

Higher genus mapping class group invariants from factorizable Hopf algebras

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ABSTRACT

Lyubashenko's construction associates representations of mapping class groups $\text{Map}_{g;n}$ of Riemann surfaces of any genus g with any number n of holes to a factorizable ribbon category. We consider this construction as applied to the category of bimodules over a finite-dimensional factorizable ribbon Hopf algebra H . For any such Hopf algebra we find an invariant of $\text{Map}_{g;n}$ for all values of g and n . More generally, we obtain such invariants for any pair (H, ω) , where ω is a ribbon automorphism of H .

Our results are motivated by the quest to understand higher genus correlation functions of bulk fields in two-dimensional conformal field theories with chiral algebras that are not necessarily semisimple, so-called logarithmic conformal field theories.

1 Introduction

The mapping class groups of Riemann surfaces with holes form an interesting system with deep properties and rich relations to geometry and arithmetic. It is therefore remarkable that a relatively simple algebraic structure – a finite-dimensional factorizable ribbon Hopf algebra H – gives rise [Ly1] to a family of (projective) representations of all these mapping class groups. The construction of mapping class group representations in [Ly1] does not require H to be semisimple. For semisimple H the so obtained system of representations obeys even tighter constraints: it is part of a so-called modular functor, or a three-dimensional topological field theory. In the present article we do *not* require semisimplicity.

Another algebraic structure leading to a system of representations of mapping class groups are vertex algebras which arise in chiral conformal field theories. More specifically, in this case the mapping class group representations are derived from the monodromies of the conformal blocks [FB] associated with the vertex algebra. In fortunate situations the representation category of a vertex algebra is equivalent to the one of a factorizable ribbon Hopf algebra at least as an abelian category. Surprisingly, such a “Kazhdan-Lusztig correspondence” seems to work particularly well for some classes of vertex algebras for which the category in question is *not* semisimple [FGST2, NT]. The chiral conformal field theories associated with these cases are known as *logarithmic* theories.

In a full, local conformal field theory, one aims in particular at constructing correlators of bulk fields as specific *bilinear* combinations of conformal blocks. Translating the situation to the Hopf algebra setting, this amounts to considering the mapping class group representations coming from the factorizable ribbon Hopf algebra $H \otimes H^{\text{op}}$, the enveloping algebra of H . (By factorizability, the category of $H \otimes H^{\text{op}}$ -modules is equivalent as a ribbon category to the Drinfeld center of $H\text{-Mod}$.) In [FSS1] we have constructed, for any ribbon automorphism ω of H , an invariant of the mapping class group of the torus with one hole. The construction in [FSS1] is based on a family of symmetric Frobenius algebras F^ω in the braided monoidal category $H\text{-Bimod}$, which is braided equivalent to $(H \otimes H^{\text{op}})\text{-Mod}$. In [FSS2] we have in addition derived, for the case that the automorphism ω is the identity morphism, integrality properties of the partition function, i.e. of the correlator for the torus without holes, relating it to the Cartan matrix of the category $H\text{-Mod}$. This shows in particular that the so obtained invariants are non-zero.

In the present paper we solve the general problem of obtaining mapping class group invariants at *arbitrary* genus. Given a factorizable Hopf algebra H and a ribbon automorphism ω of H , we identify, for each non-negative value of g and of n , a natural invariant $\text{Cor}_{g;n}^\omega$ under the action of the mapping class group $\text{Map}_{g;n}$ of Riemann surfaces of genus g with n holes on a space of morphisms that is obtained by the construction of Lyubashenko [Ly1] when taking all n insertions to be given by the H -bimodule F^ω . Rephrased in conformal field theory terms, we identify natural candidates for bulk correlation functions in a full, local conformal field theory and prove their modular invariance, for any number of insertions and at any genus.

This paper is organized as follows. In Section 2 we introduce pertinent concepts and notations that are needed to describe the morphisms $\text{Cor}_{g;n}^\omega$ and to state our main result. This assertion, Theorem 3.2, is formulated in Section 3. To establish it requires quite a few detailed calculations which, for the case $\omega = \text{id}_H$, take up Sections 4 and 5. Section 4 is essentially a collection of lemmas that are instrumental in Section 5; their proofs can be safely skipped by readers primarily interested in the results. Finally, in Section 6 we complete our main result

by extending the analysis of Sections 4 and 5, and thus the proof of Theorem 3.2, to the case of non-trivial ribbon automorphisms.

We expect that our considerations generalize from the categories $H\text{-Bimod}$ to a larger class of braided finite tensor categories \mathcal{C} . In particular, the analogue of the H -bimodule F^ω should be the coend of a natural functor from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to the enveloping category $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$. Accordingly we formulate various statements in such a more general context, e.g. we give the invariants $\text{Cor}_{g;n}^\omega$ first as morphisms in $H\text{-Bimod}$ in entirely categorical terms (see formula (3.5)) before we present their concrete expressions as linear maps (see (3.10)). However, a generalization of our main result to such a context is still elusive. Concretely, the explicit expressions for the coalgebra structure of the coend in (3.4) involve the integral of the Hopf algebra H over the field \mathbb{k} defining the category. What we are missing is a corresponding structure of the *category* $H\text{-Mod}$ that comes from the integral of H and endows the coend with a coalgebra structure.

2 Background

In this section we collect some basic definitions and notation for a class of Hopf algebras and for representations of mapping class groups associated with these Hopf algebras, which will be needed for formulating our main result, Theorem 3.2.

2.1 Factorizable Hopf algebras

Throughout this paper, the symbol \mathbb{k} stands for an algebraically closed field of characteristic zero, while H is a finite-dimensional ribbon Hopf algebra over \mathbb{k} , which in addition is factorizable. In the sequel, for brevity we will refer to H just as a *factorizable ribbon Hopf algebra*, suppressing finite-dimensionality over \mathbb{k} . All modules and bimodules in this paper will be finite-dimensional as \mathbb{k} -vector spaces as well. Similarly, all categories to be considered are assumed to be abelian and \mathbb{k} -linear, with all morphism sets being finite-dimensional \mathbb{k} -vector spaces. We denote by $m, \eta, \Delta, \varepsilon$ and s the product, unit, coproduct, counit and antipode of the Hopf algebra H .

Let us recall the meaning of a Hopf algebra to be factorizable ribbon:

Definition 2.1.

(a) A Hopf algebra $H \equiv (H, m, \eta, \Delta, \varepsilon, s)$ is called *quasitriangular* iff it comes with an invertible element R of $H \otimes_{\mathbb{k}} H$ such that the coproduct and opposite coproduct are intertwined by R , i.e. $R \Delta R^{-1} = \tau_{H,H} \circ \Delta \equiv \Delta^{\text{op}}$, and

$$(\Delta \otimes id_H) \circ R = R_{13} \cdot R_{23} \quad \text{and} \quad (id_H \otimes \Delta) \circ R = R_{13} \cdot R_{12}. \quad (2.1)$$

The element R is called the *R-matrix* of H .

(b) For (H, R) a quasitriangular Hopf algebra, the invertible element $Q := R_{21} \cdot R$ of $H \otimes_{\mathbb{k}} H$ is called the *monodromy matrix* Q of H .

(c) A quasitriangular Hopf algebra (H, R) is called a *ribbon* Hopf algebra iff it comes with a central invertible element $v \in H$ obeying

$$s \circ v = v, \quad \varepsilon \circ v = 1 \quad \text{and} \quad \Delta \circ v = (v \otimes v) \cdot Q^{-1}. \quad (2.2)$$

v is called the *ribbon element* of H .

(d) A quasitriangular Hopf algebra (H, R) is called *factorizable* iff the monodromy matrix Q can be expressed as $\sum_{\ell} h_{\ell} \otimes k_{\ell}$, where $\{h_{\ell}\}$ and $\{k_{\ell}\}$ are two vector space bases of H .

Here for \mathbb{k} -vector spaces V and W the linear map $\tau_{V,W}: V \otimes_{\mathbb{k}} W \xrightarrow{\sim} W \otimes_{\mathbb{k}} V$ is the flip map that exchanges the factors in a tensor product. Also recall that a finite-dimensional Hopf algebra H has a left integral $\Lambda \in H \equiv \text{Hom}_{\mathbb{k}}(\mathbb{k}, H)$ and a right cointegral $\lambda \in H^* \equiv \text{Hom}_{\mathbb{k}}(H, \mathbb{k})$ (as well as a right integral and a left cointegral), which are unique up to scalars. Moreover, if H is factorizable, then it is unimodular, i.e. the integral $\Lambda \in H$ is two-sided. With applications in logarithmic conformal field theory in mind [FGST1], it should be appreciated that factorizability can be formulated for Hopf algebras which do not have an R-matrix, but still a monodromy matrix.

Factorizability of H is equivalent to invertibility of the *Drinfeld map* $f_Q \in \text{Hom}_{\mathbb{k}}(H^*, H)$, which is given by $f_Q := (d_H \otimes id_H) \circ (id_{H^*} \otimes Q)$, with d the evaluation morphism in $\mathcal{Vect}_{\mathbb{k}}$. The Drinfeld map, and likewise the map $f_{Q^{-1}}$ that is obtained when replacing the monodromy matrix by its inverse, is not just a linear map from H^* to H , but it also intertwines the left-coadjoint action of H on H^* and the left-adjoint action of H on itself. If f_Q is invertible, then $f_Q(\lambda)$ is a non-zero multiple of Λ . The normalizations of the integral and cointegral can then be chosen in such a way that $\lambda \circ \Lambda = 1$ and $f_Q(\lambda) = \Lambda$ (this determines λ and Λ uniquely up to a common sign factor). Doing so one arrives at the following identities [FSS1, (5.18)], which we present graphically:

$$(2.3)$$

Such diagrams are to be read from bottom to top. Below we will often suppress the labels indicating the product and coproduct of H , as well as those for the antipode s (which is drawn as an empty circle) and its inverse s^{-1} (full circle).

The following facts about the category $H\text{-Mod}$ of finite-dimensional left modules over a factorizable ribbon Hopf algebra H are well-known: $H\text{-Mod}$ is a braided rigid monoidal category, and even a factorizable ribbon category. Moreover, it is a finite tensor category in the sense of [EO], i.e. it has finitely many isomorphism classes of simple objects, each of them has a projective cover, and every object has finite length.

We endow the category $H\text{-Bimod}$ of finite-dimensional H -bimodules in an analogous manner with the structure of a finite factorizable ribbon category: We use the pull-back of left and right actions along the coproduct to obtain the structure of a monoidal category: for H -bimodules (X, ρ_X, ϱ_X) and (Y, ρ_Y, ϱ_Y) we define the left and right actions of H on the tensor product vector space $X \otimes_{\mathbb{k}} Y$ by

$$\begin{aligned} \rho_{X \otimes Y} &:= (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (\Delta \otimes id_X \otimes id_Y) \quad \text{and} \\ \varrho_{X \otimes Y} &:= (\varrho_X \otimes \varrho_Y) \circ (id_X \otimes \tau_{Y,H} \otimes id_H) \circ (id_X \otimes id_Y \otimes \Delta). \end{aligned} \quad (2.4)$$

The monoidal unit for this tensor product is the one-dimensional vector space \mathbb{k} with left and right H -action given by the counit, $\mathbf{1}_{H\text{-Bimod}} = (\mathbb{k}, \varepsilon, \varepsilon)$. To endow the monoidal category $H\text{-Bimod}$ with a braiding, we use the action of the R-matrix of H from the right and the action of its inverse from the left. When regarding the resulting braiding morphism $c_{X,Y}$ in $H\text{-Bimod}$ as a linear map, i.e. as a morphism in the category \mathcal{Vect} of \mathbb{k} -vector spaces, we represent it pictorially as

$$c_{X,Y} = \begin{array}{c} \begin{array}{c} Y \quad X \\ \begin{array}{c} \rho_Y \\ \tau_{X,Y} \\ \rho_X \end{array} \\ X \quad Y \end{array} \end{array} \quad (2.5)$$

Here the quarter disks refer to left or right actions of the Hopf algebra H , while the crossing is just the flip map of vector spaces. (As the braiding on the category of vector spaces is symmetric, the use of over- and under-crossings in these pictures does not contain any mathematical information, but is merely for graphical convenience.)

Using the equivalence of H -bimodules with $H \otimes_{\mathbb{k}} H$ -modules, it is not hard to verify that these prescriptions endow $H\text{-Bimod}$ with the structure of a braided monoidal category. This is the braided monoidal category we are interested in in this paper. We endow it with more structure: we introduce right and left duals by associating to a bimodule $X = (X, \rho, \varrho) \in H\text{-Bimod}$ the bimodules

$$X^\vee := (X^*, \rho_\vee, \varrho_\vee) \quad \text{and} \quad {}^\vee X := (X^*, {}_\vee \rho, {}_\vee \varrho) \quad (2.6)$$

with left and right H -actions defined by

$$\begin{array}{c} \rho_\vee := \begin{array}{c} X^* \\ \begin{array}{c} \rho \\ \rho \end{array} \\ H \quad X^* \end{array} \quad \varrho_\vee := \begin{array}{c} X^* \\ \begin{array}{c} \varrho \\ \varrho \end{array} \\ X^* \quad H \end{array} \quad {}_\vee \rho := \begin{array}{c} X^* \\ \begin{array}{c} \rho \\ \rho \end{array} \\ H \quad X^* \end{array} \quad {}_\vee \varrho := \begin{array}{c} X^* \\ \begin{array}{c} \varrho \\ \varrho \end{array} \\ X^* \quad H \end{array} \end{array} \quad (2.7)$$

The left and right dualities are naturally compatible – the category $H\text{-Bimod}$ has a natural structure of a sovereign tensor category. To see this, denote by $t := uv^{-1}$ the product of the Drinfeld element

$$u := m \circ (s \otimes id_H) \circ \tau_{H,H} \circ R \in H \quad (2.8)$$

with the inverse of the ribbon element v of H ; t is a group-like element of H . Then the family

of endomorphisms

$$\pi_X := \begin{array}{c} X^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ X^* \end{array} \in \text{End}_{\mathbb{k}}(X^*) \quad (2.9)$$

is a natural monoidal isomorphism between the left and right duality functors and thus endows $H\text{-Bimod}$ with the structure of a sovereign tensor category. Being sovereign, $H\text{-Bimod}$ is also endowed with a balancing and thus has the structure of a ribbon category. From here on, the symbol $H\text{-Bimod}$ stands for this sovereign ribbon category. Explicitly, the twist endomorphism θ_X of an H -bimodule (X, ρ, ϱ) is given by acting with the ribbon element v from the left and with its inverse v^{-1} from the right [FSS1, Lemma 4.8],

$$\theta_X = \rho \circ (id_H \otimes \varrho) \circ (v \otimes id_X \otimes v^{-1}). \quad (2.10)$$

Remark 2.2. The category $H\text{-Bimod}$ with this structure of sovereign ribbon category is braided equivalent to $(H \otimes_{\mathbb{k}} H^{\text{op}})\text{-Mod}$ and thus to $H\text{-Mod} \boxtimes H\text{-Mod}^{\text{rev}}$. This is a very natural category indeed: it can be regarded as a categorification of the notion of enveloping algebra. Factorizability of $H\text{-Bimod}$ amounts to the statement that $H\text{-Bimod}$ is braided equivalent to the Drinfeld center of $H\text{-Mod}$.

2.2 The handle Hopf algebra and half-monodromies

We now recall [Ma, Ly2] that any finite sovereign braided tensor category \mathcal{C} contains a canonical Hopf algebra object. It can be constructed as the coend

$$K = \int^X F(X, X) = \int^X X^\vee \otimes X \quad (2.11)$$

of the functor F that maps a pair (X, Y) of objects of \mathcal{C} to the object $X^\vee \otimes Y \in \mathcal{C}$. As a coend, K comes with a dinatural family $(\iota_X^K)_{X \in \mathcal{C}}$ of morphisms $\iota_X^K \in \text{Hom}_{\mathcal{C}}(X^\vee \otimes X, K)$. The structure morphisms of the Hopf algebra object K are obtained with the help of the family ι^K and the braiding and duality of \mathcal{C} . They can be found e.g. in [Ly1, Vi]; we refrain from reproducing them here. We refer to K as the *handle Hopf algebra* for the category \mathcal{C} .

The Hopf algebra K is also endowed with a Hopf pairing. We call the finite sovereign braided tensor category \mathcal{C} a *factorizable finite tensor category* iff this Hopf pairing is non-degenerate. This terminology is motivated by the fact that in the case of $\mathcal{C} = H\text{-Mod}$, non-degeneracy of the Hopf pairing is equivalent to invertibility of the Drinfeld map, i.e. to factorizability of H [FSS1, Eq. (5.14)]. If and only if \mathcal{C} is factorizable in this sense, then the Hopf algebra K has (two-sided) integrals and cointegrals [KL, Prop. 5.2.9].

Let now H be a factorizable ribbon Hopf algebra and $\mathcal{C} = H\text{-Mod}$. In this case, which was already considered in [Ly2], the coend $K = H_{\triangleright}^*$ is the dual space H^* endowed with the coadjoint action of H . Similarly, if \mathcal{C} is the braided tensor category $H\text{-Bimod}$, the bimodule $K =: HH_{\triangleright\triangleleft}^*$

can be described as follows: the underlying vector space is the tensor product $H^* \otimes_{\mathbb{k}} H^*$, and introducing the morphisms

$$\rho_{\triangleright} := \text{[diagram]} \quad \text{and} \quad \rho_{\triangleleft} := \text{[diagram]} \quad (2.12)$$

the bimodule structure is given by

$$HH_{\triangleright\triangleleft}^* = (H^* \otimes_{\mathbb{k}} H^*, \rho_{\triangleright} \otimes id_{H^*}, id_{H^*} \otimes \rho_{\triangleleft}). \quad (2.13)$$

(For the dinatural family of the coend $HH_{\triangleright\triangleleft}^*$ see [FSS1, Eq. (A.29)].)

Below we will recall that representations of mapping class groups can be constructed with the help of morphisms involving the Hopf algebra K in a factorizable finite tensor category \mathcal{C} . Here we provide the main building blocks of those morphisms. It is natural to use the universal property of K as a coend to specify these building blocks in terms of dinatural families.

1. Denote by $(\theta_X)_{X \in \mathcal{C}}$ the twist on the tensor category \mathcal{C} . Then we define an endomorphism T_K of K in terms of dinatural families by

$$T_K = \text{[diagram]} = \text{[diagram]} \quad (2.14)$$

Here we indicate explicitly in the figure that the diagram has to be read in the ribbon category \mathcal{C} (rather than, as in all pictures displayed so far, in $\mathcal{Vect}_{\mathbb{k}}$), so that in particular over- and under-crossings must be carefully distinguished.

2. Similarly, the monodromies $c_{Y^v, X} \circ c_{X, Y^v}$ of \mathcal{C} allow us to deduce an endomorphism \mathcal{O}_K of $K \otimes K$ from the equality

$$\mathcal{O}_K = \text{[diagram]} = \text{[diagram]} \quad (2.15)$$

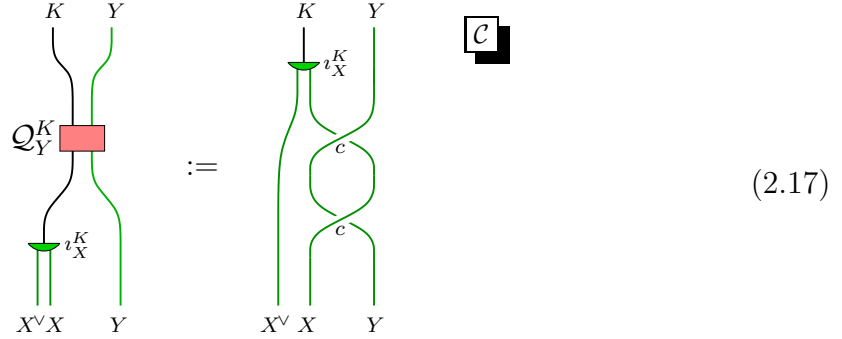
of dinatural transformations.

3. We use the endomorphism \mathcal{O}_K to define an endomorphism $S_K \in \text{End}_{\mathcal{C}}(K)$ by

$$S_K := (\varepsilon_K \otimes id_K) \circ \mathcal{O}_K \circ (id_K \otimes \Lambda_K), \quad (2.16)$$

where ε_K and Λ_K are the counit and the two-sided integral of the Hopf algebra K , respectively.

4. For any object $Y \in \mathcal{C}$ we use the monodromies $c_{Y,-} \circ c_{-,Y}$ to define an endomorphism $\mathcal{Q}_Y^K \in \text{End}_{\mathcal{C}}(K \otimes Y)$:



5. It can now be checked that for any object $Y \in \mathcal{C}$ the morphism $\rho_Y^K \in \text{Hom}_{\mathcal{C}}(K \otimes Y, Y)$ defined by

$$\rho_Y^K := (\varepsilon_K \otimes id_Y) \circ \mathcal{Q}_Y^K \quad (2.18)$$

endows Y with the structure of a left K -module.

Remark 2.3. Via the duality, the dinaturality morphisms of the coend K provide a right coaction $(id_X \otimes \iota_X^K) \circ (b_X \otimes id_X)$ of K on any object of \mathcal{C} . This coaction fits together with the action ρ_X^K to endow the object X with a structure of left-right Yetter-Drinfeld module in the monoidal category \mathcal{C} . Moreover, any morphism in \mathcal{C} is compatible with this structure, so that we indeed have a fully faithful embedding of \mathcal{C} into the category of left-right Yetter-Drinfeld modules over K .

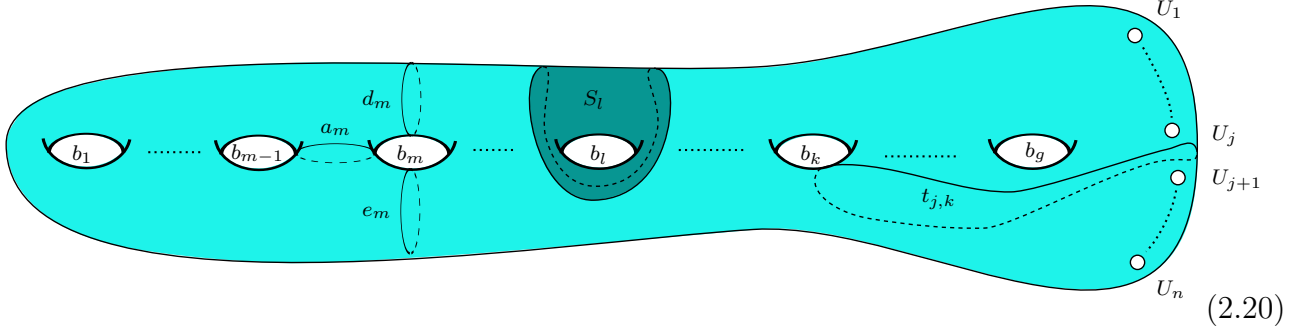
2.3 Representations of mapping class groups

One of the main results of [Ly3] is the construction of representations of mapping class groups for surfaces with holes (i.e., with open disks excised). Denote by $\text{Map}_{g;n}$ the mapping class group of closed oriented surfaces of genus g with n boundary components. Various finite presentations of $\text{Map}_{g;n}$ have been discussed in the literature. Since our purpose is to check invariance under the mapping class group, for us it is sufficient to display a finite set of generators. One such set of generators arises from the exact sequence

$$1 \rightarrow B_{g,n} \rightarrow \text{Map}_{g;n} \rightarrow \text{Map}_{g;0} \rightarrow 1, \quad (2.19)$$

(compare [FM, Thm. 9.1]), where $B_{g,n}$ is a central extension of the surface braid group by \mathbb{Z}^n . Owing to this sequence, one can take as a generating set the union of those for a presentation of $\text{Map}_{g;0}$ [Wa] and for a presentation of $B_{g,n}$ [Sc]; this particular presentation has been advocated and used in [Ly1, Ly3].

To describe these generators, we first introduce a collection of cycles a_m, b_m, d_m, e_m and $t_{j,k}$ on a genus- g surface $\Sigma_{g,n}$ with n holes. The following picture indicates these cycles on $\Sigma_{g,n}$ (see [Ly1, Figs. 2 & 6]):



For brevity we refer to the inverse Dehn twist about any of these cycles by the same symbol as for the cycle itself. The shaded region in the picture (2.20) is a neighborhood of the l th handle with the topology of a one-holed torus; we denote this region by F_l , and by F'_l the slightly smaller neighborhood that is indicated by the dotted line inside F_l . The generators considered in [Ly1, Ly3] are then the following:

1. Braidings ω_i , for $i = 1, 2, \dots, n-1$, which permute the i th and $i+1$ st boundary circle.
2. Dehn twists ϑ_i about the i th boundary circle, for $i = 1, 2, \dots, n$.
3. Homeomorphisms S_l , for $l = 1, 2, \dots, g$, which act as the identity outside the region F_l and as a modular S-transformation of the one-holed torus $F'_l \subset F_l$.
4. Inverse Dehn twists in tubular neighborhoods of the cycles a_m and e_m , for $m = 2, 3, \dots, g$.
5. Inverse Dehn twists in tubular neighborhoods of the cycles b_m and d_m , for $m = 1, 2, \dots, g$.
6. Inverse Dehn twists in tubular neighborhoods of the cycles $t_{j,k}$, for $j = 1, 2, \dots, n-1$ and $k = 1, 2, \dots, g$.

This system of generators is not minimal. Specifically, the generators S_k can be expressed in terms of the generators b_k and d_k . Nevertheless we keep the S_k in the list, because in the case $g = 1$ and $n = 0$, $S = S_k$ and $T = d_k$ are the usual S- and T-transformations generating the modular group $\text{SL}(2, \mathbb{Z})$. As already pointed out, since the aim of the present article is to determine invariants, it is sufficient to know the action of some set of generators; relations are not needed.

The representations of our interest involve decorations of the boundary circles of $\Sigma_{g,n}$ by (not necessarily distinct) objects X_1, X_2, \dots, X_n of a factorizable finite tensor category \mathcal{C} . Denote by $\mathfrak{N} = \mathfrak{N}(X_1, \dots, X_n)$ the subgroup of the symmetric group \mathfrak{S}_n that is generated by those permutations $\sigma \in \mathfrak{S}_n$ for which X_i and $X_{\sigma(i)}$ are non-isomorphic for at least one value of the label i , and set

$$X := \bigoplus_{\sigma \in \mathfrak{N}} X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)}. \quad (2.21)$$

Then the representation space for $\text{Map}_{g,n}$ relevant to us is the vector space

$$V_{g;n}^X := \text{Hom}_{\mathcal{C}}(K^{\otimes g}, X) \quad (2.22)$$

of morphisms of \mathcal{C} . In line with the designation ‘handle Hopf algebra’ for K , this involves one copy of K for each handle of $\Sigma_{g,n}$.

To describe the action of $\text{Map}_{g;n}$ on the space $V_{g;n}^X$, we introduce the following collections of morphisms: First, the endomorphisms z_{ω_i} with $i \in \{1, 2, \dots, n-1\}$ and z_{ϑ_j} with $j \in \{1, 2, \dots, n\}$ of X that act as

$$\begin{aligned} z_{\omega_i} \Big|_{X_1 \otimes X_2 \otimes \dots \otimes X_n} &:= id_{X_1 \otimes \dots \otimes X_{i-1}} \otimes c_{X_i, X_{i+1}} \otimes id_{X_{i+2} \otimes \dots \otimes X_n} \quad \text{and} \\ z_{\vartheta_j} \Big|_{X_1 \otimes X_2 \otimes \dots \otimes X_n} &:= id_{X_1 \otimes \dots \otimes X_{j-1}} \otimes \theta_{X_j} \otimes id_{X_{j+1} \otimes \dots \otimes X_n} \end{aligned} \quad (2.23)$$

on the direct summand $X_1 \otimes X_2 \otimes \dots \otimes X_n$ of X and analogously on the other summands in (2.21), with c and θ the braiding and the twist of \mathcal{C} , respectively. Second, the endomorphisms

$$\begin{aligned} z_{S_k} &:= id_{K^{\otimes g-k}} \otimes S_K \otimes id_{K^{\otimes k-1}}, \\ z_{a_l} &:= id_{K^{\otimes g-l}} \otimes [\mathcal{O}_K \circ (T_K \otimes T_K)] \otimes id_{K^{\otimes l-2}}, \\ z_{b_k} &:= id_{K^{\otimes g-k}} \otimes (S_K^{-1} \circ T_K \circ S_K) \otimes id_{K^{\otimes k-1}}, \\ z_{d_k} &:= id_{K^{\otimes g-k}} \otimes T_K \otimes id_{K^{\otimes k-1}}, \\ z_{e_l} &:= id_{K^{\otimes g-l}} \otimes [(T_K \otimes \theta_{K^{\otimes l-1}}) \circ \mathcal{Q}_{K^{\otimes l-1}}^K] \end{aligned} \quad (2.24)$$

of $K^{\otimes g}$, where $k \in \{1, 2, \dots, g\}$ and $l \in \{2, 3, \dots, g\}$, and where T_K , S_K , \mathcal{O}_K and \mathcal{Q}_Y^K are the morphisms introduced in Section 2.2. And third, for any $j \in \{1, 2, \dots, n-1\}$ and $k \in \{1, 2, \dots, g\}$ the linear map $z_{t_{j,k}}$ that maps any morphism $f \in \text{Hom}_{\mathcal{C}}(K^{\otimes g}, X_1 \otimes \dots \otimes X_n)$ to

$$\begin{aligned} z_{t_{j,k}}(f) &:= \left(\left[(id_{X_1 \otimes \dots \otimes X_j} \otimes \tilde{d}_{X_{j+1} \otimes \dots \otimes X_n}) \circ (f \otimes id_{X_n \otimes \dots \otimes X_{j+1}}) \right. \right. \\ &\quad \left. \left. \circ \{ id_{K^{\otimes g-k}} \otimes [\mathcal{Q}_{K^{\otimes k-1} \otimes X_n \otimes \dots \otimes X_{j+1}}^K \circ (T_K \otimes \theta_{K^{\otimes k-1} \otimes X_n \otimes \dots \otimes X_{j+1}})] \} \right] \right. \\ &\quad \left. \otimes id_{X_{j+1} \otimes \dots \otimes X_n} \right) \\ &\quad \left. \circ (id_{K^{\otimes g}} \otimes \tilde{b}_{X_{j+1} \otimes \dots \otimes X_n}) \right) \end{aligned} \quad (2.25)$$

in $\text{Hom}_{\mathcal{C}}(K^{\otimes g}, X_1 \otimes \dots \otimes X_n)$ and acts analogously on morphisms in $\text{Hom}_{\mathcal{C}}(K^{\otimes g}, X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \dots \otimes X_{\sigma(n)})$ for $\sigma \in \mathfrak{N}$. (A graphical description of the map $z_{t_{j,k}}$ will be given in picture (5.16) below.)

Then we can rephrase the results of [Ly1, Sect. 4] and [Ly3, Sect. 3] as follows:

Proposition 2.4. *For any collection of objects X_i of \mathcal{C} as above, the linear endomorphisms of the space $V_{g;n}^X$ that act as*

$$\pi_{g;n}^X(\gamma) := \begin{cases} (z_\gamma)_* & \text{for } \gamma = \omega_i, \vartheta_i \quad (i = 1, 2, \dots, n-1 \text{ resp. } i = 1, 2, \dots, n), \\ (z_\gamma)^* & \text{for } \gamma = S_k \quad (k = 1, 2, \dots, g) \\ & \text{or } \gamma = a_m, b_m, d_m, e_m \quad (m = 1, 2, \dots, g \text{ resp. } m = 2, 3, \dots, g), \\ z_{t_{j,k}} & \text{for } \gamma = t_{j,k} \quad (j = 1, 2, \dots, n-1 \text{ and } k = 1, 2, \dots, g) \end{cases} \quad (2.26)$$

generate a projective representation $\pi_{g;n}^X$ of the mapping class group $\text{Map}_{g;n}$ on the vector space $V_{g;n}^X$.

Remark 2.5. In [Ly1, Ly3] the role of source and target in the vector space $V_{g:n}^X$ (2.22), and correspondingly the role of pre- and post-composition in the formulas (2.26), is interchanged. Accordingly in our description the inverses of the morphisms z_γ used in [Ly1, Ly3] appear.

Remark 2.6. In applications (for details see Remark 3.3 below) one is also interested in the following variant. Partition the set of boundary circles of the surface into two subsets of sizes p and q and denote by $\text{Map}_{g:p,q}$ the subgroup of the mapping class group $\text{Map}_{g:p+q}$ that leaves each of these two subsets separately invariant. Further, denote the corresponding decorations by X_1, X_2, \dots, X_p and by Y_1, Y_2, \dots, Y_q , respectively, and define objects X and Y analogously as in (2.21). Finally note that the right duality of \mathcal{C} provides a linear isomorphism

$$\varphi : \text{Hom}_{\mathcal{C}}(K^{\otimes g} \otimes Y, X) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(K^{\otimes g}, X \otimes Y^\vee). \quad (2.27)$$

Then

$$\pi_{g,p,q}^{Y,X}(\gamma) := \varphi^{-1} \circ \pi_{g,p+q}^{X \otimes Y^\vee}(\gamma) \circ \varphi \quad (2.28)$$

defines a representation $\pi_{g,p,q}^{Y,X}$ of the group $\text{Map}_{g:p,q}$ on the space $\text{Hom}_{\mathcal{C}}(K^{\otimes g} \otimes Y, X)$.

3 Mapping class group invariants

We now consider the category $\mathcal{C} = H\text{-Bimod}$ of finite-dimensional bimodules over a factorizable ribbon Hopf algebra H . Recall that there is a canonical Hopf algebra K in $H\text{-Bimod}$, obtainable as the coend (2.11) for $H\text{-Bimod}$. Its bimodule structure is given in (2.13); for a detailed description of the structure morphisms of K as a Hopf algebra, see [FSS1, App. A.3]. Also recall the action $\pi_{g:n}^X$ of $\text{Map}_{g:n}$ described in Proposition 2.4.

Our goal is to provide, given the Hopf algebra H in $\mathcal{V}ect_{\mathbb{k}}$, and thus the Hopf algebra K in $H\text{-Bimod}$, the following collection of data:

- An object F in the category $H\text{-Bimod}$ that carries a natural structure of a commutative symmetric Frobenius algebra.
- For any choice of non-negative integers g and n a morphism

$$\text{Cor}_{g;n} \in \text{Hom}_{H|H}(K^{\otimes g}, F^{\otimes n}) \quad (3.1)$$

that is invariant under the action $\pi_{g;n}^X$ of $\text{Map}_{g;n}$ with $X = F^{\otimes n}$ (which corresponds to taking $X_1 = \dots = X_n = F$ as objects in formula (2.21)).

Remark 3.1. Any such object F is a candidate for a *space of bulk states* in a conformal quantum field theory whose chiral data are described by the category $H\text{-Mod}$ of H -modules. The morphisms $\text{Cor}_{g;n}$ are then candidates for modular invariant bulk correlation functions of the conformal field theory, for world sheets of any genus g and for any number n of bulk field insertions. The classification of spaces of bulk states for given chiral data and the construction of correlation functions for a given space of bulk states are two of the most desirable issues in the study of conformal field theory.

In order to allow for a consistent interpretation as a partition function, $\text{Cor}_{1;0}$, i.e. the zero-point correlator on a torus, should be non-zero.

It is worth stressing that the object F is by no means uniquely determined by the existence of invariants (3.1). Indeed a whole family $\{F^\omega\}$ of commutative symmetric Frobenius algebras in H -Bimod that are candidates for such an object, as well as the corresponding invariant morphisms for the case that $g=1$ and $n \in \{0, 1\}$, have already been obtained in [FSS1]. Here ω is any choice of a ribbon automorphism of the Hopf algebra H . Moreover, for the case of the identity automorphism it was shown [FSS2] that the resulting invariant for $g=1$ and $n=0$ is non-zero.

The main result of the present paper is that each of these H -bimodules F^ω indeed has all the desired properties: for any choice of ribbon automorphism ω we are able to provide the morphism $\text{Cor}_{g;n}$ and establish its $\text{Map}_{g;n}$ -invariance for arbitrary integers $g, n \geq 0$. On the other hand, generically the family $\{F^\omega\}$ can *not* be expected to exhaust all solutions to the problem posed above; our results do not suggest any concrete approach to such a classification.

As it turns out, the presence of a general ribbon automorphism ω only constitutes a minor modification of the issues that already arise in the case of the identity automorphism. Accordingly the bimodule of central importance for our discussion is F^{id_H} , the one obtained when $\omega = id_H$. Henceforth we slightly abuse notation and simply use the symbol F for this object. For the family $\{F^\omega\}$ obtained in [FSS1], setting $\omega = id_H$ yields the *coregular bimodule* in H -Bimod. By this we mean the dual of the regular bimodule (H, m, m) , i.e. the vector space H^* endowed with the dual of the regular left and right actions of H on itself. Explicitly,

$$F = (H^*, \rho_F, \varrho_F) \quad (3.2)$$

with $\rho_F \in \text{Hom}(H \otimes H^*, H^*)$ and $\varrho_F \in \text{Hom}(H^* \otimes H, H^*)$ given by

$$\begin{aligned} \rho_F &:= (d_H \otimes id_{H^*}) \circ (id_{H^*} \otimes m \otimes id_{H^*}) \circ (id_{H^*} \otimes S \otimes b_H) \circ \tau_{H, H^*} \quad \text{and} \\ \varrho_F &:= (d_H \otimes id_{H^*}) \circ (id_{H^*} \otimes m \otimes id_{H^*}) \circ (id_{H^*} \otimes id_H \otimes \tau_{H^*, H}) \circ (id_{H^*} \otimes b_H \otimes S^{-1}). \end{aligned} \quad (3.3)$$

It has been demonstrated [FSS1] that the coregular bimodule F carries a natural structure of a commutative symmetric Frobenius algebra in the ribbon category H -Bimod. Moreover, F can be characterized as the coend of a suitable functor from $H\text{-Mod}^{\text{op}} \times H\text{-Mod}$ to H -Bimod. This way the structural morphisms endowing F with the structure of an algebra in H -Bimod can be obtained with the help of the universal property of this coend. In contrast, the coalgebra structure of F involves the integral and cointegral of H . Specifically, the product m_F is the dual of the coproduct of H and the unit η_F is the dual of the counit of H , in the coproduct Δ_F the cointegral $\lambda \in H^*$ enters, and the counit ε_F is the dual of the integral $\Lambda \in H$. Explicitly, in graphical notation we have

$$m_F = \quad \eta_F = \quad \Delta_F = \quad \varepsilon_F = \quad (3.4)$$

We are now in a position to give our result for the morphisms $\text{Cor}_{g;n}$ in the category $\mathcal{C} = H\text{-Bimod}$. We first present $\text{Cor}_{g;n}$ in purely categorical terms: as an element of the morphism space $\text{Hom}_{\mathcal{C}}(K^{\otimes n}, F^{\otimes n})$ of the category \mathcal{C} it is given by

(3.5)

$\text{Cor}_{g;n} :=$

for $n > 0$, and as $\text{Cor}_{g;0} := \varepsilon_F \circ \text{Cor}_{g;1}$.

Let us describe the rationale for arriving at this expression for $\text{Cor}_{g;n}$.

1. Draw a skeleton for the surface $\Sigma_{g,n}$, including outward-oriented edges attached to the boundary components, and label each edge of this skeleton with the Frobenius algebra F in \mathcal{C} . (Instead of with edges, a priori one may want to work with ribbons, but owing to $\theta_F = id_F$ this is insignificant.)
2. Orient the internal edges in such a way that each of the vertices of the skeleton has either two incoming and one outgoing edge or vice versa. Then label each vertex with the product m_F or coproduct Δ_F of F , depending on whether two or one of its edges are incoming.
3. Further, for each handle of the surface, attach another edge labeled by the handle Hopf algebra $K \in \mathcal{C}$ to the corresponding loop of the skeleton, and label the resulting new trivalent vertex by the action ρ_F^K , as defined in (2.18), of the Hopf algebra K on the object in \mathcal{C} underlying the Frobenius algebra F .
4. The resulting graph is interpreted as a morphism in \mathcal{C} . Using the fact that the algebra F is commutative symmetric Frobenius, the vertices can be rearranged in such a way that one ends up with the morphism given in (3.5).

Our main result can now be stated as follows:

Theorem 3.2. *Let $\mathcal{C} = H\text{-Bimod}$ for H a finite-dimensional factorizable ribbon Hopf algebra. Then for any pair of integers $g, n \geq 0$ the morphism $\text{Cor}_{g;n}$ is invariant under the action $\pi_{g;n}^{F^{\otimes n}}$ of the mapping class group $\text{Map}_{g;n}$ described in Proposition 2.4.*

Remark 3.3. As already pointed out, a major motivation for our investigations are applications to conformal field theory. In that context, the morphisms $\text{Cor}_{g;n}$ describe correlation functions of bulk fields. As such they not only have to be invariant under the relevant action of the mapping class group, but must also satisfy so-called factorization constraints (for a precise formulation in the semisimple case see [FFRS]). The latter constraints relate correlators for surfaces of different topology and necessarily involve correlators with (using quantum field theory terminology) both incoming and outgoing field insertions. We do not have anything to say about factorization constraints in this paper, but including the possibility of having both incoming and outgoing insertions is easy. Indeed, for any choice of non-negative integers g, p and q , according to Remark 2.6 we have a representation of $\text{Map}_{g;p,q}$ on the space $\text{Hom}_{H|H}(K^{\otimes g} \otimes F^{\otimes q}, F^{\otimes p})$. A morphism

$$\text{Cor}_{g;p,q} \in \text{Hom}_{H|H}(K^{\otimes g} \otimes F^{\otimes q}, F^{\otimes p}) \quad (3.6)$$

that is invariant under this action is then obtained as follows. We have $\text{Cor}_{g;p,0} = \text{Cor}_{g;p}$ as given in (3.5), while for $q > 0$ the morphism $\text{Cor}_{g;p,q}$ is obtained from $\text{Cor}_{g;p}$ by just replacing the unit $\eta_F \in \text{Hom}_{H|H}(\mathbf{1}, F) \equiv \text{Hom}_{H|H}(F^{\otimes 0}, F)$ by the morphism in $\text{Hom}_{H|H}(F^{\otimes q}, F)$ that is given by a $q-1$ -fold product of F .

Remark 3.4. As already pointed out, we actually find a family of suitable objects F^ω , labeled by ribbon automorphisms ω of H , and provide invariant vectors $\text{Cor}_{g;p,q}^\omega$ in the corresponding morphism spaces $\text{Hom}_{H|H}(K^{\otimes g} \otimes (F^\omega)^{\otimes q}, (F^\omega)^{\otimes p})$ for all values of g, p and q . Thus for any ribbon automorphism ω we obtain candidates for the bulk state space and for bulk correlation functions in conformal field theory. For brevity we have concentrated above to the case of the identity automorphism. Sections 4 and 5 below contain the proof of our main result for $\omega = id_H$, while the proof for the general case will be accomplished in Section 6.

The picture (3.5) represents $\text{Cor}_{g;n}$ as a morphism of the braided tensor category of H -bimodules. We conclude this section by expressing $\text{Cor}_{g;p,q}$ as a \mathbb{k} -linear map, thereby obtaining a pictorial description in the category of vector spaces. To this end we insert explicit expressions for all the structural morphisms appearing in (3.5). The expressions for the structural morphisms of F have already been displayed in the picture (3.4). It therefore suffices to describe in addition the morphisms $\text{Cor}_{g;1,1}$ with one incoming and one outgoing insertion of F .

Let us first consider the case $g = p = 1$ and $q = 0$:

Lemma 3.5. *The morphism $\text{Cor}_{1;1,0} \equiv \text{Cor}_{1;1}$ satisfies the following chain of equalities of linear*

maps:

Here Q , λ and Λ are the monodromy matrix, the cointegral and the integral of H , respectively.

Proof. Insert the expressions for m_F , η_F and Δ_F as well as for the K -action ρ_F^K (with braidings according to (2.5) and with the formula for ρ_X^K that we will present for a general H -bimodule X in (4.4) below) into (3.5) with $g = n = 1$. Then by using associativity of the product m of H , we arrive at the first picture in the chain (3.7) of equalities. The second equality follows by using several times the anti-(co)algebra property of the antipode s of H . In the resulting morphism we can recognize the left-adjoint H -action on the right leg of the inverse monodromy matrix Q^{-1} . The third equality is then just the statement that the morphism $f_{Q^{-1}}$ intertwines the left-adjoint and left-coadjoint actions. The last equality follows with the help of the identity (2.3) together with $(s \otimes s) \circ Q = \tau_{H,H} \circ Q$, the anti-coalgebra property of the inverse antipode and the fact that, by unimodularity of H , $s \circ \Lambda = \Lambda$. \square

To proceed to $\text{Cor}_{1;1,1}$ we simply note that, by just using the unit property of η_F and the Frobenius property and associativity of F , we have

$$\text{Cor}_{1;1,1} = m_F \circ [(\text{Cor}_{1;1,1} \circ (id_K \otimes \eta_F)) \otimes id_F] = m_F \circ (\text{Cor}_{1;1,0} \otimes id_F). \quad (3.8)$$

With the result (3.7) of Lemma 2.6 and the explicit form of the multiplication m_F from formula (3.4), this amounts to

with q_{\diamond} the right-adjoint action of H on itself. This expression extends in a straightforward

manner to

$$\text{Cor}_{g;1,1} = \text{Diagram} \tag{3.10}$$

Finally, the morphisms with q or p larger than 1 are obtained by pre- and post-composition of (3.10) with an appropriate multiple product and multiple coproduct of F , respectively. Explicitly, writing $m_F^{(2)} = m_F$ and $m_F^{(l)} = m_F \circ (m_F^{(l-1)} \otimes id_F)$ for $l > 2$ as well as $m_F^{(0)} = \eta_F$ and $m_F^{(1)} = id_F$, and analogously for $\Delta_F^{(l)}$, we have

$$\text{Cor}_{g;p,q} = \Delta_F^{(p)} \circ \text{Cor}_{g;1,1} \circ (id_{K^{\otimes g}} \otimes m_F^{(q)}) \tag{3.11}$$

for all p, q and g .

4 Useful identities

In this section we present a few preliminary results that will be instrumental in the proof of Theorem 3.2. Our first task is to specialize the half-monodromies and related morphisms that we introduced, in Section 2.2, for a general factorizable finite tensor category \mathcal{C} to the specific case of $\mathcal{C} = H\text{-Bimod}$. To do so we use the specific form (2.5) of the braiding in $H\text{-Bimod}$ and the explicit expressions [Ly1, Vi] for the dinatural family and for the Hopf algebra structure morphisms of K . For the morphism \mathcal{Q}_X^K defined in formula (2.17) we then obtain, for any bimodule $X = (X, \rho_X, \varrho_X)$,

$$\mathcal{Q}_X^K = \text{Diagram} \tag{4.1}$$

Next, for further use we note that, since the R-matrix intertwines the coproduct and opposite coproduct of H , conjugating by the monodromy matrix preserves the coproduct:

$$= \Delta_H =$$
(4.6)

We will also make use of the following result that can be formulated in the general setting of ribbon categories:

Lemma 4.1. *For any commutative Frobenius algebra A in a ribbon category \mathcal{C} the equalities*

$$=$$
(4.7)

hold.

Proof. The first equality is a direct consequence of the Frobenius property of A . The second equality follows by consecutively using associativity, commutativity, and associativity combined with the Frobenius property. \square

We are now in a position to establish various convenient identities that are satisfied for any factorizable ribbon Hopf algebra H . First we obtain various equalities involving the coproduct and monodromy matrix.

Lemma 4.2. *The following identities hold:*

$$=$$
(4.8)

and

$$(4.9)$$

and

$$(4.10)$$

and

$$(4.11)$$

Proof. (i) Consider the equality between the left- and right-most expressions in (4.7) for the case of our interest, i.e. with $A=F$ and with c the braiding in H -Bimod. This involves the left-dual actions \smile^{ρ^H} and \smile^{λ^H} of H on the bimodule $\smile F$. The latter obey

$$(4.12)$$

Implementing this relation and composing the resulting equality with suitable duality morphisms yields (4.8).

(ii) On the left hand side of (4.9), push the inverse antipode that is located highest up in the

picture to the top of the picture, i.e. through a coproduct and two products, and push the two ‘lowest’ products upwards by invoking the connecting axiom of the bialgebra H . This results in the left hand side of the following chain of equalities:

(4.13)

The first of these equalities follows easily by a multiple use of associativity and coassociativity, while the second one is obtained with the help of the defining property of the antipode and of coassociativity. Now the right hand side of (4.13) equals the right hand side of (4.9), as is seen by first applying the connecting axiom to the two coproducts that come directly after the monodromy matrices and then invoking the identity (4.8).

(iii) The first equality in (4.10) is obtained by pushing the inverse antipodes down through the coproduct (together with a deformation compatible with the braid relations). The second of the equalities in (4.10) follows from coassociativity combined with (4.6), while the third equality is immediate from coassociativity and properties of the inverse antipode.

(iv) The first equality in (4.11) follows by pushing s^{-1} through the upper coproduct and products. The second equality uses coassociativity and $(s \otimes s) \circ Q = \tau_{H,H} \circ Q$. Applying first (4.6) and then undoing the manipulations performed in the first two equalities yields the third one. \square

The next two identities involve the two-sided integral Λ and the right cointegral λ of H .

Lemma 4.3. *We have*

(4.14)

where λ_0 is the right-adjoint action of H on itself (as defined in (3.9)).

Proof. This follows by combining the explicit form of λ_0 with the identities [FSS1, (2.18)]

(4.15)

(which, in turn, are obtained by combining various defining properties of the Hopf algebra structure of H with unimodularity). \square

Lemma 4.4. *We have the chain*

$$(4.16)$$

of equalities.

Proof. The second equality uses the fact that, since H is unimodular, the cointegral satisfies [Ra, Thm. 3] $\lambda \circ m = \lambda \circ m \circ \tau_{H,H} \circ (id_H \otimes S^2)$. The third follows with the help of $s \circ \Lambda = \Lambda$. The important ingredient in the last equality is the invertibility of the Frobenius map, which is equivalent to the statement that H has a natural structure of a Frobenius algebra (see e.g. [CW, Eq. (8)]). \square

Finally we have the following result involving somewhat more complicated expressions, which arise in connection with the action (2.25) of the generators $t_{j,k}$ of the mapping class group.

Lemma 4.5. *For any integer $p \geq 1$ we have*

$$(4.17)$$

where $\rho_{K^{\otimes p}}$ and $q_{K^{\otimes p}}$ are the left and right actions of the factorizable ribbon Hopf algebra H on the H -bimodule $K^{\otimes p}$, respectively.

Proof. Recalling the expression (3.10) for $\text{Cor}_{g;1,1}$, writing out the actions $\rho_{K^{\otimes p}}$ and $\mathfrak{q}_{K^{\otimes p}}$ and invoking Lemma 4.3, the left hand side of (4.17) becomes

(4.18)

Invoking the representation property of the right-adjoint action \mathfrak{q}_0 and the anti-(co)algebra morphism property of the antipode several times, this morphism can be rewritten as

(4.19)

We now observe that in (4.19) there appear two copies of the $(p-1)$ -fold coproduct of H . Using that these are bimodule morphisms from H to $H^{\otimes p}$, we conclude that (4.19), and thus the left hand side of (4.17), equals the right hand side of (4.17). \square

5 Proof of the main theorem

We are now in a position to establish invariance of the correlators $\text{Cor}_{g;n}$ under the action of the mapping class group $\text{Map}_{g;n}$. In the subsequent lemmas we treat separately the various types of generators from the presentation given in Section 2.3.

We start with the generators affording S- and T-transformations of the one-holed torus.

Lemma 5.1. *We have*

$$\text{Cor}_{1;1,0} \circ S_K = \text{Cor}_{1;1,0} \quad \text{and} \quad \text{Cor}_{1;1,0} \circ T_K = \text{Cor}_{1;1,0} \quad (5.1)$$

for S_K and T_K as given in (4.3).

Proof. This has already been shown in Lemmas 5.8 and 5.9 of [FSS1], but we find it instructive to repeat the main steps here.

(i) Composing S_K^{-1} as given in (4.5) with the first expression for $\text{Cor}_{1;1,0}$ in (3.7) results in the first equality in

$$\text{Cor}_{1;1,0} \circ S_K^{-1} = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \quad (5.2)$$

Here in the second step we get rid of the two monodromy matrices by implementing the relations (2.3) between the integral, cointegral and monodromy matrix, combined with the identity $\lambda \circ m = \lambda \circ m \circ \tau_{H,H} \circ (id_H \otimes s^2)$, while the third follows by Lemma 4.4. Finally, after pushing the ‘upper’ inverse antipode through the coproduct, the so obtained final expression in (5.2) can be recognized as the right-most one in formula (3.7). This shows $\text{Cor}_{1;1,0} \circ S_K^{-1} = \text{Cor}_{1;1,0}$.

(ii) Composing T_K as given in (4.3) with the last expression for $\text{Cor}_{1;1,0}$ in (3.7) yields

$$\text{Cor}_{1;1,0} \circ T_K = \begin{array}{c} \text{Diagram} \end{array} \quad (5.3)$$

After applying (4.15) and using that $s \circ v = v$, the central elements v and v^{-1} appearing here cancel out. Thus we arrive at $\text{Cor}_{1;1,0} \circ T_K = \text{Cor}_{1;1,0}$. \square

Using the identities (4.15) and inserting the explicit form of the right coadjoint action q_\circ , this can be rewritten as

$$\text{Cor}_{2;1,1} \circ (\mathcal{O}_K \otimes \text{id}_F) = \text{Diagram 1} = \text{Diagram 2} \quad (5.8)$$

Observing that $(S^2 \otimes S^2) \circ Q = Q$ and invoking the identity (4.10), comparison with (3.10) establishes (5.6). \square

Again this result easily generalizes:

Proposition 5.5. *For any triple of integers $g \geq 2$ and $p, q \geq 0$ we have*

$$\text{Cor}_{g;p,q} \circ (\text{id}_K^{\otimes m} \otimes \mathcal{O}_K \otimes \text{id}_K^{\otimes g-m-2} \otimes \text{id}_F^{\otimes q}) = \text{Cor}_{g;p,q} \quad (5.9)$$

for all $m \in \{0, 1, \dots, g-2\}$.

Now we present relations that will help us to show invariance under the action of the generators $t_{j,k}$ of the mapping class group.

Lemma 5.6. *For any integer $m \geq 2$ we have*

$$(\text{id}_F \otimes \tilde{b}_F) \circ (\text{Cor}_{m;2,0} \otimes \text{id}_{\vee F}) \circ \mathcal{Q}_{K^{\otimes m-1} \otimes \vee F}^K = (\text{id}_F \otimes \tilde{b}_F) \circ (\text{Cor}_{m;2,0} \otimes \text{id}_{\vee F}) \quad (5.10)$$

in $\text{Hom}_{H|H}(K^{\otimes m} \otimes \vee F, F)$. That is, graphically:

$$\text{Diagram 1} = \text{Diagram 2} \quad (5.11)$$

Proof. Denote the left hand side of (5.10) by Φ . We first insert the structural morphisms for F from (3.4) and the expressions (3.10) for $\text{Cor}_{m;1,1}$ and (4.1) for \mathcal{Q}_X^K and invoke Lemma 4.5, and in a second step we insert the explicit expression for α_\diamond and use the identities (4.15) and (4.12). This yields

(5.12)

Application of the identity (4.9) to the right hand side of (5.12) results in the right hand side of (5.10), □

Composition of (5.11) with $id_{K^{\otimes m}} \otimes \vee_{\mathcal{E}_F}$ gives

Corollary 5.7. *For any integer $g \geq 2$ we have*

$$\text{Cor}_{g;1,0} \circ \mathcal{Q}_{K^{\otimes g-1}}^K = \text{Cor}_{g;1,0}. \quad (5.13)$$

We can combine the previous results to omit not only the twist of any tensor power $K^{\otimes m}$ of K , but also the one of $K^{\otimes m} \otimes \vee F$:

Lemma 5.8. For any integer $m > 0$ the equalities

(5.14)

hold in $\text{Hom}_{H|H}(K^{\otimes m}, F \otimes F)$.

Proof. Using the compatibility between braiding and twist, as well as that F has trivial twist and that according to Lemma 5.3 the twist of $K^{\otimes m}$ can be omitted, we can replace $\theta_{K^{\otimes m} \otimes \vee F}$ by the monodromy between $K^{\otimes m}$ and $\vee F$. Using naturality of the braiding and thus of monodromy, we can push this monodromy through the F -loops and thus arrive at the middle picture. The second equality follows because F is commutative Frobenius, in the same way as in the proof of Lemma 4.1. \square

Lemma 5.9. For any pair of integers $g, n > 0$ we have

$$\text{Cor}_{g;n} = \text{Diagram} \tag{5.15}$$

for all $m = 1, 2, \dots, g$

Proof. As compared to the expression (3.5) for $\text{Cor}_{g;n}$, on the right hand side three additional pieces are present: the endomorphism T_K applied to one copy of K , a twist endomorphism of $K^{\otimes m} \otimes V F$, and a partial monodromy $\mathcal{Q}_{K^{\otimes m-1} \otimes V F}^K$. Now the latter acts trivially by Lemma 5.6. After omitting this part, we can invoke Proposition 5.2 and Lemma 5.8 (combined with the Frobenius property of F) to omit T_K and $\theta_{K^{\otimes m} \otimes V F}$, respectively, as well. \square

As a final piece of information we give the graphical description of the morphisms $z_{t_j, k}(f)$

(2.25) that were introduced in formula (2.25) for $f \in \text{Hom}_{\mathcal{C}}(K^{\otimes g}, X_1 \otimes \cdots \otimes X_n)$:

$$z_{t_{j,k}}(f) = \text{Diagram} \quad (5.16)$$

We have now all ingredients at our hands that are needed to finish the proof of Theorem 3.2.

Proof.

Invoking the presentation of the mapping class group $\text{Map}_{g;n}$ described in Section 2.3, invariance of the correlators $\text{Cor}_{g;n}$ under the action $\pi_{g;n}^{F^{\otimes n}}$ of $\text{Map}_{g;n}$ amounts to invariance under $\pi_{g;n}^{F^{\otimes n}}(\gamma)$ for $\gamma \in \text{Map}_{g;n}$ any of the generators listed in Proposition 2.4.

- (i) $\gamma = \vartheta_i$ ($i = 1, 2, \dots, n$): Invariance follows directly from the fact that F has trivial twist.
- (ii) $\gamma = \omega_i$ ($i = 1, 2, \dots, n-1$): Invariance follows directly from the fact that F is cocommutative.
- (iii) $\gamma = S_k$ or b_k or d_k ($k = 1, 2, \dots, g$): In view of the explicit expressions (2.24) for z_{S_k} , z_{b_k} and z_{d_k} , invariance is implied by Proposition 5.2.
- (iv) $\gamma = a_k$ ($k = 2, 3, \dots, g$): By using Proposition 5.2 twice we have

$$\text{Cor}_{g;n} \circ (id_K^{\otimes m} \otimes T_K \otimes T_K \otimes id_K^{\otimes g-m-2}) = \text{Cor}_{g;n} \quad (5.17)$$

for all $m = 0, 1, \dots, g-2$. Thus in view of the explicit expression (2.24) for z_{a_k} , Proposition 5.5 establishes the invariance.

- (v) $\gamma = e_k$ ($k = 2, 3, \dots, g$): Combining Proposition 5.2 and Lemma 5.3 we obtain the equality $\text{Cor}_{g;1} \circ (id_K^{\otimes g-k} \otimes T_K \otimes \theta_{K^{\otimes k-1}}) = \text{Cor}_{g;1}$ which, in turn, together with Corollary 5.7 yields

$$\text{Cor}_{g;1} \circ [(id_K^{\otimes g-k} \otimes T_K \otimes \theta_{K^{\otimes k-1}}) \circ \mathcal{Q}_{K^{\otimes k-1}}^K] = \text{Cor}_{g;1}. \quad (5.18)$$

This obviously generalizes to any number of $n \geq 0$ insertions and thus in view of the explicit expression (2.24) for z_{e_k} establishes invariance.

(vi) $\gamma = t_{j,k}$ ($j = 1, 2, \dots, n-1$ and $k = 1, 2, \dots, g$): We first note that composing the dinatural family of the coend K with the partial monodromy \mathcal{Q}^K results in an ordinary monodromy. This implies that

$$\mathcal{Q}_Y^K \circ (id_K \otimes f) = (id_K \otimes f) \circ \mathcal{Q}_X^K \quad (5.19)$$

for any morphism $f \in \text{Hom}(X, Y)$. With the help of this relation (as well as the coassociativity and Frobenius property of F) one can in particular push partial monodromies through coproducts Δ_F , and by functoriality of the twist the same can be done with twist endomorphisms. As a consequence, the morphism obtained by acting according to formula (2.25) and picture (5.16) with $t_{j,k}$ on $\text{Cor}_{g;n}$ can be rewritten as the one on the right hand side of (5.15). Thus invariance under $t_{j,k}$ reduces to the assertion of Lemma 5.9. \square

6 Invariants from ribbon automorphisms

We finally extend our result from F to similar H -bimodules for which the action of the Hopf algebra is twisted by a suitable automorphism.

Definition 6.1. A *ribbon* automorphism of a ribbon Hopf algebra H is a Hopf algebra automorphism ω of H that preserves the ribbon element and the R -matrix of H ,

$$\omega \circ v = v \quad \text{and} \quad (\omega \otimes \omega) \circ R = R. \quad (6.1)$$

For any Hopf algebra automorphism ω of a Hopf algebra H we denote by F^ω the bimodule obtained from F by twisting the right H -action by ω , i.e.

$$F^\omega := (H^*, \rho_F, \varrho_F \circ (id_{H^*} \otimes \omega)). \quad (6.2)$$

(An isomorphic bimodule is obtained when twisting instead the left H -action by ω^{-1} .) In Section 6 of [FSS1] the following is shown:

Lemma 6.2. *Let H be a factorizable ribbon Hopf algebra and ω a ribbon automorphism of H .*

(i) *The H -bimodule F^ω together with the dinatural family of morphisms*

$$i_X^{F^\omega} := (\omega^{-1})^* \circ i_X^F \quad (6.3)$$

is the coend of the functor from $H\text{-Mod}^{\text{pp}} \times H\text{-Mod}$ to $H\text{-Bimod}$ that acts on objects by assigning to a pair $((U, \rho_U), (V, \rho_V))$ of left H -modules the vector space $U^\vee \otimes_{\mathbb{k}} V$ endowed with left H -action $[(\rho_U)_\vee \circ (\omega^{-1} \otimes id_{U^})] \otimes id_V$ and right H -action $id_{U^*} \otimes [\rho_V \circ \tau_{V,H} \circ (id_V \otimes S^{-1})]$.*

(ii) *The linear maps defined in (3.4) equip the object F^ω in $H\text{-Bimod}$ with the structure of a commutative symmetric Frobenius algebra, with trivial twist, in $H\text{-Bimod}$. Furthermore, F^ω is special iff H is semisimple.*

To proceed we note the following identity:

Lemma 6.3. *For any factorizable ribbon Hopf algebra H the relation*

$$f_{Q^{-1}}(\lambda \circ m \circ (v \otimes id_H)) = (\lambda \circ v) v^{-1} \quad (6.4)$$

involving the ribbon element v , the inverse of the monodromy matrix Q and the cointegral λ holds.

Proof. Just use the fact (see formula (2.2)) that $(v \otimes v) \cdot Q^{-1} = \Delta \circ v$ and afterwards the defining property of the cointegral. \square

As a direct consequence we have

Lemma 6.4. *Every ribbon automorphism ω of H preserves the integral and cointegral of H , i.e.*

$$\lambda \circ \omega = \lambda \quad \text{and} \quad \omega \circ \Lambda = \Lambda. \quad (6.5)$$

Proof. Consider the equality obtained by composing (6.4) with ω . Using on the left hand side of this equality that ω is an algebra morphism and that it preserves v (and thus v^{-1}) as well as Q^{-1} , one arrives at an equality that differs from (6.4) only by replacing λ on the left hand side by $\lambda \circ \omega^{-1}$. Using further that the morphism $f_{Q^{-1}}$ as well as the element v of H are invertible, the first of the equalities (6.5) follows.

Further note that $\omega \circ \Lambda$ is again a non-zero integral and is thus proportional to Λ . Since $\lambda \circ \omega \circ \Lambda = \lambda \circ \Lambda \in \mathbb{k}$ is non-zero, this implies the second equality in (6.5). \square

We can now generalize the morphisms $\text{Cor}_{g;n}$ defined in (3.5) by simply replacing every occurrence of the Frobenius algebra F with F^ω . We denote the so obtained morphisms by $\text{Cor}_{g;n}^\omega$.

Proposition 6.5. *For every ribbon automorphism ω of H we have*

$$\text{Cor}_{g;n}^\omega = \text{Cor}_{g;n} \circ (id_{H^*} \otimes (\omega^{-1})^*)^{\otimes g} \quad (6.6)$$

as linear maps, for all pairs of integers $g, n \geq 0$.

Proof. Inserting the H -bimodule structure (6.2) of F^ω into the expression (4.4) for the action of K we have

$$\rho_{F^\omega}^K = \begin{array}{c} \begin{array}{c} H^* \\ \rho_F \\ \rho_F \\ \omega \\ H^* \end{array} \end{array} \quad (6.7)$$

Using $(id_H \otimes \omega) \circ Q = (\omega^{-1} \otimes id_H) \circ Q$, it follows immediately that $\rho_{F^\omega}^K = \rho_F^K \circ (id_{H^*} \otimes (\omega^{-1})^*)$ which, in turn, implies (6.6). \square

Next we note the following ω -twisted version of Lemma 4.5:

Lemma 6.6. Denoting, as in picture (4.17), by $\rho_{K^{\otimes p}}$ and $\varrho_{K^{\otimes p}}$ the left and right H -actions on $K^{\otimes p}$, we have

(6.8)

Proof. This follows by the same line of arguments as in Lemma 4.5, combined with the identity $\varrho_{\circ} \circ (\omega^{-1} \otimes id_H) = \omega^{-1} \circ \varrho_{\circ} \circ (id_H \otimes \omega)$. \square

Theorem 6.7. Let H be a finite-dimensional factorizable ribbon Hopf algebra and ω a ribbon automorphism of H . Then for any pair of integers $g, n \geq 0$ the morphism $Cor_{g;n}^{\omega}$ is invariant under the action $\pi_{g;n}^{(F^{\omega})^{\otimes n}}$ of the mapping class group $Map_{g;n}$.

Proof. Just like in the case $\omega = id_{H^*}$, invariance under the action of the generators ω_i and ϑ_i is an immediate consequence of the fact that F^{ω} is cocommutative and has trivial twist.

Next consider the generators S_k, a_k, b_k, d_k and e_k . Proposition 6.5 reduces invariance to the statement that the morphism $\pi_{g;n}^{(F^{\omega})^{\otimes n}}(\gamma)$ commutes with $(id_{H^*} \otimes (\omega^{-1})^*)^{\otimes g}$ for $\gamma = S_k, a_m, b_m, d_m$ or e_m . In particular, for the case of $\gamma = S_1$ and $g = 1$, the following chain of equalities establishes invariance:

(6.9)

The first of these equalities follows by pushing the automorphism ω through the product and using that ω commutes with the antipode of H , while the second equality follows by using $\lambda \circ \omega = \lambda$ from Lemma 6.4 and $(\omega \otimes \omega) \circ Q = Q$.

That $[\pi_{g;n}^{(F^\omega)^{\otimes n}}(\gamma), (id_{H^*} \otimes (\omega^{-1})^*)^{\otimes g}] = 0$ holds as well for any genus g and any of the generators $\gamma = S_k, a_m, b_m, d_m, e_m$ is shown in a completely analogous manner.

It remains to consider the action of the generators $t_{j,k}$. To this end we just observe that because of Proposition 6.5, a version of Proposition 5.2 holds in which F is replaced by F^ω , while Lemma 6.6 implies that there are versions of Lemma 5.6 and Lemma 5.8 in which F is replaced by F^ω . Combining these versions of Proposition 5.2 and Lemmas 5.6 and 5.8, one arrives at a corresponding ω -twisted version of Lemma 5.9. Now notice that in the case of $\omega = id_H$, invariance under the action of $t_{j,k}$ follows from Lemma 5.9 by just invoking that F is Frobenius. As a consequence, the twisted version of Lemma 5.9 allows us to deduce invariance in precisely the same manner as for $\omega = id_H$. \square

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