A Note on Computational Aspects of Farsighted Coalitional Stability

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Abstract

Farsighted stability of Chwe (1994) is discussed while attention is played on the computational framework of finding farsightedly stable coalition structures. The idea of farsightedness means that one should check for multi-step stability by comparing the profits of a coalition member after a series of deviations has come to an end. The deviation is possible only if players display a cooperate attitude by forming a coalition in order to increase their payoffs. The connections of farsighted stability with a positive, negative spillover property and profitability condition are shown. Algorithms are developed, which can find all farsighted stable coalition structures.

Keywords: game theory, farsighted stability, coalition formation.

JEL: C02, C72, H41

1 Introduction

Farsighted stability developed further the notation of stable sets of von Neumann and Morgenstern (1947). Stable sets are defined to be self-consistent. The notion is characterized by internal and external stability. Internal stability guarantees that the solution set is free from inner contradictions, that is, any two outcomes in the solution set cannot dominate each other and external stability guarantees that every outcome excluded from the solution is accounted for, that is, it is dominated by some outcome inside the solution. Harsanyi (1974) criticizes the von Neumann and Morgenstern solution also for its failing to incorporate foresight. He introduced the concept of indirect dominance to capture foresight. An outcome indirectly dominates another, if there exists a sequence of outcomes starting from the dominated outcome and leading to the dominating one, and at each stage of the sequence the group of players required to enact the inducement prefers the final outcome to its status quo. His criticism inspired a series of works on abstract environments, including among others those of Chwe (1994); Mariotti (1997) and Xue (1998). Chwe (1994) introduces the notation of farsighted stability, which is applied to the problem of IEAs by Diamantoudi and Sartzetakis (2002), Eyckmans (2003) and Osmani and Tol (2009).
We investigate what outcomes are stable, which implies that they cannot be replaced by any coalition of rational, farsighted and selfish countries. The selfishness of players shapes the aspects of *non-cooperative approach.* The idea of farsightedness means that one should check for multi-step stability by comparing the profits of a coalition member after a series of deviations has come to an end. The deviation is possible only if players display *cooperate attitude (by forming a coalition)* to each-other in order to increase their welfare.
Different from Chwe (1994), who established the coalitional farsighted stability and presented a powerful theorem which proves the existence of it, we are more interested in applying aspects of farsighted stability

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in real world problems like climate change games etc where many asymmetric players interact. As there are a lot of asymmetric players, there are far more non-profitable coalitions compared to profitable (or individual rational) coalitions. It indicates that all non-profitable farsighted stable coalitions are harder to compute.
We introduce the direct connection to farsighted stability to profitability condition. We show that there is a relation between positive spillover property and profitable farsighted stable coalitions. We show also one that it is possible to calculate the non-profitable farsighted stable coalitions starting from profitable coalitions. We build algorithms which can find farsighted stable coalition structure with many coalitions. The computational complexity is exponential in the number of players, so it is not advisable to aim finding of arbitrary farsighted stable coalitions. Other conceptual tools are necessary like focal points suggested from Schelling (1960).
There are also some minor differences compared to Chwe (1994) approach to farsighted stability. Chwe defines the indirect dominance of Harsanyi (1974) in terms of consistence set. We say that if a coalition structure is not indirectly dominated, then it is farsighted stable. Furthermore, in spirit of Consistent Set of Chwe (1994), we define Dynamic Farsighted Coalition Structure Set (DFCS); if a coalition structure does not belong to DFCS, then is is indirectly dominated by a another coalition structure which belongs to DFCS. Being aware of cycles Chwe builds a weak solution, in the sense that outcome belonging to the Large Consistent Set are only possible stable. We differently say that every outcome (or a coalition structure) which is not indirectly dominated is farsightedly stable. So a cycle is impossible any further, and we obtain a slightly different solution concept. We still think that the difference is more of technical nature than of a conceptual one.
The paper is organized as follows. The second section introduces the game, defines indirect internal, external and sub-coalition dominance, and further more expresses them in terms of partition function. Computational aspects of farsighted stability are discussed in the third and fourth section. In fourth section an example for finding multiple farsighted stable coalitions is introduced, which is taken from Osmani and Tol (2009). The section five provides the conclusions. In Appendix, algorithms for finding single and multiple farsighted stable coalitions are introduced; the numerical efforts for finding farsighted stable coalitions in a general coalition structure are discussed, and a part of numerical computations of our example of section four, are presented.

2 Definition of the Game, Direct and Indirect Domination

Similarly to Chwe (1994), a game Γ is defined as, $\Gamma = (N, O, \{\prec_i\}_{i\in N}, \{\rightarrow_C\}_{C\subset N, C\neq\emptyset})$ where N is the set of players, O is the set of all coalition structures (which are also outcomes), $N \neq \emptyset, O \neq \emptyset$; $\{\prec_i\}_{i\in N}$ are the strong preference relation of players defined on O. Before explaining further the game, let define the coalition structure:

Definition 2.1 *A coalition structure $a = \{C_1, C_2, ..., C_m\}$ is a partition of the set of players $N = \{1, 2, ..., n\} : S_i \cap S_j \neq \emptyset$ where $\bigcup_{i=1:m} C_i = N$.*

A coalition structure fully describes how many coalitions are formed, how many members they have, and also how many single players are. The relation $\rightarrow_{C_1}$ introduces the actions that are available to coalition C_1; $a_1 \rightarrow_{C_1} a_2$ indicates that if coalition structure a_1 is the status quo, coalition C_1 can make a_2 the new status quo.
The game is "played" in the following way; when the game starts, there is a coalition structure (or outcomes) status quo called a_1; the status quo is usually taken the full-noncooperative structure (a_{FNS}, definition is going to be given later). If the member of coalition C decides to change the status quo from a_1 to a_2, or $a_1 \rightarrow_C a_2$, then the new status quo becomes a_2. This change of a status quo, we call a coalition's move or deviation, from a_1 to a_2. From this new status quo a_2, other coalition might move, and so forth. If a status quo a_3 is reached, and no player prefers to move, then a_3 is called stable and game is over. The game does tell you if a move or deviation is possible, so if a coalition structure a_m is possible or not.
The game is of a cooperative and noncooperative spirit. The selfishness of players shapes the aspects of *non-cooperative approach.* The idea of farsightedness means that one should check for multi-step stability by comparing the preference of a coalition member after a series of deviations has come to an end. The deviation is possible only if players display *cooperate attitude* by forming a coalition which they mostly preferred.
As the game is defined, we will go on with discussing direct and indirect domination.
If $a_1 \prec_i a_2$, $i \in C$, we write $a_1 \prec_C a_2$.

Definition 2.2 *A coalition structure a_1 is directly dominated by the coalition structure a_2, or $a_1 < a_2$, if there exists an C_1 such that $a_1 \rightarrow_{C_1} a_2$ and $a_1 \prec_C a_2$*

The definition of indirect dominance (taken from Harsanyi (1974) is introduced below:

Definition 2.3 *A coalition structure a_1 is indirectly dominated by the coalition structure a_m, or $a_1 \ll a_m$, if there exists $a_1, a_2, a_3, ..., a_m$ and $C_1, C_2, C_3, ..., C_{m-1}$ such that $a_i \rightarrow_{C_i} a_{i+1}$, and $a_i \prec_{C_i} a_m$ where $i = 1, 2, 3, ..., m-1$.*

Henceforth and on, we will only focus on "effective relation" that leads to indirect domination. This fits to spirit of farsighted stability as the farsighted players can see all possible deviations ahead, and are going to deviate only if they see ahead, further deviations which leads to a indirect dominance. Note that if $a_1 < a_2$, then $a_1 \ll a_2$.

Definition 2.4 *A coalitions structure a_1 is farsighted stable if it is not indirectly dominated.*

Here is a difference with original farsighted coalitional stability of Chwe (1994). I simply consider any coalition, which is not indirectly dominated, as farsighted stable. Chwe is more careful, he says in this case, that the coalition is possible to be stable. My standpoint is that there is no reason for deviation, if a coalition is not indirectly dominated, then better let call it stable in stead of possible stable.
This is also an essential point, *as cycles cannot be formed*, as cycles do not lead to an indirect dominance.

Definition 2.5 *A cycle is a chain of coalitions structure $a_1 \rightarrow_{C_1} a_2 \rightarrow_{C_2} a_3 \ ... \ a_{n-1} \rightarrow_{C_{n-1}} a_n \rightarrow_{C_n} a_1$ where every coalition structure $a_i \,|\, i \in N \wedge 1 \leq i \leq n$ is not indirectly dominated.*

So cycles in our testing for farsightedly stable coalitions can not be formed, but every coalition structure, which is a part of cycle can be found, as it is not indirectly dominated, and so it is farsightedly stable.
In order to compute farsighted stable coalitions, in the beginning we have to assume that *only one coalition is formed.* There are three ways that coalition can change when one coalition is formed; coalition get smaller, get bigger or some members leave coalition and some other join it. When a coalition get smaller, this is the case for internal indirect domination; when a coalition get bigger, this is the case of external domination; when some members leave coalition and some other join it, this is the case of subcoalition domination. In order to find the farsightedly stable coalitions *all types of indirect domination (internal, external and subcoalition) are considered as combinatorial process.* The definition of internal indirect domination is introduced below.

Definition 2.6 *a_1 is internally indirectly dominated by a_m, or $a_1 \ll a_m$, if there exists $a_1, a_2, a_3, ..., a_m$ and $C_1, C_2, C_3, ..., C_{m-1}$ where $C_1 \supset C_2 \supset C_3, ..., C_{m-2} \supset C_{m-1}$ and $a_i \rightarrow_{C_i} a_{i+1}$, and $a_j \prec_{C_j} a_m$ where $i, j = 1, 2, 3, ..., m-1$.*

If a coalition gets smaller, and its remaining members prefer the final coalition compare to the initial one, we say that *an internal indirect domination* is possible.

Definition 2.7 *a_1 is externally indirectly dominated by a_m, or $a_1 \ll a_m$, if there exists $a_1, a_2, a_3, ..., a_m$ and $C_1, C_2, C_3, ..., C_{m-1}$ where $C_1 \subset C_2 \subset C_3, ..., C_{m-2} \subset C_{m-1}$ and $a_i \rightarrow_{C_i} a_{i+1}$, and $a_j \prec_{C_j} a_m$ where $i, j = 1, 2, 3, ..., m-1$.*

If a coalition gets bigger, and its remaining members prefer the final coalition compare to the initial one, we say that *an external indirect domination* is possible.

Definition 2.8 *a_1 is a sub-coalition indirectly dominated by a_m, or $a_1 \ll a_m$, if there exists $a_1, a_2, a_3, ..., a_m$ and $C_1, C_2, C_3, ..., C_{m-1}$ where $C_k \cap C_l \neq \emptyset$ where $k, l = 1, 2, 3, ..., m-1$ and $a_i \rightarrow_{C_i} a_{i+1}$, and $a_j \prec_{C_j} a_m$ where $i, j = 1, 2, 3, ..., m-1$.*

The indirect sub-coalition domination occurs when a number of old coalition members leave and a number of new members join the initial coalition. The new coalition may be larger or smaller than the original one. However, if a part of old coalition members (a sub-coalition), and the new coalition members form a coalition, and prefer it compared to the initial coalition, we say that *a sub-coalition indirect inducement* is possible.

Definition 2.9 *If a coalitions structure a_1 is not external $\vee$ internal $\vee$ sub-coalition indirectly dominated then coalition a_1 is respectively external $\vee$ internal $\vee$ sub-coalition farsightedly stable.*

Definition 2.10 *If a coalitions structure a_1 is not external $\wedge$ internal $\wedge$ sub-coalition indirectly dominated then coalition structure a_1 is farsightedly stable.*

If we can transform the preference relations to pay off comparison then it can be easily checked *by a combinatorial algorithm* if a coalition is internally, externally or sub-coalitional indirectly dominated. In order to check to be able to compare the payoff of coalition members we need to introduce the definition of partition function. Let recall that $N = \{1, ..., n\}$ is the set of players, and nonempty subsets of N are called coalitions. A partition (or coalition structure) a is a set of disjoint coalitions, $a = \{P_1, P_2, ..., P_k\}$, so that their union is N; the set of all partitions is $\mathcal{P}$, and the set of partitions of a coalition C of N (it means of all partitions where coalition C is part of them) is $\mathcal{P}(C)$.

Definition 2.11 *The partition function is a mapping $V(C, \mathcal{P}) : (C, \mathcal{P}) \mapsto \Re$ where $C \in \mathcal{P}$, that assigns a value to each coalition in every partition.*

Definition 2.12 *The per-member partition function is a mapping $v(i)_{i \in C}(C, \mathcal{P}) : (C, \mathcal{P}) \mapsto \Re$ where $C \in \mathcal{P}$, that assigns a payoff value $v(i)_{i \in C}(C, \mathcal{P})$ to every member of each coalition in every partition.*

The per-member partition function $v(i)_{i \in C}(C, \mathcal{P})$ gives a payoff value $v(i)_{i \in C}(C, \mathcal{P})$ (for now and on, I will write simply $v(i)_{i \in C}$) to every coalition member). This helps us to transform the preference relation to a comparison of coalition member payoffs.

Definition 2.13 *A coalition C prefers coalition structure a_j in comparison to a_m (or $a_j \prec_C a_m$) if and only if:*

- $v(i)_{i \in C}^{a_j} > v(i)_{i \in C}^{a_m}$
- *where $v(i)_{i \in C}^{a_j}$, $v(i)_{i \in C}^{a_m}$ are the per-member partition functions of a member i of coalition C with coalition structure a_j and a_m respectively.*

By comparing the payoff's of coalition members between different coalition structures, it can be easily checked *by a combinatorial algorithm* if a coalition is internally, externally or sub-coalitional indirectly dominated.

3 Computational aspects of finding single farsighted stable coalition

In this section, we clarify the proceeding of computing *single* farsighted stable coalition.
Let firstly, define the full-noncooperative behavior, which is necessary to define the profitable coalition, which are crucial for calculation of farsighted stable coalitions.

Definition 3.1 *The situation in which each country maximizes its own profit, and the maximum coalition size is unity is referred to as the full-noncooperative structure (or a_{FNS}).*

It is a standard Nash equilibrium. A coalition that performs better than the the full-noncooperative structure is *a profitable coalition*. Only profitable coalitions are tested, which is sufficient to find all single farsightedly stable coalitions (see Observation 3.2). The definition of a profitable coalition is introduced below:

Definition 3.2 *A coalition C in coalition structure a_c is profitable (or individual rational) if and only if it satisfies the following condition:*

- $v(i)_{i \in C}^{a_c} > v(i)_{i \in C}^{a_{FNS}}$
- *$v(i)_{i \in C}^{a_c}$, $v(i)_{i \in C}^{a_{FNS}}$ are per-member partition functions of a player i of coalition C with coalition structure a_c and a_{FNS} respectively.*

Considering only profitable coalitions also reduces the computational effort required to find farsightedly stable coalitions. Profitability condition requests more than superadditive property, which requires that two different coalitions should generate more profits (or welfare) by joining forces as by remaining separate; see definition 4.1.

Observation 3.1 *If there are no profitable coalitions than the only farsighted stable coalitions are the coalitions with unity size that are formed in the full-noncooperative structure.*

Proof: As it is simple, we omit it.

The observation makes clear that there is a close connection between profitability condition and farsighted stability. As at the moment the only single coalitions are tested, in stead of talking on coalition structure, we talk only on coalitions. Finding profitable farsighted stable coalitions is computational challenging, but a straightforward job. One find all profitable coalitions, and begin to test one by one if they are externally, internally or sub-coalitional indirectly dominated from other coalitions. The profitable coalitions, which are not indirectly dominated, are farsightedly stable. But can we find the non-profitable farsightedly stable coalitions, if there is any? It is crucial to note that, in real world problems with asymmetric countries, one expects to have far more non-profitable coalitions than profitable ones. It implies that the question has a an important computational aspect.
In order to answer those questions, we need first to define *the positive, negative and neutral spillover property.*

Definition 3.3 *If a game for any two coalitions $C_1 \subset N$ and $C_2 \subset N$ such that $C_1 \neq C_2$ satisfy:*

- $\forall\, k \notin C_1 \cup C_2 \quad v(k)^{C_1 \cup C_2} > v(k)^{C_1} \wedge v(k)^{C_1 \cup C_2} > v(k)^{C_2}$, *we say the game exhibits positive spillover property*
- $\forall\, k \notin C_1 \cup C_2 \quad v(k)^{C_1 \cup C_2} < v(k)^{C_1} \wedge v(k)^{C_1 \cup C_2} < v(k)^{C_2}$, *we say the game exhibits negative spillover property*
- $\forall\, k \notin C_1 \cup C_2 \quad v(k)^{C_1 \cup C_2} = v(k)^{C_1} \wedge v(k)^{C_1 \cup C_2} = v(k)^{C_2}$, *we say the game exhibits neutral spillover property*

Clearly if positive spillover property is not satisfied, then it does not mean the negative spillover property is satisfied. Usually one can assume that some player satisfy the positive spillover property, some others the negative or neutral spillover property.
It is reasonable to take as *starting point*, for testing coalitions if they are farsightedly stable, *the full-noncooperative structure.*
Our combinatorial proceeding realizes that all possible coalitions, which can our initial coalition (let say C_l) dominate, are considered. Clearly all possible coalitions, which can dominate our coalition C_l can be divided in three categories (C_1, C_2, C_3):

- $C_1 \subset C_l$ which are checked when internal indirect domination is examined
- $C_2 \supset C_l$ which are tested when external indirect domination is investigated
- $C_3 \cap C_l \neq \emptyset$ which are inspected when sub-coalition indirect domination is considered

As a consequence, we know if there exists a coalition which dominates our coalition C_n.
The algorithms of Table (1) and Table (2) in Appendix 6 fully describe the procedure of finding farsightedly stable coalitions. As this is a huge combinatorial effort, we often modify the algorithms in order to decrease our computational cost. In order to have an idea how much computational efforts are necessary to find farsightedly stable coalition structures with two coalitions. Let test coalition structure $b_1 = (C_1, l_1)$ which have one coalitions C_1 with i_1, and l_1 single players. We have all together m players, where $m = i_1 + l_1$. For checking the coalition C_1 if it is internally, externally and subcoalition stable, one needs to check as many coalitions as equation (3) is showing. If we take:

$$Intern_1 = \underbrace{(C_{i_1}^2 + .. + C_{i_1}^{i_1})}_{internal\ stability} = 2^{i_1} - i_1 - 1 \qquad Extern_1 = \underbrace{(C_{l_1}^1 + .. + C_{l_1}^{l_1})}_{external\ stability} = 2^{l_1} \tag{1}$$

$$Subcoal_1 = \underbrace{\sum_{t=1:(i_1-1)} C_{i_1}^t (C_{m-t}^1 + .. + C_{m-t}^{m-t})}_{subcoalition\ stability} \tag{2}$$

$$Intern_1 + Extern_1 + Subcoal_1 = 2^{i_1} - i_1 - 1 + 2^{l_1} + \sum_{t=1:(i_1-1)} C_{i_1}^t 2^{m-t} \tag{3}$$

Now we are able to state a very useful observation which makes sure that we are able to find all farsightedly stable coalitions (profitable or non-profitable) even we use as a starting point only profitable coalitions. This is a specially important to computational point of view, as in games with asymmetric players, there are far more non-profitable coalitions than profitable ones.

Observation 3.2 *If a non-profitable coalition C_n is farsighted stable if:*

1. *the positive spillover or neutral spillover property is not satisfied*
2. *$\exists$ $C_1 \subset C_m$, and C_1 is profitable; $\exists$ C_2 $\mid$ $C_2 \cap C_m \neq \emptyset$ where C_2 is profitable, and C_2 is directly or indirectly dominated from C_m, and C_m is not directly or indirectly dominated from any coalition..*

Proof, First Statement:

First direction:
If a non-profitable coalition C_m is farsightedly stable then, the positive and neutral spillover property is not satisfied.
Suppose that, there is a non-profitable farsightedly stable coalition C_n, and positive spillover property is satisfied. As C_n is not-profitable then:

$$\exists \ aplayer \ l \in C_n | v(k)^{C_n} < v(k)^{a_{FNS}} \tag{4}$$

Suppose that the player l leaves the coalition C_n and becomes a single player, then as the positive spillover property is satisfied we have:

$$v(k)^{C_n} > v(k)^{a_{FNS}} \tag{5}$$

The equation 5 is contradicting the equation 4, which proves (by contradiction) that if we have a farsightedly stable non-profitable coalition, then the positive spillover property is not satisfied.
Proof, Second direction: The prove for the second direction (and neutral spillover property) is similar with the above one, so we omit it.

Proof, Second Statement:

First direction: *If a non-profitable coalition C_m is farsightedly stable* then, there is a profitable subcoalition $C_1 \subset C_m$. Besides there is C_2 such as $C_2 \cap C \neq \emptyset$, C_2 is directly or indirectly dominated from C_m, and moreover C_m is not directly or indirectly dominated from any coalition.
Part 1, First direction :
Suppose that every sub-coalition of any non-profitable farsightedly stable (FS) coalition C_m has a country that receives a lower payoff than in the full-noncooperative behavior, then the coalition C_m is not FS. The coalition is not FS because it is possible to build an effective relation to dissolve the coalition:

$$C_m \to C_{m-1}, \ \dots, C_1 \to a_{FNS} \ as \ \forall \ \ C_l \quad 1 \leq l \leq m \quad \exists \ a \ country \ i \in C_l \ where \ v(i)^{C_l} > v(i)^{a_{FNS}} \tag{6}$$

The dissolving process is simple. Every country with lower profit than in the full-noncooperative structure leaves the coalition. As every sub-coalition has one such country, the coalition is not FS. As a consequence coalition C_1 must have a profitable sub-coalition in order to have a chance of being FS, which completes the first part of the proof.
Part 1, Second Direction: The proof of second direction is similar to the first direction, so we omit it.

The Proof, Part 2: Let suppose that we have the chain of effective relations, where C_m is a non-profitable farsightedly stable coalition:

$$a_{FNS} \to C_1, \ \dots, C_{m-1} \to C_m \tag{7}$$

As we will only focus on "effective relation" that leads to indirect domination, so a non-profitable coalition does not indirectly dominate the full non-cooperative structure. It implies that in the chain on the effective relation (7) $\exists$ C_i which is profitable $i \in N, 1 \leq i \leq m$.

Corollary 3.1 *If the positive or neutral spillover property is satisfied for all players then all farsighted stable coalitions are profitable.*

Proof: This is a corollary of Observation 3.2.

3.1 Dynamic Farsighted Coalition Structure Set

In the spirit of Chwe (1994), we characterize the set of all farsightedly stable coalitions $Coal_{fs}$ as Dynamic Farsighted Coalition Structure Set (DFCS).

Definition 3.4 *A set $\mathcal{S}$ is the Dynamic Farsighted Coalition Structure Set (DFCS) if and only if:*

- *if a coalition structure $\boldsymbol{a} \in \mathcal{S}$ then $\boldsymbol{a}$ is farsightedly stable*
- *$\forall$ coalition structure $\boldsymbol{b} \notin \mathcal{S}$ $\exists$ a coalition structure $\boldsymbol{c} \in \mathcal{S}$ $\mid$ $\boldsymbol{b} \ll \boldsymbol{c}$*

The definition firstly, indicates that any coalition structure a that belongs to DFCS is farsightedly stable. Secondly, if a coalition structure b does not belong to DFCS is not farsightedly stable. Furthermore, there exits another coalition structure $c \in DLCS$, which dominates $\mathbf{b}$ indirectly.

4 Multiple farsighted stable coalitions

When multiple coalitions are formed one has to consider two different kind of interaction between coalition and single players, and one can again talk for indirect internal, external and subcoalition domination. The second one is when coalition exchange players between each-other, which can possibly lead to another coalition structure that can dominate the initial one. During the inspection of player exchange among coalitions *we keep the number of single players, and number of coalitions fixed.*
If there exists farsightedly coalition structures $a_2 = (C_1^{'}, C_2^{'}...C_t^{'}, l_1)$, which dominate the initial coalition structure $a_1 = (C_1, C_2...C_t, l_1)$ where $C_i \cap C_i^{'} \neq \emptyset$, algorithms presented in Table 3 is able to finds it (and then coalition structure a_2 has to be tested, if there is another coalition structure a_3, which dominates it). If there is no coalition structure which dominates our initial coalition structure $a_1 = (C_1, C_2...C_t, l_1)$, algorithm will give the answer that our initial coalition structure a_1 can not be dominated, and consequently is farsightedly stable.
The algorithm is computationally very expensive, and we do not advice using it for any arbitrary coalition structures. But similar to Zermelo (1913)'s model for solving chess, we like to stress that checking for arbitrary farsighted stable coalition structure is computationally very expensive but finite.
In order to have an idea how much computational effort are necessary to find farsightedly stable coalition structures with three coalitions. Let test coalition structure $b_3 = (C_1, C_2, C_3, l_1)$. We have three coalitions C_1,C_2 and C_3 with respectfully i_1,i_2 and i_3 members, and l_1 single players. We have all together m players, where $m = i_1 + i_2 + i_3 + l_1$, and $n = i_1 + i_2 + i_3$ are all members of coalitions; let have also $n_1 = i_1 + l_1$, $n_2 = i_2 + l_1$ and $n_2 = i_2 + l_1$. For checking three coalitions if they are internally, externally and subcoalition checks one needs to check (without allowing exchange of members between coalition structures) as many coalitions as equation (11) is showing.
For checking a coalition for internal stability, one has to check all sub-coalitions of a coalition with two members or more, which are given by the expression $2^{i_2} + 2^{i_3}$ for three coalitions together. But excluding single member coalitions and the coalition itself which are equal to $i_1 + i_2 + i_3 + 3 = n + 3$ for three coalitions together.

$$Intern_3 = \underbrace{(C_{i_1-1}^{2} + .. + C_{i_1}^{i_1-1}) + (C_{i_2}^{2} + .. + C_{i_2}^{i_2-1}) + (C_{i_3}^{2} + .. + C_{i_3}^{i_3-1})}_{internal\ stability\ C_1,\ C_2\ and\ C_3} = 2^{i_1} + 2^{i_2} + 2^{i_3} - n - 3 \quad (8)$$

For checking a coalition for external stability, one has to add to the coalition all coalitions which are formed by single players (we have l_1 single players), which is given by the expression $Extern_2$ for three coalitions together.

$$Extern_3 = \underbrace{3(C_{l_1}^{1} + .. + C_{l_1}^{l_1})}_{external\ stability\ C_1,\ C_2\ and\ C_3} = 3(2^{l_1}) \quad (9)$$

For checking a coalition for sub-coalition stability, one has to check all proper sub-coalitions of our coalitions (which are equal to $\sum_{t=1:(i_1-1)} C_{i_1}^{t}$ for first coalition only) for being external farsighted stable (which is equal to $C_{n_1-t}^{1} + .. + C_{n_1-t}^{n_1-t}$ for a sub-coalition with t members) $Subcoal_3$.

$$Subcoal_3 = \sum_{t=1:(i_1-1)} C_{i_1}^{t}(C_{n_1-t}^{1} + .. + C_{n_1-t}^{n_1-t}) + \sum_{t=1:(i_2-1)} C_{i_2}^{t}(C_{n_2-t}^{1} + .. + C_{n_2-t}^{n_2-t}) +$$
$$+ \sum_{t=1:(i_3-1)} C_{i_1}^{t}(C_{n_3-t}^{1} + .. + C_{n_3-t}^{n_3-t}) = \sum_{t=1:(i_1-1)} C_{i_1}^{t} 2^{n_1-t} + \sum_{t=1:(i_2-1)} C_{i_2}^{t} 2^{n_2-t} + \sum_{t=1:(i_3-1)} C_{i_3}^{t} 2^{n_3-t} \quad (10)$$

$$Intern_3 + Extern_3 + Subcoal_3 =$$
$$= \quad 2^{i_1} + 2^{i_2} + 2^{i_3} - n - 3 + 3(2^{l_1}) + \sum_{t=1:(i_1-1)} C^t_{i_1} 2^{n_1-t} + \sum_{t=1:(i_2-1)} C^t_{i_2} 2^{n_2-t} + \sum_{t=1:(i_3-1)} C^t_{i_3} 2^{n_3-t} \quad (11)$$

If we take account the exchange of members among three coalitions (let have $i_{12} = i_1 + i_2$, $i_{13} = i_1 + i_3$ and $i_{23} = i_2 + i_3$) there are necessary to check as many as (12) is showing. Let explain the structure of the equation (12):

- in order to compute all triples sub-coalitions that can result from exchange members among coalitions C_1, C_2 and C_3, we compute first all single coalitions (which have t members, where t = 2, 3, 4 ... (n - 4)) that can be formed by $i_1 + i_2 + i_3 = n$ members of all three coalitions, which is equal to $\sum_{t=2:(n-4)} C^t_{n-4}$; the single coalitions can have at most $(n-4)$ members, so there are always left at least 4 members to form two coalitions with at least two members, and we have a triple of coalitions
- there are still to compute every couple of coalitions that can be formed by $(n-t)$ remaining members, which is equal to $\sum_{t=2:(n-4)}(C^2_{n-t} + ... + C^{n-t}_{n-t})$.

$$Exchange_3 = \sum_{t=2:(n-4)} C^t_{n-4}(C^2_{n-t} + ... + C^{n-t}_{n-t}) \quad (12)$$

The total number of check we need to perform is equal to sum of equations 11 and 12; the number is increasing exponentially with respect to the number in coalitions, and single players. This means that it is computationally very expensive to check if a coalition structure is farsightedly stable. By another side, it will also not make sense to inspect them. The last point I have to mention is that the characteristics of the concrete problem may help you so much (like focal point suggested by Thomas Schelling (Schelling, 1960)), that you will be able to find the farsighted coalition structure with a modest computational effort. This proceeding is generalized for finding farsighted stable coalition structures with t coalitions and l_p single players, in the Appendix 6

Example

The example is taken from (Osmani and Tol, 2009), where in *climate game* we use the farsighted stability concept and calculate all farsighted stable coalitions. We use the per member partition function (or simple profit function) of player i (or country i) is taken Climate Framework for Uncertainty, Negotiation and Distribution (FUND) model developed by Richard Tol (Tol, 1999a,b, 2001, 2002):

$$\pi_i = B_i - C_i = \beta_i \sum_j^n R_j E_j - \alpha_i R_i^2 Y_i \quad (13)$$

The benefit function B_i is approximated as:

$$B_i = \beta_i \sum_j^n R_j E_j \quad (14)$$

β the marginal damage costs of carbon dioxide emissions and E unabated emissions. Table (9) gives the parameters of Equations (13), (14) and (15) as estimated by FUND.
Specifically, the abatement cost function C_i is represented as:

$$C_i = \alpha_i R_i^2 Y_i \quad (15)$$

where C denotes abatement cost, R relative emission reduction, Y gross domestic product, indexes i denotes regions and α is the cost parameter.
The second derivative of $d^2\pi_i/dR_i^2 = -2\alpha_i < 0$ as $\alpha_i > 0$. It follows that the profit function of every country i is strictly concave, and as a consequence has a unique maximum. Hence, the non-cooperative optimal emission reduction is found from first order optimal condition:

$$d\pi_i/dR_i = \beta_i E_i - 2\alpha_i R_i Y_i = 0 \Rightarrow R_i = \beta_i E_i/(2\alpha_i Y_i) \quad (16)$$

If a region i is in a coalition with a region j, the optimal emission reduction is given by:

$$d\pi_{i+j}/dR_i = 0 \Rightarrow E_i(\beta_i + \beta_j) - 2\alpha_i R_i Y_i = 0 \Rightarrow R_i = (\beta_i + \beta_j)E_i/(2\alpha_i Y_i) \tag{17}$$

Thus, the price for entering a coalition is higher emission abatement at home. The return is that the coalition partners also raise their abatement efforts.
Note that our welfare functions are orthogonal. This indicates that the emissions change of a country do not affect the marginal benefits of other countries (that is the independence assumption). In our game, countries outside the coalition benefit from the reduction in emissions achieved by the cooperating countries, but they cannot affect the benefits derived by the members of the coalition.
The game satisfies the superadditivity property:

Definition 4.1 *A game is superadditive if for any two coalitions,* $C_1 \subset N_{16}$ *and* $C_2 \subset N_{16}$:
$\upsilon(C_1 \cup C_2) > \upsilon(C_1) + \upsilon(C_2) \quad C_1 \cap C_2 = \emptyset$.

The *superadditivity property* means that if C_1 and C_2 are disjoint coalitions (here C_1 and C_2 can be single players too). Clearly, they should accomplish at least as much by joining forces as by remaining separate (where N_{16} is the set of sixteen players). However, the game *very frequently (but not always)* exhibits *positive spillovers*. The positive spillover property is usually satisfied except for some coalitions that contain as members Japan & South Korea or Australia & New Zealand, which have negative marginal benefits (negative β's) from pollution abatement.

Computational results

Finding all profitable coalitions needs a simple algorithm, although the computational effort is considerable. One finds all coalitions and checks if all their members have higher profit in comparison to the atom structure. The numerical results yield fifteen profitable two-member coalitions. As there are many profitable coalitions, we have numbered them: for instance $2-13$, 2 means that coalition has 2 countries, and 13 means that it is the 13-th in the list of two-member profitable coalitions. The profitable two-member coalitions are:

$(2-1)$	(USA, CHI)	$(2-2)$	(USA, NAF)
$(2-3)$	(USA, SSA)	$(2-4)$	(CAN, SAS)
$(2-5)$	(ANZ, EEU)	$(2-6)$	(ANZ, CAM)
$(2-7)$	(ANZ, SAS)	$(2-8)$	(ANZ, SIS)
$(2-9)$	(EEU, CAM)	$(2-10)$	(EEU, SIS)
$(2-11)$	(FSU, LAM)	$(2-12)$	(CAM, SIS)
$(2-13)$	(CHI, NAF)	$(2-14)$	(CHI, SSA)
$(2-15)$	(NAF, SSA)		

The profitable three-member coalitions are introduced below (the superscript "fs" denotes farsightedly stable):

$(3-1)$	(USA, LAM, CHI)	$(3-2)$	(USA, SEA, CHI)
$(3-3)$	$(USA, CHI, NAF)^{fs}$	$(3-4)$	$(USA, CHI, SSA)^{fs}$
$(3-5)$	(USA, NAF, SSA)	$(3-6)$	$(CAN, EEU, SAS)^{fs}$
$(3-7)$	$(CAN, FSU, LAM)^{fs}$	$(3-8)$	$(CAN, CAM, SAS)^{fs}$
$(3-9)$	(CAN, CAM, SIS)	$(3-10)$	$(CAN, SAS, SIS)^{fs}$
$(3-11)$	(JPK, NAF, SSA)	$(3-12)$	$(EEU, CAM, SAS)^{fs}$
$(3-13)$	(EEU, CAM, SIS)	$(3-14)$	$(EEU, SAS, SIS)^{fs}$
$(3-15)$	$(CAM, SAS, SIS)^{fs}$	$(3-16)$	$(CHI, NAF, SSA)^{fs}$

The profitable four-member coalitions are:

$(4-1)$	(USA, LAM, SEA, CHI)	$(4-2)$	(USA, LAM, SEA, SSA)
$(4-3)$	$(USA, LAM, CHI, NAF)^{fs}$	$(4-4)$	$(USA, LAM, CHI, SSA)^{fs}$
$(4-5)$	$(USA, SEA, CHI, NAF)^{fs}$	$(4-6)$	$(USA, SEA, CHI, SSA)^{fs}$

$(4-7)$	$(USA, CHI, NAF, SSA)^{fs}$	$(4-8)$	$(CAN, EEU, CAM, SAS)^{fs}$
$(4-9)$	(CAN, EEU, CAM, SIS)	$(4-10)$	$(CAN, EEU, SAS, SIS)^{fs}$
$(4-11)$	$(CAN, CAM, SAS, SIS)^{fs}$	$(4-12)$	$(EEU, CAM, SAS, SIS)^{fs}$
$(4-13)$	(LAM, SEA, CHI, NAF)	$(4-14)$	(LAM, SEA, CHI, SSA)
$(4-15)$	$(SEA, CHI, NAF, SSA)^{fs}$		

The profitable five-member coalitions are presented below:

$(5-1)$	$(USA, LAM, SEA, CHI, NAF)^{fs}$	$(5-2)$	$(USA, LAM, SEA, CHI, SSA)^{fs}$
$(5-3)$	$(USA, LAM, SEA, NAF, SSA)^{fs}$	$(5-4)$	$(USA, LAM, CHI, NAF, SSA)^{fs}$
$(5-5)$	$(USA, SEA, CHI, NAF, SSA)^{fs}$	$(5-6)$	$(CAN, JPK, LAM, SAS, SSA)$
$(5-7)$	$(CAN, EEU, CAM, SAS, SIS)^{fs}$	$(5-8)$	$(LAM, SEA, CHI, NAF, SSA)^{fs}$

There is only one 1 six-member and only one 1 seven-member profitable coalition:

$(6-1)$ $(USA, LAM, SEA, CHI, NAF, SSA)^{fs}$
$(7-1)$ $(CAN, JPK, EEU, CAM, LAM, NAF, SIS)$

The computation is extended *to the multiple farsightedly stable coalitions* by considering coalitions $(6-1)$ and $(5-7)$ simultaneously.

$(6-1)^{fs}$ $(USA, LAM, SEA, CHI, NAF, SSA)$,
$(5-7)^{fs}$ $(CAN, EEU, CAM, SAS, SIS)$

Note that the costs of emission reduction of a region are independent of the abatement of other regions and the benefits are linear. As a consequence *in case of multiple coalitions* the changes in the pay-off of each region is independent of the behavior of other regions provided that *the two coalitions do not exchange members*. It follows that our coalitions are farsightedly stable if there is no direct or indirect domination, which results in switching members between two coalitions. This has been numerically verified. Thus we conclude that our coalitions are farsightedly stable. Therefore, we have two farsightedly stable coalitions that coexist, which are $(5-7)$ $(CAN, EEU, CAM, SAS, SIS)$, and $(6-1)$ $(USA, LAM, SEA, CHI, NAF, SSA)$.

5 Conclusions

Farsighted stability is investigated while attention is payed on the computational framework of finding farsightedly stable coalition structures. We investigate what coalition structures are stable, which implies that they cannot be replaced by any coalition of rational, farsighted and selfish countries. The selfishness of players shapes the aspects of *non-cooperative approach*. The idea of farsightedness means that one should check for multi-step stability by comparing the profits of a coalition member after a series of deviations has come to an end. The deviation is possible only if players display *cooperate attitude (by forming a coalition)* to each-other in order to increase their welfare.

Different from Chwe (1994), who firstly introduce the coalitional farsighted stability and presented a powerful theorem which proves the existence of it, we are more interested in applying aspects of farsighted stability in real world problems like climate change games, etc. where many asymmetric players interact. As there are a lot of asymmetric players, there are far more non-profitable coalitions compared to profitable (or individual rational) coalitions. It indicates that non-profitable farsighted stable coalitions are harder to compute.

We show the direct relation of farsighted stability to profitability condition and supperadditivity property. We point out that there is a connection between positive spillover property and profitable farsighted stable coalitions. We prove also one that it is feasible to find all non-profitable farsighted stable coalitions starting from profitable coalitions. Algorithms are developed, which can find all farsighted stable coalition structures. The cost of computation is high. It implies that it is more advisable to find a part of farsighted stable coalition structures but not all of them. Other conceptual tools like focal points of Schelling (1960) can be much more helpful than direct computational strategy.

6 Appendix

6.1 Simple algorithms for finding single farsightedly-stable coalitions

Here the algorithms for finding farsightedly stable coalitions are presented. The first algorithm is described in Table (1). It tests a coalition *for external farsighted stability* (EFS, see definition (2.9)). Suppose we would like to check coalition $C_n \equiv (1, 2 \ldots n)$ for EFS.

Table 1: Algorithm for finding externally farsightedly stable (EFS) coalitions in coalitions structures when one coalitions is formed

Suppose that we have a coalition structure with one coalition $C_n \equiv (1, 2 \ldots n)$, with all together **m** players, where $m \geq n$.
Calculate $\pi_1(1)..\pi_1(n)..\pi_1(m)$, the per-member partition function of this coalition structure.
for i=(n+1) to m
(the loop does not contain countries *1,2* ... and n)
Find all coalitions with i-elements $(n+1) \geq i \leq m$ where n elements are always $C_n \equiv (1, 2 \ldots n)$.
take a coalition with i countries
Calculate $\pi_2(1)..\pi_2(n)..\pi_2(m)$, the profit of this coalition structure.
if $[\pi_2(1) > \pi_1(1) \wedge \pi_2(2) > \pi_1(2) \wedge \ \ldots \ \wedge \pi_2(i) > \pi_1(i)]$ (main condition)
the coalition $C_n \equiv (1, 2 \ldots n)$ is not externally farsightedly stable (EFS).
endif
end
If the main condition is never satisfied for i=(n+1) to m, then
the coalition $C_n \equiv (1, 2 \ldots n)$ is externally farsightedly stable.

If the algorithm of Table (1) finds a $\boldsymbol{i}$ member-coalition C_i for which *the main condition* $[\pi_2(1) > \pi_1(1) \wedge \pi_2(2) > \pi_1(2) \wedge \ \ldots \ \wedge \pi_2(i-1) > \pi_1(i-1) \wedge \pi_2(i) > \pi_1(i)]$ holds, then our initial coalition C_n is not externally farsightedly stable. *But if the main condition is never satisfied (for i=(n+1) to m), then we say that no external inducement is possible.* If no external inducement is possible, then the coalition (C_n in our case) is *externally farsightedly stable (EFS).*
The algorithm for internal farsightedly stability will be presented below. Suppose again, that we have a coalition with $\boldsymbol{n}$ countries $C_n \equiv (1, 2 \ldots n)$. We need to find every sub-coalition (of 2 countries, 3 countries ... and (n-1) countries). We name by $\pi^{'}(1)...\pi^{'}(m)$ the profits when the sub-coalitions are formed, and $\pi(1)...\pi(i1)...\pi(i5)...\pi(16)$ the profits when only the n member coalition C_n is formed. Then if *the condition below* is satisfied (for any sub-coalition of C_n with i members where i=2 to (n-1)): $[\pi^{'}(1) > \pi(1) \wedge \pi^{'}(2) > \pi(2) \wedge \pi^{'}(i-1) > \pi(i-1) \wedge \pi^{'}(i) > \pi(i)]$, we say that an internal inducement is possible. If an internal inducement is possible than the coalition *is not internally farsightedly stable* (IFS, see definition (2.9)). All steps in this algorithm are presented in Table (2).

Testing *sub-coalition stability* of coalition C is similar to testing every *sub-coalition of coalition C for external farsighted stability.*

6.2 Numerical results for three-member coalitions

In this subsection, we present a small part of the numerical computations which test and find the three-member coalitions that are not farsightedly stable. Firstly, we note that all profitable coalitions are internally farsightedly stable (including the three-member coalitions). Three-member coalitions which are not externally farsightedly stable are presented in Tables 4, 5 and 6. The first column of Table 4 presents the members of five-member final coalitions. The three countries of the coalition that are inspected are labeled in bold letters, while the members who join the initial coalition are in normal typeface. The second column of Table 4, Pr_3, displays the profits (in billions of dollars) of the final coalition members when only the three-member coalition exists, while the third column Pr_5 shows

Table 2: Algorithm for finding internally farsightedly stable (EFS) coalitions in coalitions structures when one coalitions is formed

Suppose that we have a coalition structure with one coalition $C_n \equiv (1, 2 \ldots n)$, with all together **m** players, where $m \geq n$.
Calculate $\pi(1)..\pi(i1)..\pi(i5)..\pi(n)$ the per-member partition function of this coalition structure.
for i=2 to (n-1) (find all sub-coalition with i elements from coalition $C_n \equiv (1, 2 \ldots n)$)
Take a sub-coalition with i elements.
Calculate $\pi^{'}(1)..\pi^{'}(i1)..\pi^{'}(i5)..\pi^{'}(n)$ of this coalition structure.
if $[\pi^{'}(1) > \pi(1) \wedge \pi^{'}(2) > \pi(2) \wedge \pi^{'}(i-1) > \pi(i-1) \wedge \pi^{'}(i) > \pi(i)]$ (main condition)
the coalition $C_n \equiv (1, 2 \ldots n)$ is not an internally farsightedly stable coalition
endif
end
If the main condition is never satisfied for i=2 to (n-1), then
the coalition $C_n \equiv (1, 2 \ldots n)$ is internally farsightedly stable.

the profits of final coalition members when only the five-member coalition exists. The profits of each country are higher when the five-member coalition is formed (Pr_5) in comparison to the profits when the three-member coalition is formed (Pr_3). As a result, the three member coalition (USA,LAM,CHI) is not externally farsightedly stable. Columns four, five and six of Table 4 are similar to columns one, two and three,and Tables 5 and 6 are similar to Table 4. Tables 7 and 8 introduce the three-member coalitions which are not sub-coalition farsightedly stable. In the first column, the country members who change their position (join or leave the initial coalition) are placed. The three countries of a primary coalition which is inspected are labeled in bold letters, while the three members of the final coalition are marked with an asterisk in the top-right. It is clear that countries in bold letters and have an asterisk in top-right are simultaneously members of a primary and of the final coalition. The second column of Table 4, Pr_{3old} presents the profits of final coalition members when only the primary three-member coalition is formed, while the third column of Table 4, Pr_{3new} introduces the profits when the final three-member coalition is built. The profits of members of final three-member coalition (*with an asterisk in the top-right*) are greater when the final three-member coalition is formed Pr_{3new} compared to the primary three-member coalition Pr_{3old}. It follows that the three member coalition (JPK,NAF,SSA) is not sub-coalition farsightedly stable. Finally, Table 8 is similar to Table 7.

6.3 Finding multiple farsightedly stable coalition structure

If there exists farsightedly coalition structures $a_2 = (C_1^{'}, C_2^{'} \ldots C_t^{'}, l_1)$ (with t coalitions, and l_1 single players), which dominate the initial coalition structure $a_1 = (C_1, C_2 \ldots C_t, l_1)$ where $C_i \cap C_i^{'} \neq \emptyset$, the algorithm presented in Table 3 can find it (and then coalition structure a_2 has to be tested, if there is another coalition structure a_3, which dominates it). If there is no coalition structure, which dominates our initial coalition structure $a_1 = (C_1, C_2 \ldots C_t, l_1)$, algorithm will give the answer that our initial coalition structure a_1 cannot be dominated, and consequently, is farsightedly stable.
It is crucial to mention that the algorithm is computationally very expensive, and we do not advise using it for any arbitrary coalition structures. However, similar to Zermelo (1913)'s model for solving chess, we like to stress that checking for arbitrary farsighted stable coalition structure is computationally very expensive but finite.
Let generalize our proceeding for a coalition structure $b_p = (C_1, C_2 \ldots C_p, l_p)$. We have p coalitions $C_1, C_2..C_p$ with respectfully $i_1, i_2 \ldots i_p$ members, and l_p number of single players. We have all together m players, where $m = i_1 + i_2 + \ldots i_p + l_p$; let have also $n_1 = i_1 + l_p, n_2 = i_2 + l_1 \ldots n_p = i_p + l_p$. For checking p coalitions if they are internally, externally and subcoalition checks one needs to check (without allowing exchange of members among coalitions) as many coalitions as equation (21) is showing. If we take:

$$Intern_p = \underbrace{(C_{i_1}^2 + \ldots + C_{i_1}^{i_1-1}) + (C_{i_2}^2 + \ldots + C_{i_2}^{i_2-1}) + \ldots . + (C_{i_p}^2 + \ldots + C_{i_p}^{i_p-1})}_{internal\ stability\ C_1,\ C_2\ \ldots\ C_p} = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_p} - n - p \quad (18)$$

$$Extern_p = \underbrace{p(C^1_{l_p} + ... + C^{l_p}_{l_p})}_{external\ stability\ C_1,\ C_2\ ...\ C_p} = p(2^{l_p}) \tag{19}$$

$$Subcoal_p = \sum_{t=1:(i_1-1)} C^t_{i_1}(C^1_{n_1-t} + ... + C^{n_1-t}_{n_1-t}) + ... + \sum_{t=1:(i_p-1)} C^t_{i_1}(C^1_{n_p-t} + ... + C^{n_p-t}_{n_p-t}) = \sum_{t=1:(i_1-1)} C^t_{i_1} 2^{n_1-t} + $$
$$+ \quad ...... \sum_{t=1:(i_p-1)} C^t_{i_p} 2^{n_p-t} \tag{20}$$

$$\begin{aligned} & Intern_p + Extern_p + Subcoal_p = \\ & = \quad 2^{i_1} + ... + 2^{i_p} - n - p + p(2^{l_1}) + \sum_{t=1:(i_1-1)} C^t_{i_1} 2^{n_1-t} + ... + \sum_{t=1:(i_p-1)} C^t_{i_p} 2^{n_p-t} \end{aligned} \tag{21}$$

Let take into account the exchange of members between p coalitions. During the inspection of player exchange among coalitions *we keep the number of single players, and number of coalitions fixed.* Then for checking if all p coalitions are farsightedly stable, there are necessary to check as many as coalitions as equation (22) shows.

$$Exchange_p = \underbrace{\Big(\sum_{t=2:(n-2(p-1))} C^t_{n-2(p-1)} \Big) ...... \Big(\sum_{t=2:(n-6)} C^t_{n-6} \Big) \Big(\sum_{t=2:(n-4)} C^t_{n-4} \Big)}_{all\ together\ (p-2)\ sums} \Big(C^2_{n-t} + ... + C^{n-t}_{n-t} \Big) \tag{22}$$

Table 3: Algorithm for finding farsightedly stable (EFS) coalition structures when t coalitions are formed, and l_1 players are single.

Take the coalition structure $a_1 = (C_1, C_2 ... C_t, l_1)$ where l_1
is the number of single players, C_i $i \in \{1, 2, ...t\}$ are coalitions, and m is the total number of the players.
Calculate $\pi(1)..\pi(5)..\pi(m)$ the profits of this coalition structure.
First inspection: for i=1:t
Inspect if coalition C_i is internally, externally or subcoalition stable, or find the coalition C_{l_1} (where $C_{l_1} \cap C_i \neq \emptyset$).
Note at the moment exchange member among coalitions is not allowed.
endif
end
Second inspection:
Inspect if the exchange of members among C_1, C_t finds that the initial coalition structure $a_1 = (C_1, C_2 ... C_t, l_1)$
is indirectly dominated by the coalition structure $a_2 = (C_1^{'}, C_2^{'} ... C_t^{'}, l_1)$
which has to be inspected by first and second inspection (and can possibly be farsightedly stable).
If by first and second inspection we are not able to find a coalition structure which indirectly or directly dominates the coalition structure a_r (which is actually being tested), and so the coalition structure, a_r is farsightedly stable
program END

Table 4: Three member coalitions which are not externally farsightedly stable.

Coalition	Pr_3	Pr_5	*Coalition*	Pr_3	Pr_5
USA	0.5336	0.5336	**USA**	0.5765	0.6916
LAM	0.0614	0.0614	**SEA**	0.177	0.2057
CHI	0.7613	0.7613	**CHI**	0.8048	0.817
NAF	0.322	0.322	NAF	0.3533	0.3976
SSA	0.3573	0.3573	SSA	0.3921	0.4173

Table 5: Three member coalitions which are not externally farsightedly stable.

Coalition	Pr_3	Pr_5	*Coalition*	Pr_3	Pr_5
USA	0.457	0.6916	**CAN**	0.0198	0.0204
SEA	0.177	0.2057	EEU	0.0216	0.0217
CHI	0.7766	0.817	**CAM**	0.0142	0.0144
NAF	0.2203	0.3976	SAS	0.075	0.0753
SSA	0.2398	0.4173	**SIS**	0.0118	0.012

Table 6: Three member coalition which is not externally farsightedly stable.

Coalition	Pr_3	Pr_5
CAN	0.0199	0.0204
EEU	0.0216	0.0217
CAM	0.0142	0.0144
SAS	0.0751	0.0753
SIS	0.0118	0.012

Table 7: Three member coalition which is not sub-coalition farsightedly stable.

Coalition	Pr_{3old}	Pr_{3new}
USA^*	0.4476	0.457
JPK	-0.3032	-0.3467
NAF*	0.2057	0.2203
SSA*	0.2285	0.2398

Table 8: Three member coalition which is not sub-coalition farsightedly stable.

Coalition	Pr_{3old}	Pr_{3new}
USA^*	0.4824	0.5336
CAN	0.0205	0.0311
FSU	0.241	0.3945
LAM*	0.0599	0.0614
CHI^*	0.7149	0.7613

Table 9: Our data from the year 2005, where α is the abatement cost parameter (unitless), β the marginal damage costs of carbon dioxide emissions (in dollars per tonne of carbon) E the carbon dioxide emissions (in billion metric tonnes of carbon) and Y gross domestic product, in billions US dollars. Source: FUND

	α	β	E	Y
USA	0.01515466	2.19648488	1.647	10399
CAN	0.01516751	0.09315600	0.124	807
WEU	0.01568000	3.15719404	0.762	12575
JPK	0.01562780	-1.42089104	0.525	8528
ANZ	0.01510650	-0.05143806	0.079	446
EEU	0.01465218	0.10131831	0.177	407
FSU	0.01381774	1.27242378	0.811	629
MDE	0.01434659	0.04737632	0.424	614
CAM	0.01486421	0.06652486	0.115	388
LAM	0.01513700	0.26839935	0.223	1351
SAS	0.01436564	0.35566631	0.559	831
SEA	0.01484894	0.73159104	0.334	1094
CHI	0.01444354	4.35686225	1.431	2376
NAF	0.01459959	0.96627119	0.101	213
SSA	0.01459184	1.07375825	0.145	302
SIS	0.01434621	0.05549814	0.038	55

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