GOODWILLIE CALCULUS VIA ADJUNCTION AND LS COCATEGORY

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Abstract. In this paper, we show that for reduced homotopy endofunctors of spaces, F, and for all $n \geq 1$ there are adjoint functors R_n, L_n with $\mathrm{T}_nF = R_nFL_n$, where P_nF is the n-excisive approximation to F, constructed by taking the homotopy colimit over iterations of T_nF. This then endows T_n of the identity with the structure of a monad and the T_nF's are the functor version of bimodules over that monad. It follows that each T_nF (and P_nF) takes values in spaces of symmetric Lusternik-Schnirelman cocategory n, as defined by Hopkins [Hop84b]. This also recovers recent results of Chorny-Scherer[CS12]. The spaces $\mathrm{T}_nF(X)$ are in fact classically nilpotent (in the sense of Berstein-Ganea) but not nilpotent in the sense of Biedermann and Dwyer. We extend the original constructions of dual calculus to our setting and dualize our constructions to obtain analogous results concerning constructions T^n, P^n,and LS category.

1. Introduction

The Lusternik-Schnirelmann category of a space X, denoted LScat(X), is (one less than) the minimal number of open sets needed to cover X which are contractible in X[1]. This was originally defined for manifolds, and is a lower bound for the number of critical points of a function on X [LS34]. The definition was broadened to arbitrary spaces, and later definitions include an inductive version by Ganea, ind LS cat, and symmetric version, symm LScat, by Hopkins [BG61, Hop84b]. Ganea showed that inductive category agrees with the original definition in suitable cases(X path-connected, paracompact and locally contractible, e.g.) and since then all definitions of category have been shown to be equivalent(see [FHT01, §V Ch. 27]), which is not true of their duals.

Inductive and symmetric category were both defined in a way that admit natural duals, the cocategory of a space. To understand their importance, we state the following inequalities, where "W-long" is the length of Whitehead products in π_*X and nil(ΩX) is the associated Berstein-Ganea nilpotence, that is, the number n such that the homotopy commutators in ΩX of length strictly greater than n vanish [BG61, Hop84b, Gan60]:

$$\text{W-long}(X) \leq \mathrm{nil}(\Omega X) \leq \text{ind LScocat}(X) \leq \text{sym LScocat}(X). \tag{1}$$

We use constructions in Goodwillie's calculus of homotopy functors to reformulate Hopkins's definition of symmetric LScocat. We say that F is a homotopy functor if it preserves weak equivalences, and we restrict ourselves to only considering covariant homotopy functors. The analog to being polynomial of degree n in this setting is called n-excisive; homology theories (regarded as taking values in spaces) are 1-excisive functors. A homotopy functor F can be approximated by a tower of n-excisive functors P_nF. Each P_nF is defined as the homotopy colimit over a directed system of finite homotopy limit constructions, T_n^kF, that is, $\mathrm{P}_nF(X) := \mathrm{hocolim}(\mathrm{T}_nF(X) \to \mathrm{T}_n^2F(X) \to \cdots)$.

In this language, denoting by $\mathbb{I}$ the identity functor, we may restate Hopkins's definition as

Definition 1.1. *For a space X, symmetric LS cocat(X) $\leq n$ iff X is a retract of* $\mathrm{T}_n\mathbb{I}(X)$.

We prove the following:

Theorem 1.2. *For a reduced homotopy endofunctor of spaces and for all $n \geq 1$, there exist (Quillen) adjoint functors R_n, L_n such that $\mathrm{T}_nF = R_nFL_n$. In particular, $\mathrm{T}_n\mathbb{I} = R_nL_n$.*

Which has as a consequence

Corollary 1.3. T_nF *are left and right* $\mathrm{T}_n\mathbb{I}$*-functors, as are the* P_nF.

[1]Typically, LScat is renormalized by subtracting 1, so that contractible spaces have LScat 0.

We leave the rigorous definition of being left or right M-functors (for M a monad[2]) to the background section. It is exactly the functor analog of being a left or right M-module. To avoid confusion with bi-functors, we choose to use the term "left and right M-functor" instead of "bi-M-functor" to denote having the structure of both a left and right M-functor.

This structure gives us maps which express $\mathrm{T}_nF(X)$ as a retract of $\mathrm{T}_n\mathbb{I}(\mathrm{T}_nF(X))$, combining with Prop 2.6, so, we also have

Corollary 1.4. *T_nF naturally takes values in spaces of symmetric LS cocat $\leq n$, as do the P_nF.*

Theorem 2.1 of [CS12] states that the Whitehead products of length $\geq n+1$ vanish in $\mathrm{P}_nF(X)$ for any space X. Combining 1 with Corollary 1.4, we recover and extends this (to the T_n),

Corollary 1.5. *The Whitehead products of length $\geq n+1$ vanish for $\mathrm{T}_nF(X)$ and $\mathrm{P}_nF(X)$ for any space X.*

We also have the following interesting difference between the T_n's and P_n's, which we thank Boris Chorny and Georg Biedermann for pointing out. We first mention that Biedermann and Dwyer, in [BD10], define X to be a homotopy nilpotent group of class n if its loopspace is an algebra over a certain theory (in the sense of Lawvere) whose free objects are of the form $\Omega\mathrm{P}_n\mathbb{I}(\bigvee_k S^1)$.

Corollary 1.6. *For F not n-excisive, T_nF takes values in spaces which are classically nilpotent (in the sense of Berstein-Ganea) but not nilpotent in the sense of Biedermann and Dwyer (that is, not homotopy nilpotent groups once Ω is applied).*

Proof. Classical nilpotence follows from the inequalities of 1 and the fact that T_nF take values in spaces of symm LScocat $\leq n$.

There is an equivalence of categories [BD]

$$\{\text{values of functors } \Omega F,\ F\ n-\text{excisive}\} \sim \{\text{Homotopy Nilpotent Groups of class } \leq n\}.$$

Since T_nF is not n-excisive unless F is (i.e. unless it equals P_nF), $\Omega\mathrm{T}_nF$ will not be a homotopy nilpotent group (using the above equivalence of categories) in the sense of Biedermann and Dwyer. □

We point out that the constructions used in the definition of inductive LS category were proven by Deligiannis to have the structure of comonads [Del00]. Our proof is necessarily significantly different than a dualization of this result, as we lack an inductive definition of the T_n's.

We also construct duals of the adjoints in the statement of Theorem 1.2 which we will call R^n and L^n. These give rise to an alternate formulation of the dual Taylor tower, extending it to endofunctors of spaces. Dual to the normal case, we would then like to construct the n-co-excisive approximation to a functor, P^nF, as the homotopy limit over iterations of $\mathrm{T}^nF = L^nFR^n$. We establish in the Appendix the counterpart of [Goo03, Lemma 1.9], which shows that the map $t^n : \mathrm{T}^nF \to F$ factors through some co-cartesian cube. In [Goo03], the original Lemma was then combined with commutativity of finite pullbacks with filtered colimits to conclude that $\mathrm{hocolim}(\mathrm{T}_nF \to \mathrm{T}_n^2F \to \cdots)$ produced a homotopy limit cube from a strongly cocartesian $(n+1)-$cube.

The dual situation does not always happen. That is, we cannot always commute finite pushouts with (co)filtered homotopy limits of spaces. Since our current aim is not a complete re-write of the dual calculus theory to endofunctors of spaces, we choose to resolve the issue of commuting finite pushouts with (co)filtered homotopy limits by restricting to functors landing in spectra if we need to consider P^nF. Then these approximations P^nF do take strongly cartesian cubes to cocartesian ones.[3].

In this language, we may re-state another of Hopkins's definitions as

[2]We would like to point out that monads are also sometimes called "triples", especially in the more algebraic literature, and in some of the Goodwillie calculus constructions such as those of [JM04, BEJM11, BJME10]

[3]For the original usage, defined using cotriple/triple calculus, see [McC01, BM04, Kuh04]

Definition 1.7. *For a space X, symmetric LS cat(X) $\leq n$ iff the natural map $\mathrm{T}^n\mathbb{I}(X) \to X$ has a section (up to homotopy).*

The dualized theorems are then

Theorem 1.8. *For all $n \geq 1$, there exist adjoint functors R^n, L^n such that $\mathrm{T}^n F = L^n F R^n$.*

Which has as a consequence

Corollary 1.9. *$\mathrm{T}_n F$ are left and right $\mathrm{T}^n\mathbb{I}$-functors, as are the $\mathrm{P}^n F$.*

As before, this extra structures implies that

Corollary 1.10. *$\mathrm{T}^n F$ naturally takes values in spaces of symmetric LS cat $\leq n$, as do the higher iterates $(\mathrm{T}^n)^k F$.*

Due to the following inequality ([BG61])

$$\text{cup-length (X)} \leq \text{LS cat(X)}$$

we conclude that

Corollary 1.11. *The cup products of length $\geq n+1$ vanish for $\mathrm{T}^n F(X)$ and the higher iterates $(\mathrm{T}^n)^k F$.*

1.1. **Some remarks.** First, in analogy to the result in the "normal" case (see [AK98, Eld11], etc), we would expect that the homotopy colimit of the (first) partial approximation dual calculus tower

$$\mathrm{hocolim}(\mathrm{T}^1\mathbb{I}(X) \to \mathrm{T}^2\mathbb{I}(X) \to \cdots)$$

is the conilpotent analogue of the "$\mathbb{Z}$-nilpotent completion of X", for X 0-connected. However, any 0-connected space is recoverable by its loopspace, which is reflected by the fact that the homotopy colimit of the partial approximation dual calculus tower for X 0-connected is just X again (see [Hop84b]).

We would like to additionally point out that some caution should be made in statements about the dual tower. Its previous incarnations landed in Spectra, and for such it tends to be sensitive only to phantom phenomena. The version of $\mathrm{P}^n F$ for any functor from spaces to spectra and any $n \geq 0$ found in [McC01] will be contractible on any space with a finite Postnikov tower. For example, it will vanish on S^1 (though not necessarily on S^2).

Assuming a good definition of co-analyticity of a functor, one would expect a dual to [Eld12, Cor 1.4], which would give an equivalence between $\mathrm{P}^\infty F(X)$ and $\mathrm{hocolim}_\Delta F(\mathrm{Hom}(\mathrm{sk}_j \Delta^*, X))$ for j bounded (below) by the co-analyticity of F, with bounds being improved when one inputs spaces with lower connectivity. This would then lend support to viewing spectra as a more natural place for the constructions, at least when we would like to consider $\mathrm{P}^n F$ or $\mathrm{P}^\infty F$.

For functors landing in spectra, the dual tower for spectra is a reasonable object to study (as we then may define negative homotopy groups), although the identity is linear, and all spectra would thus be considered of LS category 1.

As a consequence of this formally dual tower, something like the following should be true:

Conjecture 1.12. *There is, for each n, a theory $\mathcal{P}^n$, in he sense of Lawvere, with objects the free "homotopy conilpotent groups of class n", products of $\Sigma\mathrm{P}^n\mathbb{I}$ applied to a product of 1-spheres. Then there should be a weak equivalence of categories between values taken on by functors of the form ΣF, where F is n-co-excisive, and homotopy conilpotent groups of class n.*

1.2. **Organization.** The remainder of this paper is organized as follows. Section 2 contains some background on Goodwillie Calculus, Lusternik-Schnirelmann (co)Category and the salient bits from the theory of model categories which we need. Section 3 contains proofs of our results with Section 4 containing proofs of the duals. There is also an appendix where we develop some of the dual calculus theory for endofunctors of spaces, by dualizing the proof which was provided by Charles Rezk [Rez08] of Lemma 1.9 of [Goo03], which is the first step to showing that the n-co-excisive approximations do take strongly cartesian cubes to cocartesian ones.

1.3. **Acknowledgements.** We would like to thank Tom Goodwillie for conversations which resulted in greatly simplified forms for the adjunctions and Randy McCarthy for information about and discussions of the dual calculus.

2. Background

This section contains a variety of information useful for non-experts. We first introduce terminology about cubical diagrams in section 2.1 and descriptions of our models for ho(co)lim of (co)punctured cubes in section 2.2. These are necessary for the following constructions of Goodwillie calculus in section 2.3. We follow with an overview in section 2.4 of the relationship (thus far) between Goodwillie calculus and nilpotence of a space. Shifting gears, we introduce the definition of a monads M, as well as left/right M-modules and the functor analog, left/right M-functors in section 2.5. We provide in section 2.6 a definition of a Quillen adjunction and a proposition which we find useful in our proofs later. We then say a little about LS cocategory in section 2.7.

2.1. **Cubes and cubical diagrams.** We take Δ to be the category of finite ordered sets and monotone maps, with elements $[n] = \{0, 1, \dots n\}$. If S is a set, we denote by Δ^S the topological simplicial complex $\Delta^{|S-1|}$, so that $\Delta^{[n]} = \Delta^n$. We denote by $\mathscr{P}(S)$ the power set of the set S, which we will freely use to also mean the corresponding category with morphisms given by inclusion and objects the subsets of S. We can also use $\mathscr{P}(S)$ to mean its diagrammatic representation; the following is a diagrammatic representation of the category $\mathscr{P}([1])$.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \{1\} \arrow[r] & \{0,1\} \end{tikzcd}$$

We will denote by $\mathscr{P}_0(S)$ the subcategory without the emptyset and $\mathscr{P}^1(S)$ the subcategory of $\mathscr{P}(S)$ with S removed. An $(n+1)$-*cube* of spaces is then a functor from $\mathscr{P}([n])$ to spaces, with sub-diagrams given by restricting to $\mathscr{P}_0([n])$ or $\mathscr{P}^1([n])$, the punctured or co-punctured $(n+1)$-cube, respectively. For $\mathcal{X}$ a $\mathscr{P}^1([n])$-diagram, rather than index $\mathcal{X}$ by the subsets $S \in \mathscr{P}^1([n])$, it is customary to consider instead $\mathcal{X}([n] - U)$, where $U \in \mathscr{P}_0([n])$.

We will commonly refer to a homotopy pullback square as *cartesian* and a homotopy pushout square as *cocartesian*. An n-cube $\mathcal{X}$ is *cartesian* if its initial point, $\mathcal{X}(\emptyset)$, is equivalent to the homotopy limit of the rest of the diagram, i.e. if $\mathcal{X}(\emptyset) = \mathrm{holim}_{U \in \mathscr{P}_0([n])} \mathcal{X}(U)$ and *cocartesian* if $\mathcal{X}([n]) = \mathrm{hocolim}_{U \in \mathscr{P}_0([n])} \mathcal{X}([n] - U)$. The terms *strongly cocartesian* and *strongly cartesian* imply that every sub-2-face (i.e. every sub-square) is *cocartesian* (or, respectively, *cartesian*).

2.2. **Ho(co)lim for n-cubes.** The standard model we use for the homotopy limit of a punctured cube of spaces, $\mathcal{X}$, is $\mathrm{Hom}_{\mathrm{Top}^{\mathscr{P}_0([n])}}(\Delta^S|_{S \in \mathscr{P}_0([n])}, \mathcal{X})$. For $\mathcal{X}$ a punctured square, an element of this Hom-space is a map from the left diagram to the right diagram, where Δ^i are topological simplicies (the realizations of Δ^i, by common abuse of notation):

$$\begin{tikzcd} & \Delta^0 \arrow[d, "d^1"] \\ \Delta^0 \arrow[r, "d^0"'] & \Delta^1 \end{tikzcd} \qquad\qquad \begin{tikzcd} & \mathcal{X}(0) \arrow[d, "f"] \\ \mathcal{X}(1) \arrow[r, "g"'] & \mathcal{X}(\{0,1\}) \end{tikzcd}$$

That is, a tuple $(x_0, x_1, \gamma) \in \mathcal{X}(0) \times \mathcal{X}(1) \times \mathcal{X}(\{0, 1\})^I$ such that the path γ in $\mathcal{X}(\{0, 1\})$ has $\gamma(0) = f(x_0)$ and $\gamma(1) = g(x_1)$.

This has a natural left adjoint, which takes a space X and sends it to the punctured cubical diagram $S \mapsto X \times \Delta^S$.

2.3. **Goodwillie Calculus.** Not much background in Goodwillie calculus is needed to understand our results. Information regarding the dual calculus may be found in the appendix.

2.3.1. *Definitions and constructions.* In [Goo90, Goo91], Goodwillie establishes the following definition, in analogy with a function being polynomial of degree 1 or n:

Definition 2.1. *A functor F is excisive (i.e. 1-excisive) if it takes cocartesian squares to cartesian squares and n-excisive if it takes strongly cocartesian $(n+1)$-cubes to cartesian ones.*

Homology theories (viewed as functors $X \mapsto \Omega^\infty(\Sigma^\infty X \wedge E)$ for some spectrum E) are then nice excisive functors. In particular, a functor F is excisive, reduced, and preserves filtered colimits if an only if it is a reduced homology theory in this sense.

We would like to point out that if a functor is n-excisive, it is also $(n+1)$-excisive. If F is n-excisive and $\mathcal{X}$ a strongly co-cartesian $(n+2)$-cube, then we can view $\mathcal{X}$ as a map of two strongly co-cartesian $(n+1)$-cubes, $\mathcal{X} : \mathcal{X}_0 \to \mathcal{X}_1$. Then, $F(\mathcal{X}) : F(\mathcal{X}_0) \to F(\mathcal{X}_1)$ is cartesian because $F(\mathcal{X}_i), i = 0, 1$ are cartesian, and a map of cartesian cubes is cartesian.

We will now give the constructions necessary to produce the n-excisive approximations to a functor F, P_nF, which are assembled from finite limit constructions, T_nF. We let $*$ denote the topological join and make the following definition,

We begin with a diagrammatical representation of the power set category, $\mathscr{P}([n])$, which is an $(n+1)$-cube indexed by subsets of $[n]$ and make the following definition:

$$\mathrm{T}_nF(X) := \mathrm{holim}_{U \in \mathscr{P}_0([n])} F(U * X)$$

As a result, we have a natural transformation $F(X) \to \mathrm{T}_1F(X)$, given by the natural map

$$F(X) = F(\emptyset * X) \to \underset{U \in \mathscr{P}_0([n])}{\mathrm{holim}} (U \mapsto F(U * X)).$$

That is, the map from the initial object of the square, $F(X)$ to the homotopy pullback of the rest, $\mathrm{T}_nF(X)$. We can take T_n of T_nF, and also have the same natural transformation from initial to homotopy pullback, now $\mathrm{T}_nF(X) \to \mathrm{T}_n(\mathrm{T}_nF(X)) =: \mathrm{T}_n^2F(X)$. For $n = 1$, see Figure 1.

The degree n polynomial approximation to F, P_nF, is constructed as the homotopy colimit, $\mathrm{P}_nF(X) := \mathrm{hocolim}(\mathrm{T}_nF(X) \to \mathrm{T}_n^2F(X) \to \cdots)$.

It is not immediately obvious that this is in fact n-excisive and universal (up to homotopy). We refer the reader to [Goo90, Goo03] for the details both in this case and for arbitrary n, especially Lemma 1.9 of [Goo03] with alternate proof provided by Charles Rezk [Rez08].

$$\mathrm{T}_1^2F(X) \quad := \quad \mathrm{holim}\left(\begin{tikzcd} & \mathrm{T}_1F(\{0\} * X) \arrow[d] \\ \mathrm{T}_1F(\{1\} * X) \arrow[r] & \mathrm{T}_1F(\{0,1\} * X) \end{tikzcd} \right)$$

$$\simeq \quad \mathrm{holim}\left(\begin{array}{ccc} & & \left(\begin{tikzcd} & F(\{0\} * \{0\} * X) \arrow[d] \\ F(\{0\} * \{1\} * X) \arrow[r] & F(\{0\} * \{0,1\} * X) \end{tikzcd} \right) \\ & & \downarrow \\ \left(\begin{tikzcd} & F(\{1\} * \{0\} * X) \arrow[d] \\ F(\{1\} * \{1\} * X) \arrow[r] & F(\{1\} * \{0,1\} * X) \end{tikzcd} \right) & \to & \left(\begin{tikzcd} & F(\{0,1\} * \{0\} * X) \arrow[d] \\ F(\{0,1\} * \{1\} * X) \arrow[r] & F(\{0,1\} * \{0,1\} * X) \end{tikzcd} \right) \end{array} \right)$$

FIGURE 1. $\mathrm{T}_1^2F(X)$

The collection of polynomial approximations to a functor F, $\{\mathrm{P}_nF\}_{n \geq 0}$, come with natural maps $\mathrm{P}_nF(X) \to \mathrm{P}_{n-1}F(X)$ for all $n \geq 1$.

With these maps we form a tower, the Goodwillie (Taylor) tower of $F(X)$:

$$\cdots \to \mathrm{P}_nF(X) \to \mathrm{P}_{n-1}F(X) \to \cdots \longrightarrow \mathrm{P}_1F(X) \to \mathrm{P}_0F(X).$$

As defined by Goodwillie, calculus works for spaces over a fixed space x, or over and under a space, the "basepoint". $\mathrm{P}_0F(X)$ is F applied to this fixed space which we are working over (or over and under).

We denote by $\mathrm{P}_\infty F(X)$ the homotopy inverse limit of this tower.

2.3.2. *Analyticity and convergence.* Heuristically, we say that a functor F is ρ-analytic if its failure to be n-excisive for all n is bounded with a bound depending on ρ; ρ-analytic implies $(\rho + 1)$ analytic, which is a weaker condition. This gives rise to a notion of "radius of convergence of a functor".

Proposition 2.2. [Goo91, Goo03] *If F is at least ρ-analytic and X is k-connected for k at least ρ (i.e. if X is in F's "radius of convergence") , then $F(X) \simeq \mathrm{P}_\infty(X)$.*

As towers give rise to spectral sequences, so does the Goodwillie tower, and can be read as a statement about convergence of the spectral sequence associated to the Goodwillie tower of F.

Some of the earliest and most powerful results of Goodwillie calculus relate to analyticity and other properties which follow. Examples of 1-analytic functors include $\mathbb{I}_{\mathrm{Top}}$,Waldhausen's algebraic K-theory functor, and TC the topological cyclic homology of a space. For a ρ-connected CW complex K, the functor $X \mapsto \Omega^\infty\Sigma^\infty \mathrm{Map}(K, X)$ is ρ-analytic.

2.4. **Goodwillie Calculus and nilpotence.** There is a beautiful paper by Arone and Kankaanrinta ([AK98]) which is likely the first link in the literature between Goodwillie calculus and nilpotence. The identity functor of spaces is a priori 1-analytic [Goo91], which implies that $\mathrm{P}_\infty\mathbb{I}(X) \sim \mathbb{I}(X)$ for X at least 1-connected, and this work of Arone and Kankaanrinta extends the equivalence to include X nilpotent by showing that the $n$the degree polynomial approximations of the identity functor are equivalent to those of $\mathbb{Z}_\infty X$, the $\mathbb{Z}$-nilpotent completion of X. Additionally, for X a 0-connected space, $\mathrm{P}_\infty id(X) \simeq \mathbb{Z}_\infty X$.

This connection was then strengthened via work of Biedermann and Dwyer [BD10]. They defined a theory (in the sense of Lawvere) for each n which whose algebras they call "homotopy nilpotent groups" of class n. These theories, $\mathcal{P}_n$, are built from P_n of the identity functor and allow one to now see that classical nilpotence is roughly a π_0-level version of this kind of homotopy nilpotence; $\pi_0\mathcal{P}_n = Nil_n$, the theory governing classical nilpotence. In this paper they also show one half of an equivalence of categories (the other half is [BD]), between values of functors ΩF for F $n-$excisive and $\mathcal{P}_n$-algebras.

More work in this direction has also been done recently by Boris and Chorny[CS12], who showed through a direct construction that the iterated Whitehead products iterated Whitehead products of length $(n + 1)$ vanish in any value of an n-excisive functor. The author has become aware that independently, Christina Costoya and Antonio Viruel have shown that the P_nF's take values in spaces with *inductive* cocategory n. The overlap between the work of Boris-Chorny, Costoya-Viruel and the author has lead to [CC12].

2.5. **Monads M and left/right M-Functors.** We first recall relevant definitions of a monad M and M-Functor, which is the functor extension of the notion of a module over the monad M:

Definition 2.3. [ML98, p.133] *A monad $M =< M, \eta, \mu >$ in a category C consists of a functor $M : C \to C$ and two natural transformations*

$$\eta : Id_C \to M \qquad\qquad \mu : M^2 \to M$$

which make the following commute

$$\begin{tikzcd} M^3 \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"] & M \end{tikzcd} \qquad \begin{tikzcd} Id_C \circ M \arrow[r, "\eta M"] \arrow[d, equal] & M^2 \arrow[d, "\mu"] & M \circ Id_C \arrow[l, "M\eta"'] \arrow[d, equal] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd}$$

Definition 2.4. [ML98, p.136] *If $M =< M, \eta, \mu >$ is a monad in a category C, we have notions of left and right M-module as follows. A left M-module* [4] *$< x, h >$ is a pair consisting of an object $x \in C$ and an*

[4]Referred to in [ML98, p.136] as an M-algebra

arrow $h : Mx \to x$ of C which makes both of the following diagrams commute (assoc law, unit law):

$$\begin{tikzcd} M^2x \arrow[r, "Mh"] \arrow[d, "\mu_x"'] & Mx \arrow[d, "h"] \\ Mx \arrow[r, "h"] & x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "1"] & Mx \arrow[d, "h"] \\ & x \end{tikzcd}$$

A right M-module $< x', h' >$ is a pair consisting of an object $x' \in C$ and an arrow $h' : xM \to x$ of C which makes both of the following diagrams commute (co-assoc law, co-unit law):

$$\begin{tikzcd} x'M^2 \arrow[r, "h'M"] \arrow[d, "\mu^*_{x'}"'] & x'M \arrow[d, "h'"] \\ x'M \arrow[r, "h'"] & x' \end{tikzcd} \qquad \begin{tikzcd} x' \arrow[r, "\eta^*_{x'}"] \arrow[dr, "1"] & x'M \arrow[d, "h'"] \\ & x' \end{tikzcd}$$

The following is a slight modification of Definition 9.4 from [May72]. It is the functor-level analog of being an M-module.

Definition 2.5. *Let (M, μ, η) be a monad in C. A right M-functor*[5] *(F, λ) in a category D is a functor $F : C \to D$ together with a natural transformation of functors $\lambda : FM \to F$ such that the following diagrams are commutative*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FM \arrow[d, "\lambda"] \\ & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} FM^2 \arrow[r, "F\mu"] \arrow[d, "\lambda"'] & FM \arrow[d, "\lambda"] \\ FM \arrow[r, "\lambda"] & M \end{tikzcd}$$

A left M-functor (F, λ') in a category D is a functor $F : C \to D$ together with a natural transformation of functors $\lambda' : MF \to F$ such that the following diagrams are commutative

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & MF \arrow[d, "\lambda'"] \\ & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} M^2F \arrow[r, "\mu F"] \arrow[d, "\lambda'"'] & MF \arrow[d, "\lambda"] \\ MF \arrow[r, "\lambda'"] & M \end{tikzcd}$$

Note that for an adjoint pair of functors $L : A \to B$, $R : B \to A$, with unit and counit $\eta : Id_A \to RL, \epsilon : LR \to Id_B$ we have a natural monad $M := RL$ with multiplication $\mu : R\epsilon L : RLRL \to RL$ and unit given by the unit of the adjunction.

Proposition 2.6. *For an adjoint pair of functors $L : A \to B$, $R : B \to A$ and an endofunctor $F : B \to B$, we have that $R \circ F \circ L$ has the structure of a left and right RL-functor, and dually $L \circ F \circ R$ has the structure of a left and right LR-functor.*

We will illustrate this for the bimodule over the monad case and the dual proof follows by dualizing our arguments and flipping the arrows in our diagrams. The structure map $h := R\epsilon FL$ and costructure map $h' := RF\epsilon L$.

Left RL-functor structure:

$$\begin{tikzcd} RL \circ RL \circ RFL \arrow[r, "RLR\epsilon FL"] \arrow[d, "R\epsilon LRFL"'] & RL \circ RFL \arrow[d, "R\epsilon FL"] \\ RL \circ RFL \arrow[r, "R\epsilon FL"] & RFL \end{tikzcd} \qquad \begin{tikzcd} RFL \arrow[r, "\eta RFL"] \arrow[dr, "1"] & RL \circ RFL \arrow[d, "R\epsilon FL"] \\ & RFL \end{tikzcd}$$

[5]Definition 9.4 of [May72] calls this an M-functor. We re-name it so as to be able to talk about functors with both right and left M "actions".

Right RL-functor structure:

$$\begin{tikzcd} RFL \circ RL \circ RL \arrow[r, "RF\epsilon LRL"] \arrow[d, "RFLR\epsilon L"'] & RFL \circ RL \arrow[d, "RF\epsilon L"] \\ RFL \circ RL \arrow[r, "RF\epsilon L"] & RFL \end{tikzcd} \qquad \begin{tikzcd} RFL \arrow[r, "RFL\eta"] \arrow[dr, "1"] & RFL \circ RL \arrow[d, "RF\epsilon L"] \\ & RFL \end{tikzcd}$$

2.6. **Quillen Functors.** We review the definitions of Quillen functor and adjunction and include a list of useful properties.

Definition 2.7. [DHKS05, 14.1.] *Given two model categories M and N, a Quillen adjunction is an adjunction*

$$f : M \rightleftarrows N : f'$$

of which

(i) *the left adjoint, f, is a left Quillen functor; a functor which preserves cofibrations and trivial cofibrations, and*
(ii) *the right adjoint, f', is a right Quillen functor; a functor which preserves fibrations and trivial fibrations.*

We also provide a list of properties which these functors satisfy.

Proposition 2.8. [DHKS05, 14.2.(ii)-(iii)] *Quillen functors satisfy the following properties:*

(i) *Every right adjoint of a left Quillen functor is a right Quillen functor and every left adjoint of a right Quillen functor is a left Quillen functor.*
(ii) *The opposite of a left Quillen functor is a right Quillen functor and the opposite of a right Quillen functor is a left Quillen functor.*

We are considering adjoint pairs (L, R) as well as their opposites (L^{op}, R^{op}). This proposition lets us reduce the work of showing both pairs are Quillen adjunctions to just showing it for one of the four functors.

2.7. **LS cocategory.**

2.7.1. *Why we have many (not necessarily equivalent) definitions of cocat.* As mentioned in [MV01, p332], there is a key element in the proof that all the definitions of category agree which fails to dualize. This is Mather's cube theorem [Mat76, Theorem 18]).

2.7.2. *The spectral sequence arising from symm cocat.* If we let $F = \mathbb{I}_{Top}$, the spectral sequence arising from the tower

$$\cdots \to \mathrm{T}_n\mathbb{I}(X) \to \mathrm{T}_{n-1}\mathbb{I}(X) \to \cdots \to \mathrm{T}_1\mathbb{I}(X)$$

this has already been studied by both Hopkins [Hop84a, p220] and Goerss [Goe93]. If you are are working with a functor which commutes with wedge and suspension, their results will extend.

An interesting note, especially in light of recent work relating the EHP sequence and Goodwillie Taylor tower[Beh10], is that Hopkins [Hop84a, Hop84b] concludes that in a range of about three times the connectivity of X, the spectral sequence collapses to an exact sequence (the EHP sequence)

$$\pi_n(\Sigma X) \to \pi_n(\Sigma X \wedge X) \to \pi_{n-2}(X) \to \pi_{n-1}(\Sigma X).$$

This spectral sequence is dual of the Rothenberg-Steenrod Spectral sequence. Hopkins makes the additional comment that one advantage of his formulation over that Barratt (which was completed by Goerss later) is that this formulation clarified that the higher differentials are the obstructions to maps $S^n \to X$ being co-h-maps compatible with higher associativity.

3. Proofs of main results

We will prove in this section the following theorem and its consequences:

Theorem 1.2 *Let F be a reduced endofunctor of topological spaces. For all $n \geq 1$, there exist adjoint functors R_n, L_n such that $\mathrm{T}_n F = R_n F L_n$. In particular, $\mathrm{T}_n \mathbb{I} = R_n L_n$, i.e. has the structure of a monad for all n.*

One key for this proof is to determine the "correct" categories to be working between.

3.1. **Motivation:** $n = 1$**.** We will first provide the proof in the case $n = 1$ to motivate the general proof. We will take our homotopy pushouts and pullbacks to be "standard" in the sense of Mather[Mat76].

Recall the definition of $\mathrm{T}_n F(X) := \operatorname*{holim}_{U \in \mathscr{P}_0([n])} F(U * X)$ For $n = 1$, this gives us the following homotopy pullback square:

$$\begin{tikzcd} \mathrm{T}_1 F(X) \arrow[r] \arrow[d] & F(*) \arrow[d] \\ F(*) \arrow[r] & F(SX) \end{tikzcd}$$

For reduced functors from based spaces to based spaces, $\mathrm{T}_1 F(X) \simeq \Omega F(\Sigma X)$, and Σ, Ω are adjoints between those categories. If we relax to reduced functors of unbased spaces, we have to be slightly more careful to get our adjunctions.

There is a (clear) equivalence of categories between spaces and diagrams of the form

$$* \leftarrow X \rightarrow *$$

where $*$ is a point and X is a space. However, category of dual diagrams,

$$* \rightarrow Y \leftarrow *$$

for Y a space, is equivalent to the category of *spaces with two basepoints*.

We can see that we have an adjunction

$$\text{Unreduced Suspension} : \mathrm{Top} \rightleftarrows \mathrm{Top}_{*_1 *_2} : \text{Paths (between } *_1 \text{ and } *_2)$$

such that for F reduced, $\mathrm{T}_1 F(X)$ is equivalent to $X \mapsto SX$ followed by F (which remains in $\mathrm{Top}_{*_1 *_2}$ because F is reduced) and then by taking paths.

The general case will not involve spaces with a multitude of basepoints, but cubical diagrams $\mathcal{X}$ which are similarly "reduced", i.e. $\mathcal{X}(S)$ is a point whenever $|S| = 1$.

3.2. **Proof of the general case, arbitrary** n**.** We will be working with the categories Top, $\mathrm{Fun}(\mathscr{P}_0([n]), \mathrm{Top}) = \mathrm{Top}^{\mathscr{P}_0([n])}$ and $\widetilde{\mathrm{Fun}}(\mathscr{P}_0([n]), \mathrm{Top})$, where the latter is the subcategory of "reduced" punctured cubical diagrams of spaces. That is, each diagram $\mathcal{X} \in \widetilde{\mathrm{Fun}}(\mathscr{P}_0([n]), \mathrm{Top})$ has the property that $\mathcal{X}(S)$ is a point, for $|S| = 1$.

There are adjunctions between these categories as follows:

$$\mathrm{Top} \underset{S \mapsto \Delta^S \times -}{\overset{\mathrm{holim}}{\leftrightarrows}} \mathrm{Fun}(\mathscr{P}_0([n]), \mathrm{Top}) \underset{\mathrm{red}}{\overset{\mathrm{inc}}{\leftrightarrows}} \widetilde{\mathrm{Fun}}(\mathscr{P}_0([n]), \mathrm{Top})$$

We let $L_n := \mathrm{red} \circ (S \mapsto \Delta^S \times -)$ and $R_n := \mathrm{holim} \circ \mathrm{inc}$. We will elaborate on these. The standard model which we use for the homotopy limit of a punctured cube $\mathcal{X}$ is $\mathrm{Hom}_{\mathrm{Top}^{\mathscr{P}_0([n])}}(\Delta^S|_{S \in \mathscr{P}_0([n])}, \mathcal{X})$. It has a natural left adjoint, which takes a space X and sends it to the punctured cubical diagram $S \mapsto X \times \Delta^S$.

There is also a natural left adjoint to the inclusion of reduced punctured cubical diagrams into punctured cubical diagrams. This takes a diagram Y to a diagram $\tilde{Y}$ such that

$$\tilde{Y}(S) := \mathrm{colim}(Y(S) \leftarrow \coprod_{j \in S} Y(\{j\}) \rightarrow S).$$

Note: this is not defined as a *homotopy* colimit, because we want a diagram that is honestly reduced. That is, we want $\tilde{Y}(S)$ to be a point, not just contractible, for $|S| = 1$.

The composition of the two left adjoints sends a space X to the diagram

$$S \mapsto \operatorname{colim}(X \times \Delta^S \leftarrow X \times S \to S)$$

The common model for the join of two spaces, X and Y, is the following, where C is the cone:

$$\operatorname{colim}(X \times CY \leftarrow X \times Y \to Y)$$

Since the lefthand map is a cofibration, this colim is also a homotopy colim.

For a set S considered as a discrete space, Δ^S is homotopic to CS, and the inclusion $CS \to \Delta^S$ is a cofibration. That is, we have a map of diagrams with all arrows equivalences and at least one is also a cofibration, so they have the same colimit. This suffices because, thanks to [**?**, Appendix], we do not need cofibrancy on the objects as well.

$$\begin{tikzcd}
X \times CS \arrow[d, hook] & X \times S \arrow[l] \arrow[r] \arrow[d, equal] & S \arrow[d, equal] \\
X \times \Delta^S & X \times S \arrow[l] \arrow[r] & S
\end{tikzcd}$$

As a result, we have that the composition of our two left adjoints above is not just a colim but a hocolim, and its pushout it is homotopic to $X * S$, so we may take it as a model for the join. That is, $L_n(X) = S \mapsto X * S$.

Recall that $\mathrm{T}_n F(X)$ is formed by applying F to the diagram $S \mapsto X * S$ (for $S \in \mathscr{P}_0([n])$), i.e. $F \circ L_n$, and following this with the appropriate homotopy limit functor, holim $\circ$ inc $= R_n$, its right adjoint.

That is, we have established an adjunction between Top and $\widetilde{\operatorname{Fun}}(\mathscr{P}_0([n]), \mathrm{Top})$ for each n such that $\mathrm{T}_n F(X) = R_n F(L_n X)$.

3.3. **Quillen adjunction.** By definition 2.7, to establish that our adjunctions are Quillen pairs, since we already have that they are adjunctions, we just need to check that either L_n preserves cofibrations and weak cofibrations or R_n preserves fibrations and weak fibrations.

Top has the usual model category structure with (Serre) fibrations and cofibrations and with weak equivalences as weak homotopy equivalences. Both diagram categories will be taken with the levelwise model structure induced by this model structure in Top. We have that

$$X \mapsto (S \mapsto X * S)$$

preserves cofibrations and trivial cofibrations, as follows. Given a (trivial) cofibration $X \rightarrowtail Y$, consider

$$\begin{tikzcd}
X \times \Delta^S \arrow[rr] \arrow[dr] \arrow[dd] & & S \arrow[dr] \arrow[dd] & \\
& X \arrow[rr] \arrow[dd] & & X * S \arrow[dd] \\
Y \times \Delta^S \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\
& Y \arrow[rr] & & Y * S
\end{tikzcd}$$

where the cube is cocartesian (because the top and bottom squares are cocartesian) and the three vertical maps $X \to Y$, $X \times \Delta^s \to Y \times \Delta^s$, and $S \to S$ are all (trivial) cofibrations.

As cofibrations in Top are stable under cobase change, the map $X * S \to Y * S$ is also a cofibration.

In the case of considering a trivial cofibration, homotopy invariance of homotopy colimits (i.e. topological join) gives us that the map $X * S \to Y * S$ is a weak equivalence, i.e. it is also a trivial cofibration. This will hold for all S and (trivial) cofibrations are defined levelwise, so we have shown that this is not just a left adjoint but also a left Quillen functor.

Remark 3.1. *We would like to point out that if we start with a fibrations of cosimplicial spaces and consider the induced cubical diagrams, these will still be fibrations in the levelwise structure as Reedy fibrations are also levelwise fibrations. These diagrams are obtained by precomposing with the functor* $\circ c_n : \mathscr{P}_0([n]) \to \Delta_{\leq n}$ *which sends* S *to* $[\#S - 1]$ *and inclusions to the induced coface maps. So, fibrations of cosimplicial things will go to fibrations in Top when following* $\circ c_n$ *by this Quillen adjunction.*

Using Prop 2.8(iii), we can conclude that the duals will also be Quillen adjoints since our model for hocolim is holim^{op}.

4. Proofs of dual results

We will prove in this section the following theorem and its consequences:

Theorem 1.8 *For all* $n \geq 1$*, there exist adjoint functors* R^n, L^n *such that* $\mathrm{T}^n F = L^n F R^n$*. In particular,* $\mathrm{T}^n \mathbb{I} = L^n R^n$*, i.e. has the structure of a comonad for all n and* $\mathrm{T}^n F$ *is a left and right* $\mathrm{T}^n\mathbb{I}$*-functor.*

We will be working with the categories Top, $\mathrm{Fun}(\mathscr{P}^1([n]), \mathrm{Top}) = \mathrm{Top}^{\mathscr{P}^1([n])}$ and $\underset{\sim}{\mathrm{Fun}}(\mathscr{P}^1([n]), \mathrm{Top})$, where the latter is the subcategory of "co-reduced" co-punctured cubical diagrams of spaces. That is, each diagram $\mathcal{X} \in \underset{\sim}{\mathrm{Fun}}(\mathscr{P}^1([n]), \mathrm{Top})$ has the property that $\mathcal{X}([n] - S)$ is contractable, for $|S| = 1$.

There are adjunctions between these categories as follows:

$$\underset{\sim}{\mathrm{Fun}}(\mathscr{P}^1([n]), \mathrm{Top}) \underset{inc}{\overset{cor}{\rightleftarrows}} \mathrm{Fun}(\mathscr{P}^1([n]), \mathrm{Top}) \underset{\mathrm{hocolim}}{\overset{[n]-S \mapsto X^{\Delta^S}}{\rightleftarrows}} \mathrm{Top}$$

We let L^n be the composition of the two leftward arrows and R^n be the composition of the two rightward arrows, hocolim $\circ$inc. We will elaborate on these.

The first step is to dualize the process of taking a space and producing a diagram

$$S \mapsto \mathrm{colim}(X \times \Delta^S \leftarrow X \times S \to S),$$

by dualizing the processes we composed to get this diagram. We still would like to start with a space and produce a diagram which is a sort of "co-reduced" cube.

The previous first step was to consider the natural model for a homotopy limit and its adjoint. In a similar way, we can consider the natural model for a homotopy colimit [Dug08, Example 8.12, p33]. For any "copunctured" diagram $\mathcal{X} : \mathscr{P}^1([n]) \to \mathrm{Top}$, we form its homotopy colimit by tensoring with the diagram $S \mapsto \Delta^S$ (which we will denote by $\Delta^{\leq n} \circ c_n$).

Thanks to the fact that this is an actual tensor product, we get an adjunction very similar to the homotopy limit case:

$$(\Delta^{\leq n} \circ c_n) \otimes (-) : \mathrm{Top}^{(\mathscr{P}^1([n]))} \rightleftarrows \mathrm{Top} : \mathrm{Hom}(\Delta^{\leq n} \circ c_n, -)$$

That is, the right adjoint sends $[n] - S$ to X^{Δ^S} (the dual of sending S to $X \times \Delta^S$).

This adjunction relies on observations that one can find nice exposition on (including proofs) in Dan Dugger's primer on homotopy colimits[Dug08].

We next need the universal co-reduced diagram. We will use the convention now of indexing our diagrams in $\mathscr{P}^1([n])$ by sets in $\mathscr{P}_0([n])$ by considering where we send $[n] - S$ for varying S in $\mathscr{P}_0([n])$. Dualizing the construction for reduced diagram yields

$$[n] - S \mapsto \lim \left(\begin{tikzcd} & S \arrow[d] \\ \mathcal{X}([n] - S) \arrow[r] & \prod_{s \in S} \mathcal{X}([n] - s) \end{tikzcd} \right).$$

Now, about the maps. The map into the product is the map induced by each map from the original diagram between $X([n] - S)$ and $X([n] - s)$ for all $s \in S$. The map from $S = \coprod_{s \in S} *$ is the choice of a point in each copy of $\mathcal{X}([n]-s)$ (which is the opposite of collapsing each space in $\coprod_{s\in S}\mathcal{X}(s)$ to the point indexing it), and pre-composing with the right adjoint to hocolim gives us

$$X \mapsto \left([n]-S \mapsto \lim \left(\begin{tikzcd} & S \arrow[d] \\ X^{\Delta^S} \arrow[r] & \prod_{s\in S} X \end{tikzcd} \right)\right)$$

using that $X^{\Delta^0} \cong X$, so the final element is unchanged, which is our R^n. Note that this limit is actually a homotopy limit because the bottom map is a fibrant replacement of the diagonal. Then L^n is inclusion followed by hocolim.

If we allowed the final element, i.e. $S = [n]$, this clearly yields X. Then, for every singleton $s \in S$, we get P_sX, i.e. paths in X based at whatever point was chosen by the map of s into X. If X is based already, these are all copies of the "normal" based path space, P_*X. Then we have for $S = \{i, j\}$, loop spaces. Moore loops if X is unbased, and based loops if X is based.

Example, for X based and $[n] = \{0, 1\}$, R^1X is

$$P_*X \leftarrow \Omega X \rightarrow P_*X.$$

Appendix A. Dual Goodwillie Calculus

We first point out that the original form of the dual calculus and results derived therefrom may be found in [McC01, Kuh04, BM04]. A dual tower has stages which naturally map to the functor. K-theory is natural to map out of, e.g. the trace map from K-theory to TC. McCarthy originally constructed the (algebraic/cotriple) Dual Calculus in the hopes that having an approximation to K theory which mapped into it would be illuminating.

One may form the Eckmann-Hilton dual of Goodwillie's calculus theory, switching cocartesian to cartesian everywhere. That is,

Definition A.1. *For F a homotopy endofunctor of spaces, F is* <u>*co-excisive*</u> *if it takes homotopy pullbacks to homotopy pushouts and* <u>*n-co-excisive*</u> *if it takes strongly cartesian $(n+1)$-cubes to cocartesian ones.*

In this appendix, we will establish the dual of a key lemma needed to show that the approximations do in fact take strongly cartesian cubes to cocartesian ones, and leave further development of this theory and its background to future work.

That is, we establish the counterpart of [Goo03, Lemma 1.9], which shows that the map $t^n : \mathrm{T}^nF \to F$ factors through some co-cartesian cube. In [Goo03], Goodwillie combined the original Lemma with commutativity of finite pullbacks with filtered colimits to conclude that the construction producting P_nF produces a homotopy limit cube from a strongly cocartesian $(n+1)-$cube.

However, it is important for us that we cannot always commute finite pushouts with (co)filtered homotopy limits of spaces. We choose currently to resolve the issue of commuting finite pushouts with (co)filtered homotopy limits by restricting to functors landing in spectra if we need to consider P^nF. Then these approximations P^nF do take strongly cartesian cubes to cocartesian ones.

To define T^n, rather than worry about laying out exactly what the cojoin operation should be, we will let T^n be given by our dualization of the functors we use to decompose T_n. That is, we let $\mathrm{T}^nF := L^n \circ F \circ R^n$ and $(\mathrm{T}^n)^kF := (L^n)^k \circ F \circ (R^n)^k$.

Recall that $R^n(X)$ is

$$X \mapsto \left([n]-S \mapsto \lim \left(\begin{tikzcd} & S \arrow[d] \\ X^{\Delta^S} \arrow[r] & \prod_{s\in S} X \end{tikzcd} \right)\right)$$

and L^n is the inclusion followed by hocolim.

The adjunctions are between these categories as follows:

$$R^n : \underset{\sim}{\mathrm{Fun}}(\mathscr{P}^1([n]), \mathrm{Top}) \underset{inc}{\overset{cor}{\rightleftarrows}} \mathrm{Fun}(\mathscr{P}^1([n]), \mathrm{Top}) \underset{\mathrm{hocolim}}{\overset{[n]-S \mapsto X^{\Delta^S}}{\rightleftarrows}} \mathrm{Top} : L^n$$

We then construct the n-co-excisive approximation,

$$\mathrm{P}^n F(X) := \mathrm{holim}(\cdots \to (\mathrm{T}^n)^2 F(X) \to \mathrm{T}^n F(X)).$$

To show that this should be the n-co-excisive approximation to F, we need to show that it takes strong cartesian diagrams to cartesian ones. We do this by dualizing the proof which was provided by Charles Rezk [Rez08] of Lemma 1.9 of [Goo03].

Lemma A.2. *(Dual of* [Goo03, Lemma 1.9]*) Let $\mathcal{X}$ be any strongly cartesian $(n+1)$-cube and F be any homotopy functor. The map of cubes $t^n F(\mathcal{X}) : \mathrm{T}^n F(\mathcal{X}) \to F(\mathcal{X})$ factors through some cocartesian cube.*

To prove this, we first need some setup and a sub-lemma.
Let

$$\mathcal{X}^U([n]-S) := \mathrm{holim}\left(\begin{array}{ccc} & & \prod_{u \in U} \mathcal{X}([n]-S-\{u\}) \\ & & \downarrow \\ \mathcal{X}([n]-S) & \longrightarrow & \prod_{u \in U} \mathcal{X}([n]-S) \end{array}\right)$$

Lemma A.3. *If $\mathcal{X}$ is already strongly cartesian, then $\mathcal{X}^U([n]-S) \simeq \mathcal{X}([n]-S-U)$.*

Proof. Since strongly cartesian is a property of the sub-2-faces, we will show this for an arbitrary sub-2-face of $\mathcal{X}$. Let $U = \{u_1, u_2\}$. Strongly co-cartesian gives us that the following is a homotopy pullback square

$$\mathcal{X}([n]-S-\{u_1,u_2\}) = \mathrm{holim}\left(\begin{array}{ccc} & & \mathcal{X}([n]-S-\{u_1\}) \\ & & \downarrow \\ \mathcal{X}([n]-S-\{u_2\}) & \longrightarrow & \mathcal{X}([n]-S) \end{array}\right)$$

where we take as model for the holim the space of maps from $\Delta_{\leq 1} \circ c_1$ into this diagram.

We will show that $\mathcal{X}^{\{u_1,u_2\}}([n]-S) \simeq \mathcal{X}([n]-S-\{u_1,u_2\})$.
We have that

$$\mathcal{X}^{\{u_1,u_2\}}([n]-S) = \mathrm{holim}\left(\begin{array}{ccc} & & \mathcal{X}([n]-S-\{u_1\}) \times \mathcal{X}([n]-S-\{u_2\}) \\ & & \downarrow \\ \mathcal{X}([n]-S) & \longrightarrow & \mathcal{X}([n]-S)_{\{u_1\}} \times \mathcal{X}([n]-S)_{\{u_2\}} \end{array}\right)$$

where the horizontal map is the diagonal and the vertical map is the inclusion of $\mathcal{X}([n]-S-\{u_i\})$ into the $\{u_i\}$–indexed copy of $\mathcal{X}([n]-S)$ (index denoted by subscript).

Let us examine a point in the homotopy pullback, keeping in mind that a map into a product is determined by a map into each factor. An element of the homotopy pullback is the following data

$$\begin{aligned} & x \in \mathcal{X}([n]-S) \\ & (y,z) \in \mathcal{X}([n]-S-\{u_1\}) \times \mathcal{X}([n]-S-\{u_2\}) \\ & \gamma : I \to \mathcal{X}([n]-S)_{\{u_1\}} \times \mathcal{X}([n]-S)_{\{u_2\}} \end{aligned}$$

where γ may be expressed as a path in each coordinate, $\gamma = (\gamma_1, \gamma_2)$ such that

$$\begin{array}{lll} \gamma_1 : I \to \mathcal{X}([n]-S)_{\{u_1\}} & \gamma_1(0) = y & \gamma_1(1) = x \\ \gamma_2 : I \to \mathcal{X}([n]-S)_{\{u_2\}} & \gamma_2(0) = x & \gamma_2(1) = z \end{array}$$

The point x was then effectively superfluous. Note that we now have $\widetilde{\gamma} = \gamma_1 * \gamma_2$ between b and c in $\mathcal{X}([n]-S)$, which gives the corresponding point in the homotopy limit of our first diagram,

$\mathcal{X}([n]-S-\{u_1,u_2\})$. There is a clear (up to homotopy) inverse to this process, and we have that the holims are the same. □

Proof of Lemma A.2. Given Lemma A.3, we now point out

(1) How the map t^nF factors through this cube:

Rezk observes that there is a natural map $\mathcal{X}_U(S) \to \mathcal{X}(S) * U = L_n(\mathcal{X}(S))(U)$ which induces the factorization

$$t_nF(\mathcal{X}(S)) : F(\mathcal{X}(S)) \to \mathrm{holim}_{U\in\mathscr{P}_0([n])} F(\mathcal{X}_U(S)) \to \mathrm{T}_nF(\mathcal{X}(S)).$$

The dual is a natural map $R^n(\mathcal{X}([n]-S))(U) \to \mathcal{X}^U([n]-S)$, inducing a factorization

$$\mathrm{T}^nF(\mathcal{X}([n]-S)) \to \mathrm{hocolim}_{U\in\mathscr{P}([n])} F(\mathcal{X}^U([n]-S)) \to F(\mathcal{X}([n]-S)).$$

We provide the map after recalling the two objects involved:

$$R^n(\mathcal{X}([n]-S))(U) = \mathrm{holim}\left(\begin{tikzcd} & U \arrow[d] \\ \mathcal{X}([n]-S)^{\Delta^U} \arrow[r] & \prod_{u\in U}\mathcal{X}([n]-S) \end{tikzcd}\right)$$

and

$$\mathcal{X}^U([n]-S) = \mathrm{holim}\left(\begin{tikzcd} & \prod\limits_{u\in U}\mathcal{X}([n]-S-\{u\}) \arrow[d] \\ \mathcal{X}([n]-S) \arrow[r] & \prod\limits_{u\in U}\mathcal{X}([n]-S) \end{tikzcd}\right)$$

Comparing the diagrams, we note that $\mathcal{X}([n]-S)^{\Delta^U} \to \prod_{u\in U}\mathcal{X}([n]-S)$ is a fibrant replacement of the diagonal and factors naturally through $\mathcal{X}([n]-S)$ as a result.

The map $U \to \prod_{u\in U}\mathcal{X}([n]-S)$ merely picks out a point in each copy via inclusion of $\{u\}$ into the $\{u\}$-indexed copy in the product. This factors through $\prod\limits_{u\in U}\mathcal{X}([n]-S-\{u\})$.

(2) Why this cube will be cocartesian:

We can consider $\mathcal{X}$ as two sub-cubes which differ by an element $\{u\}\in U$, we have that the maps $\mathcal{X}([n]-U-\{u\}) \to \mathcal{X}([n]-U)$ are isomorphisms; for nonempty U, the cube is cocartesian.

□

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