THETA FUNCTIONS AND MIRROR SYMMETRY

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Introduction

The classical subject of theta functions has a very rich history dating back to the nineteenth century. In modern algebraic geometry, they arise as sections of ample line bundles on abelian varieties, canonically defined after making some discrete choices of data. The definition of theta functions depends fundamentally on the group law, leaving the impression that they are a feature restricted to abelian varieties. However, new insights from mirror symmetry suggest that they exist much more generally, even on some of the most familiar varieties.

Mirror symmetry began as a phenomenon in string theory in 1989, with the suggestion that Calabi-Yau manifolds should come in pairs. Work of Greene and Plesser [GrPl] and Candelas, Lynker and Schimmrigk [CLS] gave the first hint that there was mathematical justification for this idea, with constructions given of pairs of Calabi-Yau three-folds X,

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$\check{X}$, with the property that $\chi(X) = -\chi(\check{X})$. More precisely, the Hodge numbers of these pairs obey the relation

$$h^{1,1}(X) = h^{1,2}(\check{X}), \quad h^{1,2}(X) = h^{1,1}(\check{X}).$$

In 1991, Candelas, de la Ossa, Green and Parkes [COGP] achieved an astonishing breakthrough in exploring the mathematical ramifications of some string-theoretic predictions. In particular, using string theory as a guideline, they carried out certain period integral calculations for the mirror of the quintic three-fold in $\mathbb{CP}^4$, and obtained a generating function for the numbers N_d, $d \geq 1$, where N_d is the number of rational curves of degree d in the quintic three-fold.

This immediately attracted attention from mathematicians, and there has followed twenty years of very rewarding efforts to understand the mathematics underlying mirror symmetry.

A great deal of progress has been made, but much remains to be done. In this survey article, we will discuss certain aspects of this search for understanding, guided by a relationship between mirror symmetry and theta functions. In particular, we will discuss the surprising implication of mirror symmetry that near large complex structure limits in complex moduli space, Calabi-Yau manifolds also carry theta functions. Roughly speaking, these will be canonically defined bases for the space of sections of line bundles. This implication was first suggested, as far as we know, by the late Andrei Tyurin, see [Ty99].

The constructions in fact apply much more broadly than just to Calabi-Yau manifolds. For example, in [GHKII] (see Sean Keel's lecture [Ke11]), it is proven that some of the most familiar varieties in algebraic geometry, including many familiar affine rational surfaces, carry theta functions. In addition, much stronger results apply to the surface case, with [GHKK3] proving a strong form of Tyurin's conjecture. However, in this survey we will focus only on the simplest aspects of the construction.

The discussion here represents a distillation of a number of ongoing joint projects with varying groups of coauthors. The relationship between integral affine manifolds, degenerations of Calabi-Yau varieties and mirror symmetry discussed here is based on a long-term project of the authors of this survey. The construction of theta functions as described here has come out of joint work with Paul Hacking and Sean Keel, while the relationship between theta functions and homological mirror symmetry is based on forthcoming work of Mohammed Abouzaid with Gross and Siebert.

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1. The geometry of mirror symmetry: HMS and SYZ

There are two principal approaches to the geometry underlying mirror symmetry: Kontsevich's homological mirror symmetry conjecture (HMS) [K95] and the Strominger-Yau-Zaslow (SYZ) conjecture [SYZ]. Taken together, they suggest the existence of theta functions.

These conjectures are as follows. Consider a mirror pair of Calabi-Yau manifolds, X and $\check{X}$. To be somewhat more precise, we should consider Calabi-Yau manifolds with Ricci-flat Kähler metric, so that mirror symmetry is an involution

$$(X, J, \omega) \leftrightarrow (\check{X}, \check{J}, \check{\omega}).$$

Here J is the complex structure and ω the Kähler form on X. One expects that J determines the Kähler structure $\check{\omega}$ and ω determines the complex structure $\check{J}$.[1] Kontsevich's fundamental insight is that the isomorphism that mirror symmetry predicts between the complex geometry of (X, J) and the symplectic geometry of $(\check{X}, \check{\omega})$ can be expressed in a categorical setting:

Conjecture 1.1 (Homological mirror symmetry)**.** *There is an equivalence of categories between the derived category $D^b(X)$ of bounded complexes of coherent sheaves on X and* $\mathrm{Fuk}(\check{X})$*, the Fukaya category of Lagrangian submanifolds on $\check{X}$.*

There are many technical issues hiding in this statement, the least of which is showing that the Fukaya category makes sense. In particular, $\mathrm{Fuk}(\check{X})$ is not a category in the traditional sense, as composition of morphisms is not associative. Rather, it is an A_∞-category, which essentially means that there is a sequence of higher composition maps which measure the failure of associativity; we will be more precise shortly.

To first approximation, the objects of $\mathrm{Fuk}(\check{X})$ are Lagrangian submanifolds of $\check{X}$, i.e., submanifolds $L \subseteq \check{X}$ with $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} \check{X}$ and $\check{\omega}|_L = 0$. We define the Hom between objects as follows. Let Λ be the Novikov ring, i.e., the ring of power series $\sum_{i=1}^{\infty} a_i q^{r_i}$ where $a_i \in \mathbb{C}$, $r_i \in \mathbb{R}_{\geq 0}$, $r_i \to \infty$ as $i \to \infty$. Given two Lagrangian submanifolds L_0, L_1, we define

$$\mathrm{Hom}(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda[p],$$

assuming that L_0 and L_1 intersect transversally (if not, we can perturb one of them via a generic Hamiltonian isotopy). In fact, this is a graded Λ-module, with the degree of $[p]$ being the so-called *Maslov index* of p. One can then define a series of maps

$$\mu_d : \mathrm{Hom}(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_d)$$

[1] This discussion ignores the B-field.

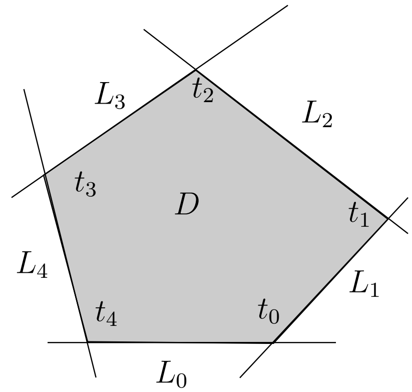

FIGURE 1.

for $d \geq 1$, $L_0, \ldots, L_d$ Lagrangian submanifolds of $\check{X}$. Roughly this map is defined by counting certain holomorphic disks:

$$\mu_d(p_{d-1,d}, \ldots, p_{0,1}) = \sum_{p_{0,d} \in L_0 \cap L_d} \sum_{\psi : D \to \check{X}} \pm q^{\int_D \psi^* \check{\omega}} [p_{0,d}]$$

where the second sum is over all holomorphic maps $\psi : D \to X$ such that there are cyclically ordered points $t_0, \ldots, t_d \in \partial D$ with $\psi(t_i) = p_{i,i+1}$, $\psi(t_d) = p_{0,d}$, and $\psi([t_i, t_{i+1}]) \subseteq L_{i+1}$ and $\psi([t_d, t_0]) \subseteq L_0$. Here $[t_i, t_{i+1}]$ denotes the interval on ∂D between t_i and t_{i+1}. See Figure 1. The contribution to $\mu_d(p_{d-1,d}, \ldots, p_{0,1})$ from $p_{0,d}$ is only counted if the expected dimension of the moduli space of disks is zero; this makes μ_d into a chain map of degree $2-d$. In suitably nice cases, these operations will satisfy the so-called A_∞-relations, which are

$$\sum_{\substack{1 \leq p \leq d \\ 0 \leq q \leq d-p}} \pm \mu_{d-p+1}(a_d, \ldots, a_{p+q+1}, \mu_p(a_{p+q}, \ldots, a_{q+1}), a_q, \ldots, a_1) = 0.$$

This tells us that μ_1 turns $\mathrm{Hom}(L_0, L_1)$ into a complex, that μ_2 is associative up to homotopy, and so on. Since μ_2 will play the role of composition of morphisms for us, this means the Fukaya category is not in general a category, because composition is not associative.

One might now object that $D^b(X)$ is a genuine category, so in what sense is it isomorphic to something which is not a genuine category? It turns out that there is a natural way to put an A_∞-category structure on $D^b(X)$ which is really enriching the structure of $D^b(X)$; the statement of HMS then says that we expect $D^b(X)$ and $\mathrm{Fuk}(\check{X})$ to be quasi-isomorphic as A_∞-categories, which is a well-defined notion.

We will not go into more technical details about HMS, as we shall only need a small part of it. Instead, we move on to discuss the Strominger-Yau-Zaslow (SYZ) conjecture [SYZ], dating from 1996. This was the first idea giving a truly geometric interpretation of mirror symmetry. Fix now an n-dimensional Calabi-Yau manifold X with a nowhere vanishing holomorphic n-form Ω and a symplectic form ω: we say a submanifold $M \subseteq X$ is *special Lagrangian* if M is Lagrangian with respect to ω and furthermore $\mathrm{Im}\,\Omega|_M = 0$. This notion was introduced by Harvey and Lawson in [HL82]; special Lagrangian submanifolds are volume minimizing in their homology class. Suppose X has a mirror $\check{X}$, with holomorphic n-form $\check{\Omega}$ and symplectic form $\check{\omega}$. We then have:

Conjecture 1.2 (The Strominger-Yau-Zaslow conjecture)**.** *There are continuous maps $f : X \to B$, $\check{f} : \check{X} \to B$ whose fibres are special Lagrangian, and whose general fibres are dual n-tori.*

This is a purposefully vague statement, partly because we are very far from a proof of anything resembling this conjecture: see [Gr98],[Gr99],[Gr00] for detailed discussion of more precise forms of this conjecture. Let us just say at this point that the duality implies that the topological monodromy of the smooth part of f is the transpose of the topological monodromy of $\check{f}$.

Early work aimed at understanding the conjecture includes [Gr98],[Gr99],[Hi97]. In particular in [Gr98], the first author conjectured that Lagrangian sections of $\check{f}$ should be expected to be mirror, under HMS, to line bundles on X. A more precise correspondence was predicted there, with specific predictions on which topological isotopy class of sections corresponded to which numerical equivalence class of line bundles. This idea was used in a number of different situations: for example, work of Polishchuk and Zaslow [PZ98] give an explicit correspondence between special Lagrangian sections on the obvious SYZ fibration on an elliptic curve and line bundles on the mirror elliptic curve. We shall say more about this in §2.

The fundamental idea we shall pursue in this paper is the following. Suppose L_0 is a Lagrangian section of $\check{f}$ corresponding to the structure sheaf $\mathcal{O}_X$, and L_1 is a Lagrangian section corresponding to an ample line bundle $\mathcal{L}$ on X. Then HMS should yield an isomorphism $\mathrm{Hom}(L_0, L_1) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{L}) \otimes_{\mathbb{C}} \Lambda$. Here the Hom's are in the Fukaya and derived categories respectively, but after taking cohomology, one expects on the right to only get a contribution from $H^0(X, \mathcal{L})$ as all higher cohomology vanishes. If all the intersection points of L_0 and L_1 are Maslov index zero, i.e., if somehow the intersection is particularly nice so that there is no μ_1, then one has of course a basis for $\mathrm{Hom}(L_0, L_1)$ given by these intersection points, and these correspond to elements of $H^0(X, \mathcal{L})$.

The moral of this is: suppose we have particularly canonical choices of Lagrangian sections corresponding to ample line bundles. Then HMS predicts the existence of a canonical basis of sections of ample line bundles. We are going to call elements of such a canonical basis *theta functions*.

To the best of our knowledge, the existence of such a canonical basis was first suggested by the late Andrei Tyurin; the first author heard him speak about these in a lectures series at the University of Warwick in 1999: see especially the remark on p. 36 of [Ty99].

We will first make more precise what these canonical sections should be, and then in the next section argue in the case of abelian varieties that the corresponding basis indeed coincides with classical theta functions. Before doing so, we need to explain more structure underlying the correct way of thinking about the SYZ conjecture.

What is actually most important about the SYZ conjecture are certain structures which should appear on the base B of the two dual fibrations. Suppose we have fibrations as in the conjecture. Let $\Delta \subseteq B$ be the set of critical values of f and $B_0 := B \setminus \Delta$, so that if $x \in B_0$, $f^{-1}(x)$ is a smooth torus. It was first observed by Hitchin in [Hi97] that the special Lagrangian fibration induces two different affine structures on B_0, one induced by ω (this affine structure arises from the Arnold-Liouville theorem) and one induced by $\operatorname{Im}\Omega$. A precise definition:

Definition 1.3. An affine structure on an n-dimensional real manifold B is a set of coordinate charts $\{\psi_i : U_i \to \mathbb{R}^n\}$ on an open cover $\{U_i\}$ of B whose transition maps $\psi_j \circ \psi_i^{-1}$ lie in $\operatorname{Aff}(\mathbb{R}^n)$, the affine linear group of $\mathbb{R}^n$. We say the structure is *tropical* if the transition maps lie in $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$, and *integral* if the transition maps lie in $\operatorname{Aff}(\mathbb{Z}^n)$.

A *(tropical, integral) affine manifold with singularities* is a manifold B along with a codimension ≥ 2 subset $\Delta \subseteq B$ and a (tropical, integral) affine structure on $B_0 := B \setminus \Delta$.

In fact, the affine structures induced by special Lagrangian fibrations are tropical, so we obtain tropical affine manifolds with singularities (except for the fact that genuine special Lagrangian fibrations are expected to have codimension one discriminant loci which retract onto a codimension two subset, as demonstrated by Joyce in many examples [J03]).

It is convenient now to largely forget about special Lagrangian fibrations, as we don't know if they exist, and instead focus on the tropical affine manifolds arising from them. It is in fact now fairly well understood what such manifolds should look like, even if the fibrations aren't known! See for example [Gr09].

In fact, tropical affine manifolds quickly give rise to a toy version of mirror symmetry:

Definition 1.4. If B is a tropical affine manifold, then let Λ be the local system contained in the tangent bundle $\mathcal{T}_B$ given locally by integral linear combinations of coordinate vector fields $\partial/\partial y_1, \ldots, \partial/\partial y_n$, where $y_1, \ldots, y_n$ are local tropical affine coordinates. The fact that transition maps lie in $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ rather than $\operatorname{Aff}(\mathbb{R}^n)$ says this local system is well-defined, independently of coordinates. Similarly, let $\check{\Lambda} \subseteq \mathcal{T}_B^*$ be the local system given locally by integral linear combinations of $dy_1, \ldots, dy_n$. Set

$$X(B) := \mathcal{T}_B/\Lambda,$$
$$\check{X}(B) := \mathcal{T}_B^*/\check{\Lambda}.$$

We have projections $f : X(B) \to B$ and $\check{f} : \check{X}(B) \to B$ which are dual torus fibrations.

Note that $X(B)$ comes along with a natural complex structure. This is most easily described by specifying the almost complex structure J. There is a natural flat connection on $\mathcal{T}_B$ such that sections of Λ are flat sections. At any point in $\mathcal{T}_B$, the horizontal and vertical tangent spaces are both isomorphic to the tangent space to B, and J interchanges these two spaces, inserting an appropriate sign to ensure $J^2 = -1$. It is easy to see that this structure is integrable, identifying $f^{-1}(U)$, for $U \subseteq B$ a small open set, with a T^n-invariant open subset of $(\mathbb{C}^*)^n$.

Furthermore, $\check{X}(B)$ carries a natural symplectic structure: as always, $\mathcal{T}_B^*$ carries a canonical symplectic form, and one checks it descends to the quotient.

As a consequence, we can view the correspondence $X(B) \leftrightarrow \check{X}(B)$ as a toy version of mirror symmetry. In this discussion we see half of mirror symmetry, as we don't have a symplectic structure on $X(B)$ or a complex structure on $\check{X}(B)$.

How close is this correspondence to actual mirror symmetry? If B is compact, e.g., $B = \mathbb{R}^n/\Gamma$ for a lattice Γ, then $X(B)$ is a complex torus, and the toy description gives a completely satisfactory description of mirror symmetry; we shall make use of this in §2. However, in general, one should work with B a tropical affine manifold with singularities, in which case one only has a subset $B_0 \subseteq B$ with an affine structure. So one can then ask to what extent can one compactify $X(B_0)$ or $\check{X}(B_0)$. One has the following general observations:

(1) In various nice cases, $X(B_0)$ and $\check{X}(B_0)$ can be compactified topologically: see [Gr01] for the three-dimensional case, and work in progress [GS13] for similar results in all dimensions. In particular, [Gr01] gives a complete description of the quintic threefold and its mirror from this point of view.
(2) In various nice cases in dimensions two and three, $\check{X}(B_0)$ can be compactified to some $\check{X}(B)$ in the symplectic category, see [CBM09].
(3) As a complex manifold, $X(B_0)$ can almost never be compactified. This is a crucial point for mirror symmetry. There are *instanton corrections* that one needs to make to the complex structure on $X(B_0)$ before one can hope to compactify this. This was first explored by Fukaya in [F05], in the two-dimensional (K3) case. That paper was the first to suggest the philosophy: *the corrections to the complex structure on* $X(B_0)$ *arise from pseudo-holomorphic disks in* $\check{X}(B)$ *with boundary on fibres of the SYZ fibration*.

The direct analytic approach of Fukaya suffers from huge technical difficulties and as such was only a heuristic. To carry out this program in a more practical way, a switch to an easier category is necessary. Kontsevich and Soibelman used the rigid analytic category, constructing in [KS06] a rigid analytic K3 surface from a tropical affine surface with 24 singular points. In parallel, we had been working on a program to address this problem in all dimensions using logarithmic geometry; combining our approach with some ideas of

[KS06], we provided a solution to this problem in all dimensions, in a somewhat different category. Roughly, our result shown in [GS11] is as follows.

Theorem 1.5. *Suppose given an integral affine manifold with singularities B. Suppose furthermore that the singularities are "nice" and B comes with a decomposition $\mathscr{P}$ into lattice polytopes. Suppose furthermore given a strictly convex, multi-valued piecewise linear function φ on B. Then one can construct a one-parameter flat family $\pi : \check{\mathcal{X}} \rightarrow \operatorname{Spec} \mathbb{C}[\![t]\!]$ from this data whose central fibre is*

$$\check{\mathcal{X}}_0 = \bigcup_{\sigma \in \mathscr{P}_{\max}} \mathbb{P}_\sigma,$$

where $\mathscr{P}_{\max}$ is the set of maximal cells in the polyhedral decomposition, and $\mathbb{P}_\sigma$ is the projective toric variety defined by the lattice polytope σ. These toric varieties are glued together along toric strata as dictated by the combinatorics of $\mathscr{P}$. Furthermore, $\check{\mathcal{X}}$ comes along with a relatively ample line bundle $\mathcal{L}$.

There are a number of important features of this construction:

(1) It is an explicit construction, giving an order-by-order algorithm for gluing standard thickenings of affine pieces of the irreducible components of $\check{\mathcal{X}}_0$. This data is described by what we call a *structure*, and as we shall see, is really controlled by counts of holomorphic disks on the mirror side.
(2) There is a notion of discrete Legendre transform which allows one to associate to the triple of data $(B, \mathscr{P}, \varphi)$ another triple $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$. The polyhedral decomposition $\check{\mathscr{P}}$ is dual to $\mathscr{P}$, and the affine structure on $\check{B}$ is dual to that on B in some precise sense, see [GS06], §1.4. Then applying the above theorem to the dual data yields the mirror Calabi-Yau.
(3) The family constructed in Theorem 1.5 can be extended to a flat family of complex analytic spaces $\mathscr{X}$ over a disk D. A general fibre of this family, $\mathscr{X}_t$, has a Kähler form repesenting $c_1(\mathcal{L})$. The expectation is that this symplectic manifold is a compactification of $\check{X}(B_0)$. Furthermore, as a complex manifold, it should roughly be a compactification of a small deformation of the complex structure on $X_\epsilon(\check{B}_0)$, where $X_\epsilon(\check{B}_0) = \mathcal{T}_{\check{B}_0}/\epsilon\Lambda$ and $\epsilon > 0$ is a real number. See [GS03] for some details of this; more details will appear in [GS13].

There is one confusing point in this discussion: the role of $X(B)$ and $\check{X}(B)$ has been interchanged. We originally said we wanted to compactify $X(B_0)$. Instead, we compactified $\check{X}(B_0)$. This makes sense from several points of view.

First, we have constructed a whole family of complex manifolds, but they are symplectomorphic as symplectic manifolds. So it makes sense that we get $\check{X}(B)$, which comes with a canonical symplectic structure. The integrality of the affine structure on B guarantees that the symplectic form on $\check{X}(B)$ represents an integral cohomology class.

Second, if one doesn't like this switch, then one can work with the Legendre dual manifold $\check{B}$. The work of Fukaya [F05] and Kontsevich and Soibelman [KS06] did precisely this. But it turns out that structures are nicer objects on B than on $\check{B}$. On B, the data controlling the family $\check{\mathcal{X}}$, the structure, is essentially tropical in nature, and can be viewed as a union of tropical trees on B. If one works on $\check{B}$, one instead needs to use trees made of gradient flow lines, and this can produce some technical difficulties. Working on B makes many aspects of our work effective.

Let us now return to theta functions. We note our construction comes with a canonical ample line bundle, whose first Chern class is represented by the symplectic form on $\check{X}(B)$. Now a line bundle should be mirror to a section of the SYZ fibration, so it is natural to ask whether $X(B) \to B$ comes with a natural section. Since we haven't given an explicit description of the compactification in this paper, let us at least answer this question over B_0. There is in fact a whole set of natural sections, indexed by $\ell \in \mathbb{Z}$, given in local integral affine coordinates by

$$\sigma_\ell : (y_1, \ldots, y_n) \mapsto -\sum_{i=1}^{n} \ell \cdot y_i \frac{\partial}{\partial y_i}. \tag{1.1}$$

Note that modulo integral vector fields, i.e., sections of Λ, this vector field is well-defined independently of the choice of integral affine coordinates. Call the image of this section L_ℓ.

Of course there is no symplectic structure on $X(B_0)$, so it doesn't quite make sense to call these Lagrangian sections, but one can imagine that one can find symplectic structures which make these sections Lagrangian, and then deduce some consequences.

The most important consequence is the description of the set $L_0 \cap L_\ell$, namely

$$f(L_0 \cap L_\ell) = B_0\left(\frac{1}{\ell}\mathbb{Z}\right),$$

where the latter denotes the set of points of B_0 whose coordinates in any (hence all) integral affine coordinate charts lie in $\frac{1}{\ell}\mathbb{Z}$.

Let us hypothesize that L_ℓ is mirror to the line bundle $\mathcal{L}^\ell$. Since for an ample line bundle on a Calabi-Yau manifold all higher cohomology vanishes, we are led to the following conjecture:

Conjecture 1.6. *For $\ell > 0$, there is a canonical basis of $\Gamma(\check{\mathcal{X}}, \mathcal{L}^\ell)$ as a $\mathbb{C}[\![t]\!]$-module indexed by elements of $B(\frac{1}{\ell}\mathbb{Z})$.*

In the sections that follow, we will first explain why theta functions for abelian varieties fit naturally into such a conjecture. Next, we outline the proof of this conjecture given in [GHKSΘ], with the precise statement given in Theorem 3.7. We finally explain various applications of the existence of such a basis.

2. Theta functions for abelian varieties and the Mumford construction

In the case that B is a torus, our construction in fact recovers Mumford's description of degenerations of abelian varieties [M72], see also [AN99]. Theorem 1.5 can be viewed as a vast generalization of this construction. We will briefly review a simple version of Mumford's construction.

The starting data is a lattice $M \cong \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, a sublattice $\Gamma \subseteq M$, a Γ-periodic polyhedral decomposition $\mathscr{P}$ of $M_{\mathbb{R}}$, and a strictly convex piecewise linear function with integral slopes $\varphi : M_{\mathbb{R}} \to \mathbb{R}$ satisfying a periodicity condition, for $\gamma \in \Gamma$,

$$\varphi(m+\gamma) = \varphi(m) + \alpha_\gamma(m)$$

for some affine linear function α_γ depending on γ. The affine manifold B in Theorem 1.5 will be $M_{\mathbb{R}}/\Gamma$ in this setup.

From this data one builds an unbounded polyhedron in $M_{\mathbb{R}} \oplus \mathbb{R}$:

$$\Delta_\varphi := \{(m,r) \,|\, m \in M_{\mathbb{R}}, r \geq \varphi(m)\}.$$

The normal fan of this polyhedron in $N_{\mathbb{R}} \oplus \mathbb{R}$ is a fan Σ_φ with an infinite number of cones, defining a toric variety X_φ which is not of finite type. Note that the one-dimensional rays of Σ_φ are in one-to-one correspondence with the maximal cells σ of $\mathscr{P}$; if $n_\sigma \in N$ is the slope of $\varphi|_\sigma$, then $(-n_\sigma, 1)$ is the corresponding ray in Σ_φ. Further, Γ acts on $N \oplus \mathbb{Z}$; indeed, $\gamma \in \Gamma$ acts by taking $(n,r) \mapsto (n - r \cdot d\alpha_\gamma, r)$, where $d\alpha_\gamma$ denotes the differential, or, slope, of α_γ. This action preserves Σ_φ.

The projection $N_{\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}$ defines a map $\pi : X_\varphi \to \mathbb{A}^1$. The fibres of this map are algebraic tori $(\mathbb{C}^*)^n$ except for $\pi^{-1}(0)$, which is an infinite union of proper toric varieties. Furthermore, the action of Γ preserves this map, and yields an action of Γ on the irreducible components of $\pi^{-1}(0)$.

While the Γ-action is global, it does not act properly discontinuously except on the subset $\pi^{-1}(D)$, where $D \subseteq \mathbb{A}^1$ is the unit disk. Thus we get a family

$$\pi : \pi^{-1}(D)/\Gamma \to D$$

whose general fibre is an abelian variety and such that the fibre over zero is a union of toric varieties.

We would actually prefer to work formally here, and instead consider

$$\mathcal{A} := (X_\Sigma \times_{\mathbb{A}^1} \operatorname{Spec} \mathbb{C}[\![t]\!])/\Gamma.$$

The quotient can be taken by dividing out the formal completion of X_φ along $\pi^{-1}(0)$ by the action of Γ, then showing that there is an ample line bundle on this quotient, and finally applying Grothendieck existence to get a scheme over $\operatorname{Spec} \mathbb{C}[\![t]\!]$. In fact, the existence of the ample line bundle will follow from the discussion below.

The family $\mathcal{A} \to \operatorname{Spec} \mathbb{C}[\![t]\!]$ is precisely the family produced by Theorem 1.5 from the data $B = M_{\mathbb{R}}/\Gamma$, polyhedral decomposition given by the image of $\mathscr{P}$ in B, and multi-valued piecewise linear function φ as given.

We now would like to understand traditional theta functions in this context. As already understood in [M72], one observes that the polyhedron Δ_φ induces a line bundle $\mathcal{L}$ on X_Σ, and the bundle $\mathcal{L}$ descends to the quotients

$$\mathcal{A}_k := (X_\Sigma \times_{\mathbb{A}^1} \operatorname{Spec} \mathbb{C}[t]/(t^{k+1}))/\Gamma$$

of the k-th order thickenings of the central fibre of $X_\Sigma \to \mathbb{A}^1$. To show $\mathcal{L}$ descends, one just needs to define an integral linear action of Γ on the cone $C(\Delta_\varphi) \subseteq M_{\mathbb{R}} \oplus \mathbb{R} \oplus \mathbb{R}$ defined as

$$C(\Delta_\varphi) = \overline{\{(\ell m, \ell r, \ell) \,|\, (m, r) \in \Delta_\varphi, \ell \in \mathbb{R}_{\geq 0}\}}.$$

Note taking the closure just adds $\{0\} \times \mathbb{R} \times \{0\}$ to the set. If c_γ is the constant part of α_γ and $d\alpha_\gamma$ the differential of α_γ, (or equivalently, $d\alpha_\gamma$ is the linear part of α_γ), one checks such an action is given by $\gamma \mapsto \psi_\gamma \in \operatorname{Aut}(M \oplus \mathbb{Z} \oplus \mathbb{Z})$ with

$$\psi_\gamma(m, r, \ell) = (m + \ell\gamma, (d\alpha_\gamma)(m) + \ell c_\gamma + r, \ell). \tag{2.1}$$

A basis of monomial sections of $\Gamma(X_\Sigma, \mathcal{L}^{\otimes \ell})$ is indexed by the set $C(\Delta_\varphi) \cap (M \times \mathbb{Z} \times \{\ell\})$: for p in this set, we write z^p for the corresponding section of $\mathcal{L}^{\otimes \ell}$. The above action on $C(\Delta_\varphi)$ lifts the Γ-action on X_Σ to a Γ-action on each $\mathcal{L}^{\otimes \ell}$. To write down sections of $\mathcal{L}^{\otimes \ell}$ on the quotient, one only need write down Γ-invariant sections of $\mathcal{L}^{\otimes \ell}$ on X_Σ, and this can be done by taking, for any $m \in \frac{1}{\ell}M$, the infinite sums

$$\vartheta_m = \vartheta_m^{[\ell]} := \sum_{\gamma \in \Gamma} z^{\psi_\gamma(\ell m, \ell\varphi(m), \ell)}.$$

We use the superscript $[\ell]$ to indicate the power of $\mathcal{L}$ when ambiguities can arise. We call ℓ the *level* of the theta function.

To see such an expression makes sense on the formal completion of the zero fibre of $X_\Sigma \to \mathbb{A}^1$, one focuses on an affine chart of X_Σ defined by a vertex $v = (m, \varphi(m))$ of Δ_φ: this affine chart is $\operatorname{Spec} \mathbb{C}[T_v\Delta_\varphi \cap (M \oplus \mathbb{Z})]$, where $T_v\Delta_\varphi$ denotes the tangent cone to Δ_φ at the vertex v. One trivializes the line bundle $\mathcal{L}^{\otimes \ell}$ in this chart using $z^{(\ell v, \ell)} \mapsto 1$, so that ϑ_m coincides with the regular function

$$\sum_{\gamma \in \Gamma} z^{\psi_\gamma(\ell m, \ell\varphi(m), \ell) - (\ell v, \ell)}.$$

Observe by the convexity of φ that with $t = z^{(0,1)} \in \mathbb{C}[T_v\Delta_\varphi \cap (M \oplus \mathbb{Z})]$, for any $k > 0$ all but a finite number of monomials in this sum lie in the ideal (t^{k+1}). Thus ϑ_m makes sense as a section of $\mathcal{L}^{\otimes \ell}$ on the k-th order thickening of the zero fibre of $X_\Sigma \to \mathbb{A}^1$, and since invariant under the Γ-action, descends to a section on $\mathcal{A}_k$.

Furthermore, one sees that $\vartheta_m = \vartheta_{m+\gamma}$, so if we set $B = M_{\mathbb{R}}/\Gamma$, we obtain a set of theta functions indexed by the points of $B(\frac{1}{\ell}\mathbb{Z})$. One can show the following facts:

(1) The functions ϑ_m extend as holomorphic functions to give the usual canonical theta functions on non-zero fibres of $\pi^{-1}(D)/\Gamma \to D$.
(2) The set $\{\vartheta_m \,|\, m \in B(\frac{1}{\ell}\mathbb{Z})\}$ form a basis for $\Gamma(\mathcal{A}, \mathcal{L}^{\otimes \ell})$ as a $\mathbb{C}[\![t]\!]$-module.
(3) Denote by $\mathscr{P}$ also the polyhedral decomposition of B induced by the Γ-periodic decomposition of $M_{\mathbb{R}}$. Assume no cells of $\mathscr{P}$ are self-intersecting: this is equivalent to all irreducible components of the central fibre $\mathcal{A}_0$ being normal. Each maximal cell $\sigma \in \mathscr{P}$ (thought of as a subset of B) then corresponds to an irreducible component of the central fibre $\mathcal{A}_0$ isomorphic to $\mathbb{P}_\sigma$, the projective toric variety determined by the lattice polytope σ. Then if $m \in \sigma$, the restriction of ϑ_m to $\mathbb{P}_\sigma$ is precisely the section of $\mathcal{O}_{\mathbb{P}_\sigma}(\ell) \cong \mathcal{L}^{\otimes \ell}|_{\mathbb{P}_\sigma}$ determined by $\ell m \in \ell\sigma$. If $m \notin \sigma$, then $\vartheta_m|_{\mathbb{P}_\sigma} = 0$. Thus ϑ_m can be viewed as a lifting of the natural monomial section of $\Gamma(\mathcal{A}_0, \mathcal{L}^{\otimes \ell}|_{\mathcal{A}_0})$ which is non-zero on those irreducible components indexed by $\sigma \in \mathscr{P}$ with $m \in \sigma$ and is given by $\ell m \in \ell\sigma$ on those components.

This construction is particularly easy to describe as there is a global description coming from the universal cover $M_{\mathbb{R}} \to B$. For more general B, we shall not have such a nice global construction, and as a consequence, it is beneficial to give here a more local description of theta functions.

We can in fact use the Γ-action (2.1) on $M \oplus \mathbb{Z} \oplus \mathbb{Z}$ to define a local system with fibres $M \oplus \mathbb{Z} \oplus \mathbb{Z}$ on B which we shall call $\widetilde{\mathcal{P}}$. The monodromy of the local system is given by (2.1); this uniquely determines the local system. One checks one has the following commutative diagram of local systems on B:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \underline{\mathbb{Z}} \arrow[r, "="] \arrow[d] & \underline{\mathbb{Z}} \arrow[d] & & \\
0 \arrow[r] & \mathcal{P} \arrow[r] \arrow[d, "r"] & \widetilde{\mathcal{P}} \arrow[r, "\deg"] \arrow[d, "\tilde{r}"] & \underline{\mathbb{Z}} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & \mathcal{A}\mathit{ff}(B, \mathbb{Z})^* \arrow[r] \arrow[d] & \underline{\mathbb{Z}} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{2.2}$$

Here $\underline{\mathbb{Z}}$ denotes the constant local system with stalks $\mathbb{Z}$, the map deg is induced by the projection $M \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ onto the last component, and $\mathcal{P}$ is defined to be the kernel of this map (hence has monodromy given by the restriction of the action (2.1) to the first two components $M \oplus \mathbb{Z}$). The inclusions of $\underline{\mathbb{Z}}$ in $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are induced by the inclusions $\mathbb{Z} \to M \oplus \mathbb{Z}, M \oplus \mathbb{Z} \oplus \mathbb{Z}$ into the second component. Here the quotient $\mathcal{P}/\underline{\mathbb{Z}}$ is just the constant sheaf $\underline{M}$, and since $M_{\mathbb{R}}$ is canonically the tangent space to any point of B, we identify $\underline{M}$ with Λ, the local system of integral vector fields on M. Finally, $\mathcal{A}\!f\!f(B, \mathbb{Z})$ denotes the local system of integral affine linear functions on B (functions with integral slope and integral constant part), and $\mathcal{A}\!f\!f(B, \mathbb{Z})^*$ denotes the dual local system. To see the identification of $\widetilde{\mathcal{P}}/\underline{\mathbb{Z}}$ with $\mathcal{A}\!f\!f(B, \mathbb{Z})^*$, write $\mathrm{Aff}(M, \mathbb{Z}) = N \oplus \mathbb{Z}$, with $(n, c) \in N \oplus \mathbb{Z}$ defining the affine linear map $m \mapsto \langle n, m\rangle + c$. Thus $M \oplus \mathbb{Z}$ is canonically $\mathrm{Aff}(M, \mathbb{Z})^*$. The action of Γ on $\mathrm{Aff}(M, \mathbb{Z})$ via pull-back of affine linear functions is given by $(n, c) \mapsto (n, c + \langle n, \gamma\rangle)$, and the transpose action on $\mathrm{Aff}(M, \mathbb{Z})^*$ is then precisely the restriction of (2.1) to the first and third components of $M \oplus \mathbb{Z} \oplus \mathbb{Z}$.

We can then describe a theta function as follows. Let $m \in B(\frac{1}{\ell}\mathbb{Z})$, and we want to describe ϑ_m. We previously described ϑ_m as a sum of monomials z^p with $p \in M \times \mathbb{Z} \times \{\ell\}$. Choose a point $x \in B$. We can identify $\widetilde{\mathcal{P}}_x$ with $M \oplus \mathbb{Z} \oplus \mathbb{Z}$ by choosing a lift $\tilde{x} \in M_{\mathbb{R}}$ of x, so we can identify ϑ_m with a sum of monomials z^p with $p \in \widetilde{\mathcal{P}}_x$. Note that the choice of lifting is irrelevant, as a different lifting gives an identification related by the transformation ψ_γ, and ϑ_m is invariant under the action of ψ_γ.

We can then write

$$\vartheta_m = \sum_{\delta} \mathrm{Mono}(\delta), \tag{2.3}$$

where we sum over all affine linear maps $\delta : [0, 1] \to B$ with the property that $\delta(0) = m$ and $\delta(1) = x$. We define $\mathrm{Mono}(\delta)$ as follows. We have a canonical element of the stalk of $\mathcal{A}\!f\!f(B, \mathbb{Z})^*$ at m given by $\ell \cdot \mathrm{ev}_m$, where ev_m denotes evaluation of integral affine functions at the point m. This element can then be lifted to the stalk of $\widetilde{\mathcal{P}}_m$ in a canonical way determined by φ: choosing a lifting $\tilde{m}$ of m to $M_{\mathbb{R}}$, we take $(\ell\tilde{m}, \ell\varphi(\tilde{m}), \ell) \in M \oplus \mathbb{Z} \oplus \mathbb{Z}$; this defines a well-defined element of $\widetilde{\mathcal{P}}_m$ independent of the choice of lift $\tilde{m}$. This element is indeed a lift of $\ell \cdot \mathrm{ev}_m$. We call this element $m_\varphi \in \widetilde{\mathcal{P}}_m$; note that by construction $\deg(m_\varphi) = \ell$. So far this is independent of the choice of δ. But now parallel transport m_φ along the path δ to get an element $m_\varphi^\delta \in \widetilde{\mathcal{P}}_x$, and define $\mathrm{Mono}(\delta) = z^{m_\varphi^\delta}$.

It is not hard to check that (2.3) then coincides with our original description of ϑ_m.

This does not represent anything radical: we are simply reinterpreting the action of Γ which led to theta functions in terms of the fundamental group of B in the guise of different choices of paths between m and x (there is one such linear path for every choice of lift $\tilde{x}$ of x to $M_{\mathbb{R}}$). However, now the description of theta functions will generalize nicely. In particular, we can see the connection between theta functions and homological mirror symmetry in a more direct manner.

To see this, we define a map

$$\mathbf{vect} : \mathcal{A}\!f\!f(B,\mathbb{Z})^* \to \mathcal{T}_B$$

as follows. An element of $\mathcal{A}\!f\!f(B,\mathbb{Z})^*_x$ is an integral linear functional on the vector space $\mathcal{A}\!f\!f(B,\mathbb{R})_x$ of germs of affine linear functions at x (with no integrality restriction). Restricting to the subspace of functions which vanish at x, one obtains a derivation, yielding a tangent vector at x. This defines the map.

For example, at $m \in B(\frac{1}{\ell}\mathbb{Z})$, $\mathbf{vect}(\ell\cdot \mathrm{ev}_m) = 0$ in $\mathcal{T}_{B,m}$, simply because ev_m evaluates functions at m. However, if $\delta : [0,1] \to B$ is a path with $\delta(0) = m$, let $\tilde{\delta} : [0,1] \to M_{\mathbb{R}}$ be a lifting with $\tilde{\delta}(0) = \tilde{m}$. Let $\alpha(t)$ denote the parallel transport of $\ell\cdot \mathrm{ev}_m$ along δ to $\mathcal{A}\!f\!f(B,\mathbb{Z})^*_{\delta(t)}$. Define

$$\mathbf{v}(t) = \mathbf{vect}(\alpha(t)). \tag{2.4}$$

Then one calculates that $\mathbf{v}(t)$ is the tangent vector $\ell(\tilde{m}-\tilde{\delta}(t))$. In particular, provided δ is in fact linear, **vect** applied to the parallel transport of $\ell\cdot \mathrm{ev}_m$ provides a vector field along δ which is always tangent to the path δ, always points towards the initial point of the path, and increases in length as we move away from the initial point at a rate proportional to ℓ.

The vector field $\mathbf{v}(t)$ gives rise to a holomorphic triangle in $X(B)$ via

$$\begin{aligned}\psi : [0,1]\times[0,1] &\to X(B)\\ (t,s) &\mapsto s\cdot\mathbf{v}(t) \in \mathcal{T}_{B,\delta(t)} \mod \Lambda_{\delta(t)}.\end{aligned}$$

Note this map contracts the edge of the square $\{0\}\times[0,1]$, giving the triangle. This triangle is depicted in Figure 2. Here L_x is the fibre $\mathcal{T}_{B,x}/\Lambda_x$ of the SYZ fibration $X(B)\to B$. This triangle can be seen as a contribution to the Floer multiplication

$$\mu_2 : \mathrm{Hom}(L_\ell, L_x)\times \mathrm{Hom}(L_0,L_\ell) \to \mathrm{Hom}(L_0,L_x).$$

In particular, it yields a contribution to the product of $p\in L_0\cap L_\ell$, corresponding to the point $m\in B(\frac{1}{\ell}\mathbb{Z})$, with the unique point of $L_\ell\cap L_x$.

This can be interpreted on the mirror side using homological mirror symmetry, where we assume L_ℓ corresponds to $\mathcal{L}^{\otimes\ell}$ and L_x to the structure sheaf of a point, as the composition map

$$\mathrm{Hom}(\mathcal{L}^{\otimes\ell},\mathcal{O}_x)\otimes\mathrm{Hom}(\mathcal{O}_{\mathcal{A}},\mathcal{L}^{\otimes\ell})\to\mathrm{Hom}(\mathcal{O}_{\mathcal{A}},\mathcal{O}_x).$$

Here m determines the theta function $\vartheta_m\in\mathrm{Hom}(\mathcal{O}_{\mathcal{A}},\mathcal{L}^{\otimes\ell})$ and a non-zero element of $\mathrm{Hom}(\mathcal{L}^{\otimes\ell},\mathcal{O}_x)$ can be interpreted as specifying an identification $\mathcal{L}^{\otimes\ell}\otimes\mathcal{O}_x\cong\mathcal{O}_x$. The composition of ϑ_m with this identification can then be viewed as specifying the value of the section ϑ_m at the point x. Thus the description of ϑ_m as a sum over paths δ then corresponds, naturally, via this association of triangles to paths δ, to the Floer theoretic description on the mirror side.

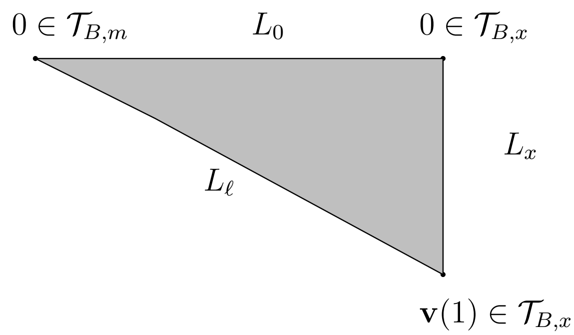

Figure 2.

It is also important to describe multiplication of theta functions. Indeed, this allows us to describe the homogeneous coordinate ring $\bigoplus_{\ell \geq 0} H^0(\mathcal{A}, \mathcal{L}^{\otimes \ell})$. Given $m_i \in B(\frac{1}{\ell_i}\mathbb{Z})$, $i = 1, 2$, we wish to describe the coefficients of the expansion

$$\vartheta_{m_1} \cdot \vartheta_{m_2} = \sum_{m \in B(\frac{1}{\ell_1+\ell_2}\mathbb{Z})} c_{m_1,m_2,m} \vartheta_m. \tag{2.5}$$

It is not difficult to see that the coefficients are given by

$$c_{m_1,m_2,m} = \sum_{\delta_1,\delta_2} t^{c(\delta_1,\delta_2)}$$

where we sum over all straight lines $\delta_1, \delta_2 : [0,1] \to B$ connecting m_1, m_2 to m respectively, with the property that, if $\mathbf{v}_1$, $\mathbf{v}_2$ are defined by (2.4) using δ_1, δ_2 respectively, then $\mathbf{v}_1(1) + \mathbf{v}_2(1) = 0$. We leave it to the reader to determine the exponent $c(\delta_1, \delta_2) \in \mathbb{N}$, depending on δ_1, δ_2: see [DBr], page 625, in the case of the elliptic curve.

Again, this description of multiplication can be interpreted in terms of Floer homology. Each pair δ_1, δ_2 contributing to the sum gives rise to a triangle as depicted in Figure 3. Here, the triangle is a union of two triangles in $X(B)$, fibering over δ_1, δ_2, as depicted in that figure. The triangle on the left is determined by δ_1 as before, while the triangle on the right is the image of the map

$$\begin{aligned} \psi : [0,1] \times [0,1] &\to X(B) \\ (t, s) &\mapsto \sigma_{\ell_1}(\delta_2(t)) + s\mathbf{v}_2(t), \end{aligned}$$

where σ_ℓ is given by (1.1). The fact that these two triangles match up along the dotted line, which lies over m, is just the statement that $\mathbf{v}_1(1) + \mathbf{v}_2(1) = 0$. The number $c(\delta_1, \delta_2)$ can be seen to be related (but not equal to) the symplectic area of this triangle, again see [DBr], pp. 626–628.

The multiplication formula (2.5) can be viewed as a kind of global generalization of a much simpler rule for multiplying sections of powers of a given ample line bundle on a toric variety. Indeed, such a line bundle $\mathcal{L}$ determines a lattice polytope $B \subset \mathbb{R}^n$, and the points

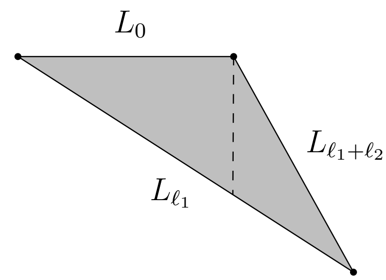

FIGURE 3.

of $B(\frac{1}{\ell}\mathbb{Z})$ correspond to a monomial basis for the global sections of $\mathcal{L}^{\otimes \ell}$. The product of the sections corresponding to $m_i \in B(\frac{1}{\ell_i}\mathbb{Z})$, $i = 1, 2$, is just the section corresponding to the weighted average $m = (\ell_1 m_1 + \ell_2 m_2)/(\ell_1 + \ell_2) \in B(\frac{1}{\ell_1+\ell_2}\mathbb{Z})$. This can be interpreted in terms of paths δ_1, δ_2 joining m_1 and m_2 to m, as in (2.5).

3. SINGULARITIES, THETA FUNCTIONS, JAGGED PATHS

We now would like to generalize these constructions to affine manifolds with singularities, as is necessary if we are to obtain any interesting examples. We will begin with some simple examples to provide guidance. In particular, we will take B to be a compact affine manifold with boundary, analogous to the very simple case where B is just a lattice polytope, but allow a few simple singularities to appear in B. We always assume B is locally convex along ∂B.

3.1. **The basic example.** We will revisit some examples introduced in [GSInv]. For now, we consider the simplest example, the affine manifold B_1 given in Figure 4. The points of $B_1(\mathbb{Z})$ are labelled in the diagram as X, Y, Z and W. The affine structure has one singularity, the point P, and the point P can be chosen freely within the line segment joining W and Z. The piecewise linear function φ takes the value 0 at X, W and Z and 1 at Y.

If we were in the purely toric case, say B_1 being either the polygon pictured on the left or the right in Figure 4, then each integral point would represent a purely monomial section of the line bundle on the toric variety corresponding to B_1. Further, the multiplication law for monomials would be either $XY = Z^2$ (on the left) or $XY = WZ$ (on the right). Each such product can be viewed by giving paths δ_1, δ_2 with initial endpoints X and Y respectively and terminating at either the point Z or the point $\frac{1}{2}(W + Z)$ (viewing these points as elements of $\mathbb{R}^2$ rather than as variables). If we also took into account the polyhedral decomposition and the choice of φ, one obtains a degeneration of one of these two toric varieties into a union of two planes, given by the equation $XY = tZ^2$ or $XY = tWZ$ in the two cases, where t is the deformation parameter.

However, we are not in the purely toric case, and if we follow the philosophy of the previous section, the description of the sections specified by the points of $B_1(\mathbb{Z})$ and their multiplication rule should be determined by drawing straight lines. Let us first consider heuristically how lines should contribute to the description of sections, and then consider how we should think of the product rule.

First looking at the points labelled W or Z, we note that given any reference point $x \in B_1 \setminus \{P\}$, there is a unique line segment joining W or Z to x. In analogy with the abelian variety case, we would expect this to tell us that the corresponding sections are still represented by monomials at x. Next, consider the point X. If x is contained in σ_1, then there is again a unique line segment joining X and x, so we expect a monomial representative for this section. On the other hand, if x lies in σ_2, we may have either one or two straight lines, depending on the precise location of x: In Figure 5, there is a line joining X and x in the first chart but not in the second, as the line drawn in the right-hand chart crosses the cut. On the other hand, in Figure 6, there are in fact two distinct line segments joining X and x.

One solution to this ambiguity is to simply include in the sum a contribution from the line segment in the right-hand chart of Figure 5. However, this is not a straight line: if drawn in the correct chart, it becomes bent, as depicted in Figure 7.

How do we justify counting this bent line, or as we shall call it, *jagged path*? The explanation is that when we introduce singularities, as explained in [GSInv], we need to introduce walls emanating from the singularity, in this case rays heading in the direction of the points W and Z. Each ray has a function attached to it; the precise role that this function plays will be explained later. But the essential point is that we no longer need to use straight lines to join X and x. We will allow our lines to bend in specified ways when lines cross walls. In the case of B_1, this bending exactly accounts for the jagged path in Figure 7. As a consequence, we should expect the section corresponding to X will be represented as a sum of two monomials in any event in a chart corresponding to the right-hand side of B_1, regardless of the position of x.

As we have not yet been very clear what these charts mean and how we are representing sections in general, it is perhaps more informative to loook at the product XY. This product should be given as a sum over all suitable choices of paths δ_1, δ_2. To realise this, let us first assume the point P lies below $\frac{1}{2}(W+Z)$. Then Figure 8 shows two possible choices of the pairs δ_1, δ_2, and the product XY should be determined as a sum over these two ways of averaging X and Y: the presence of the singularity has created this ambiguity. Taking the PL function φ into account, the suggestion then is that we should have the multiplication rule

$$XY = t(Z^2 + WZ).$$

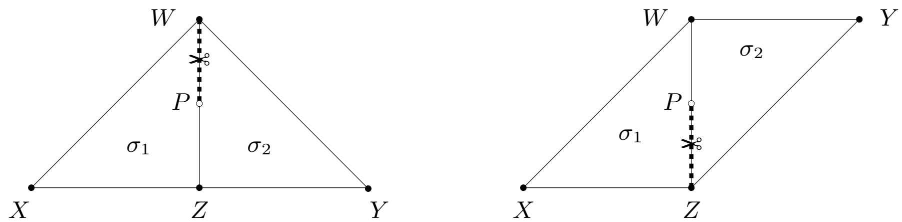

FIGURE 4. The affine manifold B_1. The diagram shows the affine embeddings of two charts, obtained by cutting the union of two triangles as indicated in two different ways. Each triangle is a standard simplex.

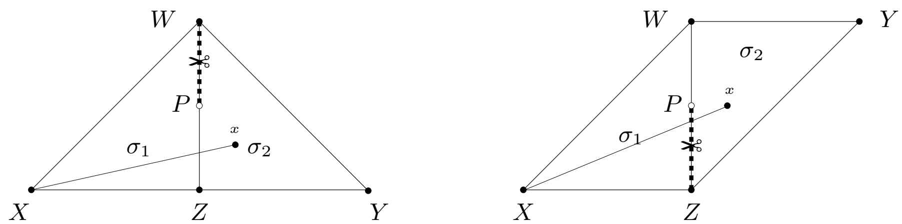

FIGURE 5.

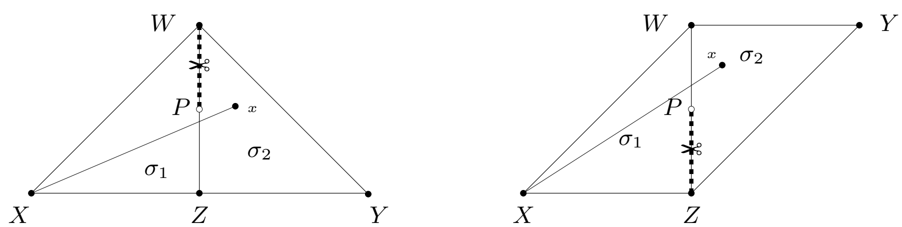

FIGURE 6.

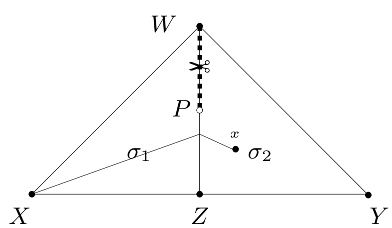

FIGURE 7.

Note that this gives a family over $\operatorname{Spec}\mathbb{C}[t]$ whose fibre over $t = 0$ is a union of two $\mathbb{P}^2$'s in $\mathbb{P}^3$ (determined by the two standard simplices σ_1 and σ_2), and for general t, we obtain a non-singular quadric surface in $\mathbb{P}^3$.

Now this argument depended on the fact that the point P was chosen below the half-integral point $\frac{1}{2}(W + Z)$. Since there is no sense that this singular point has a natural location, this is not particularly satisfactory. If we move P above this point, the second choice of δ_1, δ_2 seems to disappear.

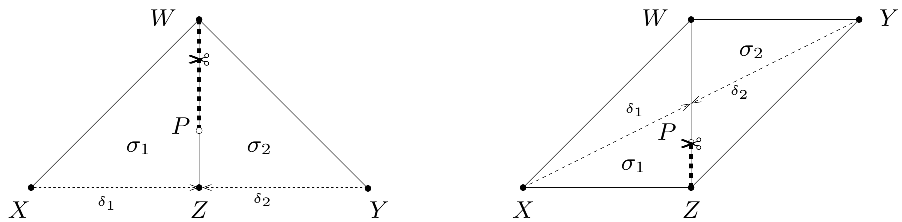

FIGURE 8.

The solution again is to use the walls to provide corrections. If P lies above the point $\frac{1}{2}(W+Z)$, then we will again obtain two contributions as depicted in Figure 9, where this time a piece of the wall emanating from P is used to correct for the fact that for the labelled δ_1, δ_2, we do not have $\mathbf{v}_1(1)+\mathbf{v}_2(1)=0$ (where $\mathbf{v}_i$ is defined using (2.4)). Rather, we have $\mathbf{v}_1(1)+\mathbf{v}_2(1)+w=0$, where w is the unit tangent vector pointing from P to W.

These pictures can be justified heuristically in terms of holomorphic disks contributing to Floer multiplication. If we think of a space X_1 fibering in tori over B_1, there is a singular fibre over P, a two-torus with a circle pinched to a point. We expect that X_1 will contain holomorphic disks fibering over the two walls emanating from P. In particular, for any point y in $B_1 \setminus \{P\}$ on the line segment $\overline{WZ}$, there is a holomorphic disk in X_1 with boundary contained in the fibre over y. We can use this to build a piecewise linear disk as follows.

Let $\delta : [0,1] \to B_1$ be the parameterized jagged path of Figure 7, bending at time $t_0 \in (0,1)$. Modify the definition of $\mathbf{v} : [0,1] \to \mathcal{T}_{B_1}$ defined using (2.4) by taking it to coincide with $\mathbf{v}$ of (2.4) for $0 \le t < t_0$ and with $\mathbf{v}+w$ for $t_0 \le t \le 1$. One checks that $\mathbf{v}(t)$ is always tangent to δ.

As in §2, we use $\mathbf{v}$ to define a polygon in $\mathcal{T}_{B_1}$, but because of the discontinuity in $\mathbf{v}$ we obtain a picture as in Figure 10. Recall that $X(B_1\setminus\{P\})$ is obtained by dividing the tangent spaces of $B_1 \setminus \{P\}$ by integral vector fields. If $\mathbf{w}_\pm = \lim_{t\to t_0^\pm} \mathbf{v}(t)$, then $\mathbf{w}_+ - \mathbf{w}_- = w$, so in fact $\mathbf{w}_+$ and $\mathbf{w}_-$ are identified in $X(B_1 \setminus \{P\})$. So the line segment joining $\mathbf{w}_+$ and $\mathbf{w}_-$ becomes a loop. We can then glue in a holomorphic disk emanating from P, attaching its boundary to the loop. This gives the triangle which is, roughly speaking, the contribution to Floer multiplication describing the section X.

A similar picture explains the contributions to the product XY: the failure of $\mathbf{v}_1(1)+\mathbf{v}_2(1)=0$ is dealt with by gluing in the holomorphic disk.

This is just a heuristic: these disks are not actual holomorphic disks. However, a variant of this example is considered in great detail in [P11] and the result here agrees with the actual result from Floer multiplication. The advantage for us is that we can describe everything combinatorially.

Before looking at some more complex examples, let us give a more precise description of what we are doing. Unfortunately, doing so requires a number of technical details; we

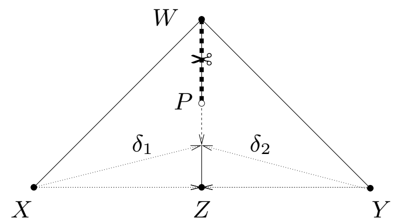

Figure 9.

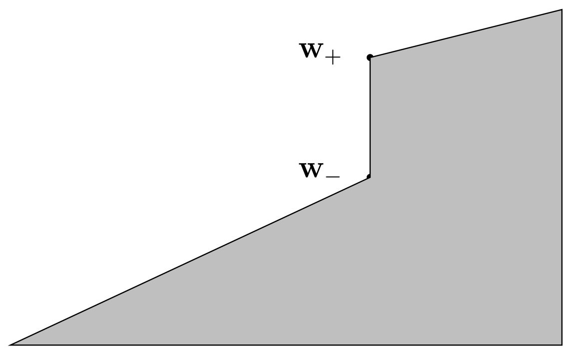

Figure 10.

shall try to avoid the most unpleasant aspects, but the reader should be advised in what follows that the definitions are only approximately correct!

3.2. **Structures and jagged paths.** Before we get into details, suppose we are given data $(B, \mathscr{P}, \varphi)$, where B is an integral affine manifold with singularities, $\mathscr{P}$ is a polyhedral decomposition of B, and φ is an integral multi-valued PL function on B. We then obtain a generalization of the diagram (2.2) of sheaves on B_0. This is a direct generalization of the torus case. Saying that φ is a multi-valued PL function on B means that there is an open cover $\{U_i\}$ of B_0 such that φ is represented by a single-valued function φ_i on U_i, with $\varphi_i - \varphi_j$ affine linear on $U_i \cap U_j$. We then construct the local system $\widetilde{\mathcal{P}}$ on B_0 as follows. On U_i, $\widetilde{\mathcal{P}}$ is isomorphic to $\underline{\mathbb{Z}} \oplus \mathcal{A}\!f\!f(B_0, \mathbb{Z})^*|_{U_i}$, and on $U_i \cap U_j$, $\underline{\mathbb{Z}} \oplus \mathcal{A}\!f\!f(B_0, \mathbb{Z})^*|_{U_i}$ is identified with $\underline{\mathbb{Z}} \oplus \mathcal{A}\!f\!f(B_0, \mathbb{Z})^*|_{U_j}$ via

$$(\ell, \alpha) \mapsto (\ell + \alpha(\varphi_j - \varphi_i), \alpha)$$

for $\alpha \in \Gamma(U_i \cap U_j, \mathcal{A}\!f\!f(B_0, \mathbb{Z})^*)$ and $\ell \in \mathbb{Z}$. We then have the projection map $\tilde{r} : \widetilde{\mathcal{P}} \to \mathcal{A}\!f\!f(B_0, \mathbb{Z})^*$, and dualizing the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{A}\!f\!f(B_0, \mathbb{Z}) \to \check{\Lambda} \to 0$$

(with the third arrow given by exterior derivative) gives the bottom row of (2.2). From this follows the whole diagram, with $\mathcal{P}$ defined as the kernel of the map deg.

In [GS11], we gave the definition of a *structure* for an integral affine manifold with singularities $(B, \mathscr{P})$. Structures were used for the proof of Theorem 1.5 to encode the explicit data necessary to describe the smoothing. For B of arbitrary dimension, a structure $\mathfrak{D}$ is a collection of *slabs* and *walls*. These are codimension 1 polyhedra in B which are either contained in codimension one cells of $\mathscr{P}$ (slabs) or contained in maximal cells of $\mathscr{P}$ but not contained in codimension one cells (walls). Slabs and walls carry additional data, certain formal power series which are used to describe gluing automorphisms. In order for the gluing to be well-defined, a structure must satisfy the notion of *compatibility*. Producing a compatible structure is the main work of [GS11]. This procedure is described at greater length in [GSInv] and is covered in full detail in [Gr11], Chapter 6 in the two-dimensional case. Slabs and walls need to be treated somewhat differently in the algorithm of [GS11] for producing compatible structures for technical reasons, but in the context here we will essentially be able to ignore these issues.

A number of details of this construction are surveyed in [GSInv]. The crucial points to know are the following:

(1) The deformation $\check{\mathcal{X}} \to \operatorname{Spec}\mathbb{C}[\![t]\!]$ is given order-by-order, with $\check{\mathcal{X}}_k := \check{\mathcal{X}} \times_{\mathbb{C}[\![t]\!]} \mathbb{C}[t]/(t^{k+1})$ constructed explicitly from a structure.

(2) $\check{\mathcal{X}}_k$ is a thickening of $\check{\mathcal{X}}_0$, and hence has the same set of irreducible components, indexed by $\sigma \in \mathscr{P}_{\max}$. The function φ determines some standard toric thickenings of affine open subsets of $\mathbb{P}_\sigma$ for $\sigma \in \mathscr{P}_{\max}$. Then $\check{\mathcal{X}}_k$ is obtained by using the structure to glue together these standard pieces in non-toric ways.

(3) If $x \in B_0$ there is a monoid $P_x \subseteq \mathcal{P}_x$, defined using φ, along with an inclusion $\mathbb{N} \to P_x$, yielding a family $\pi_x : \operatorname{Spec}\mathbb{C}[P_x] \to \operatorname{Spec}\mathbb{C}[t]$. This provides a local model for the smoothing, as follows. Let $\tau \in \mathscr{P}$ be the smallest cell containing x. There is a one-to-one correspondence between cells $\sigma \in \mathscr{P}_{\max}$ containing x and irreducible components of $\pi_x^{-1}(0)$. Furthermore, the irreducible component corresponding to σ is isomorphic to the affine open subset of $\mathbb{P}_\sigma$ determined by the face τ of σ. Then the corresponding irreducible component of $\operatorname{Spec}\mathbb{C}[P_x] \times_{\mathbb{A}^1} \operatorname{Spec}\mathbb{C}[t]/(t^{k+1})$ is the standard toric thickening of this affine open subset of $\mathbb{P}_\sigma$.

(4) In [GHKSΘ], we will explain how the line bundle $\mathcal{L}^{\otimes d}$ is obtained by gluing standard line bundles on these standard pieces, again with gluing dictated by the structure. For any given ℓ, there is a P_x-torsor $Q_x^\ell \subseteq \widetilde{\mathcal{P}}_x \cap \deg^{-1}(\ell)$ defining a line bundle on $\operatorname{Spec}\mathbb{C}[P_x]$; its restriction to the irreducible components of $\operatorname{Spec}\mathbb{C}[P_x] \times_{\mathbb{A}^1} \operatorname{Spec}\mathbb{C}[t]/(t^{k+1})$ describes these standard line bundles.

The definition of the monoid P_x is discussed in [GSInv], §3.1; the details will not be so important here. The P_x-torsor Q_x^ℓ has not yet been discussed in the literature.

As most of the conceptual issues are already present in the two-dimensional case, instead of using walls and slabs, we can use rays, much as [KS06] had done. For precise details of what follows, see [Gr11], Chapter 6.

Assume B is two-dimensional. Roughly, a ray consists of:

(1) a parameterized ray or line segment $\mathfrak{d} : [0,\infty) \to B$ or $\mathfrak{d} : [0,1] \to B$, with image a straight line of rational slope. A ray will continue until it hits a singularity or the boundary of B; otherwise it continues indefinitely.
(2) A formal power series

$$f_{\mathfrak{d}} = 1 + \sum_p c_p z^p$$

where p runs over the set of global sections of $\mathfrak{d}^{-1}\mathcal{P}$ with the property that $r(p)$ is always tangent to the image of $\mathfrak{d}$, *negatively* proportional to the derivative $\mathfrak{d}'$. There are more constraints on $f_{\mathfrak{d}}$ necessary to guarantee convergence of the gluing construction considered below, but we won't worry about these technical details as all examples considered here will be quite simple.

A *structure* $\mathfrak{D}$ is then a collection of rays $\{(\mathfrak{d}, f_{\mathfrak{d}})\}$.

Example 3.1. In the basic example of §3.1, the relevant structure consists of two rays: $\mathfrak{d}_1$, a line segment from P to W, and $\mathfrak{d}_2$, a line segment from P to Z. Since φ is single-valued, we can write $\mathcal{P} = \underline{\mathbb{Z}} \oplus \Lambda$, and we take

$$f_{\mathfrak{d}_1} = 1 + z, \quad f_{\mathfrak{d}_2} = 1 + w,$$

where z and w are the monomials corresponding to $(0,(0,-1))$ and $(0,(0,1))$ respectively. Note applying r to these two elements gives the primitive tangent vectors to the segment $\overline{ZW}$ pointing towards Z and W respectively. We shall by abuse of notation refer to these tangent vectors as z and w also. □

Note that a point $m \in B(\frac{1}{\ell}\mathbb{Z})$ defines an element $\ell \cdot \mathrm{ev}_m$ of $\mathcal{A}\!f\!f(B_0,\mathbb{Z})^*_m$ as in §2. In addition, choosing a local representative φ_m for φ in a neighbourhood of m gives a splitting $\widetilde{\mathcal{P}} = \underline{\mathbb{Z}} \oplus \mathcal{A}\!f\!f(B_0,\mathbb{Z})^*$ in a neighbourhood of m, and we define an element $m_\varphi \in \widetilde{\mathcal{P}}_m$ by

$$m_\varphi = (\ell \cdot \varphi_m(m), \ell \cdot \mathrm{ev}_m) \in \mathbb{Z} \oplus \mathcal{A}\!f\!f(B_0,\mathbb{Z})^*_m.$$

One notes this is independent of the choice of φ_m. In fact, m_φ lies in the P_m-torsor Q^ℓ_m, and hence defines a local section of the line bundle $\mathcal{L}^{\otimes \ell}$.

For example, in the case of $B = B_1$, the points X, Y, Z and W define elements of the stalks $\widetilde{\mathcal{P}}_X, \ldots, \widetilde{\mathcal{P}}_W$ of degree 1. However, as parallel transport of X and Y is not well-defined because of monodromy of the local system $\widetilde{\mathcal{P}}$ around P, we cannot directly view these as defining global sections of $\mathcal{L}$.

To do so requires the precise notion of jagged path. This definition will appear in [GHKSΘ].

Definition 3.2. A *jagged path in B with respect to a structure* $\mathfrak{D}$ consists of the following data:

(a) A continuous piecewise linear path $\gamma : [0,1] \to B$.

(b) For every maximal domain of linearity $L \subseteq [0,1]$ of γ we are given a monomial

$$m_L = c_L z^{q_L} \in \mathbb{C}[\Gamma(L, \gamma^{-1}(\widetilde{\mathcal{P}})|_L)]$$

satisfying the following two properties:

(1) If $t \in (0,1)$ is a point contained in the interior of L a maximal domain of linearity, then $\gamma'(t)$ is negatively proportional to $\mathbf{vect}(\tilde{r}(q_L))$, where $\tilde{r}$ is defined in (2.2).

(2) Let $t \in (0,1)$ be a point at which γ is not affine linear, passing from a domain of linearity L to L', with $y = \gamma(t)$. Let $\{(\mathfrak{d}_j, x_j)\}$ be the set of pairs $\mathfrak{d}_j \in \mathfrak{D}$, x_j in the domain of $\mathfrak{d}_j$ such that $\mathfrak{d}_j(x_j) = y$. Let $n_{\mathfrak{d}_j} \in \mathcal{A}\mathit{ff}(B_0, \mathbb{Z})_y$ be the germ of a primitive integral affine linear function which vanishes on the image of $\mathfrak{d}_j$ near y and is positive on the image of L near y. We assume that $n = n_{\mathfrak{d}_j}$ can be chosen independently of j; this is an assumption on the genericity of γ and can always be achieved by perturbing the endpoint $\gamma(1)$. Expand

$$\prod_j f_{\mathfrak{d}_j}^{\langle n, \tilde{r}(q_L)\rangle} \tag{3.1}$$

as a sum of monomials with distinct exponents. Note each monomial can be viewed as an element of $\mathbb{C}[\mathcal{P}_y] \subseteq \mathbb{C}[\widetilde{\mathcal{P}}_y]$.

Then there is a term cz^q in this expansion with

$$m_{L'} = m_L \cdot (cz^q) = c_L c z^{q_L + q}.$$

There are several important features of this definition. First, item (1) says the monomial attached to the line segment always tells us the direction of travel of the line segment. Since (2) tells us how these monomials change at bends, it also tells us precisely how jagged paths bend.

Second, the exponents in (3.1) are by construction always non-negative. This is important as $f_{\mathfrak{d}_j}$ need not be invertible in the relevant rings. In fact, more naive approaches to writing down sections of $\mathcal{L}$ founder on precisely this point.

Definition 3.3. For $m \in B(\frac{1}{\ell}\mathbb{Z})$ and $x \in B_0$ general, a *jagged path* from m to x is a jagged path satisfying

(1) $\gamma(0) = m$;
(2) $\gamma(1) = x$;
(3) If L is the first domain of linearity of γ, then $m_L = z^{m_\varphi}$.

Example 3.4. Let's examine this in detail with $B = B_1$, $\mathfrak{D}$ as in Example 3.1. Take $m = X \in B_1(\mathbb{Z})$. If $x \in \mathrm{Int}(\sigma_1)$, there is one jagged path from X to x, which just serves to parallel transport m_φ to x.

On the other hand, to be explicit, let's take x to be the point with coordinates $(1/8, 1/4)$ in σ_2, in the left-hand chart of Figure 4, assuming Z, Y and W have coordinates $(0,0)$,

$(1,0)$ and $(0,1)$ respectively. Suppose a jagged path from X to x crosses the segment $\overline{WZ}$ below P (which we will take to lie at $(0,1/2)$ to be explicit). Then we are crossing the ray $\mathfrak{d}_2$, and we take $n_{\mathfrak{d}_2}$ to be the linear function $(a,b)\mapsto -a$. Then $\tilde{r}(m_\varphi)=\mathrm{ev}_X$ takes the value 1 on $n_{\mathfrak{d}_2}$, precisely because $n_{\mathfrak{d}_2}$ takes the value 1 at X. Thus (3.1) is just given by $1+w$. So we can take $cz^q=1$, in which case there is no change to the monomial and no bend in the jagged path; this just yields the parallel transport of X into σ_2. Otherwise, we take $cz^q=w$. This replaces X with Xw, and changes the direction by noting that $\mathbf{vect}(\mathrm{ev}_X+w)=\mathbf{vect}(\mathrm{ev}_X)+w$. Recall here that we are using the same notation w for the tangent vector and corresponding monomial. In particular, at the point of intersection of the segment $\overline{WZ}$ with the jagged path, say at $(0,h)$, we have

$$\mathbf{vect}(\mathrm{ev}_X)=(-1,0)-(0,h)=(-1,-h),$$

so the new direction is $-(-1,-h+1)$. In order for this path to then pass through x, we need to take $h=1/3$.

A bit of experimentation shows that as we move the point x around inside σ_2, we always have *precisely two* jagged paths from X to x, and the sum of the final attached monomials is independent of the location of x. However, it is possible that both such jagged paths are in fact straight, when viewed in the correct charts, as happens in Figure 6. To describe this sum, let us continue to denote by X the monomial parallel transported from σ_1 into σ_2 below P. (Parallel transport is carried out in the local system $\widetilde{\mathcal{P}}$). Then the sum is $X+Xw$.

It is this invariance which is the crucial property of jagged paths. To get our hands on this invariance, we use:

Definition 3.5. Let $m\in B(\frac{1}{\ell}\mathbb{Z})$. For a jagged path γ from m to x, let $\mathrm{Mono}(\gamma)\in\mathbb{C}[\widetilde{\mathcal{P}}_x]$ be the monomial attached to the final domain of linearity of γ. Let

$$\mathrm{Lift}_x(m):=\sum_\gamma \mathrm{Mono}(\gamma)$$

be the sum over all distinct jagged paths from m to x. Here we view two jagged paths to be the same if they just differ by a reparametrization of their domains.

Given a path γ in B_0 connecting two points x_1 and x_2, we can define a transformation $\theta_{\gamma,\mathfrak{D}}$ which, roughly speaking, is a map $\mathbb{C}[\widetilde{\mathcal{P}}_{x_1}]\to\mathbb{C}[\widetilde{\mathcal{P}}_{x_2}]$ which is given by parallel transport and a composition of wall-crossing automorphisms: if we cross a ray $\mathfrak{d}$ at time t, we apply the transformation

$$z^q\mapsto z^q f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}},\tilde{r}(q)\rangle}$$

for $q\in\widetilde{\mathcal{P}}_{\gamma(t)}$, $n_{\mathfrak{d}}$ a primitive integral affine linear function vanishing along the image of $\mathfrak{d}$ near $\gamma(t)$ and positive on $\gamma(t-\epsilon)$.

Definition 3.6. A structure $\mathfrak{D}$ is *consistent* if for any path γ from x_1 to x_2 general points in B_0, $m \in B(\frac{1}{\ell}\mathbb{Z})$,

$$\mathrm{Lift}_{x_2}(m) = \theta_{\gamma,\mathfrak{D}}(\mathrm{Lift}_{x_1}(m)).$$

In particular if γ does not cross any ray of $\mathfrak{D}$, then $\mathrm{Lift}_x(m)$ is invariant under parallel transport.

If we have a consistent structure $\mathfrak{D}$, we can then use the lifts $\mathrm{Lift}_x(m)$ to define a global section ϑ_m of $\mathcal{L}^{\otimes \ell}$. We can write down well-defined local descriptions of the section on the various affine pieces of the irreducible components of $\check{\mathcal{X}}_k$, and consistency then guarantees that these local descriptions glue.

A main result of [GHKSΘ] is then:

Theorem 3.7. (1) *The compatible structures constructed in* [GS11] *are in fact consistent.*

(2) *Given a compatible and consistent structure, giving a formal degeneration $\mathfrak{X} \to \mathrm{Spf}\,\mathbb{C}[\![t]\!]$, the above construction gives for every $m \in B(\frac{1}{\ell}\mathbb{Z})$ a section $\vartheta_m^{[\ell]}$ of the line bundle $\mathcal{L}^{\otimes \ell}$. This section has the property that for any $\sigma \in \mathscr{P}_{\max}$, $\vartheta_m^{[\ell]}|_{\mathbb{P}_\sigma}$ is 0 if $m \notin \sigma$ and otherwise coincides with the monomial section of $\mathcal{O}_{\mathbb{P}_\sigma}(\ell)$ defined by m.*

These are our theta functions. We call $\vartheta_m^{[\ell]}$ a theta function of *level* ℓ. Keeping in mind that m can lie in $B(\frac{1}{\ell}\mathbb{Z})$ for various ℓ, $\vartheta_m^{[\ell]}$ may depend on the level. However, we will write ϑ_m when not ambiguous.

The proof of this result is a fairly straightforward extension of arguments given in [GHKI] and [CPS].

We can also use jagged paths to describe multiplication. In general, for $m_1 \in B(\frac{1}{\ell_1}\mathbb{Z})$, $m_2 \in B(\frac{1}{\ell_2}\mathbb{Z})$, we should have a multiplication rule

$$\vartheta_{m_1}^{[\ell_1]} \cdot \vartheta_{m_2}^{[\ell_2]} = \sum_{m \in B(\frac{1}{\ell_1+\ell_2}\mathbb{Z})} c_{m_1,m_2,m} \vartheta_m^{[\ell_1+\ell_2]}. \tag{3.2}$$

As before, the coefficient should be determined as a sum over pairs of jagged paths δ_1, δ_2, with δ_i a jagged path from m_i to m, and with balancing $\mathbf{v}_1(1)+\mathbf{v}_2(1) = 0$. There is a slight subtlety which we have already seen in §3.1: since m is not free to be chosen generally, it may lie on a ray of $\mathfrak{D}$. Indeed, this happens in §3.1. So it is possible that balancing fails, and this is corrected by using a contribution from the ray. Essentially, this can be accomplished by perturbing the point m a little bit, so that one of δ_1 or δ_2 has a chance to have an addditional bend along that ray. This needs to be done with a bit of care, so we omit the details of this.

3.3. **Additional examples.** In [GSInv], we considered a number of other two- and three-dimensional examples. Here, we will describe their homogeneous coordinate rings in terms of jagged paths.

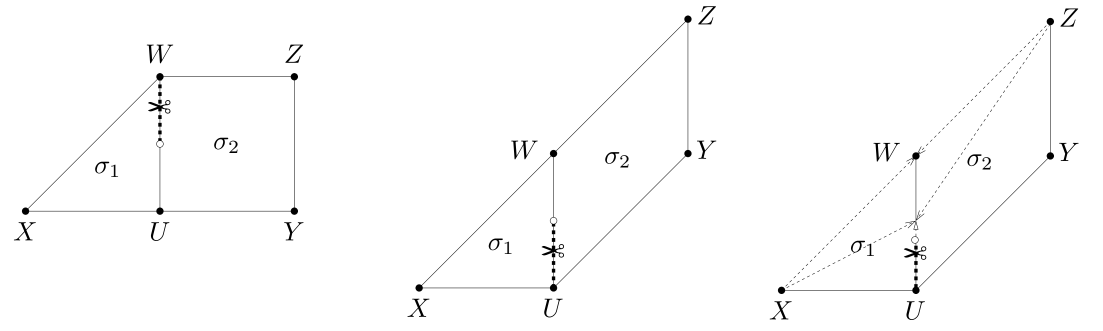

FIGURE 11.

Example 3.8. The affine manifold B_2 is depicted in two left-hand diagrams in Figure 11. The required structure $\mathfrak{D}$ is exactly as in the case of B_1, with two rays, one from P to W and one from P to U. We write X instead of ϑ_X etc. for the theta functions of level 1. One sees that the products of theta functions WY and UZ each correspond to the theta function of level 2 determined by the barycenter of the square σ_2, so one obtains the purely toric relation $WY - ZU = 0$. Much as in the case of B_1, one obtains products $XY = t(U^2 + UW)$ and $XZ = t(W^2 + WU)$; the two choices of pairs of jagged paths contributing to the latter product are indicated in the right-hand diagram of Figure 11, if we assume that the singular point occurs below the midpoint of the segment $\overline{WU}$.

Example 3.9. In Figure 12, we have an example with two singularities, with both charts shown. Note here we have four rays in the relevant structure, two each emanating from the singular points. We take the functions attached to the rays to be $1 + u$ and $1 + r$ as appropriate, where r and u are the monomials corresponding to the tangent vectors $(0, 1)$ and $(0, -1)$ respectively. We also take φ to take the value 0 on the square and the value 1 at X and Y. One sees easily, using the same strategies as above, various quadratic relations on the level 1 theta functions. First, we have the purely toric relation $RV = SU$, as neither of these products involve jagged paths which cross rays. Next, the products XS, XV, RY and UY all behave as in the previous example, and we can write

$$XS = t(R^2 + UR), \quad XV = t(U^2 + UR), \quad RY = t(S^2 + SV), \quad UY = t(V^2 + VS).$$

Finally, the product XY is the most interesting, with four contributions,

$$XY = t^2(UV + US + RV + RS).$$

Figure 13 shows all four pairs of jagged paths contributing to these terms. Note in [GSInv], Example 3.4, this relation was obtained from the previous ones by saturation of ideals.

Example 3.10. We consider next B_3 of §4.2 of [GSInv], see Figure 14. There are two singularities; the second diagram shows the affine embedding of an open set containing the

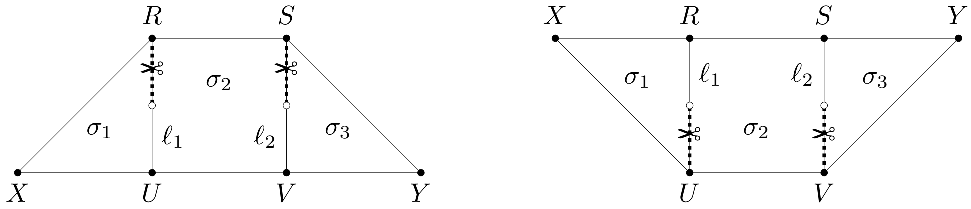

FIGURE 12.

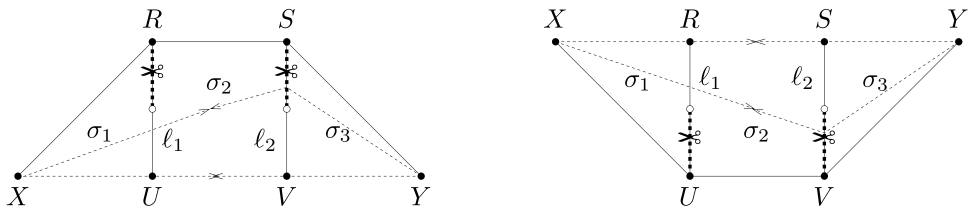

FIGURE 13. Two of the pairs of jagged paths contributing to XY are drawn in each chart. Contrary to appearances, none of the eight jagged paths appearing here bend!

two cuts of the first chart. Here φ takes the value 0 at U, Z and X, and the value 1 at Y and W.

The structure $\mathfrak{D}$ defining the deformation is as follows. Each singularity produces two rays contained in the line segments containing them; thus two of these rays intersect at U. Because of this intersection, an extra ray, $\mathfrak{p}$ as drawn, is necessary to produce compatibility. The function attached to $\mathfrak{p}$ is $f_{\mathfrak{p}} = 1 + t^2x^{-1}z^{-1}$, where x and z are the monomials corresponding to the tangent vectors $(1,0)$ and $(0,1)$ respectively. The relations are then given by

$$XY = t(U^2 + UW), \quad ZW = t(U^2 + YU).$$

The term U^2 in the expression for XY and ZW is the expected toric one from the first chart; the left-hand diagram of Figure 15 exhibits the diagrams for the other two relations. In this figure we take the singularities to be closer to the boundary of B_3 than to U.

More interesting in this case is how consistency arises. Let us consider jagged paths from Y into the chamber between the ray $\mathfrak{p}$ and the segment $\overline{UX}$. In particular, consider a jagged path that bends at $\mathfrak{p}$, e.g., δ_1 as depicted in the right-hand diagram of Figure 15. If we take U to be the origin, and if δ_1 hits $\mathfrak{p}$ at the point (r,r), then the direction of the jagged path after the bend is $(r+2, r+1)$, and hence always has slope $\geq 1/2$. Thus such a jagged path can never end at a point of the chamber in question below the line of slope $1/2$ passing through the origin. In order for the lift of Y to be independent of the endpoint inside this chamber, there must be some other jagged paths contributing the same monomial at points below the slope $1/2$ line. One sees that a path of type δ_2 does the trick. Note also that if we had omitted the ray $\mathfrak{p}$, the jagged path δ_2 would still exist,

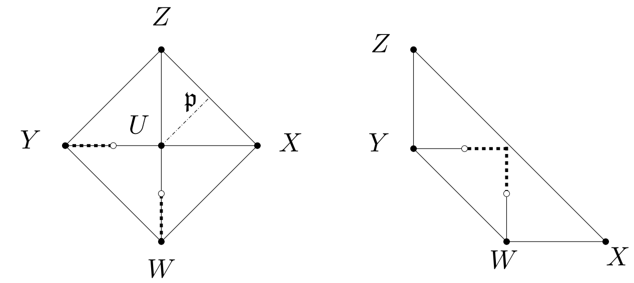

FIGURE 14.

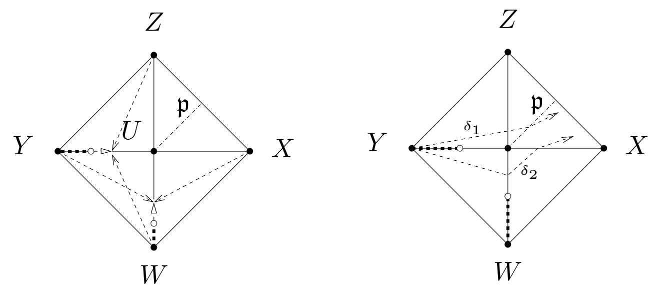

FIGURE 15.

but consistency would fail because there would be no substitute for δ_1 above the slope 1/2 line.

Example 3.11. Our final surface example is as depicted on the left-hand diagram of Figure 16; the singularities along the edges are of the same type as the previous example. We take φ to have value 0 at X, Y, and U, and 1 at Z. There are 9 elements of the structure: six of these are the usual ones emanating from the singularities (the dotted lines in the figure indicates those rays emanating from the singularities which pass into the interiors of two-cells), and three additional rays emanate from U in the directions of X, Y, and Z, to produce a consistent structure. For example, there is a ray stretching from U to X with attached function $1+yz$. See [GSInv], Examples 4.3 and 4.4, for the details. We would like to determine an equation for this family in $\mathbb{P}^3$ by computing the product XYZ. Keep in mind that we are using the notation X, Y, etc. for $\vartheta_X^{[1]}$, $\vartheta_Y^{[1]}$ etc. We do this in two steps. First we compute, say, XY. There will be two level two theta functions contributing to this product, which we shall write as $\vartheta^{[2]}_{(X+Y)/2}$, the theta function given by the mid-point of the line segment XY, and $\vartheta^{[2]}_U$, the degree two theta function corresponding to the point U, thought of as a half-integral point. In this notation, we in fact have

$$XY = \vartheta^{[2]}_{(X+Y)/2} + t\vartheta^{[2]}_U.$$

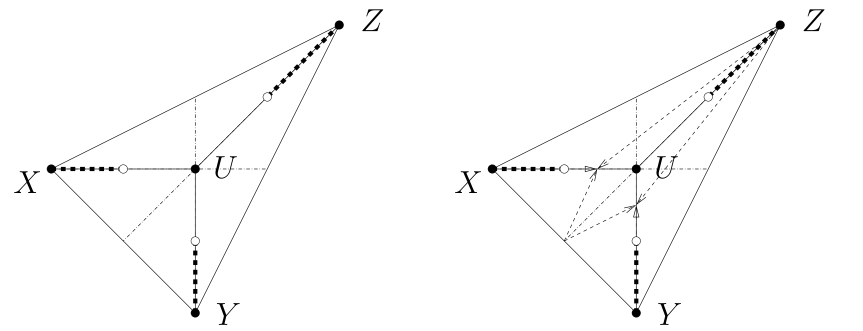

FIGURE 16.

One checks that $U^2 = \vartheta_U^{[2]}$ (such a statement need not in general be true). We then compute $\vartheta_{(X+Y)/2}^{[2]} \cdot Z$, and find this has four terms:

$$(t+t^2)\vartheta_U^{[3]} + t(\vartheta_{(2U+X)/3}^{[3]} + \vartheta_{(2U+Y)/3}^{[3]}) = (t+t^2)U^3 + tU^2(X+Y).$$

The first term tU^3 is purely toric; the second term t^2U^3 appears for the following reason. In order to correctly compute the product using jagged paths, since U appears on a number of rays, we need to perturb the endpoint a little bit. If we perturb the endpoint so the jagged path from Z now crosses the edge $\overline{UX}$, we need to consider possible bending along this line segment. The expression (3.1) arising from crossing the rays on this line segment near U is $(1+x)(1+yz) = 1+x+yz+t$. The monomial t does not change the direction of the jagged path, but produces the contribution t^2U^3. The third and fourth terms arise as shown in the right-hand diagram of Figure 16. Finally, $\vartheta_U^{[2]} \cdot Z = U^2Z$ is purely toric. This gives the equation

$$XYZ = t\big((1+t)U^3 + (X+Y+Z)U^2\big).$$

Example 3.12. We consider one crucial three-dimensional example, see Example 5.2 of [GSInv]. See Figure 17. As explained in [GSInv], the structure now consists of three slabs, the two-dimensional cells depicted on the right in the figure. The function attached to the slabs depends on which connected component of the complement of the discriminant locus we are on. The crucial point is the function attached to the central component, which can be written as

$$f = 1 + x + y + z + g(xyz).$$

Here x, y and z are the monomials corresponding to primitive tangent vectors pointing from W to the points X, Y and Z respectively. The formal power series g is determined by a procedure called *normalization* in [GS11]. It must be chosen so that $\log f$ is free of pure powers of $t = xyz$. This is expressed in terms of power series, i.e., that

$$\sum_{k\ge 1} \frac{(-1)^{k+1}}{k}(x+y+z+g(xyz))^k \in \mathbb{C}[\![x,y,z]\!]$$

does not contain any monomial $(xyz)^l = t^l$. This determines g uniquely, and one can compute $g(t)$ inductively:

$$g(t) = -2t + 5t^2 - 32t^3 + 286t^4 - 3038t^5 + \cdots.$$

One can then use jagged paths to determine products. The product XYZ is purely toric, giving tW^3, while the product UV can then be written as

$$UV = t^2(X + Y + Z + (1 + g(t))W)W.$$

Each of these terms correspond to a different term in f.

Those terms of the form $g_n t^n W^2$ will correspond to two jagged paths which do not bend, the line segments $\overline{VW}$ and $\overline{UW}$. As we know, these two jagged paths should correspond to a holomorphic triangle in the mirror manifold. As explained in [GSInv], the mirror manifold is an open subset X of the total space of the canonical bundle of $\mathbb{P}^2$. In fact X contains the zero-section of the canonical bundle, isomorphic to $\mathbb{P}^2$. The holomorphic triangle in question can be seen to intersect the $\mathbb{P}^2$ in one point, say x. How then do we explain the adjustments coming from the terms of g? The point is that $\mathbb{P}^2$ contains many holomorphic rational curves of degree n passing through x; by gluing any one of these curves to the holomorphic triangle, we get a (degenerate) triangle which should also contribute to the Floer product of U and V. This led us to conjecture that the coefficient of t^d in g should represent a type of 1-point invariant for rational curves of degree d in $\mathbb{P}^2$. This conjecture was supported by the observation that the sequence of numbers $-2, 5, -32, \ldots$ already appeared in several places in the literature. First, they appeared as Gromov-Witten invariants for certain curve classes on the total space of the canonical bundle of the blow up of $\mathbb{P}^2$ as given in [CKYZ]. Second, they appeared in Table 6 of [AKV] as open Gromov-Witten invariants for the total space of the anti-canonical bundle of $\mathbb{P}^2$, and these numbers arose precisely from relative homology classes of holomorphic disks where the holomorphic disks were likely to be represented by a single disk meeting $\mathbb{P}^2 \subset K_{\mathbb{P}^2}$ along with a sphere contained in the $\mathbb{P}^2$ attached to the disk.

More recently, this was argued from a different point of view, that of Landau-Ginzburg potentials, in [CLL]. This conjecture has now been proved by Chan, Lau and Tseng in [CLT].

3.4. **Broken lines and functions on $\mathcal{L}^{-1}$.** A jagged path is designed to organize the propagation of local monomial sections of $\mathcal{L}^{\otimes \ell}$ for any $\ell \neq 0$. Indeed, a jagged path carries a section of $\widetilde{\mathcal{P}}$, which in turn defines a monomial section of $\mathcal{L}^{\otimes \ell}$ in a local chart. At a bend the changes to the section of $\widetilde{\mathcal{P}}$ lie in $\mathcal{P}$, the kernel of the degree homomorphism $\widetilde{\mathcal{P}} \to \underline{\mathbb{Z}}$. In a local chart for $\mathcal{X}$ this means multiplication of the monomial section by a monomial. Now specializing to the case $\ell = 0$ we arrive at the notion of *broken lines*, which control the propagation of monomials on $\mathcal{X}$. Broken lines were introduced in the

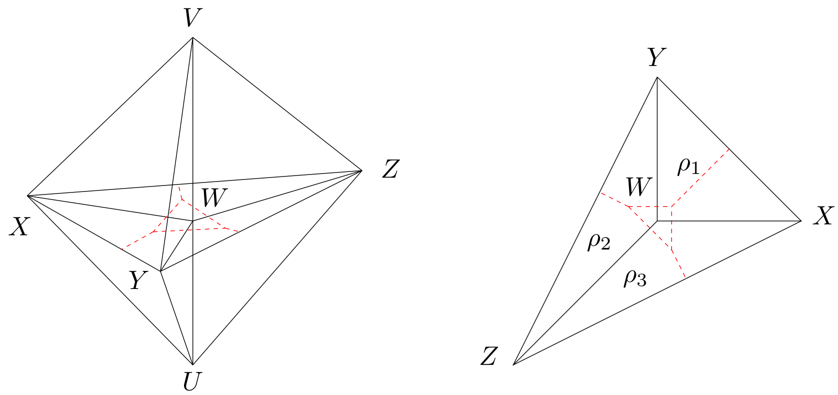

FIGURE 17. The tropical manifold appears on the left, with the central triangle containing the discriminant locus appearing on the right. These three cells comprise the slabs.

literature before jagged paths ([Gr10],[CPS],[GHKI]), where they have been used notably in the construction of Landau-Ginzburg potentials. However, jagged paths first appeared in discussions between the two authors and Mohammed Abouzaid in 2007. For the following definition we use the notation from Definition 3.2. We now also admit unbounded jagged paths, but still with only finitely many bends. The domain of definition of γ is then an interval $I \subsetneq \mathbb{R}$ rather than $[0,1]$. Note that in the unbounded case there is a unique maximal unbounded domain of linearity $(-\infty, t_0)$.

Definition 3.13. A (bounded or unbounded) jagged path $\big(\gamma : I \to B, (m_L)\big)$ in B with respect to the structure $\mathfrak{D}$ is called a *broken line* if for one (hence for every) domain of linearity $L \subset I$ it holds $\deg(m_L) = 0$.

Broken lines are conceptually somewhat easier since it suffices to work with the sheaf $\mathcal{P}$ rather than with $\mathcal{P}$ and $\widetilde{\mathcal{P}}$. In particular, this removes one layer of notation. Note also that $\mathbf{vect}(m_L)$ lies in Λ and stays constant along a domain of linearity. Since $\mathbf{vect}(m_L)$ is negatively proportional to $\gamma'(t)$ this shows that unlike jagged paths, broken lines can travel only in rational directions.

Now a little trick allows one to completely replace jagged paths by broken lines on the technical level, and this is what is done at most places in [GHKSΘ]. The trick is based on the observation that a section of $\mathcal{L}^{\otimes \ell}$ over an open set U is the same as a regular function over the preimage of U on the total space $\mathrm{Tot}(\mathcal{L}^{-1})$ that is fibrewise homogeneous of degree d.

The point is that $\mathrm{Tot}(\mathcal{L}^{-1})$ has a simple realization in terms of our program. Let $(B, \mathscr{P})$ be an integral affine manifold with singularities with polyhedral decomposition $\mathscr{P}$.

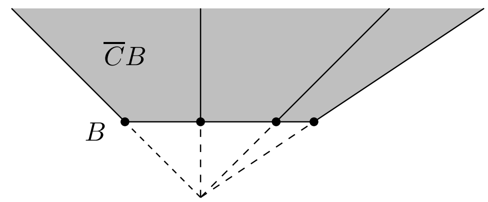

FIGURE 18. The truncated cone.

Definition 3.14. The *truncated cone over* B is the integral affine manifold defined as a set by

$$\overline{C}B := B \times [1, \infty),$$

endowed with the following affine structure. For $\psi : U \to \mathbb{R}^n$ an affine chart for B defined on an open set $U \subset B$, we define the chart

$$\tilde{\psi} : \overline{C}U \longrightarrow \mathbb{R}^{n+1}, \quad (x, h) \longmapsto (h \cdot \psi(x), h) \tag{3.3}$$

for $\overline{C}B$. The polyhedral decomposition $\overline{C}\mathscr{P}$ is given by the cells $\overline{C}\sigma := \sigma \times [1, \infty)$ for $\sigma \in \mathscr{P}$.

While $\overline{C}B$ is topologically a product, the affine structure is that of a cone over B with tip chopped off, see Figure 18. Note that $\overline{C}B$ has boundary $\overline{C}(\partial B) \cup B \times \{1\}$. As a manifestation of the cone structure let us look at parallel transport of the vertical tangent vector $(0, d) \in T_{(x,h)}\overline{C}B = T_x B \oplus \mathbb{R}$ along a straight line to $(y, h) \in \overline{C}B$. In a chart (3.3) centered at x this tangent vector maps to $(0, d)$, but at y the preimage of $(0, d)$ is $\big(-dv/h, d\big)$ where $v = \psi(y) - \psi(x)$. For $h = 1$, this is just the parallel transport underlying the sheaf $\mathcal{A}\!f\!f(B, \mathbb{Z})^*$! To discuss the relation with B let us use $\overline{C}B$ as index for $\mathcal{P}$ and Λ to distinguish these sheaves from the corresponding sheaves on B. The discussion of parallel transport shows that $\Lambda_{\overline{C}B}$ restricted to $B \times \{1\} \subset \overline{C}(\partial B)$ is canonically isomorphic to the sheaf $\mathcal{A}\!f\!f(B, \mathbb{Z})^*$ on B. Similarly, $\widetilde{\mathcal{P}}$, viewed as a sheaf on $B \times \{1\}$, extends to the sheaf $\mathcal{P}_{\overline{C}B}$.

Diagram (2.2) also has a simple interpretation in terms of $\overline{C}B$. While there is no natural affine map from $\overline{C}B$ to B, the projection to the $\mathbb{R}$-factor defines an affine map $\overline{C}B \to [1, \infty)$. This induces a homomorphism $\Lambda_{\overline{C}B} \to \underline{\mathbb{Z}}$ and, by composition with $\mathcal{P}_{\overline{C}B} \to \Lambda_{\overline{C}B}$, a homomorphism $\mathcal{P}_{\overline{C}B} \to \underline{\mathbb{Z}}$. Thus the second column of Diagram (2.2) on B is just the restriction of the first column on $\overline{C}B$ to $B \times \{1\}$. The lower two rows are defined by projection to $[1, \infty)$. The kernel of this projection defines horizontal elements in $\mathcal{P}_{\overline{C}B}$ and on $\Lambda_{\overline{C}B}$. The left column is thus just the restriction of the middle column to horizontal elements.

It is then immediate that there is a one-to-one correspondence between broken lines on $\overline{C}B$ and jagged paths on B, simply by composing with the projection $\overline{C}B \to B$ along the lines emanating from the tip of the cone. The geometric interpretation of this correspondence is very transparent on the complex side.

Proposition 3.15. *If $\pi : \check{\mathcal{X}} \to \operatorname{Spec} \mathbb{C}[\![t]\!]$ is a deformation associated to $(B, \mathscr{P})$ from Theorem 1.5 and $\mathcal{L}$ the relatively ample line bundle, then the total space of $\mathcal{L}^{-1}$ can be constructed by applying our construction to $(\overline{C}B, \overline{C}\mathscr{P})$.*

To define a regular function via broken lines one needs to look at the asymptotic integral affine manifold of a non-compact affine manifold, defined by equivalence classes of affine rays in unbounded cells, the equivalence generated by affine translations, see [CPS]. In the case of $\overline{C}B$ the asymptotic integral affine manifold is just B. Then given $\ell \in \mathbb{N} \setminus \{0\}$ and a $1/\ell$-integral point m in the asymptotic affine manifold, consider broken lines $\big(\gamma : (-\infty, 0] \to \overline{C}B, (m_L)\big)$ whose unbounded asymptotic direction is m, and for L the unbounded domain of linearity, m_L is a ℓ-fold multiple of an primitive element of vanishing t-order. Note this fixes m_L uniquely. Now the same procedure as in the construction of ϑ_m (Definition 3.5), but with jagged paths from m to x replaced by broken lines with the asymptotics defined by m and ending at x, defines a regular function on the total space of the associated degeneration of non-compact varieties. In the situation of $\overline{C}B$ this defines ϑ_m, viewed as a regular function on $\operatorname{Tot}(\mathcal{L}^{-1})$. Note that this regular function is fibrewise homogeneous of degree ℓ since for any domain of linearity L, the projection of m_L to $[1, \infty)$ has length ℓ.

4. Tropical Morse trees

We have seen that jagged paths can be used to compute products of theta functions, emulating the use of holomorphic triangles to compute the product in Floer homology. This raises the question as to whether one can compute the higher A_∞ operations using a similar strategy. This is explored in work in progress of Abouzaid, Gross, and Siebert [AGS]. This idea was already explored in [DBr], Chapter 8 in the case of elliptic curves (where $B = \mathbb{R}/d\mathbb{Z}$ for some positive integer d). In that case, as there are no singularities, one could just use trees composed of straight lines; in the general case, one needs to use jagged paths. We outline this here.

Let us return to the situation of §1, where given an integral affine manifold with singularities B, we assume we have $X(B) \to B$ and sections L_ℓ (the image of σ_ℓ) which hopefully become Lagrangian after a suitable choice of symplectic structure on $X(B)$. Further, we expect L_ℓ to be mirror to $\mathcal{L}^{\otimes \ell}$ on $\check{\mathcal{X}}$.

Suppose we wish to compute

$$\mu_d : \operatorname{Hom}(L_{\ell_{d-1}}, L_{\ell_d}) \otimes \cdots \otimes \operatorname{Hom}(L_{\ell_0},\ L_{\ell_1}) \to \operatorname{Hom}(L_{\ell_0}, L_{\ell_d})[2-d]. \tag{4.1}$$

As this should be defined using holomorphic disks, we hope to be able to compute this using jagged paths instead, using the philosophy that jagged paths correspond to holomorphic objects which can be glued together. This leads us to the definition of *tropical Morse tree*.

To give this definition, assume given B and a consistent structure $\mathfrak{D}$, so that we can talk about jagged paths on B with respect to $\mathfrak{D}$. Further, recall that a *ribbon tree* is a tree S

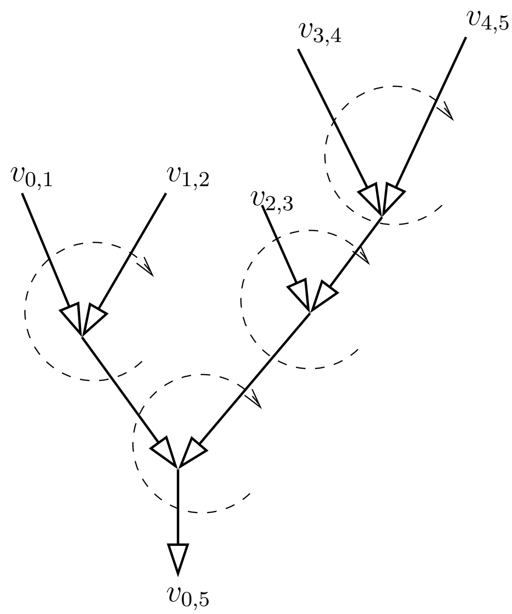

FIGURE 19.

with a cyclic ordering of edges adjacent to each vertex. This provides a cyclic ordering of leaves of the tree, and by choosing one leaf as an output, labelled $v_{0,d}$, and orienting each edge towards this output, we obtain a directed tree and canonical labelling of all other leaves as $v_{0,1}, \ldots, v_{d-1,d}$, see Figure 19. In addition, given a sequence of integers $\ell_0, \ldots, \ell_d$, we can assign an integer ℓ_e to each edge e of S: for e adjacent to $v_{i,i+1}$, $\ell_e = \ell_{i+1} - \ell_i$. If e is the outgoing edge at an interior vertex with incoming edges $e_1, \ldots, e_p$, then $\ell_e = \sum_{i=1}^p \ell_{e_i}$.

Definition 4.1. Suppose given distinct integers $\ell_0, \ldots, \ell_d$. Then a tropical Morse tree with respect to the data $B, \mathfrak{D}$ and $\ell_0, \ldots, \ell_d$ is a map $\psi : S \to B$ with S a ribbon tree with $d+1$ leaves whose restriction to any edge is a jagged path (and hence comes with the additional data of attached monomials). This data should satisfy the following conditions:

(1)
$$\psi(v_{i,i+1}) = p_{i,i+1} \in B\left(\frac{1}{\ell_{i+1} - \ell_i}\mathbb{Z}\right)$$
and the initial monomial attached to the edge $e_{i,i+1}$ adjacent to $v_{i,i+1}$, viewing $\psi|_{e_{i,i+1}}$ as a jagged path, is $z^{(p_{i,i+1})_\varphi}$.

(2) Let v be an internal vertex of S with incoming edges $e_1, \ldots, e_p$ and outgoing edge e_{out}. Let $c_1 z^{m_1}, \ldots, c_p z^{m_p}$ be the monomials attached to the last linear segment of each jagged path $\psi|_{e_1}, \ldots, \psi|_{e_p}$. Then the monomial $c_{\text{out}} z^{m_{\text{out}}}$ attached to the initial linear segment of $\psi|_{e_{\text{out}}}$ is $\prod_{i=1}^p c_i z^{m_i}$.

(3) Let $e_{0,d}$ be the edge adjacent to $v_{0,d}$, and let $c_{0,d} z^{m_{0,d}}$ be the monomial attached to the last linear segment of $\psi|_{e_{0,d}}$ as a jagged path. Then at $\psi(v_{0,d})$, we have $\mathbf{vect}(\tilde{r}(m_{0,d})) = 0$.

Let us note a number of features of this definition. First, morally such a tropical Morse tree should contribute to μ_d as in (4.1) where the inputs are intersections points of Lagrangian sections corresponding to $p_{0,1},\ldots,p_{d-1,d}$. However, because (4.1) is a map of degree $2-d$, this will always be zero if $d>2$ and all inputs are of degree 0. However, we will allow $\ell_{i+1}-\ell_i$ to be negative, so that $\mathrm{Hom}(L_{\ell_i},L_{\ell_{i+1}})\cong H^*(\check{\mathcal{X}},\mathcal{L}^{\otimes(\ell_{i+1}-\ell_i)})$ consists only of top degree cohomology. Indeed, as $\mathcal{L}$ is ample, it follows by Kodaira vanishing and Serre duality that for $d<0$

$$H^i(\mathcal{X},\mathcal{L}^{\otimes\ell})\cong H^{\dim B-i}(\check{\mathcal{X}},\mathcal{L}^{\otimes(-\ell)})=\begin{cases}H^0(\check{\mathcal{X}},\mathcal{L}^{\otimes(-\ell)}) & i=\dim B\\ 0 & i<\dim B\end{cases}$$

Thus we can write $\{\vartheta_p\,|\,p\in B(\frac{1}{\ell}\mathbb{Z})\}$ as a basis of $H^{\dim B}(\mathcal{X},\mathcal{L}^{\otimes\ell})$ Serre dual to the basis $\{\vartheta_p\,|\,p\in B(\frac{1}{-\ell}\mathbb{Z})=B(\frac{1}{\ell}\mathbb{Z})\}$ of theta functions for $H^0(\mathcal{X},\mathcal{L}^{\otimes(-\ell)})$. This allows us to treat negative and positive powers of $\mathcal{L}$, and so we will get non-trivial possibilities for μ_d for many different d.

The next point to observe is that if $\ell_{i+1}-\ell_i<0$, then in fact $\psi|_{e_{i,i+1}}$ needs to be viewed as a trivial jagged path which is just a point rather than a line segment. Indeed, in this case, near $p_{i,i+1}$, $\mathbf{vect}(\tilde{r}((p_{i,i+1})_\varphi))$ points *away* from $p_{i,i+1}$. But property (1) of Definition 3.2 says that this vector must be negatively proportional to ψ', so it is impossible for such a jagged path to move away from $p_{i,i+1}$.

A similar argument show that if $\ell_d-\ell_0>0$, then $\psi|_{e_{0,d}}$ must also be a trivial jagged path, because $\mathbf{vect}(\tilde{r}(\cdot))$ gets bigger, not smaller, along jagged paths with attached monomials being of positive degree. Thus in this case we cannot achieve condition (3) of Definition 4.1 unless $\psi|_{e_{\mathrm{out}}}$ is contracted.

Note that the condition that $\mathbf{vect}(\tilde{r}(m_{0,d}))=0$ implies that $\tilde{r}(m_{0,d})$, thought of as a local section of $\mathcal{A}\!f\!f(B,\mathbb{Z})^*$, is given by $(\ell_d-\ell_0)\cdot\mathrm{ev}_{p_{0,d}}$. Thus $m_{0,d}-(p_{0,d})_\varphi$ lies in $\ker(\tilde{r})\cong\mathbb{Z}$; denote this difference by $\mathrm{ord}(\psi)$.

Finally, we note that condition (2) of Definition 4.1 imposes a kind of balancing condition at the vertices, namely,

$$\mathbf{vect}(\tilde{r}(m_{\mathrm{out}}))=\sum_{i=1}^{p}\mathbf{vect}(\tilde{r}(m_i)).$$

So the vectors on the incoming edges determine the tangent direction of the outgoing edge.

In theory, we would like to define

$$\mu_d(\vartheta_{p_{d-1,d}},\ldots,\vartheta_{p_{0,1}})=\sum_{\psi}c_{0,d}t^{\mathrm{ord}(\psi)}\vartheta_{p_{0,d}},$$

where the sum is over all tropical Morse trees with $\psi(v_{i,i+1})=p_{i,i+1}$ and $p_{0,d}$ defined to be $\psi(v_{0,d})$. We note the conditions imply that $p_{0,d}\in B\left(\frac{1}{\ell_d-\ell_0}\mathbb{Z}\right)$.

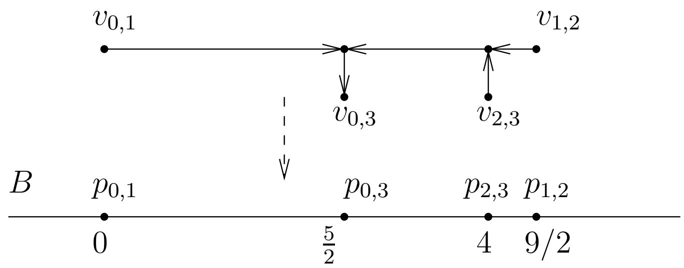

Figure 20.

For $d = 2$, this formula in fact recovers the theta multiplication formula of (3.2). Indeed, in this case S is just a trivalent tree with 3 leaves and one vertex; the outgoing edge is necessarily contracted, and the balancing condition $\mathbf{v}_1(1) + \mathbf{v}_2(1) = 0$ is enforced by (2) and (3) of Definition 4.1.

There is a problem with this definition for $d > 2$, however. The moduli spaces of tropical Morse trees need not be the correct dimension. This happens, for example, even in the case that $B = \mathbb{R}^n/\Gamma$ is a torus, and all the input points are contained inside a hyperplane in $\mathbb{R}^n$. As pointed out to us by M. Slawinski, this is already a problem when $n = 1$, the case of an elliptic curve, if all inputs are the same point in B. (So in fact the arguments in [DBr], Chap. 8, are not complete.)

As a consequence, in order to properly define μ_d for $d \geq 3$, one needs to perturb the moduli problem so we get finite counts of trees when needed. We hope to find a way of doing this which preserves the combinatorial nature of the construction, allowing for actual computations of A_∞-structures. This will certainly involve choices, but the resulting A_∞-structures should be impervious to these choices, up to quasi-isomorphism.

Once such a choice is made, it should be easy to show that the A_∞-precategory whose objects are powers of $\mathcal{L}$ and whose morphisms are given, say, by Čech complexes computing cohomology of powers of $\mathcal{L}$ (see [DBr], §8.4.5 for the elliptic curve case) is quasi-isomorphic to an A_∞-precategory defined using the μ_d's along the lines of [DBr]. A much more challenging problem is to then relate this category to the actual Fukaya category of the mirror.

We also note that M. Slawinski, in his thesis [Sl12], introduced the notion of tropical Morse graph with an aim of identifying a quantum A_∞-category structure [B07] on the category of powers of $\mathcal{L}$.

Examples 4.2. (1) Consider the tree in Figure 20 with either $B = \mathbb{R}$ or $B = \mathbb{R}/m\mathbb{Z}$ for some positive integer m, the latter by considering the lift of ψ to the universal cover. This tree contributes to the coefficient of $p_{0,3}$ in $\mu_3(p_{2,3}, p_{1,2}, p_{0,1})$. Here we take $\ell_0 = 0$, $\ell_1 = 1$, $\ell_2 = 3$ and $\ell_3 = 2$, noting $e_{2,3}$ and $e_{0,3}$ are contracted to points.

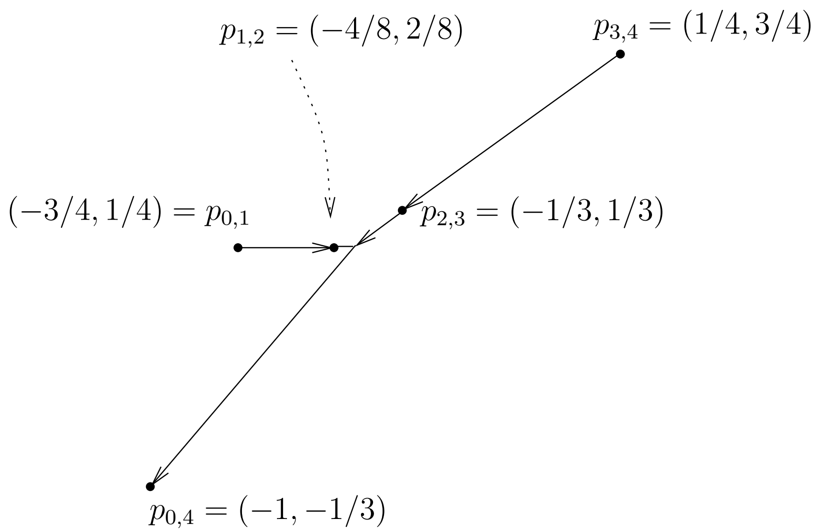

FIGURE 21.

(2) In Figure 21, we give a two-dimensional example, again in $B = \mathbb{R}^2$ or $\mathbb{R}^2/\Gamma$ for a lattice Γ, contributing to μ_4. In this example we take $\ell_0 = 0$, $\ell_1 = 4$, $\ell_2 = -4$, $\ell_3 = -7$, $\ell_4 = -3$. Again, $e_{1,2}$ and $e_{2,3}$ are contracted. $\square$

5. Applications of theta functions to mirror constructions

Theta functions play a crucial role in extensive new work, partly of the first author jointly with Hacking and Keel, [GHKI],[GHKII] and partly of both authors again jointly with Hacking and Keel [GHKSΘ]. We will only discuss this work briefly here, and only one aspect of this work.

In particular, we explain how theta functions allow us to greatly expand the class of singularities of affine manifolds we can treat. In this survey, we have almost exclusively considered only one two-dimensional singularity known from the integrable systems literature as a *focus-focus singularity*. This singularity has the feature that the monodromy in the local system Λ about the singularity is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and the invariant tangent direction is tangent to a line passing through the singularity. Further, the singularity must appear in the interior of an edge, and not at a vertex. In higher dimensions, the allowable singularities are the same generically: Example 3.12 is a typical three-dimensional example, with trivalent vertices for the discriminant locus. Theorem 1.5 as proved in [GS11] holds for this sufficiently nice class (and a little more generally), called "simple" in [GS06]. This class of singularities is related to the structure of the degenerations constructed by Theorem 1.5. These degenerations are special kinds of degenerations we call *toric degenerations*. These are degenerations of Calabi-Yau varieties $f : \mathcal{X} \to D$ such that the central fibre $\mathcal{X}_0$ is a union of toric varieties glued along toric strata, and f is given locally in a neighbourhood of the most singular points of $\mathcal{X}_0$ by a monomial in a toric variety.

This is an ideal class of degenerations for studying mirror symmetry, as it exhibits the greatest level of symmetry (under mirror symmetry, the data of the irreducible components is exchanged with the structure of the family at the most singular points of the central fibre). Furthermore, it works very well for complete intersections in toric varieties, see e.g., [Gr05]. However, one would ideally like to construct a mirror for any maximally unipotent degenerating family $\mathcal{X} \to D$, and it might be difficult to find a birationally equivalent family which is a toric degeneration. Thus it is desirable to expand the class of allowable degenerations, and this is equivalent to expanding the class of allowable singularities that our program can handle.

Let us consider the setup of [GHKI] by way of example. Consider (Y, D) with Y a non-singular projective rational surface and $D \in |-K_Y|$ a cycle of rational curves. We call such data a *Looijenga pair*. We can construct an integral affine manifold homeomorphic to $\mathbb{R}^2$ with one singularity associated to (Y, D), as follows. Let $D = D_1 + \cdots + D_n$, with $D_1, \ldots, D_n$ cyclically ordered.

For each node $p_{i,i+1} := D_i \cap D_{i+1}$ of D we take a rank two lattice $M_{i,i+1}$ with basis v_i, v_{i+1}, and the cone $\sigma_{i,i+1} \subset M_{i,i+1} \otimes_{\mathbb{Z}} \mathbb{R}$ generated by v_i and v_{i+1}. We then glue $\sigma_{i,i+1}$ to $\sigma_{i-1,i}$ along the rays $\rho_i := \mathbb{R}_{\geq 0} v_i$ to obtain a piecewise-linear manifold B homeomorphic to $\mathbb{R}^2$ and a decomposition

$$\Sigma = \{\sigma_{i,i+1} \,|\, 1 \leq i \leq n\} \cup \{\rho_i \,|\, 1 \leq i \leq n\} \cup \{0\}.$$

We define an integral affine structure on $B \setminus \{0\}$. We do this by defining charts $\psi_i : U_i \to M_{\mathbb{R}}$ (where $M = \mathbb{Z}^2$). Here

$$U_i = \mathrm{Int}(\sigma_{i-1,i} \cup \sigma_{i,i+1})$$

and ψ_i is defined on the closure of U_i by

$$\psi_i(v_{i-1}) = (1, 0), \quad \psi_i(v_i) = (0, 1), \quad \psi_i(v_{i+1}) = (-1, -D_i^2),$$

with ψ_i linear on $\sigma_{i-1,i}$ and $\sigma_{i,i+1}$. The idea behind this formula is that we are pretending that (Y, D) is in fact a toric pair. Given a ray in a two-dimensional fan generated by $(0, 1)$ corresponding to a divisor C, with adjacent rays generated by $(1, 0)$ and $(-1, -D_i^2)$ respectively, one has $C^2 = -D_i^2$. In particular, if (Y, D) were in fact toric, the above construction would just yield $B \cong \mathbb{R}^2$ as an affine manifold, with Σ the fan defining Y. If (Y, D) is not toric, B has a non-trivial singularity at the origin.

The reader can check a simple example: if Y is a del Pezzo surface of degree 5, one can find a cycle of 5 -1-curves on Y giving D. In this case, the monodromy of Λ about the resulting singularity is $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$; see [GHKI], Example 1.8 for details.

Here (B, Σ) can be thought of as a dual intersection complex of (Y, D). If one reinterprets (B, Σ) as an intersection complex for a degeneration, one would hope to find a flat family $\check{\mathcal{X}} \to \operatorname{Spf} \mathbb{C}[\![t]\!]$ whose central fibre is a union of n copies of $\mathbb{A}^2$. Specifically, $\check{\mathcal{X}}_0$ should be the *n-vertex*, the union of coordinate planes (if $n \geq 3$)

$$\mathbb{V}_n = \mathbb{A}^2_{x_1,x_2} \cup \mathbb{A}^2_{x_2,x_3} \cup \cdots \cup \mathbb{A}^2_{x_n,x_1} \subseteq \mathbb{A}^n,$$

where the subscripts denote the non-zero coordinates on each plane.

One can attempt to use the techniques of [GS11] to produce such a deformation. The problem is there is no local model for a smoothing in a neighbourhood of $0 \in \mathbb{V}_n$, and the arguments of [GS11] work by gluing together local models, which are required in codimension ≤ 2. However, if we throw away $0 \in \mathbb{V}_n$, a choice of a strictly convex piecewise linear function φ on B gives rise to a k-th order deformation of $\mathbb{V}_n^o := \mathbb{V}_n \setminus \{0\}$, denoted $\check{\mathcal{X}}_k^o \to \operatorname{Spec} \mathbb{C}[t]/(t^{k+1})$. This deformation looks purely toric in a neighbourhood of each connected component of $\operatorname{Sing}(\mathbb{V}_n^o)$. Since $\mathbb{V}_n$ is affine, we can try to recover a k-th order deformation $\check{\mathcal{X}}_k$ of $\mathbb{V}_n$ as follows. Suppose the coordinates $x_1, \ldots, x_n$ on $\mathbb{V}_n$ lift to functions on $\check{\mathcal{X}}_k^o$. Then we can embed $\check{\mathcal{X}}_k^o$ into $\mathbb{A}^n \times \operatorname{Spec} \mathbb{C}[t]/(t^{k+1})$ using these lifts, and take the closure of the image to be $\check{\mathcal{X}}_k$. The problem is that for the naive family $\check{\mathcal{X}}_k^o$, the x_i won't lift.

The solution is to modify the construction of $\check{\mathcal{X}}_k^o$ via a structure as in [GS11]: this structure should consist of lines radiating from the origin. The functions attached to these lines determine automorphisms, and these automorphisms are then used to modify the standard gluings, giving a different deformation $\check{\mathcal{X}}_k^o$. Any structure will provide such a deformation, as there is no meaningful compatibility of automorphisms which can be checked at the origin, as we have deleted the origin. If we had not deleted it, there is no local model for a smoothing there in which we could have checked such compatibility.

However, one can instead insist on using a structure which is consistent, which allows us to define theta functions on $\check{\mathcal{X}}_k^o$, which yield lifts of the variables x_i. To do so requires one to make a careful guess for the structure. In [GHKI], we define a canonical choice of structure motivated by [GPS]: this structure encodes certain relative Gromov-Witten invariants of the pair (Y, D). Importantly, one needs the original surface to construct the structure; it depends on more than just the singularity. In particular, it is possible to have two different pairs (Y, D), (Y', D') giving rise to the same affine singularity but to different structures.

This allows one to construct the deformations $\check{\mathcal{X}}_k \to \operatorname{Spec} \mathbb{C}[t]/(t^{k+1})$, and taking the limit, one obtains a formal deformation $\check{\mathcal{X}} \to \operatorname{Spf} \mathbb{C}[\![t]\!]$.

The following is a more precise statement of the main result of [GHKI]:

Theorem 5.1. *Let (Y, D) be a Looijenga pair and let $\sigma_P \subseteq H_2(Y, \mathbb{R})$ be a strictly convex rational polyhedral cone containing the Mori cone (the cone of effective curves) of Y. Let*

$P = \sigma_P \cap H_2(Y,\mathbb{Z})$, and $\mathfrak{m}_P$ the monomial ideal in $\mathbb{C}[P]$ generated by $\{z^p \,|\, p \in P \setminus \{0\}\}$. Let $\widehat{\mathbb{C}[P]}$ be the formal completion of $\mathbb{C}[P]$ with respect to $\mathfrak{m}_P$. Then there is a formal smoothing of the n-vertex $\check{\mathcal{X}} \to \operatorname{Spf}\widehat{\mathbb{C}[P]}$ canonically associated to (Y,D). This family can be viewed as the mirror family to the pair (Y,D).

If D supports a divisor ample on Y, then we can take σ_P to be the Mori cone, and the mirror family extends to a family $\check{\mathcal{X}} \to \operatorname{Spec}\mathbb{C}[P]$.

Example 5.2. The cubic surface with a triangle of -1-curves as D provides a particularly attractive example. In this case the monodromy of the singularity is minus the identity. The relevant structure controlling the deformation is extremely complicated, with there being a non-trivial ray of every possible rational slope. Nevertheless, it can be shown that the theta function lifts of x_1, x_2, x_3 satisfy a very simple cubic algebraic equation, see [GHKI], Example 5.12. Taking the closure of this family in $\mathbb{P}^3$ gives a universal family of cubic surfaces constructed by Cayley.

The above results have an application to a conjecture of Looijenga [L81] predating mirror symmetry, concerning the deformation theory of cusp singularities. A cusp singularity is a normal surface singularity whose minimal resolution has exceptional divisor a cycle of rational curves.

It had been observed that cusp singularities come in *dual pairs*. This can be explained as follows.

Let $M = \mathbb{Z}^2$. Let $T \in \mathrm{SL}(M)$ be a hyperbolic matrix, i.e., having a real eigenvalue $\lambda > 1$. Let $w_1, w_2 \in M_\mathbb{R}$ be eigenvectors with eigenvalues $\lambda_1 = 1/\lambda$, $\lambda_2 = \lambda$, chosen so that $w_1 \wedge w_2 > 0$ (in the standard counter-clockwise orientation of $\mathbb{R}^2$). Let $\bar{C}, \bar{C}'$ be the strictly convex cones spanned by w_1, w_2 and $w_2, -w_1$, and let C, C' be their interiors, either of which is preserved by T. Let $U_C, U_{C'}$ be the corresponding tube domains, i.e.,

$$U_C := \{z \in M_\mathbb{C} \,|\, \operatorname{Im}(z) \in C\}/M \subset M_\mathbb{C}/M = M \otimes \mathbb{G}_m.$$

T acts freely and properly discontinuously on $U_C, U_{C'}$. The holomorphic hulls of the quotients $U_C/\langle T\rangle, U_{C'}/\langle T\rangle$ are normal surface germs. These are each a cusp singularity, and they are considered dual to each other. All cusps (and their duals) arise this way.

We then obtain a proof of Looijenga's conjecture concerning smoothability of cusp singularities:

Theorem 5.3. *A germ of a cusp singularity (X,p) is smoothable if and only if there is a Looijenga pair (Y,D) along with a birational morphism $Y \to \bar{Y}$ contracting D to the dual cusp singularity.*

Looijenga proved smoothability of (X,p) implies the existence of (Y,D). This is done by realising a cusp singularity and its dual inside an Inoue surface, and then deforming the Inoue surface so that p smooths but the dual cusp remains untouched. The resulting surface resolves to (Y, D). The converse is proved in [GHKI], by studying the mirror family to (Y, D). One can show with some additional effort that the family extends (analytically) to one which contains as a fibre the dual cusp to p. Since the mirror family is a smoothing, the dual cusp is thus smoothable.

There are also connections between this construction and cluster varieties associated to skew symmetric matrices of rank 2 (see [FG06] for definitions). In particular, theta functions suggest a vast generalisation of the (conjectural) Fock-Goncharov dual bases for cluster varieties. In particular, the above construction leads to a proof of the Fock-Goncharov conjecture for the X-cluster variety associated to such a rank 2 skew-symmetric matrix.

The ideas used to prove Theorem 5.1 can currently be generalized to the K3 case. One starts with a one-parameter maximally unipotent degeneration $\mathcal{Y} \to D$ of K3 surfaces. We assume $\mathcal{Y}$ is non-singular and the map to D is normal crossings and relatively minimal. We can then build a dual intersection complex $(B, \mathscr{P})$ of this degeneration, where $\mathscr{P}$ is a decomposition of B into standard simplices. All the singularities of B now lie at vertices, reflecting the geometry of the irreducible components of $\mathcal{Y}_0$. The mirror family is constructed by first building a union $\check{\mathcal{X}}_0$ of projective planes whose intersection complex is $(B, \mathscr{P})$. We smooth by constructing a suitable structure on B. This will define deformations of $\check{\mathcal{X}}_0^o$ obtained from $\check{\mathcal{X}}$ by deleting zero-dimensional strata. The correct choice of structure will be consistent, yielding well-defined theta functions on these deformations, enabling us to compactify the deformations much as before. As a consequence, we obtain theta functions on K3 surfaces which enjoy many nice properties; see [GHKK3] for details. Crucially, we can show that theta functions are essentially independent of the choice of birational model of $\mathcal{Y} \to D$. This leads to a proof of a strong form of Tyurin's conjecture.

This general construction of mirrors for K3 surfaces gives encouragement that a similar construction will work in all dimensions. At the moment, the techniques available rely heavily on [GPS], which is a two-dimensional result. However, we anticipate that a suitable understanding of log Gromov-Witten invariants [GSlog],[Ch],[ACh] will allow us to create consistent structures in general. Certain types of Gromov-Witten invariants associated to the degeneration $\mathcal{Y} \to D$ will be used to construct the structure defining the mirror. Morally, these Gromov-Witten invariants will count holomorphic disks with boundary contained in fibres of the SYZ fibration, but we expect a purely algebro-geometric description of these invariants.

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