SOME PROPERTIES OF THE THOM SPECTRUM OVER $\Omega\Sigma\mathbb{C}P^\infty$

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Abstract. This note provides a reference for some properties of the Thom spectrum $M\xi$ over $\Omega\Sigma\mathbb{C}P^\infty$. Some of this material is used in recent work of Kitchloo and Morava. We determine the $M\xi$-cohomology of $\mathbb{C}P^\infty$ and show that $M\xi^*(\mathbb{C}P^\infty)$ injects into power series over the algebra of non-symmetric functions. We show that $M\xi$ gives rise to a commutative formal group law over the non-commutative ring $\pi_*M\xi$. We also discuss how $M\xi$ and some real and quaternionic analogues behave with respect to spectra that are related to these Thom spectra by splittings and by maps.

Introduction

The map $\mathbb{C}P^\infty = BU(1) \to BU$ gives rise to a canonical loop map from $\Omega\Sigma\mathbb{C}P^\infty$ to BU. Therefore the associated Thom spectrum has a strictly associative multiplication. But as is visible from its homology, which is a tensor algebra on the reduced homology of $\mathbb{C}P^\infty$, it is not even homotopy commutative. This homology ring coincides with the ring of non-symmetric functions, NSymm. We show that there is a map from the $M\xi$-cohomology of $\mathbb{C}P^\infty$ to the power series over the ring of non-symmetric functions, NSymm. This result is used in [MK] in an application of $M\xi$ to quasitoric manifolds.

As $M\xi$ maps to MU but as it does not receive a map from MU, it is a priori not clear, that there is a formal group law associated to $M\xi$. However, the classical Atiyah-Hirzebruch spectral sequence calculations carry over to $M\xi$ and this yields a commutative formal group, but the coefficients $M\xi_*$ are not commutative and they don't commute with the variables coming from the choices of complex orientations.

For MU the p-local splitting gives rise to a map of ring spectra $BP \to MU_{(p)}$. We show that despite the fact that $M\xi_{(p)}$ splits into (suspended) copies of BP, there is no map of ring spectra $BP \to M\xi_{(p)}$. For the canonical Thom spectrum over $\Omega\Sigma\mathbb{R}P^\infty$, $M\xi_\mathbb{R}$, we show that the map of E_2-algebra spectra $H\mathbb{F}_2 \to MO$ does not give rise to a ring map $H\mathbb{F}_2 \to M\xi_\mathbb{R}$.

For a map of ring spectra $MU \to E$ to some commutative S-algebra E one can ask whether a map of commutative S-algebras $S \wedge_{\mathbb{P}(S)} \mathbb{P}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to E$ factors over MU. Here, $\mathbb{P}(-)$ denotes the free commutative S-algebra functor. It is easy to see that MU is not equivalent to $S \wedge_{\mathbb{P}(S)} \mathbb{P}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$. We show that there are commutative S-algebras for which that is not the case. In the associative setting, the analoguous universal gadget would be $S \amalg_{\mathbb{A}(S)} \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$, where $\mathbb{A}(-)$ is the free associative S-algebra functor and $\amalg$ denotes the coproduct in the category of associative S-algebras. It is obvious that the homology of $S \amalg_{\mathbb{A}(S)} \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$ is much bigger than the one of $M\xi$. If we replace the coproduct by the smash product, there is still a canonical map $S \wedge_{\mathbb{A}(S)} \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to M\xi$ due to the coequalizer property of the smash product. However, this smash product has still homology that is larger than that of $M\xi$. Therefore the freeness of $\Omega\Sigma\mathbb{C}P^\infty$ is not reflected on the level of Thom spectra.

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1. The Thom spectrum of ξ

Lewis showed [LMSM, Theorem IX.7.1] that an n-fold loop map to BF gives rise to an E_n-structure on the associated Thom spectrum. Here E_n is the product of the little n-cubes operad with the linear isometries operad. For a more recent account in the setting of symmetric spectra see work of Christian Schlichtkrull [Sch].

The map $j\colon \Omega\Sigma\mathbb{C}P^\infty \to BU$ is a loop map and so the Thom spectrum $M\xi$ associated to that map is an A_∞ ring spectrum and the natural map $M\xi \to MU$ is one of A_∞ ring spectra, or equivalently of S-algebras in the sense of [EKMM]. Since the homology $H_*(M\xi)$ is isomorphic as a ring to $H_*(\Omega\Sigma\mathbb{C}P^\infty)$, we see that $M\xi$ is not even homotopy commutative. We investigated some of the properties of $M\xi$ in [BR].

For any commutative ring R, under the Thom isomorphism $H_*(\Omega\Sigma\mathbb{C}P^\infty; R) \cong H_*(M\xi; R)$, the generator Z_i corresponds to an element $z_i \in H_{2i}(M\xi; R)$ (we set $z_0 = 1$). Thomifying the map $i\colon \mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$, we obtain a map $Mi\colon MU(1) \to \Sigma^2 M\xi$ and it is easy to see that

$$Mi_*\beta_{i+1} = z_i. \tag{1.1}$$

1.1. **Classifying negatives of bundles.**

For every based space X, time-reversal of loops is a loop-map from $(\Omega X)^{\mathrm{op}}$ to ΩX, *i.e.*,

$$\bar{(\,\cdot\,)}\colon (\Omega X)^{\mathrm{op}} \to \Omega X; \quad w \mapsto \bar{w},$$

where $\bar{w}(t) = w(1-t)$. Here $(\Omega X)^{\mathrm{op}}$ is the space of loops on X with the opposite multiplication of loops.

We consider BU with the H-space structure coming from the Whitney sum of vector bundles and denote this space by $BU_\oplus$. A complex vector bundle of finite rank on a decent space Y is represented by a map $f\colon Y \to BU$ and the composition

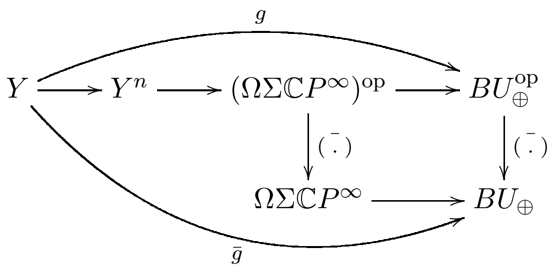
$$Y \xrightarrow{f} BU_\oplus^{\mathrm{op}} \xrightarrow{\bar{(\,\cdot\,)}} BU_\oplus$$

classifies the negative of that bundle, switching the role of stable normal bundles and stable tangent bundles for smooth manifolds.

For line bundles $g_i\colon Y \to \mathbb{C}P^\infty \to BU_\oplus$ $(i = 1, \dots, n)$ we obtain a map

$$g = (g_n, \dots, g_1)\colon Y \to Y^n \to (\mathbb{C}P^\infty)^n \to (\Omega\Sigma\mathbb{C}P^\infty)^{\mathrm{op}} \to BU_\oplus^{\mathrm{op}}$$

and the composition with loop reversal classifies the negative of the sum $g_n \oplus \cdots \oplus g_1$ as indicated in the following diagram.



In this way, the splitting of the stable tangent bundle of a toric manifold into a sum of line bundles can be classified by $\Omega\Sigma\mathbb{C}P^\infty$.

2. $M\xi$-(co)homology

We note that the natural map $i\colon \mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$ composed with $j\colon \Omega\Sigma\mathbb{C}P^\infty \to BU$ classifies the reduced line bundle $\eta - 1$ over $\mathbb{C}P^\infty$. The associated map $Mi\colon \Sigma^\infty MU(1) \to \Sigma^2 M\xi$ gives a distinguished choice of complex orientation $x_\xi \in M\xi^2(\mathbb{C}P^\infty)$, since the zero-section $\mathbb{C}P^\infty \to MU(1)$ is an equivalence.

We use the Atiyah-Hirzebruch spectral sequence

$$\mathrm{E}_2^{*,*} = H^*(\mathbb{C}P^\infty; M\xi^*) \Longrightarrow M\xi^*(\mathbb{C}P^\infty). \tag{2.1}$$

As $M\xi$ is an associative ring spectrum, this spectral sequence is multiplicative and its E_2-page is $\mathbb{Z}[[x]] \otimes M\xi^*$. As the spectral sequence collapses, the associated graded is of the same form and we can deduce the following:

Lemma 2.1. *As a left $M\xi^* = M\xi_{-*}$-module we have*

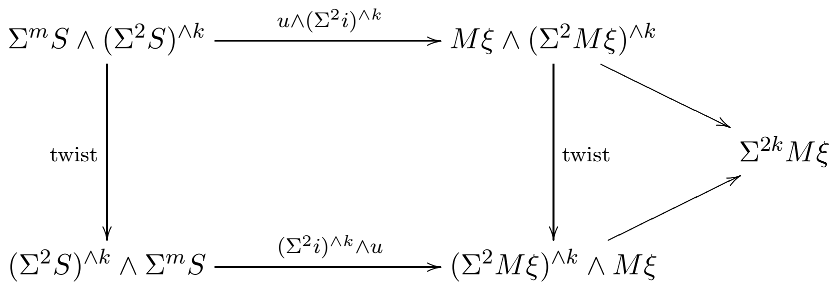
$$M\xi^*(\mathbb{C}P^\infty) = \{\sum_{i\geqslant 0} a_i x_\xi^i : a_i \in M\xi_*\}. \tag{2.2}$$

The filtration in the spectral sequence (2.1) comes from the skeleton filtration of $\mathbb{C}P^\infty$ and corresponds to powers of the augmentation ideal $\widetilde{M\xi}^*(\mathbb{C}P^\infty)$ in $M\xi^*(\mathbb{C}P^\infty)$. Of course the product structure in the ring $M\xi^*(\mathbb{C}P^\infty)$ is more complicated than in the case of $MU^*(\mathbb{C}P^\infty)$ since x_ξ is not a central element.

In order to understand a difference of the form $ux_\xi^k - x_\xi^k u$ with $u \in M\xi_m$ and $k \geqslant 1$ we consider the cofibre sequence

$$\Sigma^m \mathbb{C}P^{k-1} \subseteq \Sigma^m \mathbb{C}P^k \to \Sigma^m \mathbb{S}^{2k}.$$

Both elements ux_ξ^k and $x_\xi^k u$ restrict to the trivial map on $\Sigma^m\mathbb{C}P^{k-1}$. The orientation x_ξ restricted to $\mathbb{S}^2$ is the 2-fold suspension of the unit of $M\xi$, $\Sigma^2 i \in M\xi^2(\mathbb{S}^2)$. Centrality of the unit ensures that the diagram



commutes, so the difference $ux_\xi^k - x_\xi^k u$ is trivial. This yields

Lemma 2.2. *For every $u \in M\xi_m$ and $k \geqslant 1$, u and x_ξ^k commute up to elements of filtration at least $2k+2$,* i.e.,

$$ux_\xi^k - x_\xi^k u \in (\widetilde{M\xi}^*(\mathbb{C}P^\infty))^{[2k+2]}. \tag{2.3}$$

Let E be any associative S-algebra with an orientation class $u_E \in E^2(\mathbb{C}P^\infty)$. The Atiyah-Hirzebruch spectral sequence for $E^*\mathbb{C}P^\infty$ identifies $E^*(\mathbb{C}P^\infty)$ with the left E_*-module of power series in u_E as in the case of $M\xi$:

$$E^*(\mathbb{C}P^\infty) = \{\sum_{i\geqslant 0} \theta_i u_E^i : \theta_i \in E_*\}.$$

The orientation class $u_E \in E^2(\mathbb{C}P^\infty)$ restricts to the double suspension of the unit of E, $\Sigma^2 i_E \in E^2(\mathbb{C}P^1)$. Induction on the skeleta shows that for all n, $E_*(\mathbb{C}P^n)$ is free over E^* and we obtain that

$$E_*(\mathbb{C}P^\infty) \cong E_*\{\beta_0, \beta_1, \ldots\}$$

with $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ being dual to u_E^i.

Making use of (1.1), and passing to $E = H\wedge M\xi$-cohomology using the map $\varphi\colon M\xi \to H\wedge M\xi$ induced by the unit of H, we see that in $(H\wedge M\xi)^*(\mathbb{C}P^\infty)$,

$$x_\xi = \sum_{i\geqslant 0} z_i x_H^{i+1} = z(x_H), \tag{2.4}$$

where $x_H \in (H \wedge M\xi)^2(\mathbb{C}P^\infty)$ is the orientation coming from the canonical generator of $H^2(\mathbb{C}P^\infty)$. The proof is analogous to that for MU in [Ad]. Note that $H \wedge M\xi$ is an algebra spectrum over the commutative S-algebra H which acts centrally on $H \wedge M\xi$. Hence x_H is a central element of $(H \wedge M\xi)^*(\mathbb{C}P^\infty)$. This contrasts with the image of x_ξ in $(H \wedge M\xi)^*(\mathbb{C}P^\infty)$ which does not commute with all elements of $(H \wedge M\xi)_*$.

Let NSymm denote the ring of non-symmetric functions. This ring can be identified with $H_*(\Omega\Sigma\mathbb{C}P^\infty)$. Using this and the above orientation we obtain

Proposition 2.3. *There is a monomorphism* $\Theta\colon M\xi^*(\mathbb{C}P^\infty) \to \mathsf{NSymm}[[x_H]]$.

Proof. The right-hand side is isomorphic to the $H \wedge M\xi$-cohomology of $\mathbb{C}P^\infty$ and Θ is the map $\Theta\colon (M\xi)^*(\mathbb{C}P^\infty) \to (H \wedge M\xi)^*(\mathbb{C}P^\infty)$ that is induced by $\varphi\colon M\xi \to H \wedge M\xi$. As $M\xi$ is a sum of suspensions of BP at every prime and as the map is also rationally injective, we obtain the injectivity of Θ. $\square$

Note that for any $\lambda \in M\xi_*$ we can express $x_\xi\lambda$ in the form

$$\sum_{i \geqslant 0} z_i x_H^{i+1}\lambda = \sum_{i \geqslant 0} z_i \lambda x_H^{i+1},$$

but as the coefficients are non-commutative, we cannot pass λ to the left-hand side, so care has to be taken when calculating in $\mathsf{NSymm}[[x_H]]$.

3. A formal group law over $M\xi_*$

The two evident line bundles η_1, η_2 over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ can be tensored together to give a line bundle $\eta_1 \otimes \eta_2$ classified by a map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ and by naturality we obtain an element $\mu^* x_\xi \in M\xi^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. We also have

$$M\xi^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \left\{ \sum_{i,j \geqslant 0} a_{i,j}(x'_\xi)^i (x''_\xi)^j : a_{i,j} \in M\xi_* \right\} \tag{3.1}$$

as a left $M\xi^* = M\xi_{-*}$-module, where $x'_\xi, x''_\xi \in M\xi^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ are obtained by pulling back x_ξ along the two projections. We have

$$\mu^* x_\xi = F_\xi(x'_\xi, x''_\xi) = x'_\xi + x''_\xi + \sum_{i,j \geqslant 1} a_{i,j}(x'_\xi)^i (x''_\xi)^j,$$

where $a_{i,j} \in M\xi_{2(i+j)-2}$. The notation $F_\xi(x'_\xi, x''_\xi)$ is meant to suggest a power series, but care needs to be taken over the use of such notation. For example, since the tensor product of line bundles is associative up to isomorphism, the formula

$$F_\xi(F_\xi(x'_\xi, x''_\xi), x'''_\xi) = F_\xi(x'_\xi, F_\xi(x''_\xi, x'''_\xi)) \tag{3.2a}$$

holds in $M\xi^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, where $x'_\xi, x''_\xi, x'''_\xi$ denote the pullbacks of x_ξ along the three projections. When considering this formula, we have to bear in mind that the inserted expressions do not commute with each other or coefficients. We also have the identities

$$F_\xi(0, x_\xi) = x_\xi = F_\xi(0, x_\xi), \tag{3.2b}$$

$$F_\xi(x'_\xi, x''_\xi) = F_\xi(x''_\xi, x'_\xi). \tag{3.2c}$$

Let $\bar{x}_\xi = \gamma^* x_\xi$ denotes the pullback of x_ξ along the map $\gamma\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the inverse $\eta^{-1} = \bar{\eta}$ of the canonical line bundle η. Then $\bar{x}_\xi \in M\xi^2(\mathbb{C}P^\infty)$ and there is a unique expansion

$$\bar{x}_\xi = -x_\xi + \sum_{k \geqslant 1} c_k x_\xi^{k+1}$$

with $c_k \in M\xi_{2k}$. Since $\eta \otimes \bar{\eta}$ is trivial, this gives the identities

$$F_\xi(x_\xi, \bar{x}_\xi) = 0 = F_\xi(\bar{x}_\xi, x_\xi)$$

and so

(3.2d) $$F_\xi(x_\xi, -x_\xi + \sum_{k\geqslant 1} c_k x_\xi^{k+1}) = 0 = F_\xi(-x_\xi + \sum_{k\geqslant 1} c_k x_\xi^{k+1}, x_\xi).$$

To summarize, we obtain the following result.

Proposition 3.1. *The identities* (3.2) *together show that* $F_\xi(x'_\xi, x''_\xi)$ *defines a commutative formal group law over the non-commutative ring* $M\xi_*$.

Remark 3.2. Note however, that most of the classical structure theory for formal group laws over (graded) commutative rings does *not* carry over to the general non-commutative setting. For power series rings over associative rings where the variable commutes with the coefficients most of the theory works as usual. If the variable commutes with the coefficients up to a controlled deviation, then the ring of skew power series still behave reasonably (see for example [D]), but our case is more general.

4. The splitting of $M\xi$ into wedges of suspensions of BP

In [BR] we showed that there is a splitting of $M\xi$ into a wedge of copies of suspensions of BP locally at each prime p. In the case of MU the inclusion of the bottom summand is given by a map of ring spectra $BP \to MU_{(p)}$. However, for $M\xi$ this is not the case.

Proposition 4.1. *For each prime* p*, there is no map of ring spectra* $BP \to M\xi_{(p)}$

Proof. We give the proof for an odd prime p, the case $p = 2$ is similar. We set $H_* = (H\mathbb{F}_p)_*$.
Recall that

$$H_*(BP) = \mathbb{F}_p[t_1, t_2, \ldots]$$

where $t_r \in H_{2p^r-2}(BP)$ and the $\mathcal{A}_*$-coaction on these generators is given by

$$\psi(t_n) = \sum_{k=0}^{n} \zeta_k \otimes t_{n-k}^{p^k},$$

where $\zeta_r \in \mathcal{A}_{2p^r-2}$ is the conjugate of the usual Milnor generator ξ_r as in Adams [Ad]. The right action of the Steenrod algebra satisfies

$$\mathcal{P}^1_* t_1 = -1, \quad \mathcal{P}^1_* t_2 = -t_1^p, \quad \mathcal{P}^p_* t_2 = 0.$$

Assume that a map of ring spectra $u\colon BP \to M\xi_{(p)}$ exists. Then $\mathcal{P}^1_* u_*(t_1) = u_*(-1) = -1$, hence $w := u_*(t_1) \neq 0$. Notice that

$$\mathcal{P}^1_*(w^{p+1}) = -w^p, \quad \mathcal{P}^p_*(w^{p+1}) = -w.$$

Also, $\mathcal{P}^p_* u_*(t_2) = 0$. This shows that $u_*(t_2)$ cannot be equal to a non-zero multiple of w^{p+1}. Therefore it is not contained in the polynomial subalgebra of $H_*(M\xi_{(p)})$ generated by w^{p+1} and thus it cannot commute with w. This shows that the image of u_* is not a commutative subalgebra of $H_*M\xi_{(p)}$ which contradicts the commutativity of H_*BP. □

Remark 4.2. Note that Proposition 4.1 implies that there is no map of ring spectra from MU to $M\xi$, because if such a map existed, we could precompose it p-locally with the ring map $BP \to MU_{(p)}$ to get a map of ring spectra $BP \to M\xi_{(p)}$.

5. The real and the quaternionic cases

Analogous to the complex case, the map $\mathbb{R}P^\infty = BO(1) \to BO$ gives rise to a loop map $\xi_\mathbb{R}\colon \Omega\Sigma\mathbb{R}P^\infty \to BO$, and hence there is an associated map of associative S-algebras $M\xi_\mathbb{R} \to MO$ on the level of Thom spectra. There is a splitting of MO into copies of suspensions of $H\mathbb{F}_2$. In fact a stronger result holds.

Proposition 5.1. *There is a map of* E_2*-spectra* $H\mathbb{F}_2 \to MO$.

Proof. The map $\alpha\colon \mathbb{S}^1 \to BO$ that detects the generator of the fundamental group of BO gives rise to a double-loop map

$$\Omega^2\Sigma^2\mathbb{S}^1 = \Omega^2\mathbb{S}^3 \to BO.$$

As the Thom spectrum associated to $\Omega^2\mathbb{S}^3$ is an E_2-model of $H\mathbb{F}_2$ by [Mah], the claim follows. □

Generalizing an argument by Hu-Kriz-May [HKM], Gilmour [G] showed that there is no map of commutative S-algebras $H\mathbb{F}_2 \to MO$.

The E_2-structure on the map from Proposition 5.1 cannot be extended to $\xi_{\mathbb{R}}$. On the space level,

$$H_*(\Omega\Sigma\mathbb{R}P^\infty; \mathbb{F}_2) \cong T_{\mathbb{F}_2}(\bar{H}_*(\mathbb{R}P^\infty; \mathbb{F}_2)),$$

where $H_n(\mathbb{R}P^\infty; \mathbb{F}_2)$ is generated by an element x_n.

Proposition 5.2. *There is no map of ring spectra $H\mathbb{F}_2 \to M\xi_{\mathbb{R}}$.*

Proof. Assume $\gamma\colon H\mathbb{F}_2 \to M\xi_{\mathbb{R}}$ were a map of ring spectra. We consider $\gamma_*\colon (H\mathbb{F}_2)_*H\mathbb{F}_2 \to (H\mathbb{F}_2)_*M\xi_{\mathbb{R}}$. Note that $(H\mathbb{F}_2)_*M\xi_{\mathbb{R}}$ is the free associative $\mathbb{F}_2$-algebra generated by $z_1, z_2, \ldots$ with z_i in degree i being the image of x_i under the Thom-isomorphism.

Under the action of the Steenrod-algebra on $H\mathbb{F}_2$-homology $Sq^1_*(z_1) = 1$ and hence $Sq^1_*(z_1^3) = z_1^2$ by the derivation property of Sq^1_*.

In the dual Steenrod algebra we have $Sq^1_*(\xi_1) = 1$ and $Sq^2_*(\xi_2) = \xi_1$ and $Sq^1_*(\xi_2) = 0$.

Combining these facts we obtain

$$Sq^1_*(\gamma_*(\xi_1)) = \gamma_*(Sq^1_*\xi_1) = \gamma_*(1) = 1, \tag{5.1}$$

in particular $\gamma_*(\xi_1) \neq 0$ and thus $\gamma_*(\xi_1) = z_1$.

Similarly,

$$Sq^2_*\gamma_*(\xi_2) = \gamma_*(Sq^2_*\xi_2) = \gamma_*\xi_1 = z_1 \neq 0.$$

The image of γ_* generates a commutative sub-$\mathbb{F}_2$-algebra of $(H\mathbb{F}_2)_*M\xi_{\mathbb{R}}$. The only elements in $(H\mathbb{F}_2)_*M\xi_{\mathbb{R}}$ that commute with z_1 are polynomials in z_1. Assume that $\gamma_*\xi_2 = z_1^3$. Then

$$0 = \gamma_*Sq^1_*\xi_2 = Sq^1_*(z_1^3) = z_1^2 \neq 0,$$

which is impossible. Therefore, $\gamma_*\xi_2$ does not commute with z_1, so we get a contradiction. □

Note that Proposition 5.2 implies that there is no loop map $\Omega^2\mathbb{S}^3 \to \Omega\Sigma\mathbb{R}P^\infty$, since such a map would induce a map of associative S-algebras $H\mathbb{F}_2 \to M\xi_{\mathbb{R}}$.

A quaternionic model of quasisymmetric functions is given by $H^*(\Omega\Sigma\mathbb{H}P^\infty)$. Here, the algebraic generators are concentrated in degrees that are divisible by 4. The canonical map $\mathbb{H}P^\infty = BSp(1) \to BSp$ induces a loop-map $\xi_{\mathbb{H}}\colon \Omega\Sigma\mathbb{H}P^\infty \to BSp$ and thus gives rise to a map of associative S-algebras on the level of Thom spectra $M\xi_{\mathbb{H}} \to MSp$.

Of course, the spectrum MSp is not as well understood as MO and MU. There is a commutative S-algebra structure on MSp [May, pp. 22, 76], but for instance the homotopy groups of MSp are not known in an explicit form.

6. Associative versus commutative orientations

We work with the second desuspension of the suspension spectrum of $\mathbb{C}P^\infty$. Such spectra are inclusion prespectra [EKMM, X.4.1] and thus a map of S-modules from $S = \Sigma^\infty\mathbb{S}^0$ to $\Sigma^{\infty-2}\mathbb{C}P^\infty := \Sigma^{-2}\Sigma^\infty\mathbb{C}P^\infty$ is given by a map from the zeroth space of the sphere spectrum to the zeroth space of $\Sigma^{\infty-2}\mathbb{C}P^\infty$ which in turn is a colimit, namely $\operatorname{colim}_{\mathbb{R}^2\subset W}\Omega^W\Sigma^{W-\mathbb{R}^2}\mathbb{C}P^\infty$. As a map $\varrho\colon S \to \Sigma^{\infty-2}\mathbb{C}P^\infty$ we take the one that is induced by the inclusion $\mathbb{S}^2 = \mathbb{C}P^1 \subset \mathbb{C}P^\infty$. The fold map

$$f\colon \mathbb{A}(S) \simeq \bigvee_{i\geqslant 0} S \to S$$

is a map of S-algebras.

The commutative S-algebra $S \wedge (\mathbb{N}_0)_+ = S[\mathbb{N}_0]$ has a canonical map $S[\mathbb{N}_0] \to S$ which is given by the fold map. We can model this via the map of monoids that sends the additive monoid $(\mathbb{N}_0, 0, +)$ to the monoid $(0, 0, +)$; thus $S[\mathbb{N}_0] \to S$ is a map of commutative S-algebras.

We get a map $S[\mathbb{N}_0] \to \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$ by taking the following map on the nth copy of S in $S[\mathbb{N}_0]$. We can view S as $S \wedge \{*\}_+$ where $\{*\}$ is a one-point space. The n-fold space diagonal gives a map

$$\delta_n \colon S = S \wedge \{*\}_+ \to S \wedge \{\underbrace{(*, \ldots, *)}_{n}\}_+ \simeq S^{\wedge n}$$

which fixes an equivalence of S with $S^{\wedge n}$. We compose this map with the n-fold smash product of the map $\varrho\colon S \to \Sigma^{\infty-2}\mathbb{C}P^\infty$. The maps

$$\varrho^{\wedge n} \circ \delta_n \colon S \to (\Sigma^{\infty-2}\mathbb{C}P^\infty)^{\wedge n} \to \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$$

together give a map of S-algebras

$$\tau\colon S[\mathbb{N}_0] \to \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty).$$

Note, however, that $S[\mathbb{N}_0]$ is not central in $\mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$. Thus the coequalizer

$$S \wedge_{S[\mathbb{N}_0]} \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$$

does not possess any obvious S-algebra structure. Furthermore, there is a natural map

$$S \wedge_{S[\mathbb{N}_0]} \mathbb{A}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to M\xi,$$

but this is not a weak equivalence since the $H\mathbb{Z}$-homology of the left-hand side is bigger than $H\mathbb{Z}_* M\xi$.

In the commutative context the pushout of commutative S-algebras is given by the smash product and hence there is a natural map of commutative S-algebras

$$\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty) = S \wedge_{\mathbb{P}(S)} \mathbb{P}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to MU,$$

where the pushout is defined by the following diagram of commutative S-algebras

$$\begin{array}{ccc} \mathbb{P}(S) & \longrightarrow & \mathbb{P}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \\ \downarrow & \ulcorner & \downarrow \\ S & \longrightarrow & \widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \end{array}$$

using the identity map on S to induce the left hand vertical map of commutative S-algebras and the inclusion of the bottom cell of $\Sigma^{\infty-2}\mathbb{C}P^\infty$ to induce the top map which is a cofibration and therefore $\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$ is cofibrant. However, this map is not a weak equivalence.

Lemma 6.1. *The canonical map of commutative S-algebras*

$$\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to MU$$

is an equivalence rationally, but not globally. Furthermore, there is a morphism of ring spectra

$$MU \to \widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$$

which turns MU into a retract of $\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty)$.

Proof. Let $\boldsymbol{k}$ be a field. The Künneth spectral sequence for the homotopy groups of

$$H\boldsymbol{k} \wedge (\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty)) \simeq H\boldsymbol{k} \wedge_{\mathbb{P}_{H\boldsymbol{k}}(H\boldsymbol{k})} \mathbb{P}_{Hk}(\Sigma^{-2}H\boldsymbol{k} \wedge \mathbb{C}P^\infty)$$

has E^2-term

$$\mathrm{E}^2_{*,*} = \mathrm{Tor}_{*,*}^{\pi_*(\mathbb{P}_{H\boldsymbol{k}}(H\boldsymbol{k}))}(\boldsymbol{k}, \pi_*(\mathbb{P}_{H\boldsymbol{k}}(\Sigma^{-2}H\boldsymbol{k} \wedge \mathbb{C}P^\infty))).$$

When $\boldsymbol{k} = \mathbb{Q}$, $\pi_*(\mathbb{P}_{H\mathbb{Q}}(H\mathbb{Q}))$ is a polynomial algebra on a zero-dimensional class x_0 and

$$\pi_*(\mathbb{P}_{H\mathbb{Q}}(\Sigma^{-2}H\mathbb{Q} \wedge \mathbb{C}P^\infty)) \cong \mathbb{Q}[x_0, x_1, \ldots], \tag{6.1}$$

where $|x_i| = 2i$. Thus

$$\pi_*(H\mathbb{Q} \wedge (\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty)) \cong \mathbb{Q}[x_1, x_2, \ldots] \cong H\mathbb{Q}_*(MU).$$

However, when $\boldsymbol{k} = \mathbb{F}_p$ for a prime p, the freeness of the commutative S-algebras $\mathbb{P}(S)$ and $\mathbb{P}(\Sigma^{\infty-2}\mathbb{C}P^\infty))$ implies that $(H\mathbb{F}_p)_*(\mathbb{P}(\Sigma^{\infty-2}\mathbb{C}P^\infty))$ is a free $(H\mathbb{F}_p)_*(\mathbb{P}(S))$-module and thus the E^2-term reduces to the tensor product in homological degree zero. Note that this tensor product contains elements of odd degree, but $(H\mathbb{F}_p)_*(MU)$ doesn't.

Using the orientation for line bundles given by the canonical inclusion

$$\Sigma^{\infty-2}\mathbb{C}P^\infty \to \widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty),$$

we have a map of ring spectra

$$\varphi\colon MU \to \widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty).$$

The inclusion map $\mathbb{C}P^\infty = BU(1) \to BU$ gives rise to the canonical map $\sigma\colon \Sigma^{\infty-2}\mathbb{C}P^\infty \to MU$ and with this orientation we get a morphism of commutative S-algebras

$$\theta\colon \widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to MU,$$

such that the composite $\theta \circ \varphi \circ \sigma$ agrees with σ, hence $\theta \circ \varphi$ is homotopic to the identity on MU. □

Using topological André-Quillen homology, $TAQ_*(-)$, we can show that the map of ring spectra φ cannot be rigidified to a map $\widetilde{\varphi}$ of commutative S-algebras in such a way that the composite $\theta \circ \widetilde{\varphi}$ is homotopic to the identity. By Basterra-Mandell [BM],

$$TAQ_*(MU|S; H\mathbb{F}_p) \cong (H\mathbb{F}_p)_*(\Sigma^2 ku),$$

while [BGR, proposition 1.6] together with subsequent work of the first named author (in preparation) gives

$$TAQ_*(\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty)|S; H\mathbb{F}_p) \cong (H\mathbb{F}_p)_*(\Sigma^{\infty-2}\mathbb{C}P_2^\infty),$$

where $\mathbb{C}P_2^\infty$ is the cofiber of the inclusion of the bottom cell.

Proposition 6.2. *For a prime p, there can be no morphism of commutative $S_{(p)}$-algebras*

$$\theta\colon MU \to (\widetilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)_{(p)}$$

for which $\sigma \circ \theta$ is homotopic to the identity. Hence there can be no morphism of commutative S-algebras

$$\theta\colon MU \longrightarrow \widetilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty$$

for which $\sigma \circ \theta$ is homotopic to the identity.

Proof. It suffices to prove the result for a prime p, and we will assume all spectra are localised at p. Assume such a morphism θ existed. Then by naturality of Ω_S, there are (derived) morphisms of MU-modules and a commutative diagram

$$\Omega_S(MU) \xrightarrow[\theta_*]{} \Omega_S(\widetilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty) \xrightarrow[\sigma_*]{} \Omega_S(MU), \qquad \sigma_* \circ \theta_* = \mathrm{id}$$

which induces a commutative diagram in $TAQ_*(-; H\mathbb{F}_p)$

$$H_*(\Sigma^2 ku; \mathbb{F}_p) \xrightarrow[\theta_*]{} H_*(\Sigma^{\infty-2}\mathbb{C}P_2^\infty; \mathbb{F}_p) \xrightarrow[\sigma_*]{} H_*(\Sigma^2 ku; \mathbb{F}_p), \qquad \sigma_* \circ \theta_* = \mathrm{id}$$

where $\mathbb{C}P_2^\infty = \mathbb{C}P^\infty/\mathbb{C}P^1$.

It is standard that

$$H_n(\Sigma^{\infty-2}\mathbb{C}P_2^\infty; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } n \geqslant 2 \text{ and is even,} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, when $p = 2$,

$$H_*(ku; \mathbb{F}_2) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \subseteq \mathcal{A}(2)_*$$

with $|\zeta_s| = 2^s - 1$, while when p is odd

$$\Sigma^2 ku_{(p)} \sim \bigvee_{1\leqslant r\leqslant p-1} \Sigma^{2r}\ell,$$

where ℓ is the Adams summand with

$$H_*(\ell;\mathbb{F}_2) = \mathbb{F}_p[\zeta_1,\zeta_2,\zeta_3,\ldots] \otimes \Lambda(\bar{\tau}_r : r \geqslant 2),$$

for $|\zeta_s| = 2p^s - 2$ and $|\bar{\tau}_s| = 2p^s - 1$. Clearly this means that no such θ can exist. □

Proposition 6.3. *There are commutative S-algebras E which possess a map of commutative S-algebras*

$$\widetilde{\mathbb{P}}(\Sigma^{\infty-2}\mathbb{C}P^\infty) \to E$$

that cannot be extended to a map of commutative S-algebras $MU \to E$.

Proof. Matthew Ando [An] constructed complex orientations for the Lubin-Tate spectra E_n which are H_∞-maps $MU \to E_n$. However, in [JN], Niles Johnson and Justin Noël showed that none of these are p-typical for all primes up to at least 13 (and subsequently verified for primes up to 61). For any p-typical orientation there is a map of ring spectra $MU \to E_n$, but this map cannot be an H_∞-map and therefore is not a map of commutative S-algebras. □

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