Conification of Kähler and hyper-Kähler manifolds

D.V. Alekseevsky[1], V. Cortés[2] and T. Mohaupt[3]

[1]Institute for Information Transmission Problems
B. Karetny per. 19
101447 Moscow, Russia
daleksee@staffmail.ed.ac.uk

[2]Department of Mathematics
and Center for Mathematical Physics
University of Hamburg
Bundesstraße 55, D-20146 Hamburg, Germany
cortes@math.uni-hamburg.de

[3]Department of Mathematical Sciences
University of Liverpool
Liverpool L69 3BX, UK
Thomas.Mohaupt@liv.ac.uk

July 13, 2012

Abstract

Given a Kähler manifold M endowed with a Hamiltonian Killing vector field Z, we construct a conical Kähler manifold $\hat{M}$ such that M is recovered as a Kähler quotient of $\hat{M}$. Similarly, given a hyper-Kähler manifold (M, g, J_1, J_2, J_3) endowed with a Killing vector field Z, Hamiltonian with respect to the Kähler form of J_1 and satisfying $\mathcal{L}_Z J_2 = -2J_3$, we construct a hyper-Kähler cone $\hat{M}$ such that M is a certain hyper-Kähler quotient of $\hat{M}$. In this way, we recover a theorem by Haydys. Our work is motivated by the problem of relating the supergravity c-map to the rigid c-map. We show that any hyper-Kähler manifold in the image of the c-map admits a Killing vector field with the above properties. Therefore, it gives rise to a hyper-Kähler cone, which in turn defines a quaternionic Kähler manifold. Our results for the signature of the metric and the sign of the scalar curvature are consistent with what we know about the supergravity c-map.

Introduction

Let us recall that there is an interesting geometric construction called the c-map, which was found by theoretical physicists. There are in fact two versions of the c-map: the supergravity c-map and the rigid c-map. The *supergravity c-map* associates a quaternionic Kähler manifold of negative scalar curvature with any projective special Kähler manifold, see [FS, H2, CM]. The metric is explicit but rather complicated. The *rigid c-map* is much simpler, see [CFG, H1, ACD]. It associates a hyper-Kähler manifold with any affine special Kähler manifold. The initial motivation for this work was our idea to reduce the supergravity c-map to the rigid c-map by means of a conification of the hyper-Kähler manifold obtained from the rigid c-map. Let us explain this idea in more detail.

Since any projective special Kähler manifold $\bar{M}$ is the base of a $\mathbb{C}^*$-bundle with the total space a conical affine special Kähler manifold M, we have the following diagram:

$$\begin{tikzcd} M^{2n} \arrow[r, mapsto, "c"] \arrow[d, "\mathbb{C}^*"'] & N^{4n} \\ \bar{M}^{2n-2} \arrow[r, mapsto, "\bar{c}"] & \bar{N}^{4n} \end{tikzcd}$$

where c stands for the rigid c-map, $\bar{c}$ for the supergravity c-map and N, $\bar{N}$ are the resulting (pseudo-)hyper-Kähler and quaternionic Kähler manifolds, respectively. We have indicated the real dimension. This shows that N cannot simply be the Swann bundle $\hat{N}$ over $\bar{N}$. In fact, N is in general not conical and the (pseudo-)hyper-Kähler cone $\hat{N}$ should be obtained from N by a certain conification procedure $N^{4n} \overset{con}{\mapsto} \hat{N}^{4n+4}$ such that the following diagramm commutes:

$$\begin{tikzcd} M^{2n} \arrow[r, mapsto, "c"] \arrow[d, "\mathbb{C}^*"'] & N^{4n} \arrow[r, mapsto, "con"] & \hat{N}^{4n+4} \arrow[d, "\mathbb{H}^*/\pm 1"] \\ \bar{M}^{2n-2} \arrow[rr, mapsto, "\bar{c}"] & & \bar{N}^{4n} \end{tikzcd}$$

We are also interested in the analogous problem for the r-map, where we have a diagramm of the form:

$$\begin{tikzcd} M^{n} \arrow[r, mapsto, "r"] \arrow[d, "\mathbb{R}^{>0}"'] & N^{2n} \arrow[r, mapsto, "con"] & \hat{N}^{2n+2} \arrow[d, "\mathbb{C}^*"] \\ \bar{M}^{n-1} \arrow[rr, mapsto, "\bar{r}"] & & \bar{N}^{2n} \end{tikzcd}$$

Now M is an affine special real manifold with homogeneous cubic prepotential, $\bar{M}$ is the corresponding projective special real manifold, r is the rigid r-map [CMMS1, AC], $\bar{r}$ is the supergravity r-map [DV, CM] and $\hat{N}$ is the conical affine special Kähler manifold over the projective special Kähler manifold $\bar{N}$.

An important inspiration for our work has been the paper [Ha] by Haydys, see also [APP] in which Haydys construction is called *QK/HK correspondence*. The construction has two parts. The first part is the hyper-Kähler reduction of a hyper-Kähler cone with respect to a Hamiltonian Killing vector field which is compatible with the cone structure. The hyper-Kähler manifold (M, g, J_1, J_2, J_3) obtained by such a reduction inherits a Killing vector field Z which preserves one of the three complex structures J_1 of the hyper-Kähler triplet (J_α) and rotates the two other ones. The second part is the inversion of the reduction, which is much more involved than the first part. As a result of our careful analysis, we are able to give our own proof of the inversion recovering and extending the results by Haydys. Under the assumptions stated precisely in Section 2, the conical hyper-Kähler structure is rigorously established in Theorem 2. The final formulas are explicit enough to allow for further progress in the study of hyper-Kähler manifolds obtained by such a conification. As an example, we can easily compute the signature and scalar curvature of the resulting quaternionic Kähler manifolds, see Corollary 1 and Corollary 2. These results are new even in the case when the initial hyper-Kähler metric is positive definite, as considered in [Ha]. We show that (positive definite) quaternionic Kähler manifolds of negative scalar curvature can be obtained from indefinite as well as from positive definite hyper-Kähler manifolds, whereas quaternionic Kähler manifolds of positive scalar curvature do always require a positive definite initial metric.

We prove that a similar, but simpler, conification result holds for any Kähler manifold endowed with a Hamiltonian Killing vector field, see Theorem 1. This construction is new and may provide the needed conification procedure for the r-map. We will study this problem in the future.

For the c-map we prove the existence of a canonical Killing vector field satisfying the assumptions of Theorem 2. In this way we can associate a family of (possibly indefinite) conical hyper-Kähler metrics and, hence, a family of quaternionic Kähler metrics to any projective special Kähler manifold. In view of the results of [APP] Section 2.4, this family should contain the Ferrara-Sabharwal metric as well as the (locally defined) one-parameter deformation discussed in [APSV] and in the papers cited there. The parameter should be related to the choice of Hamiltonian function f, which is unique up to a constant. This will be the topic of future investigation.

Acknowledgments We thank Stefan Vandoren for discussions and for his notes about examples of the QK/HK correspondence. We also thank Malte Dyckmanns for useful comments. This work is part of a research project within the RTG 1670 "Mathematics inspired by String Theory", funded by the Deutsche Forschungsgemeinschaft (DFG). The work of T.M. is supported in part by STFC grant ST/J000493/1. T.M. thanks the

Department of Mathematics and the SFB 676 for hospitality and support during several stages of this work.

1 Conification of Kähler manifolds

Definition 1 *An* almost pseudo-Kähler manifold (M, g, J) *is a pseudo-Riemannian manifold* (M, g) *endowed with a skew-symmetric almost complex structure with closed fundamental form* $\omega := g(J\cdot, \cdot)$*. It is called a* pseudo-Kähler manifold *if the almost complex structure is integrable. In that case* ω *is called the* Kähler form*.*

Let (M, g, J, Z) be a pseudo-Kähler manifold endowed with a time-like or space-like Hamiltonian Killing vector field Z. Let $-f$ be a corresponding Hamiltonian function, that is $df = -\omega(Z, \cdot)$, where ω is the Kähler form. We will assume that f and $f_1 := f - \frac{1}{2}g(Z, Z)$ are nowhere vanishing.

Lemma 1 *Let* Z *be a Killing vector field on a pseudo-Kähler manifold* (M, g, J) *and put* $h = \frac{g(Z,Z)}{2}$*. Then*

$$dh = \omega(JD_Z Z, \cdot),$$

where D *is the Levi-Civita connection. In particular,* $dh = -\omega(Z, \cdot)$ *holds if and only if* $D_Z Z = JZ$*.*

Proof:

$$dh = g(DZ, Z) = -g(D_Z Z, \cdot) = \omega(JD_Z Z, \cdot).$$

□

The Lemma implies that

$$df_1 = d(f - h) = -\omega(Z + JD_Z Z, \cdot) = -g(J(Z + JD_Z Z), \cdot). \tag{1.1}$$

Let $\pi : P \to M$ be an S^1-principal bundle endowed with a principal connection η with the curvature $d\eta = \pi^*(\omega - \frac{1}{2}d\beta)$, where $\beta = g(Z, \cdot)$. Notice that locally we can always assume $P = M \times S^1$ and $\eta = ds + \eta_M$, where $\eta_M \in \Omega^1(M)$ and s is the angular coordinate on $S^1 = \{e^{is} | s \in \mathbb{R}\}$. We will denote the fundamental vector field of P by X_P. It coincides with the vertical coordinate vector field ∂_s in any local trivialisation of the principal bundle. We define a pseudo-Riemannian metric on P by

$$g_P := \frac{2}{f_1}\eta^2 + \pi^* g.$$

Next we consider $\hat{M} := P \times \mathbb{R}$ with the coordinate t on the $\mathbb{R}$-factor and the projection $\hat{\pi} : \hat{M} \to M$ defined as $\hat{\pi}(p,t) = \pi(p)$, for all $(p,t) \in \hat{M}$. On $\hat{M}$ we introduce the following tensor fields.

$$
\begin{aligned}
\xi &:= \partial_t \in \mathfrak{X}(\hat{M}), && (1.2)\\
\hat{g} &:= e^{2t}(g_P + 2f dt^2 + 2\alpha dt) \in \Gamma(S^2 T^* \hat{M}), && (1.3)\\
\theta &:= e^{2t}(\eta + \frac{1}{2}\beta) \in \Omega^1(\hat{M}), && (1.4)\\
\hat{\omega} &:= d\theta \in \Omega^2(\hat{M}), && (1.5)
\end{aligned}
$$

where $\alpha := df$ and covariant tensor fields on M and P are identified with their pullbacks to tensor fields on $\hat{M}$.

Definition 2 *A* conical *pseudo-Riemannian manifold (M, g, ξ) is a pseudo-Riemannian manifold (M, g) endowed with a time-like or space-like vector field ξ such that $D\xi = \mathrm{Id}$.*

Theorem 1 *Given (M, g, J, Z) as above, the tensor field $\hat{g}$ defines a pseudo-Riemannian metric such that $(\hat{M}, \hat{g}, \hat{J} := \hat{g}^{-1}\hat{\omega}, \xi)$ is a conical pseudo-Kähler manifold. The induced CR-structure on the hypersurface $P \subset \hat{M}$ coincides with the horizontal distribution T^hP for the connection η and $\pi : P \to M$ is holomorphic, that is $d\pi \hat{J} = J d\pi$ on T^hP. The projection $\hat{\pi} : \hat{M} \to M$ is not holomorphic, since* $\ker d\hat{\pi} = \mathrm{span}\{X_P, \xi\}$ *is not $\hat{J}$-invariant. The metric $\hat{g}$ has signature $(2k+2, 2\ell)$ if $f_1 > 0$ and $(2k, 2\ell+2)$ if $f_1 < 0$, where $(2k, 2\ell)$ is the signature of the metric g.*

Proof: It is clear that the restriction of $\hat{g}$ to the horizontal distribution $T^hP = \ker \eta \subset TP$ is nondegenerate. Let us denote by E the orthogonal complement of the 2-dimensional distribution $\mathrm{span}\{\widetilde{Z}, \widetilde{JZ}\} \subset T^hP$, where $\widetilde{X} \in \mathfrak{X}(P)$ stands for the horizontal lift of a vector field $X \in \mathfrak{X}(M)$. Since $g(Z,Z) \neq 0$, we see that $E \oplus \mathbb{R}\widetilde{Z} \subset T^hP$ is nondegenerate. The orthogonal distribution in $\hat{M}$ is precisely

$$\mathcal{D} = \mathrm{span}\{\widetilde{JZ}, X_P, \xi\},$$

as follows from $\alpha(Z) = df(Z) = -\omega(Z,Z) = 0$. The matrix representing the bilinear form $\hat{g}|_{\mathcal{D}}$ with respect to the frame $(\widetilde{JZ}, X_P, \xi)$ is given by

$$e^{2t}\begin{pmatrix} g(Z,Z) & 0 & -g(Z,Z) \\ 0 & \frac{2}{f_1} & 0 \\ -g(Z,Z) & 0 & 2f \end{pmatrix},$$

which has the determinant $4e^{6t}g(Z,Z) \neq 0$. This proves that $\hat{g}$ is nondegenerate. The signature of $\hat{g}$ can be easily read off from the above matrix.

Let us prove next that the skew-symmetric endomorphism field $\hat{J} = \hat{g}^{-1}\hat{\omega}$ is also nondegenerate. Calculating the differential of θ we obtain

$$\hat{\omega} = 2e^{2t}dt \wedge (\eta + \frac{1}{2}\beta) + e^{2t}\omega. \tag{1.6}$$

This formula immediately implies that $\hat{J}$ preserves the distribution E and $\hat{J}\widetilde{X} = \widetilde{JX}$ for all $X \in \mathfrak{X}(M)$ which are perpendicular to Z and JZ.

Claim 1: $\hat{J}$ preserves the distribution T^hP and

$$\hat{J}\widetilde{X} = \widetilde{JX} \quad \text{for all} \quad X \in \mathfrak{X}(M). \tag{1.7}$$

It remains to check (1.7), or equivalently, that $\hat{\omega}(\widetilde{X}, \cdot) = \hat{g}(\widetilde{JX}, \cdot)$, for $X \in \{Z, JZ\}$. Using the formulas (1.3) and (1.6), we have

$$\hat{\omega}(\widetilde{Z}, \cdot) = -e^{2t}(\beta(Z)dt + \alpha), \quad \hat{\omega}(\widetilde{JZ}, \cdot) = -e^{2t}\beta,$$

$$\hat{g}(\widetilde{JZ}, \cdot) = e^{2t}(-\alpha + \alpha(JZ)dt), \quad \hat{g}(\widetilde{Z}, \cdot) = e^{2t}\beta.$$

This proves Claim 1, since $\alpha(JZ) = -\beta(Z)$.

Claim 2:

$$\hat{J}X_P = -\frac{1}{f_1}(\widetilde{JZ} + \xi).$$

It suffices to check that

$$\hat{\omega}(X_P, \cdot) = -\frac{1}{f_1}\hat{g}(\widetilde{JZ} + \xi, \cdot).$$

Using (1.6), we see that the left-hand side is simply $-2e^{2t}dt$. The right-hand side yields

$$-\frac{e^{2t}}{f_1}(-\alpha + \alpha(JZ)dt + 2f dt + \alpha) = -\frac{e^{2t}}{f_1}(-g(Z,Z) + 2f)dt = -2e^{2t}dt.$$

This proves Claim 2.

Claim 3:

$$T\hat{M} = T^hP \overset{\perp}{\oplus} \operatorname{span}\{X_P, \hat{J}X_P\}. \tag{1.8}$$

In view of Claim 2, is clear that $X_P \perp T^hP$ and $\hat{J}X_P \perp E \oplus \mathbb{R}\widetilde{Z}$. Therefore it suffices to show that $\hat{J}X_P$ is perpendicular to $\widetilde{JZ}$. We compute

$$-f_1\hat{g}(\hat{J}X_P, \widetilde{JZ}) = \hat{g}(\widetilde{JZ} + \xi, \widetilde{JZ}) = e^{2t}(g(Z,Z) + \alpha(JZ)) = 0.$$

This proves Claim 3.

Claim 4: The distributions $T^hP, \operatorname{span}\{X_P, \hat{J}X_P\} \subset T\hat{M}$ are nondegenerate and orthogonal with respect to $\hat{\omega}$.

In fact,

$$\hat{\omega}|_{T^hP} = e^{2t}\omega \tag{1.9}$$

is nondegenerate and also

$$\hat{\omega}(X_P, \hat{J}X_P) = 2e^{2t}\left(dt \wedge (\eta + \frac{1}{2}\beta)\right)(X_P, \hat{J}X_P) = -2e^{2t}dt(\hat{J}X_P) = \frac{2e^{2t}}{f_1} \neq 0.$$

The $\hat{\omega}$-orthogonality of the distributions follows from Claim 3 and the $\hat{J}$-invariance of T^hP. This proves Claim 4.

Claim 5: $\hat{J}$ is an almost complex structure.

Recall that, by Claim 1, $\hat{J}|_{T^hP}$ corresponds to the complex structure J by means of the identification $T^hP \cong \pi^*TM$. Therefore, it suffices to check that $\hat{J}$ squares to $-\mathrm{Id}$ on $\mathrm{span}\{X_P, \hat{J}X_P\}$. Using Claim 1 and 2, we compute

$$\hat{J}^2 X_P = \frac{1}{f_1}(\widetilde{Z} - \hat{J}\xi).$$

So we need to check that

$$X_P = -\frac{1}{f_1}(\widetilde{Z} - \hat{J}\xi), \tag{1.10}$$

or, equivalently, that

$$\hat{g}(X_P, \cdot) = -\frac{1}{f_1}(\hat{g}(\widetilde{Z}, \cdot) - \hat{\omega}(\xi, \cdot)).$$

The left-hand side is simply $\frac{2e^{2t}}{f_1}\eta$ and the right-hand side

$$-\frac{e^{2t}}{f_1}(g(Z, \cdot) - 2\eta - \beta) = \frac{2e^{2t}}{f_1}\eta.$$

This proves Claim 5.

So far we have proven that $\hat{J}$ is a skew-symmetric almost complex structure with closed fundamental form, in other words that $(\hat{M}, \hat{g}, \hat{J})$ is an almost pseudo-Kähler manifold. Notice that Claim 1 implies that the induced CR-structure on P coincides with the horizontal distribution and that $\pi : P \to M$ is holomorphic. Claim 2 shows that $\hat{\pi} : \hat{M} \to M$ is not holomorphic.

Next we prove that $D\xi = \mathrm{Id}$. By the Koszul formula, we have

$$\begin{aligned} 2\hat{g}(D_{X_1}\xi, X_2) &= X_1\hat{g}(\xi, X_2) + \xi\hat{g}(X_1, X_2) - X_2\hat{g}(X_1, \xi) \\ &\quad + \hat{g}([X_1, \xi], X_2) - \hat{g}(X_1, [\xi, X_2]) - \hat{g}(\xi, [X_1, X_2]) \end{aligned}$$

for all vector fields X_1, X_2 on $\hat{M}$. If X_1, X_2 are horizontal lifts of commuting vector fields on M, the right-hand side yields

$$e^{2t}d\alpha(X_1, X_2) + 2\hat{g}(X_1, X_2) = 2\hat{g}(X_1, X_2).$$

Similarly, if $X_1, X_2 \in \{X_P, \xi\}$, the right-hand side is also

$$2f(X_1e^{2t}dt(X_2) - X_2e^{2t}dt(X_1)) + 2\hat{g}(X_1, X_2) = 2\hat{g}(X_1, X_2).$$

Next we consider the case where X_1 is a horizontal lift and $X_2 = \xi$. The Koszul formula gives again

$$2\hat{g}(D_{X_1}\xi, \xi) = 2e^{2t}X_1 f = 2\hat{g}(X_1, \xi)$$

and, similarly, for $X_2 = X_P$:

$$2\hat{g}(D_{X_1}\xi, X_P) = 0 = 2\hat{g}(X_1, X_P).$$

Next, let X_2 be a horizontal lift and $X_1 = \xi$. Then

$$2\hat{g}(D_\xi\xi, X_2) = 2\xi e^{2t}\alpha(X_2) - 2e^{2t}X_2 f = 2e^{2t}\alpha(X_2) = 2\hat{g}(\xi, X_2).$$

Finally, for $X_1 = X_P$ we get

$$2\hat{g}(D_{X_P}\xi, X_2) = 0 = 2\hat{g}(X_P, X_2).$$

Next we prove that $\hat{J}$ is integrable. In order to apply the Newlander-Nirenberg theorem, let us first recall that the decomposition (1.8) is $\hat{J}$-invariant, in virtue of Claim 4. Therefore, Claim 1 implies that

$$T_p^{1,0}\hat{M} = \mathrm{span}\{(\tilde{X} - i\widetilde{JX})_p | X \in \mathfrak{X}(M)\} \oplus \mathbb{C}(X_P - i\hat{J}X_P)_p,$$

for all $p \in \hat{M}$. By the integrability of the complex structure on M, we know that for all $X, Y \in \mathfrak{X}(M)$ there exists $W \in \mathfrak{X}(M)$ such that

$$[X - iJX, Y - iJY] = W - iJW.$$

Therefore,

$$\begin{aligned} [\tilde{X} - i\widetilde{JX}, \tilde{Y} - i\widetilde{JY}] &= \tilde{W} - i\widetilde{JW} - d\eta(\tilde{X} - i\widetilde{JX}, \tilde{Y} - i\widetilde{JY})X_P \\ &= \tilde{W} - i\widetilde{JW} - (\omega - \frac{1}{2}d\beta)(X - iJX, Y - iJY)X_P. \end{aligned}$$

Here we have used, the well known fact that the vertical part of the commutator of two horizontal vector fields on a principal bundle with connection is given by minus the curvature. We claim that not only ω but also $d\beta$ is of type $(1,1)$, which finally implies $[\tilde{X} - i\widetilde{JX}, \tilde{Y} - i\widetilde{JY}] = \tilde{W} - i\widetilde{JW}$. In fact,

$$d\beta = dg(Z, \cdot) = -d\omega(JZ, \cdot) = -\mathcal{L}_{JZ}\omega$$

is the Lie derivative of a form of type $(1,1)$ with respect to a holomorphic (thus type-preserving) vector field. Finally, with the help of Claim 2, for $X \in \mathfrak{X}(M)$, we compute

$$\begin{aligned} & [\tilde{X} - i\widetilde{JX}, X_P - i\hat{J}X_P] = -i[\tilde{X} - i\widetilde{JX}, \hat{J}X_P] \\ = \; & id(\frac{1}{f_1})(X - iJX)(\widetilde{JZ} + \xi) + \frac{i}{f_1}[\tilde{X} - i\widetilde{JX}, \widetilde{JZ}] \\ = \; & i\frac{df_1}{f_1}(X - iJX)\hat{J}X_P - \frac{i}{f_1}d\eta(X - iJX, JZ)X_P + \frac{i}{f_1}[X - iJX, JZ]\tilde{}. \end{aligned}$$

Notice that the last term is the horizontal lift of a vector field of type $(1,0)$. In fact, it suffices to observe that the Lie derivative with respect to the holomorphic vector field JZ preserves the type. The remaining part is of type $(1,0)$ if and only if

$$d\eta(X - iJX, JZ) = -idf_1(X - iJX). \tag{1.11}$$

for all $X \in TM$. Now

$$\begin{aligned}
d\eta(\cdot, JZ) &= g(Z,\cdot) + \frac{1}{2}d\beta(JZ,\cdot), \\
d\beta(JZ,\cdot) &= \mathcal{L}_{JZ}\beta = g(D_Z(JZ),\cdot) + g(Z, D(JZ)) \\
g(Z, D(JZ)) &= -g(JZ, DZ) = g(D_{JZ}Z,\cdot) = g(D_Z(JZ),\cdot) = g(JD_ZZ,\cdot)
\end{aligned}$$

Therefore,

$$d\eta(\cdot, JZ) = g(Z + JD_ZZ, \cdot).$$

Comparing with (1.1) we see that that (1.11) is equivalent to

$$\begin{aligned}
& g(Z + JD_ZZ, X - iJX) = ig(J(Z + JD_ZZ), X - iJX) \\
= \;& g(J(Z + JD_ZZ), J(X - iJX)) = g(Z + JD_ZZ, X - iJX),
\end{aligned}$$

which is always satisfied. □

Definition 3 *Let (M,g) be any pseudo-Riemannian manifold. Then $C_\pm(M) := (\mathbb{R}^{>0} \times M, \pm dr^2 + r^2 g)$ is called the* space-like or time-like cone *over (M,g), respectively.*

The vector field $\xi = r\partial_r$ defines on $C_\pm(M)$ the structure of a conical pseudo-Riemannian manifold and any conical pseudo-Riemannian manifold is locally isomorphic to a space-like or time-like cone.

Definition 4 *A* pseudo-Sasakian structure *on a pseudo-Riemannian manifold (M,g) is a unit Killing vector field Z such that $J := DZ|_{Z^\perp}$ defines an integrable CR-structure $H = Z^\perp \subset TM$ with the Levi form $2g$. The* Levi-form *is the symmetric bilinear form L on H defined by $L(X,Y) = \frac{g(Z,[JX,Y])}{g(Z,Z)}$.*

It is well known that (M,g) admits a space-like or time-like pseudo-Sasakian structure Z if and only if the space-like or time-like cone over (M,g) admits a Kähler structure $\hat{J}$ compatible with the cone metric.

Example 1 *In Theorem 1 we have assumed that Z is nowhere light-like. However, one can verify that the construction remains meaningful if we put $Z = 0$. Taking $Z = 0$ and $f = const = c \neq 0$ in the construction of Theorem 1, yields a conical pseudo-Kähler manifold $(\hat{M}, \hat{g}, \hat{J} = \hat{g}^{-1}\hat{\omega}, \xi)$. It is precisely the space-like ($c > 0$) or time-like ($c < 0$) cone over $(P, \frac{1}{2|c|}g_P)$, where $r = \sqrt{2|c|}e^t$. The unit Killing vector field $\zeta := |c|X_P$ defines a pseudo-Sasakian structure on $(P, \frac{1}{2|c|}g_P)$. Notice that $(P, \frac{1}{2|c|}g_P)$ is a pseudo-Riemannian submersion over the pseudo-Kähler manifold $(M, \frac{1}{2|c|}g)$. In particular, we can take $f = \pm\frac{1}{2}$ and $r = e^t$, which yields $(\hat{M}, \hat{g})$ as the space-like or time-like cone over the pseudo-Sasaki manifold $(P, g_P, \zeta = \frac{1}{2}X_P)$ and the latter fibers as a pseudo-Riemannian submersion over (M, g). Alternatively, we may take $c = \pm 1$, for which X_P is the Sasaki structure. In that case $(\hat{M}, \hat{g})$ is the space-like or time-like cone over the pseudo-Sasaki manifold $(P, \frac{1}{2}g_P = \pm\eta^2 + \frac{1}{2}g, X_P)$ and the latter fibers as a pseudo-Riemannian submersion over $(M, \frac{1}{2}g)$.*

2 Conification of hyper-Kähler manifolds

Let (M, g, J_1, J_2, J_3) be a pseudo-hyper-Kähler manifold with the three Kähler forms $\omega_\alpha := gJ_\alpha := g(J_\alpha\cdot, \cdot)$, $\alpha = 1, 2, 3$. We will assume that Z is a time-like or space-like Killing vector field and that f is a nowhere vanishing function such that $df = -\omega_1 Z := -\omega_1(Z, \cdot)$. Following the notation of the previous section, we put $f_1 := f - h$, where $h := \frac{g(Z,Z)}{2}$. We will also assume that f_1 is nowhere zero. Applying Theorem 1 to the pseudo-Kähler manifold (M, g, J_1) endowed with the ω_1-Hamiltonian Killing vector field Z, we obtain the principal bundle $\pi : P \to M$ with the connection η and the pseudo-Riemannian metric g_P such that $\hat{M}_1 := P \times \mathbb{R}$ is endowed with the structure of a conical pseudo-Kähler manifold. Our aim is to construct a conical pseudo-hyper-Kähler manifold $(\hat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ such that $\hat{M}_1 \subset \hat{M}$ with the conical pseudo-Kähler structure induced by $(\hat{g}, \hat{J}_1, \xi)$. As a first step, we define the vector field

$$Z_1 := \tilde{Z} + f_1 X_P$$

and the one-forms

$$\begin{aligned} \theta_1^P &:= \eta + \frac{1}{2}gZ \\ \theta_2^P &:= \frac{1}{2}\omega_3 Z \\ \theta_3^P &:= -\frac{1}{2}\omega_2 Z \end{aligned} \tag{2.1}$$

on P. We consider $\theta_\alpha := f^{-1}\theta_\alpha^P$ as the components of a one-form $\theta := \sum_\alpha \theta_\alpha i_\alpha$ with values in the imaginary quaternions, where $(i_1, i_2, i_3) = (i, j, k)$. Then we extend θ to a one-form $\tilde{\theta}$ on $\tilde{M} := \mathbb{H}^* \times P \supset \{1\} \times P \cong P$ by

$$\tilde{\theta}_\alpha(q,p) := \varphi_\alpha(q) + (\mathrm{Ad}_q\theta(p))_\alpha, \quad (q,p) \in \tilde{M},$$

where $\varphi = \varphi_0 + \sum_\alpha \varphi_\alpha i_\alpha$ is the right-invariant Maurer-Cartan form of $\mathbb{H}^*$ and $\mathrm{Ad}_q x = qxq^{-1} = x_0 + \sum_\alpha (\mathrm{Ad}_q x)_\alpha i_\alpha$ for all $x = x_0 + \sum x_\alpha i_\alpha \in \mathbb{H}$. Notice that

$$\varphi_a(e_b) = \delta_{ab},$$

where $(e_0, \ldots, e_3)$, is the right-invariant frame of $\mathbb{H}^*$ which coincides with the standard basis of $\mathbb{H} = \mathrm{Lie}(\mathbb{H}^*)$ at $q = 1$. Next we define

$$\tilde{\omega}_\alpha := d(\rho^2 \tilde{\theta}^P_\alpha),$$

where $\tilde{\theta}^P_\alpha := f\tilde{\theta}_\alpha$ and $\rho := |q|$. Let us denote by e_1^L the left-invariant vector field on $\mathbb{H}^*$ which coincides with e_1 at $q = 1$ and by $\hat{M}$ the space of integral curves of the vector field $V_1 := e_1^L - Z_1$. We will assume that the quotient map $\tilde{\pi} : \tilde{M} \to \hat{M}$ is a submersion onto a Hausdorff manifold. (Locally this is always the case, since the vector field has no zeroes.)

Theorem 2 *Let (M, g, J_1, J_2, J_3, Z) be a pseudo-hyper-Kähler manifold endowed with a Killing vector field Z satisfying the above assumptions and $\mathcal{L}_Z J_2 = -2J_3$. Then there exists a pseudo-hyper-Kähler structure $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3)$ on $\hat{M}$ with exact Kähler forms $\hat{\omega}_\alpha$ determined by*

$$\tilde{\pi}^* \hat{\omega}_\alpha = \tilde{\omega}_\alpha. \tag{2.2}$$

The vector field $\rho\partial_\rho$ on $\tilde{M}$ projects to a vector field ξ on $\hat{M}$ such that $(\hat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ is a conical pseudo-hyper-Kähler manifold. The signature of the metric $\hat{g}$ is $(4k, 4\ell + 4)$ if $f_1 < 0$ and $(4k+4, 4\ell)$ if $f_1 > 0$, where $(4k, 4\ell)$ is the signature of the metric g.

Proof: We first show that the one-forms $\tilde{\theta}_\alpha$ on $\tilde{M}$ induce one-forms $\hat{\theta}_\alpha$ on the quotient $\hat{M}$.

Lemma 2 *There exist one-forms $\hat{\theta}_\alpha$ on $\hat{M}$ such that $\tilde{\theta}_\alpha = \tilde{\pi}^* \hat{\theta}_\alpha$.*

Proof: Let us first observe that the above definitions imply that $\theta(Z_1) = i_1 = i$. To compute $\varphi(e_1^L)$, we use the equivariance of the right-invariant Maurer-Cartan form with respect to left-multiplication:

$$\varphi(dL_q v) = \mathrm{Ad}_q \varphi(v),$$

for all $q \in \mathbb{H}^*$, $v \in T\mathbb{H}^*$. Using that $\varphi(v) = v$ for all $v \in T_e\mathbb{H}^*$, we conclude that

$$\varphi(e_\alpha^L) = \varphi(dL_q i_\alpha) = \mathrm{Ad}_q(i_\alpha).$$

Combining these facts, we get

$$\tilde{\theta}(V_1) = \varphi(e_1^L) - \mathrm{Ad}_q(\theta(Z_1)) = \mathrm{Ad}_q i - \mathrm{Ad}_q i = 0.$$

This shows that the one-forms $\tilde{\theta}_\alpha$ on $\tilde{M}$ are horizontal with respect to the projection $\tilde{\pi} : \tilde{M} \to \hat{M}$. To prove the lemma, it now suffices to check that $\mathcal{L}_{V_1}\tilde{\theta} = 0$. First of all, the right-invariance of φ implies the invariance under any left-invariant vector field. So $\mathcal{L}_{V_1}\varphi = \mathcal{L}_{e_1^L}\varphi = 0$ and we are left with

$$\mathcal{L}_{V_1}\tilde{\theta} = \mathcal{L}_{e_1^L}\mathrm{Ad}_q\theta - \mathrm{Ad}_q\mathcal{L}_{Z_1}\theta. \tag{2.3}$$

The first term is easily computed as follows:

$$\mathcal{L}_{e_1^L}\mathrm{Ad}_q x = \frac{d}{dt}\Big|_{t=0} q\exp(ti)x\exp(-ti)q^{-1} = \mathrm{Ad}_q[i,x],$$

for all $x \in \mathbb{H}$. This shows that

$$\mathcal{L}_{e_1^L}\mathrm{Ad}_q\theta = \mathrm{Ad}_q[i,\theta]. \tag{2.4}$$

For the computation of $\mathcal{L}_{Z_1}\theta$ we first remark that $\mathcal{L}_{X_P}\theta = 0$, such that $\mathcal{L}_{f_1X_P}\theta = f^{-1}df_1\theta(X_P) = if^{-1}df_1$ and $\mathcal{L}_{Z_1}\theta = \mathcal{L}_{\tilde{Z}}\theta + if^{-1}df_1$. We compute each component $\mathcal{L}_{\tilde{Z}}\theta_\alpha$. Using that Z is Killing, we get

$$\mathcal{L}_{\tilde{Z}}\theta_1^P = \mathcal{L}_{\tilde{Z}}\eta + \frac{1}{2}\mathcal{L}_Z(gZ) = \mathcal{L}_{\tilde{Z}}\eta = \omega_1 Z - \frac{1}{2}d(gZ)Z = \omega_1 Z + dh = -df_1.$$

Here we used that $d(gZ)Z = \mathcal{L}_Z(gZ) - d(g(Z,Z)) = 0 - 2dh = -2dh$. The hypothesis $\mathcal{L}_Z J_2 = -2J_3$ on the ω_1-Hamiltonian Killing vector field Z immediately implies

$$\mathcal{L}_{\tilde{Z}}\theta_2^P = \frac{1}{2}\mathcal{L}_Z\omega_3 Z = \frac{1}{2}\mathcal{L}_Z\omega_1 J_2 Z = -\omega_1 J_3 Z = \omega_2 Z = -2\theta_3^P$$

and, similarly, $\mathcal{L}_{\tilde{Z}}\theta_3^P = 2\theta_2^P$. (Notice that $\mathcal{L}_Z J_2 = -2J_3$ implies $\mathcal{L}_Z J_3 = 2J_2$, because Z is J_1-holomorphic.) Since $\mathcal{L}_{Z_1}f = 0$, this shows that $\mathcal{L}_{Z_1}\theta_1 = \mathcal{L}_{Z_1}(f^{-1}\theta_1^P) = 0$, $\mathcal{L}_{Z_1}\theta_2 = -2\theta_3$ and $\mathcal{L}_{Z_1}\theta_3 = 2\theta_2$. Summarising, we have

$$\mathcal{L}_{Z_1}\theta = [i,\theta]. \tag{2.5}$$

The equations (2.3), (2.4) and (2.5) show that

$$\mathcal{L}_{V_1}\tilde{\theta} = \mathrm{Ad}_q[i,\theta] - \mathrm{Ad}_q[i,\theta] = 0.$$

□

Since $\mathcal{L}_{V_1}\rho = \mathcal{L}_{V_1}f = 0$, the functions f and ρ are well defined on the quotient $\hat{M}$. Therefore, the lemma shows that

$$\hat{\omega}_\alpha := d(\rho^2 f\hat{\theta}_\alpha)$$

are two-forms on $\hat{M}$, which satisfy (2.2). To prove that the triplet $(\hat{\omega}_\alpha)$ defines a pseudo-hyper-Kähler structure we will prove that the $\hat{\omega}_\alpha$ are nondegenerate such that we can consider the nondegenerate endomorphisms $\hat{J}_\alpha$ defined by

$$\hat{\omega}_\alpha \hat{J}_\beta = \hat{\omega}_\gamma \tag{2.6}$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. In the following (α, β, γ) will be always a cyclic permutation. We have to show that $(\hat{J}_\alpha)$ is an almost hyper-complex structure. The integrability then follows from the closure of the $\hat{\omega}_\alpha$, in virtue of the Hitchin lemma. The pseudo-hyper-Kähler metric is then given by $\hat{g} = -\hat{\omega}_\alpha \hat{J}_\alpha$. Notice that this expression is independent of α, due to the relations $\hat{J}_\alpha \hat{J}_\beta = \hat{J}_\gamma$. The skew-symmetry of $\hat{J}_\beta$ with respect to $\hat{g}$ follows from the symmetry of $\hat{J}_\beta$ with respect to $\hat{\omega}_\alpha$ (a consequence of (2.6)) and the relation $\hat{J}_\alpha \hat{J}_\beta = -\hat{J}_\beta \hat{J}_\alpha$. The symmetry of $\hat{g}$ is then obtained from $\hat{J}_\alpha^2 = -\mathrm{Id}$. The nondegeneracy of $\hat{g}$ is a consequence of that of $\hat{\omega}_\alpha$ and $\hat{J}_\alpha$.

Lemma 3 *The two-forms $\tilde{\omega}_\alpha$ on $\tilde{M}$ are given by:*

$$\tilde{\omega}_\alpha = 2f\rho^2(\varphi_0 \wedge \varphi_\alpha + \varphi_\beta \wedge \varphi_\gamma + \varphi_0 \wedge \theta'_\alpha - \theta_0 \wedge \varphi_\alpha + \varphi_\beta \wedge \theta'_\gamma - \varphi_\gamma \wedge \theta'_\beta) + \rho^2 \omega', \tag{2.7}$$

where $\theta_0 := -\frac{1}{2} f^{-1} df$, $\theta' := \mathrm{Ad}_q \theta$, $\omega := \sum \omega_\alpha i_\alpha$ and $\omega' = \mathrm{Ad}_q \omega$,

Proof: We first calculate the differential of $\tilde{\theta}^P = f\tilde{\theta} = f\varphi + \mathrm{Ad}_q \theta^P$, where $\theta^P = f\theta$. Using the Maurer Cartan equation

$$d\varphi_\alpha = 2\varphi_\beta \wedge \varphi_\gamma,$$

we obtain

$$d(f\varphi_\alpha) = 2f(-\theta_0 \wedge \varphi_\alpha + \varphi_\beta \wedge \varphi_\gamma).$$

Using the fact that $\varphi = dqq^{-1}$, we see that

$$\begin{aligned} d(\mathrm{Ad}_q \theta^P) &= dq \wedge \theta^P q^{-1} + q d\theta^P q^{-1} - q\theta^P \wedge d(q^{-1}) \\ &= \varphi \wedge \mathrm{Ad}_q \theta^P + \mathrm{Ad}_q(\theta^P) \wedge \varphi + \mathrm{Ad}_q d\theta^P \\ &= f(\varphi \wedge \theta' + \theta' \wedge \varphi) + \mathrm{Ad}_q d\theta^P. \end{aligned}$$

The components are given by

$$\begin{aligned} d(\mathrm{Ad}_q \theta^P)_\alpha &= f(\varphi_0 \wedge \theta'_\alpha + \varphi_\beta \wedge \theta'_\gamma - \varphi_\gamma \wedge \theta'_\beta + \theta'_\beta \wedge \varphi_\gamma - \theta'_\gamma \wedge \varphi_\beta + \theta'_\alpha \wedge \varphi_0) + (\mathrm{Ad}_q d\theta^P)_\alpha \\ &= 2f(\varphi_\beta \wedge \theta'_\gamma - \varphi_\gamma \wedge \theta'_\beta) + (\mathrm{Ad}_q d\theta^P)_\alpha. \end{aligned}$$

Using $\varphi_0 = \rho^{-1} d\rho$ and the above equations, we get

$$\begin{aligned} \tilde{\omega}_\alpha &= d(\rho^2 \tilde{\theta}^P_\alpha) = 2\rho^2 \varphi_0 \wedge \tilde{\theta}^P_\alpha + \rho^2 d\tilde{\theta}^P_\alpha = 2f\rho^2 \varphi_0 \wedge (\varphi_\alpha + \theta'_\alpha) + \rho^2 d\tilde{\theta}^P_\alpha \\ &= 2f\rho^2(\varphi_0 \wedge \varphi_\alpha + \varphi_\beta \wedge \varphi_\gamma + \varphi_0 \wedge \theta'_\alpha - \theta_0 \wedge \varphi_\alpha + \varphi_\beta \wedge \theta'_\gamma - \varphi_\gamma \wedge \theta'_\beta) + \rho^2 (\mathrm{Ad}_q d\theta^P)_\alpha. \end{aligned}$$

Finally, we claim that

$$d\theta^P = \omega,$$

which implies the lemma. In fact,

$$\begin{aligned}
d\theta_1^P &= d\eta + \frac{1}{2}d(gZ) = \omega_1 \\
d\theta_2^P &= \frac{1}{2}d(\omega_3 Z) = \frac{1}{2}\mathcal{L}_Z\omega_3 = \omega_2 \\
d\theta_3^P &= -\frac{1}{2}d(\omega_2 Z) = -\frac{1}{2}\mathcal{L}_Z\omega_2 = \omega_3.
\end{aligned}$$

□

Next we will show that the two-forms $\tilde{\omega}_\alpha$ computed in Lemma 3 are nondegenerate on any distribution complementary to $\mathbb{R}V_1 \subset T\tilde{M}$. Let us denote by $\mathcal{D}_1, \mathcal{D}_2 \subset T\tilde{M}$ the distributions which correspond to the factors of the product $\tilde{M} = \mathbb{H}^* \times P$. The second distribution can be decomposed as

$$\mathcal{D}_2 = \mathbb{R}X_P \overset{\perp}{\oplus} \mathcal{D}_2^h, \quad \mathcal{D}_2^h = E \overset{\perp}{\oplus} E', \quad E' := \mathrm{span}\{\tilde{Z}, \widetilde{J_1 Z}, \widetilde{J_2 Z}, \widetilde{J_3 Z}\},$$

with respect to the metric g_P on the leaves $\{q\}\times P \cong P$ of $\mathcal{D}_2$. Notice that $\mathcal{D}_2^h|_{(q,p)} \cong T_p^h P$. We will study the restriction of $\tilde{\omega}_\alpha$ to the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2^h$, which is complementary to $\mathbb{R}V_1$. From (2.7) we first see that the distributions E and $\mathcal{D} := \mathcal{D}_1 \oplus E'$ are orthogonal with respect to $\tilde{\omega}_\alpha$. Furthermore,

$$\tilde{\omega}_\alpha|_E = \rho^2\omega'_\alpha|_E = \rho^2 g J'_\alpha|_E, \quad J'_\alpha = \sum A_{\alpha\beta}J_\beta,$$

where $(A_{\alpha\beta}) \in \mathrm{SO}(3)$ is the matrix $A = A(q)$ representing $\mathrm{Ad}_q|_{\mathrm{Im}\,\mathbb{H}}$ in the basis (i_1, i_2, i_3). This shows that $\tilde{\omega}_\alpha|_E$ is nondegenerate and that

$$\tilde{\omega}_\alpha J'_\beta|_E = \tilde{\omega}_\gamma|_E.$$

Notice that (J'_α) coincides with the hyper-complex structure (J_α) of M up to a rotation, which depends on q. It remains to analyse $\tilde{\omega}_\alpha$ on the eight-dimensional distribution $\mathcal{D} = \mathcal{D}_1 \oplus E'$. We put

$$W_0 := J_1 Z, \quad W_\alpha := -J_\alpha W_0$$

and

$$W'_\alpha = \sum_{\beta=1}^{3} A_{\alpha\beta}W_\beta.$$

From (2.1) one can check that

$$\theta_a^P(\tilde{W}_b) = h\delta_{ab}, \quad a, b \in \{0, \ldots, 3\},$$

where $\theta_0^P := f\theta_0 = -\frac{1}{2}df$. This implies that

$$\omega_\alpha|_{E'} = -2h^{-1}(\theta_0^P \wedge \theta_\alpha^P - \theta_\beta^P \wedge \theta_\gamma^P) = -2f^2h^{-1}(\theta_0 \wedge \theta_\alpha - \theta_\beta \wedge \theta_\gamma)$$

and, hence,

$$\tilde{\omega}_\alpha|_{E'} = -2f^2h^{-1}\rho^2(\theta_0 \wedge \theta'_\alpha - \theta'_\beta \wedge \theta'_\gamma).$$

Now we can write the matrix $\mathcal{M}(\tilde{\omega}_\alpha)$ which represents $\tilde{\omega}_\alpha|_{\mathcal{D}}$ in the basis $(e_0, e_\alpha, e_\beta, e_\gamma, \tilde{W}_0, \tilde{W}'_\alpha, \tilde{W}'_\beta, \tilde{W}'_\gamma)$:

$$\mathcal{M}(\tilde{\omega}_\alpha) = 2\rho^2 \begin{pmatrix} f\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} & h\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \\ -h\begin{pmatrix} I & 0 \\ 0 & -J \end{pmatrix} & -h\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \end{pmatrix}, \tag{2.8}$$

where

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The invertibility of this matrix follows from the assumption $f_1 = f - h \neq 0$. This proves that the two-forms $\tilde{\omega}_\alpha$ are nondegenerate on any complement of $\mathbb{R}V_1$, which implies the nondegeneracy of the induced forms $\hat{\omega}_\alpha$ on $\hat{M}$. Now we compute the three endomorphisms

$$\tilde{J}_\alpha = \tilde{J}_\alpha|_{\mathcal{D}} \oplus \tilde{J}_\alpha|_E = \tilde{J}_\alpha|_{\mathcal{D}} \oplus J'_\alpha|_E$$

of $\mathcal{D} \oplus E \cong T\tilde{M}/\mathbb{R}V_1$ defined by

$$\tilde{\omega}_\alpha \tilde{J}_\beta = \tilde{\omega}_\gamma|_{\mathcal{D}\oplus E}.$$

Under the projection $\tilde{M} \to \hat{M}$, they correspond to the three endomorphism fields $\hat{J}_\alpha$ on $\hat{M}$ such that

$$\hat{\omega}_\alpha \hat{J}_\beta = \hat{\omega}_\gamma.$$

Using the expression (2.8) one can check that the matrix representing $\tilde{J}_\alpha|_{\mathcal{D}}$ in the basis $(e_0, e_\alpha, e_\beta, e_\gamma, \tilde{W}_0, \tilde{W}'_\alpha, \tilde{W}'_\beta, \tilde{W}'_\gamma)$ is given by

$$\mathcal{M}(\tilde{J}_\alpha) = \begin{pmatrix} -J & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & -J \end{pmatrix}, \tag{2.9}$$

or, equivalently,

$$\begin{array}{rclcrcl} \tilde{J}_\alpha e_0 & = & e_\alpha & \quad & \tilde{J}_\alpha \tilde{W}_0 & = & -\tilde{W}'_\alpha \\ \tilde{J}_\alpha e_\alpha & = & -e_0 & & \tilde{J}_\alpha \tilde{W}'_\alpha & = & \tilde{W}_0 \\ \tilde{J}_\alpha e_\beta & = & e_\gamma & & \tilde{J}_\alpha \tilde{W}'_\beta & = & \tilde{W}'_\gamma \\ \tilde{J}_\alpha e_\gamma & = & -e_\beta & & \tilde{J}_\alpha \tilde{W}'_\gamma & = & -\tilde{W}'_\beta. \end{array}$$

This proves that the $\tilde{J}_\alpha$ satisfy the standard quaternionic relations and that $\tilde{J}_\alpha|_{\mathcal{D}_2^h}$ corresponds to J'_α under the isomorphism $\mathcal{D}_2^h|_{(q,p)} \cong T_p^h P \cong T_{\pi(p)}M$, $(q,p) \in \tilde{M}$. Therefore, we have proven that the three symplectic forms $\hat{\omega}_\alpha$ on $\hat{M}$ define a hyper-Kähler structure $(\hat{M}, \hat{g}, \hat{J}_\alpha, \alpha = 1, 2, 3)$.

Next we calculate the explicit expression for the pseudo-hyper-Kähler metric $\hat{g}$. It amounts to calculating the metric

$$\tilde{g} := -\tilde{\omega}_\alpha \tilde{J}_\alpha,$$

which is defined on the codimension one distribution $\mathcal{D} \oplus E \subset T\tilde{M}$.

Proposition 1

$$\tilde{g} = \tilde{g}|_{\mathcal{D}} \oplus \tilde{g}_E, \quad \tilde{g}|_{\mathcal{D}} = 2\rho^2 f(\sum_{a=0}^{3} \varphi_a^2 + h^{-1} \sum_{a=0}^{3} (\theta'_a)^2 - 2\varphi_0\theta_0 + 2\sum_{\alpha=1}^{3} \varphi_\alpha \theta'_\alpha), \quad \tilde{g}_E = \rho^2 g|_E.$$

Proof: It suffices to calculate the matrix $\mathcal{M}(\tilde{g})$ which represents $\tilde{g} = -\tilde{\omega}_\alpha \tilde{J}_\alpha$ in the basis $(e_0, e_\alpha, e_\beta, e_\gamma, \tilde{W}_0, \tilde{W}'_\alpha, \tilde{W}'_\beta, \tilde{W}'_\gamma)$ of $\mathcal{D}$. In view of (2.8) and (2.9), it is given by

$$\mathcal{M}(\tilde{g}) = -\mathcal{M}(\tilde{J}_\alpha)^t \mathcal{M}(\tilde{\omega}_\alpha) = \mathcal{M}(\tilde{\omega}_\alpha)\mathcal{M}(\tilde{J}_\alpha) = 2\rho^2 \begin{pmatrix} f & & & & -h & & & \\ & f & & & & h & & \\ & & f & & & & h & \\ & & & f & & & & h \\ -h & & & & h & & & \\ & h & & & & h & & \\ & & h & & & & h & \\ & & & h & & & & h \end{pmatrix}, \tag{2.10}$$

where only the nonzero entries are written. This proves the above formula for $\tilde{g}_{\mathcal{D}}$, since $\theta'_a(\tilde{W}'_b) = f^{-1}h\delta_{ab}$. □

Let us now extend $\tilde{g}$ from a metric defined on the distribution $\mathcal{D} \oplus E = \mathcal{D}_1 \oplus \mathcal{D}_2^h \subset T\tilde{M}$ to a pseudo-Riemannian metric on $\tilde{M}$ such that V_1 is perpendicular to $\mathcal{D} \oplus E$. Then $\tilde{\pi} : (\tilde{M}, \tilde{g}) \to (\hat{M}, \hat{g})$ is a pseudo-Riemannian submersion and we can calculate the covariant derivative of $\xi = \tilde{\pi}_* e_0$ by calculating $\tilde{g}(D_X e_0, Y)$ for all $X, Y \in \mathcal{D} \oplus E$. In order to show that $D\xi = \mathrm{Id}$, we have to check that $\tilde{g}(D_X e_0, Y) = \tilde{g}(X, Y)$. Using the Koszul formula and the commutator relations of the vector fields e_a, we obtain

$$2\tilde{g}(D_{e_a} e_0, e_b) = 2f(\delta_{0b}e_a + \delta_{ab}e_0 - \delta_{0a}e_b)\rho^2 = 2f\delta_{ab}e_0\rho^2 = 4f\rho^2\delta_{ab} = 2\tilde{g}(e_a, e_b),$$

for all $a, b \in \{0, \ldots, 3\}$. Let us next observe that

$$\tilde{g}(e_0, \cdot)|_{\mathcal{D}_2} = -2\rho^2\theta_0^p, \tag{2.11}$$

as follows from $\tilde{g}(e_0,\cdot)|_{\mathcal{D}_2^h} = -2\rho^2\theta_0^p$ and $\tilde{g}(e_0, X_P) = \tilde{g}(e_0, \frac{e_1^L - \tilde{Z}}{f_1}) = 0$. Now let $X, Y \in \Gamma(T^h P) \subset \Gamma(\mathcal{D}_2^h)$ be horizontal lifts of vector fields in M. Then using (2.11), $[e_0, X] = [e_0, Y] = 0$ and $d\theta_0^P = 0$ we get

$$\begin{aligned}
2\tilde{g}(D_X e_0, Y) &= X\tilde{g}(e_0, Y) + e_0\tilde{g}(X, Y) - Y\tilde{g}(e_0, X) - \tilde{g}(e_0, [X, Y]) \\
&= 2\tilde{g}(X, Y) - 2\rho^2(X\theta_0^P(Y) - Y\theta_0^P(X) - \theta_0^P([X, Y])) \\
&= 2\tilde{g}(X, Y) - 2\rho^2 d\theta_0^P(X, Y) = 2\tilde{g}(X, Y).
\end{aligned}$$

To compute $\tilde{g}(D_{e_a} e_0, X)$, we observe that $[e_0, e_a] = [e_0, X] = [e_a, X] = 0$, such that

$$\begin{aligned}
2\tilde{g}(D_{e_a} e_0, X) &= e_a\tilde{g}(e_0, X) + e_0\tilde{g}(e_a, X) - X\tilde{g}(e_0, e_a) \\
&= 2\tilde{g}(e_a, X) + 2\delta_{0a}(\tilde{g}(e_0, X) - \rho^2 Xf) \\
&= 2\tilde{g}(e_a, X) + 2\delta_{0a}(-2\rho^2\theta_0^P(X) - \rho^2 Xf) = 2\tilde{g}(e_a, X).
\end{aligned}$$

Here we have used(2.11) and $\theta_0^P = -\frac{1}{2}df$.

Similarly, we get

$$2\tilde{g}(D_X e_0, e_a) = X\tilde{g}(e_0, e_a) + e_0\tilde{g}(X, e_a) - e_a\tilde{g}(e_0, X) = 2\tilde{g}(X, e_a).$$

To finishes the proof of Theorem 2 it remains to compute the signature of $\hat{g}$. One can easily check that the matrix (2.10) has signature $(4, 4)$ if $f_1 h < 0$, signature $(8, 0)$ if $h > 0$ and $f_1 > 0$ and signature $(0, 8)$ if $h < 0$ and $f_1 < 0$. This implies that $\hat{g}$ has signature $(4k, 4\ell + 4)$ if $f_1 < 0$ and signature $(4k + 4, 4\ell)$ if $f_1 > 0$. □

Any conical pseudo-hyper-Kähler manifold $(M, g, J_1, J_2, J_3, \xi)$ is foliated by the leaves of the four-dimensional integrable distribution defined by the vector fields ξ, $J_1\xi$, $J_2\xi$, $J_3\xi$. The space of leaves inherits a quaternionic pseudo-Kähler structure, at least if we restrict the foliation to a suitable open subset of M. Let us denote by $(\bar{M}, \bar{g}, Q)$ the (at least locally defined) quaternionic pseudo-Kähler manifold associated with the conical pseudo-hyper-Kähler manifold $(\hat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ of Theorem 2.

Corollary 1 *The signature of the quaternionic pseudo-Kähler manifold $(\bar{M}, \bar{g}, Q)$ resulting from Theorem 2 depends only on the signature $(4k, 4\ell)$ of the original pseudo-hyper-Kähler manifold (M, g, J_1, J_2, J_3, Z) and on the signs of the functions f and $f_1 = f - h$, where $-f$ is the Hamiltonian chosen for the construction (unique up to an additive constant) and $h = g(Z, Z)/2$. It is $(4k, 4\ell)$ if $f_1 f > 0$, $(4k - 4, 4\ell + 4)$ if $f > 0$ and $f_1 < 0$ and $(4k + 4, 4\ell - 4)$ if $f < 0$ and $f_1 > 0$.*

Proof: This follows from the fact that the signature of $\bar{g}$ is obtained from that of $\hat{g}$ by subtracting $(4, 0)$ if $f > 0$ and $(0, 4)$ if $f < 0$. □

Corollary 2 *The construction of Theorem 2 yields a (positive definite) quaternionic Kähler manifold* $(\bar{M}, \bar{g}, Q)$ *of positive scalar curvature if and only if the metric* g *of the original pseudo-hyper-Kähler manifold* (M, g, J_1, J_2, J_3, Z) *is positive definite and* $f_1 > 0$. *It yields a quaternionic Kähler manifold* $(\bar{M}, \bar{g}, Q)$ *of negative scalar curvature if and only if* g *is either positive definite and* $f < 0$ *or if it has signature* $(4k, 4)$, $f < 0$ *and* $f_1 > 0$.

Remark: Notice that Theorem 2 provides us with a quaternionic pseudo-Kähler manifold for any choice of Hamiltonian f for Z with respect to ω_1 and any choice of S^1-principal bundle (P, η) with connection η such that the curvature is $\omega_1 - \frac{1}{2}d(g_N\zeta)$. Locally any two S^1-principal bundles with the same curvature are equivalent such that, for the local geometry, the only essential choice is the Hamiltonian f, which is unique up to a constant c. It follows from [HKLR] p. 553-554 that, up to a constant factor, f is a Kähler potential with respect to J_2. The Hamiltonian will be explicitly computed in the examples of the next section.

3 Application to the c-map

Definition 5 *A* conical (affine) special Kähler manifold (M, J, g, ∇, ξ) *is a pseudo-Kähler manifold* (M, J, g) *endowed with a flat torsionfree connection* ∇ *and a vector field* ξ *such that*

(i) $\nabla\omega = 0$, *where* $\omega = gJ$ *is the Kähler form,*

(ii) $d^\nabla J = 0$, *where* J *is considered as a one-form with values in the tangent bundle,*

(iii) $\nabla\xi = D\xi = \mathrm{Id}$, *where* D *is the Levi-Civita connection,*

(iv) g *is definite on the plane* $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$.

The above definition is slightly more general than the definition of a conical special Kähler manifold in [CM], for instance, since here we prefer not to restrict the signature of the metric. The rigid c-map associates with M the pseudo-hyper-Kähler manifold $(N = T^*M, g_N, J_1, J_2, J_3)$, with the geometric data defined as follows, cf. [ACD]. Using the connection ∇ we can identify $TN = T^hN \oplus T^vN = \pi^*TM \oplus \pi^*T^*M$, where $\pi : N = T^*M \to M$ is the canonical projection, $T^vN = \ker d\pi$ is the vertical distribution and T^hN is the horizontal distribution defined by ∇. Using these identifications, we have

$$g_N = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad J_1 = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad J_3 = J_1 J_2.$$

The vector field $Z = J\xi$ is a Hamiltonian Killing vector field on M. In fact, $DZ = JD\xi = J$ is skew-symmetric and the function $h = \frac{g(Z,Z)}{2}$ satisfies $dh = g(DZ, Z) = g(J\cdot, Z) = -\omega Z$. We extend Z to a vector field Z_N on N by

$$Z_N(\pi^* q^i) := \pi^* Z(q^i), \quad Z_N(p_i) = 0,$$

where (q^i) are ∇-affine local coordinates on M and $(\pi^* q^i, p_i)$ are the corresponding coordinates of $N = T^*M$. One can easily check that this extension does not depend on the choice of affine coordinates.

Proposition 2 *For any pseudo-hyper-Kähler manifold (N, g_N, J_1, J_2, J_3) obtained from the rigid c-map, the vector field Z_N is a time-like or space-like ω_1-Hamiltonian Killing vector field, which satisfies $D_{Z_N} Z_N = J_1 Z_N$ and $\mathcal{L}_{Z_N} J_2 = -J_3$.*

Proof: Z is time-like or space-like by (iv) of Definition 5. This implies that Z_N is time-like or space-like. Let us recall that there exist ∇-affine local coordinates (q^i) on M such that $\xi = \sum q^i \partial_i$, see [CM1] Prop. 5. The following calculations will be always in such coordinates. Notice that $Z = J\xi = \sum J^i_j q^j \partial_i$. Since $\mathcal{L}_Z g = 0$, we have

$$\mathcal{L}_{Z_N} g_N = \pi^* \mathcal{L}_Z g + \sum Z(g^{ij}) dp_i dp_j = \sum Z(g^{ij}) dp_i dp_j.$$

Since $Z(g^{ij}) = -\sum g^{ik} Z(g_{kl}) g^{lj}$, it suffices to show that $Z(g_{kl}) = 0$. Let us first recall[1] that ∇g is totally symmetric, which follows from Definition 5 (i-ii) using the skew-symmetry of J. Using this property and Definition 5 (i-ii), we obtain

$$\begin{aligned} Z(g_{kl}) &= \sum J^i_j q^j g_{kl,i} = \sum J^i_j q^j g_{ki,l} = -\sum q^j (J^i_j)_{,l} g_{ki} = -\sum q^j (J^i_l)_{,j} g_{ki} \\ &= \sum q^j J^i_l g_{ki,j} = \sum J^i_l \xi(g_{ki}). \end{aligned}$$

We claim that $\xi(g_{ki}) = 0$. Let us first observe that $D\xi = \mathrm{Id}$ implies $\mathcal{L}_\xi g = 2g$. This shows that $\xi(g_{ij}) = 0$, since $\mathcal{L}_\xi q^i = q^i$. Summarizing, we have proven that Z_N is a Killing vector field.

Next we prove that Z_N is Hamiltonian with respect to $\omega_1 = g_N J_1$. We consider the function $h = \frac{g(Z,Z)}{2}$. Then the calculation

$$d(\pi^* h) = \pi^* dh = -\pi^*(\omega Z) = -\omega_1 Z_N \tag{3.1}$$

proves that $-h$ is a Hamiltonian function for Z_N. This implies the equation $D_{Z_N} Z_N = J_1 Z_N$, as follows from Lemma 1.

[1] (M, g, ∇) is in fact an intrinsic affine hypersphere [BC], which implies the symmetry of ∇g.

Finally, we check that $\mathcal{L}_{Z_N} J_2 = -J_3$ or, equivalently, that $\mathcal{L}_{Z_N}\omega_2 = -\omega_3$. Notice that

$$J_3 = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}.$$

So

$$\omega_2 = \begin{pmatrix} 0 & -J^* \\ J & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

that is

$$\begin{aligned} \omega_2 &= \sum J_i^j dq^i \wedge dp_j \\ \omega_3 &= \sum dq^i \wedge dp_i. \end{aligned}$$

It is sufficient to check that $\mathcal{L}_{Z_N}\omega_3 = \omega_2$. Now

$$\mathcal{L}_{Z_N}\omega_3 = \sum d(Jq)^i \wedge dp_i = \omega_2 + \sum J^i_{k,l} q^k dq^k \wedge dp_i = \omega_2,$$

where $(Jq)^i = \sum J^i_j q^j$ and we have used that

$$\sum J^i_{k,l} q^k = \sum q^k J^i_{l,k} = \xi(J^i_l) = \xi(g^{ij}\omega_{jl}) = 0.$$

□

Corollary 3 *For any pseudo-hyper-Kähler manifold (N, g_N, J_1, J_2, J_3) obtained from the rigid c-map, the assumptions for the conification construction of Theorem (2) are satisfied for the Killing vector field $\zeta = 2Z_N$. Therefore any choice of Hamiltonian f for ζ with respect to ω_1 and any choice of S^1-principal bundle (P, η) with connection η with the curvature $\omega_1 - \frac{1}{2}d(g_N\zeta)$ defines a conical pseudo-hyper-Kähler manifold $(\hat{N}, g_{\hat{N}}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$.*

Consider now the function $h := \frac{1}{2}g_N(\zeta, \zeta) = 2g_N(Z_N, Z_N)$ associated with the rescaled vector field $\zeta = 2Z_N$. Then (3.1) shows that any function f satisfying $df = -\omega_1\zeta$ is of the form

$$f = \frac{1}{2}h + c,$$

for some constant c. The choice $c = 0$ will be called the *canonical choice of Hamiltonian.* Thus the function $f_1 := f - h$ is now given by

$$f_1 = -\frac{1}{2}h + c.$$

Now we specialise to the cases where the resulting quaternionic Kähler manifold $(\bar{N}, \bar{g}, Q)$ is positive definite, see Corollary 2.

Corollary 4 *If the conical special Kähler manifold M is positive definite, then the resulting quaternionic Kähler manifold $\bar{N}$ has positive scalar curvature if $-\frac{1}{2}h + c > 0$ and negative scalar curvature if $\frac{1}{2}h + c < 0$. If the conical special Kähler manifold M has signature $(2k, 2)$ with time-like Euler field and if $-\frac{1}{2}h + c > 0$ and $\frac{1}{2}h + c < 0$, then $\bar{N}$ has negative scalar curvature. In particular, for the canonical choice of Hamiltonian the scalar curvature of the quaternionic Kähler manifold $\bar{N}$ is always negative.*

The last result is consistent with our conjecture that the canonical choice of Hamiltonian yields the Ferrara-Sabharwal metric (up to a factor). The deformation by the constant c should correspond to the one-loop correction of the metric considered in [APSV]. The determination of the precise relation between the constant c and the one-loop parameter is left for the future.

References

[AC] D.V. Alekseevsky and V. Cortés, *Geometric construction of the r-map: from affine special real to special Kähler manifolds*, Comm. Math. Phys. **291** (2009), no. 2, p. 579 [arXiv:0811.1658].

[ACD] D.V. Alekseevsky, V. Cortés and C. Devchand, *Special complex manifolds*, J. Geom. Phys. **42** (2002) 85 [math.dg/9910091].

[APP] S. Alexandrov, D. Persson and B. Pioline, *Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence*, arXiv:1110.0466.

[APSV] S. Alexandrov, B. Pioline, F. Saueressig and S. Vandoren, *Linear perturbations of quaternionic metrics*, Commun. Math. Phys. **296** (2010), no. 2, 353-403.

[BC] O. Baues and V. Cortés, *Realisation of special Kähler manifolds as parabolic spheres*, Proc. Amer. Math. Soc., **129** (2001), no. 8, 2403-2407.

[CFG] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, Int. J. Mod. Phys. **A 4** (1989) 2475.

[CM1] V. Cortés and T. Mohaupt, *Special Geometry of Euclidean Supersymmetry III: the local r-map, instantons and black holes*, JHEP 07, 066 (2009).

[CM] V. Cortés, X. Han and T. Mohaupt, *Completeness in supergravity constructions*, Comm. Math. Phys. **311** (2012), no. 1, 191-213.

[CMMS1] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry I: vector multiplets*, J. High Energy Phys. **2004**, no. 3, 028, 73 pp.

[DV] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. **149** (1992), 307-333.

[FS] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nucl. Phys. **B332** (1990), 317-332.

[Ha] A. Haydys, *Hyper-Kähler and quaternionic Kähler manifolds with S^1-symmetries*, J. Geom. Phys. **58** (2008), no. 3, 293-306.

[H1] N. Hitchin, *The moduli space of complex Lagrangian submanifolds*, Asian J. Math. **3** (1999), no. 1, 77-91.

[H2] N. Hitchin, *Quaternionic Kähler moduli spaces*, Riemannian Topology and Geometric Structures on Manifolds, 49-61, K. Galicki, S. Simanca (eds.), Progress in Mathematics 271, Birkhäuser, 2009.

[HKLR] N. Hitchin, A. Karlhede, U. Lindström and Roček, *Hyperkähler Metrics and Supersymmetry*, Commun. Math. Phys. **108** (1987), 535-589.