COCALIBRATED G_2-STRUCTURES ON PRODUCTS OF FOUR- AND THREE-DIMENSIONAL LIE GROUPS

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Abstract. Cocalibrated G_2-structures are structures naturally induced on hypersurfaces in $\mathrm{Spin}(7)$-manifolds. Conversely, such structures can be used to construct $\mathrm{Spin}(7)$-manifolds via the Hitchin flow. In this article, we concentrate as in [8] on left-invariant cocalibrated G_2-structures on Lie groups, but now on those Lie groups G which are a direct product $\mathrm{G} = \mathrm{G}_4 \times \mathrm{G}_3$ of a four-dimensional Lie group G_4 and a three-dimensional Lie group G_3. We achieve a full classification of the Lie groups $\mathrm{G} = \mathrm{G}_4 \times \mathrm{G}_3$ which admit such structures.

1. Introduction

A G_2-structure on a seven-dimensional manifold M is a three-form $\varphi \in \Omega^3 M$ which pointwise looks like a certain standard form. Such a three-form naturally induces a Riemannian metric and an orientation and so a Hodge star operator $\star_\varphi : \Lambda^\bullet \mathfrak{g}^* \to \Lambda^\bullet \mathfrak{g}^*$. We call φ *cocalibrated* if

$$d \star_\varphi \varphi = 0.$$

Interest on cocalibrated real-analytic G_2-structures has arised for some years due to the fact that they are initial values for a time-dependent partial differential equation for three-forms, introduced by Hitchin [12], whose solution defines a Riemannian metric on a neighboorhood of $M \times \{0\}$ in $M \times \mathbb{R}$ with holonomy contained in $\mathrm{Spin}(7)$, see [12], [5].

Classification results for manifolds admitting real-analytic cocalibrated G_2-structures have recently been obtained for certain subclasses. In [17], the compact homogeneous spaces admitting homogeneous cocalibrated G_2-structures were obtained and in [8] the author identified the seven-dimensional Lie groups which admit left-invariant cocalibrated G_2-structures in the class of seven-dimensional Lie groups such that the associated Lie algebra has a six-dimensional Abelian ideal.

In this paper we look again at left-invariant cocalibrated G_2-structures on Lie groups G, namely on those G which are a direct product of a three-dimensional Lie group G_3 and a four-dimensional Lie group G_4. We classify which of these Lie groups admit left-invariant cocalibrated G_2-structures.

Identifying as usual left-invariant k-forms on the Lie group with k-forms on the Lie algebra and introducing a differential on $\Lambda^\bullet \mathfrak{g}^*$ by this identification, we may speak of *cocalibrated* G_2*-structures on a seven-dimensional Lie algebra* and these forms are in one-to-one correspondence to left-invariant cocalibrated G_2-structures on each corresponding Lie group. Our main result can now be formulated as follows, where we refer the reader for the names of the appearing Lie algebras to the Tables 1 and 2.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional Lie algebra $\mathfrak{g}_3$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if one of the following four conditions is fulfilled:*

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(a) $\mathfrak{g}_4$ *is not unimodular,* $\mathfrak{g}_3$ *is unimodular and* $h^1(\mathfrak{g}_4)+h^1(\mathfrak{u})-h^2(\mathfrak{g}_4)+h^2(\mathfrak{g}_3)\leq 4$*, where* $\mathfrak{u}$ *is the unimodular kernel of* $\mathfrak{g}_4$*.*

(b) $\mathfrak{g}_4$ *is unimodular,* $\mathfrak{g}_3$ *is unimodular and at least one of the following conditions is true:*

(i) $\mathfrak{g}_3\in\{\mathfrak{so}(3),\mathfrak{so}(2,1)\}$

(ii) $\mathfrak{g}_4=\mathfrak{h}\oplus\mathbb{R}$ *for a three-dimensional unimodular Lie algebra* $\mathfrak{h}$*.*

(iii) $\mathfrak{g}\in\{A_{4,1}\oplus e(2),A_{4,1}\oplus e(1,1),A_{4,8}\oplus e(1,1)\}$*.*

(c) $\mathfrak{g}_4$ *is unimodular,* $\mathfrak{g}_3$ *is not unimodular and at least one of the following conditions is true:*

(i) $\mathfrak{g}_4$ *has an Abelian ideal of codimension one,* $\mathfrak{g}_4\notin\{\mathbb{R}^4,\mathfrak{h}_3\oplus\mathbb{R}\}$ *and* $\mathfrak{g}_3=\mathfrak{r}_2\oplus\mathbb{R}$*.*

(ii) $[\mathfrak{g}_4,\mathfrak{g}_4]\in\{\mathfrak{h}_3,\mathfrak{so}(3),\mathfrak{so}(2,1)\}$*.*

(d) $\mathfrak{g}_4$ *is not unimodular,* $\mathfrak{g}_3$ *is not unimodular and at least one of the following conditions is true:*

(i) The unimodular kernel $\mathfrak{u}$ *of* $\mathfrak{g}_4$ *is isomorphic to* $e(2)$ *or* $e(1,1)$*.*

(ii) $\mathfrak{g}=A_{4,9}^{-\frac{1}{2}}\oplus\mathfrak{r}_2\oplus\mathbb{R}$*.*

(iii) The unimodular kernel $\mathfrak{u}$ *of* $\mathfrak{g}_4$ *is isomorphic to* $\mathfrak{h}_3$*,* $\mathfrak{g}_3\neq\mathfrak{r}_2\oplus\mathbb{R}$ *and*
$$\mathfrak{g}\notin\left\{A_{4,9}^1\oplus\mathfrak{r}_{3,\mu},A_{4,9}^{\alpha}\oplus\mathfrak{r}_{3,1}\,\middle|\,\mu\in\left[-\tfrac{1}{3},0\right),\ \alpha\in\left(-1,-\tfrac{1}{3}\right]\right\}.$$

For the proof of Theorem 1.1 we use as in [8] the algebraic invariants introduced by Westwick [19]. In contrast to [8], these algebraic invariants only lead to obstructions. The construction of cocalibrated G_2-structures relies on the following two properties of G_2-structures. Firstly, from a decomposition $\mathfrak{g}=V_4\oplus V_3$ of $\mathfrak{g}$ into a four-dimensional subspace V_4 and a three-dimensional subspace V_3 and certain two-forms on V_4 and V_3 one can build the Hodge dual of a G_2-structure. Note that in the concrete applications later these subspaces may not always coincide with $\mathfrak{g}_4$ and $\mathfrak{g}_3$. Secondly, we use the openness of the orbit of all Hodge duals. Therefore, we write down the Hodge dual $\Psi\in\Lambda^4\mathfrak{g}^*$ of a G_2-structure "well-adapted" to the structure of the Lie algebra $\mathfrak{g}$, add some term $\Phi\in\Lambda^4\mathfrak{g}^*$ such that $\Psi+\Phi$ is closed and rescale Ψ and Φ such that the sum stays closed and Φ gets small in comparison to Ψ. Then $\Psi+\Phi$ is the Hodge dual of a cocalibrated G_2-structure.

The work is organized as follows: Section 2 deals with preliminaries on G_2-structures, four- and three-dimensional Lie algebras and the mentioned algebraic invariants. We begin in subsection 2.1 by recalling the definition and basic properties of a G_2-structure on a seven-dimensional real vector space. Moreover, we show that the orbit of all Hodge duals of such structures is "uniformly" open in a sense made precise in that subsection. In the following subsection, we expand our definition to G_2-structures on manifolds and introduce cocalibrated G_2-structures on Lie algebras. Subsections 2.3 and 2.4 are devoted to recalling basic facts about three-dimensional and four-dimensional Lie algebras. In subsection 2.5 we recall the algebraic invariants for k-vectors introduced partly by Westwick [19] and the values of these invariants for certain k-forms associated to G_2-structures obtained in [19] and [8]. Moreover, we investigate under which circumstances a subspace of the space of all two-forms on a four-dimensional vector space consists entirely of non-degenerate two-forms and how one can build from such two-forms the Hodge dual of a G_2-structure on a seven-dimensional vector space.

In section 3 we give the classification. For that purpose we use in subsection 3.1 the "uniform" openness of the orbit of all Hodge duals to show that, under certain assumptions, one may deform a given G_2-structure on a seven-dimensional manifold in a particular way to obtain a whole family of cocalibrated G_2-structures on M. We apply this result to our situation, namely G_2-structures on Lie algebras which are direct sums of a four-dimensional and a three-dimensional Lie algebra, to get existence results for certain classes of such Lie algebras.

In subsection 3.2 we use the algebraic invariants to obtain obstructions to the existence of cocalibrated G_2-structures on the Lie algebras in question and exclude such structures for large classes. In the subsections 3.3 - 3.6 we apply the results of the subsections 3.1 and 3.2 to the direct sums $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ and deal separately with the four cases which naturally appear by distinguishing whether $\mathfrak{g}_4$ or $\mathfrak{g}_3$ is unimodular or not.
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2. Preliminaries

2.1. G_2-structures on vector spaces.

We give a short introduction into G_2-structures on vector spaces. More thorough introductions may be found in [4] and in [8].

Definition 2.1. Let V be a seven-dimensional real vector space. A G_2-*structure* on V is a three-form $\varphi \in \Lambda^3 V^*$ for which there exists a basis $e_1, \ldots, e_7$ of V with

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

Thereby, $e^1, \ldots, e^7 \in V^*$ denotes the dual basis of $e_1, \ldots, e_7$. We call the seven-tupel $(e_1, \ldots e_7) \in V^7$ an *adapted basis* for the G_2-structure φ.

Remark 2.2. All G_2-structures lie in one orbit under the natural action of $\mathrm{GL}(V)$ on $\Lambda^3 V^*$. The isotropy group of a G_2-structure in $\mathrm{GL}(V)$ under this action is isomorphic to G_2, which is in our context the simply-connected compact real form of the complex simple Lie group $(G_2)_{\mathbb{C}}$. Since $\dim(\mathrm{GL}(V)) = 49$, $\dim(G_2) = 14$ and $\dim(\Lambda^3 V^*) = 35$, the orbit is open, i.e. a G_2-structure is a *stable form* [12]. Note that there is another open orbit in $\Lambda^3 V^*$ whose stabilizer is G_2^*, the split real form of $(G_2)_{\mathbb{C}}$ with $\pi_1(G_2^*) = \mathbb{Z}_2$ [4].

Since $G_2 \subseteq \mathrm{SO}(7)$, a G_2-structure induces a Euclidean metric and an orientation on V as follows [5]:

Lemma 2.3. *Let V be a seven-dimensional real vector space and φ be a G_2-structure on V. Then φ induces a unique Euclidean metric g_φ and a unique metric volume form vol_φ on V such that each adapted basis $(e_1, \ldots, e_7)$ for φ is an oriented orthonormal basis of V. For all $v, w \in V$ the Euclidean metric g_φ and the metric volume form $vvol_\varphi$ are given by the formula*

$$g_\varphi(v, w) vol_\varphi = (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi.$$

Remark 2.4. G_2-structures may be understood through the division algebra $(\mathbb{O}, \langle \cdot, \cdot \rangle)$ of the octonions. Therefore, let $1 \in \mathbb{O}$ be the unit element of $\mathbb{O}$ and let $\mathrm{Im}\,\mathbb{O} := \mathrm{span}(1)^\perp$ be the imaginary octonions. Then $\varphi \in \Lambda^3 \mathrm{Im}\,\mathbb{O}^*$ given by $\varphi(u, v, w) := \langle u \cdot v, w \rangle$ for $u, v, w \in \mathrm{Im}\,\mathbb{O}$ is a G_2-structure on the seven-dimensional vector space $\mathrm{Im}\,\mathbb{O}$. Moreover, φ induces in the sense of Lemma 2.3 exactly the Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathrm{Im}\,\mathbb{O}$. For more details and for the relation of our definition to other definitions in the literature, we refer the reader to [8].

Lemma 2.3 tells us that a G_2-structure $\varphi \in \Lambda^3 V$ induces naturally a Euclidean metric g_φ and a volume form vol_φ on V. Thus we can define a Hodge star operator $\star_\varphi : \Lambda^\bullet V^* \to \Lambda^\bullet V^*$ by the usual requirement that for a k-form $\phi \in \Lambda^k V^*$ the $(n-k)$-form $\star_\varphi \phi \in \Lambda^{n-k} V^*$ is the unique $(n-k)$-form on V such that for all $\psi \in \Lambda^k V^*$ the identity

$$\phi \wedge \psi = g_\varphi(\star_\varphi \phi, \psi) \mathrm{vol}_\varphi$$

is true. A short computation shows that the Hodge dual $\star_\varphi \varphi$ of the G_2-structure φ is given by

$$\star_\varphi \varphi = e^{1234} + e^{1256} + e^{3456} - e^{2467} + e^{2357} + e^{1457} + e^{1367},$$

where $e^1, \ldots, e^7$ is a dual basis of an adapted basis $(e_1, \ldots, e_7)$ for φ.

Conversely, a four-form $\Psi \in \Lambda^4 V^*$ of this kind has stabilizer G_2 in $\mathrm{GL}^+(V)$. So if we fix an orientation on V, it gives rise to a Euclidean metric g_Ψ and a G_2-structure φ. In this case, $g_\varphi = g_\Psi$, $\star_\varphi \varphi = \Psi$, $\star_\varphi \Psi = \varphi$ and the orientation induced by φ is the one fixed before [12]. Hence, alternatively, it would also be possible to call such a four-form Ψ together with an orientation a G_2-structure. Even though this alternative definition is more appropriate in our case, we follow the convention in the literature and only call the three-form φ a G_2-structure. The set of all Hodge duals $\star_\varphi \varphi$ forms again an open orbit under $\mathrm{GL}(V)$ [12]. So for each Hodge dual $\star_\varphi \varphi$ there exists a small ball of radius ϵ_φ in $(\Lambda^4 V^*, g_\varphi)$ such that each four-form in this ball is again the Hodge dual of a G_2-structure. In fact, the sizes of these balls do not depend on the G_2-structure φ and the orbit is in this sense "uniformly" open. Namely, for two different G_2-structures $\varphi_1, \varphi_2 \in \Lambda^3 V^*$ on V the automorphism of V which maps an adapted basis of φ_1 onto an adapted basis of φ_2 induces an isometric isomorphism between $(\Lambda^4 V^*, g_{\varphi_2})$ and $(\Lambda^4 V^*, g_{\varphi_1})$. Hence, if a ball of radius ϵ with respect to g_{φ_2} around $\star_{\varphi_2}\varphi_2$ lies in the orbit of all Hodge duals of G_2-structures, then also a ball of radius ϵ with respect to g_{φ_1} around $\star_{\varphi_1}\varphi_1$ lies in the orbit of all Hodge duals of G_2-structures. This is the next

Lemma 2.5. *There exists a universal constant $\epsilon_0 > 0$ such that if $\varphi \in \Lambda^3 V^*$ is a G_2-structure on a seven-dimensional real vector space V and $\Psi \in \Lambda^4 V^*$ is a four-form on V which fulfills*

$$\left\| \Psi - \star_\varphi \varphi \right\|_\varphi < \epsilon_0$$

for the norm $\|\cdot\|_\varphi$ induced by the Euclidean metric g_φ on V, then Ψ is the Hodge dual of a G_2-structure on V.

For a G_2-structure on a seven-dimensional vector space there are distinguished three- and four-planes:

Definition 2.6. Let φ be a G_2-structure on a seven-dimensional vector space V. An *associative three-plane* U is a three-dimensional subspace of V such that $\varphi|_U = \mathrm{vol}_U$, where vol_U is the metric volume form on U induced by g_φ and an appropriate orientation on U. Similarly, a *coassociative four-plane* W is a four-dimensional subspace of V such that $\star_\varphi \varphi|_W = \mathrm{vol}_W$ for an appropriate orientation on W. This is equivalent to $\varphi|_W = 0$. A *coassociative/associative splitting of* V is a vector space decomposition $V = W \oplus U$ into a coassociative four-plane W and an associative three-plane U.

Remark 2.7. If $V = W \oplus U$ is a coassociative/associative splitting of V, then the splitting is orthogonal. Moreover there exists an adapted basis $e_1, \ldots, e_7$ for φ such that $e_1, \ldots, e_4$ is a basis of W and $e_5, \ldots, e_7$ is a basis of U and $\star_\varphi \varphi \in \Lambda^4 W^* \oplus \Lambda^2 W^* \wedge \Lambda^2 U^*$ [13].

2.2. **Cocalibrated G_2-structures on manifolds and Lie algebras.** A G_2*-structure on a seven-dimensional manifold* M is by definition a reduction of the frame bundle $\mathrm{GL}(M)$ to $\mathrm{G}_2 \subseteq \mathrm{GL}_7(\mathbb{R})$. Since G_2 is conjugated to the stabilizer of a G_2-structure $\varphi \in \Lambda^3 \left(\mathbb{R}^7\right)^*$, there exists a one-to-one correspondence between G_2-structures on M and three-forms $\varphi \in \Omega^3 M$ such that $\varphi_p \in \Lambda^3 T_p M^*$ is a G_2-structure on $T_p M$ for all $p \in M$. Therefore, we call in the following the three-form $\varphi \in \Omega^3 M$ a G_2*-structure*. One can show that G_2-structures exist exactly when M is orientable and spin [13].
An *associative distribution* E is a three-dimensional differentiable distribution which is pointwise associative. Analogously, a *coassociative distribution* is defined. Note that if E is an associative distribution, then $E^\perp$ is a coassociative distribution. Since associative distributions always exist [2], a *coassociative/associative splitting of* M, i.e. a decomposition $TM = E_4 \oplus E_3$ where E_4 is a coassociative distribution and E_3 is an associative distribution, always exists.

A G_2-structure carries in general intrinsic torsion. It is well-known that it is torsion-free exactly when $Hol(M,g) \subseteq G_2$ and this is equivalent to $d\varphi = 0 = d\star_\varphi \varphi$ [7]. In this case φ and $\star_\varphi\varphi$ are calibrations on M, see [11], and the calibrated submanifolds are precisely those whose tangent spaces are pointwise associative or coassociative, respectively.
We consider a weakened condition in this article, namely *cocalibrated* G_2-structures, i.e. G_2-structures $\varphi \in \Omega^3 M$ for which $d\star_\varphi \varphi = 0$. Then still $\star_\varphi\varphi$ is a calibration on M but φ is in general not.
Moreover, we concentrate on left-invariant G_2-structures on Lie groups G. These are in one-to-one correspondence to G_2-structures on the corresponding Lie algebra $\mathfrak{g}$. If we use this identification as usual to define a differential $d_\mathfrak{g}$ on $\Lambda^\bullet\mathfrak{g}^*$, we are able to speak of *cocalibrated* G_2*-structures on a Lie algebra.*

Convention 2.8. If $V = W \oplus U$ is a vector space which is the vector space direct sum of two vector spaces W and U and $\pi_W : V \to W$ is the projection onto W along U, then $\pi_W^* : \Lambda^\bullet W^* \to \Lambda^\bullet V^*$ is injective. The image of π_W^* is $\Lambda^\bullet U^0$. Thus it is justifiable to identify in this situation $\Lambda^k U^0$ with $\Lambda^k W^*$, which we will often do in the following.
Moreover, if $\mathfrak{g} = \mathfrak{u} \oplus U$ is a Lie algebra which is the vector space direct sum of an ideal $\mathfrak{u}$ in $\mathfrak{g}$ and a vector subspace $U \subseteq \mathfrak{g}$, then the above injection also identifies the cochain complexes $(\Lambda^\bullet U^0, \pi_{\Lambda^\bullet U^0} \circ d_\mathfrak{g}|_{\Lambda^\bullet U^0})$ and $(\Lambda^\bullet \mathfrak{u}^*, d_\mathfrak{u})$, where $\pi_{\Lambda^\bullet U^0} : \Lambda^\bullet \mathfrak{g}^* \to \Lambda^\bullet U^0$ is the projection onto $\Lambda^\bullet U^0$ along $\mathfrak{u}^0 \wedge \Lambda^\bullet\mathfrak{g}^*$. Note that the former is a cochain subcomplex of $(\Lambda^\bullet\mathfrak{g}^*, d_\mathfrak{g})$. Using this identification, we will often write $d_\mathfrak{u}$ instead of $\pi_{\Lambda^\bullet U^0} \circ d_\mathfrak{g}|_{\Lambda^\bullet U^0}$. Note that if U is also an ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{u} \oplus U$ is a Lie algebra direct sum, then $\pi_{\Lambda^\bullet U^0} \circ d_\mathfrak{g}|_{\Lambda^\bullet U^0} = d_\mathfrak{g}|_{\Lambda^\bullet \mathfrak{u}^*} = d_\mathfrak{u}$ in our identification. In this case we will often omit the index and simply write d.

All four-dimensional and all three-dimensional Lie algebras with the exception of the two simple Lie algebras admit a codimension one unimodular ideal. So it is not surprising that the following lemma turns out to be useful in our study and we use it often without further noting:

Lemma 2.9. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra which admits a codimension one unimodular ideal $\mathfrak{u} \subseteq \mathfrak{g}$. Let $e_n \in \mathfrak{g}\backslash\mathfrak{u}$ and $e^n \in \mathfrak{u}^0$, $e^n(e_n) = 1$. Then:*

(a) $de^n = 0$ and there exists a linear map $f : \mathfrak{u}^* \to \mathfrak{u}^*$ such that $d_\mathfrak{g}\alpha = d_\mathfrak{u}\alpha + f(\alpha) \wedge e^n$ for all $\alpha \in \mathfrak{u}^*$.
(b) $d_\mathfrak{g}(\omega \wedge e^n) = d_\mathfrak{u}(\omega) \wedge e^n$ for all $\omega \in \Lambda^\bullet\mathfrak{u}^*$.
(c) $d_\mathfrak{g}(\Lambda^{n-2}\mathfrak{u}^*) \subseteq \Lambda^{n-2}\mathfrak{u}^* \wedge e^n$.
(d) $d_\mathfrak{g}(\Lambda^{n-2}\mathfrak{u}^* \wedge e^n) = \{0\}$. Moreover, $d_\mathfrak{g}(\Lambda^{n-1}\mathfrak{u}^*) = \{0\}$ exactly when $\mathfrak{g}$ is unimodular.

Proof. (a) For arbitrary $X, Y \in \mathfrak{g}$, the commutator $[X,Y]$ is in $\mathfrak{u}$. Hence $de^n(X,Y) = -e^n([X,Y]) = 0$.
It is clear that there are linear maps $f : \mathfrak{u}^* \to \mathfrak{u}^*$ and $g : \mathfrak{u}^* \to \Lambda^2\mathfrak{u}^*$ such that $d_\mathfrak{g}(\alpha) = g(\alpha) + f(\alpha) \wedge e^n$ for all $\alpha \in \mathfrak{u}^*$. For $Z, W \in \mathfrak{u}$, we have $[Z,W] \in \mathfrak{u}$ and

$$g(\alpha)(Z,W) = (d_\mathfrak{g}\alpha)(Z,W) = -\alpha([Z,W]) = (d_\mathfrak{u}\alpha)(Z,W)$$

by Convention 2.8. This shows $g(\alpha) = d_\mathfrak{u}(\alpha)$ and finishes the proof of part (a).
(b) Part (a) implies, for each $k \in \{0,\ldots,n-1\}$, the existence of a linear map $f_k : \Lambda^k\mathfrak{u}^* \to \Lambda^k\mathfrak{u}^*$ such that $d_\mathfrak{g}\omega = d_\mathfrak{u}\omega + f_k(\omega) \wedge e^n$ for all $\omega \in \Lambda^k\mathfrak{u}^*$. Then (a) implies $d_\mathfrak{g}(\omega \wedge e^n) = d_\mathfrak{g}(\omega) \wedge e^n = d_\mathfrak{u}(\omega) \wedge e^n$ as claimed.
(c) We have $d_\mathfrak{g}\omega = d_\mathfrak{u}\omega + f_{n-2}(\omega) \wedge e^n$ for all $\omega \in \Lambda^{n-2}\mathfrak{u}^*$. But $\mathfrak{u}$ is unimodular, which is equivalent to the fact that all $(n-2)$-forms on $\mathfrak{u}$ are $d_\mathfrak{u}$-closed. Hence $d_\mathfrak{g}\omega = f_{n-2}(\omega) \wedge e^n \in \Lambda^{n-2}\mathfrak{u}^* \wedge e^n$ as claimed.

(d) Part (a) and (c) directly imply $d_{\mathfrak{g}}(\Lambda^{n-2}\mathfrak{u}^* \wedge e^n) = \{0\}$. Since $\mathfrak{g}$ is unimodular exactly when all $(n-1)$-forms are $d_{\mathfrak{g}}$-closed, the first part implies that $d_{\mathfrak{g}}(\Lambda^{n-1}\mathfrak{u}^*) = \{0\}$ exactly when $\mathfrak{g}$ is unimodular.

□

2.3. **Three-dimensional Lie algebras.** The classification of three-dimensional Lie algebras is well-known [3] and given in the appendix in Table 1. We highlight some aspects of the classification which we use later on in this article.

Lemma 2.10. *Let $\mathfrak{g}$ be a three-dimensional unimodular Lie algebra.*

(a) There exists a basis e_1, e_2, e_3 of $\mathfrak{g}$ and $\tau_1, \tau_2, \tau_3 \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$ such that $de^i = \tau_i \sum_{j,k=1}^3 \epsilon_{ijk} e^{jk}$ for $i = 1, 2, 3$.

(b) $d(\mathfrak{g}^*) \wedge \ker d|_{\mathfrak{g}^*} = \{0\}$.

(c) There exists a linear map $g : \Lambda^2\mathfrak{g}^* \to \ker d|_{\mathfrak{g}^*}$ such that the kernel of the map $G : \Lambda^2\mathfrak{g}^* \to \Lambda^3\mathfrak{g}^*$, $G(\omega) := \omega \wedge g(\omega)$ is exactly $d(\mathfrak{g}^*)$.

(d) If $\tau_i\tau_j \geq 0$ for all $i, j \in \{1, 2, 3\}$, i.e. $\mathfrak{g} \notin \{e(1, 1), \mathfrak{so}(2, 1)\}$, then the kernel of the map $F : \mathfrak{g}^* \to \Lambda^3\mathfrak{g}^*$, $F(v) := d(v) \wedge v$ is exactly $\ker d|_{\mathfrak{g}^*}$.

Proof. We use the well-known part (a) [3] to show (b)-(d).

(b) Let $\omega = d\alpha$, $\alpha = \sum_{i=1}^3 a_i e^i \in \mathfrak{g}^*$ and $\beta = \sum_{i=1}^3 b_i e^i \in \mathfrak{g}^*$. Then

$$\omega = \sum_{i,j,k=1}^{3} \tau_i a_i \epsilon_{ijk} e^{jk} \tag{2.1}$$

and so

$$\omega \wedge \beta = \sum_{i,j,k,l=1}^{3} \tau_i a_i b_l \epsilon_{ijk} e^{jkl} = \sum_{i,j,k,l=1}^{3} \tau_i a_i b_l \epsilon_{ijk} \epsilon_{jkl} e^{123} = \left(\sum_{i=1}^{3} 2\tau_i a_i b_i\right) e^{123}. \tag{2.2}$$

If $d\beta = \sum_{i,j,k=1}^3 \tau_i b_i \epsilon_{ijk} e^{jk} = 0$, then $\tau_i b_i = 0$ for all $i = 1, 2, 3$ and so $\omega \wedge \beta = 0$. This shows (b).

(c) Let $\omega \in \Lambda^2\mathfrak{g}^*$. Then $\omega = \sum_{i,j,k=1}^3 a_i \epsilon_{ijk} e^{jk}$ for unique $a_1, a_2, a_3 \in \mathbb{R}$. Set $g(\omega) := \sum_{i=1,\tau_i=0}^3 a_i e^i$. Then Equation (2.1) shows that $g(\omega) \in \ker d|_{\mathfrak{g}^*}$. Moreover,

$$\begin{aligned} \omega \wedge g(\omega) &= \sum_{i,j,k,l=1,\tau_l=0}^{3} a_i a_l \epsilon_{ijk} e^{jkl} = \left(\sum_{i,j,k,l=1,\tau_l=0}^{3} a_i a_l \epsilon_{jki} \epsilon_{jkl}\right) e^{123} \\ &= \left(\sum_{i,l=1,\tau_l=0}^{3} 2 a_i a_l \delta_{il}\right) e^{123} = \left(\sum_{l=1,\tau_l=0}^{3} a_l^2\right) e^{123} = 0 \end{aligned}$$

if and only if $\tau_l = 0$ implies $a_l = 0$ for $l = 1, 2, 3$. But Equation (2.1) shows that this is equivalent to $\omega \in d(\mathfrak{g}^*)$.

(d) The signs of the non-zero τ_i are all the same due to the assertion. Let $\alpha = \sum_{i=1}^3 a_i e^i \in \mathfrak{g}^*$, $a_1, a_2, a_3 \in \mathbb{R}$. Then Equation (2.2) implies that $d\alpha \wedge \alpha = 0$ if and only if $\sum_{i=1}^3 \tau_i a_i^2 = 0$ and this is the case if and only if $\tau_i a_i = 0$ for all $i = 1, 2, 3$. But Equation (2.1) states that this is equivalent to $\alpha \in \ker d|_{\mathfrak{g}^*}$.

□

The only two non-solvable three-dimensional Lie algebras are the simple ones, namely $\mathfrak{so}(3)$ and $\mathfrak{so}(2, 1)$. If $\mathfrak{g}$ is solvable and unimodular, then, by elementary Lie theory, there exists a codimension one ideal, which then has to be unimodular and so Abelian. If $\mathfrak{g}$ is not unimodular, then the unimodular kernel gives a codimension one Abelian ideal. Thus Lemma 2.9 implies

Lemma 2.11. *Let $\mathfrak{g}$ be a three-dimensional solvable Lie algebra. Then $\mathfrak{g}^*$ admits a vector space decomposition $\mathfrak{g}^* = W_2 \oplus \mathrm{span}(e^3)$, W_2 two-dimensional, and a linear map $f : W_2 \to W_2$ such that $d\alpha = f(\alpha) \wedge e^3$ for all $\alpha \in W_2$ and $de^3 = 0$. If $\mathrm{tr}(f) \neq 0$, $\frac{\det(f)}{\mathrm{tr}(f)^2}$ only depends on the Lie algebra $\mathfrak{g}$. Moreover, $\mathrm{tr}(f) = 0$ exactly when $\mathfrak{g}$ is unimodular.*

We recapitulate the definition of a contact form on an odd-dimensional Lie algebra.

Definition 2.12. Let $\mathfrak{g}$ be a $(2m+1)$-dimensional Lie algebra. A *contact form* on $\mathfrak{g}$ is a one-form $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge (d\alpha)^m \neq 0$. For $m = 1$, the case we will be mainly interested in, the condition simply is $\alpha \wedge d\alpha \neq 0$.

In section 3 we need a classification of the three-dimensional Lie algebras which do admit a contact form. This classification is well-known [6] and straightforward to prove:

Lemma 2.13. *A three-dimensional Lie algebra does not admit a contact form if and only if $\mathfrak{g}$ is solvable and f as in Lemma 2.11 is a multiple of the identity. So $\mathfrak{g}$ admits a contact-form if and only if $\mathfrak{g} \notin \{\mathbb{R}^3, \mathfrak{r}_{3,1}\}$.*

2.4. **Four-dimensional Lie algebras.** A classification of all four-dimensional Lie algebra has first been achieved by Mubarakzyanov [14]. We give a complete list in Table 2.
In [1] it is proven that each four-dimensional solvable Lie algebra admits a codimension one unimodular ideal. Since the only simple Lie algebras up to dimension four are $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$, it is an immediate consequence of Levi's decomposition theorem that the non-solvable four-dimensional Lie algebras are exactly $\mathfrak{so}(3) \oplus \mathbb{R}$ and $\mathfrak{so}(2,1) \oplus \mathbb{R}$. This shows the first part of

Lemma 2.14. *Let $\mathfrak{g}$ be a four-dimensional Lie algebra. Then $\mathfrak{g}$ admits a codimension one unimodular ideal $\mathfrak{u}$. $\mathfrak{u}$ is unique if and only if $\mathfrak{g}$ is not unimodular or $\dim([\mathfrak{g},\mathfrak{g}]) = 3$. In these cases $\mathfrak{u}$ is the unimodular kernel or the commutator ideal $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$, respectively.*

Proof. If $\mathfrak{g}$ is not unimodular, then the unimodular kernel has codimension one and each unimodular ideal of $\mathfrak{g}$ is an ideal of the unimodular kernel. Thus a codimension one unimodular ideal has to coincide with the unimodular kernel. The commutator ideal $[\mathfrak{g},\mathfrak{g}]$ is a unimodular ideal and contained in each codimension one ideal. Thus the uniqueness statement follows if $\dim([\mathfrak{g},\mathfrak{g}]) = 3$.
If $\mathfrak{g}$ is unimodular and $\dim([\mathfrak{g},\mathfrak{g}]) < 3$, then, by inspecting Table 2, we see that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ with a three-dimensional unimodular solvable Lie algebra $\mathfrak{h}$ or $\mathfrak{g} = A_{4,1}$. In the former cases, the first summand $\mathfrak{h}$ in $\mathfrak{h} \oplus \mathbb{R}$ is a unimodular codimension one ideal and the direct sum of an Abelian codimension one ideal of $\mathfrak{h}$ and the $\mathbb{R}$ summand gives a different unimodular codimension one ideal. For $\mathfrak{g} = A_{4,1}$, in the dual basis e_1, e_2, e_3, e_4 of the basis e^1, e^2, e^3, e^4 of $A_{4,1}^*$ given in Table 2, the subspace $\mathrm{span}(e_1, e_2, e_3)$ is an Abelian codimension one ideal whereas $\mathrm{span}(e_1, e_2, e_4)$ is a codimension one ideal isomorphic to $\mathfrak{h}_3$. □

We recapitulate the definition of a symplectic two-form on an even-dimensional Lie algebra:

Definition 2.15. Let $\mathfrak{g}$ be a Lie algebra of dimension $2m$. A closed two-form $\omega \in \Lambda^2\mathfrak{g}^*$ is called *symplectic* if it is *non-degenerate*, i.e. $\omega^m \neq 0$. For the case we are interested in, namely $m = 2$, this simply means $\omega^2 \neq 0$.

All symplectic four-dimensional Lie algebras have been identified and also all symplectic two-forms (up to isomorphisms) have been determined by Ovando in [15]. We give a new proof of some part of the results in order to relate the existence of one or more symplectic two-forms satisfying certain compatibility relations to the dimensions of the cohomology groups of $\mathfrak{g}$ and of a codimension one unimodular ideal $\mathfrak{u}$.

Lemma 2.16. *Let $\mathfrak{g}$ be a four-dimensional Lie algebra and assume that $\mathfrak{g}$ admits an Abelian codimension one ideal $\mathfrak{u}$ or $\mathfrak{g}$ is not unimodular and the unimodular kernel $\mathfrak{u}$ is not isomorphic to $e(1,1)$. Then $\mathfrak{g}$ admits a*

$$D := h^2(\mathfrak{g}) - h^1(\mathfrak{g}) - h^1(\mathfrak{u}) + 4$$

-dimensional subspace of $\Lambda^2\mathfrak{g}^*$ in which each non-zero element is symplectic.

Remark 2.17. Lemma 2.16 applies to all but five Lie algebras:

- The only non-unimodular four-dimensional Lie algebra with unimodular kernel $\mathfrak{u}$ isomorphic to $e(1,1)$ is $\mathfrak{r}_2 \oplus \mathfrak{r}_2$. In the basis given in Table 2 the two-form $e^{14} + e^{23}$ is symplectic. One can show that the maximal dimension of a subspace $V \subseteq \Lambda^2(\mathfrak{r}_2 \oplus \mathfrak{r}_2)^*$ in which each non-zero element is symplectic is one.
- The unimodular four-dimensional Lie algebras which do not admit a codimension one Abelian ideal are the two non-solvable ones $\mathfrak{so}(3) \oplus \mathbb{R}$ and $\mathfrak{so}(2,1) \oplus \mathbb{R}$ and two other Lie algebras, namely $A_{4,8}$ and $A_{4,10}$. All four do not admit any symplectic two-form.

Proof of Lemma 2.16. Choose an element $e_4 \in \mathfrak{g} \backslash \mathfrak{u}$ and let $e^4 \in \mathfrak{g}^*$ be such that $e^4(e_4) = 1$, $e^4 \in \mathfrak{u}^0$. For the following let $f : \mathfrak{u}^* \to \mathfrak{u}^*$ be the linear map such that $d_\mathfrak{g}\beta = d_\mathfrak{u}\beta + f(\beta) \wedge e^4$ for all $\beta \in \mathfrak{u}^*$. For the proof we fix a norm $\|\cdot\|$ on $\Lambda^2\mathfrak{g}^*$ and choose an isomorphism $\Lambda^4\mathfrak{g}^* \cong \mathbb{R}$. We fix a complement V of $\ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$, set $W_\lambda := \{\omega + \lambda g(\omega) \wedge e^4 | \, \omega \in \ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{u}^*}\}$ for $\lambda \neq 0$, where $g : \Lambda^2\mathfrak{u}^* \to \ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ is the map of Lemma 2.10 (c), and claim that there is $\lambda \neq 0$ such that $U := d_\mathfrak{g}(V) + W_\lambda$ consists, with the exception of the origin, entirely of symplectic two-forms and that the dimension of U is equal to $D = h^2(\mathfrak{g}) - h^1(\mathfrak{g}) - h^1(\mathfrak{u}) + 4$. Note that the closedness of all elements in U is clear. We divide the proof into six steps.

Step I: All non-zero elements in $d_\mathfrak{g}(V)$ are symplectic and $d_\mathfrak{g}|_V : V \to d_\mathfrak{g}(V)$ is an isomorphism:

If $V = \{0\}$, then there is nothing to show. Otherwise our assumptions imply that $\mathfrak{g}$ is not unimodular and so $d_\mathfrak{g}(\Lambda^3\mathfrak{u}^*) \neq \{0\}$. Let $\alpha \in V\backslash\{0\}$. By definition of V, $d_\mathfrak{u}\alpha \neq 0$ and so Lemma 2.10 (d) tells us that $\Lambda^3\mathfrak{u}^* \ni d_\mathfrak{u}\alpha \wedge \alpha \neq 0$. Hence $d_\mathfrak{g}(d_\mathfrak{u}\alpha \wedge \alpha) \neq 0$ and so

$$d_\mathfrak{g}\alpha \wedge d_\mathfrak{g}\alpha = d_\mathfrak{g}(\alpha \wedge d_\mathfrak{g}\alpha) = d_\mathfrak{g}(\alpha \wedge d_\mathfrak{u}\alpha + \alpha \wedge f(\alpha) \wedge e^4) = d_\mathfrak{g}(\alpha \wedge d_\mathfrak{u}\alpha) \neq 0.$$

So $d_\mathfrak{g}\alpha$ is non-degenerate and, in particular, $d_\mathfrak{g}\alpha \neq 0$. This proves Step I.

Step II: $f(V)$ is a complement of $\ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$ and for all $\lambda \neq 0$: $d_\mathfrak{g}(V) \cap W_\lambda = d_\mathfrak{g}(V) \cap (\ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_\mathfrak{u}|_{\mathfrak{u}^*} \wedge e^4) = \{0\}$:

The inequality $0 \neq d_\mathfrak{g}\alpha \wedge d_\mathfrak{g}\alpha = 2 d_\mathfrak{u}\alpha \wedge f(\alpha) \wedge e^4$ for $\alpha \in V\backslash\{0\}$ implies that $f|_V$ is injective and so $\dim(V) = \dim(f(V))$. By Lemma 2.10 (b), $\ker d_\mathfrak{u}|_{\mathfrak{u}^*} \wedge d_\mathfrak{u}(\mathfrak{u}^*) = \{0\}$. Thus $f(V)$ is a complement of $\ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$. Let $\omega \in d_\mathfrak{g}(V) \cap (\ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_\mathfrak{u}|_{\mathfrak{u}^*} \wedge e^4)$. Then there are $\alpha \in V, \omega_1 \in \ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{u}^*}$ and $\beta \in \ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ such that

$$\omega = d_\mathfrak{u}\alpha + f(\alpha) \wedge e^4 = \omega_1 + \beta \wedge e^4.$$

This implies $f(\alpha) = \beta \in \ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ and so, since $f(V)$ is a complement of $\ker d_\mathfrak{u}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$, $\beta = 0$. Now $f|_V$ is injective and so we must have $\alpha = 0$, which ultimately implies $\omega = 0$. This finishes the proof of Step II.

Step III: $\dim(d_\mathfrak{g}(V) \oplus W_\lambda) = h^2(\mathfrak{g}) - h^1(\mathfrak{g}) - h^1(\mathfrak{u}) + 4$:

Note that the dimension of W_λ is equal to the dimension of $\ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{u}^*}$ and that the dimension of $\ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{g}^*}$ is $h^2(\mathfrak{g}) + 4 - h^1(\mathfrak{g})$. Therefore it suffices to show

$$\ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{g}^*} = \ker d_\mathfrak{g}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_\mathfrak{u}|_{\mathfrak{u}^*} \wedge e^4 \oplus d_\mathfrak{g}(V)$$

to get the statement about the dimension of $d_{\mathfrak{g}}(V) \oplus W_\lambda$. The inclusion "$\supseteq$" is clear. For the other inclusion, let $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{g}^*}$. Then there exists $\omega_1 \in \Lambda^2\mathfrak{u}^*$ and $\beta \in \mathfrak{u}^*$ such that $\omega = \omega_1 + \beta \wedge e^4$. Since $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$, there exists $\alpha \in V$ with $\beta - f(\alpha) \in \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$. Then

$$\omega - (\beta - f(\alpha)) \wedge e^4 - d_{\mathfrak{g}}\alpha = \omega_1 + \beta \wedge e^4 - (\beta - f(\alpha)) \wedge e^4 - d_{\mathfrak{u}}\alpha - f(\alpha) \wedge e^4 = \omega_1 - d_{\mathfrak{u}}\alpha \in \Lambda^2\mathfrak{u}^*$$

and $\omega - (\beta - f(\alpha)) \wedge e^4 - d_{\mathfrak{g}}\alpha$ is $d_{\mathfrak{g}}$-closed. Hence $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4 \oplus d_{\mathfrak{g}}(V)$.
Step IV: $\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \cap d_{\mathfrak{u}}(\mathfrak{u}^*) = \{0\}$:
Let $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \cap d_{\mathfrak{u}}(\mathfrak{u}^*)$. Then $\omega = d_{\mathfrak{u}}\beta$ for some $\beta \in \mathfrak{u}^*$ and $d_{\mathfrak{g}}\omega = 0$. We may assume that $\beta \in V$. But then

$$0 = d_{\mathfrak{g}}\omega = d_{\mathfrak{g}}(d_{\mathfrak{g}}\beta - f(\beta) \wedge e^4) = -d_{\mathfrak{u}}(f(\beta)) \wedge e^4.$$

Since $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$ and $f|_V$ is injective we get $\beta = 0$ and so $\omega = 0$ as claimed.
Step V: Norm estimates:
Note first that the identity

$$(d_{\mathfrak{g}}\alpha)^2 = 2 d_{\mathfrak{u}}\alpha \wedge f(\alpha) \wedge e^4$$

and the fact that $f|_V$ and $d_{\mathfrak{u}}|_V$ are injective imply the existence of a constant $A > 0$ such that

$$|(d_{\mathfrak{g}}\alpha)^2| \geq A \|\alpha\|^2. \tag{2.3}$$

Note further the sign of $(d_{\mathfrak{g}}\alpha)^2 \in \Lambda^4\mathfrak{g}^* \cong \mathbb{R}$ for $\alpha \neq 0$ does not depend on α. Namely, let $F : V \to \mathbb{R}$, $F(\alpha) := (d_{\mathfrak{g}}\alpha)^2$. For $\dim(V) > 1$ the set $V\backslash\{0\}$ is connected, while $F(V\backslash\{0\})$ would be disconnected if the sign would depend on $\alpha \neq 0$, contradicting the continuity of F. If $\dim(V) = 1$ then the statement follows from the fact that F is homogeneous of degree two in α.
Next we consider the space W_λ for an arbitrary $\lambda \neq 0$. Lemma 2.10 (c) tells us that

$$(\omega + \lambda g(\omega) \wedge e^4)^2 = 2\lambda\, \omega \wedge g(\omega) \wedge e^4 = 0$$

for $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$ implies $\omega \in d_{\mathfrak{u}}(\mathfrak{u}^*)$. But Step IV tells us that then $\omega = 0$. Thus there exists $C > 0$, independent of λ, such that

$$|(\omega + \lambda g(\omega) \wedge e^4)^2| \geq C|\lambda| \|\omega\|^2 \tag{2.4}$$

for all $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$. Note that for fixed $\lambda \neq 0$, argueing as above, we see that the sign of $(\omega + \lambda g(\omega) \wedge e^4)^2 \in \mathbb{R}$ does not depend on ω. But it gets reversed if we reverse the sign of λ. Hence we may assume that it is choosen such that $\omega_1^2 \cdot \omega_2^2 > 0$ for all $\omega_1 \in d_{\mathfrak{g}}(V)\backslash\{0\}$, $\omega_2 \in W_\lambda\backslash\{0\}$.
For all $\alpha \in V$ and $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$ Lemma 2.10 (b) tells us that $d_{\mathfrak{u}}\alpha \wedge g(\omega) = 0$. Thus

$$2 d_{\mathfrak{g}}\alpha \wedge (\omega + \lambda g(\omega) \wedge e^4) = 2(d_{\mathfrak{u}}\alpha + f(\alpha) \wedge e^4) \wedge (\omega + \lambda g(\omega) \wedge e^4) = 2 f(\alpha) \wedge e^4 \wedge \omega$$

and there exists a constant $B > 0$ such that

$$|2 d_{\mathfrak{g}}\alpha \wedge (\omega + \lambda g(\omega) \wedge e^4)| \leq B \|\alpha\| \|\omega\|. \tag{2.5}$$

Step VI: All non-zero elements in $d_{\mathfrak{g}}(V) \oplus W_\lambda$ are symplectic for appropriate $\lambda \neq 0$:
Let $0 \neq \omega_0 = \omega_1 + \omega_2 \in d_{\mathfrak{g}}(V) \oplus W_\lambda$ with $\omega_1 = d_{\mathfrak{g}}\alpha \in d_{\mathfrak{g}}(V)$ for some $\alpha \in V$ and $\omega_2 = \omega + \lambda g(\omega) \wedge e^4 \in W_\lambda$ for some $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$. Then we may assume that both α and ω are not zero by the equations (2.3) and (2.4). The discriminant of the polynomial $\omega_0^2 = (\omega_1 + X\omega_2)^2 = \omega_2^2 + 2X\, \omega_1 \wedge \omega_2 + X^2\, \omega_1^2$ is given by

$$(2\omega_1 \wedge \omega_2)^2 - 4\, \omega_1^2 \cdot \omega_2^2 \leq B^2 \|\alpha\|^2 \|\omega\|^2 - 4|\lambda| A\, C \|\alpha\|^2 \|\omega\|^2 = (B^2 - 4|\lambda| A\, C) \|\alpha\|^2 \|\omega\|^2,$$

where we used equations (2.3), (2.4) and (2.5) and the fact that the sign of $\omega_1^2 \cdot \omega_2^2$ may be assumed to be positive. But for sufficiently large $|\lambda|$, independent of α and ω, this is negative and the quadratic polynomial in X does not have a real root. In particular $X = 1$ is not a real root. But this is equivalent to saying that $\omega_0 = \omega_1 + \omega_2$ is non-degenerate. This finishes the proof.

□

A four-dimensional Lie algebra $\mathfrak{g}$ which admits an ideal of codimension one which is isomorphic to $\mathfrak{h}_3$ admits a certain decomposition into subspaces which turns out to be useful for computations. A proof that such a decomposition exists may be found in [1].

Lemma 2.18. *If $\mathfrak{g}$ is a four-dimensional Lie algebra $\mathfrak{g}$ which possesses an ideal $\mathfrak{u}$ isomorphic to $\mathfrak{h}_3$, then there exist an element $e_4 \in \mathfrak{g} \backslash \mathfrak{u}$, an element $e^1 \in \mathfrak{u}^*$, a two-dimensional subspace $V_2 \subseteq \mathfrak{u}^*$ with $\mathrm{span}(e^1) \oplus V_2 = \mathfrak{u}^*$, a linear map $F : V_2 \to V_2$ and a non-zero two-form $\nu \in \Lambda^2 V_2 \backslash \{0\}$ such that $de^1 = \mathrm{tr}(F) e^{14} + \nu$, $d\alpha = F(\alpha) \wedge e^4$ for all $\alpha \in V_2$ and $de^4 = 0$. In this case, $\mathrm{tr}(F) = 0$ if and only if $\mathfrak{g}$ is unimodular.*

2.5. **Algebraic invariants.** Westwick introduced certain kinds of algebraic invariants to classifiy the orbits of three-forms on a seven-dimensional real vector space V under $\mathrm{GL}(V)$ [19]. In [8], we already used these invariants to get obstructions to the existence of G_2-structures. For that reason we determined the values of these invariants for the orbit of all Hodge duals in $\Lambda^4 V^*$. Here we briefly recapitulate the definitions and results and restrict ourselves, without further notification, to *real* vector spaces:

Definition 2.19. Let V be an n-dimensional vector space. The *Grassman cone* $G_k(V)$ consists of all *decomposable* k-forms on V, i.e. of all those k-forms $\psi \in \Lambda^k V^*$ such that there are k one-forms $\alpha_1, \ldots, \alpha_k$ with $\psi = \alpha_1 \wedge \ldots \wedge \alpha_k$. The *length* $l(\phi)$ of an arbitrary k-form $\phi \in \Lambda^k V^*$ is defined as the minimal number m of decomposable k-forms $\phi_1, \ldots, \phi_m$ which is needed to write ϕ as the sum of $\phi_1, \ldots, \phi_m$, i.e. as $\phi = \sum_{i=1}^m \phi_i$. The *rank* $\mathrm{rk}(\phi)$ of ϕ is the dimension of the subspace

$$[\phi] := \bigcap \left\{ \phi \in \Lambda^k U | U \text{ is a subspace of } V^* \right\}$$

or, equivalently, the rank of the linear map $T : V \to \Lambda^{k-1} V^*$, $T(v) = v \lrcorner \phi$. $[\phi]$ is also called the *support (of ϕ)*. For a vector $v \notin \ker T$ and a subspace $W \subseteq V$ such that $W \oplus \mathrm{span}(v) \oplus \ker T = V$ is a direct vector space sum, we set $\rho(v, W) := (v \lrcorner \phi)|_W \in \Lambda^{k-1} W^*$ and $\Omega(W) := \phi|_W \in \Lambda^k W^*$. We introduce two more algebraic invariants by

$$r(\phi) := \min \left\{ l(\Omega) | \Omega = \Omega(W) \in \Lambda^k W^*, \dim(W) = (\mathrm{rk}(\phi) - 1), W \cap \ker T = \{0\} \right\},$$
$$m(\phi) := \min \left\{ l(\rho) | \rho = \rho(v, W) \in \Lambda^{k-1} W^*, v \notin \ker T, W \oplus \mathrm{span}(v) \oplus \ker T = V \right\}.$$

Remark 2.20. An equivalent description of the numbers $r(\phi)$ and $m(\phi)$ is obtained as follows: Let $\alpha \in [\phi]$, $\alpha \neq 0$ and U be a complement of $\mathrm{span}(\alpha)$ in $[\phi]$. Denote by $\rho(\alpha, U) \in \Lambda^{k-1} U$ and $\Omega(\alpha, U) \in \Lambda^k U$ the unique three- and four-form on V such that

$$\phi = \rho(\alpha, U) \wedge \alpha + \Omega(\alpha, U).$$

Then

$$r(\phi) = \min\{l(\Omega) | \Omega = \Omega(\alpha, U) \in \Lambda^k U, \alpha \in [\phi] \backslash \{0\}, U \oplus \mathrm{span}(\alpha) = [\phi]\},$$
$$m(\phi) = \min\{l(\rho) | \rho = \rho(\alpha, U) \in \Lambda^{k-1} U, \alpha \in [\phi] \backslash \{0\}, U \oplus \mathrm{span}(\alpha) = [\phi]\}.$$

We will mostly work with this description.

Remark 2.21.
- The numbers $l(\phi), \mathrm{rk}(\phi), r(\phi)$ and $m(\phi)$ for a k-form $\phi \in \Lambda^k V^*$ are invariant under isomorphisms $f^* : \Lambda^k V^* \to \Lambda^k W^*$ induced by isomorphisms $f : W \to V$ or, more generally, under monomorphisms $g_* : \Lambda^k V^* \to \Lambda^k W^*$ induced by monomorphisms $g : V^* \to W^*$. In particular, these four numbers are invariants of orbits under the natural action of $\mathrm{GL}(V)$ on $\Lambda^k V^*$. Moreover, if $W := V \oplus \mathrm{span}(w)$, $w \neq 0$ and $\alpha \in \mathrm{span}(w)^*$, $\alpha \neq 0$, then $l(\alpha \wedge \phi) = l(\phi)$.
- Let $\phi \in \Lambda^k V^*$ be a k-form and set $T : V \to \Lambda^{k-1} V^*$, $T(w) := w \lrcorner \phi$ as above. Let $v \notin \ker T$ and let W_1, W_2 be two subspaces of V such that $V = \mathrm{span}(v) \oplus W_i \oplus \ker T$ for $i = 1, 2$. Let $\rho(v, W_i) := (v \lrcorner \phi)|_{W_i}$ for $i = 1, 2$ and denote by $\mathrm{pr}_{W_2} : V \to W_2$ the projection of V onto W_2 along $\mathrm{span}(v) \oplus \ker T$. Then $f : W_1 \to W_2$, $f := \mathrm{pr}_{W_2}|_{W_1}$ is an isomorphism with $f^* \rho(v, W_2) = \rho(v, W_1)$. In this sense, $\rho(v, W_i)$ essentially only depends on v and the values of the above introduced algebraic invariants coincide for $\rho(v, W_1)$ and $\rho(v, W_2)$.
- A two-form $\omega \in \Lambda^2 V^*$ has length l if and only if $\omega^l \neq 0$ and ω^{l+1} is zero. If the dimension n of V is even, i.e. $n = 2m$, then the non-degenerate two-forms are exactly those of length m.
- There exists an isomorphism $\delta : \Lambda^k V^* \to \Lambda^{n-k} V^*$ such that $l(\phi) = l(\delta(\phi))$ for all $\phi \in \Lambda^k V^*$ [8]. Moreover, if $V = V_1 \oplus V_2$ as vector spaces then we may assume that $\delta : \Lambda^{k_1} V_1^* \wedge \Lambda^{k_2} V_2^* \to \Lambda^{n_1 - k_1} V_1^* \wedge \Lambda^{n_2 - k_2} V_2^*$, where $n_i := \dim(V_i)$, $i = 1, 2$ (e.g. we may choose an appropriate Hodge star operator).

The following lemma was proven in [8].

Lemma 2.22. *Let φ be a G_2-structure on a seven-dimensional vector space V. Let $v \in V \backslash \{0\}$ and W be a complement of $\mathrm{span}(v)$ in V. Then*

(a) $(\mathrm{rk}(\star_\varphi \varphi), l(\star_\varphi \varphi), r(\star_\varphi \varphi), m(\star_\varphi \varphi)) = (\mathrm{rk}(\varphi), l(\varphi), r(\varphi), m(\varphi)) = (7, 5, 3, 3)$.

(b) The three-form $\rho := (v \lrcorner \star_\varphi \varphi)|_W \in \Lambda^3 W^*$ fulfills

$$(\mathrm{rk}(\rho), l(\rho), r(\rho), m(\rho)) = (6, 3, 2, 2).$$

(c) The four-form $\Omega := \star_\varphi \varphi|_W \in \Lambda^4 W^*$ fulfills

$$(\mathrm{rk}(\Omega), l(\Omega), r(\Omega), m(\Omega)) = (6, 3, 1, 2).$$

Remark 2.23. We like to note that Lemma 2.22 may also be proved more directly. Therefore, note that by Remark 2.21 we may assume that the decomposition $\mathfrak{g} = \mathrm{span}(v) \oplus W$ is orthogonal with respect to the induced metric. It is well-known, see e.g. [5], that then $\Omega(v, W) = \frac{1}{2}\omega^2$ for some $\omega \in \Lambda^2 W^*$ such that $(\omega, \rho(v, W)) \in \Lambda^2 W^* \times \Lambda^3 W^*$ is an SU(3)-structure on W. [19] gives us now the values of the algebraic invariants for $\rho(v, W)$ and the ones for $\Omega(v, W) = \frac{1}{2}\omega^2$ are easily computed.

The next technical lemma will be used in some of the proofs of the next section.

Lemma 2.24. *Let V be a six-dimensional vector space.*

(a) Let $V = V_3 \oplus W_3$ be a decomposition into two vector spaces of dimension three and let $\Omega = \Omega_1 + \Omega_2 \in \Lambda^4 V^*$ with $\Omega_1 \in \Lambda^2 V_3^* \wedge \Lambda^2 W_3^*$ and $\Omega_2 \in V_3^* \wedge \Lambda^3 W_3^*$ be a four-form of length three. Then the length of Ω_1 is also three.

(b) Let $V = V_4 \oplus V_2$ be a decomposition into a vector space V_4 of dimension four and a vector space V_2 of dimension two. Let ρ be a three-form of rank six with $r(\rho) = 2$ such that $\rho \in \Lambda^2 V_4^* \wedge V_2^* \oplus V_4^* \wedge \Lambda^2 V_2^*$. Then, for any basis α_1, α_2 of V_2^*, the unique two-forms $\omega_1, \omega_2 \in \Lambda^2 V_4^*$ such that $\rho - \sum_{i=1}^2 \omega_i \wedge \alpha_i \in V_4^* \wedge \Lambda^2 V_2^*$ span a two-dimensional subspace in $\Lambda^2 V_4^*$ in which each non-zero element is of length two.

Proof. (a) We use a dual isomorphism δ adapted to the splitting as explained above. Then $\delta(\Omega_1) \in V_3^* \wedge W_3^*$ and $\delta(\Omega_2) \in \Lambda^2 V_3^*$. Since the length of $\delta(\Omega)$ is three, we have $0 \neq \delta(\Omega)^3 = (\delta(\Omega_1) + \delta(\Omega_2))^3 = \delta(\Omega_1)^3$. Thus $\delta(\Omega_1)$ and so Ω_1 has length three.

(b) There is $\beta \in V_4^*$ such that $\rho = \omega_1 \wedge \alpha_1 + \omega_2 \wedge \alpha_2 + \beta \wedge \alpha_1 \wedge \alpha_2$. We have to show that $l(a\omega_1 + b\omega_2) = 2$ for all $(a, b) \neq (0, 0)$. Without loss of generality, we may assume $a \neq 0$ and then even $a = 1$. If we rewrite ρ as

$$\rho = (\omega_2 + \beta \wedge \alpha_1) \wedge (\alpha_2 - b\alpha_1) + (\omega_1 + b\omega_2) \wedge \alpha_1$$

we see that $(\omega_1 + b\omega_2) \wedge \alpha_1 \in \Lambda^3(V_4^* \oplus \mathrm{span}(\alpha_1))$ and $(\omega_2 + \beta \wedge \alpha_1) \in \Lambda^2(V_4^* \oplus \mathrm{span}(\alpha_1))$. Thus $r(\rho) = 2$ implies $l((\omega_1 + b\omega_2) \wedge \alpha_1) = 2$ (consider $V^* = (V_4^* \oplus \mathrm{span}(\alpha_1)) \oplus \mathrm{span}(\alpha_2 - b\alpha_1)$) and so $l(\omega_1 + b\omega_2) = 2$.

□

The next lemma provides us with different criterions when a subspace of the two-forms in four dimensions consists, with the exception of the origin, entirely of two-forms of length two.

Lemma 2.25. *Let V be a four-dimensional vector space and $D \in \{0, 1, 2, 3\}$.*

(a) Let $W \subseteq \Lambda^2 V^*$ be a D-dimensional subspace such that each non-zero element in W is of length two. For each subset $I \subseteq \{1, 2, 3\}$ of cardinality D there exists a basis e^1, e^2, e^3, e^4 of V^* such that for $\omega_1 := e^{14} + e^{23}$, $\omega_2 := e^{13} - e^{24}$, $\omega_3 := e^{12} + e^{34}$ the set $\{\omega_i | i \in I\}$ is a basis of W.

(b) A subspace $W \subseteq \Lambda^2 V^*$ consists, with the exception of the origin, entirely of two-forms of length two if and only if there exists a Euclidean metric g and an orientation on V such that W is a subspace of the space of all self-dual two-forms on V.

(c) A two-dimensional subspace $W \subseteq \Lambda^2 V^*$ consists, with the exception of the origin, entirely of two-forms of length two if and only if there exist two two-forms $\tilde{\omega}_1, \tilde{\omega}_2 \in W$ such that $0 \neq \tilde{\omega}_1^2$ and the numbers $B \in \mathbb{R}$ and $C \in \mathbb{R}$ defined by $2\tilde{\omega}_1 \wedge \tilde{\omega}_2 = B\tilde{\omega}_1^2$ and $\tilde{\omega}_2^2 = C\tilde{\omega}_1^2$ fulfill $B^2 - 4C < 0$.

Proof. (a) is [19]. (b) follows directly from (a) by the observation that if e_1, e_2, e_3, e_4 is an oriented orthonormal basis of V, then a basis of all self-dual two-forms on V is given by $\omega_1, \omega_2, \omega_3$ as in (a). One direction in (c) follows directly from (a) since the two-forms ω_1, ω_2, ω_3 in (a) fulfill $\omega_1^2 = \omega_2^2 = \omega_3^2 \neq 0$ and $\omega_i \wedge \omega_j = 0$ for $i \neq j$.

For the other direction, let $\tilde{\omega}_1, \tilde{\omega}_2 \in W$ be such that $\tilde{\omega}_1^2 \neq 0$ and such that the numbers B, $C \in \mathbb{R}$, defined by $2\tilde{\omega}_1 \wedge \tilde{\omega}_2 = B\tilde{\omega}_1^2$ and $\tilde{\omega}_2^2 = C\tilde{\omega}_1^2$, fulfill $B^2 - 4C < 0$. Then, necessarily, $\tilde{\omega}_1$, $\tilde{\omega}_2$ are linearly independent. If there was $(a, b) \neq (0, 0)$ such that $a\tilde{\omega}_1 + b\tilde{\omega}_2$ is of length less than two, then $b \neq 0$ due to $\tilde{\omega}_1^2 \neq 0$. But so

$$0 = (a\tilde{\omega}_1 + b\tilde{\omega}_2)^2 = b^2 \left(\left(\frac{a}{b}\right)^2 \tilde{\omega}_1^2 + 2\frac{a}{b}\tilde{\omega}_1 \wedge \tilde{\omega}_2 + \tilde{\omega}_2^2 \right) = b^2 \left(\left(\frac{a}{b}\right)^2 + B\frac{a}{b} + C \right) \tilde{\omega}_1^2$$

implies that the quadratic polynomial $X^2 + BX + C$ in X has a real root. But the discriminant of this polynomial is $B^2 - 4C < 0$ and so the roots are not real, a contradiction. □

The final lemma in this subsection tells us how we may build a G_2-structure on a seven-dimensional vector space using two-forms of length two on a four-dimensional subspace $V_4 \subseteq V$. This lemma allows us in some occasions to reduce the construction of a cocalibrated G_2-structure on a Lie algebra to the construction of certain closed two-forms of length two on a four-dimensional subspace.

Lemma 2.26. *Let $V = V_4 \oplus V_3$ be a vector space decomposition into a four-dimensional subspace V_4 and a three-dimensional subspace V_3. Let $\nu_1, \nu_2, \nu_3 \in \Lambda^2 V_3^*$ be a basis of $\Lambda^2 V_3^*$.*

Assume that there is a $k \in \{0,1,2,3\}$ *and* k *linearly independent two-forms* $\omega_1,\ldots,\omega_k \in \Lambda^2 V_4^*$ *such that* $\operatorname{span}(\omega_1,\ldots,\omega_k)$ *consists with the exception of the origin entirely of two-forms of length two. Then there exist* $(3-k)$ *linearly independent two-forms* $\omega_{k+1},\ldots,\omega_3 \in \Lambda^2 V_4^*$ *such that*

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2*-structure with coassociative/associative splitting* $V_4 \oplus V_3$.

Proof. By Lemma 2.25, we may assume that $\omega_i = \sum_{j=1}^k a_{ij}\tilde{\omega}_j$ for $i=1,\ldots,k$, where

$$\tilde{\omega}_1 = e^{14}+e^{23},\ \tilde{\omega}_2 = e^{13}-e^{24},\ \tilde{\omega}_3 = e^{12}+e^{34},$$

e^1,e^2,e^3,e^4 is a basis of V_4^* and $A=(a_{ij})_{i,j} \in \mathrm{GL}_k(\mathbb{R})$. Moreover, we may assume that $\omega_1 = \tilde{\omega}_1$, i.e. $a_{1j}=0$ for $j=2,\ldots,k$, $a_{11}=1$ if $k>0$. Set $\omega_i := \tilde{\omega}_i$ for $i=k+1,\ldots,3$ and

$$B := \begin{pmatrix} A & 0 \\ 0 & I_{3-k} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{R}).$$

Then

$$\sum_{i=1}^{3} \omega_i \wedge \nu_i = \sum_{i,j=1}^{3} b_{ij}\tilde{\omega}_j \wedge \nu_i = \sum_{j=1}^{3} \tilde{\omega}_j \wedge \tilde{\nu}_j$$

with $\tilde{\nu}_j := \sum_{i=1}^3 b_{ij}\nu_i \in \Lambda^2 V_3^*$. Since $B \in \mathrm{GL}_3(\mathbb{R})$, $\tilde{\nu}_1,\tilde{\nu}_2,\tilde{\nu}_3$ is again a basis of $\Lambda^2 V_3^*$. Therefore, there exists a basis e^5,e^6,e^7 of V_3^* such that $\tilde{\nu}_1 = e^{57}$, $\tilde{\nu}_2 = e^{67}$ and $\tilde{\nu}_3 = e^{56}$. But then

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2 = \sum_{i=1}^{3} \tilde{\omega}_i \wedge \tilde{\nu}_i + \frac{1}{2}\tilde{\omega}_1^2 = e^{1457}+e^{2357}+e^{1367}-e^{2467}+e^{1256}+e^{3456}+e^{1234}$$

is the Hodge dual of a G_2-structure with adapted basis $e_1,\ldots,e_7$. Moreover, $V_4 \oplus V_3$ is a coassociative/associative splitting of V. □

3. Classification Results

3.1. **Existence.** In this subsection we state different existence results which will be used in subsections 3.3-3.6 to prove Theorem 1.1. We begin with a general proposition which is true for any seven-manifold. This proposition is used afterwards to derive different more specific existence results for left-invariant cocalibrated G_2-structures on Lie groups.

Proposition 3.1. *Let* M *be a seven-dimensional manifold. Assume that there exists a* G_2*-structure* φ *on* M *which admits a coassociative/associative splitting* $TM = E_4 \oplus E_3$ *such that the following is true:*

(i) $\Omega_1 := (\star_\varphi \varphi)|_{E_4} \in \Gamma(\Lambda^4 E_4^*) \cong \Gamma(\Lambda^4 {E_3}^0) \subseteq \Gamma(\Lambda^4 T^*M)$ *is closed.*

(ii) There exists a bounded four-form $\Phi \in \Gamma(\Lambda^3 E_4^* \wedge E_3^*)$ *(i.e.* $\|\Phi\|_{C_0} < \infty$*) with* $d\Phi = d\Omega_2$ *for the four-form* $\Omega_2 := \star_\varphi\varphi - \Omega_1 \in \Gamma(\Lambda^2 E_4^* \wedge \Lambda^2 E_3^*)$.

Then M *admits a cocalibrated* G_2*-structure, e.g. a* G_2*-structures* $\varphi_\lambda \in \Omega^3(M)$ *whose Hodge dual is given by*

$$\Psi_\lambda := \lambda^4\Omega_1 + \lambda^2\Omega_2 - \lambda^2\Phi$$

for $\lambda \in \mathbb{R}$ *with* $|\lambda| > \frac{\|\Phi\|_{C_0}}{\epsilon_0}$. *Thereby,* ϵ_0 *is the constant in Lemma 2.5.*

Proof. Let $p \in M$. Due to the coassociative/associative splitting, Remark 2.7 ensures the existence of an adapted basis $e_1, \ldots, e_7$ for φ_p such that $e_1, \ldots, e_4$ is a basis of $(E_4)_p$ and $e_5, \ldots, e_7$ is a basis of $(E_3)_p$. Thus, in fact, $(\Omega_2)_p$ is in $\Lambda^2 (E_4)_p^* \wedge \Lambda^2 (E_3)_p^*$ and $(\Omega_1)_p = e^{1234}$. Hence

$$\sigma_\lambda := \lambda^4 (\Omega_1)_p + \lambda^2 (\Omega_2)_p$$

is, for each $\lambda \neq 0$, the Hodge-Dual of a G_2-structure on T_pM with adapted basis $\frac{1}{\lambda}e_1$, $\frac{1}{\lambda}e_2$, $\frac{1}{\lambda}e_3$, $\frac{1}{\lambda}e_4$, e_5, e_6, e_7. This implies $\|\lambda^3 \Phi_p\|_\lambda = \|\Phi_p\|_1 = \|\Phi_p\|_{\varphi_p}$ for all $\lambda \neq 0$, where $\|\cdot\|_\lambda$ is the norm on T_pM induced by σ_λ. Thus

$$\|(\Psi_\lambda)_p - \sigma_\lambda\|_\lambda = \left\|\lambda^2 \Phi_p\right\|_\lambda = \frac{\|\Phi_p\|_{\varphi_p}}{|\lambda|} \leq \frac{\|\Phi\|_{C^0}}{|\lambda|} < \epsilon_0.$$

for all $|\lambda| > \frac{\|\Phi\|_{C^0}}{\epsilon_0}$. Hence Lemma 2.5 shows that Ψ_λ is the Hodge dual of a cocalibrated G_2-structure on M. □

Remark 3.2.
- Proposition 3.1 can easily be generalized to include other splittings of the tangent bundle such that the G_2-structure has everywhere an adapted basis which is also adapted to this splitting, even with more summands. However, we only use the version above in this article.
- The condition on the boundedness of Φ is trivially fulfilled if Φ is left-invariant or M is compact. In the left-invariant case, if the initial G_2-structure φ, the splitting $E_4 \oplus E_3$ and Φ are left-invariant, so is the cocalibrated G_2-structure we get by Proposition 3.1.
- To prove an analog of Proposition 3.1 in the left-invariant case for G_2- and also for G_2^*-structures we do not need at all a metric. We only need that the orbit of all Hodge duals is open in both cases. For the proof we note that on a seven-dimensional Lie algebra $\mathfrak{g}$ the openness of the orbit implies that for any sequence $(A_n)_n$, $A_n \in \mathrm{GL}(\mathfrak{g})$, any Hodge dual $\Psi \in \Lambda^4 \mathfrak{g}^*$ and any sequence $(\Phi_n)_n$, $\Phi_n \in \Lambda^4 \mathfrak{g}^*$ with $\lim_{n\to\infty} \Phi_n = 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ the four-form $A_n^*(\Psi + \Phi_n)$ is again a Hodge dual of the same type.

  Let now $\varphi \in \Lambda^3 \mathfrak{g}^*$ be a G_2- or G_2^*-structure and $\mathfrak{g} = E_4 \oplus E_3$ be a splitting into a four-dimensional subspace E_4 and a three-dimensional subspace E_3 such that $\Psi := \star_\varphi \varphi = \Omega_1 + \Omega_2$ with $\Omega_1 \in \Lambda^4 E_4^*$, $\Omega_2 \in \Lambda^2 E_4^* \wedge \Lambda^2 E_3^*$, $d\Omega_1 = 0$ and such that there exists $\Phi \in \Lambda^3 E_4^* \wedge E_3^*$ with $d\Omega_2 = d\Phi$. Define $A_n \in \mathrm{GL}(\mathfrak{g})$ such that it acts by multiplication with n on E_4 and by the identity map on E_3 and set $\Phi_n := -\frac{\Phi}{n} \in \Lambda^3 E_4^* \wedge E_3^*$. Then our previous considerations show that

  $$\Psi_n := A_n^*(\star_\varphi \varphi + \Phi_n) = A_n^* \left(\Omega_1 + \Omega_2 - \frac{\Phi}{n} \right) = n^4 \Omega_1 + n^2 \Omega_2 - n^2 \Phi$$

  is, for n large enough, a Hodge dual of the same type as Ψ. Moreover, our assumptions imply that it is closed and so defines a cocalibrated G_2- or G_2^*-structure on $\mathfrak{g}$, respectively.

Proposition 3.3. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional Lie algebra $\mathfrak{g}_3$.*

(a) If $\mathfrak{g}_3$ is unimodular and there exists a $D := h^2(\mathfrak{g}_3)$-dimensional subspace W of $\Lambda^2 \mathfrak{g}_4^*$ such that each non-zero element in W is a symplectic two-form, then $\mathfrak{g}$ admits a cocalibrated G_2-structure.

(b) If $\mathfrak{g}_4 \in \{A_{4,12}, \mathfrak{r}_2 \oplus \mathfrak{r}_2\}$ and $\mathfrak{g}_3$ admits a contact-form α, then $\mathfrak{g}$ admits a cocalibrated G_2-structure.

(c) *If $\mathfrak{g}_4$ is unimodular, admits a codimension one ideal $\mathfrak{u} \cong \mathfrak{h}_3$, $\mathfrak{g}_3$ is not unimodular and $h^1(\mathfrak{g}_4) + h^1(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \geq 2$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure.*

Proof. (a) Choose a basis ν_1, ν_2, ν_3 of $\Lambda^2\mathfrak{g}_3^*$ such that $\nu_{D+1} = d\alpha_{D+1}, \ldots, \nu_3 = d\alpha_3$ is a basis of $d(\mathfrak{g}_3^*)$, $\alpha_{D+1}, \ldots, \alpha_3 \in \mathfrak{g}_3^*$. Note that this is possible since the unimodularity of $\mathfrak{g}_3$ exactly means that all two-forms on $\mathfrak{g}_3$ are closed. Furthermore, choose a basis $\omega_1, \ldots, \omega_D$ of W. Then Lemma 2.26 implies that there exist two-forms $\omega_{D+1}, \ldots, \omega_3 \in \Lambda^2\mathfrak{g}_4^*$ such that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure with coassociative/associative splitting $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$. Since $d(\Lambda^2\mathfrak{g}_3^*) = 0$, the identity $d(\sum_{i=1}^3 \omega_i \wedge \nu_i) = d(-\sum_{i=D+1}^3 d\omega_i \wedge \alpha_i)$ is true and $\sum_{i=D+1}^3 d\omega_i \wedge \alpha_i \in \Lambda^3\mathfrak{g}_4^* \wedge \mathfrak{g}_3^*$. Hence Proposition 3.1 implies the result.

(b) Let e^1, e^2, e^3, e^4 be a basis of $\mathfrak{g}_4^* \in \{A_{4,12}^*, (\mathfrak{r}_2 \oplus \mathfrak{r}_2)^*\}$ as in Table 2, i.e. $de^1 = e^{14} + e^{23}$, $de^2 = e^{24} - \epsilon e^{13}$, $de^3 = 0 = de^4$, where $\epsilon = 1$ if $\mathfrak{g}_4 = A_{4,12}$ and $\epsilon = -1$ if $\mathfrak{g}_4 = \mathfrak{r}_2 \oplus \mathfrak{r}_2$. Set $V_4^* := \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*$, $V_3^* := \mathrm{span}(e^1, e^2, e^3)$. Then $d(\Lambda^4 V_4^*) = \{0\}$. Let $\alpha_1 \in \mathfrak{g}_3^*$ be a contact form and set $\omega_1 := 2e^4 \wedge \alpha_1 - d\alpha_1 \in \Lambda^2 V_4^*$. Then ω_1 is of length two. Hence, if we set $\nu_1 := e^{12}$, $\nu_2 := e^{13}$, $\nu_3 := e^{23}$, Lemma 2.26 implies the existence of two-forms $\omega_2, \omega_3 \in \Lambda^2 V_4^*$ such that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure with coassociative/associative splitting $\mathfrak{g} = V_4 \oplus V_3$. Decompose $\omega_i = e^4 \wedge \alpha_i + \theta_i$ with $\alpha_i \in \mathfrak{g}_3^*$, $\theta_i \in \Lambda^2\mathfrak{g}_3^*$ for $i = 2, 3$. Then $d(\omega_1 \wedge \nu_1) = d(2e^4 \wedge \alpha_1 \wedge e^{12} - d\alpha_1 \wedge e^{12}) = 0$ and so the differential of the four-form $\sum_{i=1}^3 \omega_i \wedge \nu_i$ is given by

$$\begin{aligned} d\left(\sum_{i=1}^{3} \omega_i \wedge \nu_i\right) &= 0 + d(e^{134} \wedge \alpha_2 + e^{234} \wedge \alpha_3) + d(e^{13} \wedge \theta_2 + e^{23} \wedge \theta_3) \\ &= d(\epsilon e^{24} \wedge d\alpha_2 - e^{14} \wedge d\alpha_3) \\ &\quad + d(\epsilon(e^{24} \wedge \theta_2 - e^2 \wedge d\theta_2) - e^{14} \wedge \theta_3 + e^1 \wedge d\theta_3) \\ &= d(e^1 \wedge \rho_1 - \epsilon e^2 \wedge \rho_2). \end{aligned}$$

with $\rho_1 := -e^4 \wedge (d\alpha_3 + \theta_3) + d\theta_3$, $\rho_2 := -e^4 \wedge (d\alpha_2 + \theta_2) + d\theta_2 \in \Lambda^3 V_4^*$. Thus $e^1 \wedge \rho_1 - \epsilon e^2 \wedge \rho_2$ is in $V_3^* \wedge \Lambda^3 V_4^*$ and Proposition 3.1 implies the result.

(c) By Lemma 2.18 we may decompose $\mathfrak{g}_4^*$ into $\mathrm{span}(e^1) \oplus V_2 \oplus \mathrm{span}(e^4)$ for $e^1, e^4 \in \mathfrak{g}_4^*$ and a two-dimensional subspace V_2 such that $0 \neq de^1 \in \Lambda^2 V_2$, $d\beta = F(\beta) \wedge e^4$ for all $\beta \in V_2$, $F : V_2 \to V_2$ a trace-free linear map, and $de^4 = 0$. Moreover, by Lemma 2.11 we may decompose $\mathfrak{g}_3^* = W_2 \oplus \mathrm{span}(e^7)$ with $e^7 \in \mathfrak{g}_3^*$ and a two-dimensional subspace W_2 such that $d\beta = G(\beta) \wedge e^7$ for all $\beta \in W_2$, $G : W_2 \to W_2$ a linear map which is not trace-free, and $de^7 = 0$. By rescaling e^7 we may assume that $\mathrm{tr}(G) = 1$.

We have $\ker d|_{\Lambda^2\mathfrak{g}_4^*} = \mathrm{span}(de^1) \oplus V_2 \wedge e^4 \oplus \ker(F) \wedge e^1$. Thus the identity

$$2 - \mathrm{rk}(F) + 3 = \dim(\ker(F)) + 3 = \dim(\ker d|_{\Lambda^2\mathfrak{g}_4^*}) = h^2(\mathfrak{g}_4) + 4 - h^1(\mathfrak{g}_4)$$

is true. Moreover, $\dim(\ker G) = h^1(\mathfrak{g}_3) - 1$ and so the condition in the statement is equivalent to $\dim(\ker G) \geq 2 - \mathrm{rk}(F)$. Hence we may choose a basis α_1, α_2 of V_2, elements $\gamma_i \in V_2$, $1 \leq i \leq \mathrm{rk}(F)$, and a basis β_1, β_2 of W_2 such that $de^1 = \alpha_1 \wedge \alpha_2$, such that $\alpha_i = F(\gamma_i)$, $1 \leq i \leq \mathrm{rk}(F)$, is a basis of $F(V_2)$ and such that $\mathrm{span}(\beta_j|\mathrm{rk}(F)+1 \leq j \leq 2)$ is a subspace of $\ker G$. Set $V_4^* := \mathrm{span}(e^1) \oplus V_2 \oplus \mathrm{span}(e^7)$, $V_3^* := W_2 \oplus \mathrm{span}(e^4)$ and

$$\begin{aligned}
\nu_1 :=& \beta_1 \wedge \beta_2, \quad \nu_2 := \beta_1 \wedge e^4, \quad \nu_3 := -\beta_2 \wedge e^4,\\
\omega_1 :=& e^{71} - de^1 = e^{71} - \alpha_1 \wedge \alpha_2, \quad \omega_2 := e^7 \wedge \alpha_2 - e^1 \wedge \alpha_1,\\
\omega_3 :=& e^7 \wedge \alpha_1 + e^1 \wedge \alpha_2.
\end{aligned}$$

Since ω_1, ω_2, ω_3 span a three-dimensional subspace in $\Lambda^2 V_4^*$ in which each non-zero element has length two by Lemma 2.25 and ν_1, ν_2, ν_3 is a basis of $\Lambda^2 V_3^*$, Lemma 2.26 implies that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure with coassociative/associative splitting $V_4 \oplus V_3$. Moreover,

$$\begin{aligned}
d(\omega_1 \wedge \nu_1) &= d(e^{71} \wedge \beta_1 \wedge \beta_2 - de^1 \wedge \beta_1 \wedge \beta_2)\\
&= -e^7 \wedge de^1 \wedge \beta_1 \wedge \beta_2 + \mathrm{tr}(G) de^1 \wedge \beta_1 \wedge \beta_2 \wedge e^7 = 0
\end{aligned}$$

and so

$$\begin{aligned}
d\left(\sum_{i=1}^{3} \omega_i \wedge \nu_i\right) &= d\left(-\sum_{i=1}^{2} e^1 \wedge \alpha_i \wedge \beta_i \wedge e^4\right)\\
&= -\sum_{i=1}^{\mathrm{rk}(F)} F(\gamma_i) \wedge e^4 \wedge e^1 \wedge G(\beta_i) \wedge e^7\\
&= d\left(-\sum_{i=1}^{\mathrm{rk}(F)} \gamma_i \wedge e^1 \wedge G(\beta_i) \wedge e^7\right).
\end{aligned}$$

But $-\sum_{i=1}^{\mathrm{rk}(F)} \gamma_i \wedge e^1 \wedge G(\beta_i) \wedge e^7$ is in $V_3^* \wedge \Lambda^3 V_4^*$ and $d(\Lambda^4 V_4^*) = \{0\}$. So again Proposition 3.1 implies the result.

□

Remark 3.4. The following generalization of Proposition 3.3 (a) may be proved easily with the aid of Proposition 3.1 and Lemma 2.25 (b):
Let $M = N \times G$ be a seven-dimensional manifold such that N is a four-dimensional compact Riemannian manifold with trivial bundle of self-dual two-forms and such that G is a unimodular three-dimensional Lie group. If N admits $D := h^2(\mathfrak{g})$ ($\mathfrak{g}$ being the Lie algebra of G) self-dual, closed two-forms $\omega_i \in \Omega^2 N$ such that $\omega_i \wedge \omega_j = 0$ and $\omega_i^2 = \omega_j^2$ for $i \neq j$, then M admits a cocalibrated G_2-structure which is invariant under the left-action of G on $M = N \times G$ given by left-translation on the second factor.

$D = 0$ is allowed in Proposition 3.3 (a). Since each non-solvable four-dimensional Lie algebra $\mathfrak{g}$ is a Lie algebra direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ with $\mathfrak{h} \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$, $h^2(\mathfrak{so}(3)) = h^2(\mathfrak{so}(2,1)) = 0$ and $\mathfrak{so}(3), \mathfrak{so}(2,1)$ are the only three-dimensional non-solvable Lie algebras we get

Corollary 3.5. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional Lie algebra $\mathfrak{g}_3$. If $\mathfrak{g}$ is not solvable, then $\mathfrak{g}$ admits a cocalibrated* G_2*-structure.*

3.2. **Obstructions.** In this subsection we derive obstructions to the existence of cocalibrated G_2-structures on Lie algebras, which we use in subsections 3.3 - 3.6 to prove Theorem 1.1. We start with

Proposition 3.6. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$. Assume that $\mathfrak{g}$ admits a cocalibrated G_2-structure and $\mathfrak{g}_4$ admits a unique unimodular ideal $\mathfrak{u}$ of codimension one. Then*

$$h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) \leq 4.$$

Proof. Let Ψ be the Hodge dual of a cocalibrated G_2-structure. Fix an element $e_4 \in \mathfrak{g}\backslash\mathfrak{u}$ and let $e^4 \in \mathfrak{u}^0$ be such that $e^4(e_4) = 1$. We set

$$\Lambda^{i,j,k} := \Lambda^i\mathfrak{u}^* \wedge \Lambda^j\mathfrak{g}_3^* \wedge \Lambda^k\mathrm{span}(e^4)$$

and denote by $\theta^{i,j,k}$ the projection of θ into $\Lambda^{i,j,k}$ for all $i, j, k \in \mathbb{N}_0$ and all $(i+j+k)$-forms $\theta \in \Lambda^{i+j+k}\mathfrak{g}^*$.
Generally, by Lemma 2.9, we have

$$d(\Lambda^{i,j,0}) \subseteq \Lambda^{i+1,j,0} \oplus \Lambda^{i,j,1} \oplus \Lambda^{i,j+1,0}, \quad d(\Lambda^{i,j,1}) \subseteq \Lambda^{i+1,j,1} \oplus \Lambda^{i,j+1,1}$$

for all $i, j \in \mathbb{N}_0$ and the unimodularity of $\mathfrak{u}$ and $\mathfrak{g}_3$ imply that for all $i \in \mathbb{N}_0$:

$$d(\Lambda^{2,i,0}) \subseteq \Lambda^{2,i,1} \oplus \Lambda^{2,i+1,0},\ d(\Lambda^{2,i,1}) \subseteq \Lambda^{2,i+1,1},\ d(\Lambda^{i,2,0}) \subseteq \Lambda^{i+1,2,0} \oplus \Lambda^{i,2,1},\ d(\Lambda^{i,2,1}) \subseteq \Lambda^{i+1,2,1}.$$

We show that there are $D := h^2(\mathfrak{g}_3)$ linearly independent closed two-forms $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ such that $\mathrm{span}(\omega_1, \ldots, \omega_D) \cap \Lambda^{1,0,1} = \{0\}$. Note that $\dim(\ker d|_{\Lambda^{1,0,1}}) = h^1(\mathfrak{u})$ since $\ker d|_{\Lambda^{1,0,1}} = \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4$ by Lemma 2.9. Hence the existence of such $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ implies

$$\begin{aligned} &h^2(\mathfrak{g}_4) + 4 - h^1(\mathfrak{g}_4) = \dim(\ker d|_{\Lambda^2\mathfrak{g}^*}) \geq D + h^1(\mathfrak{u}) = h^2(\mathfrak{g}_3) + h^1(\mathfrak{u}) \\ &\Leftrightarrow\ h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \leq 4. \end{aligned}$$

We will obtain that the two-forms $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ are certain parts of $\Psi^{2,2,0} + \Psi^{1,2,1}$. Therefore, we decompose as

$$\Psi = \Omega + \rho \wedge e^4$$

with $\Omega \in \Lambda^4(\mathfrak{u}^* \oplus \mathfrak{g}_3^*)$, $\rho \in \Lambda^3(\mathfrak{u}^* \oplus \mathfrak{g}_3^*)$.
The first step of the proof is to show that the length of $\Omega^{2,2,0}$ is three. For that purpose, note that the identities

$$0 = (d\Psi)^{3,1,1} + (d\Psi)^{3,2,0} = d(\Omega^{3,1,0}), \quad 0 = (d\Psi)^{1,3,1} + (d\Psi)^{2,3,0} = d(\Omega^{1,3,0})$$

are true. If $\mathfrak{g}_4$ is not unimodular, then $d(\Lambda^{3,0,0}) = \Lambda^{3,0,1}$. Hence $\Omega^{3,1,0} = 0$ in this case. If $\dim([\mathfrak{g}_4, \mathfrak{g}_4]) = 3$, then $d|_{\Lambda^{1,0,0}}$ and so $d|_{\Lambda^{1,3,0}}$ is injective. Hence $\Omega^{1,3,0} = 0$ in this case. Now we know by Lemma 2.14 that the uniqueness of the unimodular ideal $\mathfrak{u}$ implies that $\mathfrak{g}_4$ is not unimodular or $\dim([\mathfrak{g}_4, \mathfrak{g}_4]) = 3$. In both cases, Lemma 2.24 shows that then $l(\Omega^{2,2,0}) = 3$.
Next we look at the $(2, 2, 1)$-component of $d\Psi$. This component is given by

$$0 = (d\Psi)^{2,2,1} = d(\Omega^{2,2,0}) + d(\rho^{2,1,0} \wedge e^4) + d(\rho^{1,2,0} \wedge e^4)$$

and we obtain

$$d(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4) \in d(\Lambda^2\mathfrak{g}_4^*) \wedge \Lambda^2\mathfrak{g}_3^* \cap \Lambda^3\mathfrak{g}_4^* \wedge d(\mathfrak{g}_3^*) = d(\Lambda^2\mathfrak{g}_4^*) \wedge d(\mathfrak{g}_3^*).$$

Let

$$\pi_k : \Lambda^k\mathfrak{g}_4^* \wedge \Lambda^2\mathfrak{g}_3^* \to (\Lambda^k\mathfrak{g}_4^* \wedge \Lambda^2\mathfrak{g}_3^*)/(\Lambda^k\mathfrak{g}_4^* \wedge d(\mathfrak{g}_3^*)) \cong \Lambda^k\mathfrak{g}_4^* \otimes H^2(\mathfrak{g}_3)$$

be the natural projection for $k \in \mathbb{N}$, where the last canonical isomorphism holds since $\mathfrak{g}_3$ is unimodular. Then we have $\pi_3 \circ d = (d \otimes \mathrm{id}) \circ \pi_2$. If we set $\Phi := \pi_2(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4)$, then

$$(d \otimes \mathrm{id})(\Phi) = \pi_3(d(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4)) = 0.$$

Thus, if we write

$$\Phi = \sum_{i=1}^{D} \omega_i \otimes \nu_i$$

for $\omega_1, \ldots, \omega_D \in \Lambda^2 \mathfrak{g}_4^*$ and some basis $\nu_1, \ldots, \nu_D$ of $H^2(\mathfrak{g}_3)$, then $\omega_1, \ldots, \omega_D$ are all closed. By choosing a complement V of $d(\mathfrak{g}_3^*)$ in $\Lambda^2 \mathfrak{g}_3^*$, we may identify $\nu_1, \ldots, \nu_D$ with elements in V and get

$$\Omega^{2,2,0} = \psi + \sum_{i=1}^{D} \omega_i^{2,0,0} \wedge \nu_i$$

with $\psi \in \Lambda^2 \mathfrak{g}_4^* \wedge d(\mathfrak{g}_3^*)$. Since the length of $\Omega^{2,2,0}$ is three and the length of ψ is at most $\dim(d(\mathfrak{g}_3^*))$, the length of $\sum_{i=1}^{D} \omega_i^{2,0,0} \wedge \nu_i$ has to be $3 - \dim(d(\mathfrak{g}_3^*)) = D$ and so $\omega_1^{2,0,0}, \ldots, \omega_D^{2,0,0}$ have to be linearly independent. Thus $\omega_1, \ldots \omega_D$ are linearly independent and $\mathrm{span}(\omega_1, \ldots, \omega_D) \cap \Lambda^{1,0,1} = \{0\}$. This finishes the proof. □

Proposition 3.6 gives us an obstruction if the three-dimensional part is unimodular, whereas the next proposition gives us an obstruction if the three-dimensional part is not unimodular.

Proposition 3.7. *(a) Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional non-unimodular Lie algebra $\mathfrak{g}_3$. Assume that $\mathfrak{g}$ admits a cocalibrated G_2-structure and that $\mathfrak{g}_4$ admits a codimension one Abelian ideal $\mathfrak{u}_3$. Then $\mathfrak{g}_4$ is unimodular and $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$.*

(b) Let $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathfrak{r}_2$ be a Lie algebra direct sum of a five-dimensional Lie algebra $\mathfrak{g}_5$ which admits a codimension one Abelian ideal $\mathfrak{u}$ and of the two-dimensional Lie algebra $\mathfrak{r}_2$. If $\mathfrak{g}$ admits a cocalibrated G_2-structure, then $\mathfrak{g}_5$ is unimodular

Proof. (a) Choose an element $e_4 \in \mathfrak{g}_4 \backslash \mathfrak{u}_3$ and an element $e_7 \in \mathfrak{g}_3 \backslash \mathfrak{u}_2$, where $\mathfrak{u}_2$ is a codimension one Abelian ideal in $\mathfrak{g}_3$. Let $e^4 \in {\mathfrak{u}_3}^0 \subseteq \mathfrak{g}_4^*$, $e^4(e_4) = 1$ and $e^7 \in {\mathfrak{u}_2}^0 \subseteq \mathfrak{g}_3^*$, $e^7(e_7) = 1$. Let $\Psi \in \Lambda^4 \mathfrak{g}^*$ be the Hodge dual of a cocalibrated G_2-structure, set $\Lambda^{i,j,k,l} := \Lambda^i \mathfrak{u}_3^* \wedge \Lambda^j \mathfrak{u}_2^* \wedge \Lambda^k \mathrm{span}(e^4) \wedge \Lambda^l \mathrm{span}(e^7)$ and denote by $\theta^{i,j,k,l}$ for each $s := (i+j+k+l)$-form $\theta \in \Lambda^s \mathfrak{g}^*$ the projection of θ onto $\Lambda^{i,j,k,l}$.

By Lemma 2.9,

$$d(\Lambda^{i,j,k,l}) \subseteq \Lambda^{i,j,k+1,l} + \Lambda^{i,j,k,l+1}$$

for all $i, j, k, l \in \mathbb{N}_0$. By Lemma 2.22 (c) (consider $\mathfrak{g}^* = ((\mathfrak{u}_3^* \oplus \mathfrak{u}_2^*) \oplus \mathrm{span}(e^4)) \oplus \mathrm{span}(e^7))$, $l(\Psi^{2,2,0,0} + \Psi^{3,1,0,0}) \geq 1$, i.e. $\Psi^{2,2,0,0} + \Psi^{3,1,0,0} \neq 0$. Moreover, the closedness of Ψ implies

$$0 = (d\Psi)^{2,2,0,1} = d(\Psi^{2,2,0,0})^{2,2,0,1}, \; 0 = (d\Psi)^{3,1,1,0} = d(\Psi^{3,1,0,0})^{3,1,1,0},$$
$$0 = (d\Psi)^{3,1,0,1} = d(\Psi^{3,1,0,0})^{3,1,0,1}.$$

Since $\mathfrak{g}_3$ is not unimodular, $d(\Lambda^2 \mathfrak{u}_2^*) = \Lambda^3 \mathfrak{g}_3^*$ and so $d(\Psi^{2,2,0,0})^{2,2,0,1} = 0$ implies $\Psi^{2,2,0,0} = 0$. Thus $\Psi^{3,1,0,0} \neq 0$. If $\mathfrak{g}_4$ was non-unimodular, then $d(\Lambda^3 \mathfrak{u}_3^*) = \Lambda^4 \mathfrak{g}_4^*$ and so $d(\Psi^{3,1,0,0})^{3,1,1,0} \neq 0$, a contradiction. Hence $\mathfrak{g}_4$ is unimodular. Similarly, if $d|_{\mathfrak{u}_2^*}$ was injective, then $d(\Psi^{3,1,0,0})^{3,1,0,1} \neq 0$, a contradiction. Thus $d|_{\mathfrak{u}_2^*}$ is not injective and so $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$.

(b) The proof of part (b) is completely analogous to (a). Therefore, let $\Psi \in \Lambda^4\mathfrak{g}^*$ be the Hodge dual of a cocalibrated G_2-structure, let $\mathfrak{u}$ be the Abelian ideal of dimension four in $\mathfrak{g}_5$, $e_5 \in \mathfrak{g}_5\backslash\mathfrak{u}$, $e^5 \in \mathfrak{u}^0 \subseteq \mathfrak{g}_5^*$ with $e^5(e_5) = 1$ and e^6, e^7 a basis of $\mathfrak{r}_2^*$ such that $de^6 = e^{67}$ and $de^7 = 0$. Set similar to (a)

$$\Lambda^{i,j,k,l} := \Lambda^i\mathfrak{u}^* \wedge \Lambda^j\mathrm{span}(e^6) \wedge \Lambda^k\mathrm{span}(e^5) \wedge \Lambda^l\mathrm{span}(e^7),$$

and denote for all $s := (i+j+k+l)$-forms $\theta \in \Lambda^s\mathfrak{g}^*$ the projection of θ onto $\Lambda^{i,j,k,l}$ by $\theta^{i,j,k,l}$. Then $d(\Lambda^{i,j,k,l}) \subseteq \Lambda^{i,j,k+1,l} + \Lambda^{i,j,k,l+1}$ as in (a). Moreover, $\Psi^{4,0,0,0} + \Psi^{3,1,0,0} \neq 0$, again by Lemma 2.22 (c), and $de^6 \neq 0$ shows that

$$0 = (d\Psi)^{3,1,0,1} = d(\Psi^{3,1,0,0})^{3,1,0,1}$$

only if $\Psi^{3,1,0,0} = 0$. Thus $\Psi^{4,0,0,0} \neq 0$. But then

$$0 = (d\Psi)^{4,0,1,0} = d(\Psi^{4,0,0,0})$$

implies that $\mathfrak{g}_5$ is unimodular since otherwise $d(\Lambda^4\mathfrak{u}^*)$ would be equal to $\Lambda^5\mathfrak{g}_5^*$.

□

3.3. $\mathfrak{g}_4$ **not unimodular,** $\mathfrak{g}_3$ **unimodular.** In this subsection we prove Theorem 1.1 (a). In the following, $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always denotes a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$. Furthermore, $\mathfrak{u}$ denotes the unimodular ideal of $\mathfrak{g}_4$.

Proposition 3.6 shows that if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) > 4$, then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure, giving us one direction of Theorem 1.1 (a).

For the other direction, Lemma 2.16 and Proposition 3.3 (a) tell us that if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \leq 4$ and $\mathfrak{u} \neq e(1,1)$, then $\mathfrak{g}$ does admit a cocalibrated G_2-structure. By Table 2 or Remark 2.17, the only four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ with unimodular ideal $\mathfrak{u} = e(1,1)$ is $\mathfrak{g}_4 = \mathfrak{r}_2 \oplus \mathfrak{r}_2$. For $\mathfrak{g}_4 = \mathfrak{r}_2 \oplus \mathfrak{r}_2$, Lemma 2.13 and Proposition 3.3 (b) imply that $\mathfrak{g}_4 \oplus \mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{g}_3$ does admit a cocalibrated G_2-structure if $\mathfrak{g}_3 \neq \mathbb{R}^3$, i.e. if $h^2(\mathfrak{g}_3) \leq 2$. But $h^1(\mathfrak{r}_2 \oplus \mathfrak{r}_2) + h^1(e(1,1)) - h^2(\mathfrak{r}_2 \oplus \mathfrak{r}_2) = 2$. Hence, also in this case, $\mathfrak{g}_4 \oplus \mathfrak{g}_3$ admits a cocalibrated G_2-structure if and only if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \leq 4$. This proves Theorem 1.1 (a).

3.4. $\mathfrak{g}_4$ **unimodular,** $\mathfrak{g}_3$ **unimodular.** Here we prove Theorem 1.1 (b) and denote by $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$.

We begin with the case that $\mathfrak{g}_4$ is indecomposable. If $[\mathfrak{g}_4, \mathfrak{g}_4] = \mathbb{R}^3$, then Lemma 2.16, Proposition 3.3 (a) and Proposition 3.6 tell us that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if

$$h^1(\mathfrak{g}_4) + 3 - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) = h^1(\mathfrak{g}_4) + h^1(\mathbb{R}^3) - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) \leq 4.$$

Table 2 tells us that always $h^1(\mathfrak{g}_4) - h^2(\mathfrak{g}_4) = 1$ in the considered cases. Hence $\mathfrak{g}$ admits for these cases a cocalibrated G_2-structure exactly when $h^2(\mathfrak{g}_3) = 0$, i.e. when $\mathfrak{g}_3 \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$.

Next, we assume that $\mathfrak{g}_4$ is indecomposable but $[\mathfrak{g}_4, \mathfrak{g}_4] \neq \mathbb{R}^3$. By inspection (see Table 2), $\mathfrak{g}_4 \in \{A_{4,1}, A_{4,8}, A_{4,10}\}$.

Let us begin with $\mathfrak{g}_4 \in \{A_{4,8}, A_{4,10}\}$. Then, in both cases, $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) - h^2(\mathfrak{g}_4) = 3$, where $\mathfrak{u}$ is the unique unimodular ideal in $\mathfrak{g}_4$ which is isomorphic to $\mathfrak{h}_3$. Thus we may apply Proposition 3.6 to show that if $h^2(\mathfrak{g}_3) \geq 2$, then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure. Conversely, Corollary 3.5 tells us that if $h^2(\mathfrak{g}_3) = 0$, i.e. $\mathfrak{g}_3$ is not solvable, then $\mathfrak{g}$ does admit a cocalibrated G_2-structure. So we are left with the case that $h^2(\mathfrak{g}_3) = 1$, i.e. $\mathfrak{g}_3 \in \{e(2), e(1,1)\}$. For $\mathfrak{g} = A_{4,8} \oplus e(1,1)$, a cocalibrated G_2-structure is given in Table 3. All other cases do not admit a cocalibrated G_2-structure:

Lemma 3.8. *Let $\mathfrak{g} \in \{A_{4,8} \oplus e(2), A_{4,10} \oplus e(2), A_{4,10} \oplus e(1,1)\}$. Then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure.*

Proof. Let e^1, e^2, e^3, e^4 be the basis of $\mathfrak{g}_4^*$, $\mathfrak{g}_4 \in \{A_{4,8}, A_{4,10}\}$ as in Table 2. Then there exists a linear, trace-free, invertible map $F : \mathrm{span}(e^2, e^3) \to \mathrm{span}(e^2, e^3)$ such that $de^1 = e^{23}$, $d\alpha = F(\alpha) \wedge e^4$, $de^4 = 0$ for all $\alpha \in \mathrm{span}(e^2, e^3)$. For $\mathfrak{g}_4 = A_{4,8}$ we have $F(e^2) = e^2$, $F(e^3) = -e^3$ whereas for $\mathfrak{g}_4 = A_{4,10}$ we have $F(e^2) = e^3$ and $F(e^3) = -e^2$. In particular, $\det(F) = -1$ if $\mathfrak{g}_4 = A_{4,8}$ and $\det(F) = 1$ if $\mathfrak{g}_4 = A_{4,10}$.
Let e^5, e^6, e^7 be a basis of $\mathfrak{g}_3^*$, $\mathfrak{g}_3 \in \{e(2), e(1,1)\}$ as in Table 1. Then there exists a linear, trace-free, invertible map $G : \mathrm{span}(e^5, e^6) \to \mathrm{span}(e^5, e^6)$ such that $d\beta = G(\beta) \wedge e^7$, $de^7 = 0$ for all $\beta \in \mathrm{span}(e^5, e^6)$. In both cases we have $G(e^5) = e^6$, whereas $G(e^6) = e^5$ if $\mathfrak{g}_3 = e(1,1)$ and $G(e^6) = -e^5$ if $\mathfrak{g}_3 = e(2)$. In particular, $\det(G) = -1$ if $\mathfrak{g}_3 = e(1,1)$ and $\det(G) = 1$ if $\mathfrak{g}_3 = e(2)$.
Let us now assume that $\Psi \in \Lambda^4\mathfrak{g}^*$ is a (closed) Hodge dual of a cocalibrated G_2-structure $\varphi \in \Lambda^3\mathfrak{g}^*$. We decompose Ψ uniquely into

$$\Psi = \rho \wedge e^1 + \Omega$$

with $\rho \in \Lambda^3(\mathrm{span}(e^2, e^3, e^4) \oplus \mathfrak{g}_3^*)$, $\Omega \in \Lambda^4(\mathrm{span}(e^2, e^3, e^4) \oplus \mathfrak{g}_3^*)$. Then

$$0 = d\Psi = d\rho \wedge e^1 - \rho \wedge e^{23} + d\Omega,$$

$d\Omega \in \Lambda^3\mathrm{span}(e^2, e^3, e^5, e^6) \wedge e^{47}$ (note that $de^{2356} = 0$) and $d\rho \in \Lambda^4(\mathrm{span}(e^2, e^3, e^4) \oplus \mathfrak{g}_3^*)$ imply $d\rho = 0$ and $\mathrm{proj}_{\mathrm{span}(e^{456}, e^{567})}(\rho) = 0$. Moreover, $\ker F = \{0\} = \ker G$ and $d\rho = 0$ imply $\mathrm{proj}_{\Lambda^3\mathrm{span}(e^2,e^3,e^5,e^6)}(\rho) = 0$.
Thus $\rho = (\omega_1 + ae^{23}) \wedge e^4 + (\omega_2 + be^{23}) \wedge e^7 + \beta \wedge e^{47}$ for certain $\omega_1, \omega_2 \in \mathrm{span}(e^2, e^3) \wedge \mathrm{span}(e^5, e^6)$, $a, b \in \mathbb{R}$ and $\beta \in \mathrm{span}(e^2, e^3, e^5, e^6)$. Now Lemma 2.22 (b) and Lemma 2.24 (b) tell us that $\omega_1 + ae^{23}$ and $\omega_2 + be^{23}$ span a two-dimensional subspace in $\Lambda^2\mathrm{span}(e^2, e^3, e^5, e^6)$ in which each non-zero element has length two. This is equivalent to the requirement that ω_1 and ω_2 span such a two-dimensional subspace of $\Lambda^2\mathrm{span}(e^2, e^3, e^5, e^6)$ and Lemma 2.25 (c) shows that this is equivalent to $\omega_1^2 \neq 0$ and $B^2 - 4C < 0$ for the numbers $B, C \in \mathbb{R}$, defined by $2\omega_1 \wedge \omega_2 = B\omega_2^2$, $\omega_2^2 = C\omega_1^2$.
Since ω_1 is of length two, by Lemma 2.25 (a) there exists a basis α_1, α_2 of $\mathrm{span}(e^2, e^3)$ and α_3, α_4 of $\mathrm{span}(e^5, e^6)$ such that $\omega_1 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3$. Since $d(\omega_1 \wedge e^4 + \omega_2 \wedge e^7) = d\rho = 0$, we must have $\omega_2 = F^{-1}(\alpha_1) \wedge G(\alpha_4) + F^{-1}(\alpha_2) \wedge G(\alpha_3)$. Then $C = \frac{\det(G)}{\det(F)}$ and $C < 0$ if $\mathfrak{g} \in \{A_{4,8} \oplus e(2), A_{4,10} \oplus e(1,1)\}$ leading to $B^2 - 4C > 0$. Thus, for these cases, there cannot exist a cocalibrated G_2-structure.
For the missing case $\mathfrak{g} = A_{4,10} \oplus e(2)$, let $\omega_1 := c_1e^{25} + c_2e^{26} + c_3e^{35} + c_4e^{36}$ be a general two-form in $\mathrm{span}(e^2, e^3) \wedge \mathrm{span}(e^5, e^6)$ of length two, i.e. with $c_1c_4 - c_2c_3 \neq 0$. Then $\omega_2 = -c_4e^{25} + c_3e^{26} + c_2e^{35} - c_1e^{36}$, $B = -\frac{c_1^2+c_2^2+c_3^2+c_4^2}{c_1c_4-c_2c_3}$, $C = 1$ and so

$$\begin{aligned} B^2 - 4C &= \frac{(c_1^2 + c_2^2 + c_3^2 + c_4^2)^2 - 4(c_1c_4 - c_2c_3)^2}{(c_1c_4 - c_2c_3)^2} \\ &= \frac{((c_1 - c_4)^2 + (c_2 + c_3)^2)((c_1 + c_4)^2 + (c_2 - c_3)^2)}{(c_1c_4 - c_2c_3)^2} > 0. \end{aligned}$$

Thus $A_{4,10} \oplus e(2)$ does not admit a cocalibrated G_2-structure. □

For $\mathfrak{g}_4 = A_{4,1}$ note that $\mathfrak{g}_4$ admits a codimension one Abelian ideal and a symplectic two-form, e.g. $\omega = e^{14} + e^{23}$ in the basis e^1, e^2, e^3, e^4 given in Table 2. Hence Proposition 3.3 (a) shows that $A_{4,1} \oplus \mathfrak{g}_3$ admits a cocalibrated G_2-structure if $h^2(\mathfrak{g}_3) \leq 1$, i.e. if $\mathfrak{g}_3 \notin \{\mathbb{R}^3, \mathfrak{h}_3\}$. For the missing cases we have

Lemma 3.9. *Let $\mathfrak{g} = A_{4,1} \oplus \mathfrak{g}_3$ with $\mathfrak{g}_3 \in \{\mathfrak{h}_3, \mathbb{R}^3\}$. Then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure.*

Proof. Choose a basis $e^1, e^2, e^3, e^4, e^5, e^6, e^7$ of $A_{4,1} \oplus \mathfrak{g}_2$, $\mathfrak{g}_2 \in \{\mathbb{R}^3, \mathfrak{h}_3\}$ as in Table 2 and Table 1, i.e.

$$de^1 = e^{24},\ de^2 = e^{34},\ de^3 = 0,\ de^4 = 0,\ de^5 = Ae^{67}\ de^6 = 0,\ de^7 = 0,$$

where $A = 1$ if $\mathfrak{g}_2 = \mathfrak{h}_3$ and $A = 0$ if $\mathfrak{g}_2 = \mathbb{R}^3$. Asssume that there exists a cocalibrated G_2-structure and let

$$\Psi = \sum_{1 \leq i < j < k < l \leq 7} a_{ijkl} e^{ijkl}$$

be its (closed) Hodge dual. Then a short computation shows that $a_{1567} = a_{2567} = a_{1256} = a_{1356} = a_{1257} = a_{1357} = 0$. If we decompose Ψ uniquely into

$$\Psi = \Omega + e^1 \wedge \nu + e^{14} \wedge \omega,$$

$\Omega \in \Lambda^4 \mathrm{span}(e^2, e^3, e^4, e^5, e^6, e^7)$, $\nu \in \Lambda^3 \mathrm{span}(e^2, e^3, e^5, e^6, e^7)$, $\omega \in \Lambda^2 \mathrm{span}(e^2, e^3, e^5, e^6, e^7)$, then ν is given by

$$\nu = a_{1235} e^{235} + a_{1236} e^{236} + a_{1237} e^{237} + a_{1267} e^{267} + a_{1367} e^{367}$$

and is of length two by Lemma 2.22 (b) (consider the decomposition $\big(\mathrm{span}(e^2, e^3, e^5, e^6, e^7) \oplus \mathrm{span}(e^4)\big) \oplus \mathrm{span}(e^1) = \mathfrak{g}^*$).
If $A = 1$, also $a_{1235} = 0$ and $\nu \in \Lambda^3 \mathrm{span}(e^2, e^3, e^6, e^7)$. Thus $l(\nu) \leq 1$, a contradiction.
If $A = 0$, also $a_{1267} = a_{1367} = 0$ and $\nu \in e^{23} \wedge \mathrm{span}(e^5, e^6, e^7)$. Thus $l(\nu) \leq 1$, a contradiction. $\square$

So we are left with the case that $\mathfrak{g}_4$ is decomposable. Then $\mathfrak{g}_4$ is the Lie algebra direct sum of a three-dimensional unimodular Lie algebra $\mathfrak{h}$ and $\mathbb{R}$ and $\mathfrak{g}$ always admits a cocalibrated G_2-structure.

Proposition 3.10. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a Lie algebra direct sum of a four-dimensional unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$. Moreover, let $\mathfrak{g}_4 = \mathfrak{h} \oplus \mathbb{R}$ be a Lie algebra direct sum of a three-dimensional unimodular Lie algebra $\mathfrak{h}$ and $\mathbb{R}$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure.*

Proof. We may assume that $h^2(\mathfrak{h}) \geq h^2(\mathfrak{g}_3)$. Moreover, we may assume that $\mathfrak{g}_4 = \mathfrak{h} \oplus \mathbb{R}$ does admit an Abelian ideal $\mathfrak{u}$ of codimension 1 since otherwise $\mathfrak{h} \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$ and Corollary 3.5 gives us the affirmative answer. By Künneth's formula, $h^1(\mathfrak{h} \oplus \mathbb{R}) = h^1(\mathfrak{h}) + 1$ and $h^2(\mathfrak{h} \oplus \mathbb{R}) = h^2(\mathfrak{h}) + h^1(\mathfrak{h})$. Thus

$$\begin{aligned} h^1(\mathfrak{h} \oplus \mathbb{R}) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{h} \oplus \mathbb{R}) &= h^1(\mathfrak{h}) + 1 + 3 + h^2(\mathfrak{g}_3) - h^2(\mathfrak{h}) - h^1(\mathfrak{h}) \\ &= h^2(\mathfrak{g}_3) - h^2(\mathfrak{h}) + 4 \leq 4, \end{aligned}$$

and Proposition 3.3 (a) implies the statement. $\square$

3.5. $\mathfrak{g}_4$ **unimodular,** $\mathfrak{g}_3$ **not unimodular.** In this subsection we prove Theorem 1.1 (c). In the following, $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always denotes a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional non-unimodular Lie algebra $\mathfrak{g}_3$.
We start with the case that $\mathfrak{g}_4$ admits a codimension one Abelian ideal. Then Proposition 3.7 (a) implies that if $\mathfrak{g}_3 \neq \mathfrak{r}_2 \oplus \mathbb{R}$ then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure. So, in this case, it remains to consider sums of the form $\mathfrak{g}_4 \oplus \mathfrak{r}_2 \oplus \mathbb{R}$. This is done in the next theorem which tells us more generally when a sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}_2$ with a five-dimensonal Lie algebra $\mathfrak{h}$ which admits a codimension one Abelian ideal possesses a cocalibrated G_2-structure. For the proof of this theorem, we need the following

Lemma 3.11. *Let $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathfrak{r}_2$ with a five-dimensional unimodular Lie algebra which admits a codimension one Abelian ideal $\mathfrak{a}$. Choose $e_5 \in \mathfrak{h} \backslash \mathfrak{a}$ and $e^5 \in \mathfrak{a}^0 \subseteq \mathfrak{h}^*$, $e^5(e_5) = 0$. Then $\mathfrak{g}_5$ admits a cocalibrated G_2-structure if and only if there exist two linearly independent two-forms $\omega_1, \omega_2 \in \Lambda^2\mathfrak{a}^*$ such that each non-zero linear combination is of length two and such that $d\omega_1 = \omega_2 \wedge e^5$.*

Proof. Let e^6, e^7 be a basis of $\mathfrak{r}_2^*$ such that $de^6 = e^{67}$, $de^7 = 0$.
Assume first that $\mathfrak{g}$ admits a cocalibrated G_2-structure $\varphi \in \Lambda^3\mathfrak{g}^*$ with (closed) Hodge dual $\Psi \in \Lambda^4\mathfrak{g}^*$. Decompose Ψ uniquely into

$$\Psi = \Omega + \rho \wedge e^6$$

with $\Omega \in \Lambda^4(\mathfrak{g}_5^* \oplus \operatorname{span}(e^7))$, $\rho \in \Lambda^3(\mathfrak{g}_5^* \oplus \operatorname{span}(e^7))$. Since $d\Omega \in \Lambda^5(\mathfrak{g}_5^* \oplus \operatorname{span}(e^7))$ and $d(\rho \wedge e^6) \in \Lambda^4(\mathfrak{g}_5^* \oplus \operatorname{span}(e^7)) \wedge e^6$, the identities $d\Omega = 0 = d(\rho \wedge e^6)$ are true.
Set $\Lambda^{i,j,k} := \Lambda^i\mathfrak{a}^* \wedge \Lambda^j \operatorname{span}(e^5) \wedge \Lambda^k \operatorname{span}(e^7)$. For an $s := (i+j+k)$-form $\theta \in \Lambda^s(\mathfrak{g}_5^* \oplus \operatorname{span}(e^7))$ let $\theta^{i,j,k}$ be the projection of θ onto $\Lambda^{i,j,k}$. Lemma 2.9 implies $d(\Lambda^{i,0,k}) \subseteq \Lambda^{i,1,k}$ and $d(\Lambda^{i,1,k}) = 0$ for all $i, k \in \mathbb{N}_0$.
The closedness of $\rho \wedge e^6$ implies $0 = d(\rho \wedge e^6) = d\rho \wedge e^6 - \rho \wedge e^{67}$ and so $0 = d\rho + \rho \wedge e^7$. Then the identities

$$0 = (d\rho + \rho \wedge e^7)^{3,0,1} = \rho^{3,0,0} \wedge e^7,\ 0 = (d\rho + \rho \wedge e^7)^{2,1,1} = d(\rho^{2,0,1}) + \rho^{2,1,0} \wedge e^7$$

are true. Thus $\rho^{3,0,0} = 0$ and $d(\rho^{2,0,1}) = -\rho^{2,1,0} \wedge e^7$. This shows that

$$\rho = \omega_1 \wedge e^7 - \omega_2 \wedge e^5 + \alpha \wedge e^{57}$$

for ω_1, $\omega_2 \in \Lambda^{2,0,0}$, $\alpha \in \Lambda^{1,0,0}$ and that

$$\omega_2 \wedge e^{57} = -\rho^{2,1,0} \wedge e^7 = d(\rho^{2,0,1}) = d(\omega_1 \wedge e^7) = d\omega_1 \wedge e^7 \ \Leftrightarrow\ d\omega_1 = \omega_2 \wedge e^5.$$

By Lemma 2.22 (b) and Lemma 2.24 (b), ω_1 and ω_2 span a two-dimensional subspace in which each non-zero element is of length two.
Conversely, let $\omega_1, \omega_2 \in \Lambda^2\mathfrak{a}^*$ be such that $d\omega_1 = \omega_2 \wedge e^5$ and such that ω_1, ω_2 are linearly independent and each non-zero linear combination of them is of length two. Set $V_4 := \mathfrak{a}^*$, $V_3 := \operatorname{span}(e^5) \oplus \mathfrak{r}_2^*$, $\nu_1 := e^{67} \in \Lambda^2 V_3$, $\nu_2 := e^{56} \in \Lambda^2 V_3$, $\nu_3 := e^{57} \in \Lambda^2 V_3$. Moreover, by Lemma 2.26 there exists a two-form $\omega_3 \in \Lambda^2\mathfrak{a}^*$ such that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure. By Lemma 2.9, $d(\Lambda^4\mathfrak{a}^*) = 0$ and $d(\Lambda^k\mathfrak{a}^* \wedge e^5) = 0$ for all $k \in \mathbb{N}_0$. Using these properties of d, a short computation shows that Ψ is closed. $\square$

Lemma 3.11 allows us to prove

Theorem 3.12. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}_2$ be a Lie algebra direct sum of a five-dimensional Lie algebra $\mathfrak{h}$ admitting a codimension one Abelian ideal $\mathfrak{a}$ and of $\mathfrak{r}_2$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $\mathfrak{h}$ is unimodular and $\mathfrak{h} \notin \{\mathbb{R}^5, \mathfrak{h}_3 \oplus \mathbb{R}^2, A_{5,7}^{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}}\}$, where $A_{5,7}^{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}}$ is the Lie algebra for which there exists an element $e_5 \in \mathfrak{h}\backslash\mathfrak{a}$ which acts diagonally on $\mathfrak{a}$ with eigenvalues $\left(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$.*

Proof. By Proposition 3.7 (b), $\mathfrak{h}$ has to be unimodular if $\mathfrak{g}$ admits a cocalibrated G_2-structure. So, for the rest we assume that $\mathfrak{h}$ is unimodular and let $e_5 \in \mathfrak{h}\backslash\mathfrak{a}$, $e^5 \in \mathfrak{a}^0 \subseteq \mathfrak{h}^*$, $e^5(e_5) = 1$. By Lemma 2.9 there exists a linear trace-free map $H : \mathfrak{a}^* \to \mathfrak{a}^*$ such that $d\alpha = H(\alpha) \wedge e^5$, $de^5 = 0$ for all $\alpha \in \mathfrak{a}^*$.

Let e^6, e^7 be a basis of $\mathfrak{r}_2^*$ with $de^6 = e^{67}$, $de^7 = 0$. Then Lemma 3.11 tells us that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there are two linearly independent two-forms $\omega_1, \omega_2 \in \Lambda^2\mathfrak{a}^*$ such that $d\omega_1 = \omega_2 \wedge e^5$ and such that each non-zero linear combination is of length two.

We first prove that such a pair of two-forms always exists if there is a vector decomposition $\mathfrak{a}^* = V_2 \oplus W_2$ into two two-dimensional H-invariant subspaces such that the restrictions of H to V_2 and to W_2 are both not a multiple of the identity. In this case we may choose, for all $\lambda \neq 0$, a basis e^1, e^2 of V_2 and a basis e^3, e^4 of W_2 such that the restrictions of H to V_2 and W_2 with respect to the corresponding bases are given by

$$\begin{pmatrix} 0 & -\frac{\det(H|_{V_2})}{\lambda} \\ \lambda & \operatorname{tr}(H|_{V_2}) \end{pmatrix} \text{ and } \begin{pmatrix} \operatorname{tr}(H|_{W_2}) & -\lambda \\ \frac{\det(H|_{W_2})}{\lambda} & 0 \end{pmatrix},$$

respectively. Set $\omega_1 := e^{14} + e^{23}$. Then ω_1 is of length two and $d\omega_1 = \left(\lambda(e^{13} - e^{24}) + \omega_3\right) \wedge e^5$ with $\omega_3 := de^{23} \in \Lambda^2\mathfrak{a}^*$. Set $\omega_2 := \lambda(e^{13} - e^{24}) + \omega_3$ and observe that $d\omega_1 = \omega_2 \wedge e^5$ and

$$\omega_1 \wedge \omega_2 = e_5 \lrcorner (\omega_1 \wedge d\omega_1) = e_5 \lrcorner \left(d\left(\frac{1}{2}\omega_1^2\right)\right) = 0$$

since $\mathfrak{g}_5$ is unimodular. Furthermore, observe that $C(\lambda)$ defined by

$$\omega_2^2 = \lambda^2 e^{1234} + 2\lambda(e^{13} - e^{24}) \wedge \omega_3 + \omega_3^2 = C(\lambda)\omega_1^2,$$

fulfills $C(\lambda) = \lambda^2 + \mathcal{O}(\lambda)$ as $\lambda \to \infty$. Thus, for $|\lambda|$ sufficiently large, $C(\lambda) > 0$ and Lemma 2.25 (c) tells us that then ω_1, ω_2 span a two-dimensional subspace in which each non-zero element is of length two. So all considered Lie algebras which admit such a splitting do admit a cocalibrated G_2-structure.

Looking at the possible real Jordan normal forms of F, we therefore may, after rescaling e^5, assume for the rest of the proof that there is a basis e^1, e^2, e^3, e^4 of $\mathfrak{a}^*$ such that F acts with respect to this basis as one of the following matrices:

$$\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & A \\ & & & -3a \end{pmatrix}, \begin{pmatrix} b & 1 & & \\ -1 & b & & \\ & & -b & \\ & & & -b \end{pmatrix}, \begin{pmatrix} c & 1 & & \\ & c & & \\ & & -c & \\ & & & -c \end{pmatrix}, \begin{pmatrix} f & & & \\ & -\frac{f}{3} & & \\ & & -\frac{f}{3} & \\ & & & -\frac{f}{3} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad a,\ c,\ f,\ A \in \{0, 1\},\ b \in \mathbb{R}^+.$$

In the first case, $\omega_1 := e^{12} + e^{34} - 5e^{23}$ and $\omega_2 := -e^{24} + 2a(-e^{12} + e^{34}) + 10ae^{23} + 5e^{13}$ fulfill all desired conditions.

In the second case, $\omega_1 := e^{13} - e^{24}$ and $\omega_2 := e^{14} + e^{23}$ fulfill all desired conditions.

In the third case, we start with $c = 1$. Then $\omega_1 := e^{13} - e^{24} - \frac{1}{2}\left(e^{12} - e^{34}\right)$, $\omega_2 := e^{12} + e^{34} + e^{14}$ fulfill all desired conditions. If $c = 0$, then $\mathfrak{h} = \mathfrak{h}_3 \oplus \mathbb{R}^2$ and we already know by Proposition 3.6 that $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R}^2 \oplus \mathfrak{h}_3$ does not admit a cocalibrated G_2-structure. However, this also follows easily from the fact that in this case $d(\Lambda^2\mathfrak{a}^*) = \mathrm{span}(e^{135}, e^{145})$.
In the fourth case, let $\omega_1 \in \Lambda^2\mathfrak{a}^*$ be of length two. Then there exist $\alpha \in \mathrm{span}(e^2, e^3, e^4)$ and $\omega \in \Lambda^2\mathrm{span}(e^2, e^3, e^4)$ such that $\omega_1 = \omega + \alpha \wedge e^1$. But then $d\omega_1 = \frac{2}{3} f\left(\omega - \alpha \wedge e^1\right) \wedge e^5$, i.e. $\omega_2 = \frac{2}{3} f\left(\omega - \alpha \wedge e^1\right)$ and so $\frac{2}{3} f\omega_1 + \omega_2 = \frac{4}{3} f\omega$ is of length one. Thus $\mathfrak{g}$ does not admit a cocalibrated G_2-structure in this case, i.e. if $\mathfrak{h} \in \left\{\mathbb{R}^5, A_{5,7}^{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}}\right\}$.
In the last case, $\omega_1 := e^{12} - e^{34}$ and $\omega_2 := e^{14} - e^{23}$ fulfill all desired conditions.

□

Remark 3.13. We like to note an interesting consequence of Theorem 3.12 concerning the connection to half-flat SU(3)-structures on six-dimensional Lie algebras $\mathfrak{b}$. It is well-known, see e.g. [18], that if there exists a half-flat SU(3)-structure on $\mathfrak{b}$, then one can define naturally a cocalibrated G_2-structure on the seven-dimensional Lie algebra $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}$ such that $\mathfrak{b}$ and $\mathbb{R}$ are orthogonal to each other. Conversely, a cocalibrated G_2-structure on a seven-dimensional Lie algebra $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}$ for which $\mathfrak{b}$ and $\mathbb{R}$ are orthogonal defines a half-flat SU(3)-structure on $\mathfrak{b}$. So far there seems to be no example known in the literature for a seven-dimensional Lie algebra $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}$ which admits a cocalibrated G_2-structure such that $\mathfrak{b}$ does not admit a half-flat SU(3)-structure. But now Theorem 3.12 provides us with an example. Namely $\mathfrak{g} = A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ admits a cocalibrated G_2-structure due to Theorem 3.12 but in [9] it is shown that $\mathfrak{b} = A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ does not admit a half-flat SU(3)-structure. Note that this shows that $\mathfrak{g} = A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ cannot admit a cocalibrated G_2-structure such that $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ and $\mathbb{R}$ are orthogonal.

The only unimodular four-dimensional Lie algebras which do not admit an Abelian ideal are the two non-solvable ones $\mathfrak{so}(3) \oplus \mathbb{R}$, $\mathfrak{so}(2,1) \oplus \mathbb{R}$ and the two whose commutator ideal $\mathfrak{u}$ is isomorphic to $\mathfrak{h}_3$, namely $A_{4,8}$, $A_{4,10}$. Direct sums with the non-solvable four-dimensional Lie algebras admit cocalibrated G_2-structures by Corollary 3.5. Direct sums with $A_{4,8}$, $A_{4,10}$ admit cocalibrated G_2-structures by Proposition 3.3 (c) if $h^1(\mathfrak{g}_3) \geq 1$ and by Corollary 3.5 if $h^1(\mathfrak{g}_3) = 0$, i.e. $\mathfrak{g}_3 \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$. This finishes the proof of Theorem 1.1 (c).

3.6. **$\mathfrak{g}_4$ not unimodular, $\mathfrak{g}_3$ not unimodular.** In this subsection we prove Theorem 1.1 (d). In the following, $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always denotes a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional non-unimodular Lie algebra $\mathfrak{g}_3$. Furthermore, $\mathfrak{u}$ should always denote the unimodular kernel of $\mathfrak{g}_4$
By Proposition 3.7 (a), $\mathfrak{g}$ does not admit a cocalibrated G_2-structure if $\mathfrak{u}$ is Abelian. If $\mathfrak{u} \in \{e(2), e(1,1)\}$, then $\mathfrak{g}_4 \in \{A_{4,1}, \mathfrak{r}_2 \oplus \mathfrak{r}_2\}$ and Proposition 3.3 (b) and Lemma 2.13 imply that $\mathfrak{g}$ admits a cocalibrated G_2-structure unless $\mathfrak{g}_3 = \mathfrak{r}_{3,1}$. But for $\mathfrak{g} = A_{4,12} \oplus \mathfrak{r}_{3,1}$ and $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_{3,1}$ cocalibrated G_2-structures can be found in Table 3.
Therefore, it remains to consider the case when the unimodular ideal $\mathfrak{u}$ is isomorphic to $\mathfrak{h}_3$. Then Lemma 2.18 tells us that we may decompose $\mathfrak{g}_4^* = \mathrm{span}(e^1) \oplus V_2 \oplus \mathrm{span}(e^4)$ with $e^1, e^4 \neq 0$ and $\dim(V_2) = 2$ such that $de^1 = \mathrm{tr}(F)e^{14} + \nu$ for $0 \neq \nu \in \Lambda^2 V_2$, $d\alpha = F(\alpha) \wedge e^4$ for a linear map $F : V_2 \to V_2$ with trace and all $\alpha \in V_2$ and $de^4 = 0$. Moreover, by Lemma 2.11, we may decompose $\mathfrak{g}_3^* = W_2 \oplus \mathrm{span}(e^7)$ with $0 \neq e^7$ and W_2 two-dimensional such that $d\beta = G(\beta) \wedge e^7$ for a linear map $G : W_2 \to W_2$ with trace and all $\beta \in W_2$ and $de^7 = 0$.

Proposition 3.14. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $\mathfrak{u}$, e^1, $e^4 \in \mathfrak{g}_4^*\setminus\{0\}$, $e^7 \in \mathfrak{g}_3^*\setminus\{0\}$, $V_2 \subseteq \mathfrak{g}_4^*$, $W_2 \subseteq \mathfrak{g}_3^*$ and $\nu \in \Lambda^2 V_2$ as above. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there are two linearly independent two-forms $\omega_1, \omega_2 \in V_2 \wedge W_2$, a non-zero two-form $\hat{\nu} \in \Lambda^2 W_2$ and some $\lambda \in \mathbb{R}$ such that the following conditions are fulfilled:*

(i) $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$.

(ii) The two-forms $\tilde{\omega}_1 := \hat{\nu} + \omega_1$, $\tilde{\omega}_2 := \frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}\hat{\nu} + \lambda\nu + \omega_2$ are linearly independent and each non-zero linear combination is of length two.

Proof. "$\Rightarrow$":
We set
$$\Lambda^{i,j,k,l} := \Lambda^i V_2 \wedge \Lambda^j W_2 \wedge \Lambda^k \mathrm{span}(e^4) \wedge \Lambda^l \mathrm{span}(e^7)$$
and denote, for an $s := (i+j+k+l)$-form $\Phi \in \Lambda^s(V_2 \oplus \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*)$, by $\Phi^{i,j,k,l}$ the projection of Φ into $\Lambda^{i,j,k,l}$. Then we have
$$d(\Lambda^{i,j,0,0}) \subseteq \Lambda^{i,j,1,0} + \Lambda^{i,j,0,1}, \quad d(\Lambda^{i,j,1,0}) \subseteq \Lambda^{i,j,1,1}, \quad d(\Lambda^{i,j,0,1}) \subseteq \Lambda^{i,j,1,1}, \quad d(\Lambda^{i,j,1,1}) = \{0\}$$
for all $i, j \in \mathbb{N}_0$. Moreover, $d(\hat{\mu}) = -\mathrm{tr}(F)\hat{\mu} \wedge e^4$ for all $\hat{\mu} \in \Lambda^{2,0,0,0}$ and $d(\tilde{\mu}) = -\mathrm{tr}(G)\tilde{\mu} \wedge e^7$ for all $\tilde{\mu} \in \Lambda^{0,2,0,0}$.
Let $\Psi \in \Lambda^4(\mathfrak{g}_4 \oplus \mathfrak{g}_3)^*$ be the Hodge dual of a cocalibrated G_2-structure. Decompose Ψ into
$$\Psi = \Omega + e^1 \wedge \rho$$
with $\Omega \in \Lambda^4(V_2 \oplus \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*)$, $\rho \in \Lambda^3(V_2 \oplus \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*)$. Then
$$0 = d\Psi = d\Omega + (\mathrm{tr}(F)e^{14} + \nu) \wedge \rho - e^1 \wedge d\rho = e^1 \wedge (\mathrm{tr}(F)e^4 \wedge \rho - d\rho) + d\Omega + \nu \wedge \rho \quad (3.1)$$
implies $\Phi := \mathrm{tr}(F)e^4 \wedge \rho - d\rho = 0$. We look at different components of Φ. We have the identities
$$\begin{aligned}
0 = \Phi^{2,1,1,0} &= \mathrm{tr}(F)e^4 \wedge \rho^{2,1,0,0} - d(\rho^{2,1,0,0})^{2,1,1,0} = \mathrm{tr}(F)e^4 \wedge \rho^{2,1,0,0} - \mathrm{tr}(F)\rho^{2,1,0,0} \wedge e^4, \\
&= 2\mathrm{tr}(F)e^4 \wedge \rho^{2,1,0,0}, \\
0 = \Phi^{1,2,0,1} &= -d(\rho^{1,2,0,0})^{1,2,0,1} = -\mathrm{tr}(G)\rho^{1,2,0,0} \wedge e^7, \\
0 = \Phi^{2,0,1,1} &= \mathrm{tr}(F)e^4 \wedge \rho^{2,0,0,1} - d(\rho^{2,0,0,1}) = 2\mathrm{tr}(F)e^4 \wedge \rho^{2,0,0,1},
\end{aligned}$$
which imply $\rho^{2,1,0,0} = \rho^{1,2,0,0} = \rho^{2,0,0,1} = 0$. Moreover,
$$0 = \Phi^{0,2,1,1} = \mathrm{tr}(F)e^4 \wedge \rho^{0,2,0,1} - d(\rho^{0,2,1,0}) = \mathrm{tr}(F)e^4 \wedge \rho^{0,2,0,1} + \mathrm{tr}(G)e^7 \wedge \rho^{0,2,1,0},$$
i.e. $e^4 \wedge \rho^{0,2,0,1} = -\frac{\mathrm{tr}(G)}{\mathrm{tr}(F)} e^7 \wedge \rho^{0,2,1,0}$. Thus ρ decomposes as
$$\rho = e^7 \wedge (\omega_1 + \hat{\nu}) + e^4 \wedge \left(\omega_2 + \frac{\mathrm{tr}(G)}{\mathrm{tr}(F)}\hat{\nu} + \lambda\nu\right) + e^{47} \wedge \alpha$$
with $\omega_1, \omega_2 \in \Lambda^{1,1,0,0}$, $\hat{\nu} \in \Lambda^{0,2,0,0}$, $\lambda \in \mathbb{R}$, $\alpha \in \Lambda^{1,0,0,0} \oplus \Lambda^{0,1,0,0}$.
By Lemma 2.22 (b) and Lemma 2.24 (b), $\tilde{\omega}_1 := \omega_1 + \hat{\nu}$ and $\tilde{\omega}_2 := \omega_2 + \frac{\mathrm{tr}(G)}{\mathrm{tr}(F)}\hat{\nu} + \lambda\nu$ span a two-dimensional subspace in which each non-zero element is of length two.
The $(1,1,1,1)$-component of Φ is given by
$$0 = \Phi^{1,1,1,1} = \mathrm{tr}(F)e^4 \wedge \rho^{1,1,0,1} - d(\rho^{1,1,1,0}) - d(\rho^{1,1,0,1})$$
which shows that
$$\begin{aligned}
d(e^1 \wedge (\rho^{1,1,1,0} + \rho^{1,1,0,1})) &= (\nu + \mathrm{tr}(F)e^{14}) \wedge (\rho^{1,1,1,0} + \rho^{1,1,0,1}) - e^1 \wedge d(\rho^{1,1,1,0} + \rho^{1,1,0,1}) \\
&= \mathrm{tr}(F)e^{14} \wedge \rho^{1,1,0,1} - e^1 \wedge d(\rho^{1,1,1,0}) - e^1 \wedge d(\rho^{1,1,0,1}) = e^1 \wedge \Phi^{1,1,1,1} \\
&= 0.
\end{aligned}$$
Since $\rho^{1,1,1,0} = e^4 \wedge \omega_2$ and $\rho^{1,1,0,1} = e^7 \wedge \omega_1$, we get $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$.

What is left to show is that $\hat{\nu} \neq 0$. Therefore, let $\tilde{\Omega}$ be the projection of Ψ onto the subspace $\Lambda^4(\mathrm{span}(e^1) \oplus V_2 \oplus W_2)$ (along $\sum_{i=1}^{2} \Lambda^i(\mathrm{span}(e^1) \oplus V_2 \oplus W_2) \wedge \Lambda^{2-i}\mathrm{span}(e^4, e^7)$). By Lemma 2.22 (c), $\tilde{\Omega} \neq 0$. We may write $\tilde{\Omega}$ in terms of the components of ρ and Ω as

$$\tilde{\Omega} = e^1 \wedge \rho^{2,1,0,0} + e^1 \wedge \rho^{1,2,0,0} + \Omega^{2,2,0,0} = \Omega^{2,2,0,0}.$$

This means that $\Omega^{2,2,0,0} \neq 0$. Equation (3.1) gives us

$$0 = (d\Omega + \nu \wedge \rho)^{2,2,0,1} = (d\Omega^{2,2,0,0})^{2,2,0,1} + \nu \wedge \rho^{0,2,0,1} = -\mathrm{tr}(G)\Omega^{2,2,0,0} \wedge e^7 + \nu \wedge \rho^{0,2,0,1}$$

and so $e^7 \wedge \hat{\nu} = \rho^{0,2,0,1} \neq 0$, i.e. $\hat{\nu} \neq 0$.
"$\Leftarrow$":
Assume that there exist two-forms ω_1, $\omega_2 \in V_2 \wedge W_2$, $0 \neq \hat{\nu} \in \Lambda^2 W_2$ and $\lambda \in \mathbb{R}$ fulfilling all the conditions. Then $\tilde{\omega}_1 = \hat{\nu} + \omega_1$ fulfills $0 \neq \tilde{\omega}_1^2 \in \Lambda^2 V_2 \wedge \Lambda^2 W_2$. Hence there exists $0 \neq \tilde{\lambda} \in \mathbb{R}$ such that $\frac{\tilde{\lambda}}{2}\tilde{\omega}_1^2 = -\frac{1}{\mathrm{tr}(G)}\nu \wedge \hat{\nu}$. Set now $\theta_1 := \frac{1}{\tilde{\lambda}}e^{71}$, $\theta_2 := \frac{1}{\tilde{\lambda}}e^{41}$, $\theta_3 := e^{74} \in \Lambda^2\mathrm{span}(e^1, e^4, e^7)$. By assumption, $\tilde{\omega}_1$, $\tilde{\omega}_2$ span a two-dimensional space in which each non-zero element has length two. Thus we may apply Lemma 2.26 to $V_4^* := V_2 \oplus W_2$, $V_3^* := \mathrm{span}(e^1, e^4, e^7)$ and get the existence of a two-form $\tilde{\omega}_3 \in \Lambda^2 V_4^*$ such that

$$\Psi := \sum_{i=1}^{3} \tilde{\omega}_i \wedge \theta_i + \frac{1}{2}\tilde{\omega}_1^2$$

is the Hodge dual of a G_2-structure. Using $d\nu = -\mathrm{tr}(F)\nu \wedge e^4$, $d\hat{\nu} = -\mathrm{tr}(G)\hat{\nu} \wedge e^7$, we compute

$$\begin{aligned}
d\Psi &= \frac{1}{\tilde{\lambda}}d(\tilde{\omega}_1 \wedge e^{71} + \tilde{\omega}_2 \wedge e^{41}) + d(\tilde{\omega}_3 \wedge e^{74}) - \frac{1}{\tilde{\lambda} \cdot \mathrm{tr}(G)}d(\nu \wedge \hat{\nu}) \\
&= \frac{1}{\tilde{\lambda}}d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) + \frac{1}{\tilde{\lambda}}d(\hat{\nu} \wedge e^{71}) + \frac{1}{\tilde{\lambda}}d\left(\frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}\hat{\nu} \wedge e^{41} + \lambda\nu \wedge e^{41}\right) \\
&\quad + \frac{\mathrm{tr}(F)}{\tilde{\lambda} \cdot \mathrm{tr}(G)}\nu \wedge \hat{\nu} \wedge e^4 + \frac{1}{\tilde{\lambda}}\nu \wedge \hat{\nu} \wedge e^7 \\
&= 0 - \frac{\mathrm{tr}(F)}{\tilde{\lambda}}\hat{\nu} \wedge e^{714} - \frac{1}{\tilde{\lambda}}\hat{\nu} \wedge e^7 \wedge \nu - \frac{\mathrm{tr}(F)}{\tilde{\lambda}}\hat{\nu} \wedge e^{741} - \frac{\mathrm{tr}(F)}{\tilde{\lambda} \cdot \mathrm{tr}(G)}\hat{\nu} \wedge e^4 \wedge \nu \\
&\quad + \frac{\mathrm{tr}(F)}{\tilde{\lambda} \cdot \mathrm{tr}(G)}\nu \wedge \hat{\nu} \wedge e^4 + \frac{1}{\tilde{\lambda}}\nu \wedge \hat{\nu} \wedge e^7 \\
&= 0.
\end{aligned}$$

□

Remark 3.15. The two-form $\omega_1 \in V_2 \wedge W_2$ in Proposition 3.14 has to be of length two since $\tilde{\omega}_1 = \omega_1 + \hat{\nu}$ is of length two. By Lemma 2.25 (a) there exists a basis e^2, e^3 of V_2 and a basis e^5, e^6 of W_2 such that $\omega_1 = e^{26} + e^{35}$. In the case $\det(G) \neq 0$ the condition $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$ implies that $\omega_2 = (F + \mathrm{tr}(F)\mathrm{id})(e^2) \wedge G^{-1}(e^6) + (F + \mathrm{tr}(F)\mathrm{id})(e^3) \wedge G^{-1}(e^5)$.

Let us, nevertheless, start with $\det(G) = 0$.

Lemma 3.16. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $e^1, e^4 \in \mathfrak{g}_4^*$, $e^7 \in \mathfrak{g}_3^*$, V_2, $F : V_2 \to V_2$, W_2 and $G : W_2 \to W_2$ as in Proposition 3.14. Assume further that* $\det(G) = 0$, *i.e.* $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$. *Then $\mathfrak{g}$ admits a cocalibrated* G_2*-structure if and only if* $\det(F + \mathrm{tr}(F)\mathrm{id}) = 0$, *i.e.* $\mathfrak{g}_4 = A_{4,9}^{-\frac{1}{2}}$.

Proof. "$\Rightarrow$:"
If $\mathfrak{g}$ admits a cocalibrated G_2-structure, then, by Proposition 3.14 and Remark 3.15 there

exists a basis e^2, e^3 of V_2 and a basis e^5, e^6 of W_2 such that, in particular, $\omega_1 := e^{26} + e^{35}$ fulfills $d(\omega_1 \wedge e^{71}) \in d(V_2 \wedge W_2 \wedge e^{41}) = V_2 \wedge G(W_2) \wedge e^{741}$. Each element in $V_2 \wedge G(W_2) \wedge e^{741}$ is of length at most one due to $\det(G) = 0$. But

$$d(\omega_1 \wedge e^{71}) = ((F + \mathrm{tr}(F)\mathrm{id})(e^2) \wedge e^6 + (F + \mathrm{tr}(F)\mathrm{id})(e^3) \wedge e^5) \wedge e^{741}$$

is of length less than two if and only if $\det(F + \mathrm{tr}(F)\mathrm{id}) = 0$. Thus $\det(F + \mathrm{tr}(F)\mathrm{id}) = 0$.
"$\Leftarrow$:"
$\det(F + \mathrm{tr}(F)\mathrm{id}) = 0 = \det(G)$ and $\mathrm{tr}(F + \mathrm{tr}(F)\mathrm{id}) = 3\mathrm{tr}(F) \neq 0, \mathrm{tr}(G) \neq 0$ show that $F + \mathrm{tr}(F)\mathrm{id}$ and G are diagonalizable with one zero eigenvalue and one non-zero eigenvalue from which we may assume, after rescaling e^4 and e^7, that it is equal one in both cases. Since $d(e^1 \wedge \alpha) = -e^1 \wedge (F + \mathrm{tr}(F)\mathrm{id})(\alpha) \wedge e^4$ for all $\alpha \in V_2$, there exists a basis e^2, e^3 of V_2 such that $de^{12} = 0$ and $de^{13} = -e^{134}$. Moreover, we may choose a basis e^5, e^6 of W_2 with $de^5 = 0$ and $de^6 = e^{67}$.
Set

$$\omega_1 := e^{25} - e^{36} + e^{26}, \quad \omega_2 := e^{25} - e^{36} - 2e^{35}, \quad \tilde{\omega}_1 := e^{56} + \omega_1, \quad \tilde{\omega}_2 := \frac{1}{3}e^{56} + \omega_2.$$

Then $\tilde{\omega}_1$, $\tilde{\omega}_2$ fulfill all the conditions in Proposition 3.14 since $\mathrm{tr}(F) = \frac{1}{3}$ and $\mathrm{tr}(G) = 1$ by our choice. Note that the fact that $\tilde{\omega}_1$, $\tilde{\omega}_2$ span a two-dimensional subspace of $\Lambda^2(V_2 \oplus W_2)$ in which each non-zero element is of length two directly follows from $\tilde{\omega}_1^2 = \tilde{\omega}_2^2 \neq 0$ and $\tilde{\omega}_1 \wedge \tilde{\omega}_2 = 0$. □

Next we consider the case when $\det(G) \neq 0$ but F and G are both not multiples of the identity:

Lemma 3.17. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $e^1, e^4 \in \mathfrak{g}_4^*$, $e^7 \in \mathfrak{g}_3^*$, V_2, $F : V_2 \to V_2$, W_2 and $G : W_2 \to W_2$ as in Proposition 3.14. Assume further that F and G are both not multiples of the identity, i.e. $\mathfrak{g}_4 \neq A_{4,9}^1$ and $\mathfrak{g}_3 \neq \mathfrak{r}_{3,1}$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure.*

Proof. Set $H := -(F + \mathrm{tr}(F)\mathrm{id})$. Then also $H : V_2 \to V_2$ is not a multiple of the identity, not trace-free and $d(e^1 \wedge \alpha) = e^1 \wedge H(\alpha) \wedge e^4$ for all $\alpha \in V_2$. By rescaling e^4 appropriately, we may assume that $\mathrm{tr}(H) = -3$, i.e. $\mathrm{tr}(F) = 1$. Hence we may choose a basis e^2, e^3 of V_2 such that the transformation matrix of H with respect to this basis is given by

$$\begin{pmatrix} 0 & \frac{\det(H)}{\det(G)} \\ -\det(G) & -3 \end{pmatrix}.$$

Moreover, by rescaling e^7 appropriately, we may assume that $\mathrm{tr}(G) = 1$. Hence, for all $a \in \mathbb{R}\backslash\{0\}$, we may choose a basis e^5, e^6 of W_2 such that the transformation matrix of G with respect to this basis is given by

$$\begin{pmatrix} 0 & -\frac{\det(G)}{a} \\ a & 1 \end{pmatrix}.$$

Set

$$\omega_1 := e^{25} + e^{36}, \quad \omega_2 := -\frac{\det(H)}{\det(G)a}e^{25} + \frac{3+a}{a}e^{35} - a\, e^{36}.$$

A short computation shows $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$. Set

$$\tilde{\omega}_1 := e^{56} + \omega_1, \quad \tilde{\omega}_2 := e^{56} - a\, e^{23} + \omega_2.$$

Then $\tilde{\omega}_1^2 = 2e^{2536} \neq 0$ and $2\tilde{\omega}_1 \wedge \tilde{\omega}_2 = B\tilde{\omega}_1^2$, $\tilde{\omega}_2^2 = C\tilde{\omega}_1^2$ with $B = -\frac{\det(H)}{a\det(G)}$ and $C = \frac{\det(H)+a\det(G)}{\det(G)}$. Thus

$$B^2 - 4C = \frac{\det(H)^2}{a^2\det(G)^2} - 4\frac{\det(H)}{\det(G)} - 4a < 0$$

for $a > 0$ large enough. Thus $\mathfrak{g}$ admits a cocalibrated G_2-structure. □

Therefore it remains to consider the cases when at least one of the maps F and G is (a multiple of) the identity:

Lemma 3.18. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $e^1, e^4 \in \mathfrak{g}_4^*$, $e^7 \in \mathfrak{g}_3^*$, V_2, $F : V_2 \to V_2$, W_2 and $G : W_2 \to W_2$ as in Proposition 3.14.*

(a) If F is a multiple of the identity, i.e. $\mathfrak{g}_4 = A_{4,9}^1$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $-\frac{3}{4}\mathrm{tr}(G)^2 > \det(G)$ or $\det(G) > 0$.

(b) If G is a multiple of the identity, i.e. $\mathfrak{g}_3 = \mathfrak{r}_{3,1}$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $\det(F) > -\frac{3}{4}\mathrm{tr}(F)^2$.

Proof. (a) By rescaling e^4 we may assume that $\mathrm{tr}(F) = 2$, i.e. $F = \mathrm{id}$. Hence Proposition 3.14 and Remark 3.15 tell us that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there exists a basis e^2, e^3 of V_2, a basis e^5, e^6 of W_2, $\lambda, \alpha \in \mathbb{R}$, $\alpha \neq 0$ such that each non-zero linear combination of

$$\tilde{\omega}_{1,\alpha,\lambda} := \alpha e^{56} + e^{26} + e^{35}, \quad \tilde{\omega}_{2,\alpha,\lambda} := \frac{2}{\mathrm{tr}(G)}\alpha e^{56} + \lambda e^{23} + 3e^2 \wedge G^{-1}(e^6) + 3e^3 \wedge G^{-1}(e^5)$$

is of length two. A short computation shows

$$\tilde{\omega}_{1,\alpha,\lambda}^2 = 2e^{2356}, \quad 2\tilde{\omega}_{1,\alpha,\lambda} \wedge \tilde{\omega}_{2,\alpha,\lambda} = \left(2\alpha\lambda + \frac{6\mathrm{tr}(G)}{\det(G)}\right) e^{2356},$$
$$\tilde{\omega}_{2,\alpha,\lambda}^2 = \left(4\frac{\alpha\lambda}{\mathrm{tr}(G)} + 18\frac{1}{\det(G)}\right) e^{2356}$$

since for an invertible two-by-two matrix $\mathrm{tr}\left(G^{-1}\right) = \frac{\mathrm{tr}(G)}{\det(G)}$. Set $X := \alpha\lambda$. Then Lemma 2.25 (c) tells us that each non-zero linear combination of $\tilde{\omega}_{1,\alpha,\lambda}$ and $\tilde{\omega}_{2,\alpha,\lambda}$ is of length two if and only if the quadratic polynomial

$$\left(X + \frac{3\mathrm{tr}(G)}{\det(G)}\right)^2 - 4 \cdot \left(2\frac{X}{\mathrm{tr}(G)} + 9\frac{1}{\det(G)}\right)$$
$$= X^2 + \left(6\frac{\mathrm{tr}(G)}{\det(G)} - 8\frac{1}{\mathrm{tr}(G)}\right) X + 9\frac{\mathrm{tr}(G)^2}{\det(G)^2} - 36\frac{1}{\det(G)}$$

in X with leading positive coefficient is negative for some $X \in \mathbb{R}$. Note that this expression does not depend on the basis we have chosen. Hence $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if this quadratic polynomial is negative for some $X \in \mathbb{R}$ and this is true if and only if its discriminant is positive. The discriminant is given by

$$\left(6\frac{\mathrm{tr}(G)}{\det(G)} - 8\frac{1}{\mathrm{tr}(G)}\right)^2 - 4 \cdot \left(9\frac{\mathrm{tr}(G)^2}{\det(G)^2} - 36\frac{1}{\det(G)}\right) = \frac{16(3\mathrm{tr}(G)^2 + 4\det(G))}{\det(G)\mathrm{tr}(G)^2}$$

and it is positive if and only if

$$-\frac{3}{4}\mathrm{tr}(G)^2 > \det(G) \quad \text{or} \quad \det(G) > 0.$$

(b) By rescaling e^7 we may assume $\mathrm{tr}(G) = 2$, i.e. $G = \mathrm{id}$. Then we see similarly as in the proof of part (a) that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there exists a basis e^2, e^3 of V_2, a basis e^5, e^6 of W_2, $\lambda, \alpha \in \mathbb{R}$, $\alpha \neq 0$ such that each non-zero linear combination of

$$\tilde{\omega}_{1,\alpha,\lambda} := \alpha e^{56} + e^{26} + e^{35},$$

$$\tilde{\omega}_{2,\alpha,\lambda} := \frac{\mathrm{tr}(F)}{2}\alpha e^{56} + \lambda e^{23} + (F + \mathrm{tr}(F)\mathrm{id})(e^2) \wedge e^6 + (F + \mathrm{tr}(F)\mathrm{id})(e^3) \wedge e^5$$

is of length two. If we set $X := \alpha\lambda$ as before, we can argue as in part (a) that the existence of a cocalibrated G_2-structure on $\mathfrak{g}$ is equivalent to the existence of $X \in \mathbb{R}$ such that $X^2 + 4\mathrm{tr}(F)X + \mathrm{tr}(F)^2 - 4\det(F)$ is negative. Note therefore that for a two-by-two matrix $A \in \mathbb{R}^{2\times 2}$ we generally have $\det(A + \mathrm{tr}(A)I_2) = \det(A) + 2\mathrm{tr}(A)^2$. Now $X^2 + 4\mathrm{tr}(F)X + \mathrm{tr}(F)^2 - 4\det(F)$ is negative for some $X \in \mathbb{R}$ exactly when the discriminant of this quadratic polynomial in X, which is given by $12\mathrm{tr}(F)^2 + 16\det(F)$, is positive. And this is the case if and only if

$$\det(F) > -\frac{3}{4}\mathrm{tr}(F)^2.$$

$\square$

Note that a real two-by-two matrix with negative determinant is always diagonalizable in the reals. The determinant of G is negative if the condition in Lemma 3.18 (a) is not fulfilled and the determinant of F is negative if the condition in Lemma 3.18 (b) is not fulfilled. Hence it is easily checked that the condition on $\mathfrak{g}_3$ in Lemma 3.18 (a) is not fulfilled exactly when $\mathfrak{g}_3 \in \left\{\mathfrak{r}_{3,\mu} \,\middle|\, \mu \in \left[-\frac{1}{3}, 0\right)\right\}$ and that the condition on $\mathfrak{g}_4$ in Lemma 3.18 (b) is not fulfilled exactly when $\mathfrak{g}_4 \in \left\{A_{4,9}^{\alpha} \,\middle|\, \alpha \in \left(-1, -\frac{1}{3}\right]\right\}$. This finishes the proof of Theorem 1.1.

4. Appendix

Table 1 contains all three-dimensional Lie algebras. The list is further subdivided into the unimodular and the non-unimodular three-dimensional Lie algebras. The names for the non-unimodular Lie algebras in the first column have been adopted from [10]. In the second column the Lie bracket is encoded dually. Thereby, e^5, e^6, e^7 is a basis of $\mathfrak{g}^*$ and we write down the vector (de^5, de^6, de^7) and use the abbreviation $e^{ij} := e^i \wedge e^j$. Note that, instead of the more natural denotation of the basis of $\mathfrak{g}^*$ by e^1, e^2, e^3, we denote it by e^5, e^6, e^7 since these one-forms are always the last three basis elements in the dual basis of the seven-dimensional Lie algebras we consider. In the last column the vector $(h^1(\mathfrak{g}), h^2(\mathfrak{g}), h^3(\mathfrak{g}))$ of the dimensions of the corresponding Lie algebra cohomology groups is given. We omitted $h^0(\mathfrak{g})$ since it is always equal one.
Table 2 contains all four-dimensional Lie algebras and it is, as before, further subdived into the unimodular and the non-unimodular ones. The names for the Lie algebras in the first column have been adopted from [16]. In the second column the Lie bracket is encoded dually for a basis e^1, e^2, e^3, e^4 of $\mathfrak{g}^*$ as in Table 1. The next column contains the vector $(h^1(\mathfrak{g}), h^2(\mathfrak{g}), h^3(\mathfrak{g}), h^4(\mathfrak{g}))$ of the dimensions of the corresponding Lie algebra cohomology groups, where we again omit $h^0(\mathfrak{g}) = 1$. The column labelled "$\mathfrak{u}$" contains all isomorphism classes of unimodular codimension one ideals in $\mathfrak{g}$. If there are different isomorphic codimension one unimodular ideals we remark it in a footnote. The next column, labelled $[\mathfrak{g}, \mathfrak{g}]$ contains the commutator ideal of $\mathfrak{g}$. Finally, in the last column the number $h^1(\mathfrak{g}) + h^1(\mathfrak{u}) - h^2(\mathfrak{g})$ is computed. If there is more than one isomorphism class of codimension one unimodular ideals $\mathfrak{u}$, then the different numbers are written next to each other, ordered according to the order in the column "$\mathfrak{u}$".

Table 3 contains (the dual bases of) adapted bases for cocalibrated G_2-structures on three different seven-dimensional Lie algebras $\mathfrak{g}$ which are Lie algebra direct sums of a four and a three-dimensional Lie algebra. These three cases are exceptional in the sense that they do not fulfill any of the different conditions we obtained in this article which ensure the existence of a cocalibrated G_2-structure.

Table 1: Three-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$h^*(\mathfrak{g})$
	unimodular	
$\mathfrak{so}(3)$	$(e^{67}, -e^{57}, e^{56})$	$(0,0,1)$
$\mathfrak{so}(2,1)$	(e^{67}, e^{57}, e^{56})	$(0,0,1)$
$e(2)$	$(e^{67}, -e^{57}, 0)$	$(1,1,1)$
$e(1,1)$	$(e^{67}, e^{57}, 0)$	$(1,1,1)$
$\mathfrak{h}_3$	$(e^{67}, 0, 0)$	$(2,2,1)$
$\mathbb{R}^3$	$(0,0,0)$	$(3,3,1)$
	non-unimodular	
$\mathfrak{r}_2 \oplus \mathbb{R}$	$(e^{57}, 0, 0)$	$(2,1,0)$
$\mathfrak{r}_3$	$(e^{57}+e^{67}, e^{67}, 0)$	$(1,0,0)$
$\mathfrak{r}_{3,\mu}$	$(e^{57}, \mu e^{67}, 0)$, $-1 < \mu \leq 1$, $\mu \neq 0$	$(1,0,0)$
$\mathfrak{r}'_{3,\mu}$	$(\mu e^{57}+e^{67}, \mu e^{67}-e^{57}, 0)$, $\mu > 0$	$(1,0,0)$

Table 2: Four-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$h^*(\mathfrak{g})$	$\mathfrak{u}$	$[\mathfrak{g},\mathfrak{g}]$	$h^1(\mathfrak{g})+h^1(\mathfrak{u})-h^2(\mathfrak{g})$
	unimodular				
$\mathfrak{so}(3)\oplus\mathbb{R}$	$(e^{23}, -e^{13}, e^{12}, 0)$	$(1,0,1,1)$	$\mathfrak{so}(3)$	$\mathfrak{so}(3)$	1
$\mathfrak{so}(2,1)\oplus\mathbb{R}$	$(e^{23}, e^{13}, e^{12}, 0)$	$(1,0,1,1)$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(2,1)$	1
$e(2)\oplus\mathbb{R}$	$(e^{23}, -e^{13}, 0, 0)$	$(2,2,2,1)$	$\mathbb{R}^3$, $e(2)$	$\mathbb{R}^2$	3, 1
$e(1,1)\oplus\mathbb{R}$	$(e^{23}, e^{13}, 0, 0)$	$(2,2,2,1)$	$\mathbb{R}^3$, $e(1,1)$	$\mathbb{R}^2$	3, 1
$\mathfrak{h}_3\oplus\mathbb{R}$	$(e^{23}, 0, 0, 0)$	$(3,4,3,1)$	$\mathbb{R}^{3\dagger}$, $\mathfrak{h}_3$	$\mathbb{R}$	2, 1
$\mathbb{R}^4$	$(0,0,0,0)$	$(4,6,4,1)$	$\mathbb{R}^{3\ \ddagger}$	$\{0\}$	1
$A_{4,1}$	$(e^{24}, e^{34}, 0, 0)$	$(2,2,2,1)$	$\mathbb{R}^3$, $\mathfrak{h}_3$	$\mathbb{R}^2$	3, 2
$A_{4,2}^{-2}$	$(-2e^{14}, e^{24}+e^{34}, e^{34}, 0)$	$(1,0,1,1)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4

†There are several Abelian codimension one ideals, namely for all $(a,b) \neq 0$, $\mathrm{span}(e_1, ae_2+be_3, e_4)$ is one.

‡Although all codimension one unimodular ideals are isomorphic, there are of course different ones. Namely, all three-dimensional subspaces.

Table 2: Four-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$h^*(\mathfrak{g})$	$\mathfrak{u}$	$[\mathfrak{g},\mathfrak{g}]$	$h^1(\mathfrak{g})+h^1(\mathfrak{u})-h^2(\mathfrak{g})$
$A_{4,5}^{\alpha,-(\alpha+1)}$	$(e^{14},\alpha e^{24},-(\alpha+1)e^{34},0)$, $-1<\alpha\leq-\frac{1}{2}$	$(1,0,1,1)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4
$A_{4,6}^{\alpha,-\frac{1}{2}\alpha}$	$\big(\alpha e^{14},-\frac{1}{2}\alpha e^{24}+e^{34}$, $-\frac{1}{2}\alpha e^{34}-e^{24},0\big)$, $\alpha>0$	$(1,0,1,1)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4
$A_{4,8}$	$(e^{23},e^{24},-e^{34},0)$	$(1,0,1,1)$	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3
$A_{4,10}$	$(e^{23},e^{34},-e^{24},0)$	$(1,0,1,1)$	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3
	non-unimodular				
$\mathfrak{r}_2\oplus\mathbb{R}^2$	$(e^{14},0,0,0)$	$(3,3,1,0)$	$\mathbb{R}^3$	$\mathbb{R}$	3
$\mathfrak{r}_3\oplus\mathbb{R}$	$(e^{14}+e^{24},e^{24},0,0)$	$(2,1,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	4
$\mathfrak{r}_{3,\mu}\oplus\mathbb{R}$	$(e^{14},\mu e^{24},0,0)$, $-1<\mu\leq 1$, $\mu\neq 0$	$(2,1,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	4
$\mathfrak{r}'_{3,\mu}\oplus\mathbb{R}$	$(\mu e^{14}+e^{24},-e^{14}+\mu e^{24},0,0)$, $\mu>0$	$(2,1,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	4
$A_{4,2}^{\alpha}$	$(\alpha e^{14},e^{24}+e^{34},e^{34},0)$				
	$\alpha\neq 0,-1,-2$	$(1,0,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4
	$\alpha=-1$	$(1,1,1,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	3
$A_{4,3}$	$(e^{14},e^{34},0,0)$	$(2,2,1,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	3
$A_{4,4}$	$(e^{14}+e^{24},e^{24}+e^{34},e^{34},0)$	$(1,0,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4
$A_{4,5}^{\alpha,\beta}$	$(e^{14},\alpha e^{24},\beta e^{34},0)$				
	$-1<\alpha\leq\beta\leq 1$, $\alpha\beta\neq 0$, $\beta\neq-\alpha,-\alpha-1$	$(1,0,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4
	$\alpha=-1$, $\beta>0$, $\beta\neq 1$	$(1,1,1,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	3
	$\alpha=-1$, $\beta=1$	$(1,2,2,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	2
$A_{4,6}^{\alpha,\beta}$	$(\alpha e^{14},\beta e^{24}+e^{34},\beta e^{34}-e^{24},0)$				
	$\alpha>0$, $\beta\neq 0,-\frac{1}{2}\alpha$	$(1,0,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	4
	$\beta=0$, $\alpha>0$	$(1,1,1,0)$	$\mathbb{R}^3$	$\mathbb{R}^3$	3
$A_{4,7}$	$(2e^{14}+e^{23},e^{24}+e^{34},e^{34},0)$	$(1,0,0,0)$	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3
$A_{4,9}^{\alpha}$	$((\alpha+1)e^{14}+e^{23},e^{24},\alpha e^{34},0)$				
	$-1<\alpha\leq 1$, $\alpha\neq-\frac{1}{2},0$	$(1,0,0,0)$	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3
	$\alpha=-\frac{1}{2}$	$(1,1,1,0)$	$\mathfrak{h}_3$	$\mathfrak{h}_3$	2
	$\alpha=0$	$(2,1,0,0)$	$\mathfrak{h}_3$	$\mathbb{R}^2$	3
$A_{4,11}^{\alpha}$	$\big(2\alpha e^{14}+e^{23},\alpha e^{24}+e^{34}$, $\alpha e^{34}-e^{24},0\big)$, $\alpha>0$	$(1,0,0,0)$	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3

Table 2: Four-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathrm{h}^*(\mathfrak{g})$	$\mathfrak{u}$	$[\mathfrak{g},\mathfrak{g}]$	$h^1(\mathfrak{g})+h^1(\mathfrak{u})-h^2(\mathfrak{g})$
$A_{4,12}$	$\left(e^{14}+e^{23},e^{24}-e^{13},0,0\right)$	$(2,1,0,0)$	$e(2)$	$\mathbb{R}^2$	2
$\mathfrak{r}_2\oplus\mathfrak{r}_2$	$\left(e^{14}+e^{23},e^{24}+e^{13},0,0\right)$[1]	$(2,1,0,0)$	$e(1,1)$	$\mathbb{R}^2$	2

Table 3: Dual adapted bases for cocalibrated G_2-structures for some exceptional cases

Lie algebra	dual adapted basis [2]
$A_{4,8}\oplus e(1,1)$	$\left(\mathrm{e}^5,\mathrm{e}^6,\mathrm{e}^7,\mathrm{e}^4,\mathrm{e}^2,\mathrm{e}^3,\mathrm{e}^1\right)$
$A_{4,12}\oplus\mathfrak{r}_{3,1}$	$\left(-\frac{1}{3}\sqrt{5}\,\mathrm{e}^1,\sqrt{5}\,\mathrm{e}^4,\mathrm{e}^2-\frac{4}{5}\sqrt{5}\,\mathrm{e}^5,\mathrm{e}^3+\frac{2}{5}\sqrt{5}\,\mathrm{e}^6,\mathrm{e}^5,\mathrm{e}^6,\mathrm{e}^7\right)$
$\mathfrak{r}_2\oplus\mathfrak{r}_2\oplus\mathfrak{r}_{3,1}$	$\left(\mathrm{e}^2+\frac{13}{9}\mathrm{e}^5,\mathrm{e}^5,\mathrm{e}^3+3\mathrm{e}^6,\mathrm{e}^6,\frac{1}{2\sqrt{10}}\mathrm{e}^7,\frac{1}{3\sqrt{10}}\mathrm{e}^4,\frac{9}{\sqrt{10}}\mathrm{e}^1\right)$

[1]A relation of the standard basis f^1, f^2, f^3, f^4 of $\mathfrak{r}_2^*\oplus\mathfrak{r}_2^*$ with $(df^1,df^2,df^3,df^4)=(f^{12},0,f^{34},0)$ to our basis e^1, e^2, e^3, e^4 is given by $e^1=f^1+f^3$, $e^2=f^1-f^3$, $e^3=\frac{1}{2}\left(f^2-f^4\right)$, $e^4=\frac{1}{2}\left(f^2+f^4\right)$.

[2]In each case, $(\mathrm{e}^1,\ldots,\mathrm{e}^7)$ denotes a basis such that $\mathrm{e}^1,\ldots,\mathrm{e}^4$ satisfy the Lie algebra structure given in Table 2 and $\mathrm{e}^5,\ldots,e^7$ satisfy the Lie algebra structure given in Table 1

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