n]$, where one sums over fundamental classes in a suitable sense. One such framework is given in [HVZ] where geometric chains of Joyce [J] are used.

The boundary in this case is essentially the geometric boundary of the fundamental class viewed as an orbifold with corners. Notice that while in the DM setting the compactification was with a divisor i.e. of complex codimension one, after blowing up in the KSV setting the compactification is done by a real codimension one bordification. Thus dS is the sum over these boundaries, which are exactly given by the blow ups of the divisors and these correspond exactly to the surfaces with one double point, either self glued or non–self glued. Working this out one finds that S satisfies the master equation.

8.2.2. **Open gluing case/orientation version.** Likewise there is a construction in the open/closed case in [HVZ]. Here the relevant moduli spaces are the real blow–ups $\overline{\mathcal{M}}_{g,n}^{KSV\, b,\vec{m}}$ of the moduli space $\overline{\mathcal{M}}_{g,n}^{b,\vec{m}}$ introduced in [Liu]. These are the moduli spaces of genus g curves with n marked labeled points, b boundary components and $\vec{m}$ marked labeled points on the boundary. In the closed case the blow up inherits an orientation because before compactifying the moduli space has a natural complex structure. In the open/closed case one can define iteratively the orientation by lifting or pushing the natural orientation of $M_{g,n}^{HVZb,(1,\dots,1)}$ (see [IS]) along fibre bundles that at the end reach any open/closed moduli space

Whereas the degree 1 in the closed case came from the fundamental class, here the grading comes from a grading by codimension in the corresponding moduli space. This agrees with the geometric dimension concept in the closed case.

For instance, if a geometric chain has degree d and it is constructed from $\overline{\mathcal{M}}_{g,n}^{KSV}$, the real blow-up of the DM-compactification of the moduli space as

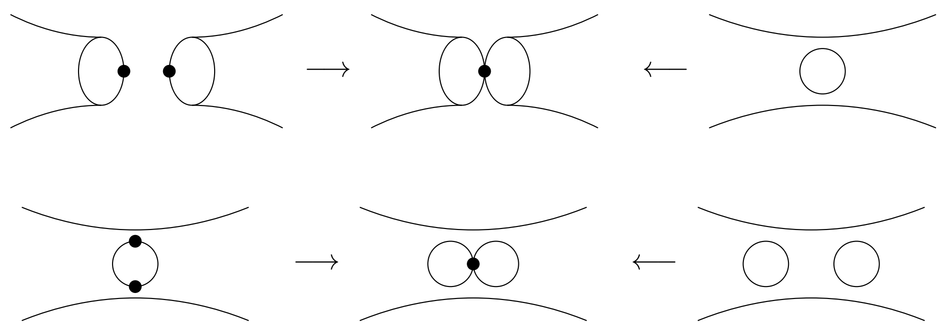

FIGURE 3. Boundary degenerations for the open case.

in [HVZ], we assign it a new degree: $6g - 6 + 2n - d$. In this new grading we also obtain a degree one map. Indeed, if we have two chains of degrees d_1 and d_2 constructed from $\overline{\mathcal{M}}^{KSV}_{g_1,n_1}$ and $\overline{\mathcal{M}}^{KSV}_{g_2,n_2}$ respectively, their corresponding codimensions are $6g_1 - 6 - 2n_1 - d_1$ and $6g_2 - 6 - 2n_2 - d_2$. After twist gluing we obtain a chain of degree $d_1 + d_2 + 1$ which lives in $\overline{\mathcal{M}}^{KSV}_{g_1+g_2,n_1+n_2-2}$ and therefore has codimension

$$6g_1 + 6g_2 - 6 - 2n_1 - 2n_2 - 4 - d_1 - d_2 - 1 = 6g_1 + 6g_2 - 2n_1 - 2n_2 - d_1 - d_2 - 11.$$

However, the sum of the original codimensions is $6g_1 + 6g_2 - 2n_1 - 2n_2 - d_1 - d_2 - 12$ which shows that the change in degrees is exactly 1. In the self-twist gluing picture something similar happens and the change in degree is 1 as well.

This grading by codimension may seem odd but it is exactly what we need in the open case. Recall that the twist gluing appeared in the closed case because of the different choices one has to attach surfaces along labeled points in the interior of the surface (different angles). This is not the case for labeled points in the boundary.

If we consider surfaces with at least one marked point in all boundary components we have essentially two cases for the boundary degeneration shown in Figure 3. In the first one we have two labeled points in different boundary components and in the second we have two labeled points in the same boundary component. The surface on the center is the result of attaching the labeled points represented on the left. The surface on the right is the desingularized version of the one in the center. Since there are no ambiguities in how to attach the labeled points this operation induces a degree zero map. However, grading by codimension is a completely different story. In the first case we have two chains of dimensions d_1 and d_2 respectively. Recall that the dimension of the moduli space $\overline{\mathcal{M}}^{HVZ\,b,\vec{m}}_{g,n}$ is $6g - 6 + 2n + 3b + m$ where b correspond to the number of boundary components and m is the number of labeled points in this boundary as in [HVZ]. The codimensions are then $6g_1 - 6 + 2n_1 + 3b_1 + m_1 - d_1$ and $6g_2 - 6 + 2n_2 + 3b_2 + m_2 - d_2$ respectively and their sum is

$$6g_1 + 6g_2 + 2n_1 + 2n_2 + 3b_1 + 3b_2 + m_1 + m_2 - d_1 - d_2 - 12.$$

After attaching, the new chain lives in $\overline{\mathcal{M}}^{HVZ\, b_1+b_2-1,\vec{m}'}_{g_1+g_2,n_1+n_2}$, where $\vec{m}'$ has $m_1 + m_2 - 2$ components, and therefore its codimension is

$$6g_1 + 6g_2 - 6 + 2n_1 + 2n_2 + 3b_1 + 3b_2 - 3 + m_1 + m_2 - 2 - d_1 - d_2$$

which is equal to

$$6g_1 + 6g_2 + 2n_1 + 2n_2 + 3b_1 + 3b_2 + m_1 + m_2 - d_1 - d_2 - 11$$

and therefore we get a degree one map again. Similar calculations take care of the self attaching operation and the second case.

Geometrically, the grading reflects the chosen orientations. And it is this choice of orientation [HVZ] that makes the gluing odd.

Intuitively, in the closed case there is an extra vector being added in the tangent bundle due to the circle. But there is also another vector being added in the normal bundle. In the open case there is an additional vector being added only in the normal bundle so grading by codimension gives us an odd gluing.

8.2.3. **Open/closed interaction; adding a derivation.** This idea is also the guide if we consider surfaces without marked points in some of their boundary components. In this case there is a new phenomenon that occurs in the boundary. Namely, as a boundary component degenerates it actually turns into something that looks like a marked point (a puncture in fact). Therefore it is essential to consider a new operation that simply re-labels a marked point as a degenerate boundary component in order to balance the quantum master equation.

If we make the same computation we did before for chains using codimension we also encounter a degree one map. However it is very clear in this case that we are not really changing the chain, we are just placing it in a different moduli space and hence changing the codimension. This is an interesting interaction between the closed and open operations and it is like twist gluing a surface at an interior (closed) marked point with a disc with only one interior marked point at such point giving a sort of degenerate boundary.

This open/closed interaction given by this degeneration leads to a contribution Δ_{oc} which is not only a derivation, but also a derivation of degree 1. Of course adding a degree 1 derivation to a BV operator which anti–commutes with it results in a new BV operator.

8.2.4. **Metatheorem 4.** In the above cases, we see that the fact that S which is composed out of fundamental classes, satisfies the QME is equivalent to the fact that the boundary divisors are either given by twist gluing two curves ${}_i\bullet_j$ or self–gluing the curves $\bullet_{ij}$ or the open gluing.

Question 8.7. What is the meaning of the ME or QME in the context of the $\mathcal{A}rc$ operad, the framed little discs and the Sullivan PROP?

There are two things which have to be solved (1) what kind of chains (2) what is the correct notion of fundamental chains.

For $\mathcal{A}rc$ there is a partial compactification, while the Sullivan PROP retracts to a CW complex, so one can use cellular chains. A clue might be provided by the Stasheff polytopes and the A_∞ Deligne conjecture [KSch], see below §8.3.

It seems that a fundamental role for the $\mathcal{A}rc$ or Sullivan PROP is played exactly by the arc families whose arcs do not quasi–fill the surface. Recall that an arc family quasi–fills the surface if its complement are finitely many polygons which contain at most one puncture, see [K4, K5].

8.3. **Other examples: A_∞ and A_∞ Deligne.** The Stasheff polytopes are also a geometric incarnation of the master equation. This follows e.g. from [HVZ], where discs with boundary points are used. But even classically the boundary of an associahedron, is precisely given by all possible compositions of lower order associahedra. This is precisely the compactification one would get for the non–sigma bracket and the corresponding master equation.

The link to the algebraic world is then to take a chain model where the usual power series of fundamental classes rel boundary gives a solution to the ME,

This is taken a step further in [KSch] where a product of cyclohedra and associahedra was given as the topological operad lying above the minimal operad of [KS] which in our framework is a Feynman transform of the Poisson operad $Assoc \circ Lie$.

8.4. **Topological Feynman transform?** One question that remains is what is the general theory of a topological Feynman transform.

For the closed type the set could be:

$$F\mathcal{O}((g,S)) = \coprod_{colim(\mathcal{G}raph(g,S)\downarrow *_{g,S})} \coprod_{v \in V_{\mathcal{G}raph}} \mathcal{O}(*_v) \coprod_{e \in E_{\mathcal{G}raph}} S^1 \tag{8.5}$$

This could be considered as a real blow up of the DM compactification. However, it is the way that this set is topologized, which is not clear.

Furthermore there are the open examples, where the S^1 factors disappear in favor of more structure at the vertices. In all one could make the following tentative definition.

Definition 8.8. A topological Feynman transform of a modular operad $\mathcal{O}$ is a collection of spaces $\overline{\mathcal{O}}((g,n))$ with $\mathcal{O}((g,n)) \subset \bar{\mathcal{O}}((g,n))$ such that there are fundamental classes coming from the relative fundamental classes which satisfy the quantum master equation.

Examples are then the moduli spaces above and the associahedra as well as the topological model for the minimal operad of Kontsevich and Soibelman [KS].

This is essentially equivalent to the cut–off view of Sullivan [S1–S3]. Here the cut off is given by removing a tubular neighborhood of the compactification divisor which amounts to a real blow–up of that divisor.

Remark 8.9. Notice that in more involved cases, like the open/closed version, there might be several terms in the master equation. Basically there is one term for each type of elementary operation. Closed self–, closed non–self–, open self–, open non–self–gluing and open/closed degeneration. This theme is explained in [Fey] where we define a Feynman transform relative to a set of generating morphisms.

Remark 8.10. Considering the master equations from the chain level, the master equation here could be interpreted as giving a morphism to the trivial modular operad. This of course can be viewed as pushing forward to a point, which is what integration is.

A. Appendix: Graphs

A.1. **The category of graphs.**

A.1.1. **Abstract graphs.** An abstract graph Γ is a quadruple $(V_\Gamma, F_\Gamma, \imath_\Gamma, \partial_\Gamma)$ of a finite set of vertices V_Γ a finite set of half edges or flags F_Γ and involution on flags $\imath_\Gamma : F_\Gamma \to F_\Gamma; \imath_\Gamma^2 = id$ and a map $\partial_\Gamma : F_\Gamma \to V_\Gamma$. We will omit the subscripts Γ if no confusion arises.

Since the map $\imath$ is an involution, it has orbits of order one or two. We will call the flags in an orbit of order one *tails*. We will call an orbit of order two an *edge*. The flags of an edge are its elements.

It is clear that the set of vertices and edges form a 1–dim simplicial complex. The realization of a graph is the realization of this simplicial complex.

Example A.1. A graph with one vertex is called a corolla. Such a graph only has tails and no edges. Any set S gives rise to a corolla. Let p be a one point set then the corolla is $*_{p,S} = (p, S, id, \partial)$ where ∂ is the constant map.

Given a vertex v of $\mathcal{G}raph$ we set $F_v = F_v(\Gamma) = \partial^{-1}(v)$ and call it *the flags incident to v*. This set naturally gives rise to a corolla. The *tails* at v is the subset of tails of F_v.

As remarked above F_v defines a corolla $*_v = *_{\{v\},F_v}$.

Remark A.2. The way things are set up, we are talking about finite sets, so changing the sets even by bijections changes the graphs.

An S labeling of a graph is a map from its tails to S.

An orientation for a graph Γ is a map $F_\Gamma \to \{in, out\}$ such that the two flags of each edge are mapped to different values. This allows one to speak about the "in" and the "out" edges, flags or tails at a vertex.

Example A.3. A tree is a contractible graph. It is rooted if it has a distinguished vertex, called the root. A tree has an induced orientation with the out edges being the ones pointing toward the root.

As usual there are edge paths on a graph and the natural notion of an oriented edge path. An edge path is a (oriented) cycle if it starts and stops at the same vertex and all the edges are pairwise distinct. An oriented cycle with pairwise distinct vertices is sometimes called a wheel. A cycle of length one is a loop.

A *naïve morphism* of graphs $\psi : \Gamma \to \Gamma'$ is given by a pair of maps ($\psi_F : F_\Gamma \to F_{\Gamma'}, \psi_V : V_\Gamma, V_{\Gamma'}$) compatible with the maps i and ∂ in the obvious fashion. This notion is good to define subgraphs and automorphism.

It turns out that this data is not enough to capture all the needed aspects for composing along graphs. For instance it is not possible to contract edges with such a map or graft two flags into one edge. The basic operations of composition in an operad viewed in graphs is however exactly grafting two flags and then contracting. There is a more sophisticated version of maps given in [BoM] which we will use in the sequel [Fey]. For now we wish to add the following morphisms.

Grafting. Given two graphs Γ and Γ', a tail s of Γ and a tail t of Γ' then $\Gamma\,{}_s\circ_t\Gamma'$ is the graph with the same vertices, flags, ∂, but where $\imath(s) = t$, and the rest of i is unchanged.

The *contraction of an edge* e of Γ is the graph where the two flags of e are omitted from the set of flags and the vertices of e are identified. It is denoted by Γ/e.

Merger. Given two graphs Γ and Γ' merging the vertex v of Γ with the vertex v' of Γ' means that these two vertices are identified and the rest of the structures just descend.

Remark A.4. One thing that is not so obvious is how S–labeling behave under these operations. If S are arbitrary sets (the unbiased case) this is clear. If one uses enumerations however (the biased case), one must specify how to re–enumerate. This is usually built into the definition of the composition type gadget.

A.2. **Standard algebras.** For the readers' convenience, we list the definition of the algebras we talk about. Let A be a graded vector space over k and let $|a|$ be the degree of an element a. Let's fix char $k = 0$ or at least $\neq 2$.

(1) Pre–Lie algebra. $(A, \circ : A \times A \to A)$ s.t.
$$a \circ (b \circ c) - (a \circ b) \circ c = (-1)^{|a||b|}[a \circ (c \circ b) - (a \circ c) \circ b]$$
(2) Odd Lie. $(A, \{\bullet\} : A \otimes A \to A)$
$\{a \bullet b\} = (-1)^{|a|-1)(|b|-1)}\{b \bullet a\}$ and Jacobi with appropriate signs
(3) Odd Poisson or Gerstenhaber. $(A, \{\bullet\}, \cdot)$ is odd Lie plus another associative multiplication for which the bracket is a derivation with the appropriate signs. (Sometimes Gerstenhaber also is defined to be super–commutative.)
$$\{a \bullet bc\} = \{a \bullet b\}c + (-1)^{(|a|-1)|b|}b\{a \bullet c\} \;\forall\; a, b, c \in A \tag{A-1}$$
(4) (dg)BV. $(A, \cdot, \Delta)$. $(A, \cdot)$ associative (differential graded) supercommutative algebra, Δ a differential of degree 1: $\Delta^2 = 0$ and
$$\{a \bullet b\} := (-1)^{|a|}\Delta(ab) - a\Delta(b) - (-1)^{|a|}\Delta(a)b \tag{A-2}$$
is a Gerstenhaber bracket.

An equivalent condition for a BV operator is

$$\begin{aligned}\Delta(abc) &= \Delta(abc)\Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) - \Delta(a)bc \\ &\quad -(-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c) \qquad \text{(A-3)}\end{aligned}$$

(5) (dg)GBV. This name is used if *a priori* there is a BV operator and a given Gerstenhaber bracket and *a posteriori* the given Gerstenhaber bracket coincides with the one induced by the BV operator.

References

[ASZK] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich. *The geometry of the master equation and topological quantum field theory. Internat.* J. Modern Phys. A 12 (1997), no. 7, 1405â1429

[Bar] S. Barannikov. *Modular operads and Batalin-Vilkovisky geometry.* Int. Math. Res. Not. IMRN (2007), no.19.

[BF] C. Berger and B. Fresse Une décomposition prismatique de l'opérade de Barratt-Eccles. C. R. Math. Acad. Sci. Paris 335 (2002), no. 4, 365-370

[BLB] R. Bocklandt and L. Le Bruyn. *Necklace Lie algebras and noncommutative symplectic geometry.* Math. Z. 240 (2002), no. 1, 141–167.

[BoM] D.V. Borisov and Yu.I. Manin. *Generalized operads and their inner cohomomorphisms.* Geometry and dynamics of groups and spaces *in* Progr. Math. 265, p247–308 (2008).

[BV] J. M. Boardman and R. M. Vogt. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[Ch] F. Chapoton. *On some anticyclic operads.* Algebr. Geom. Topol. 5 (2005), 53â69

[CS] M. Chas and D. Sullivan *String Topology.* arxiv.org/abs/math/9911159, (1999).

[CV] J. Conant and K. Vogtmann. *On a theorem of Kontsevich.* Algebr. Geom. Topol. 3, (2003), p1167–1224.

[EM] S. Eilenberg and J.C. Moore. *Adjoint functors and triples.* Illinois J. Math. 9, (1965), p381–398.

[Fey] *Feynman categories.* In preparation.

[G] M. Gerstenhaber. *The cohomology structure of an associative ring.* Ann. of Math. (2) 78 (1963) p267–288.

[Ga] W. Gan. *Koszul duality for dioperads*, Math. Res. Lett. 10 (2003), no. 1, p. 109–124.

[GeK1] E. Getzler and M.M. Kapranov. *Cyclic operads and cyclic homology.* Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV p167–201, Int. Press, Cambridge, MA, 1995.

[GeK2] E. Getzler and M.M. Kapranov. *Modular operads.* Compositio Math. 110 (1998), no.1, p65–126.

[Gi] V. Ginzburg. *Non-commutative symplectic geometry, quiver varieties, and operads.* Math. Res. Lett. 8 (2001), no. 3, p377–400.

[GiK] V. Ginzburg and M. Kapranov. *Koszul duality for operads.* Duke Math. J. 76 (1994), no.1, p203–272.

[GM] S. I. Gelfand, and Yu. I. Manin. Methods of homological algebra. Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[GV] M. Gerstenhaber and A. A. Voronov. *Homotopy G-algebras and moduli space operad.* Internat. Math. Res. Notices 1995, 141â153

[HVZ] E. Harrelson, A.A. Voronov and J.J. Zuniga. *Open-closed moduli spaces and related algebraic structures.* Lett. Math. Phys. 94 (2010), no.1, p1–26.

[IS] S. Ivashkovich and V. Shevchishin.*Holomorphic Structure on the Space of Riemann Surfaces with marked boundary.* Tr. Mat. Inst. Steklova 235 (2001), 98109, Anal. i Geom. Vopr. Kompleks. Analiza.

[J] D. Joyce. *Kuranishi homology and Kuranishi cohomology.* Prerpint, 2007, arXiv: 0707.3572

[KM] M. Kapranov and Yu.I. Manin. *Modules and Morita theorem for operads.* Amer. J. Math. 123 (2001), no.5, p811–838.

[K1] R.M. Kaufmann. "On several varieties of cacti and their relations". Algebraic & Geometric Topology 5 (2005), 237-300.

[K2] R.M. Kaufmann. "On Spineless Cacti, Deligne's Conjecture and Co.nes–Kreimer's Hopf Algebra." Topology 46, 1 (2007), 39-88.

[K3] R.M. Kaufmann. *A proof of a cyclic version of Deligne's conjecture via cacti.* Math. Res. Lett. 15 (2008), no.5 p901–921.

[K4] R.M. Kaufmann. "Moduli space actions on the Hochschild cochain complex I: cell models". Journal of Noncommutative Geometry 1, 3 (2007), 333-384.

[K5] R.M. Kaufmann. "Moduli space actions on the Hochschild cochain complex II: correlators". Journal of Noncommutative Geometry 2, 3 (2008), 283-332.

[K6] R.M. Kaufmann. "Graphs, strings and actions". in: Algebra, Arithmetic and Geometry Volume II: In Honor of Yu. I. Manin. Progress in Mathematics 270, 127–178. Birhauser, Boston (2010).

[K7] R.M. Kaufmann. "Arc Geometry and Algebra: Foliations, Moduli Spaces, String Topology and Field Theory" Handbook of Teichmueller Theory to appear.

[KLP] R. Kaufmann, M. Livernet and R.C. Penner. "Arc Operads and Arc Algebras". Geometry and Topology 7 (2003), 511-568.

[KSch] R.M. Kaufmann and R. Schwell. *Associahedra, cyclohedra and a topological solution to the A_∞ Deligne conjecture.* Adv. Math. 223 (2010), no.6, p2166–2199.

[KSV] T. Kimura, J. Stasheff and A.A. Voronov. *On operad structures of moduli spaces and string theory.* Comm. Math. Phys. 171 (1995), no.1 p1–25.

[Kl] H. Kleisli. *Every standard construction is induced by a pair of adjoint functors.* Proc. Amer. Math. Soc. 16 (1965), p544–546.

[K] M. Kontsevich. *Formal (non)commutative symplectic geometry.* The GelÊfand Mathematical Seminars, 1990–1992, p173–187, Birkhauser Boston, Boston, MA, 1993.

[KS] M. Kontsevich and Y. Soibelman. *Deformations of algebras over operads and the Deligne conjecture.* Conférence Moshé Flato 1999, Vol. I (Dijon), *in* Math. Phys. Stud. 21 p255–307, Kluwer Acad. Publ. Dordrecht, 2000.

[Liu] C.C. M. Liu *Moduli of J–holomorphic curves with Lagrangian boundary conditions and open gromov–Witten invariants for an S^1 equivariant pair.* Preprint arXiv math.DG/0210257

[Mar] Markl, Martin Operads and PROPs. Handbook of algebra. Vol. 5, 87â140, Handb. Algebr., 5, Elsevier/North-Holland, Amsterdam, 2008.

[McCS] J. E. McClure and J. H. Smith, Jeffrey H. A solution of Deligne's Hochschild cohomology conjecture. Recent progress in homotopy theory (Baltimore, MD, 2000), 153-193, Contemp. Math., 293, Amer. Math. Soc., Providence, RI, 2002.

[McL1] S. Mac Lane. *Categorical algebra.* Bull. Amer. Math. Soc. 71, (1965) 40-106.

[McL2] S. Mac Lane. Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998

[MMS] M. Markl, S. Merkulov and S. Shadrin. *Wheeled PROPs, graph complexes and the master equation* J. Pure Appl. Algebra 213 (2009), no.4 p496–535.

[MSS] M. Markl, S. Shnider and J. Stasheff. *Operads in algebra, topology and physics.* Mathematical Surveys and Monograps 96, American Mathematical Society, Providence, RI, 2002.

[S1] D. Sullivan. *Sigma models and string topology.* Graphs and patterns in mathematics and theoretical physics, 1â11, Proc. Sympos. Pure Math., 73, Amer. Math. Soc., Providence, RI, 2005.

[S2] D. Sullivan. *String topology background and present state.* Current developments in mathematics, 2005, 41â88, Int. Press, Somerville, MA, 2007

[S3] D. Sullivan. *Homotopy theory of the master equation package applied to algebra and geometry: a sketch of two interlocking programs.* Algebraic topologyâold and new, 297â305, Banach Center Publ., 85, Polish Acad. Sci. Inst. Math., Warsaw, 2009

[Sch] T. Schedler. *A Hopf algebra quantizing a necklace Lie algebra canonically associated to a quiver*. Int. Math. Res. Not. 12 (2005), p725–760.

[Schw] A. Schwarz *Grassmannian and string theory.* Comm. Math. Phys. 199 (1998)

[T] D. Tamarkin. Another proof of M. Kontsevich formality theorem. Peprint math/9803025. Formality of Chain Operad of Small Squares. Lett. Math. Phys. 66 (2003), no. 1-2, 65–72.

[TZ] T. Tradler and M. Zeinalian. *On the cyclic Deligne conjecture.* J. Pure Appl. Algebra 204 (2006), no.2 p280–299.

[V] A. A. Voronov. Homotopy Gerstenhaber algebras. Conférence Moshé Flato 1999, Vol. II (Dijon), 307-331, Math. Phys. Stud., 22, Kluwer Acad. Publ., Dordrecht, 2000

[Va] B. Vallette. *A Koszul duality for PROPs.* Trans. Amer. Math. Soc. 359 (2007), no. 10 p4865–4943.

[Wa] B.C. Ward. *Cyclic A_∞ structures and Deligne's conjecture.* Algebr. Geom. Topol. 12 (2012), p1487–1551.

[SZ] A. Sen and B. Zwiebach. *Quantum background independence of closed-string field theory.* Nuclear Phys. B 423 (1994), 580–630.

E-mail address: rkaufman@math.purdue.edu

Purdue University Department of Mathematics, West Lafayette, IN 47907

E-mail address: wardbc@math.purdue.edu

Purdue University Department of Mathematics, West Lafayette, IN 47907

E-mail address: zuniga_jj@up.edu.pe

Departamento Académico de Economía Universidad del Pacífico, Lima 21 Peru