STRUCTURAL CONNECTIONS BETWEEN A FORCING CLASS AND ITS MODAL LOGIC

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Abstract. Every definable forcing class Γ gives rise to a corresponding forcing modality, for which $\Box_\Gamma \varphi$ means that φ is true in all Γ extensions, and the valid principles of Γ forcing are the modal assertions that are valid for this forcing interpretation. For example, [9] shows that if ZFC is consistent, then the ZFC-provably valid principles of the class of all forcing are precisely the assertions of the modal theory S4.2. In this article, we prove similarly that the provably valid principles of collapse forcing, Cohen forcing and other classes are in each case exactly S4.3; the provably valid principles of c.c.c. forcing, proper forcing, and others are each contained within S4.3 and do not contain S4.2; the provably valid principles of countably closed forcing, CH-preserving forcing and others are each exactly S4.2; and the provably valid principles of ω_1-preserving forcing are contained within S4.tBA. All these results arise from general structural connections we have identified between a forcing class and the modal logic of forcing to which it gives rise.

1. Introduction

In [9], we considered the *modal logic of forcing*, which arises when one considers a model of set theory in the context of all its forcing extensions, interpreting $\Box$ as "in all forcing extensions" and $\Diamond$ as "in some forcing extension". This modal language allows one easily to express sweeping general principles concerning forcing absoluteness and the effect of forcing on truth in set theory, such as the assertion $\Diamond \Box \varphi \to \Box \Diamond \varphi$, expressing that every possibly necessary statement is necessarily possible, which the reader may verify is valid for the forcing interpretation, or the assertion $\Diamond \Box \varphi \to \varphi$, that every possibly necessary statement is true, which is an equivalent formulation of the maximality principle [8, 17], a forcing axiom independent of but equiconsistent with ZFC. It was known from [8] that the valid principles of forcing include all the assertions of the modal theory S4.2, and the main theorem of [9] established that if ZFC is consistent, then in fact the ZFC-provably valid principles of forcing are exactly the assertions of S4.2.

In this article, we consider more generally the modal logic of various particular kinds of forcing, such as c.c.c. forcing or ω_1-preserving forcing, by relativizing the modal operators to such a class. In [9], a number of connections between various modal axioms and corresponding operations on the class of forcing notions had surfaced: for instance, the validity of

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the axiom .2 mentioned above is linked to the operation of finite products of forcing notions [9, p. 1794]. In this article, we shall uncover further such structural connections between a class of forcing notions and the modal logic of forcing to which it gives rise, and use these connections to settle the exact modal logic of several natural forcing classes, while providing new bounds on several others. We have aimed to provide general tools, including the control statements of §4 and their connection to the forcing validities, which seem promising to enable set theorists to undertake an investigation of the modal logic of additional forcing classes by means of a forcing-only analysis (requiring no substantial modal logic), by discovering which kinds of control statements their forcing class supports.

For example, using these methods we prove in this article that the provably valid principles of collapse forcing, Cohen forcing and other classes are in each case exactly $\mathsf{S4.3}$ (theorems 24 and 31); the provably valid principles of c.c.c. forcing, proper forcing, and others are each contained within $\mathsf{S4.3}$ and do not contain $\mathsf{S4.2}$ (corollary 33); the provably valid principles of countably closed forcing, CH-preserving forcing and others are each exactly $\mathsf{S4.2}$ (theorems 34 and 37); and the provably valid principles of ω_1-preserving forcing are contained within $\mathsf{S4.tBA}$ (theorem 36).

The forcing interpretation of modal logic was introduced in [8], with a deeper more explicit investigation in [9]. Various other aspects of the modal logic of forcing are considered in [13, 10, 5, 6, 14, 16, 4, 3]. We consider this article a natural successor to [9].

2. Modal logic background

The modal logic of forcing involves an interplay between two formal languages, which we sharply distinguish, namely (i) the language of propositional modal logic $\mathcal{L}_\Box$, which has propositional variables, logical connectives and the modal operators $\Box$ and $\Diamond$ (where $\Diamond\varphi$ is defined as $\neg\Box\neg\varphi$; and (ii) the usual first-order language of set theory $\mathcal{L}_\in$, which has variables, quantifiers, logical connectives and the relations $=$ and $\in$. One may regard a modal assertion $\varphi(p_1,\ldots,p_n)$ as a template for the scheme of set-theoretic assertions $\varphi(\psi_1,\ldots,\psi_n)$ which arise by the substitution of set-theoretic assertions ψ_i for the propositional variables p_i and the forcing interpretation of the modal operators. The nature of this translation will be further investigated in §3. Meanwhile, in this section, let us introduce the modal theories that will arise in that analysis. Using the following axioms (with their established nomenclature)

$$\begin{array}{cl}
\mathrm{K} & \Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)\\
\mathrm{T} & \Box\varphi\to\varphi\\
4 & \Box\varphi\to\Box\Box\varphi\\
.2 & \Diamond\Box\varphi\to\Box\Diamond\varphi\\
.3 & (\Diamond\varphi\wedge\Diamond\psi)\to\Diamond[(\varphi\wedge\Diamond\psi)\vee(\psi\wedge\Diamond\varphi)]\\
5 & \Diamond\Box\varphi\to\varphi,
\end{array}$$

we define the desired modal theories, where in each case we close the axioms under modus ponens and necessitation:

$$\begin{array}{rcl}
\mathsf{S4} & = & \mathrm{K}+\mathrm{T}+4\\
\mathsf{S4.2} & = & \mathrm{K}+\mathrm{T}+4+.2\\
\mathsf{S4.3} & = & \mathrm{K}+\mathrm{T}+4+.3\\
\mathsf{S5} & = & \mathrm{K}+\mathrm{T}+4+5.
\end{array}$$

It is not difficult to see that $\mathsf{S4} \vdash 5 \to .3$ and $\mathsf{S4} \vdash .3 \to .2$, and consequently $\mathsf{S4} \subseteq \mathsf{S4.2} \subseteq \mathsf{S4.3} \subseteq \mathsf{S5}$.

The Kripke model concept provides a robust semantics for modal logic: a *Kripke model* is a collection M of possible worlds, each providing truth values for the propositional variables, together with a relation on the worlds called the accessibility relation. The *frame* of such a model is this accessibility relation, disregarding the truth assignments of the worlds. Modal truth is defined by induction in the natural way, so that $\Box\varphi$ is true at a world in M, if φ is true in all accessible worlds, and similarly $\Diamond\varphi$ is true at a world if φ is true at some accessible world. We write $[\![v]\!]_M$ for the set of $\mathcal{L}_\Box$-formulas true at v in M and note that this is a set closed under modus ponens.

If F is a frame, a modal assertion is *valid for F* if it is true at all worlds of all Kripke models having F as a frame. If $\mathcal{C}$ is a class of frames, a modal theory is *sound with respect to $\mathcal{C}$* if every assertion in the theory is valid for every frame in $\mathcal{C}$. A modal theory is *complete with respect to $\mathcal{C}$* if every assertion valid for every frame in $\mathcal{C}$ is in the theory. Finally, a modal theory is *characterized by $\mathcal{C}$* (equivalently, $\mathcal{C}$ *characterizes* the modal theory) if it is both sound and complete with respect to $\mathcal{C}$ [11, p. 40].

It turns out that all the modal theories we mentioned are characterized by certain classes of finite Kripke frames. For the rest of this section, we mention the completeness results of which we shall subsequently make use. Numerous further completeness results can be found in [2].

A *pre-order* is a reflexive transitive relation. Every pre-order $\leq$ admits a quotient by the equivalence $x \equiv y \leftrightarrow x \leq y \leq x$, and this quotient will be a partial order. Thus, a pre-order is obtained from a partial order by replacing each node with a cluster of equivalent nodes. A *linear pre-order* is a pre-order in which any two nodes are comparable.

Theorem 1. *The modal logic* $\mathsf{S4.3}$ *is characterized by the class of finite linear pre-order frames. That is, a modal assertion is derivable in* $\mathsf{S4.3}$ *if and only if it holds in all Kripke models having a finite linear pre-ordered frame.*

Proof. Cf. [2, corollary 5.18 & theorem 5.33] or [1, exercise 4.33, theorem 4.96, & lemma 6.40]. □

Every finite Boolean algebra partial order is isomorphic to the set of subsets of a finite set, ordered by inclusion. Of course, this algebra also has the structure of meets (intersection), joins (unions) and negation (complements), with a smallest element $\mathbb{0}$ and a largest element $\mathbb{1}$. A *pre-Boolean algebra* is a partial pre-order $\langle \mathbb{B}, \leq \rangle$, such that the quotient by the relation $x \equiv y \leftrightarrow x \leq y \leq x$ is a Boolean algebra. Thus, every pre-Boolean algebra is obtained from a Boolean algebra by replacing each node in the Boolean algebra by a cluster of equivalent nodes.

Theorem 2 ([9, theorem 11])**.** *The modal logic* $\mathsf{S4.2}$ *is characterized by the class of finite pre-Boolean algebras. That is, a modal assertion is derivable in* $\mathsf{S4.2}$ *if and only if it holds in all Kripke models having a finite pre-Boolean algebra frame.*

Theorem 3. *The modal logic* $\mathsf{S5}$ *is characterized by the class of finite equivalence relations with one equivalence class (a single cluster).*

Proof. Cf. [2, corollaries 5.19 & 5.29] or [1, theorems 4.29, 4.96 & exercise 6.6.4]. □

We have observed another modal theory to arise several times in the modal logic of various forcing classes, and so let us now introduce it. This is the logic of finite topless pre-Boolean algebras. A *topless* Boolean algebra is obtained from a Boolean algebra by omitting the top element $\mathbb{1}$. Thus, a finite topless Boolean algebra is isomorphic to the collection of strictly proper subsets of a given finite set. A topless pre-Boolean algebra is a partial pre-order (transitive and reflexive), such that the natural quotient by the relation $x \equiv y \leftrightarrow x \leq y \leq x$ is a topless Boolean algebra. We define the modal theory $\mathsf{S4.tBA}$, or *topless pre-Boolean algebra logic*, to be the collection of all modal assertions that are true in all Kripke models whose frame is a finite topless pre-Boolean algebra. Thus, by definition, this logic is complete with respect to the class of finite topless pre-Boolean algebras. This logic is the *smallest modal companion* of a well-known intermediate logic called *Medvedev's Logic* ML.[1]

Observation 4. $\mathsf{S4.tBA}$ *is properly contained within* $\mathsf{S4.2}$.

Proof. First, we argue that $\mathsf{S4.tBA}$ is contained in $\mathsf{S4.2}$ by arguing that every Kripke model M with a finite pre-Boolean algebra frame admits a model bisimilarity with a Kripke model M^+ having a finite topless pre-Boolean algebra frame. Consider first the case where the frame underlying M is actually a Boolean algebra. We construct M^+ as follows: take the Boolean algebra that is the frame of M, add a new atom and consider the resulting generated Boolean algebra; then remove the top to create a topless Boolean algebra. In the resulting frame, keep the original worlds for the nodes that came from M and place the top world of M at each of the newly created nodes of the frame of M^+. It is easy to check that M and M^+ have a model bisimulation associating each of the new worlds to the top world of M, and the other worlds to themselves. It follows that the truths of M and M^+ at their initial worlds are the same. Similarly, for the general case where the frame of M is a finite pre-Boolean algebra rather than a Boolean algebra, then there is a cluster of mutually accessible nodes at the top of M, and it is this entire cluster that we duplicate in M^+ at each of the new positions in the topless Boolean algebra. Again there is a model bisimulation of M and M^+ associating each of these newly created worlds with their corresponding duplicate in the top cluster of M and the other worlds to themselves. Thus, again the truths of M and M^+ at their initial worlds are the same. Finally, since any statement outside $\mathsf{S4.2}$ must fail in such a Kripke model M with a finite pre-Boolean algebra frame, it follows that the statement also fails in the corresponding Kripke model M^+ with a finite topless pre-Boolean algebra frame, and so it is also outside $\mathsf{S4.tBA}$, as desired. So $\mathsf{S4.tBA}$ is included in $\mathsf{S4.2}$.

Second, we argue that this inclusion is strict. As the topless Boolean algebras are not directed, it is easy to construct violations of .2 by making a statement true on one co-atom and false on another. So .2 is not in $\mathsf{S4.tBA}$, and consequently $\mathsf{S4.tBA} \subsetneq \mathsf{S4.2}$, as desired. □

Since $\mathsf{S4}$ is valid in any topless Boolean algebra, it follows that $\mathsf{S4} \subseteq \mathsf{S4.tBA}$. This conclusion is strict in light of the fact that all topless Boolean algebras satisfy the principle that whenever three mutually incompatible assertions are possibly necessary, then it is possible

[1] An intermediate logic is a propositional logic between intuitionistic and classical logic. A modal logic Λ is a *modal companion* of an intermediate logic Λ' if Λ consists of the Gödel translations of formulas in Λ'. If Λ is characterized by a class $\mathcal{C}$ in the above sense, and Λ' is characterized in the sense of Kripke semantics for intermediate logics by the same class $\mathcal{C}$, then Λ is the smallest modal companion of Λ'. Medvedev's Logic ML is characterized by the class of finite topless pre-Boolean algebras and known not to be finitely axiomatizable [7, 15].

to exclude one of them without yet deciding between the other two. This is expressible in the modal language as

$$\Diamond\Box\varphi_1 \wedge \Diamond\Box\varphi_2 \wedge \Diamond\Box\varphi_3 \wedge \neg\Diamond[(\varphi_1\wedge\varphi_2)\vee(\varphi_1\wedge\varphi_3)\vee(\varphi_2\wedge\varphi_3)]$$
$$\to \Diamond[\Diamond\Box\varphi_1 \wedge \Diamond\Box\varphi_2 \wedge \neg\Diamond\Box\varphi_3],$$

and there are similar assertions for four possibilities and five and so on. The assertion above is valid for S4.tBA but not for S4 (consider a frame that is a tree with one root and three leaves), and thus we may summarize the situation as

$$\mathsf{S4} \quad \subsetneq \quad \mathsf{S4.tBA} \quad \subsetneq \quad \mathsf{S4.2} \quad \subsetneq \quad \mathsf{S4.3} \quad \subsetneq \quad \mathsf{S5}.$$

We close this section with two simple modal logic observations that will be of use in §§ 5.7 and 5.8.

Observation 5. *The modal logic* S4 *proves* $.2 \to \Box .2$.

Proof. The following simple chain of transformations proves the claim, using the fact that $\Diamond\Diamond\varphi$ implies $\Diamond\varphi$ in S4:

$$\begin{aligned}\neg\Box .2 = \neg\Box(\Diamond\Box\varphi\to\Box\Diamond\varphi) &\Rightarrow \Diamond(\Diamond\Box\varphi\wedge\neg\Box\Diamond\varphi)\\ &\Rightarrow \Diamond\Diamond\Box\varphi\wedge\Diamond\neg\Box\Diamond\varphi\\ &\Rightarrow \Diamond\Diamond\Box\varphi\wedge\Diamond\Diamond\Box\neg\varphi\\ &\Rightarrow \Diamond\Box\varphi\wedge\Diamond\Box\neg\varphi\\ &\Rightarrow \Diamond\Box\varphi\wedge\neg\Box\Diamond\varphi\\ &\Rightarrow \neg(\Diamond\Box\varphi\to\Box\Diamond\varphi) = \neg .2\end{aligned}$$

□

Observation 6. *Let* Λ *and* A *be sets of* $\mathcal{L}_\Box$*-formulas, and let* Λ^* *be the closure of* $\Lambda\cup\mathrm{A}$ *under modus ponens and necessitation. Let* M *be a transitive Kripke model and* $v\in M$. *Assume that (1) for each* $w\in M$, *we have* $\Lambda\subseteq[\![w]\!]_M$, *and (2) for each* $\alpha\in\mathrm{A}$, *we have* $\Box\alpha\in[\![v]\!]_M$. *Then for every* w *accessible from* v, *we have* $\Lambda^*\subseteq[\![w]\!]_M$.

Proof. The set Λ^* can be written as $\bigcup_{n\in\omega}\Lambda_n$ where $\Lambda_0 := \Lambda\cup\mathrm{A}$, Λ_{2n+1} is the closure of Λ_{2n} under modus ponens, and $\Lambda_{2n+2} := \Lambda_{2n+1}\cup\{\,\Box\varphi \mid \varphi\in\Lambda_{2n+1}\,\}$.

We'll prove by induction that $\Lambda^*\subseteq[\![w]\!]_M$ for all w accessible from v. The base case follows from assumptions (1) and (2). Since each $[\![w]\!]_M$ is closed under modus ponens, the step $2n\mapsto 2n+1$ is trivial. For the step $2n+1\mapsto 2n+2$, fix w accessible from v and φ in Λ_{2n+1}, and show that $\Box\varphi\in[\![w]\!]_M$. Let w' be accessible from w; by transitivity, we have that w' is accessible from v, and thus, $\varphi\in[\![w']\!]$. Since w' was arbitrary, $\Box\varphi\in[\![w]\!]$. □

3. Forcing with a forcing class Γ

Suppose that Γ is a fixed definable class of forcing notions, with a fixed definition. We freely interpret this definition in any model of set theory, thereby reading Γ *de dicto* rather than *de re*. We say that Γ is a *forcing class* if in every such model, this definition defines a class of forcing notions, closed under the equivalence of forcing. A forcing class Γ is *reflexive* if in every model of set theory, Γ contains the trivial forcing poset. In this case, every model of set theory is a Γ forcing extension of itself. The class Γ is *transitive* if it closed under finite iterations, in the sense that if $\mathbb{Q}\in\Gamma$ and $\dot{\mathbb{R}}\in\Gamma^{V^{\mathbb{Q}}}$, then $\mathbb{Q}*\dot{\mathbb{R}}\in\Gamma$. Thus, any Γ forcing extension of a Γ forcing extension is a Γ forcing extension. The class Γ is *closed under product forcing* if, necessarily, whenever $\mathbb{Q}$ and $\mathbb{R}$ are in Γ, then so is $\mathbb{Q} \times \mathbb{R}$. Related to this, Γ is *persistent* if, necessarily, members of Γ are Γ necessarily in Γ; that is, if $\mathbb{P}, \mathbb{Q} \in \Gamma$ implies $\mathbb{P} \in \Gamma^{V^{\mathbb{Q}}}$ in all models. These properties are related in that every transitive persistent class is closed under products, since $\mathbb{Q} \times \mathbb{R}$ is forcing equivalent to $\mathbb{Q} * \check{\mathbb{R}}$. The class Γ is *directed* if whenever $\mathbb{P}, \mathbb{Q} \in \Gamma$, then there is $\mathbb{R} \in \Gamma$, such that both $\mathbb{P}$ and $\mathbb{Q}$ are factors of $\mathbb{R}$ by further Γ forcing, that is, if $\mathbb{R}$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{S}}$ for some $\dot{\mathbb{S}} \in \Gamma^{V^{\mathbb{P}}}$ and also equivalent to $\mathbb{Q} * \dot{\mathbb{T}}$ for some $\dot{\mathbb{T}} \in \Gamma^{V^{\mathbb{Q}}}$. This is stronger than asserting merely that any two members of Γ are absorbed by some other forcing in Γ, since we require that the quotient forcing is also in Γ. Note that if Γ is transitive and persistent, then product forcing shows that Γ is directed. The class Γ has the *linearity property* if for any two forcing notions $\mathbb{P}, \mathbb{Q}$, then one of them is forcing equivalent to the other one followed by additional Γ forcing; that is, either $\mathbb{P}$ is forcing equivalent to $\mathbb{Q} * \dot{\mathbb{R}}$ for some $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$ or $\mathbb{Q}$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{R}}$ for some $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$. Combining these notions, we define that Γ is a *linear forcing class* if Γ is reflexive, transitive and has the linearity property. Any linear forcing class is clearly directed. Similar related definitions can be found in [13].

Any forcing class Γ leads to the corresponding Γ forcing modalities. Namely, a set-theoretic sentence ψ is Γ-*forceable* or Γ-*possible*, written $\Diamond_\Gamma \psi$, if ψ holds in a forcing extension by some forcing notion in Γ, and ψ is Γ-*necessary*, written $\Box_\Gamma \psi$, if ψ holds in all forcing extensions by forcing notions in Γ. These modal operators are easily seen to obey certain modal validities, such as the following, for any class Γ.

$$\begin{array}{rl} \text{K} & \Box_\Gamma(\varphi \to \psi) \to (\Box_\Gamma \varphi \to \Box_\Gamma \psi), \quad \text{and} \\ \text{Dual} & \Diamond_\Gamma \varphi \leftrightarrow \neg \Box_\Gamma \neg\varphi. \end{array}$$

The validity of other statements will depend on the class Γ. A modal assertion $\varphi(p_0, \ldots, p_n)$ is a *valid principle of* Γ *forcing* if all substitution instances $\varphi(\psi_0, \ldots, \psi_n)$ hold under the Γ-forcing interpretation, where we substitute arbitrary set-theoretic assertions ψ_i for the propositional variables p_i and interpret $\Box$ as $\Box_\Gamma$. More formally, for any forcing class Γ, every assignment $p_i \mapsto \psi_i$ of the propositional variables p_i to set-theoretical assertions ψ_i extends recursively to a Γ *forcing translation*, a function $H : \mathcal{L}_\Box \to \mathcal{L}_\in$ satisfying $H(p_i) = \psi_i$ and the obvious recursive rules for conjunction $H(\varphi \wedge \eta) = H(\varphi) \wedge H(\eta)$, negation $H(\neg\varphi) = \neg H(\varphi)$ and modality $H(\Box \varphi) = \Box_\Gamma H(\varphi)$, where in this last case we mean the assertion in the language of set theory asserting that $H(\varphi)$ has Boolean value one for any forcing notion satisfying the definition of Γ. In this terminology, the *modal logic of* Γ *forcing* over a model of set theory W is the set

$$\{\, \varphi \in \mathcal{L}_\Box \mid W \models H(\varphi) \text{ for all } \Gamma \text{ forcing translations } H \,\}.$$

Note that the interpretation of iterated modal expressions, such as $\Diamond_\Gamma \Box_\Gamma \psi$, leads to the situation where Γ is reinterpreted in the appropriate forcing extensions by forcing in Γ. For example, the c.c.c. forcing notions of a c.c.c. forcing extension do not necessarily include all the c.c.c. forcing notions of the original ground model. To improve readability, we will sometimes omit the subscripts from $\Diamond_\Gamma$ and $\Box_\Gamma$ when the class Γ is clear.

Theorem 7 ([13])**.**

(1) S4 *is valid for any reflexive transitive forcing class.*
(2) S4.2 *is valid for any reflexive transitive directed forcing class.*

(3) S4.3 *is valid for any linear forcing class.*

Proof. The reader is encouraged to work through the details as an exercise in understanding the forcing modalities. Axiom K is valid for any forcing class. Axiom T is valid for any reflexive forcing class. Axiom 4 is valid for any transitive forcing class. Axiom .2 is valid for any directed forcing class. And axiom .3 is valid for any forcing class with the linearity property. We note that one subtlety of the argument is that it is not sufficient merely to check the validity of the axioms K, T, 4 and .3 alone, since the theory S4.3, for example, is defined to be the closure of these axioms under modus ponens *and necessitation*. Thus, one needs to know that the axioms remain valid in any Γ extension. In our case here, the definition of what it means to be a linear class requires that the desired property holds in all models, which is more than sufficient. □

The construction used in the next lemma lies at the heart of our method, serving as the principal technique connecting modal truth in Kripke models with set-theoretic truth in models of set theory. It is a general version of [9, lemma 7.1], employing the notion of a Γ-labeling, which we introduce here.

Definition 8. A Γ-*labeling* of a frame F for a model of set theory W is an assignment to each node w in F an assertion Φ_w in the language of set theory, such that

(1) The statements Φ_w form a mutually exclusive partition of truth in the Γ forcing extensions of W, meaning that every such extension $W[G]$ satisfies exactly one Φ_w.
(2) Any Γ forcing extension $W[G]$ in which Φ_w is true satisfies $\Diamond\,\Phi_u$ if and only if $w \leq_F u$.
(3) $W \models \Phi_{w_0}$, where w_0 is a given initial element of F.

The formal assertion of these properties is called the Jankov-Fine formula for F (cf. [9, § 7]).

Lemma 9. *Suppose that $w \mapsto \Phi_w$ is a Γ-labeling of a finite frame F for a model of set theory W and that w_0 is an initial world of F. Then for any Kripke model M having frame F, there is an assignment of the propositional variables to set-theoretic assertions $p \mapsto \psi_p$ such that for any modal assertion $\varphi(p_0, \ldots, p_k)$,*

$$(M, w_0) \models \varphi(p_0, \ldots, p_k) \quad \textit{iff} \quad W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

In particular, any modal assertion φ that fails at w_0 in M also fails in W under the Γ forcing interpretation. Consequently, the modal logic of Γ forcing over W is contained in the modal logic of assertions valid in F.

Proof. Suppose that $w \mapsto \Phi_w$ is a Γ-labeling of F for W, and suppose that M is a Kripke model with frame F. Thus, we may view each $w \in F$ as a propositional world in M. For each propositional variable p, let $\psi_p = \bigvee\{\, \Phi_w \mid (M, w) \models p \,\}$, the join of the set-theoretic statements Φ_w associated with each world w where p is true in M. We shall prove the lemma by proving the more uniform claim that whenever $W[G]$ is a Γ forcing extension of W and $W[G] \models \Phi_w$, then

$$(M, w) \models \varphi(p_0, \ldots, p_k) \quad \text{iff} \quad W[G] \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

We prove this for all such $W[G]$ simultaneously by induction on the complexity of φ. The atomic case follows immediately from the definition of ψ_p. Boolean combinations go through easily. Consider now the modal operator case. If $W[G] \models \Diamond\,\varphi(\psi_{p_0}, \ldots, \psi_{p_k})$, then there is a further Γ extension $W[G][H]$ satisfying $\varphi(\psi_{p_0},\ldots,\psi_{p_k})$. This extension $W[G][H]$ must satisfy some Φ_u, and consequently by induction we know $(M,u)\models\varphi(p_0,\ldots,p_k)$. Since Φ_u was Γ forceable over $W[G]$, where Φ_w was true, it follows by the labeling properties that $w\leq_F u$. Thus, $(M,w)\models\Diamond\varphi(p_0,\ldots,p_k)$, as desired. Conversely, if $(M,w)\models\Diamond\varphi(p_0,\ldots,p_k)$, then there is a u with $w\leq_F u$ and $(M,u)\models\varphi(p_0,\ldots,p_k)$. Thus, by induction, any Γ forcing extension with Φ_u will satisfy $\varphi(\psi_{p_0},\ldots,\psi_{p_k})$. Since Φ_u is forceable over any Γ extension $W[G]$ with Φ_w, it follows that any such $W[G]$ will satisfy $\Diamond\varphi(\psi_{p_0},\ldots,\psi_{p_k})$, as desired.

The further final claims in the lemma now follow immediately. □

4. Control statements: buttons, switches and ratchets

In the previous section, we reduced much of the problem of determining the modal logic of Γ forcing to the question of whether certain kinds of frames admit Γ-labelings. In this section, we shall prove that the existence of such labelings for large classes of finite frames often breaks down into simpler, more modular control statements, such as what we call buttons, switches and ratchets. Since labelings of complex frames can be constructed from these more fundamental control statements, the question of whether a given class of frames admit labelings often reduces to the question of whether the forcing class allows for independent families of these control statements.

Buttons and switches were introduced in [9]; here, we shall augment them with ratchets, weak buttons and other types of control statements. Suppose that Γ is a reflexive transitive forcing class. A *switch* for Γ is a statement s such that both s and $\neg s$ are Γ necessarily possible. A *button* for Γ is a statement b that is Γ necessarily possibly necessary. In the case that S4.2 is valid for Γ, this is equivalent to saying that b is possibly necessary. The button b is *pushed* when $\Box b$ holds, and otherwise it is *unpushed*. A finite collection of buttons and switches (or other controls of this type) is *independent* if necessarily, each can be operated without affecting the truth of the others. For more details, cf. [9, p. 1798]. A button b is *pure* if whenever it becomes true, it becomes necessarily true, that is, if $\Box(b\to\Box b)$. Every (unpushed) button b has a corresponding (unpushed) pure button $\Box b$, and pure buttons are sometimes more convenient.

A sequence of first-order statements $r_1, r_2, \ldots r_n$ is a *ratchet* for Γ of length n if each is an unpushed pure button for Γ, each necessarily implies the previous, and each can be pushed without pushing the next (this notion was called a *volume control* in [9]). This is expressed formally as follows:

$$\begin{array}{l}\neg r_i\\ \Box(r_i\to\Box r_i)\\ \Box(r_{i+1}\to r_i)\\ \Box[\neg r_{i+1}\to\Diamond(r_i\wedge\neg r_{i+1})]\end{array}$$

The key idea of a ratchet is that it is unidirectional, any further Γ forcing can only increase the ratchet value or leave it the same. It is sometimes convenient to introduce the ratchet statement r_0 as any tautological statement $\top$ (an already-pushed button). A model has ratchet value (or volume) at least i when r_i holds and exactly i when $r_i\wedge\neg r_{i+1}$ holds, for $i<n$; ratchet value exactly n means r_n is true. Further Γ forcing pushes the value only higher, and any higher value is precisely attainable in an appropriate extension. A ratchet of length n partitions the Γ forcing extensions into $n+1$ equivalence classes—those having the same ratchet value—and from any model in a lower ratchet class, one can perform further Γ forcing

to arrive at a model in any desired higher class. If $r_1, r_2, \ldots r_n$ is a ratchet and $s_0, s_1, \ldots s_k$ are switches, then this combined family is independent if in any extension, any finite pattern of the switches is obtainable without increasing the ratchet value. A transfinite sequence of set-theoretic statements $\langle r_\alpha \mid 0 < \alpha < \delta\rangle$, perhaps involving parameters, is a ratchet for Γ of length δ if each is an unpushed pure button for Γ, each necessarily implies the previous, and each can be pushed without pushing the next. The ratchet is *uniform* if there is a formula $r(x)$ with one free variable, such that $r_\alpha = r(\alpha)$. Every finite length ratchet is uniform. The ratchet is *continuous*, if for every limit ordinal $\lambda < \delta$, the statement r_λ is equivalent to $\forall\alpha{<}\lambda\, r_\alpha$. Any uniform ratchet can be made continuous by reindexing, replacing each r_β by the assertion "$\forall\alpha{<}\beta\, r_{\alpha+1}$." A *long ratchet* is a uniform ratchet $\langle r_\alpha \mid 0 < \alpha < \mathrm{ORD}\rangle$ of length ORD, with the additional property that no Γ forcing extension satisfies all r_α, so that every Γ extension exhibits some ordinal ratchet value. We will explain in theorem 12 that from a long ratchet, one may construct a mutually independent family of switches and a ratchet of any desired length.

A *weak button* is a statement b that is possibly necessary. We have mentioned that under S4.2, every weak button is a button, but without S4.2, this conclusion does not follow, and it can be that a statement b and its negation $\neg b$ are both weak buttons for a given class of forcing. For example, if Γ is c.c.c. forcing, then the assertion "the L-least Souslin tree has a branch" and its negation are both weak buttons, since one can either add a branch, which pushes the button, or specialize the tree, which prevents branches and therefore pushes the negation. A sequence of weak buttons $b_0, \ldots, b_{n-1}$ is *strongly independent* if no extension has all of them pushed, but in any extension, any additional one of them can be pushed without pushing any of the others, as long as this wouldn't push them all. Such a family is similar to an independent family of buttons, except that one cannot push them all.

The next three theorems were implicit in [9], but we revisit them here explicitly in the context of an arbitrary forcing class Γ. Let us begin with the easiest case.

Theorem 10. *If Γ is a reflexive transitive forcing class having arbitrarily large finite independent families of switches over a model of set theory W, then the valid principles of Γ forcing over W are contained within the modal theory* S5.

Proof. Suppose that Γ is a reflexive transitive forcing class and that W is a model of set theory having arbitrarily large finite independent families of switches. We may assume that the switches are all off in W. By theorem 3, any modal assertion not in S5 fails in a Kripke model M built on a frame F consisting of a single cluster of worlds $w_0, w_1, \ldots, w_{n-1}$, each accessible from all of them. By adding dummy copies of worlds in this Kripke model (which does not affect modal truth), we may assume that $n = 2^m$ for some natural number m. Let $s_0, s_1, \ldots s_{m-1}$ be an independent family of m switches over W. We will provide a Γ-labeling of this frame. For any $j < 2^m$, let Φ_{w_j} be the assertion that the overall pattern of truth values for the switches conforms with the m binary digits of j. Thus, we have associated each world w_j of the frame of M with a set-theoretic assertion Φ_{w_j}. These assertions are mutually exclusive, since different j will lead to different incompatible patterns for the switches in Φ_j, and exhaustive, since every model must exhibit some pattern of switches. Since the switches are independent over W, any pattern of switches is Γ necessarily Γ forceable, and so every Γ extension $W[G]$ satisfies $\Diamond\, \Phi_{w_j}$ for all j. Since the switches are all off in W, we have $W \models \Phi_{w_0}$. Thus, we have verified the three labeling requirements, and so by lemma 9, there is an assignment of the propositional variables p to set-theoretic assertions ψ_p such that

$(M, w_0) \models \varphi(p_0, \ldots, p_k)$ if and only if $W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k})$. In particular, any statement φ that fails in M at w_0 has a set-theoretic substitution instance $\varphi(\psi_{p_0}, \ldots, \psi_{p_k})$ failing in W. Since any statement outside S5 fails in such an (M, w_0), it follows that the modal logic of Γ forcing over W is contained within S5, as desired. □

Theorem 11. *If Γ is a reflexive transitive forcing class having arbitrarily long finite ratchets over a model of set theory W, mutually independent with arbitrarily large finite families of switches, then the valid principles of Γ forcing over W are contained within the modal theory* S4.3.

Proof. Suppose that Γ is a reflexive transitive forcing class with arbitrarily long finite ratchets, mutually independent of switches over a model of set theory W. By theorem 1, any modal assertion not in S4.3 must fail in a Kripke model M built on a finite pre-linear order frame. Thus, by lemma 9, it suffices to provide a Γ labeling of the frame of M. This frame consists of a finite increasing sequence of n clusters of mutually accessible worlds. That is, the k^{th} cluster consists of n_k many worlds $w_0^k, w_1^k, \ldots, w_{n_k-1}^k$, and the frame order is simply $w_i^k \leq w_j^s$ if and only if $k \leq s$. By adding dummy copies of worlds in each cluster, which does not affect truth in the Kripke model, we may assume that all clusters have the same size and furthermore, that $n_k = 2^m$ for some fixed natural number m.

Let $r_1, \ldots, r_n$ be a ratchet of length n for Γ over W, mutually independent from the m many switches $s_0, \ldots, s_{m-1}$. We may assume that all switches are off in W. Let $\bar{r}_k$ be the assertion that the ratchet value is exactly k, so that $\bar{r}_0 = \neg r_1$, $\bar{r}_k = r_k \wedge \neg r_{k+1}$ for $1 \leq k < n$ and $\bar{r}_n = r_n$, and let $\bar{s}_j$ assert for $j < 2^m$ that the pattern of switches accords with the m binary digits of j. We associate each world w_j^k, where $k < n$ and $j < 2^m$, with the assertion $\Phi_{w_j^k} = \bar{r}_k \wedge \bar{s}_j$, which asserts that the ratchet value is exactly k and the switches exhibit pattern j. Since the ratchet values cannot go down, any pattern of switches is possible without increasing the ratchet value, and any value is possible, it follows that the Γ possibility of Φ_w corresponds exactly with the order in the frame. That is, whenever $W[G]$ satisfies $\Phi_{w_j^k}$, then it satisfies $\Diamond_\Gamma \Phi_{w_i^s}$ if and only if $w_j^k \leq w_i^s$, which is to say, if and only if $k \leq s$. Also, since the ratchet value is 0 in W itself and the switches are off, we have that $W \models \Phi_{w_0^0}$, and so we have provided a Γ-labeling of the frame of M. It follows by lemma 9 that there is an assignment of the propositional variables p to set-theoretic assertions ψ_p such that for any modal assertion φ we have $(M, w_0) \models \varphi(p_0, \ldots, p_t)$ if and only if $W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_t})$. In particular, by theorem 1, any statement outside S4.3 will fail in such an M, and consequently will have a failing substitution instance in W under the Γ forcing interpretation. Thus, the valid principles of Γ forcing over W are contained within S4.3, as desired. □

It turns out that in most of the set-theoretic situations where we are able to build ratchets, we are also able to build a long ratchet, and in this case the next theorem allows us to simplify things by avoiding the need to consider switches.

Theorem 12. *If Γ is a reflexive transitive forcing class having a long ratchet over a model of set theory W, then the valid principles of Γ forcing over W are contained within the modal theory* S4.3.

Proof. Suppose that $\langle r_\alpha \mid 0 < \alpha < \mathrm{ORD} \rangle$ is a long ratchet over W, that is, a uniform ratchet control of length ORD, such that no Γ extension satisfies every r_α. We may assume the ratchet is continuous. It suffices by theorem 11 to produce arbitrarily long finite ratchets independent from arbitrarily large finite families of switches. To do this, we shall divide the ordinals into blocks of length ω, and think of the position within one such a block as determining a switch pattern and the choice of block itself as another ratchet. Specifically, every ordinal can be uniquely expressed in the form $\omega \cdot \alpha + k$, where $k < \omega$, and we think of this ordinal as being the kth element in the αth block. Let s_i be the statement that if the current ratchet value is exactly $\omega \cdot \alpha + k$, then the ith binary bit of k is 1. Let v_α be the assertion $r_{\omega\cdot\alpha}$, which expresses that the current ratchet value is in the αth block of ordinals of length ω or higher. Since we may freely increase the ratchet value to any higher value, we may increase the value of k while staying in the same block of ordinals, and so the v_α form themselves a ratchet, mutually independent of the switches s_i. Thus, by theorem 11, the valid principles of Γ forcing over W are contained within S4.3. □

Theorem 13. *If Γ is a reflexive transitive forcing class and there are arbitrarily large finite families of mutually independent buttons and switches over a model of set theory W, then the valid principles of Γ forcing over W are contained within* S4.2.

Proof. This was the main application of this technique in [9], applied to the class of all forcing. The point is that when you have mutually independent buttons and switches, you can label any finite pre-Boolean algebra.

Suppose that Γ is a reflexive transitive forcing class having arbitrarily large finite families of mutually independent buttons and switches over a model of set theory W. Suppose that M is a Kripke model whose frame F is a finite pre-Boolean algebra. Thus, the quotient of F by the equivalence relation $w \equiv v \leftrightarrow w \leq v \leq w$ is a finite Boolean algebra B, which must be a power set $P(A)$ of a finite set A. Each element $a \in B$ is associated with a cluster of worlds $w_0^a, \ldots, w_{k_a}^a$. By adding dummy worlds to each cluster, we may assume that all the clusters have size $k_a = 2^m$ for some fixed m. Suppose that A has size n, so that there are n atoms in the Boolean algebra. Thus, the frame F can be thought of as worlds w_j^a, where $a \subseteq A$ and $j < 2^m$, with the order $w_j^a \leq w_i^c$ if and only if $a \subseteq c$.

Associate each element $i \in A$ with a pure button b_i, such that these form a mutually independent family with m many switches $s_0, \ldots, s_{m-1}$. For $j < 2^m$, let $\bar{s}_j$ be the assertion that the pattern of switches corresponds to the binary digits of j. We label the world w_j^a with the assertion $\Phi_{w_j^a} = (\bigwedge_{i\in a} b_i) \wedge \bar{s}_j$. If $W[G]$ satisfies $\Phi_{w_j^a}$, then by the mutual independence of the buttons and switches, we conclude that $W[G]$ satisfies $\Diamond\, \Phi_{w_r^c}$ if and only if $a \subseteq c$, since any prescribed larger collection of buttons can be pushed and the switches can be set to any pattern without pushing any additional buttons. Also, since none of the buttons is pushed and all the switches are off in W, we have $W \models \Phi_{w_0^\varnothing}$. Thus, we have provided a Γ labeling of the frame for W.

By lemma 9, therefore, there is an assignment of the propositional variables $p \mapsto \psi_p$ such that $(M, w_0^\varnothing) \models \varphi(p_0, \ldots, p_k)$ if and only if $W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k})$. In particular, any φ failing at $(M, w_0^\varnothing)$ will have a substitutions instance failing in W. By theorem 2, any modal assertion outside S4.2 fails in such a Kripke model $(M, w_0^\varnothing)$, and so the valid principles of Γ forcing over W will be contained in S4.2, as desired. □

Corollary 14. *If Γ is a reflexive transitive directed forcing class and there are arbitrarily large finite families of mutually independent buttons and switches over a model of set theory W, then the valid principles of Γ forcing over W are exactly* S4.2.

Proof. The lower bound is provided by theorem 7, and the upper bound by theorem 13. □

Let us say that a model W of set theory admits a *uniform family of* ORD *many independent buttons* for Γ forcing, if there is a formula φ in the language of set theory such that the assertions $b_\alpha = \varphi(\alpha)$, for each ordinal α in W, form an independent family of buttons for Γ forcing, and furthermore any forcing extension of W pushes at most boundedly many of the buttons. With such a large collection of buttons, as in theorem 12, we may avoid the need in theorem 13 to also have independent switches.

Theorem 15. *If Γ is a reflexive transitive forcing class and W is a model of set theory having a uniform family of* ORD *many independent buttons, then the valid principles of Γ forcing over W are contained within* S4.2*.*

Proof. The idea is simply to keep the first ω many buttons as buttons and to use the rest of the buttons to form a long ratchet by looking at the supremum of the pushed buttons beyond ω. This ratchet gives rise to independent switches as in theorem 12, and so one has a family of independent buttons and switches for Γ forcing over W. It follows by theorem 13 that the valid principles of Γ forcing over W is contained in S4.2. □

Theorem 16. *If Γ is a reflexive transitive forcing class and there are arbitrarily large finite families of strongly independent weak buttons and switches over a model of set theory W, then the valid principles of Γ forcing over W are contained within* S4.tBA*.*

Proof. This argument proceeds almost identically to the proof of theorem 13, except that here the pre-Boolean algebras will be topless. The impossibility of pushing all the weak buttons corresponds exactly to the absence of the top element in the Boolean algebra, so the corresponding labeling works here for the topless pre-Boolean algebra. □

Let us conclude this section with an aside, briefly correcting a flaw in our paper [9] concerning the existence of independent buttons for forcing over L. These families exist, as explained in the following, but Jakob Rittberg noticed that the particular buttons we had proposed in the proof of [9, lemma 6.1] were problematic. Specifically, we had claimed there that the buttons b_n stating that "ω_n^{L} is not a cardinal" form a family of independent buttons for forcing over L. Although these are indeed buttons and one can push them in any finite pattern by forcing over L, for true independence one must be able to continue to control the buttons independently also in all the forcing extensions of L. And while any two of the buttons can be controlled independently in this way, what Rittberg had noticed was that if $\mathrm{L}[G]$ is a generic extension of L in which 2^ω is at least $\aleph_3$ and cardinals have not been collapsed, then the usual collapse forcing of $\aleph_2^{\mathrm{L}}$ to $\aleph_1^{\mathrm{L}}$ will also collapse $\aleph_3^{\mathrm{L}}$. So it isn't clear how to push b_2 over such a model while pushing neither b_1 nor b_3. It appears to be a subtle question in forcing whether it is always possible to do so. So we do not actually know whether the original buttons of [9, lemma 6.1] are independent over L or not.

Once this problem came to light, a variety of other families of independent buttons were provided. Indeed, we had already in the original article provided an alternative correct family of independent buttons and switches just after [9, theorem 29], and we shall presently explain these again more fully below. Rittberg himself provided an independent family of buttons in his Master's thesis [16, § 2.4.2] with a full proof of their independence over L:

$$b_n^{\mathrm{R}} := \text{at least one of } \aleph_{3n}^{\mathrm{L}}, \aleph_{3n+1}^{\mathrm{L}}, \text{ and } \aleph_{3n+2}^{\mathrm{L}} \text{ is not a cardinal, or } |\wp(\aleph_{3n}^{\mathrm{L}})| > |\aleph_{3n+1}^{\mathrm{L}}|.$$

Friedman, Fuchino and Sakai [4, § 5] have another family of independent buttons, namely, if T^{L}_n denotes the L-least $\aleph_n$-Suslin tree, then any finite subfamily of the assertions

$$b^{\mathrm{FSS},1}_n := \aleph^{\mathrm{L}}_n \text{ is not a cardinal or } T^{\mathrm{L}}_n \text{ is not an } \aleph_n\text{-Suslin tree}$$

is an independent family of buttons over L. Furthermore, they show that the simpler statements

$$b^{\mathrm{FSS},2}_n := \text{there is an injection from } \aleph^{\mathrm{L}}_{n+2} \text{ to } \wp(\aleph^{\mathrm{L}}_n)$$

form an infinite family of independent buttons over L. For the sake of completeness, let us give a full proof here of the independence of the alternative buttons from our original paper:

$$b^*_n := S_n \text{ is no longer stationary,}$$

where $\omega^{\mathrm{L}}_1 = \bigsqcup_{n\in\omega} S_n$ is the L-least partition of ω^{L}_1 into ω many disjoint stationary sets (cf. [9, theorem 29]). The independence of these buttons is based on our ability to control whether a set or its complement remains stationary in a forcing extension.

Lemma 17 (Baumgartner, Harrington, Kleinberg [12, thm 23.8, ex 23.6])**.** *If $S \subseteq \omega_1$ is stationary, then the club-shooting forcing for S, consisting of the closed bounded subsets of S ordered by end-extension, adds a club subset of S, is countably distributive, and preserves all stationary subsets of S.*

Proof. Let $\mathbb{Q}_S$ be the club-shooting forcing for S. Since S is unbounded, it is dense that the conditions become unbounded in S, and so the union of the generic filter in $\mathbb{Q}_S$ is a closed unbounded subset of S. Suppose that $T \subseteq S$ is stationary in the ground model, and suppose $c_0 \Vdash \tau$ is a club subset of $\check{\omega}_1$. Since T is stationary, we may find a countable elementary substructure $X \prec H_\theta$ for some large θ, with $c_0, \tau, S, T \in X$ and $\delta = X \cap \omega_1 \in T$. Since X is countable, we may build a descending sequence $c_0 > c_1 > \cdots$ of conditions in $\mathbb{Q}_S \cap X$ that get inside every open dense set in X. The limit condition $c = \bigcup_n c_n \cup \{\delta\}$ is a condition in $\mathbb{Q}_S$ precisely because $\delta \in T \subseteq S$. Since c is X-generic, it follows that c decides the values of any name in X for a countable sequence of ordinals, and consequently the forcing is countably distributive. Similarly, c forces that τ is unbounded in δ, and consequently c forces that $\delta \in \tau$ and hence forces that τ meets $\check{T}$. Thus, every stationary subset of S is preserved to the extension. □

The independence of the buttons b^*_n for forcing over L now follows as an easy consequence. Namely, each b^*_n is a pure button, since non-stationarity is upward absolute, and we may in any case push all the buttons by collapsing ω_1. More delicately, we may push just b^*_n by forcing as in the lemma to shoot a club through the complement of S_n, that is, with the club-shooting forcing for $\bigsqcup_{m\neq n} S_m$. Lemma 17 exactly ensures that this forcing preserves the stationarity of any remaining stationary S_m for $m \neq n$, and thus ensures that we may push b^*_n while not inadvertently pushing any other unpushed b^*_m. Meanwhile, the statements s_k asserting "$2^{\aleph_k} = \aleph_{k+1}$" are switches that can be controlled independently for $k > 1$ by countably closed forcing, which does not affect the stationarity of any subset of ω_1 and therefore does not interfere with the buttons b^*_n. So we have the desired independent family of buttons and switches for forcing over L.

Higher analogues of these buttons exist on higher cardinals, where one has a stationary subset $S \subseteq \mathrm{Cof}_\kappa \cap \kappa^+$ for κ regular and seeks to add by forcing a club set C such that $C \cap \mathrm{Cof}_\kappa \subseteq S$. The natural forcing to accomplish this is $<\kappa$-closed and $\leq \kappa$-distributive, and the analogue of lemma 17 goes through.

5. Applications to various specific classes of forcing

In this section, we apply the results of §§ 3 and 4 and determine the valid principles of forcing for various specific natural classes of forcing.

5.1. **Our classes of forcing notions.** For any ordinal θ, the *collapse* poset $\mathrm{Coll}(\omega,\theta)$ consists of the finite partial functions from ω to θ, ordered by inclusion. For any nonzero ordinal θ, forcing with this poset adds a function from ω onto θ, making it countable in the forcing extension. Similarly, for any ordinal θ, let $\mathrm{Add}(\omega,\theta)$ be the set of finite partial functions from $\theta\times\omega$ into 2 and adds θ many Cohen reals. Let Coll be the class of all forcing notions $\mathbb{Q}$ that are forcing equivalent to $\mathrm{Coll}(\omega,\theta)$ for some ordinal θ and Add be the class of all forcing notions $\mathbb{Q}$ that are forcing equivalent to $\mathrm{Add}(\omega,\theta)$ for some ordinal θ. Note that both Coll and Add include trivial forcing, since $\mathrm{Add}(\omega,0)=\mathrm{Coll}(\omega,0)=\{\varnothing\}$ is trivial (and also $\mathrm{Coll}(\omega,1)$ is forcing equivalent to trivial forcing). Also note that $\mathrm{Add}(\omega,1)$ is isomorphic to $\mathrm{Coll}(\omega,2)$.

Let us say that a forcing notion $\mathbb{Q}$ *necessarily collapses* θ *to* ω if every forcing extension by $\mathbb{Q}$ has a function from ω onto θ that is not in the ground model. A forcing notion $\mathbb{Q}$ *absorbs* a forcing notion $\mathbb{R}$, if $\mathbb{Q}$ is forcing equivalent to $\mathbb{R}*\dot{\mathbb{S}}$ for some (quotient) forcing $\dot{\mathbb{S}}$. This is equivalent to saying that $\mathbb{R}$ is forcing equivalent to a complete subalgebra of the Boolean algebra of $\mathbb{Q}$.

Lemma 18 (Folklore)**.** *Suppose that θ is any infinite ordinal.*

1. *Up to forcing equivalence,* $\mathrm{Coll}(\omega,\theta)$ *is the unique forcing notion of size $|\theta|$ necessarily collapsing θ to ω.*
2. $\mathrm{Coll}(\omega,\theta)$ *absorbs every forcing notion of size $|\theta|$.*
3. $\mathrm{Coll}(\omega,\theta)*\mathrm{Coll}(\omega,\lambda)$ *is forcing equivalent to* $\mathrm{Coll}(\omega,\max\{\theta,\lambda\})$.

Proof. (1). Suppose that $\mathbb{Q}$ is a forcing notion of size $|\theta|$ that necessarily collapses θ to ω. We may assume without loss of generality that $\mathbb{Q}$ is separative, since the separative quotient of $\mathbb{Q}$ is forcing equivalent to it and no larger in size. Below every condition in $\mathbb{Q}$, we claim that there is an antichain of size θ. If θ is countable, this is immediate since every nontrivial forcing notion has infinite antichains; if θ is uncountable, then any failure of this claim would mean that $\mathbb{Q}$ is θ-c.c. below some condition and consequently unable to collapse θ to ω below that condition, contrary to our assumption. Since forcing with $\mathbb{Q}$ adds a function from ω onto θ and $\mathbb{Q}$ has size θ, there is a name $\dot{g}$ forced to be a function from ω onto the generic filter $\dot{G}$. We build a refining sequence of maximal antichains $A_n\subseteq\mathbb{Q}$ as follows. Begin with $A_0=\{\mathbb{1}\}$. If A_n is defined, then let A_{n+1} be a maximal antichain of conditions such that every condition in A_n splits into θ many elements of A_{n+1}, and such that every element of A_{n+1} decides the value $\dot{g}(\check{n})$. The union $\mathbb{R}=\bigcup_n A_n$ is clearly isomorphic as a subposet of $\mathbb{Q}$ to the tree $\theta^{<\omega}$, and so it is forcing equivalent to $\mathrm{Coll}(\omega,\theta)$. Furthermore, we claim that $\mathbb{R}$ is dense in $\mathbb{Q}$. To see this, fix any condition $q\in\mathbb{Q}$. Since q forces that q is in $\dot{G}$, there is some $p\leq q$ and natural number n such that $p\Vdash_{\mathbb{Q}}\dot{g}(\check{n})=\check{q}$. Since A_{n+1} is a maximal antichain, there is some condition $r\in A_{n+1}$ that is compatible with p. Since r also decides the value of $\dot{g}(\check{n})$ and is compatible with p, it must be that $r\Vdash\dot{g}(\check{n})=\check{q}$ also. In particular, r forces $\check{q}\in\dot{G}$, and so by separativity it must be that $r\leq q$. So $\mathbb{R}$ is dense in $\mathbb{Q}$, as desired. Thus, $\mathbb{Q}$ is forcing equivalent to $\mathbb{R}$, which we have said is forcing equivalent to $\mathrm{Coll}(\omega,\theta)$, as desired.

(2). Suppose that $\mathbb{Q}$ has size θ. Since $\mathbb{Q}\times\mathrm{Coll}(\omega,\theta)$ also has size θ and necessarily collapses θ to ω, it follows that it is forcing equivalent to $\mathrm{Coll}(\omega,\theta)$. Thus, $\mathbb{Q}$ is absorbed by $\mathrm{Coll}(\omega,\theta)$. We point out also that the quotient forcing is $\mathrm{Coll}(\omega,\theta)$, a fact of which we shall later make use.

(3). For any other ordinal λ, the poset $\mathrm{Coll}(\omega,\theta) * \mathrm{Coll}(\omega,\lambda)$ has size $\max\{\theta,\lambda\}$ and necessarily collapses this ordinal to ω. □

Lemma 19. *The collapse forcing class* Coll *is a linear forcing class, which is also persistent and closed under products.*

Proof. We have already pointed out that Coll includes trivial forcing, and so Coll is reflexive. It is transitive by lemma 18 (3). The same fact shows that it is linear, since the larger of $\mathrm{Coll}(\omega,\lambda)$ and $\mathrm{Coll}(\omega,\theta)$ factors through the smaller, and the quotient is collapse forcing. Finally, we observe that because the conditions are finite, the poset $\mathrm{Coll}(\omega,\theta)$ is absolute to all models of set theory having the ordinal θ, and so Coll is persistent. It is closed under products, since $\mathrm{Coll}(\omega,\theta)\times\mathrm{Coll}(\omega,\lambda)$ has size $\max\{\theta,\lambda\}$ and necessarily collapses $\max\{\theta,\lambda\}$ to ω, so by lemma 18 it is forcing equivalent to $\mathrm{Coll}(\omega,\max\{\theta,\lambda\})$. □

Lemma 20. *The class of Cohen forcing* Add *is a linear forcing class, which is also persistent and closed under products.*

Proof. This class is reflexive since $\mathrm{Add}(\omega,0)$ is trivial and transitive since $\mathrm{Add}(\omega,\theta)*\mathrm{Add}(\omega,\lambda)$ is forcing equivalent to $\mathrm{Add}(\omega,\theta+\lambda)$. It has the linearity property because if $\theta<\lambda$, then $\mathrm{Add}(\omega,\theta)*\mathrm{Add}(\omega,\lambda)$ is forcing equivalent to $\mathrm{Add}(\omega,\lambda)$. It is persistent since the definition of $\mathrm{Add}(\omega,\theta)$ is absolute to all models having θ as an ordinal. It is closed under products since $\mathrm{Add}(\omega,\theta)\times\mathrm{Add}(\omega,\lambda)\cong\mathrm{Add}(\omega,\theta+\lambda)$. □

The Lévy collapse poset $\mathrm{Coll}(\omega,{<}\lambda)$ is defined to be the finite support product $\prod_{\alpha<\lambda}\mathrm{Coll}(\omega,\alpha)$ and collapses all ordinals below λ to ω. This has particularly nice features when λ is an inaccessible cardinal, but we may consider it for any ordinal λ. The class of *Lévy collapse forcing*, denoted $\mathrm{Coll}^{<}$, is the class of all forcing notions that are forcing equivalent to $\mathrm{Coll}(\omega,{<}\lambda)$ for some ordinal λ.

Lemma 18 (3) implies that $\mathrm{Coll}(\omega,\theta)$ is forcing equivalent to $\mathrm{Coll}(\omega,{<}(\theta+1))$ for any ordinal θ, and so $\mathrm{Coll}\subseteq\mathrm{Coll}^{<}$. Note that $\mathrm{Coll}^{<}$ includes the forcing to add ω_1 many Cohen reals $\mathrm{Add}(\omega,\omega_1)\cong\mathrm{Coll}(\omega,{<}\omega_1)$, as well as the Lévy collapse $\mathrm{Coll}(\omega,{<}\kappa)$ of any inaccessible cardinal κ, if there are such cardinals.

Lemma 21. *Let θ be an ordinal.*

1. *If θ is not a cardinal or is singular, then* $\mathrm{Coll}(\omega,{<}\theta)$ *is forcing equivalent to* $\mathrm{Coll}(\omega,\theta)$.
2. $\mathrm{Coll}(\omega,{<}\theta)*\mathrm{Coll}(\omega,{<}\lambda)$ *is forcing equivalent to* $\mathrm{Coll}(\omega,<\max\{\theta,\lambda\})$.

Proof. (1). If θ is not a cardinal or is singular, then $\mathrm{Coll}(\omega,{<}\theta)$ necessarily collapses θ, and since this poset has size $|\theta|$, the conclusion follows from lemma 18.

(2). This follows from lemma 18 by combining the posets at the common stages. □

In general, if κ is a cardinal, we can consider collapse-to-κ forcing class Coll_κ, which consists of all forcing of the form $\mathrm{Coll}(\kappa,\theta)$, where θ is any ordinal. Since we want to interpret the class in any model of set theory, we shall consider Coll_κ only when κ is a definable regular cardinal, whose definition is absolute to all Coll_κ extensions. For example, we shall consider Coll_{ω_1} and Coll_{ω_2}, and so on.

Lemma 22. *For any absolutely definable regular cardinal κ, the class Coll_κ is a linear forcing class, which is also persistent and closed under products.*

Proof. The class Coll_κ is reflexive, since $\mathrm{Coll}(\kappa, 0) = \{\varnothing\}$ is trivial. The class is transitive, since $\mathrm{Coll}(\kappa, \lambda) * \mathrm{Coll}(\kappa, \theta)$ is forcing equivalent to $\mathrm{Coll}(\kappa, \max\{\,\lambda, \theta\,\})$ by an argument analogous to lemma 18. The same fact shows that Coll_κ is linear. Since the definition of κ is absolute and posets in Coll_κ do not add new sets of size less than κ, it follows that Coll_κ is persistent. From this and transitivity, it follows that Coll_κ is closed under products. □

In the same spirit, we define the class of all κ-Cohen forcing, denoted by Add_κ, consisting of all forcing notions of the form $\mathrm{Add}(\kappa, \theta)$, the poset to add θ many Cohen subsets to κ, having conditions that are partial functions from $\theta \times \kappa$ to 2 of size less than κ. We consider this class only when κ is a definable regular cardinal, whose definition is absolute to all Add_κ extensions. In this case, the class Add_κ is a linear forcing class by essentially the same argument as in lemma 20.

A forcing notion $\mathbb{P}$ has *essential size* δ, if the complete Boolean algebra corresponding to $\mathbb{P}$ has size δ and $\mathbb{P}$ is not forcing equivalent to any $\mathbb{Q}$ whose complete Boolean algebra has smaller size. If W is any model of set theory, we can define the *forcing distance from* L as the least L-cardinal δ such that W can be written as $\mathrm{L}[G]$ where G is $\mathbb{P}$-generic over L for a forcing notion $\mathbb{P}$ of essential size δ. We write $\mathrm{fd}_\mathrm{L} = \delta$ in this case.

Lemma 23. *If $W \models \mathrm{fd}_\mathrm{L} = \delta$ and H is $\mathbb{Q}$-generic over W for some $\mathbb{Q} \in W$, then $W[H] \models \mathrm{fd}_\mathrm{L} \geq \delta$.*

Proof. Let $W[H] = \mathrm{L}[G]$ for some G which is $\mathbb{P}$-generic over L for some $\mathbb{P} \in \mathrm{L}$ and let $\mathbb{B}$ be the complete Boolean algebra corresponding to $\mathbb{P}$. Since $\mathrm{L} \subseteq W \subseteq W[H] = \mathrm{L}[G]$, we know (by [12, lemma 15.43]) that there is a subalgebra $\mathbb{C}$ of $\mathbb{B}$ such that $W = \mathrm{L}[G']$ for some G' which is $\mathbb{C}$-generic over L. We assumed that $\mathrm{fd}_\mathrm{L}^W = \delta$, and thus $|\mathbb{C}|^\mathrm{L} \geq \delta$. Since $\mathbb{C}$ was a subalgebra of $\mathbb{B}$, we have that $|\mathbb{B}|^\mathrm{L} \geq \delta$. □

5.2. **The modal logic of collapse forcing.** We aim to determine the provably valid principles of collapse forcing. Lemma 19 and theorem 7 give us S4.3 as a lower bound.

Theorem 24. *If* ZFC *is consistent, then the* ZFC*-provably valid principles of collapse forcing* Coll *are exactly those in* S4.3.

Proof. For the upper bound, we shall show that Coll admits a long ratchet over the constructible universe L. For each non-zero ordinal α, let r_α be the statement "$\aleph_\alpha^\mathrm{L}$ is countable." These statements form a long ratchet for collapse forcing over the constructible universe L, since any collapse extension $\mathrm{L}[G]$ collapses an initial segment of the cardinals of L to ω, and in any such extension in which $\aleph_\alpha^\mathrm{L}$ is not yet collapsed, the forcing to collapse it will not yet collapse $\aleph_{\alpha+1}^\mathrm{L}$. Thus, by theorem 12, the valid principles of collapse forcing over L are contained within S4.3. So the valid principles of collapse forcing over L are precisely S4.3, and if ZFC is consistent, then the ZFC-provably valid principles of collapse forcing are exactly S4.3. □

Theorem 25. *If Γ is a transitive reflexive forcing class and necessarily* $\mathrm{Coll} \subseteq \Gamma$*, then the valid principles of Γ forcing over* L *contain* S4.2 *and are contained within* S4.3.

Proof. Suppose that Γ is a transitive reflexive forcing class necessarily containing collapse forcing. By lemma 18, it follows that every poset is absorbed by any sufficiently large collapse poset, and so Γ is directed. Thus, by theorem 7, the modal theory S4.2 is valid for Γ forcing over any model of set theory.

Let us now establish the upper bound of S4.3 by finding a long ratchet over the constructible universe L. For each nonzero ordinal α, let r_α be the assertion "Some L cardinal above $\aleph_\alpha^{\mathrm{L}}$ is collapsed." Clearly, these statements are all false in L, each implies its own necessity, each implies the previous, and if $r_{\alpha+1}$ is not yet true in a forcing extension $\mathrm{L}[G]$, then we may collapse $\aleph_{\alpha+1}^{\mathrm{L}}$ without collapsing any larger cardinal, making $r_\alpha \wedge \neg r_{\alpha+1}$ true. So these statements form a long ratchet over L, and therefore, by theorem 12, the valid principles of Γ forcing over L are included within S4.3. □

Corollary 26. *If* ZFC *is consistent and* Γ *is a linear forcing class containing collapse forcing* Coll*, then the* ZFC*-provably valid principles of* Γ *forcing are exactly* S4.3*.*

Proof. If Γ is a linear forcing class, then S4.3 is valid by theorem 7. And the validities are contained with S4.3 by the previous theorem. □

Theorem 27. *If* ZFC *is consistent, then the* ZFC*-provably valid principles of Lévy collapse forcing* $\mathrm{Coll}^{<}$ *is exactly* S4.3*.*

Proof. It follows by lemma 21 that $\mathrm{Coll}^{<}$ is a linear forcing class containing Coll, and so this theorem is an instance of corollary 26. □

Theorem 28. *If* ZFC *is consistent, then the* ZFC*-provably valid principles of collapse-to-*κ *forcing, for any absolutely definable regular cardinal* κ*, are exactly* S4.3*.*

Proof. By lemma 22, the collapse-to-κ forcing class Coll_κ is a linear forcing class, and so by theorem 7, every S4.3 assertion is valid for collapse-to-κ forcing. Conversely, consider κ as it is defined in L, namely, $\kappa^{\mathrm{L}} = \aleph_\beta^{\mathrm{L}}$ for some ordinal β. For each nonzero ordinal α, let r_α be the assertion "some L-cardinal above $\aleph_{\beta+\alpha}^{\mathrm{L}}$ is collapsed." Just as in the previous section, these form a long ratchet for collapse-to-κ forcing over L, because in any Coll_κ forcing extension $\mathrm{L}[G]$, if no cardinals above $\aleph_{\beta+\alpha}^{\mathrm{L}}$ are collapsed, then we can collapse $\aleph_{\beta+\alpha}^{\mathrm{L}}$ to κ without collapsing any larger L-cardinals. By theorem 12, it follows that the valid principles of collapse-to-κ forcing over L are contained within S4.3. Thus, if ZFC is consistent, it follows that the ZFC-provably valid principles of collapse-to-κ forcing are exactly S4.3. □

Theorem 29. *Suppose that* Γ *is a transitive reflexive forcing class, with the property that there is a definable proper class* C *of cardinals in* L*, such that in any forcing extension* $\mathrm{L}[G]$ *by a forcing notion in* Γ *having essential size* δ_0 *and any larger* $\delta \in C$*, there is a forcing notion in* $\Gamma^{\mathrm{L}[G]}$ *having essential size* δ*. Then the valid principles of* Γ *forcing over* L *are contained within* S4.3*.*

Proof. We shall construct a long ratchet. For each nonzero ordinal α, let w_α be the assertion "fd_{L} is bigger than the αth element of C". By lemma 23, each statement w_α is an unpushed pure button in L. Our assumptions on Γ and C ensure that if a set forcing extension $\mathrm{L}[G]$ does not yet satisfy w_α, then $w_\alpha \wedge \neg w_{\alpha+1}$ is Γ forceable. Finally, since any set forcing extension $\mathrm{L}[G]$ was obtained by forcing of *some* size, no such extension can satisfy all w_α. Thus, this is indeed a long ratchet, and so the valid principles of Γ forcing over L are contained within S4.3 by theorem 12. □

Theorem 30. *If* Γ *is any reflexive transitive forcing class necessarily containing* Coll_κ *for some* Γ*-absolutely definable regular cardinal* κ*, then the valid principles for* Γ *forcing over* L *are contained within* S4.3*.*

Proof. Suppose that Γ and κ are as stated. Check that the conditions of theorem 29 are satisfied: The fact that κ is Γ-absolutely definable guarantees that $C := \{(\kappa^{+(\alpha)})^{\mathrm{L}} : \alpha \in \mathrm{Ord}\}$ is a definable proper class; if $\mathrm{L}[G]$ is a Γ extension of essential size $\delta_0 < \delta \in C$, then $\mathrm{Coll}(\kappa, \delta) \in \Gamma$ has essential size δ. Now the claim follows from theorem 29. □

5.3. **The modal logic of Cohen forcing.**

Theorem 31. *If* ZFC *is consistent, then the* ZFC*-provably valid principles of Cohen forcing* Add *are exactly* S4.3*.*

Proof. By lemma 20, the class Add is a linear forcing class, and so the valid principles of Cohen forcing include S4.3 by theorem 7. For the upper bound, we shall construct a long ratchet for Cohen forcing over L. For each nonzero ordinal α, let r_α be the statement that $2^\omega > \aleph_\alpha^{\mathrm{L}}$. By adding additional subsets to ω, we can push up the value of the continuum to any such prescribed degree. And in any Cohen extension of L, all cardinals have been preserved, so the continuum can be pushed up so as to realize any particular $\aleph_{\beta+1}^{\mathrm{L}}$ above the current size of the continuum, thereby forcing $r_\beta \wedge \neg r_{\beta+1}$. Thus, this is indeed a long ratchet for Cohen forcing Add over L, and so by theorem 12, the valid principles of Cohen forcing over L are contained within S4.3, and consequently are exactly equal to S4.3. In particular, if ZFC is consistent, then the ZFC-provably valid principles of Cohen forcing are exactly S4.3. □

As mentioned above, the class Add_κ is a linear forcing class, and therefore S4.3 is always valid for Add_κ forcing.

Theorem 32. *If* κ *is an absolutely definable regular cardinal, then the valid principles of* κ*-Cohen forcing* Add_κ *over* L *are exactly* S4.3*. Consequently, if* ZFC *is consistent, then the* ZFC*-provably valid principles of* κ*-Cohen forcing are exactly* S4.3*.*

Proof. This argument proceeds as in theorem 28, but using the analogue of the long ratchet of theorem 31. As mentioned, S4.3 is always valid for Add_κ forcing. Conversely, consider κ as defined in L, which must be $\aleph_\beta^{\mathrm{L}}$ for some ordinal β. For each nonzero ordinal α, let r_α be the statement that $2^\kappa > \aleph_{\beta+\alpha}^{\mathrm{L}}$. This is a long ratchet over L with respect to Add_κ, since we can always push up 2^κ to attain any particular higher cardinal value, and since Add_κ forcing does not collapse cardinals over Add_κ extensions $\mathrm{L}[G]$, once r_α is true in an Add_κ extension, it remains true in all further extensions. Thus, by theorem 12, the valid principles of Add_κ forcing over L are contained within S4.3, and therefore are in fact equal to S4.3. It follows that if ZFC is consistent, then the ZFC-provably valid principles of Add_κ forcing are precisely those in S4.3. □

5.4. **Upper bounds for other classes of forcing.** We shall now apply theorem 29 to other forcing classes:

Corollary 33. *The valid principles of forcing over* L *for each of the following forcing classes is contained within* S4.3*. For none of the forcing classes do the validities over* L *include* S4.2*.*

(1) *c.c.c. forcing.*

(2) *Proper forcing.*
(3) *Semi-proper forcing.*
(4) *Stationary-preserving forcing.*
(5) ω_1*-preserving forcing.*
(6) *Cardinal-preserving forcing.*
(7) *Countably distributive forcing.*

Proof. Each of these classes is easily seen to be transitive and reflexive. Let C be the class of uncountable successor cardinals of L. For any $\delta \in C$, each class contains forcing of essential size δ. For example, $\mathrm{Add}(\omega, \delta)$ is in classes (1) through (6), and $\mathrm{Add}(\delta, 1)$ is in class (7). The same is true in any forcing extension $\mathrm{L}[G]$ by set forcing of essential size below δ. Thus, each class satisfies the requirements of theorem 29, and so the valid principles of forcing in each case are included within S4.3.

We complete the proof by showing for each of the classes that .2 is not valid over L. Let φ be the assertion "the L-least Souslin tree is a special Aronszajn tree." This statement is c.c.c. forceable over L, simply by specializing the tree, and once the statement is true, it remains true in all ω_1-preserving extensions. Consider now the c.c.c. extension $\mathrm{L}[b]$ obtained by forcing to add a branch b through T. In this extension and all subsequent ω_1-preserving extensions, φ is false. Since each of the forcing classes in (1) through (6) contains c.c.c. forcing and is contained within ω_1-preserving forcing, this statement φ is a violation of axiom .2 with respect to each of the forcing classes. For class (7), we may consider the forcing to kill the stationary of the L-least stationary co-stationary subset of ω_1, or to kill its complement, and the impossibility of doing both while adding no reals provides a violation of .2. $\square$

The reader can probably extend this list with additional natural forcing classes satisfying the hypothesis of theorem 29 (the classes of countably closed forcing and more generally, κ-closed forcing for any fixed absolutely definable cardinal κ, will be covered in § 5.5). Let us state for the record that we do not currently know the exact forcing validities for any of the classes listed in corollary 33. Nevertheless, our analysis of the case of ω_1-preserving forcing and cardinal-preserving will be improved in theorem 36.

5.5. **Countably closed and κ-closed forcing.** We turn now to the class of countably closed forcing, a highly natural forcing class with robust closure properties, along with its generalization to $<\kappa$-closed forcing. As usual, a forcing notion $\mathbb{Q}$ is *countably closed* if every decreasing sequence $q_0 \geq q_1 \geq \cdots$ has a lower bound in $\mathbb{Q}$. This property is not preserved by equivalence of forcing (for example, an atomless complete Boolean algebra is *never* countably complete, since one may take the supremum of a countable antichain, stripping off one element in each step), so we understand the class of countably closed forcing to refer to the class of all forcing notions that are forcing equivalent to a countably closed forcing notion.

Theorem 34. *If* ZFC *is consistent, then the* ZFC*-provably valid principles of the class of countably closed forcing are exactly those in* S4.2.

Proof. Since any countably closed forcing notion remains countably closed in any countably closed extension, it follows that the class of countably closed forcing is persistent. Thus, by theorem 7, the validities always include S4.2. For the upper bound, consider countably closed forcing over L. Consider the higher buttons and switches introduced in § 3, either the higher analogues we mentioned of our own buttons b_n^* or the higher buttons of Rittberg, or of Friedman, Fuchino and Sakai, which can in each case be controlled independently via countably closed forcing. Thus, by theorem 13, the valid principles of countably closed forcing over L are contained within S4.2. In summary, if ZFC is consistent, then the ZFC-provably valid principles of countably closed forcing are precisely the assertions of S4.2. □

A forcing notion $\mathbb{Q}$ is $<\kappa$-closed if every descending sequence in $\mathbb{Q}$ of length less than κ has a lower bound in $\mathbb{Q}$. Thus, the countably closed posets are exactly the $<\aleph_1$-closed. A cardinal κ is *absolutely defined* by a formula φ in the context of a reflexive transitive forcing class Γ, if κ is the only object satisfying $\varphi(x)$ in any Γ-forcing extension.

Theorem 35. *More generally, if* ZFC *is consistent, then for any absolutely definable regular cardinal κ, the* ZFC*-provably valid principles of κ-closed forcing are exactly those in* S4.2.

Proof. Since the class is persistent, we again obtain S4.2 as a lower bound. And for the upper bound, one may use the higher analogues of the buttons and switches used in theorem 34, by translating them above κ. □

5.6. ω_1**-preserving forcing.** Surely the class of cardinal-preserving forcing and the larger class of ω_1-preserving forcing have been focal points of forcing theory. We have already observed in corollary 33 that the valid principles of ω_1-preserving forcing over L is contained within S4.3. Here, we prove a stronger conclusion, although the exact modal theory of validities for this class remains an open question.

Theorem 36. *The valid principles of ω_1-preserving forcing over* L *are included within* S4.tBA $\subsetneq$ S4.2.

Proof. By theorem 16, it suffices to find arbitrarily large finite strongly independent families of weak buttons and switches. For this, we may use the same buttons b_n^* mentioned in the context of lemma 17, which are described also in [9, thm 29]. These were a independent family of buttons for the class of all forcing, but for the class of ω_1-preserving forcing, they become a strongly independent family of weak buttons, since in any ω_1-preserving extension $L[G]$, at least one of the sets S_n must still remain stationary, and so not all buttons are pushed. But subject to this requirement, lemma 17 and the subsequent argument shows that the buttons can be controlled independently by ω_1-preserving forcing. For a family of switches that is mutually independent of these weak buttons, consider $(2^\omega)^{L[G]}$, which must be $\aleph^L_{\omega\cdot\xi+k}$ for some ordinal ξ and some natural number k, and let s_j assert that the jth binary digit of k is 1. Since we can force any desired value for k by adding Cohen reals, it follows that any finite pattern of the s_j switches is realizable, by forcing that preserves all stationary sets, and hence does not inadvertently push any of the unpushed weak buttons b_n. Since we have a strongly independent family of weak buttons, with a mutually independent family of switches, it follows by theorem 16 that the valid principles of ω_1-preserving forcing over L is contained within S4.tBA. □

We are unsure whether this argument can be extended to the class of cardinal-preserving forcing, since it seems that in some extensions, the club-shooting forcing could conceivably collapse cardinals above ω_1. Nevertheless, this issue may be solvable by the alternative method of shooting clubs, via finite conditions, which does preserve all cardinals.

5.7. CH**-preserving forcing.** In any model of ZFC $+\varphi$, we say that forcing notion $\mathbb{Q}$ is φ-*preserving*, if every forcing extension by $\mathbb{Q}$ satisfies φ. Note that we consider only φ-preserving forcing in models that actually satisfy φ, so by "provably valid principles of φ-preserving forcing", we mean the ZFC $+\varphi$-provably valid principles of φ-preserving forcing.

Theorem 37. *If* ZFC *is consistent, then the provably valid principles of* CH*-preserving forcing, of* GCH*-preserving forcing and of* $\neg$CH*-preserving forcing are all exactly* S4.2*.*

Proof. Let us treat the case of CH first. The class of CH-preserving forcing, over any model satisfying CH, is easily seen to be reflexive and transitive. So S4 is valid. Let us argue that .2 is also valid for CH-preserving forcing.

Suppose that $V \models$ CH and $V \models \Diamond\Box\varphi$ as well. Assume that V does not satisfy .2, that is, that $V \models \neg\Box\Diamond\varphi$, or, equivalently, $V \models \Diamond\Box\neg\varphi$. Then there are CH-preserving extensions $V[G]$ and $V[H]$ where $V[G] \models \Box\varphi$ and $V[H] \models \Box\neg\varphi$. We may assume without loss of generality that G and H are mutually V-generic. We don't know that $V[G*H]$ satisfies CH, so we can't just directly combine them. But let $V[G*H][I]$ be a further forcing extension of $V[G*H]$ that does satisfy CH. We claim that $V[G*H][I]$ is the result of CH-preserving forcing over $V[G]$. Consider some partial order $\mathbb{P}$ such that there is some $\mathbb{P}$-generic J such that $V[G*H][I] = V[G][J]$. Since this model satisfies CH, there is some $p \in J$ such that p forces CH. Let $\mathbb{P}_p$ be the poset $\mathbb{P}$ below p, i.e., $\mathbb{P}_p$ is CH-preserving. Without loss of generality, we can assume that J is $\mathbb{P}_p$-generic, thus yielding the claim. Thus, $V[G*H][I]$ is obtained from $V[G]$ by CH-preserving forcing, satisfying both φ and $\neg\varphi$, a contradiction. Thus, V satisfies .2 as desired, and by observation 5, V satisfied $\Box$.2. Since the class of CH-preserving forcings is closed under iterations, the Kripke model corresponding to every forcing translation is transitive. Hence, by observation 6, this means that S4.2 is valid in V for CH-preserving forcing, thus (since V was arbitrary) giving the lower bound.

For the upper bound, consider CH-preserving forcing over L. We may use the independent family of buttons and switches for L presented in § 4, which work just as well for CH-preserving forcing, and so by theorem 13 the valid principles of CH-preserving forcing over L are contained within S4.2. So they are exactly S4.2.

Essentially the same argument works in the case of $\neg$CH-preserving forcing, by means of the higher analogues of the buttons and switches, which can be controlled with highly closed and therefore $\neg$CH-preserving forcing. For GCH-preserving forcing, however, we shall need to use different switches. To find them, consider any forcing extension L$[G]$, and let β be the least ordinal such that there are no L-generic Cohen subsets in L$[G]$ of any cardinal above $\aleph_\beta^{\mathrm{L}}$. The ordinal β can be written uniquely in the form $\omega\cdot\alpha+k$ for some ordinal α and $k<\omega$. Let s_m be the statement that the mth binary digit of k is 1. These form an independent family of switches (and independent from the buttons b_n above), because by adding a Cohen set up high, we can realize any desired natural number k, and hence realize any desired finite pattern in the switches. (Alternatively, we could build a long ratchet with a similar method, mutually independent from the buttons b_n, and this also suffices.) ☐

5.8. **The modal logic of c.c.c. forcing.** The class of c.c.c. forcing is the pre-eminent forcing class, noticed already in the earliest days of forcing as a central notion. Since the class of c.c.c. partial orders is not closed under the equivalence of forcing (for example, every poset is forcing equivalent to very large lottery sums of the poset with itself), let us understand the class of c.c.c. forcing to consist of all forcing notions that are forcing equivalent to a c.c.c. partial order. In corollary 33, we observed that the valid principles of c.c.c. forcing over L are contained within S4.3 and do not contain S4.2. But we do not know the exact modal theory of c.c.c. forcing validities over L or the class of ZFC-provably valid principles of c.c.c. forcing.

Theorem 38. *If* MA_{ω_1} *holds, then* S4.2 *is valid for c.c.c. forcing.*

Proof. Since the class of c.c.c. forcing is clearly reflexive and transitive, it follows by theorem 7 at least that S4 is valid. Martin's axiom MA_{ω_1} implies that c.c.c. forcing in V is persistent, that is, every c.c.c. poset in V remains c.c.c. in every c.c.c. extension of V [12, theorem 16.21]. This exactly shows that axiom .2 is valid for c.c.c. forcing in V. As in the proof of theorem 37, observations 5 and 6 give us that S4.2 is valid for c.c.c. forcing in every model of MA_{ω_1}. □

It should be pointed out that the (modal logic) proof of theorem 38 hides the set-theoretic fact that the MA_{ω_1} hypothesis is rather fragile, and easily destroyed by c.c.c. forcing (e.g., adding a Cohen real creates Souslin trees and therefore destroys MA). Our theorem tells us that S4.2 will continue to be valid in all further c.c.c. forcing extensions of a model of MA_{ω_1}, even though such extensions may no longer have MA_{ω_1}. (In particular, it means that while Souslin trees are created, they cannot be definable in a way that allows us to create the counterexamples to .2, as in corollary 33).

The argument generalizes beyond c.c.c. forcing to any transitive reflexive forcing class Γ. Namely, if such a Γ is persistent in some model V, then it will continue to satisfy S4.2 in all Γ extensions $V[G]$, even if the interpretation of Γ in those later models is no longer persistent.

Finally, we remark that in order to prove that the $\mathrm{ZFC} + \mathrm{MA}_{\omega_1}$-provably valid principles of c.c.c. forcing are exactly S4.2, it would be sufficient to identify arbitrarily large finite families of independent c.c.c.-buttons and switches over the Solovay-Tennenbaum model $\mathrm{L}[G]$ of $\mathrm{MA} + \neg\mathrm{CH}$.

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