ZMP-HH/11-9
Hamburger Beiträge zur Mathematik Nr. 408
June 2011

MODULAR INVARIANT FROBENIUS ALGEBRAS FROM RIBBON HOPF ALGEBRA AUTOMORPHISMS

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Abstract

For any finite-dimensional factorizable ribbon Hopf algebra H and any ribbon automorphism of H, we establish the existence of the following structure: an H-bimodule F_ω and a bimodule morphism Z_ω from Lyubashenko's Hopf algebra object K for the bimodule category to F_ω that is invariant under the natural action of the modular group on the space of bimodule morphisms from K to F_ω. We further show that the bimodule F_ω can be endowed with a natural structure of a commutative symmetric Frobenius algebra in the monoidal category of H-bimodules, and that it is a special Frobenius algebra iff H is semisimple.

The bimodules K and F_ω can both be characterized as coends of suitable bifunctors. The morphism Z_ω is obtained by applying a monodromy operation to the coproduct of F_ω; a similar construction for the product of F_ω exists as well.

Our results are motivated by the quest to understand the bulk state space and the bulk partition function in two-dimensional conformal field theories.

1 Introduction

One remarkable feature of complex Hopf algebras is their intimate connection with low-dimensional topology, including invariants of knots, links and three-manifolds. These connections are particularly well understood for semisimple Hopf algebras. The representation category of a semisimple factorizable finite-dimensional (weak) Hopf algebra is a modular tensor category [NTV] and thus allows one to construct a three-dimensional topological field theory. As a consequence, it provides finite-dimensional representations of mapping class groups of punctured surfaces.

It has been shown by Lyubashenko [Ly1, Ly3] that such representations of mapping class groups can be constructed for non-semisimple factorizable Hopf algebras H as well. This construction is in fact purely categorical, in the sense that it only uses the representation category as an abstract ribbon category with certain non-degeneracy properties. In the present paper we apply this construction not to the category of left H-modules, but rather to the category of H-*bimodules*. To this end we endow this category H-Bimod with the structure of a monoidal category using the coproduct of H (rather than by taking the tensor product $\otimes_H$ over H as an associative algebra). With this tensor product, the category H-Bimod can be endowed with further structure such that it becomes a sovereign braided monoidal category.

For our present purposes we restrict to the case that the punctured surface in question is a one-punctured torus, whose mapping class group is the modular group $\mathrm{SL}(2,\mathbb{Z})$. Specializing the results of [Ly1], we obtain a Hopf algebra object K in the monoidal category H-Bimod. For any H-bimodule X the vector space $\mathrm{Hom}_{H|H}(K,X)$ of bimodule morphisms then carries a representation of the modular group. The main result of this paper is the following assertion:

Theorem
Let H be a (not necessarily semisimple) finite-dimensional factorizable ribbon Hopf algebra over an algebraically closed field of characteristic zero, and let $\omega\colon H \to H$ be an automorphism of H as a ribbon Hopf algebra. Then there is an object F_ω in the category H-Bimod and a morphism

$$Z_\omega \in \mathrm{Hom}_{H|H}(K, F_\omega)$$

that is invariant under the action of the modular group.

The considerations leading to this result are inspired by structure one hopes to encounter in certain two-dimensional conformal field theories that are based on non-semisimple representation categories. More information about this motivation can be found in appendix B; here it suffices to remark that F_ω is a candidate for what in conformal field theory is called the *algebra of bulk fields*, and that the morphism Z_ω is a candidate for a modular invariant *partition function*.

To arrive at our result we show in fact first that the object F_ω actually carries a lot more natural structure: F_ω is a commutative symmetric Frobenius algebra in H-Bimod. Furthermore, the Frobenius algebra F_ω is a special [1] Frobenius algebra if and only if the Hopf algebra H is semisimple. A Frobenius algebra carries a natural coalgebra structure; the invariant morphism Z_ω is obtained by applying a monodromy operation to the coproduct of F_ω.

[1] A Frobenius is called special iff, up to non-zero scalars, the counit is a left inverse of the unit and the coproduct is a right inverse of the product, see Def. 4.6.

This paper is organized as follows. In section 2 we introduce the relevant structure of a monoidal category on H-Bimod and construct, for the case $\omega = id_H$, the bimodule $F = F_{id_H}$ as a Frobenius algebra in H-Bimod. In section 3 we endow the monoidal category H-Bimod with a natural braiding and show that with respect to this braiding the Frobenius algebra F is commutative, while in section 4 it is established that F is symmetric, has trivial twist, and is special iff H is semisimple. Modular invariance of Z_{id_H} is proven in section 5. Section 6 is finally devoted to the case of a general ribbon Hopf algebra automorphism ω of H, which can actually be treated by modest modifications of the arguments of sections 2 – 5. In appendix A we gather some notions from category theory and explain how the bimodules K and F_ω can be characterized as coends of suitable bifunctors. The latter shows that the objects in our constructions are canonically associated with the category of H-bimodules as an abstract category. Appendix B contains some motivation from (logarithmic) conformal field theory.

2 A Frobenius algebra in the bimodule category

2.1 Finite-dimensional ribbon Hopf algebras

In this section we collect some basic definitions and notation for Hopf algebras and recall that finite-dimensional Hopf algebras admit a canonical Frobenius algebra structure.

Throughout this paper, $\Bbbk$ is an algebraically closed field of characteristic zero and, unless noted otherwise, H is a finite-dimensional factorizable ribbon Hopf algebra over $\Bbbk$. We denote by m, η, Δ, ε and s the product, unit, coproduct, counit and antipode of the Hopf algebra H.

There exist plenty of factorizable ribbon Hopf algebras (see e.g. [Bu]). For instance, the Drinfeld double of a finite-dimensional Hopf algebra K is factorizable ribbon provided that [KaR, Thm. 3] a certain condition for the square of the antipode of K is satisfied. Let us recall what it means that a Hopf algebra is factorizable ribbon.

Definition 2.1.

(a) A Hopf algebra $H \equiv (H, m, \eta, \Delta, \varepsilon, \mathrm{s})$ is called *quasitriangular* iff it is endowed with an invertible element $R \in H \otimes H$ (called the *R-matrix*) that intertwines the coproduct and opposite coproduct, i.e. $\Delta^{\mathrm{op}} = \mathrm{ad}_R \circ \Delta$, and satisfies

$$(\Delta \otimes id_H) \circ R = R_{13} \cdot R_{23} \qquad \text{and} \qquad (id_H \otimes \Delta) \circ R = R_{13} \cdot R_{12} \,. \tag{2.1}$$

(b) The *monodromy matrix* $Q \in H \otimes H$ of a quasitriangular Hopf algebra (H, R) is the invertible element

$$Q := R_{21} \cdot R \equiv (m \otimes m) \circ (id_{\mathcal{H}} \otimes \tau_{H,H} \otimes id_{\mathcal{H}}) \circ ((\tau_{H,H} \circ R) \otimes R) \,. \tag{2.2}$$

(c) A quasitriangular Hopf algebra (H, R) is called a *ribbon* Hopf algebra iff it is endowed with a central invertible element $v \in H$, called the *ribbon element*, that satisfies

$$\mathrm{s} \circ v = v \,, \qquad \varepsilon \circ v = 1 \qquad \text{and} \qquad \Delta \circ v = (v \otimes v) \cdot Q^{-1} \,. \tag{2.3}$$

(d) A quasitriangular Hopf algebra (H, R) is called *factorizable* iff the monodromy matrix can be written as $Q = \sum_\ell h_\ell \otimes k_\ell$ with $\{h_\ell\}$ and $\{k_\ell\}$ two vector space bases of H.

Here and below, the symbol $\otimes$ denotes the tensor product over $\Bbbk$, and for vector spaces V and W the linear map $\tau_{V,W} \colon V \otimes W \xrightarrow{\cong} W \otimes V$ is the flip map which exchanges the two tensor factors. Also, we canonically identify H with $\mathrm{Hom}_\Bbbk(\Bbbk, H)$ and think of elements of (tensor products over $\Bbbk$ of) H and $H^* = \mathrm{Hom}_\Bbbk(H, \Bbbk)$ as (multi)linear maps. This has e.g. the advantage that many of our considerations still apply directly in the situation that H is a Hopf algebra, with adequate additional structure and properties, in an arbitrary $\Bbbk$-linear ribbon category instead of $\mathcal{V}ect_\Bbbk$. Various properties of the R-matrix and of the ribbon element, as well as of some further distinguished elements of H, will be recalled later on. Note that we do not assume the Hopf algebra H to be semisimple; in particular, the ribbon element does not need to be semisimple.

We also need a few further ingredients that are available for general finite-dimensional Hopf algebras, without assuming quasitriangularity, in particular the notions of (co)integrals and of a Frobenius structure for Hopf algebras.

Definition 2.2.
(a) A *left integral* of a Hopf algebra H is an element $\Lambda \in H$ satisfying

$$m \circ (id_H \otimes \Lambda) = \Lambda \circ \varepsilon \,. \tag{2.4}$$

Analogously, a *right integral* of H is an element $\tilde\Lambda \in H$ such that $m \circ (\tilde\Lambda \otimes id_H) = \tilde\Lambda \circ \varepsilon$.

(b) A *right cointegral* of H is an element $\lambda \in H^*$ satisfying

$$(\lambda \otimes id_H) \circ \Delta = \eta \circ \lambda \,. \tag{2.5}$$

A *left cointegral* is defined analogously.

Put differently, a (left or right) integral of H is nothing but a morphism of (left or right) H-modules from the trivial H-module $(\Bbbk, \varepsilon)$ to the regular H-module (H, m), while a cointegral of H is nothing but a comodule morphism from $(\Bbbk, \eta)$ to the regular H-comodule (H, Δ).

Recall [LS] that for a finite-dimensional Hopf $\Bbbk$-algebra the antipode is invertible and that H has, up to normalization, a unique non-zero left integral $\Lambda \in H$ and a unique non-zero right cointegral $\lambda \in H^*$. The number $\lambda \circ \Lambda \in \Bbbk$ is invertible. A factorizable ribbon Hopf algebra is unimodular [Ra3, Prop. 3(c)], i.e. the left integral Λ is also a right integral, implying that $\mathrm{s} \circ \Lambda = \Lambda$.

The integral and the cointegral allow one to endow Hopf $\Bbbk$-algebras with more algebraic structure. The following characterization of Frobenius algebras will be convenient.

Definition 2.3. A *Frobenius algebra* A in $\mathcal{V}ect_\Bbbk$ is a vector space A together with (bi)linear maps m_A, η_A, Δ_A and ε_A such that (A, m_A, η_A) is an algebra, $(A, \Delta_A, \varepsilon_A)$ is a coalgebra and

$$(m_A \otimes id_A) \circ (id_A \otimes \Delta_A) = \Delta_A \otimes m_A = (id_A \otimes m_A) \circ (\Delta_A \otimes id_A) \,, \tag{2.6}$$

i.e. the coproduct Δ_A is a morphism of A-bimodules.

We have

Lemma 2.4. *A finite-dimensional Hopf* $\Bbbk$*-algebra* $(H, m, \eta, \Delta, \varepsilon, \mathrm{s})$ *carries a canonical structure of a Frobenius algebra* A*, with the same algebra structure on* $A = H$*, and with Frobenius coproduct and Frobenius counit given by*

$$\Delta_A = (m \otimes \mathrm{s}) \circ (id_A \otimes (\Delta \circ \Lambda)) \qquad \text{and} \qquad \varepsilon_A = (\lambda \circ \Lambda)^{-1} \, \lambda \,. \tag{2.7}$$

This actually holds more generally for finitely generated projective Hopf algebras over commutative rings (see e.g. [Pa, KaS]), as well as for any Hopf algebra in an additive ribbon category $\mathcal{C}$ that has an invertible antipode and a left integral $\Lambda \in \mathrm{Hom}(\mathbf{1}, H)$ and right cointegral $\lambda \in \mathrm{Hom}(H, \mathbf{1})$ such that $\lambda \circ \Lambda \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is invertible (see e.g. appendix A.2 of [FSc]).

The Frobenius algebra structure given by (2.7) is unique up to rescaling the integral Λ by an invertible scalar. In the sequel, for a given choice of (non-zero) Λ, we choose the cointegral λ such that $\lambda \circ \Lambda = 1$.

2.2 H-Bimod as a monoidal category

Our focus in this paper is on natural structures on a distinguished H-bimodule, the coregular bimodule to be described below. To formulate these we need to endow the abelian category H-Bimod of H-bimodules with the structure of a sovereign braided monoidal category.

To prepare the stage, let us express a few well-known aspects of the underlying category $\mathcal{V}ect_{\Bbbk}$ of finite-dimensional $\Bbbk$-vector spaces in the language of ribbon categories. $\mathcal{V}ect_{\Bbbk}$ is a symmetric rigid monoidal category. The symmetry is furnished by the flip map. The (right) duality gives a contravariant endofunctor of $\mathcal{V}ect_{\Bbbk}$, mapping a vector space V to its dual space $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and a linear map $f\colon V \to W$ to the dual map $f^* := (d_W \otimes id_{V^*}) \circ (id_{W^*} \otimes b_V)\colon W^* \to V^*$, with $d_V\colon V^* \otimes V \to \Bbbk$ the evaluation map and $b_V\colon \Bbbk \to V \otimes V^*$ the coevaluation. Together with the flip map the right duality also gives a left duality, with evaluation and coevaluation maps $\tilde{d}_V\colon V \otimes V^* \to \Bbbk$ and $\tilde{b}_V\colon \Bbbk \to V^* \otimes V$ (in terms of dual bases $\{v_i\}$ of V and $\{\varphi^i\}$ of V^* one has $b_V(1) = \sum_i v_i \otimes \varphi^i$ and $\tilde{b}_V(1) = \sum_i \varphi^i \otimes v_i$). The left duality coincides with the right duality as a functor (i.e., $\mathcal{V}ect_{\Bbbk}$ is strictly sovereign), so there is no need to treat it separately; but it will occasionally be convenient to work with both the right duality maps d and b and with their left counterparts $\tilde{d}$ and $\tilde{b}$.

The objects of the $\Bbbk$-linear abelian category H-Bimod of bimodules over a Hopf $\Bbbk$-algebra H are triples $(X, \rho, \text{\reflectbox{ρ}})$ such that (X, ρ) is a left H-module and $(X, \text{\reflectbox{$\rho$}})$ is a right H-module and the left and right actions of H commute, $\rho \circ (id_H \otimes \text{\reflectbox{ρ}}) = \text{\reflectbox{ρ}} \circ (\rho \otimes id_H)$. Morphisms are $\Bbbk$-linear maps commuting with both actions. We denote the morphism spaces of H-Mod and H-Bimod by $\mathrm{Hom}_H(-,-)$ and $\mathrm{Hom}_{H|H}(-,-)$, respectively, while $\mathrm{Hom}(-,-) \equiv \mathrm{Hom}_{\Bbbk}(-,-)$ is reserved for linear maps.

Just like the bimodules over any unital associative algebra, H-Bimod carries a monoidal structure for which the tensor product is the one over H, for which the vector space underlying a tensor product bimodule $X \otimes_H Y$ is a non-trivial quotient of the vector space tensor product $X \otimes Y \equiv X \otimes_{\Bbbk} Y$. But for our purposes, we need instead a different monoidal structure on H-Bimod for which also the coalgebra structure of H is relevant. This is obtained by pulling back the natural $H \otimes H$-bimodule structure on $X \otimes Y$ along the coproduct to the structure of an H-bimodule. Thus if $(X, \rho_X, \text{\reflectbox{ρ}}_X)$ and $(Y, \rho_Y, \text{\reflectbox{ρ}}_Y)$ are H-bimodules, then their tensor product is $X \otimes Y$ together with the left and right actions

$$\begin{aligned} \rho_{X\otimes Y} &:= (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (\Delta \otimes id_X \otimes id_Y) \qquad \text{and} \\ \text{\reflectbox{ρ}}_{X\otimes Y} &:= (\text{\reflectbox{ρ}}_X \otimes \text{\reflectbox{ρ}}_Y) \circ (id_X \otimes \tau_{Y,H} \otimes id_H) \circ (id_X \otimes id_Y \otimes \Delta)\,. \end{aligned} \tag{2.8}$$

of H. The monoidal unit for this tensor product is the one-dimensional vector space $\Bbbk$ with both left and right H-action given by the counit, $\mathbf{1}_{H\text{-Bimod}} = (\Bbbk, \varepsilon, \varepsilon)$.

Obviously, (2.8) is just the standard tensor product

$$(X,\rho_X)\otimes^{H\text{-Mod}}(Y,\rho_Y) = (\,X\otimes Y,\, (\rho_X\otimes\rho_Y)\circ(id_H\otimes\tau_{H,X}\otimes id_Y)\circ(\Delta\otimes id_X\otimes id_Y)\,) \tag{2.9}$$

of the category H-Mod of left H-modules together with the corresponding tensor product of the category of right H-modules. For both monoidal structures the ground field $\Bbbk$, endowed with a left, respectively right, action via the counit, is the monoidal unit.

If H is a *ribbon* Hopf algebra, then (see e.g. Section XIV.6 of [Ka]) H-Mod carries the structure of a ribbon category. Analogous further structure on H-Bimod will become relevant later on, and we will introduce it in due course: a braiding on H-Bimod in Section 3, and left and right dualities and a twist in Section 4.

2.3 The coregular bimodule

We now identify an object of the monoidal category H-Bimod that is distinguished by the fact (see Appendix A.2) that it can be determined, up to unique isomorphism, by a universal property formulated in H-Bimod, and thus may be thought of as being canonically associated with H-Bimod as a rigid monoidal category. Afterwards we will endow this object F with the structure of a Frobenius algebra in the monoidal category defined by the tensor product (2.8). As a vector space, F is the dual H^* of H.

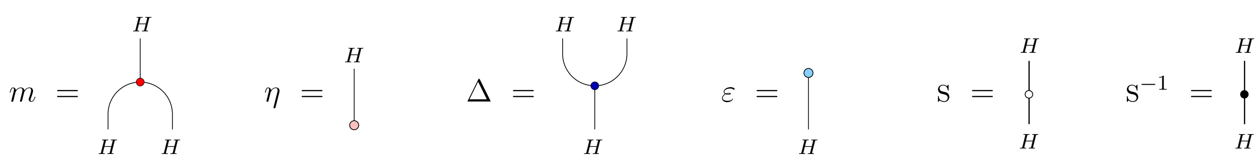
Definition 2.5. The *coregular bimodule* $F\in H$-Bimod is the vector space H^* endowed with the dual of the regular left and right actions of H on itself. Explicitly,

$$F = (H^*, \rho_F, \reflectbox{ρ}_F)\,, \tag{2.10}$$

with $\rho_F\in\mathrm{Hom}(H\otimes H^*,H^*)$ and $\reflectbox{$\rho$}_F\in\mathrm{Hom}(H^*\otimes H,H^*)$ given by

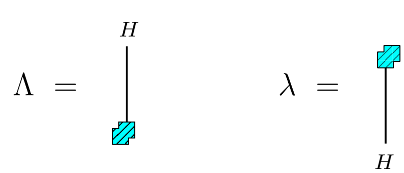
$$\begin{aligned} \rho_F &:= (d_H\otimes id_{H^*})\circ(id_{H^*}\otimes m\otimes id_{H^*})\circ(id_{H^*}\otimes \mathrm{s}\otimes b_H)\circ\tau_{H,H^*}) \qquad \text{and} \\ \reflectbox{ρ}_F &:= (d_H\otimes id_{H^*})\circ(id_{H^*}\otimes m\otimes id_{H^*})\circ(id_{H^*}\otimes id_H\otimes\tau_{H^*,H})\circ(id_{H^*}\otimes b_H\otimes \mathrm{s}^{-1})\,. \end{aligned} \tag{2.11}$$

Expressions involving maps like ρ_F and $\reflectbox{$\rho$}_F$ tend to become unwieldy, at least for the present authors. It is therefore convenient to resort to a pictorial description. We depict the structure maps of the Hopf algebra H as [2]



(2.12)

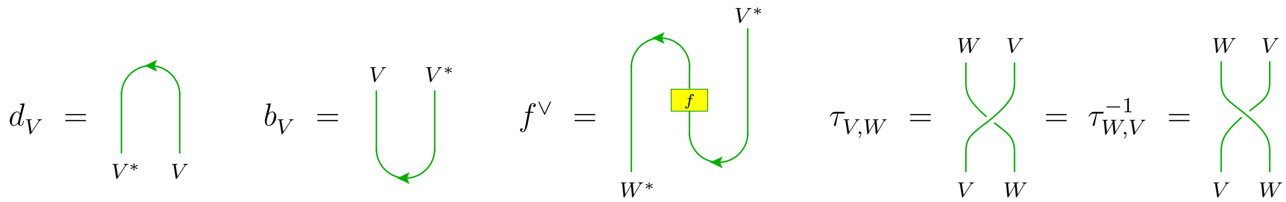
the integral and cointegral as



(2.13)

[2] It is worth stressing that these pictures refer to the category $\mathcal{V}ect_\Bbbk$ of finite-dimensional $\Bbbk$-vector spaces. Later on, we will occasionally also work with pictures for morphisms in more general monoidal categories $\mathcal{C}$; to avoid confusion we will mark pictures of the latter type with the symbol $\boxed{\mathcal{C}}$.

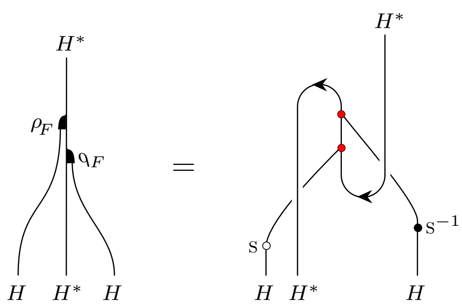
and the evaluation and coevaluation maps, dual maps, and flip maps of $\mathcal{V}ect_{\Bbbk}$ as



(2.14)

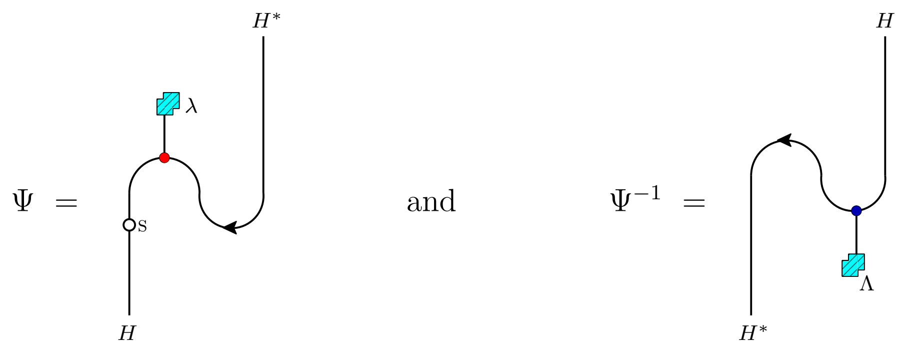
The left-pointing arrows in the pictures for the evaluation and coevaluation indicate that they refer to the *right* duality of $\mathcal{V}ect_{\Bbbk}$. The evaluation and coevaluation $\tilde{d}$ and $\tilde{b}$ for the *left* duality of $\mathcal{V}ect_{\Bbbk}$ are analogously drawn with arrows pointing to the right. Also, for better readability we indicate the flip by either an over- or underbraiding, even though in the present context of the *symmetric* monoidal category $\mathcal{V}ect_{\Bbbk}$ both of them describe the same map.

In this graphical description the left and right actions (2.11) of H on F are given by



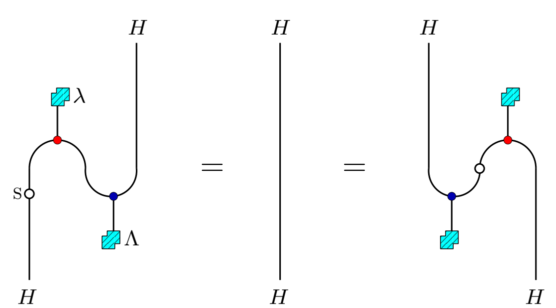
(2.15)

Let us also mention that *Frobenius map* $\Psi\colon H \to H^*$ and its inverse $\Psi^{-1}\colon H^* \to H$ are given by $\Psi(h) = \lambda \leftharpoonup \mathrm{S}(h)$ and $\Psi^{-1}(p) = \Lambda \leftharpoonup p$, respectively (see e.g. [CW3]), i.e.



(2.16)

The statement that H is a Frobenius algebra (see Lemma 2.4) is equivalent to the invertibility of Ψ. That the two maps (2.16) are indeed each others' inverses means that



(2.17)

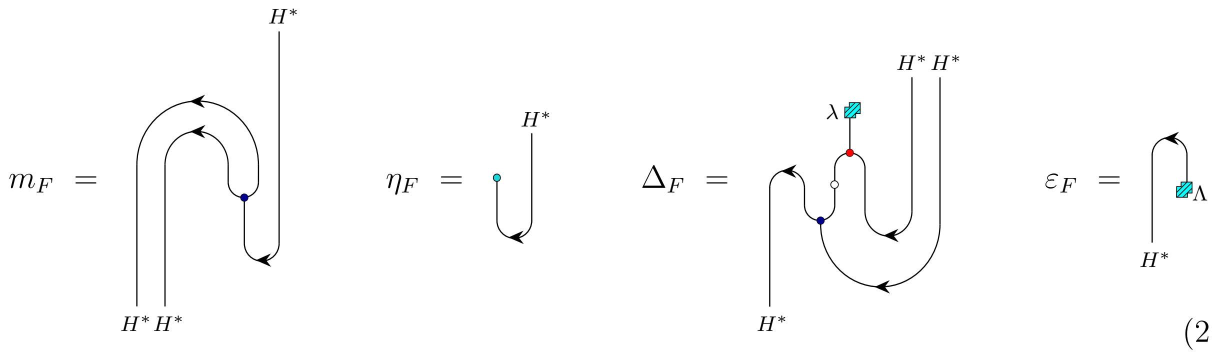
2.4 Morphisms for algebraic structure on the bimodule F

We now introduce the linear maps that furnish the structural morphisms for the Frobenius algebra structure on F as an object of the monoidal category H-Bimod. Very much like the coregular bimodule F itself, the algebra structure on F is fixed by universal properties that can be formulated within H-Bimod (see Appendix A.2). The Frobenius algebra F can thus be thought of as being canonically associated with the (abstract) monoidal category H-Bimod.

Definition 2.6. For H a finite-dimensional Hopf algebra, we introduce the following linear maps $m_F\colon H^*\otimes H^* \to H^*$, $\eta_F\colon \Bbbk \to H^*$, $\Delta_F\colon H^* \to H^*\otimes H^*$ and $\varepsilon_F\colon H^* \to \Bbbk$:

$$
\begin{aligned}
& m_F := \Delta^* \,, \qquad \eta_F := \varepsilon^* \,, \\
& \Delta_F := [(id_H \otimes (\lambda \circ m)) \circ (id_H \otimes \mathrm{s} \otimes id_H) \circ (\Delta \otimes id_H)]^* \,, \qquad \varepsilon_F := \Lambda^* \,.
\end{aligned}
\tag{2.18}
$$

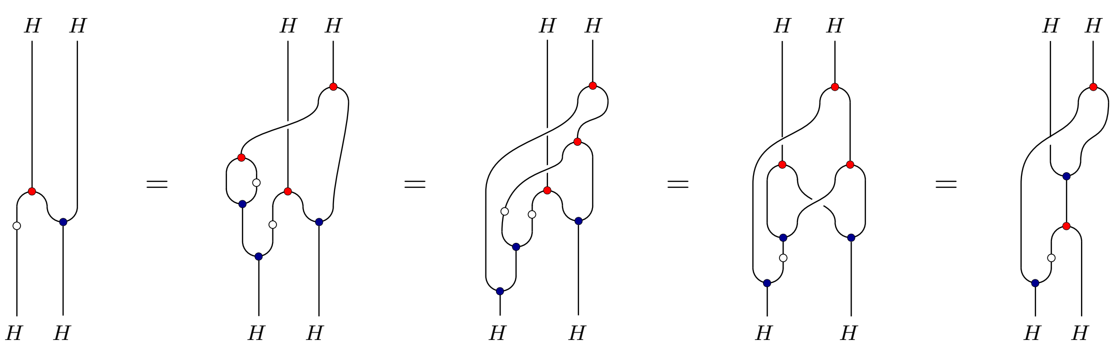
Again the graphical description appears to be convenient:



(2.19)

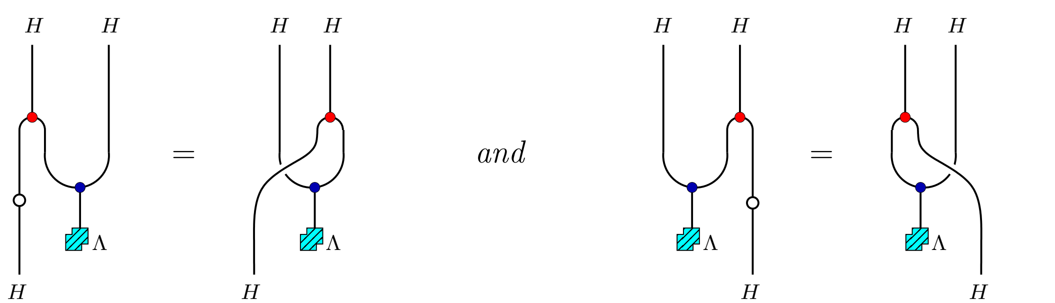
We would like to interpret the maps (2.19) as the structural morphisms of a Frobenius algebra in H-Bimod. To this end we must first show that these maps are actually morphisms of bimodules. We start with a few general observations.

Lemma 2.7. *(i) For any Hopf algebra H we have*



(2.20)

(ii) Further, if H is unimodular with integral Λ, we have



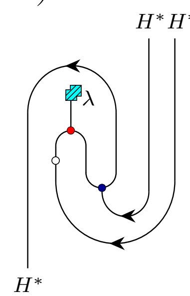
(2.21)

Proof. (i) The first equality holds by the defining properties of the antipode, unit and counit of H. The second equality follows by associativity and coassociativity, the third by the anti-coalgebra morphism property of the antipode, and the last by the connecting axiom for product and coproduct of the bialgebra underlying H.

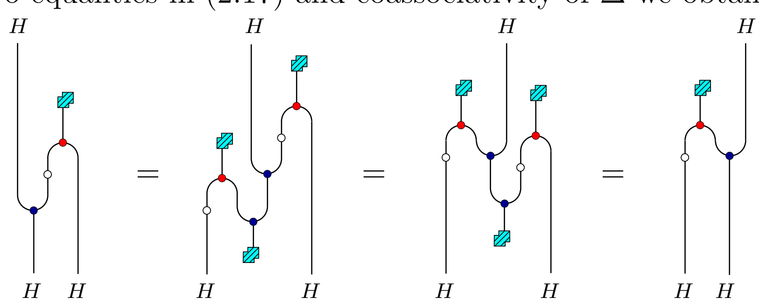
(ii) The first equality in (2.21) follows by composing (2.20) with $id_H \otimes \Lambda$ and using that Λ is a left integral. The second equality in (2.21) follows by composing the left-right-mirrored version of (2.20) (which is proven in the same way as in (i)) with $\Lambda \otimes id_H$ and using that Λ is a right integral. □

We will refer to the equality of the left and right hand sides of (2.20) as the *Hopf-Frobenius trick*.

Lemma 2.8. *The map Δ_F introduced in (2.19) can alternatively be expressed as $\Delta_F = \Delta'_F$ with*

$\Delta'_F :=$  (2.22)

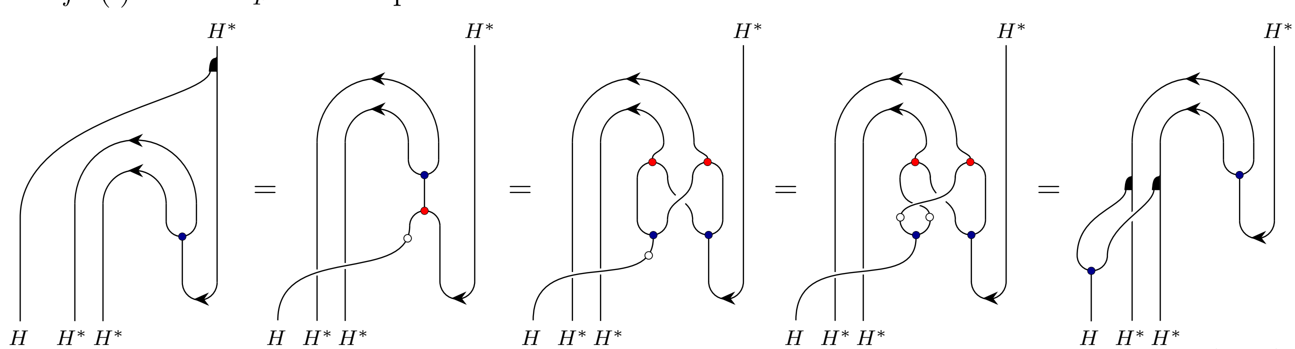
Proof. Using the two equalities in (2.17) and coassociativity of Δ we obtain

 (2.23)

Dualizing the expressions on the left and right hand sides of (2.23) establishes the claimed equality. □

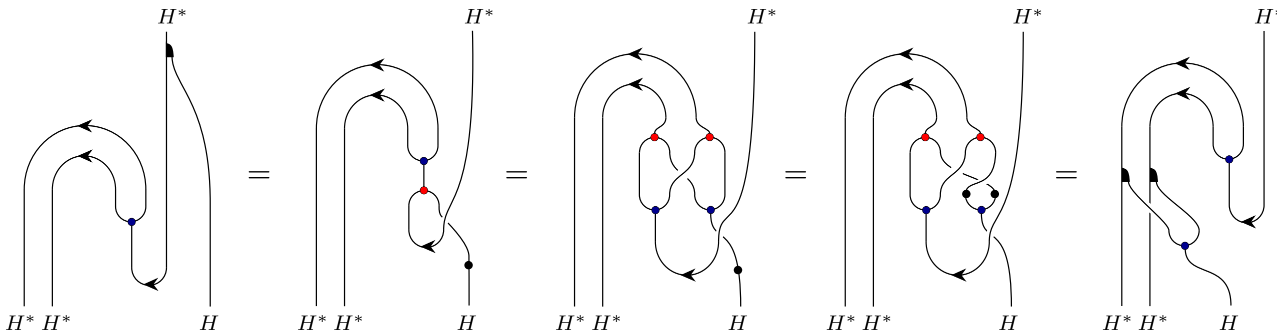
Proposition 2.9. *When H-Bimod is endowed with the tensor product (2.8), $\Bbbk$ is given the structure of the trivial H-bimodule $\Bbbk_\varepsilon = (\Bbbk, \varepsilon, \varepsilon)$ (the monoidal unit of H-Bimod) and H^* the H-bimodule structure (2.11), then the maps (2.18) are morphisms of H-bimodules.*

Proof. (i) That m_F is a morphism of left H-modules is seen as



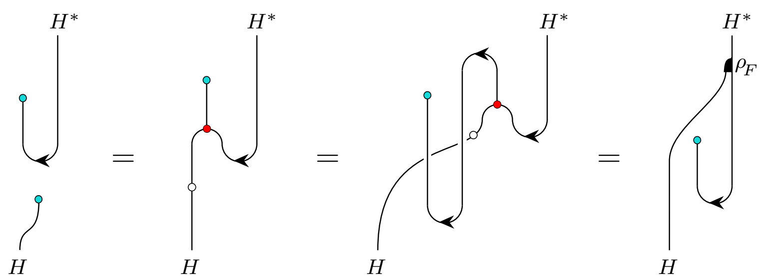
(2.24)

Here the first and last equalities just implement the definition (2.15) of the H-action, the second is the connecting axiom of H, and the third the anti-algebra morphism property of the antipode.
Similarly, that m_F is also a right module morphism follows as



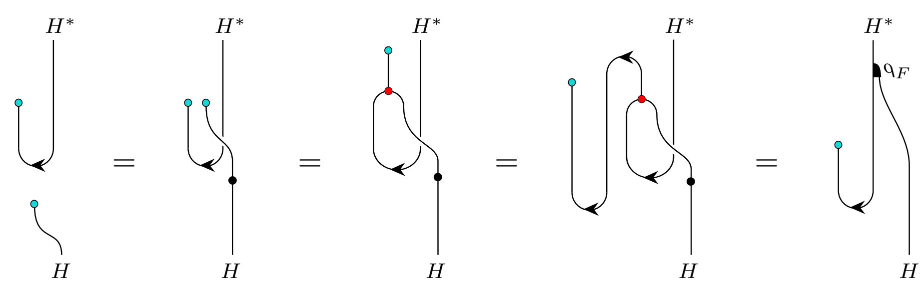
(2.25)

(ii) That η_F is a left and right module morphism follows with the help of the properties $\varepsilon \circ m = \varepsilon \otimes \varepsilon$ and $\varepsilon \circ \mathrm{S} = \varepsilon$ of the antipode. We have



(2.26)

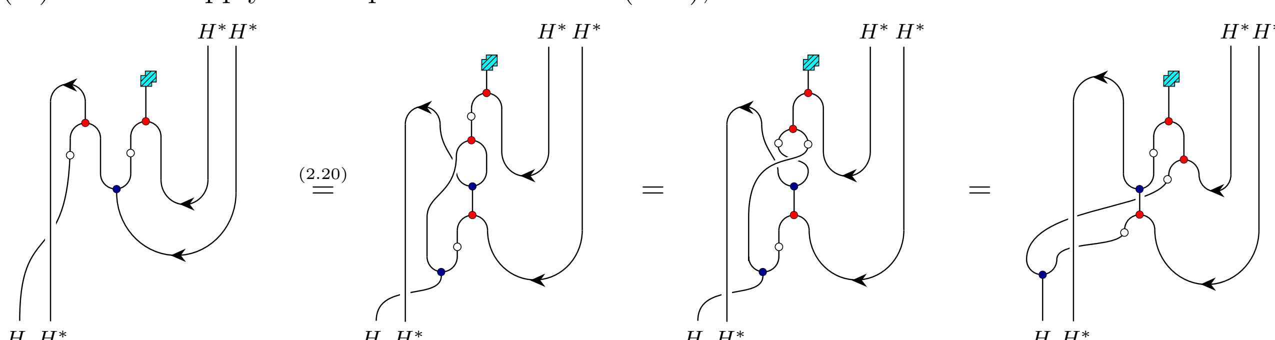
and



(2.27)

respectively. This uses in particular the homomorphism property of the counit ε of H and the fact that $\varepsilon \circ \mathrm{S} = \varepsilon$.

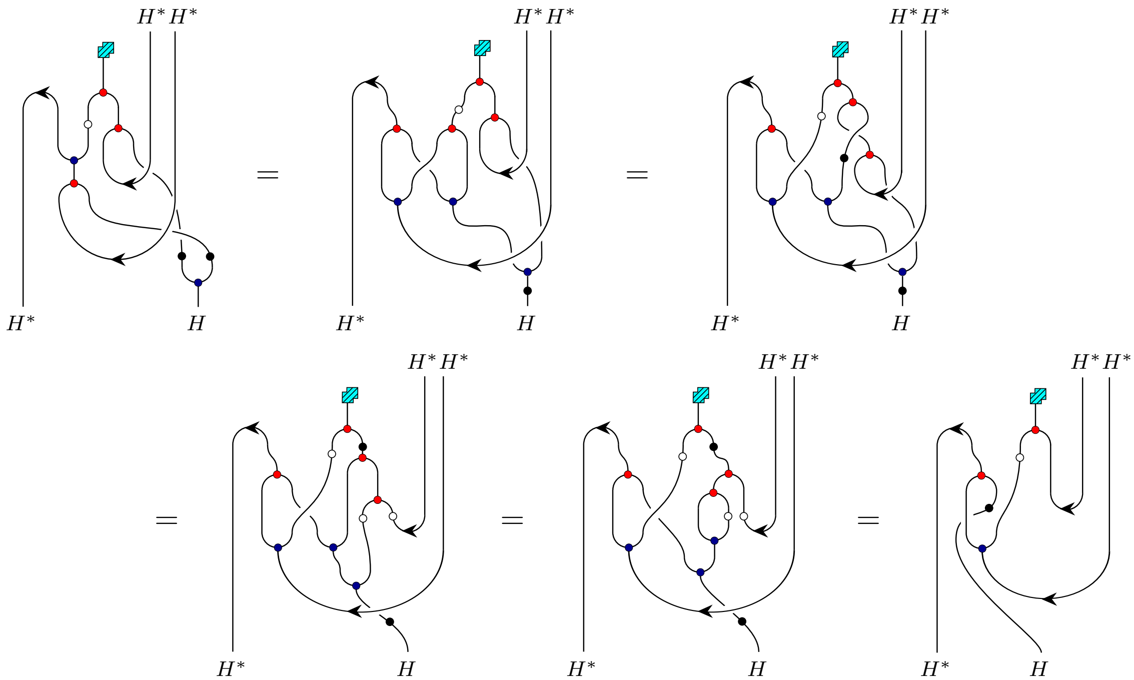
(iii) Next we apply the Hopf-Frobenius trick (2.20), which allows us to write



(2.28)

This tells us that Δ_F is a left module morphism.

(iv) For establishing the right module morphism property of Δ_F we recall from Lemma 2.8 that $\Delta_F = \Delta'_F$ with Δ'_F given by (2.22). The following chain of equalities shows that Δ'_F is a morphism of right H-modules:



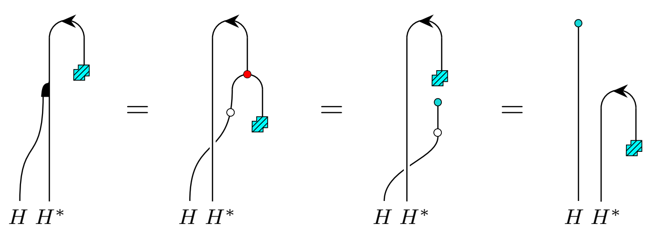
(2.29)

Here the first equality combines the anti-coalgebra morphism property of the antipode and the connecting axiom, while the second equality uses that $\lambda \circ m = \lambda \circ \tau_{H,H} \circ (id_H \otimes \mathrm{s}^2)$ and hence

$$\lambda \circ m \circ ((\mathrm{s} \circ m) \otimes id_H) = \lambda \circ m \circ [\mathrm{s} \otimes (m \circ \tau_{H,H} \circ (\mathrm{s}^{-1} \otimes id_H))] \,, \tag{2.30}$$

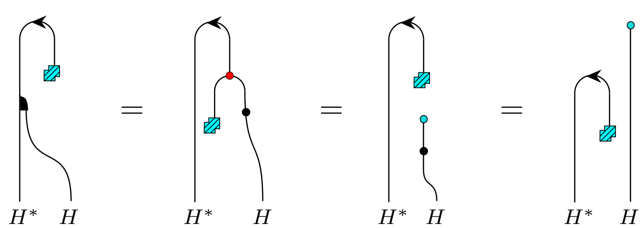
which can be shown (see [CW3, p. 4306]) by using that H is unimodular.

(v) Finally, the proof of the bimodule morphism property for ε_F is similar to the one for η_F:



(2.31)

and



(2.32)

Note that here it is again essential that H is unimodular: The second equality in (2.31) holds because Λ is a left integral, while the second equality in (2.32) holds because it is also a right integral. □

2.5 The Frobenius algebra structure of F

Proposition 2.10. *The morphisms (2.18) endow the object* $F = (H^*, \rho_F, \varsigma_F)$ *with the structure of a Frobenius algebra in H-Bimod (with tensor product (2.8)). That is,* (F, m_F, η_F) *is a (unital associative) algebra,* $(F, \Delta_F, \varepsilon_F)$ *is a (counital coassociative) coalgebra, and the two structures are connected by*

$$(id_{H^*} \otimes m_F) \circ (\Delta_F \otimes id_{H^*}) = \Delta_F \circ m_F = (m_F \otimes id_{H^*}) \circ (id_{H^*} \otimes \Delta_F) . \tag{2.33}$$

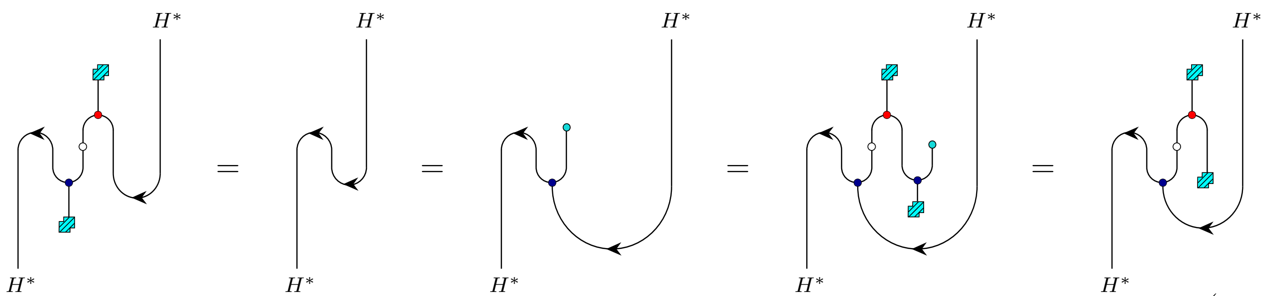
Proof. (i) That $(F, m_F, \eta_F) = (H^*, \Delta^*, \varepsilon^*)$ is a unital associative algebra just follows from (and implies) the fact that (H, Δ, ε) is a counital coassociative coalgebra.
(ii) It follows directly from the coassociativity of Δ that

$$(id_{H^*} \otimes \Delta'_F) \circ \Delta_F = (\Delta_F \otimes id_{H^*}) \circ \Delta'_F . \tag{2.34}$$

Since, as seen above, $\Delta'_F = \Delta_F$, this shows that Δ_F is a coassociative coproduct.
(iii) The coassociativity of Δ also implies directly the first of the Frobenius properties (2.33), as well as

$$(m_F \otimes id_{H^*}) \circ (id_{H^*} \otimes \Delta'_F) = \Delta'_F \circ m_F . \tag{2.35}$$

In view of $\Delta'_F = \Delta_F$, (2.35) is the second of the equalities (2.33).
(iv) That $\varepsilon_F = \Lambda^*$ is a counit for the coproduct Δ_F follows with the help of the invertibility (2.17) of the Frobenius map: we have



(2.36)

Here the left hand side is $(id_{H^*} \otimes \varepsilon_F) \circ \Delta_F$, while the right hand side is $(\varepsilon_F \otimes id_{H^*}) \circ \Delta_F$. □

Remark 2.11. The spaces of left- and right-module morphisms, respectively, from F to $\mathbf{1}$ are given by $\Bbbk\,\Lambda_{\mathrm{r}}$ and by $\Bbbk\,\Lambda_{\mathrm{l}}$, respectively, with Λ_{r} and Λ_{l} non-zero left and right integrals of H. Thus a non-zero bimodule morphism from F to $\mathbf{1}$ exists iff H is unimodular, and in this case it is unique up to a non-zero scalar. In particular, up to a non-zero scalar the Frobenius counit ε_F is already completely determined by the requirement that it is a morphism of bimodules.

3 Commutativity

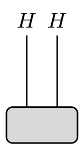
The conventional tensor product (2.9) of bimodules generically does not admit a braiding. In contrast, the monoidal category H-Bimod, with tensor product as defined in (2.8), over a quasitriangular Hopf algebra admits braidings, and in fact can generically be endowed with several inequivalent ones. We will select one such braiding c and then show that with respect to the braiding c the algebra (F, m_F, η_F) is commutative.

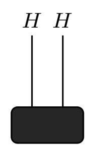
The R-matrix $R \in H \otimes H$ is equivalent to a braiding $c^{H\text{-Mod}}$ on the category H-Mod of left H-modules, consisting of a natural family of isomorphisms in $\mathrm{Hom}_H(X \otimes Y, Y \otimes X)$ for each pair $(X, \rho_X), (Y, \rho_Y)$ of H-modules. These braiding isomorphisms are given by

$$c^{H\text{-Mod}}_{X,Y} = \tau_{X,Y} \circ (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (R \otimes id_X \otimes id_Y)\,, \tag{3.1}$$

where τ is the flip map. (When written in terms of elements $x \in X$ and $y \in Y$, this amounts to $x \otimes y \mapsto \sum_i s_i\, y \otimes r_i\, x$ for $R = \sum_i r_i \otimes s_i$, but recall that we largely refrain from working with elements.) The inverse braiding is given by a similar formula, with R replaced by $R_{21}^{-1} \equiv \tau_{H,H} \circ R^{-1}$. Besides R, also the inverse R_{21}^{-1} endows the category H-Mod with the structures of a braided tensor category; the two braidings are inequivalent unless R_{21}^{-1} equals R, in which case the category is symmetric.

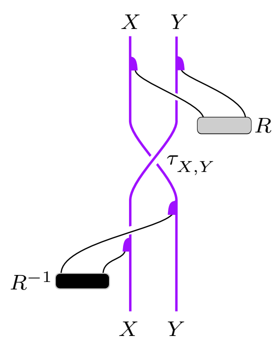
Likewise one can act with R and with R_{21}^{-1} from the right to obtain two different braidings on the category of right H-modules. As a consequence, with respect to the chosen tensor product on H-Bimod there are two inequivalent natural braidings obtained by either using R both on the left and on the right, or else using (say) R_{21}^{-1} on the left and R on the right. For our present purposes (compare Lemma A.4(iii)) the second of these possibilities turns out to be the relevant braiding c. Pictorially, describing the R-matrix and its inverse by

R = 

and R^{-1} = 

(3.2)

the braiding on H-Bimod looks as follows:

$c_{X,Y}$ = 

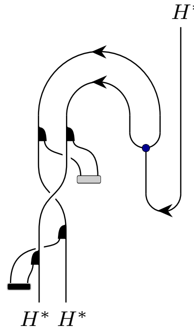
(3.3)

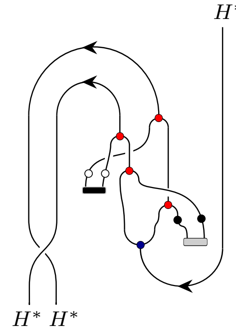
We are now in a position to state

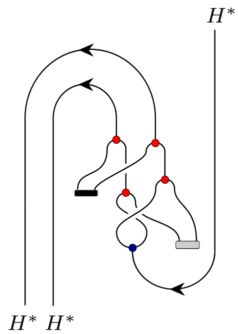
Proposition 3.1. *The product m_F of the Frobenius algebra F in H-Bimod is commutative with respect to the braiding (3.3):*

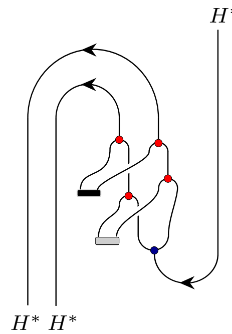
$$m_F \circ c_{F,F} = m_F\,. \tag{3.4}$$

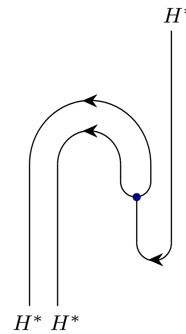
Proof. We have

$m_F \circ c_{F,F} =$ 

$=$ 

$=$ 

$=$ 

$=$ 

$= m_F \,.$ (3.5)

Here in the second equality the definition (2.11) of the H-actions on F is inserted. The third equality holds because the R-matrix satisfies

$$(\mathrm{s} \otimes id_H) \circ R = R^{-1} = (id_H \otimes \mathrm{s}^{-1}) \circ R \,, \tag{3.6}$$

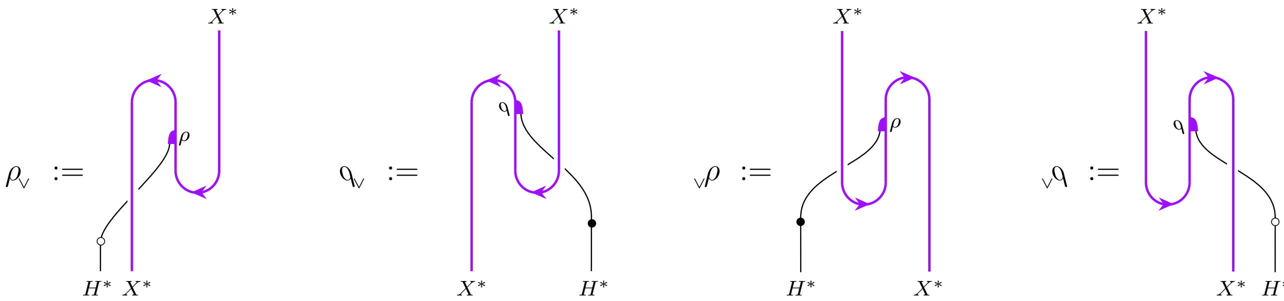
which implies $(\mathrm{s} \otimes \mathrm{s}) \circ R^{-1} = R^{-1}$ as well as $(\mathrm{s}^{-1} \otimes \mathrm{s}^{-1}) \circ R = R$. The fourth equality follows by the defining property of R to intertwine the coproduct and opposite coproduct of H. □

4 Symmetry, specialness and twist

By combining the dualities of $\mathcal{V}ect_{\Bbbk}$ with the antipode or its inverse, one obtains left and right dualities on the category H-Mod of left modules over a finite-dimensional Hopf algebra H, and likewise for right H-modules. In the same way we can define left and right dualities on H-Bimod. Since the monoidal unit of H-Bimod (with our choice of tensor product) is the ground field $\Bbbk$, we can actually take for the evaluation and coevaluation morphisms (and thus for the action of the functors on morphisms) just the evaluation and coevaluation maps (2.14) of $\mathcal{V}ect_{\Bbbk}$, and choose to define the action on objects $X = (X, \rho, \varrho) \in H$-Bimod by

$$X^\vee := (X^*, \rho_\vee, \varrho_\vee) \qquad \text{and} \qquad {}^\vee X := (X^*, {}_\vee\rho, {}_\vee\varrho) \tag{4.1}$$

with



(4.2)

That the morphisms (4.2) are (left respectively right) H-actions follows from the fact that the antipode is an anti-algebra morphism, and that the evaluations and coevaluations are bimodule morphisms follows from the defining property $m\circ(\mathrm{s}\otimes id_H)\circ\Delta=\eta\circ\varepsilon=m\circ(id_H\otimes\mathrm{s})\circ\Delta$ of the antipode. Note that with our definition of dualities [3] we have

$$ {}^\vee(X^\vee) = X = ({}^\vee X)^\vee \tag{4.3}$$

as equalities (not just isomorphisms) of H-bimodules.

The *canonical element* (also called *Drinfeld element*) $u\in H$ of a quasitriangular Hopf algebra (H,R) with invertible antipode is the element

$$ u := m\circ(\mathrm{s}\otimes id_H)\circ\tau_{H,H}\circ R\,. \tag{4.4}$$

u is invertible and satisfies $\mathrm{s}^2=\mathrm{ad}_u$ [Ka, prop VIII.4.1]. We denote by $t\in H$ the inverse of the so-called special group-like element, i.e. the product

$$ t := u\,v^{-1} \equiv m\circ(u\otimes v^{-1}) \tag{4.5}$$

of the Drinfeld element and the inverse of the ribbon element v. Since v is invertible and central, we have $\mathrm{ad}_t=\mathrm{s}^2$ and, as a consequence,

$$ m\circ(\mathrm{s}\otimes t) = m\circ(t\otimes\mathrm{s}^{-1}) \qquad\text{and}\qquad m\circ(\mathrm{s}^{-1}\otimes t^{-1})=m\circ(t^{-1}\otimes\mathrm{s})\,. \tag{4.6}$$

Also, since t is group-like we have $\varepsilon\circ t=1$ and

$$ \mathrm{s}\circ t = t^{-1} = \mathrm{s}^{-1}\circ t\,. \tag{4.7}$$

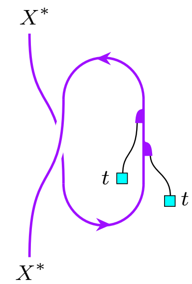
A *sovereign* category is a category with left and right dualities that are naturally isomorphic as functors and for which this natural isomorphism π is monoidal [Dr, Def. 2.7]. [4] The category H-Mod of left modules over a ribbon Hopf algebra H is sovereign iff the square of the antipode of H is inner [Bi, Dr]. Similarly, we have

[3] The left and right duals of any object in a category with dualities are unique up to distinguished isomorphism. Our choice does not make use of the fact that H is a ribbon Hopf algebra. Another realization of the dualities on H-Mod (and analogously on H-Bimod), which involves the special group-like element of H and hence does use the ribbon structure, is described e.g. in [Vi, Lemma 4.2].

[4] Equivalently [Ye1, Prop. 2.11] one may require the existence of monoidal natural isomorphisms between the (left or right) double dual functors and the identity functor. The latter is called a balanced structure (see e.g. section 1.7 of [Da]), or sometimes also a pivotal structure (see e.g. section 3 of [Sc]).

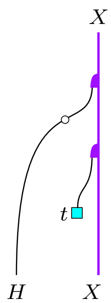
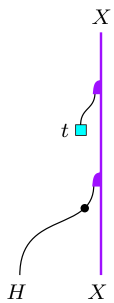
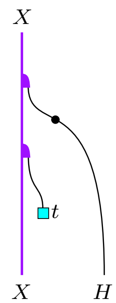
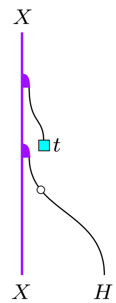
Lemma 4.1. *The category H-Bimod of bimodules over a ribbon Hopf algebra H with invertible antipode is sovereign.*

Proof. For $X \in H$-Bimod consider the linear map

$\pi_X \;:=$  $\in \mathrm{Hom}_{\Bbbk}(X^*, X^*)\,,$ (4.8)

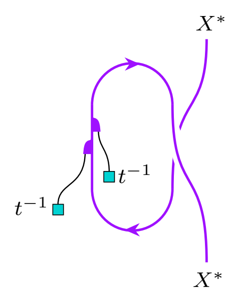
where X^* is the vector space dual of X. To establish sovereignty of H-Bimod we will show that π_X is an invertible bimodule intertwiner from $X^\vee$ to ${}^\vee X$ (i.e. the dual bimodules as defined in (4.1)), that the family $\{\pi_X\}$ is natural, and that it is monoidal, i.e. $\pi_{X\otimes Y} = \pi_X \otimes \pi_Y$.

(i) That π_X is a morphism in $\mathrm{Hom}_{H|H}(X^\vee, {}^\vee X)$ is equivalent to

 $=$  and  $=$  (4.9)

This in turn follows directly by combining (4.6) and the (left, respectively right) representation properties.

(ii) With the help of the defining properties of the evaluation and coevaluation maps it is easily checked that

$\pi_X^{-1} \;:=$  $\in \mathrm{Hom}_{\Bbbk}(X^*, X^*)$ (4.10)

is a linear two-sided inverse of π_X. That π_X^{-1} is a bimodule morphism is then automatic.

(iii) That the family $\{\pi_X\}$ of isomorphisms furnishes a natural transformation from the right to the left duality functor means that for any morphism $f \in \mathrm{Hom}_{H|H}(X,Y)$ one has

$$ {}^\vee f \circ \pi_Y = \pi_X \circ f^\vee \qquad (4.11) $$

as morphisms in $\mathrm{Hom}_{H|H}(Y^\vee, {}^\vee X)$. Now by sovereignty of $\mathcal{V}ect_{\Bbbk}$ we know that ${}^\vee f = f^\vee$ as linear maps from Y^* to X^*, and as a consequence (4.11) is equivalent to

$$ f \circ \varphi_X = \varphi_Y \circ f \qquad (4.12) $$

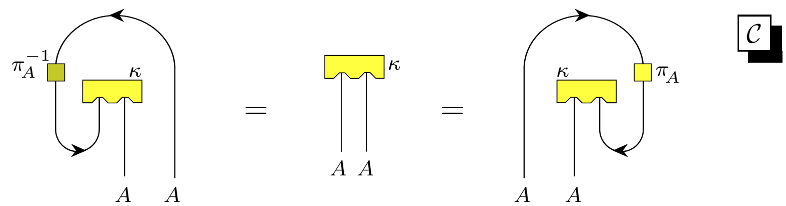
as morphisms in $\mathrm{Hom}_{H|H}(X,Y)$, where φ_X is the left action on X with $t \in H$ composed with the right action on X with t. (4.12), in turn, is a direct consequence of the fact that f is a bimodule morphism.

(iv) That π_X is monoidal follows from the fact that t is group-like. $\square$

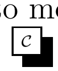
Definition 4.2.

(a) An *invariant pairing* on an algebra $A = (A, m, \eta)$ in a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is a morphism $\kappa \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, \mathbf{1})$ satisfying $\kappa \circ (m \otimes id_A) = \kappa \circ (id_A \otimes m)$.

b) A *symmetric* algebra (A, κ) in a sovereign category $\mathcal{C}$ is an algebra A in $\mathcal{C}$ together with an invariant pairing κ that is symmetric, i.e. satisfies



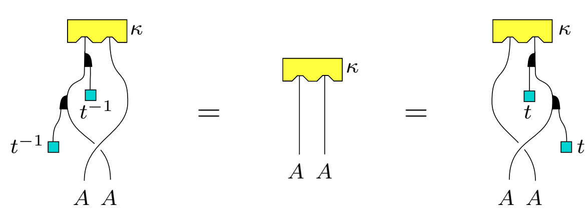
(4.13)

Remark 4.3. (i) Unlike the pictures used so far (and most of the pictures below), which describe morphisms in $\mathcal{V}ect$, (4.13) refers to morphisms in the category $\mathcal{C}$ rather than in $\mathcal{V}ect$; to emphasize this we have added the box  to the picture. Also note that the morphisms (4.13) involve the left and right dualities of $\mathcal{C}$, but do not assume a braiding. Thus the natural setting for the notion of symmetry of an algebra is the one of sovereign categories $\mathcal{C}$; a braiding on $\mathcal{C}$ is not needed.

(ii) An algebra with an invariant pairing κ is Frobenius iff κ is non-degenerate, see e.g. [FSt, Sect. 3].

(iii) The two equalities in (4.13) actually imply each other.

In the case of the category H-Bimod with sovereign structure π as defined in (4.8), the equalities (4.13) read



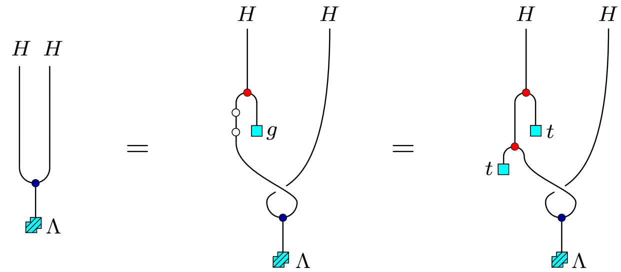
(4.14)

Theorem 4.4. *For any unimodular finite-dimensional ribbon Hopf algebra H the pair (F, κ_F) with F the coregular bimodule (with Frobenius algebra structure as defined above) and*

$$\kappa_F := \varepsilon_F \circ m_F = (\Delta \circ \Lambda)^* \tag{4.15}$$

is a symmetric Frobenius algebra in H-Bimod.

Proof. That the pairing κ_F is invariant follows directly from the coassociativity of Δ_F. To establish that κ_F is symmetric, consider the following equalities:



(4.16)

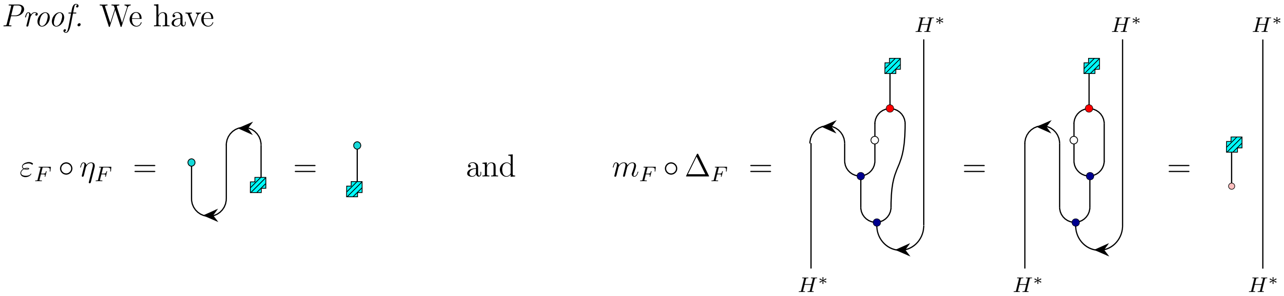
The first equality is Theorem 3(d) of [Ra2], and involves the right modular element (also known as distinguished group-like element) g of H, which by definition satisfies $g \circ \lambda = (id_H \otimes \lambda) \circ \Delta$. The second equality uses that $g = t^2$ (which holds by Theorem 2(a) and Corollary 1 of [Ra1], specialized to unimodular H) and $\mathrm{s}^2 = \mathrm{ad}_t$.
Using also the identity $\rho_F \circ (t^{-1} \otimes id_{H^*}) = \big(m \circ (t \otimes id_H)\big)^*$ (which, in turn, uses (4.7)), it follows that the equality of the left and right sides of (4.16) is nothing but the dualized version of the first of the equalities (4.14) for the case $A = F$ and $\kappa = \kappa_F$. $\square$

Next we observe:

Lemma 4.5. *The morphisms (2.18) satisfy*

$$\varepsilon_F \circ \eta_F = \varepsilon \circ \Lambda \qquad \text{and} \qquad m_F \circ \Delta_F = (\lambda \circ \varepsilon)\, id_{H^*} \,. \tag{4.17}$$

Proof. We have



(4.18)

Here the last equality uses the defining property of the antipode s of H. $\square$

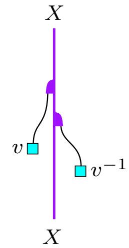
Definition 4.6. A Frobenius algebra $(A, m_A, \eta_A, \Delta_A, \varepsilon_A)$ in a $\Bbbk$-linear monoidal category is called *special* [FRS, EP] (or strongly separable [Mü]) iff $\varepsilon_A \circ \eta_A = \xi\, id_{\mathbf{1}}$ and $m_A \circ \Delta_A = \zeta\, id_A$ with $\xi, \zeta \in \Bbbk^\times$.

It is known that a finite-dimensional Hopf algebra H is semisimple iff the Maschke number $\varepsilon \circ \Lambda \in \Bbbk$ is non-zero, and it is cosemisimple iff $\lambda \circ \varepsilon \in \Bbbk$ is non-zero [LS]; also, in characteristic zero cosemisimplicity is implied by semisimplicity [LR, Thm. 3.3]. Thus we have

Corollary 4.7. *The Frobenius algebra F in H-Bimod is special iff H is semisimple.*

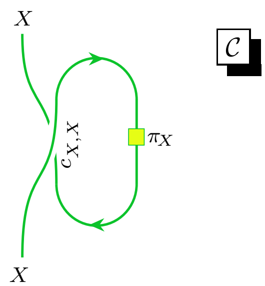
As already pointed out we do *not*, however, assume that H is semisimple. We now note a consequence of the fact that F is commutative and symmetric, irrespective of whether H is semisimple or not. We first observe:

Lemma 4.8. *The braided monoidal category H-Bimod of bimodules over a finite-dimensional ribbon Hopf $\Bbbk$-algebra H is balanced. The twist endomorphisms are given by*

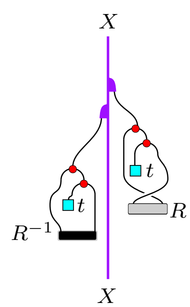
$\theta_X \;=$  (4.19)

with v the ribbon element of H.

Proof. We have seen that H-Bimod is braided, and according to lemma 4.1 it is sovereign. Now a braided monoidal category with a (left or right) duality is sovereign iff it is balanced, see e.g. prop. 2.11 of [Ye1].
For any sovereign braided monoidal category $\mathcal{C}$ the twist endomorphsms θ_X can be obtained by combining the braiding, dualities and sovereign structure according to

$\theta_X \;=$  (4.20)

With the explicit form (3.3) of the braiding and (4.8) of the sovereign structure, this results in

$\theta_X \;=$  (4.21)

Using the relations $t = v^{-1}u$ and $\mathrm{S}^2 = \mathrm{ad}_t$, the fact that v is central and the relation (4.4) between the canonical element u and the R-matrix then gives the formula (4.19). □

Remark 4.9. (i) By using that $v \in H$ is central and satisfies $\mathrm{S} \circ v = v$. it follows immediately from (4.19) that the Frobenius algebra F has trivial twist,

$$\theta_F = id_{H^*} \, . \tag{4.22}$$

(ii) In fact, a commutative symmetric Frobenius algebra in any sovereign braided category has trivial twist. This was shown in Prop. 2.25(i) of [FFRS] for the case that the category is strictly sovereign (i.e. that the sovereign structure is trivial in the sense that $\pi_X = id_X$ for all X), and the proof easily carries over to general sovereign categories.

5 Modular invariance

Our focus so far has been on a natural object in the sovereign braided finite tensor category H-Bimod, the symmetric Frobenius algebra F. But in any such category there exists another natural object K, which has been studied by Lyubashenko. K is a Hopf algebra, and it plays a central role in the construction of mapping class group actions. In fact, Lyubashenko has constructed a group homomorphism

$$\varpi_K : \quad \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{End}_{\mathcal{C}}(K) \,. \tag{5.1}$$

By pre-composition with this group homomorphism one endows a morphism space of the form $\mathrm{Hom}_{\mathcal{C}}(K \otimes X, Y)$, for X, Y any objects of $\mathcal{C}$, with a linear $\mathrm{SL}(2,\mathbb{Z})$-action

$$\begin{array}{rcl} \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{Hom}_{\mathcal{C}}(K \otimes X, Y) & \to & \mathrm{Hom}_{\mathcal{C}}(K \otimes X, Y) \\ (\gamma, f) & \mapsto & f \circ (\varpi_K(\gamma^{-1}) \otimes id_X) \,. \end{array} \tag{5.2}$$

Similarly by post-composition with $id_Y \otimes \varpi_K(\gamma)$ one obtains an $\mathrm{SL}(2,\mathbb{Z})$-action on morphism spaces of the form $\mathrm{Hom}_{\mathcal{C}}(X, Y \otimes K)$.

These general constructions apply in particular to the finite tensor category H-Bimod. The main goal in this section is to use the coproduct of the Frobenius algebra F to construct an element in $\mathrm{Hom}_{H|H}(K, F)$ that is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$. A similar construction allows one to derive an invariant element in $\mathrm{Hom}_{H|H}(K \otimes F, \mathbf{1})$ from the product of F. For the motivation to detect such elements and for possible applications we refer to Appendix B.

5.1 Coends with $\mathrm{SL}(2,\mathbb{Z})$-action

A finite tensor category [EO] is a $\Bbbk$-linear abelian rigid monoidal category with enough projectives and with finitely many simple objects up to isomorphism, with simple tensor unit, and with every object having a composition series of finite length. Both H-Mod and H-Bimod, for H a finite-dimensional unimodular ribbon Hopf algebra, belong to this class of categories, see [LM, Ly1].

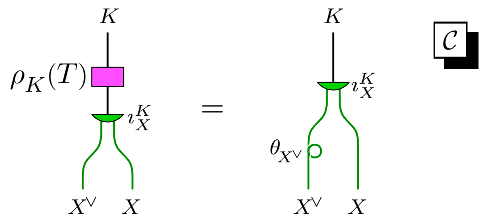
Let $\mathcal{C}$ be a sovereign braided finite tensor category. As shown in [Ly1,Ke] (compare also [Vi] or, as a review, Sections 4.3 and 4.5 of [FSc]), there exists an object K in $\mathcal{C}$ that carries a natural structure of Hopf algebra in $\mathcal{C}$. Moreover, there is a two-sided integral as well as a Hopf pairing for this Hopf algebra K, and there is a group homomorphism (5.1) from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{End}_{\mathcal{C}}(K)$ [LM, Ly1, Ly2]. The Hopf algebra K can be characterized as the *coend* [5]

$$K = \int^X F(X,X) = \int^X X^\vee \otimes X \tag{5.3}$$

of the functor F that acts on objects as $(X,Y) \mapsto X^\vee \otimes Y$. As described in some detail in appendix A.4, in the case $\mathcal{C} = H$-Bimod the object K is the *coadjoint bimodule* $H^*_{\triangleright\triangleleft}$, that is, the underlying vector space is the tensor product $H^* \otimes_\Bbbk H^*$, and this space is endowed with a left H-action by the coadjoint left action (A.27) on the first factor, and with a right H-action by the coadjoint right action on the second factor.

[5] The definition of the coend K of a functor $G\colon\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, including the associated dinatural family $\imath^K$ of morphisms $\imath^K_X \in \mathrm{Hom}_{\mathcal{C}}(F(X,X), K)$, will be recalled in Appendix A.1.

The group $\mathrm{SL}(2,\mathbb{Z})$ can be presented by two generators S and T subject to the relations $S^2 = (ST)^3$ and $S^4 = 1$. In the case of a general braided finite tensor category $\mathcal{C}$ the action of S and T on the space $\mathrm{End}_{\mathcal{C}}(K)$ is defined with the help of the braiding c and the twist θ of $\mathcal{C}$, respectively [LM, Ly1, Ly2]. The endomorphism $\rho_K(T)$ is given by the dinatural family

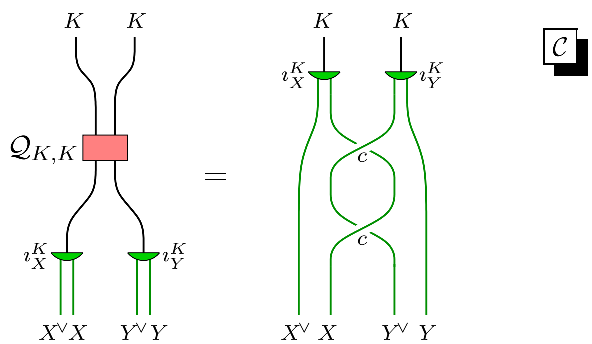


(5.4)

Here it is used that a morphism f with domain the coend K is uniquely determined by the dinatural family $\{f \circ \imath^K_X\}$ of morphisms. For S one defines

$$\rho_K(S) = (\varepsilon_K \otimes id_K) \circ \mathcal{Q}_{K,K} \circ (id_K \otimes \Lambda_K)\,, \tag{5.5}$$

where ε_K and Λ_K are the counit and the two-sided integral of the Hopf algebra K, respectively, while the morphism $\mathcal{Q}_{K,K} \in \mathrm{End}_{\mathcal{C}}(K \otimes K)$ determined through monodromies $c_{Y^\vee,X} \circ c_{X,Y^\vee}$ according to



(5.6)

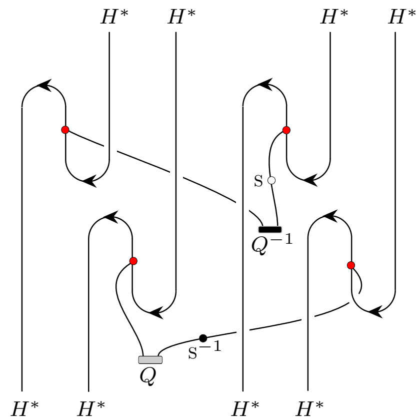
We also note that the Hopf algebra K is endowed with a Hopf pairing ω_K, given by [Ly1]

$$\omega_K = (\varepsilon_K \otimes \varepsilon_K) \circ \mathcal{Q}_{K,K}\,. \tag{5.7}$$

5.2 The Drinfeld map

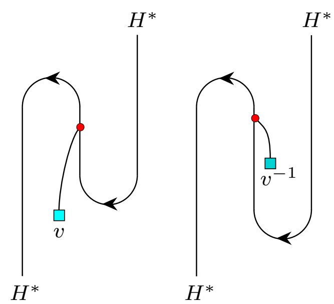
Let us now specialize the latter formulas to the case of our interest, i.e. $\mathcal{C} = H$-Bimod. Then the coend is $K = H^*_{\triangleright\triangleleft}$, with dinatural family $\imath^K = i^{\triangleright\triangleleft}$ given by (A.30), while the twist is given by (4.19) and the braiding by (3.3). Further, the structure morphisms of the categorical Hopf algebra $H^*_{\triangleright\triangleleft}$ can be expressed through those of the algebraic Hopf algebra H; in particular, the counit and integral are $\varepsilon_{\triangleright\triangleleft} = \eta^\vee \otimes \eta^\vee$ (see (A.33)) and $\Lambda_{\triangleright\triangleleft} = \lambda^\vee \otimes \lambda^\vee$ (see (A.37)). The monodromy morphism $\mathcal{Q}_{H^*_{\triangleright\triangleleft},H^*_{\triangleright\triangleleft}} \in \mathrm{End}_{H|H}(H^*_{\triangleright\triangleleft} \otimes H^*_{\triangleright\triangleleft})$ that was introduced in (5.6) then reads

$$\mathcal{Q}_{H^*_{\triangleright\triangleleft},H^*_{\triangleright\triangleleft}} \;=\; \tag{5.8}$$



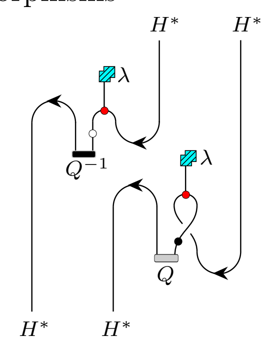
while the general formulas for $\rho_K(T)$ and $\rho_K(S)$ specialize to the morphisms

$$\rho_{\triangleright\triangleleft}(T) \;=$$



and

$$\rho_{\triangleright\triangleleft}(S) \;= \tag{5.9}$$



in $\mathrm{End}_{H|H}(H^*_{\triangleright\triangleleft})$. Note that the morphism $\rho_{\triangleright\triangleleft}(S)$ is composed of (variants of) the Frobenius map (2.16) and the *Drinfeld map*

$$f_Q := (d_H \otimes id_H) \circ (id_{H^*} \otimes Q) \;\in \mathrm{Hom}(H^*, H)\,. \tag{5.10}$$

In order that $\rho_{\triangleright\triangleleft}(S)$ is invertible, which is necessary for having an $\mathrm{SL}(2,\mathbb{Z})$-representation, it is necessary and sufficient that f_Q is invertible.

Remark 5.1. By the results for general $\mathcal{C}$, $\rho_{\triangleright\triangleleft}(S)$ is indeed a morphism in H-Bimod. But this is also easily checked directly: One just has to use that the Drinfeld map intertwines the left coadjoint action $\rho_\triangleright$ (see (A.27)) of H on H^* and the left adjoint action ρ_{ad} of H on itself [CW1, Prop. 2.5(5)], i.e. that

$$f_Q \in \mathrm{Hom}_H(H^*_\triangleright, H_{\mathrm{ad}})\,, \tag{5.11}$$

together with the fact that the cointegral λ satisfies (since H is unimodular) [Ra2, Thm. 3]

$$\lambda \circ m = \lambda \circ m \circ \tau_{H,H} \circ (id_H \otimes \mathrm{S}^2)\,. \tag{5.12}$$

Remark 5.2. The Drinfeld map f_Q of a finite-dimensional quasitriangular Hopf algebra H is invertible iff H is factorizable. In the semisimple case, factorizability is the essential ingredient for a Hopf algebra to be modular [NTV, Lemma 8.2]. Here, without any assumption of semisimplicity, we see again a direct link between invertibility of S and factorizability.

Remark 5.3. The coadjoint H-module $H^*_\triangleright$ is actually the coend (5.3) for the case that $\mathcal{C}$ is the category H-Mod of left H-modules. In this case the S-transformation (5.5) is precisely the composition

$$\rho_\triangleright(S) = \Psi \circ f_Q \tag{5.13}$$

of the Drinfeld and Frobenius maps (also called the quantum Fourier transform), see e.g. [LM, FGST]. Further, the Drinfeld map f_Q is related to the Hopf pairing $\omega_{H^*_\triangleright}$ from (5.7) for the coend $H^*_\triangleright$ in H-Mod by

$$f_Q = (\mathrm{s} \otimes \omega_{H^*_\triangleright}) \circ (id_H \otimes \tau_{H^*,H^*}) \circ (b_H \otimes id_{H^*})\,. \tag{5.14}$$

In particular, since the antipode is invertible, the Drinfeld map of H is invertible iff the Hopf pairing of $H^*_\triangleright$ is non-degenerate.

Remark 5.4. For factorizable H, the Drinfeld map f_Q maps any non-zero cointegral λ of H to a non-zero integral Λ. Thus we may (and do) choose λ and Λ (uniquely, up to a common factor ± 1) such that besides $\lambda \circ \Lambda = 1$ we also have

$$f_Q(\lambda) = \Lambda \tag{5.15}$$

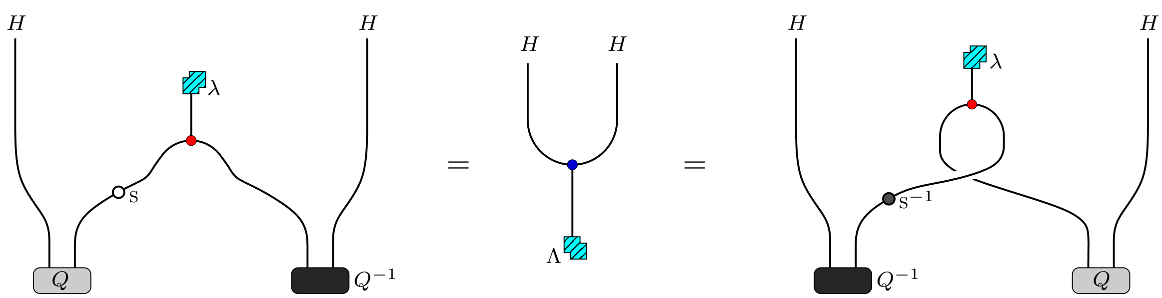
(see [GW, Thm. 2.3.2] and [CW2, Rem. 2.4]). Together with $\mathrm{s} \circ \Lambda = \Lambda$ and the property (5.12) of the cointegral one then also has

$$f_{Q^{-1}}(\lambda) = \Lambda\,, \tag{5.16}$$

where $f_{Q^{-1}}$ is the morphism (5.10) with the monodromy matrix Q replaced by its inverse Q^{-1}. Further, one has [CW2, Lemma 2.5]

$$f_Q \circ \Psi \circ f_{Q^{-1}} \circ \Psi = id_H\,, \tag{5.17}$$

which in turn by comparison with (5.13) shows that $\Psi \circ f_{Q^{-1}} = \rho_\triangleright(S^{-1})$. Further, the latter identity and (5.13) are equivalent to the relations



(5.18)

5.3 SL(2, ℤ)-action on morphism spaces

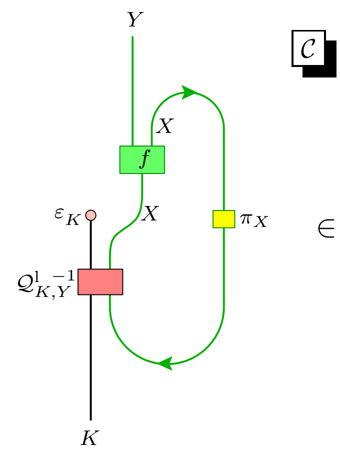
Recall from (5.2) that for any pair of objects $X, Y \in \mathcal{C}$ the morphism space $\mathrm{Hom}_\mathcal{C}(K \otimes X, Y)$ carries a representation of $\mathrm{SL}(2,\mathbb{Z})$. Of particular interest to us are the spaces $\mathrm{Hom}_\mathcal{C}(K, Y)$, and specifically $\mathrm{SL}(2,\mathbb{Z})$-*invariants* in these spaces, i.e. morphisms g satisfying

$$g \circ \varpi_K(\gamma) = g \tag{5.19}$$

for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

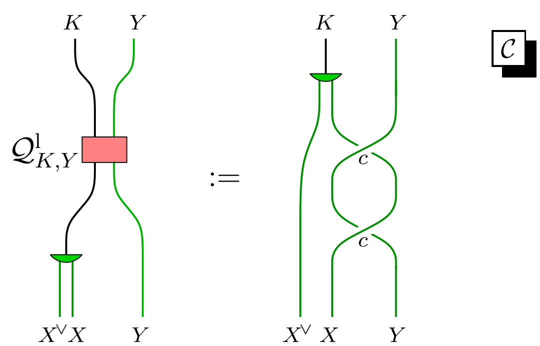
Morphisms in $\mathrm{Hom}_{\mathcal{C}}(K,Y)$ can in particular be obtained by defining a linear map $t_{\mathcal{Q}}$ from $\mathrm{Hom}_{\mathcal{C}}(X, Y\otimes X)$ to $\mathrm{Hom}_{\mathcal{C}}(K,Y)$ as a universal partial trace, to which we refer as the *partial monodromy trace*. Thus for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y\otimes X)$ we set

$t_{\mathcal{Q}}(f) \;:=$



$\in\; \mathrm{Hom}_{\mathcal{C}}(K,Y)\;,$ (5.20)

where $\mathcal{Q}^{\mathrm{l}}_{\mathcal{H},Y} \in \mathrm{End}_{\mathcal{C}}(K\otimes Y)$ is defined by



(5.21)

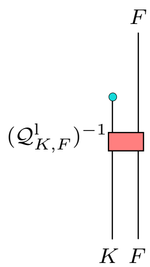
Note that, by the naturality of the braiding, the morphisms (5.21) are natural in Y.

Remark 5.5. If $\mathcal{C}$ is a (semisimple, strictly sovereign) modular tensor category, then the invariance property (5.19) for $\gamma = S$ and $g = t_{\mathcal{Q}}(f)$ is equivalent to the definition of S-invariance of morphisms in $\mathrm{Hom}_{\mathcal{C}}(Y\otimes X, X)$ that was given in [KoR, Def. 3.1(i)].

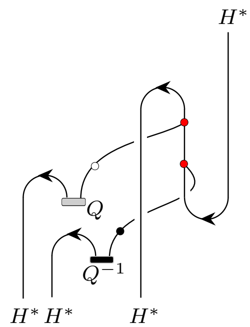
Let us now specialize to $\mathcal{C} = H$-Bimod and $X = Y$ being the Frobenius algebra F in H-Bimod.

Theorem 5.6. *The subspace of* $\mathrm{SL}(2,\mathbb{Z})$*-invariants of* $\mathrm{Hom}_{H|H}(F, F\otimes F)$ *is non-zero. Specifically, the partial monodromy trace of the coproduct* Δ_F *is* $\mathrm{SL}(2,\mathbb{Z})$*-invariant.*

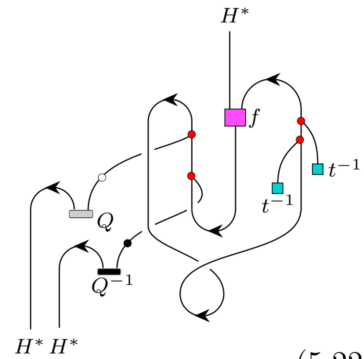
We will prove this statement by establishing separately invariance under the two generators S and T of $\mathrm{SL}(2,\mathbb{Z})$. Before investigating the action of S and T, let us first present the partial monodromy trace $t_{\mathcal{Q}}(\Delta_F)$ in a convenient form. To this end we first note that, invoking the explicit form (3.3) of the braiding and (4.8) of the sovereign structure of H-Bimod, we have



$=$



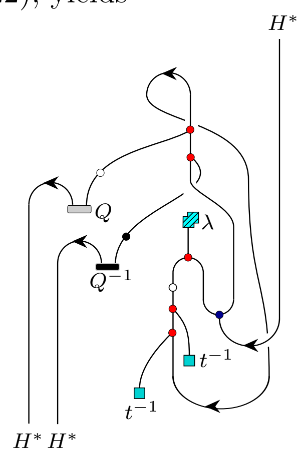
and hence $\quad t_{\mathcal{Q}}(f) \;=$



(5.22)

for any $f \in \mathrm{Hom}_{H|H}(F, F \otimes F)$. Specializing (5.22) to the partial monodromy trace of the coproduct Δ_F, i.e. inserting Δ_F from (2.22), yields

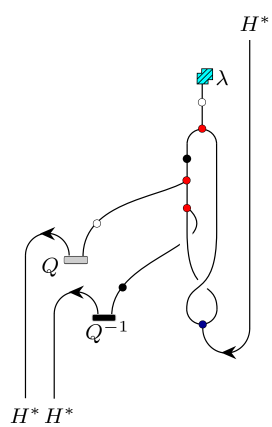
$t_{\mathcal{Q}}(\Delta_F) \;=$



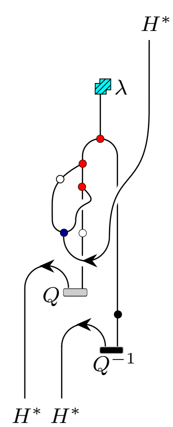
(5.23)

This can be rewritten as follows:

$t_{\mathcal{Q}}(\Delta_F) \;=$



$=$



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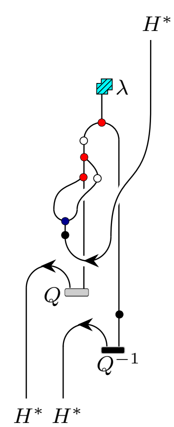
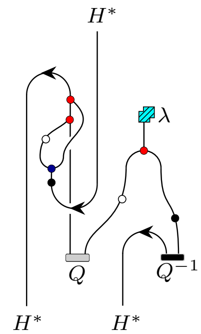


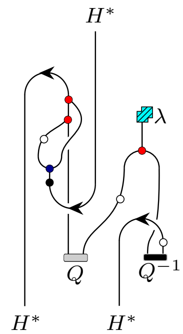
(5.24)

Here in the first step it is used that $\mathrm{s}^2 = \mathrm{ad}_t$ and that $\lambda \circ m \circ (g \otimes id_H) = \lambda \circ \mathrm{s}$ (which is a *left* cointegral), while the second equality follows by the fact that the antipode is an anti-algebra morphism and by associativity of m. We can now use the fact that the Drinfeld map intertwines the coadjoint and adjoint actions (see (5.11)); we then have

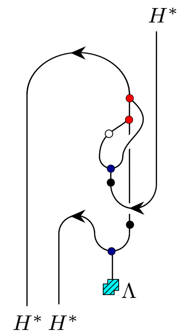
$$t_{\mathcal{Q}}(\Delta_F) \;=$$



$$=$$



$$=$$



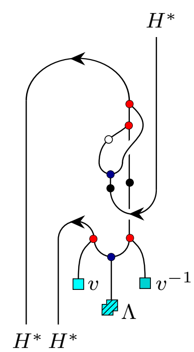
(5.25)

where the last equality uses the second of the identities (5.18) together with the fact that the antipode is an anti-coalgebra morphism and that $\mathrm{s} \circ \Lambda = \Lambda$.

Lemma 5.7. *The morphism $t_{\mathcal{Q}}(\Delta_F)$ is T-invariant, i.e. satisfies (5.19) for $\gamma = T \in \mathrm{SL}(2,\mathbb{Z})$.*

Proof. Invoking the expressions for $\rho_{\triangleright\triangleleft}(T)$ given in (5.9) and for $t_{\mathcal{Q}}(\Delta_F)$ given on the right hand side of (5.25), and using the centrality of the ribbon element $v \in H$, we have

$$t_{\mathcal{Q}}(\Delta_F) \circ \rho_{\triangleright\triangleleft}(T) \;=$$



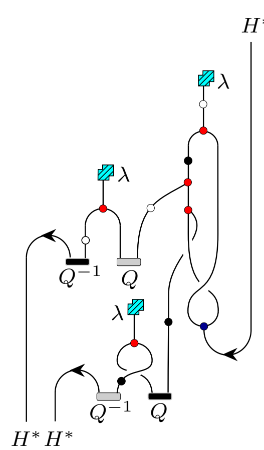
(5.26)

Recalling now the identity (2.21), the central elements v and v^{-1} cancel each other, hence (5.26) equals $t_{\mathcal{Q}}(\Delta_F)$. □

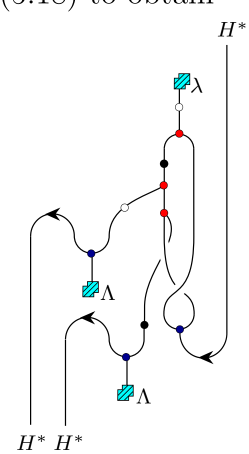
Lemma 5.8. *The morphism $t_{\mathcal{Q}}(\Delta_F)$ is S-invariant, i.e. satisfies (5.19) for $\gamma = S \in \mathrm{SL}(2,\mathbb{Z})$.*

Proof. Applying definition (5.9), we can use the identities (5.18) to obtain

$$t_{\mathcal{Q}}(\Delta_F) \circ \rho_{\triangleright\triangleleft}(S) \;=$$

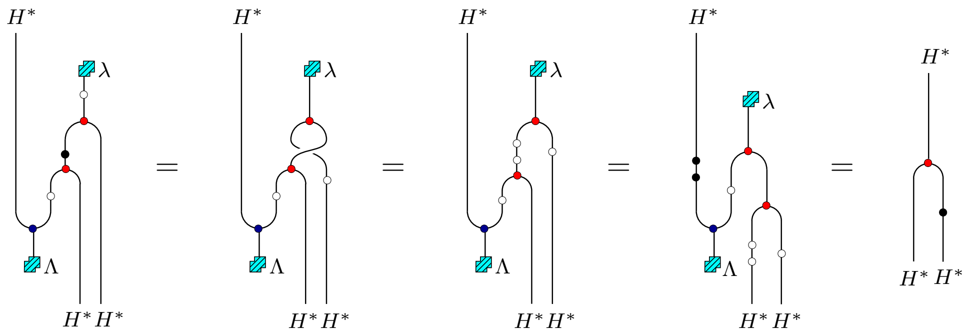


$$=$$



(5.27)

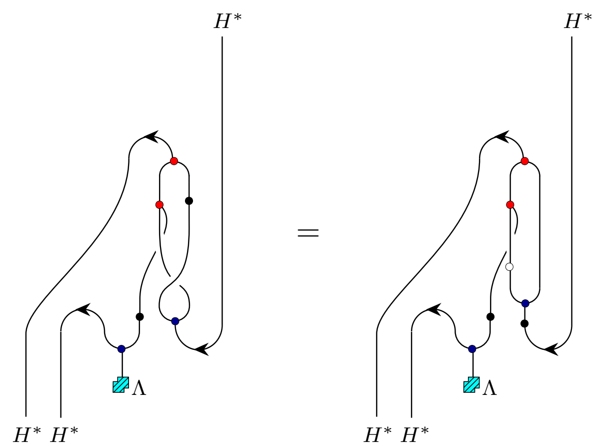
Further we note that owing to the identities (5.12), $\mathrm{s}^{-2} \circ \Lambda = \Lambda$ and (2.17) we can write



(5.28)

It follows that

$t_{\mathcal{Q}}(\Delta_F) \circ \rho_{\triangleright\triangleleft}(S) \;=$



(5.29)

This coincides with the right hand side of (5.25) and thus with $t_{\mathcal{Q}}(\Delta_F)$. $\square$

To neatly summarize the results above we state

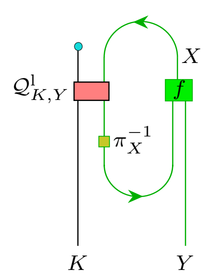
Definition 5.9. A coalgebra $(C, \Delta_C, \varepsilon_C)$ in H-Bimod is called *modular invariant* iff the morphism $t_{\mathcal{Q}}(\Delta_C)$ is $\mathrm{SL}(2,\mathbb{Z})$-invariant.

Thus what we have shown can be rephrased as

Corollary 5.10. *The Frobenius algebra $F \in H$-Bimod introduced in (2.18) is modular invariant.*

Remark 5.11. In the same way as the partial monodromy trace (5.20) associates a morphism in $\mathrm{Hom}_{\mathcal{C}}(K, Y)$ to a morphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y \otimes X)$, one may introduce another partial monodromy trace $t'_{\mathcal{Q}}$ that maps morphisms in $\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, X)$ linearly to morphism in $\mathrm{Hom}_{\mathcal{C}}(K \otimes Y, \mathbf{1})$, say as

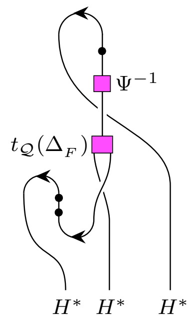
$t'_{\mathcal{Q}}(f) \;:=$



(5.30)

It is not difficult to check that the morphism $t'_{\mathcal{Q}}(m_F)$ obtained this way from the product of the Frobenius algebra F is modular invariant in the sense that $t'_{\mathcal{Q}}(m_F) \circ (\rho_K(\gamma) \otimes id_{H^*}) = t'_{\mathcal{Q}}(m_F)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Indeed, $t'_{\mathcal{Q}}(m_F)$ is related to $t_{\mathcal{Q}}(\Delta_F)$ by

$$t'_{\mathcal{Q}}(m_F) \;=\; \qquad (5.31)$$



with Ψ the Frobenius map, and as a consequence (using that $(\mathrm{S}^{-2})^* \otimes id_{H^*}$ commutes with the action of $\mathrm{SL}(2,\mathbb{Z})$ and that $\tau_{H^*,H^*} \circ \rho_{\triangleright\triangleleft}(\gamma) = \rho_{\triangleright\triangleleft}(\gamma^{-1})$ for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$) modular invariance of $t'_{\mathcal{Q}}(m_F)$ is equivalent to modular invariance of $t_{\mathcal{Q}}(\Delta_F)$. Accordingly, from the perspective of H-Bimod alone we could as well have referred to algebras rather than coalgebras in Def. 5.9. Indeed, this is the option that was chosen for the semisimple case in [KoR, Def. 3.1(ii)]. Our preference for coalgebras derives from the fact that, as described in Appendix B, the morphism space $\mathrm{Hom}_{\mathcal{C}}(K,F)$ plays a more direct role than $\mathrm{Hom}_{\mathcal{C}}(K \otimes F, \mathbf{1})$ in the motivating context of modular functors and conformal field theory.

6 The case of non-trivial Hopf algebra automorphisms

A *Hopf algebra automorphism* of a Hopf algebra H is a linear map from H to H that is both an algebra and a coalgebra automorphism and commutes with the antipode. For H a ribbon Hopf algebra with R-matrix R and ribbon element v, an automorphism ω of H is said to be a *ribbon Hopf algebra automorphism* iff $(\omega \otimes \omega)(R) = R$ and $\omega(v) = v$. For any H-bimodule (X, ρ, ϱ) and any pair of algebra automorphisms ω, ω' of H there is a corresponding (ω,ω')-*twisted bimodule* ${}^{\omega}X^{\omega'} = (X, \rho \circ (\omega \otimes id_X), \varrho \circ (id_X \otimes \omega'))$. If ω and ω' are *Hopf* algebra automorphisms, then the twisting is compatible with the monoidal structure of H-Bimod, and if they are even *ribbon* Hopf algebra automorphisms, then it is compatible with the ribbon structure of H-Bimod.

In this section we observe that to any finite-dimensional factorizable ribbon Hopf algebra H and any ribbon Hopf algebra automorphism of H there is again associated a Frobenius algebra in H-Bimod, which moreover shares all the properties, in particular modular invariance, of the Frobenius algebra F that we obtained in the previous sections. The arguments needed to establish this result are simple modifications of those used previously. Accordingly we will be quite brief.

Proposition 6.1. *(i) For H a finite-dimensional factorizable ribbon Hopf algebra over $\Bbbk$ and ω a Hopf algebra automorphism of H, the bimodule $F_\omega := {}^{id_H}(F)^{\omega}$ carries the structure of a Frobenius algebra. The structure morphisms of F_ω as a Frobenius algebra are given by the formulas (2.18) (thus as linear maps they are the same as for $F \equiv F_{id_H}$).*

(ii) F_ω is commutative and symmetric, and it is special iff H is semisimple.

(iii) If ω is a ribbon Hopf algebra automorphism, then F_ω is modular invariant.

Proof. The proofs of all statements are completely parallel to those in the case $\omega = id_H$. The only difference is that the various morphisms one deals with, albeit coinciding as linear maps with those encountered before, are now morphisms between different H-bimodules than previously. That they do intertwine the relevant bimodule structures follows by combining the simple facts that (since ω is compatible with the ribbon structure of H-Bimod) $(F\otimes F)^\omega = F_\omega \otimes F_\omega$ as a bimodule and that a linear map $f \in \mathrm{Hom}(X,Y)$ for $X,Y \in H$-Bimod lies in the subspace $\mathrm{Hom}_{H|H}(X,Y)$ iff it lies in the subspace $\mathrm{Hom}_{H|H}(X^\omega, Y^\omega)$. $\square$

Furthermore, again the Frobenius algebra F_ω is canonically associated with H-Bimod as an abstract category. Indeed, analogously as in Proposition A.3, one sees that F_ω can be constructed as a coend, namely the one of the functor $G^{H;\omega}_{\otimes_\Bbbk}\colon H\text{-Mod}^{\mathrm{op}} \times H\text{-Mod} \to H\text{-Bimod}$ that acts on morphisms as $f\times g \mapsto f^\vee \otimes_\Bbbk g$ and on objects by mapping $(X,\rho_X)\times(Y,\rho_Y)$ to

$$\big(X^* \otimes_\Bbbk Y\,,\ [\rho_{X^\vee}\circ(\omega^{-1}\otimes id_{X^*})]\otimes id_Y\,,\ id_{X^*}\otimes(\rho_Y\circ\tau_{Y,H}\circ(id_Y\otimes \mathrm{s}^{-1}))\big) \tag{6.1}$$

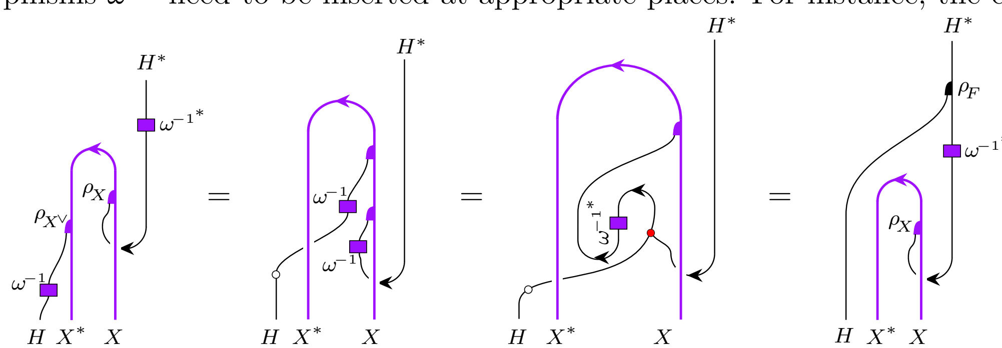
(or, in other words, $G^{H;\omega}_{\otimes_\Bbbk} = (?^{\omega^{-1}}\times Id)\circ G^H_{\otimes_\Bbbk}$ with the functor $G^H_{\otimes_\Bbbk}$ (whose coend is F) given by (A.3)):

Proposition 6.2. *The H-bimodule F_ω together with the dinatural family of morphisms*

$$\imath^{F_\omega}_X := (\omega^{-1})^* \circ \imath^F_X\,, \tag{6.2}$$

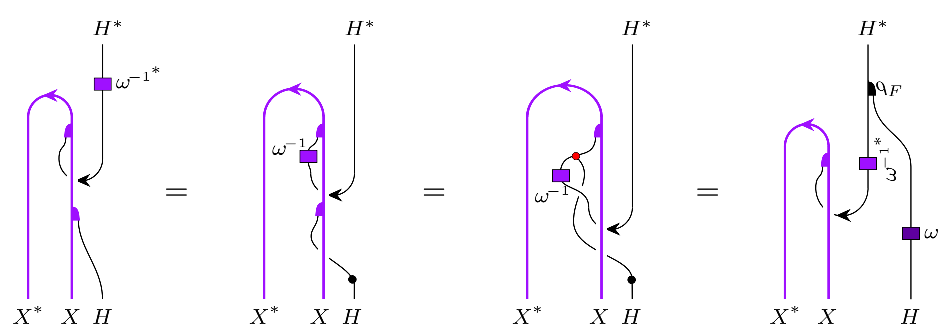
with $\imath^F_X$ as defined in (A.6), is the coend of the functor $G^{H;\omega}_{\otimes_\Bbbk}$.

Proof. Again the proof is parallel to the one for the case $\omega = id_H$, the difference being that the automorphisms $\omega^{\pm1}$ need to be inserted at appropriate places. For instance, the equalities



(6.3)

and



(6.4)

which generalize the relations (A.9) and (A.10), respectively, demonstrate that the linear maps $\imath^{F_\omega}_X \in \mathrm{Hom}(X^*\otimes_\Bbbk X, H^*)$ are indeed bimodule morphisms in $\mathrm{Hom}_{H|H}(G^{H;\omega}_{\otimes_\Bbbk}(X,X), F_\omega)$. $\square$

Remark 6.3. (i) When discussing twists of F we can restrict to the case that only the, say, right module structure is twisted, because the bimodule ${}^{\omega}H^{\omega'}$ is isomorphic to ${}^{id_H}H^{\omega^{-1}\circ\omega'}$.

(ii) It follows from the automorphism property of ω that together with Λ also $\omega(\Lambda)$ is a non-zero two-sided integral of H. As a consequence, just like in the case $\omega = id_H$ considered in Remark 2.11, the counit ε_F of F_ω is uniquely determined up to a non-zero scalar.

A Coend constructions

A.1 Dinatural transformations and coends

Here we recall a few pertinent concepts from category theory.

A *dinatural transformation* $F \Rightarrow B$ from a functor $F\colon\, \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, to an object $B \in \mathcal{D}$ is a family of morphisms $\varphi = \{\varphi_X\colon\, F(X,X) \to B\}_{X\in\mathcal{C}}$ such that the diagram

$$\begin{tikzcd}
F(Y,X) \arrow[r, "F(id_Y{,}f)"] \arrow[d, "F(f{,}id_X)"'] & F(Y,Y) \arrow[d, "\varphi_Y"] \\
F(X,X) \arrow[r, "\varphi_X"] & B
\end{tikzcd} \tag{A.1}$$

commutes for all $f \in \mathrm{Hom}(X,Y)$. For instance, the family $\{d_X\}$ of evaluation morphisms of a rigid monoidal category $\mathcal{C}$ forms a dinatural transformation from the functor that acts as $X \times Y \mapsto X^\vee \otimes Y$ to the monoidal unit $\mathbf{1} \in \mathcal{C}$.

Dinatural transformations from a given functor F to an object of $\mathcal{D}$ form a category, with the morphisms from $(F \Rightarrow B, \varphi)$ to $(F \Rightarrow B', \varphi')$ being given by morphisms $f \in \mathrm{Hom}_{\mathcal{D}}(B,B')$ satisfying $f \circ \varphi_X = \varphi'_X$ for all $X \in \mathcal{C}$. A *coend* (A,ι) for the functor F is an initial object in this category, i.e. a dinatural transformation (A,ι) such that for any dinatural transformation $(B,\varphi)\colon\, F \Rightarrow B$ there is a unique morphism $A \to B$ making the two triangles in

$$\begin{tikzcd}
F(Y,X) \arrow[r, "F(id_Y{,}f)"] \arrow[d, "F(f{,}id_X)"'] & F(Y,Y) \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "\varphi_Y"] & \\
F(X,X) \arrow[r, "\iota_X"] \arrow[drr, bend right, "\varphi_X"'] & A \arrow[dr, dashed] & \\
 & & B
\end{tikzcd} \tag{A.2}$$

commute for all $f \in \mathrm{Hom}(X,Y)$. If the coend of F exists, then it is unique up to unique isomorphism; one denotes it by $\int^X F(X,X)$. A morphism with domain $\int^X F(X,X)$ and codomain Y is equivalent to a family $\{f_X\}_{X\in\mathcal{C}}$ of morphisms from $F(X,X)$ to Y such that (Y,f) is a dinatural transformation.

A.2 The coregular bimodule as a coend

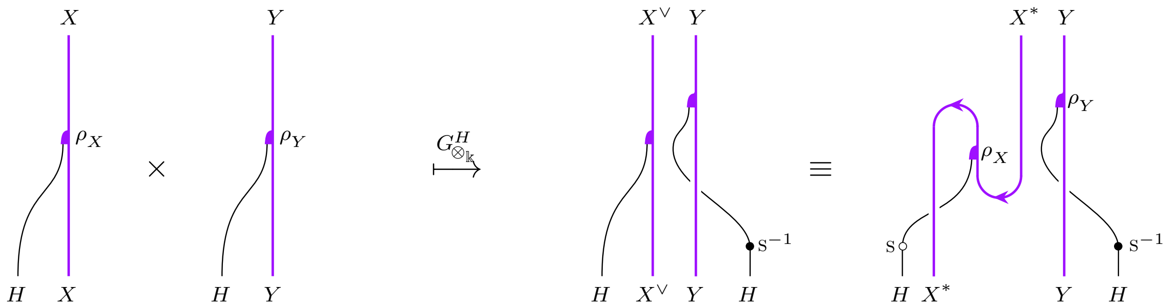
For H a finite-dimensional Hopf algebra over $\Bbbk$, endow the categories H-Mod and H-Bimod of left H-modules and of H-bimodules, respectively, with the tensor products (2.9) and (2.8) and with the dualities described at the beginning of Section 4. Consider the tensor product (bi)functor

$$G^H_{\otimes_\Bbbk}: \quad H\text{-Mod}^{\mathrm{op}} \times H\text{-Mod} \;\to\; H\text{-Bimod} \tag{A.3}$$

that acts on objects as

$$(X,\rho_X) \times (Y,\rho_Y) \;\stackrel{G^H_{\otimes_\Bbbk}}{\longmapsto}\; \big(X^* \otimes_\Bbbk Y\,,\, \rho_{X^\vee} \otimes id_Y\,,\, id_{X^*} \otimes (\rho_Y \circ \tau_{Y,H} \circ (id_Y \otimes \mathrm{S}^{-1})\big) \tag{A.4}$$

and on morphisms as $f \times g \mapsto f^{\vee} \otimes_{\Bbbk} g$. Pictorially, the action on objects is



(A.5)

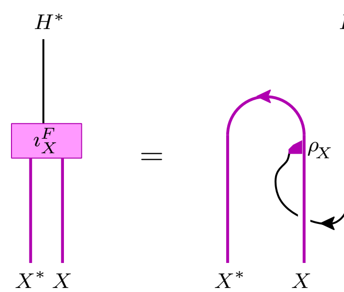
Remark A.1. The category H-Mod$^{\mathrm{op}} \times H$-Mod is naturally endowed with a tensor product, acting on objects as $(X \times Y) \times (X' \times Y') \mapsto (X \otimes^{H\text{-Mod}} X') \times (Y' \otimes^{H\text{-Mod}} Y)$. With respect to this tensor product and the tensor product (2.8) on H-Bimod, $G^H_{\otimes_\Bbbk}$ together with the associativity constraints from $\mathcal{V}ect_\Bbbk$ is a monoidal functor.

In this appendix we show that the coregular H-bimodule F introduced in Def. 2.11 is the coend of the functor $G^H_{\otimes_\Bbbk}$. We first present the appropriate dinatural family.

Lemma A.2. *The family* $(\imath^F_X)$ *of morphisms*

$$\imath^F_X := (d_X \otimes id_{H^*}) \circ [id_{X^*} \otimes (\rho_X \circ \tau_{X,H}) \otimes id_{H^*}] \circ (id_{X^*} \otimes id_X \otimes b_H) \tag{A.6}$$

in H-Bimod, pictorially given by



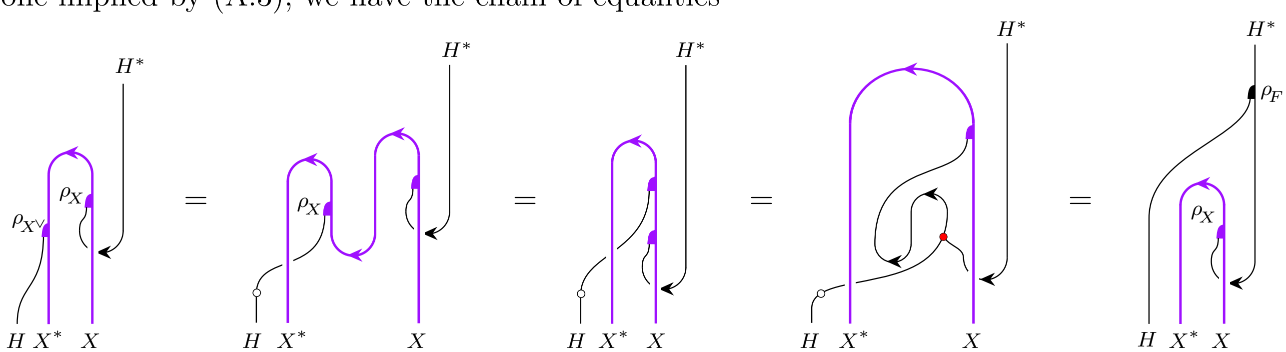
(A.7)

is dinatural for the functor $G^H_{\otimes_\Bbbk}$, *i.e.*

$$\imath^F_Y \circ G^H_{\otimes_\Bbbk}(id_Y, f) = \imath^F_x \circ G^H_{\otimes_\Bbbk}(f, id_X) \tag{A.8}$$

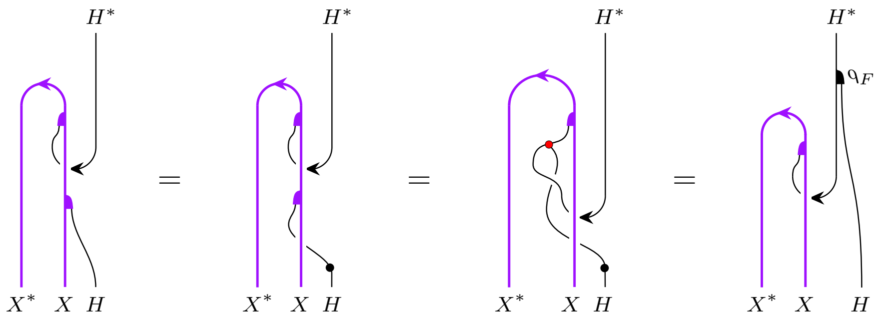
for any $f \in \mathrm{Hom}_H(X, Y)$.

Proof. (i) First note that the maps (A.6) are a priori just linear maps in $\mathrm{Hom}_\Bbbk(X^* \otimes_\Bbbk X, H^*)$. However, when H^* is endowed with the H-bimodule structure (2.11) and $X^* \otimes_\Bbbk X$ with the one implied by (A.3), we have the chain of equalities



(A.9)

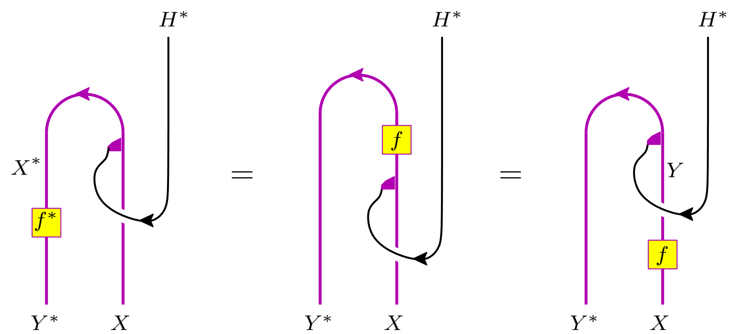
showing that (A.7) intertwines the left action of H, and



(A.10)

showing that it also intertwines the right action.

(ii) The dinaturalness property amounts to the equality of the left and right hand sides of



(A.11)

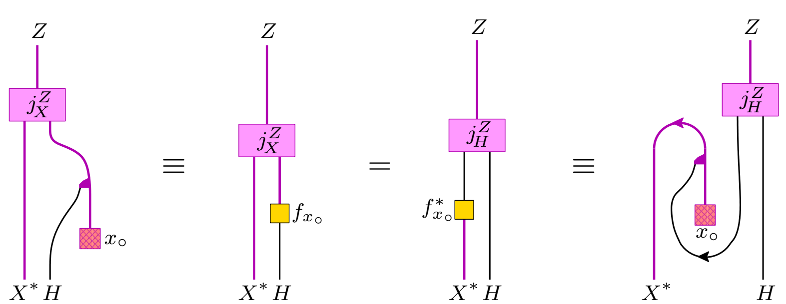
for any module morphism f from X to Y. Now the first equality in (A.11) holds by definition of f^*, and the second equality holds because f is a module morphism. $\square$

Proposition A.3. *The H-bimodule F together with the dinatural family $(\imath^F_X)$ given by (A.6) is the coend of the functor $G^H_{\otimes_\Bbbk}$,*

$$(F, \imath^F) = \int^X G^H_{\otimes_\Bbbk}(X,X)\,. \tag{A.12}$$

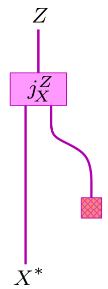
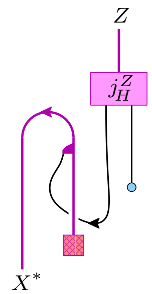
Proof. We have to show that $(F, \imath^F)$ is an initial object in the category of dinatural transformations from $G^H_{\otimes_\Bbbk}$ to a constant.

(i) Let j^Z be a dinatural transformation from $G^H_{\otimes_\Bbbk}$ to $Z \in H$-Bimod. Given any $X \in H$-Mod and any $x_\circ \in \mathrm{Hom}_\Bbbk(\Bbbk, X)$ (i.e. element of X), applying the dinaturalness property of j^Z to the morphism $f_{x_\circ} := \rho_X \circ (id_H \otimes x_\circ) \in \mathrm{Hom}_H(H, X)$ (with H regarded as an H-module via the regular left action) yields $j^Z_X \circ (id_{X^*} \otimes x_\circ) = j^Z_H \circ (\imath^F_X \otimes \eta) \circ (id_{X^*} \otimes x_\circ)$. Namely, we have



(A.13)

and thus, after composition with $id_{X^*} \otimes \eta$,

$j_X^Z \circ (id_{X^*} \otimes x_0) \;=$  $=$  $= \; j_H^Z \circ (\imath_X^F \otimes \eta) \circ (id_{X^*} \otimes x_\circ)$ (A.14)

with $\imath_X^F$ from (A.6). Since $x_\circ \in \mathrm{Hom}_{\Bbbk}(\Bbbk, X)$ is arbitrary, we actually have

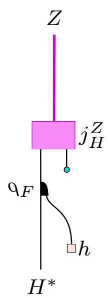
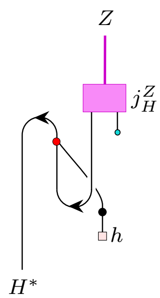
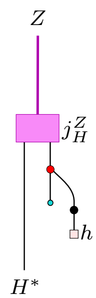
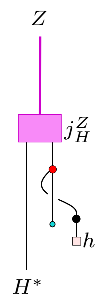
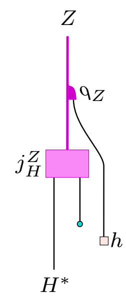
$$j_X^Z = j_H^Z \circ (\imath_X^F \otimes \eta) \tag{A.15}$$

for any bimodule Z and dinatural transformation j^Z from $G^H_{\otimes_{\Bbbk}}$ to Z.

(ii) Now consider the linear map

$$\kappa^Z := j_H^Z \circ (id_{H^*} \otimes \eta) \tag{A.16}$$

from H^* to Z. This is in fact a bimodule morphism from F to Z: Compatibility with the left H-action follows directly from the fact that j_H^Z is a morphism of bimodules, and thus in particular of left modules, while compatibility with the right H-action is seen as follows:

 $=$  $=$  $=$  $=$  (A.17)

Here the element $h \in \mathrm{Hom}_{\Bbbk}(\Bbbk, H)$ is arbitrary; the second equality invokes the dinaturalness of j^Z for the map $m \circ (id_H \otimes (\mathrm{s}^{-1} \circ h)) \in \mathrm{End}_H(H)$.

(iii) In terms of the morphism κ^Z, (A.15) amounts to

$$j_X^Z = \kappa^Z \circ \imath_X^F\,. \tag{A.18}$$

This establishes existence of the morphism from F to Z that is required for the universal property of the coend.

(iv) It remains to show that κ^Z is uniquely determined. This just follows by specializing (A.18) to the case $X = H$ and observing that $\imath_H^F$ has a right-inverse. The latter property holds because of $\imath_H^F \circ (id_{H^*} \otimes \eta) = (d_H \otimes id_{H^*}) \circ (id_{H^*} \otimes b_H) = id_{H^*}$. □

A.3 Some equivalences of braided monoidal categories

We note the following equivalences, where as usual H^{op} is H with opposite product $m \circ \tau_{H,H}$ (and with the same coproduct), and H^{coop} is H with opposite coproduct $\tau_{H,H} \circ \Delta$ (and with the same product).

Lemma A.4. *(i) For any Hopf algebra H there are equivalences*

$$H\text{-Bimod} \;\simeq\; (H{\otimes_\Bbbk}H)\text{-Mod} \;\simeq\; (H{\otimes_\Bbbk}H^{\rm op})\text{-Mod} \tag{A.19}$$

as abelian categories.

(ii) The equivalences (A.19) extend to equivalences

$$H\text{-Bimod} \;\simeq\; (H{\otimes_\Bbbk}H^{\rm coop})\text{-Mod} \;\simeq\; (H{\otimes_\Bbbk}H^{\rm op})\text{-Mod} \tag{A.20}$$

as monoidal categories, with respect to the tensor products (2.9) on H-Mod and (2.8) on H-Bimod. The constraint morphisms for the tensor functor structures of the equivalence functors are all identities.

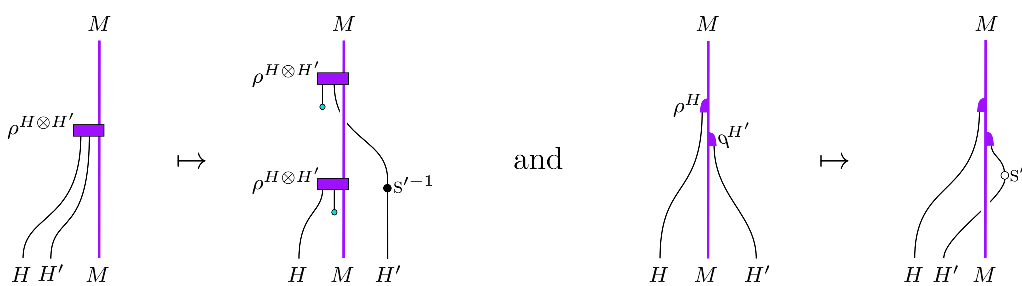
(iii) If the Hopf algebra H is quasitriangular with R-matrix R, then the equivalences (A.20) extend to equivalences

$$H\text{-Bimod} \;\simeq\; (\overline{H}{\otimes_\Bbbk}H^{\rm coop})\text{-Mod} \;\simeq\; (\overline{H}{\otimes_\Bbbk}H^{\rm op})\text{-Mod} \tag{A.21}$$

as braided monoidal categories, where H-Bimod is endowed with the braiding (3.3) and $\overline{H}$ is H with R-matrix R_{21}^{-1}. (Also, $H^{\rm op}$ is endowed with the natural quasitriangular structure inherited from H, i.e. has R-matrix R_{21}.)

Proof. (i) We derive each of the equivalences in a somewhat more general context.

For any two Hopf algebras H and H' there is an equivalence H-H'-Bimod $\simeq (H{\otimes_\Bbbk}H')$-Mod as abelian categories. The equivalence is furnished by the two functors which on morphisms are the identity and which map objects according to

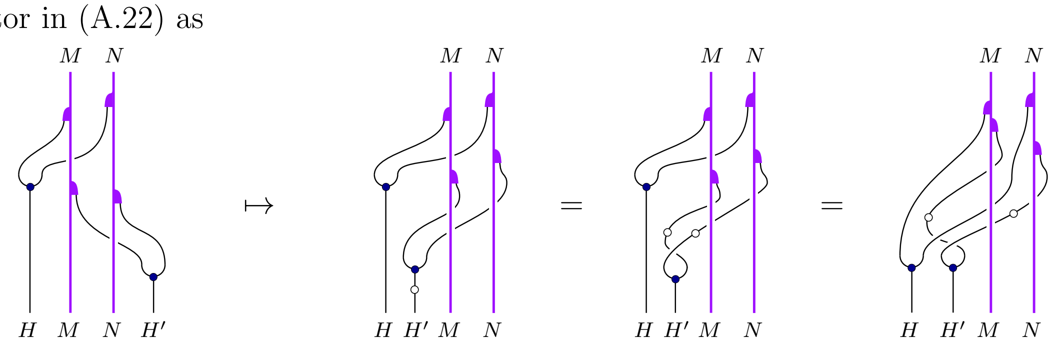


(A.22)

respectively.

Similarly, an equivalence H-H'-Bimod $\simeq (H{\otimes_\Bbbk}H'^{\rm op})$-Mod as abelian categories is furnished by functors that differ from those in (A.22) by just omitting the (inverse) antipode (compare e.g. [FRS, Prop. 4.6]).

(ii) For the first equivalence, compatibility with the tensor product follows for the second functor in (A.22) as



(A.23)

and analogously for the first functor, as well as for the second equivalence.

(iii) The Hopf algebras $\overline{H} \otimes_{\Bbbk} H^{\rm coop}$ and $\overline{H} \otimes_{\Bbbk} H^{\rm op}$ have natural quasitriangular structures, with R-matrices given by $(id_H \otimes c_{H,H} \otimes id_H) \circ (R_{21}^{-1} \otimes R^{-1})$. By direct calculation one checks that the two functors given in (A.22) (with $H' = H$), respectively the ones with the occurences of the antipode removed, not only furnish an equivalence between H-Bimod and $(\overline{H} \otimes_{\Bbbk} H^{\rm coop})$-Mod, respectively $(\overline{H} \otimes_{\Bbbk} H^{\rm op})$-Mod, as abelian monoidal categories, but map the braidings of these categories to each other as well.
Also note that the R-matrix furnishes an equivalence between $H^{\rm coop}$-Mod and H-Mod as monoidal categories, so that in the equivalences (A.21) we could as well use H instead of $H^{\rm coop}$. ☐

Remark A.5. In view of Lemma A.4, Prop. A.3 is implied by Theorem 7.4.13 of [KL].

The significance of the coend (A.12) and of the equivalences in Lemma A.4 actually transcends the framework of the (bi)module categories considered in this paper. Namely, one can consider the situation that H-Mod is replaced by a more general ribbon category $\mathcal{C}$, while the role of H-Bimod is taken over by the Deligne product of $\mathcal{C}$ with itself. Recall [De, Sect. 5] that the Deligne tensor product of two $\Bbbk$-linear abelian categories $\mathcal{C}$ and $\mathcal{D}$ that are locally finite, i.e. all morphism spaces of which are finite-dimensional and all objects of which have finite length, is a category $\mathcal{C} \boxtimes \mathcal{D}$ together with a bifunctor $\boxtimes \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ that is right exact and $\Bbbk$-linear in both variables and has the following universal property: for any bifunctor G from $\mathcal{C} \times \mathcal{D}$ to a $\Bbbk$-linear abelian category $\mathcal{E}$ being right exact and $\Bbbk$-linear in both variables there exists a unique right exact $\Bbbk$-linear functor $G_\Box \colon \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$ such that $G \cong G_\Box \circ \boxtimes$. In short, bifunctors from $\mathcal{C} \times \mathcal{D}$ become functors from $\mathcal{C} \boxtimes \mathcal{D}$. The category $\mathcal{C} \boxtimes \mathcal{D}$ is again $\Bbbk$-linear abelian and locally finite.

By the universal property of the Deligne product, there is a unique functor

$$G_\Box^H : \quad H\text{-Mod} \boxtimes H\text{-Mod} \longrightarrow H\text{-Bimod} \tag{A.24}$$

such that the bifunctor (A.3) can be written as the composition $G_{\otimes_\Bbbk}^H = G_\Box^H \circ (?^\vee \boxtimes Id)$, with the functor $?^\vee \boxtimes Id = \boxtimes \circ (?^\vee \times Id)$ acting as $X \times Y \mapsto X^\vee \boxtimes Y$ and $f \times g \mapsto f^\vee \boxtimes g$. On objects of $\mathcal{C} \boxtimes \mathcal{D}$ that are of the form $U \boxtimes V$ with $U \in \mathcal{C}$ and $V \in \mathcal{D}$, the functor $G_\Box^H$ acts as

$$(X, \rho_X) \boxtimes (Y, \rho_Y) \;\stackrel{G_\Box^H}{\longmapsto}\; \big(X \otimes Y \,,\, \rho_X \otimes id_Y \,,\, id_X \otimes (\rho_Y \circ \tau_{Y,H} \circ (id_Y \otimes \mathrm{s}^{-1}))\big)\,. \tag{A.25}$$

Now by combining Prop. 5.3 of [De] with the first equivalence in (A.19), one sees (compare also e.g. [Fr, Ex. 7.10]) that the functor $G_\Box^H$ is an equivalence of abelian categories. Further, H-Mod $\boxtimes$ H-Mod has a natural monoidal structure [De, Prop. 5.17] as well as a braiding (which on objects of the form $U \boxtimes V$ acts as $(c_{U',U}^{H\text{-Mod}})^{-1} \otimes_\Bbbk c_{V,V'}^{H\text{-Mod}}$). With respect to these the equivalence (A.24) can be endowed with the structure of an equivalence of braided monoidal categories.

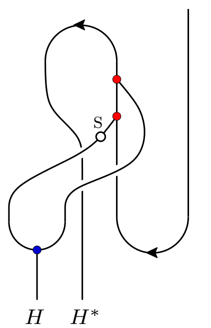
Observations analogous to those made here for the category $\mathcal{C}_H = H$-Mod in fact apply to any locally finite $\Bbbk$-linear abelian ribbon category $\mathcal{C}$. Hereby the Frobenius algebra F in H-Bimod can be understood as a particular case of the coend

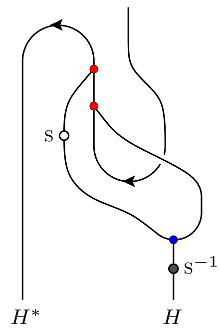
$$F_\mathcal{C} := \int^X X^\vee \boxtimes X \tag{A.26}$$

of the functor $?^\vee \boxtimes Id\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \overline{\mathcal{C}} \boxtimes \mathcal{C}$ (where $\overline{\mathcal{C}}$ is $\mathcal{C}$ with opposite braiding), which exists for any such category $\mathcal{C}$. This coend $F_{\mathcal{C}}$ has already been considered in [Ke] and [KL, Sect 5.1.3]. It is natural to expect that also in this general setting the coend $F_{\mathcal{C}}$ still carries a natural Frobenius algebra structure. However, so far we only know that that $F_{\mathcal{C}}$ is naturally a unital associative algebra in $\overline{\mathcal{C}} \boxtimes \mathcal{C}$.

A.4 The coend $H^*_{\triangleright\triangleleft}$ in H-Bimod

The coadjoint left and right actions $\rho_\triangleright \in \mathrm{Hom}(H \otimes H^*, H^*)$ and $\varrho_\triangleleft \in \mathrm{Hom}(H^* \otimes H, H^*)$ of H on its dual H^* are by definition the morphisms

$\rho_\triangleright \;=$ 

and $\varrho_\triangleleft \;=$ 

(A.27)

We call the H-bimodule that consists of the vector space $H^* \otimes_{\Bbbk} H^*$, endowed with the coadjoint left H-action on the first tensor factor and with the coadjoint right H-action on the second factor, the *coadjoint bimodule* and denote it by $H^*_{\triangleright\triangleleft}$. That is,

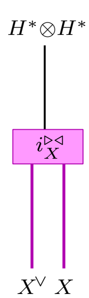
$$H^*_{\triangleright\triangleleft} = (H^* \otimes_{\Bbbk} H^*, \rho_\triangleright \otimes id_{H^*}, id_{H^*} \otimes \rho_\triangleleft)\,. \tag{A.28}$$

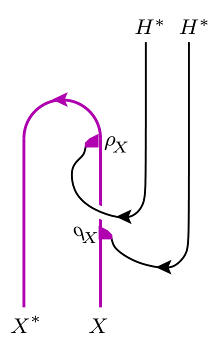
We will now show that this bimodule arises as the coend of the functor

$$\otimes \circ (?^\vee \times Id): \quad H\text{-Bimod}^{\mathrm{op}} \times H\text{-Bimod} \to H\text{-Bimod}\,, \tag{A.29}$$

where $\otimes$ and $?^\vee$ are the tensor product (2.8) and right duality (4.1) of H-Bimod. A crucial input is the braided monoidal equivalence described in Lemma A.4(iii).

Proposition A.6. *Let H be a finite-dimensional ribbon Hopf $\Bbbk$-algebra. Then the H-bimodule $H^*_{\triangleright\triangleleft}$ together with the family $i^{\triangleright\triangleleft}$ of morphisms*



$:=$ 

(A.30)

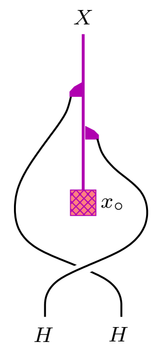
from $X^\vee \otimes X$ to $H^*_{\triangleright\triangleleft}$, for $X = (X, \rho_X, \varrho_X) \in H$-Bimod, is the coend for the functor (A.29):

$$(H^*_{\triangleright\triangleleft}, i^{\triangleright\triangleleft}) = \int^X X^\vee \otimes X\,. \tag{A.31}$$

Proof. The statement follows from the results of [Ly1, Sect. 1.2] and [Vi, Sect. 4.5] for the coend of the functor $\otimes \circ (?^\vee \times Id)$ from H'-Mod$^{\rm op} \times H'$-Mod to H'-Mod, with the Hopf algebra $H' = \overline{H} \otimes H^{\rm op}$, by transporting them via the equivalence (A.21) to H-Bimod.

We omit the details, but find it instructive to compare a few aspects of a direct proof to the corresponding parts of the proof of Lemma A.2 and of Proposition A.3:

First, dinaturalness follows by an argument completely parallel to the one used in (A.11) to show dinaturalness of the family (A.6). Second, the role of the morphism $f_{x_\circ}$ (that is, left action of H on an element $x_\circ$ of X) that appears in formula (A.13) is taken over by the map



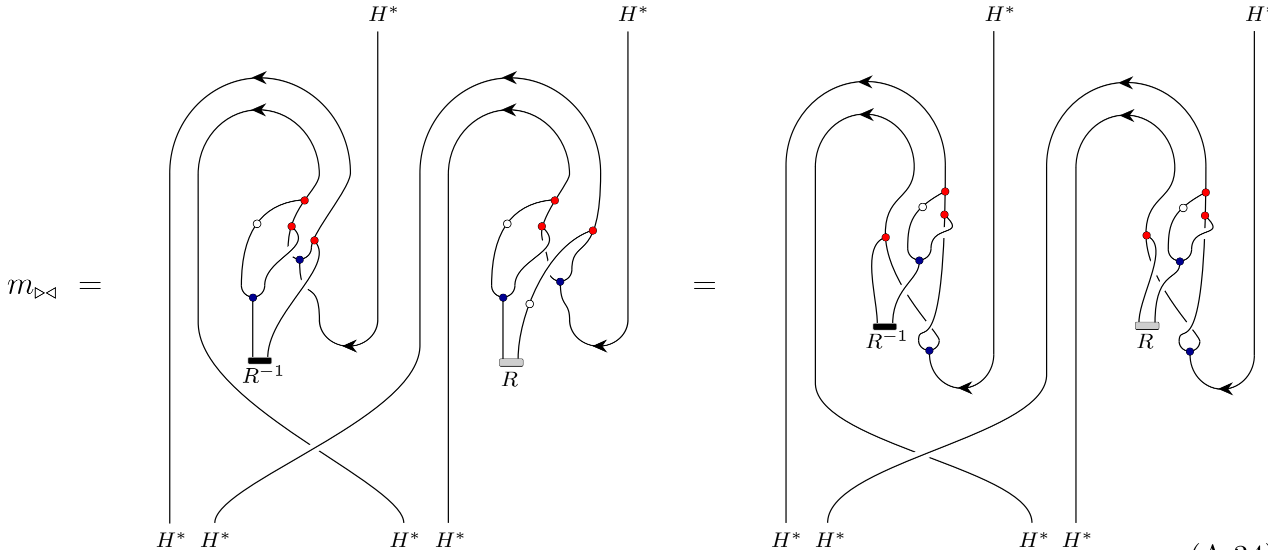
(A.32)

This map is a bimodule morphism from $H \otimes_{\Bbbk} H$ – regarded as an H-bimodule $(H \otimes_{\Bbbk} H)_{\rm reg}$ via the regular left and right actions on the second and first factor, respectively – to X. Analogously as in (A.14) one shows that for any dinatural transformation j^Z from the functor (A.29) to $Z \in H$-Bimod one has $j^Z_X \circ (id_{X^*} \otimes x_\circ) = \kappa^Z \circ i^{\triangleright\triangleleft}_X \circ (id_{X^*} \otimes x_\circ)$, with the map κ^Z defined by $\kappa^Z := j^Z_{(H \otimes_{\Bbbk} H)_{\rm reg}} \circ (id_{H^*} \otimes id_{H^*} \otimes \eta \otimes \eta)$. And again κ^Z is a bimdoule morphism, so that the existence part of the universal property of the coend is established. Uniqueness follows by specializing to the case $X = (H \otimes_{\Bbbk} H)_{\rm reg}$ and observing that $i^{\triangleright\triangleleft}_{(H \otimes_{\Bbbk} H)_{\rm reg}}$ is an epimorphism (as is e.g. seen from $i^{\triangleright\triangleleft}_{(H \otimes_{\Bbbk} H)_{\rm reg}} \circ (id_{H^*} \otimes id_{H^*} \otimes \eta \otimes \eta) = id_{H^*} \otimes id_{H^*}$). □

Corollary A.7. *The H-bimodule $H^*_{\triangleright\triangleleft}$ carries the structure of a Hopf algebra, with structure morphisms given as follows. The unit, counit and coproduct are*

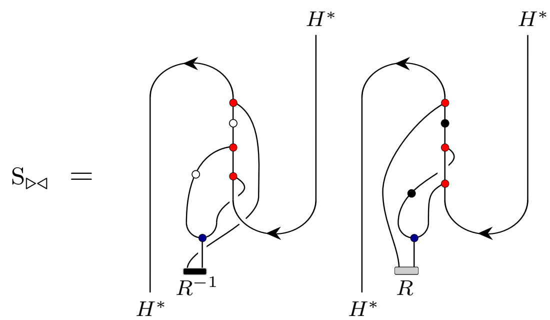
$$\begin{aligned} &\eta_{\triangleright\triangleleft} = \varepsilon^\vee \otimes \varepsilon^\vee\,, \qquad \varepsilon_{\triangleright\triangleleft} = \eta^\vee \otimes \eta^\vee\,, \\ &\Delta_{\triangleright\triangleleft} = (id_{H^*} \otimes \tau_{H^*,H^*} \otimes id_{H^*}) \circ \big((m^{\rm op})^\vee \otimes m^\vee \big)\,, \end{aligned} \tag{A.33}$$

the product is



(A.34)

and the antipode is



(A.35)

Proof. We just have to specialize the general results of [Ly2], which apply to the coend of the functor $\otimes \circ (?^\vee \otimes Id)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ in any $\Bbbk$-linear abelian ribbon category $\mathcal{C}$, to the case $\mathcal{C} = H$-Bimod. The calculations are straightforward, and except for the multiplication and the antipode they are very short.
Let us just mention that the first equality in (A.34) follows from the general results (see [Ly2, Prop. 2,3], as well as [Vi, Sect. 1.6] or [FSc, Sect. 4.3]) together with (4.2) and the defining relation (2.1) of the R-matrix. The second equality in (A.34) follows with the help of standard manipulations from the fact that the R-matrix intertwines the coproduct and the opposite coproduct. $\square$

Proposition A.8. *(i) If Λ is a two-sided integral of H, then*

$$\lambda_{\bowtie} := \Lambda^\vee \otimes \Lambda^\vee \tag{A.36}$$

is two-sided cointegral of the Hopf algebra $(H^*_{\bowtie}, m_{\bowtie}, \eta_{\bowtie}, \Delta_{\bowtie}, \varepsilon_{\bowtie}, \mathrm{S}_{\bowtie})$.
(ii) If λ is a right cointegral of H, then

$$\Lambda_{\bowtie} := \lambda^\vee \otimes \lambda^\vee \tag{A.37}$$

is a two-sided integral of $(H^*_{\bowtie}, m_{\bowtie}, \eta_{\bowtie}, \Delta_{\bowtie}, \varepsilon_{\bowtie}, \mathrm{S}_{\bowtie})$.

Proof. (i) Inserting the defininitions one has

$$\begin{aligned}(\lambda_{\bowtie} \otimes id_{H^*\otimes H^*}) \circ \Delta_{\bowtie} &= \big(m \circ (\Lambda \otimes id_{H^*})\big)^* \otimes \big(m \circ (id_{H^*} \otimes \Lambda)\big)^* \qquad \text{and} \\ (id_{H^*\otimes H^*} \otimes\,) \circ \Delta_{\bowtie} &= \big(m \circ (id_{H^*} \otimes \Lambda)\big)^* \otimes \big(m \circ (\Lambda \otimes id_{H^*})\big)^*.\end{aligned} \tag{A.38}$$

Since Λ is a two-sided integral of H, both of these expressions are equal to $\eta_{\bowtie} \circ \lambda_{\bowtie}$.

(ii) That $\Lambda_{\bowtie}$ is a left integral readily follows from the first expression for the product in (A.34) together with the fact that λ is a right cointegral and that it satisfies (5.12). That $\Lambda_{\bowtie}$ is also a right cointegral follows in the same way by using instead the second expression in (A.34) for the product. □

B Motivation from conformal field theory

A major motivation for the mathematical results of this paper comes from structures that originate in two-dimensional conformal field theory. In this appendix we briefly describe some of these structures.

In representation theoretic approaches to conformal field theory the starting point is a chiral symmetry algebra together with its category $\mathcal{C}$ of representations. For any mathematical structure that formalizes the physical concept of chiral symmetry algebra, the category $\mathcal{C}$ can be endowed with a lot of additional structure. In particular, in many cases it leads to a so-called modular functor. A modular functor actually consists of a collection of functors. Namely, to any compact Riemann surface $\Sigma_{g,n}$ of genus g and with a finite number n of marked points it assigns a functor

$$F_{\Sigma_{g,n}}: \quad \mathcal{C}^{\boxtimes n} \to \mathcal{V}ect \tag{B.1}$$

from $\mathcal{C}^{\boxtimes n}$ to the category $\mathcal{V}ect$ of finite-dimensional complex vector spaces. This collection of functors is required to obey a system of compatibility conditions, which in particular expresses factorization constraints and accommodates actions of mapping class groups of surfaces. Thus, selecting for a genus-g surface $\Sigma_{g,n}$ with n marked points any n-tuple $(V_1, V_2, ..., V_n)$ of objects of the category $\mathcal{C}$, we obtain a finite-dimensional complex vector space $F_{\Sigma_{g,n}}(V_1, V_2, ..., V_n)$ which carries an action of the mapping class group of $\Sigma_{g,n}$. In conformal field theory, this space plays the role of the space of conformal blocks with chiral insertion of type V_i at the ith marked point of $\Sigma_{g,n}$.

In the particular case that the category $\mathcal{C}$ is finitely semisimple, this structure is reasonably well understood. Specifically, precise conditions are known under which the representation category of a vertex algebra $\mathcal{V}$ is a modular tensor category. In this case the Reshetikhin-Turaev construction allows one to obtain a modular functor just on the basis of $\mathcal{C}$ as an abstract category. In a remarkable development, Lyubashenko and others (see [KL] and references cited there) have extended many aspects of this story to a larger class of monoidal categories that are not necessarily semisimple any longer. In particular, given an abstract monoidal category with adequate additional properties, one can still construct representations of mapping class groups.

Representation categories that are not semisimple are of considerable physical interest; they arise in particular in various systems of statistical mechanics. The corresponding models of conformal field theory have been termed "logarithmic" conformal field theories. A complete characterization of this class of models has not been achieved yet, but a necessary requirement ensuring tractability is that the category $\mathcal{C}$, while being non-semisimple, still possesses certain finiteness properties, e.g. each object should have a composition series of finite length.

In the present paper we consider an even more restricted, but non-empty, subclass, namely the one for which the monoidal category $\mathcal{C}$ is equivalent to the representation category of a finite-dimensional factorizable ribbon Hopf algebra. Finite-dimensional Hopf algebras H_{KL} have indeed been associated, via the Kazhdan-Lusztig correspondence, to certain classes of logarithmic conformal field theories. These Hopf algebras H_{KL} are factorizable, but do not have an R-matrix, albeit they do have a monodromy matrix (so that in particular the partial monodromy traces which we introduced in section 5 can still be defined) [FGST]. Accordingly our results do not perfectly match the presently available conformal field theoretic proposals. On the other hand, it is apparent that the Hopf algebras H_{KL} are not quite the appropriate algebraic structures: their representation categories, albeit being equivalent to the representation categories of the relevant vertex algebras as abelian categories, are not equivalent to them as monoidal categories. [6]

The Riemann surface of interest to us is $\Sigma_{1,1}$, a one-punctured torus. This surface is distinguished by the fact [Ye2] that it carries a natural Hopf algebra structure in the category of three-cobordisms. The functor $F_{\Sigma_{1,1}}$ is representable: to the category $\mathcal{C}$, one can canonically associate a Hopf algebra object $K_{\mathcal{C}} \in \mathcal{C}$ such that [Ly1]

$$F_{\Sigma_{1,1}} \cong \mathrm{Hom}_{\mathcal{C}}(K_{\mathcal{C}}, -)\,. \tag{B.2}$$

The corresponding mapping class group is the modular group $\mathrm{SL}(2,\mathbb{Z})$. As described in (5.2), it acts on each vector space $F_{\Sigma_{1,1}}(V) \cong \mathrm{Hom}_{\mathcal{C}}(K_{\mathcal{C}}, V)$ for any object $V \in \mathcal{C}$.

From the point of view of two-dimensional conformal field theory, the one-punctured torus is the surface relevant for partition functions. We are interested in this paper in a candidate for the partition function of the space of bulk fields and thus in one-point functions of bulk fields on the torus. The space $\mathcal{H}_{\mathrm{bulk}}$ of bulk fields carries the structure of a bimodule over the chiral symmetry algebra $\mathcal{V}$. In the case that the category $\mathcal{C}$ is semisimple, a particularly simple solution is given by the bulk state space $\bigoplus_i S_i^\vee \otimes_{\mathbb{C}} S_i$, where the (finite) summation is over all isomorphism classes $[S_i]$ of simple $\mathcal{V}$-modules. The corresponding partition function is the so-called charge conjugation modular invariant. It has been conjectured [QS, GR] that this type of bulk state space exists in the non-semisimple case as well, and that as a left $\mathcal{V}$-module it decomposes as

$$\mathcal{H}_{\mathrm{bulk}} \,\cong\, \bigoplus_i P_i^\vee \otimes_{\mathbb{C}} S_i\,, \tag{B.3}$$

with P_i the projective cover of the simple $\mathcal{V}$-module S_i.

According to the principle of holomorphic factorization, a correlation function for a conformal real surface is an element in the space of conformal blocks associated to the oriented double of the surface. Thus a one-point function on the torus is a specific element in the space of conformal blocks associated to the double of the torus (as a real surface), that is, of the disconnected sum of two copies of $\Sigma_{1,1}$ with opposite orientation. For any selection of a pair (V_1, V_2)

[6] Also, constructing algebras with the help of the Kazhdan-Lusztig correspondence involves some arbitrariness. It has been suggested [ST] that one should better work with Hopf algebras in a category of Yetter-Drinfeld modules built from a braided vector space, rather than Hopf algebras in $\mathcal{V}ect$.

of objects of $\mathcal{C}$ at the two points on the double cover that lie over the one insertion point on the torus, this space of conformal blocks is the tensor product $\mathrm{Hom}_{\mathcal{C}}(K_{\mathcal{C}},V_1)^* \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(K_{\mathcal{C}},V_2)$. More compactly, this space can be written as a morphism space of another braided tensor category $\mathcal{D} := \overline{\mathcal{C}} \boxtimes \mathcal{C}$, which has its own canonical Hopf algebra object $K_{\mathcal{D}}$. As we have noted in Section A.3, if $\mathcal{C}$ is the category H-Mod of left modules over a finite-dimensional factorizable ribbon Hopf algebra H, then $\mathcal{D}$ can be identified with the category of bimodules over H, with a tensor product derived from the coproduct on H.

Partition functions should be modular invariant. We are thus interested in finding an object $F \in \mathcal{D}$ corresponding to the space of bulk fields as well as a vector

$$Z \in \mathrm{Hom}_{\mathcal{D}}(K_{\mathcal{D}}, F) \tag{B.4}$$

that is invariant under the action of the modular group. Moreover, comparison with the semisimple situation, in which $\mathcal{C}$ is a modular tensor category, indicates that the object F should possess a structure of a commutative symmetric Frobenius algebra in $\mathcal{D}$.

This is precisely what the present paper achieves for the case $\mathcal{C} \simeq H$-Mod: given a ribbon Hopf algebra automorphism ω of H, we obtain a commutative symmetric Frobenius algebra F_ω in the category H-Bimod. As an object, F_ω is the twisted coregular bimodule ${}^{id_H}(F)^{\omega}$, so that e.g. its decomposition as a left H-module precisely reproduces the decomposition (B.3) above. (The conjecture (B.3) has only been made for the case corresponding to trivial automorphism $\omega = id_H$, though.) Also note that according to Remark 6.3(ii) the counit of F_ω is unique up to a non-zero scalar; in the conformal field theory context this amounts to uniqueness of the vacuum state. The partial monodromy trace (5.23) of the coproduct $\Delta\colon F_\omega \to F_\omega \otimes F_\omega$ furnishes a modular invariant morphism $Z_\omega \in \mathrm{Hom}(K_{\mathcal{D}}, F_\omega)$. This morphism, associated to H and ω, is a natural candidate for a modular invariant partition function on the torus.

Acknowledgments: JF is largely supported by VR under project no. 621-2009-3993. CSc is partially supported by the Collaborative Research Centre 676 “Particles, Strings and the Early Universe - the Structure of Matter and Space-Time” and by the DFG Priority Programme 1388 “Representation Theory”. We thank Jens Fjelstad for discussions. JF is grateful to Hamburg university, and in particular to CSc and Astrid Dörhöfer, for their hospitality during the time when this study was initiated.

References

[Bi] J. Bichon, *Cosovereign Hopf algebras*, J. Pure Appl. Alg. 157 (2001) 121–133 [math.QA/9902030]

[Bu] S. Burciu, *A class of Drinfeld doubles that are ribbon algebras*, J. Algebra 320 (2008) 2053–2078

[CW1] M. Cohen and S. Westreich, *Some interrelations between Hopf algebras and their duals*, J. Algebra 283 (2005) 42–62

[CW2] M. Cohen and S. Westreich, *Fourier transforms for Hopf algebras*, Contemp. Math. 433 (2007) 115–133

[CW3] M. Cohen and S. Westreich, *Characters and a Verlinde-type formula for symmetric Hopf algebras*, J. Algebra 320 (2008) 4300–4316

[Da] A.A. Davydov, *Monoidal categories*, J. Math. Sci. 88 (1998) 457–519

[De] P. Deligne, *Catégories tannakiennes*, in: *The Grothendieck Festschrift*, vol. II, P. Cartier et al., eds. (Birkhäuser, Boston 1990), p. 111–195

[Dr] B. Drabant, *Notes on balanced categories and Hopf algebras*, in: *Quantum Groups and Lie Theory*, A. Pressley, ed. (Cambridge University Press, Cambridge 2001), p. 63–88

[EO] P.I. Etingof and V. Ostrik, *Finite tensor categories*, Moscow Math. J. 4 (2004) 627–654 [math.QA/0301027]

[EP] D.E. Evans and P.R. Pinto, *Subfactor realisation of modular invariants*, Commun. Math. Phys. 237 (2003) 309–363 [math.OA/0309174]

[FGST] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Commun. Math. Phys. 265 (2006) 47–93 [hep-th/0504093]

[Fr] I.L. Franco, *Autonomous pseudomonoids*, Ph.D. thesis Cambridge 2008

[FFRS] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Correspondences of ribbon categories*, Adv. Math. 199 (2006) 192–329 [math.CT/0309465]

[FRS] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators I: Partition functions*, Nucl. Phys. B 646 (2002) 353–497 [hep-th/0204148]

[FSc] J. Fuchs and C. Schweigert, *Hopf algebras and finite tensor categories in conformal field theory*, Revista de la Unión Matemática Argentina 51 (2010) 41–90 [1004.3405]

[FSt] J. Fuchs and C. Stigner, *On Frobenius algebras in rigid monoidal categories*, Arabian Journal for Science and Engineering 33-2C (2008) 175–191 [0901.4886]

[GR] M.R. Gaberdiel and I. Runkel, *From boundary to bulk in logarithmic CFT*, J. Phys. A 41 (2008) 075402_1–075402_29 [0707.0388]

[GW] S. Gelaki and S. Westreich, *On semisimple Hopf algebras of dimension pq*, Proc. Amer. Math. Soc. 128 (1999) 39–47

[KaS] L. Kadison and A.A. Stolin, *Separability and Hopf algebras*, Contemp. Math. 259 (2000) 279–296

[Ka] C. Kassel, *Quantum Groups* (Springer Verlag, New York 1995)

[KaR] L.H. Kauffman and D.E. Radford, *A necessary and sufficient condition for a finite-dimensional Drinfeld double to be a ribbon Hopf algebra*, J. Algebra 159 (1993) 98–114

[Ke] T. Kerler, *Genealogy of nonperturbative quantum-invariants of 3-manifolds: The surgical family*, in: *Quantum Invariants and Low-Dimensional Topology*, J.E. Andersen et al. (Dekker, New York 1997), p. 503–547 [q-alg/9601021]

[KL] T. Kerler and V.V. Lyubashenko, *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners Springer Lecture Notes in Mathematics 1765* (Springer Verlag, New York 2001)

[KoR] L. Kong and I. Runkel, *Cardy algebras and sewing constraints, I*, Commun. Math. Phys. 292 (2009) 871–912 [math.QA/0807.3356]

[LR] R.G. Larson and D.E. Radford, *Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple*, J. Algebra 117 (1988) 267–289

[LS] R.G. Larson and M.E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. 91 (1969) 75–94

[Ly1] V.V. Lyubashenko, *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun. Math. Phys. 172 (1995) 467–516 [hep-th/9405167]

[Ly2] V.V. Lyubashenko, *Modular transformations for tensor categories*, J. Pure Appl. Alg. 98 (1995) 279–327

[Ly3] V.V. Lyubashenko, *Ribbon abelian categories as modular categories*, J. Knot Theory and its Ramif. 5 (1996) 311–403

[LM] V.V. Lyubashenko and S. Majid, *Braided groups and quantum Fourier transform*, J. Algebra 166 (1994) 506–528

[Mü] M. Müger, *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Alg. 180 (2003) 81–157 [math.CT/0111204]

[NTV] D. Nikshych, V. Turaev, and L. Vainerman, *Quantum groupoids and invariants of knots and 3-manifolds*, Topology and its Appl. 127 (2003) 91–123 [math.QA/0006078]

[Pa] B. Pareigis, *When Hopf algebras are Frobenius algebras*, J. Algebra 18 (1971) 588–596

[Ra1] D.E. Radford, *On the antipode of a quasitriangular Hopf algebra*, J. Algebra 151 (1992) 1–11

[Ra2] D.E. Radford, *The trace function and Hopf algebras*, J. Algebra 163 (1994) 583–622

[Ra3] D.E. Radford, *On Kauffman's knot invariants arising from finite-dimensional Hopf algebras*, in: *Advances in Hopf Algebras*, J. Bergen and S. Montgomery, eds. (Marcel Dekker, NewYork 1994), p. 205–266

[QS] T. Quella and V. Schomerus, *Free fermion resolution of supergroup WZNW models*, J. High Energy Phys. 0709 (2007) 085_1–085_51 [0706.0744]

[Sc] P. Schauenburg, *On the Frobenius-Schur indicators for quasi-Hopf algebras*, J. Algebra 282 (2004) 129–139

[ST] A.M. Semikhatov and I.Yu. Tipunin, *The Nichols algebra of screenings*, preprint math.QA/1101.5810

[Vi] A. Virelizier, *Kirby elements and quantum invariants*, Proc. London Math. Soc. 93 (2006) 474–514 [math.GT/0312337]

[Ye1] D.N. Yetter, *Framed tangles and a theorem of Deligne on braided deformations of Tannakian categories*, Contemp. Math. 134 (1992) 325–349

[Ye2] D.N. Yetter, *Portrait of the handle as a Hopf algebra*, in: *Physics and Geometry*, J.E. Andersen, H. Pedersen, and A. Swann, eds. (Marcel Dekker, New York 1997), p. 481–502