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Multiplicative Types

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Introduction. The purpose of this article is to make propaganda for an elementary number-theoretic concept, which of course is not unknown (it would be fairly impossible that anything of such elementary character has been overlooked up to now) but which, to the best of my knowledge, has not yet received the systematic treatment it certainly deserves; this latter claim will, as I hope, be justified by what follows. After the relevant definitions I shall go through some natural problems connected with the concept and present a number of first results (of a more or less elementary nature). No claim of originality can be made here, but I can give only a few references [1].

1. Definitions. By the *multiplicative type*, or simply type, of a natural number n > 1, I mean the unordered sequence of the exponents in the prime decomposition $n = p^a q^b \ldots$; we write

$$t(n) = (a, b, \ldots.)$$

for the type and call a,b,.. the *exponents* of the type. Type (1) means that n is a prime, type (r) denotes an r-th power of a prime, squarefree numbers have a type of the form (1,...,1), and so on. We can ascribe the type (0) to the number 1. Types will be denoted T or S, the set of all types is $\mathcal{T}$, and for a real number x we write $\mathcal{T}(x)$ for the set of all types which have occurred up to (and including) x. The sum a + b + …. of the exponents will be called the *length* of the type, denoted $l(T)$. Evidently, a type of length m is simply a partition of m. The number of summands in the partition T (the number of different prime factors of any number having type T) will be written ω(T), conforming to a common notation for this number, and may be called the *breadth* of T. Clearly, $l(T) \geq \omega(T)$ with equality exactly for the squarefree types.

2. Identification of types. The most basic problem is, of course, the determination of t(n) for a given number n. We have defined t(n) in terms of the prime decomposition. Are there other characterizations? One thinks, naturally, of Wilson's theorem, or rather one half of it, which is hardly more than the definition of prime number and which we can formulate as

$$t(n) = (1) \text{ if and only if } (n-1)! \text{ is a unit mod } n ;$$

(then automatically $(n-1)! \equiv -1 \bmod n$). Are there analogous statements for other types? A natural algebraic characterization of t(n) is via the ring-theoretic structure of the residue class ring $R = \mathbb{Z} \bmod n$. Namely, by the Chinese remainder theorem, ω(T) is the number of simple R-modules (or of primitive idempotents), and the exponents are the composition lengths of the indecomposable summands of the regular R-module or the maximal orders of nilpotency occurring in them. Are there procedures for finding out these data without explicitly calculating the prime decomposition? The type is, so to speak, the formal part of the prime decomposition, whereas the actual primes entering it make up the "content". Note that any *explicit* determination of a (nontrivial) idempotent of R (or even a nontrivial divisor of 0) leads at once (via the Euclidean algorithm) to a proper divisor of n, thus amountig to a step in the prime factorization.

2.1. Regarding the complexity, we now know that type (1) can be identified in polynomial time (though this took quite a while to establish). It seems reasonable to expect the same for arbitrary types; explicitely, there should exist, for a given type T, a polynomial algorithm which for any n decides whether or not t(n) = T. Much more ambitious, of course, is the determination of t(n). As we just pointed out, this is "half the prime factorization", which no one expects to be possible polynomially.

3. Minimal realizations and the type graph. Every type occurs for the first time somewhere, and (happily) we can say exactly where: if we write T in the natural order (with decreasing exponents),

$$T = (a, b, c, \ldots) \quad \text{with} \quad a \geq b \geq c \geq \ldots,$$

then T occurs for the first time at the number

$$m(T) = 2^a\, 3^b\, 5^c \ldots ,$$

which we call the *minimal realization* of T. These numbers will also be called *m-numbers.* The types can be listed naturally by length, and within each length lexicographically. The following table contains all types occurring up to 200 together with their minimal realizations:

(7)	(6,1)	…….			
128	192				
(6)	(5,1)	(4,2)	….		
64	96	144			
(5)	(4,1)	(3,2)	(3,1,1)	(2,2,1)	….
32	48	72	120	180	
(4)	(3,1)	(2,2)	(2,1,1)	….	
16	24	36	60		
(3)	(2,1)	(1,1,1)			
8	12	30			
(2)	(1,1)				
4	6				
(1)					
2					

In this ordering the leftmost type for each length r is (r) , having m(r) = 2^r (we omit one pair of brackets in expressions like this one), the rightmost one is (1^r): =(1,...,1) , with r coefficients 1, and m(1^r) = P(r): = $p_1 ... p_r$ = product of the first primes (more details below; we denote, here and elsewhere, by p_r the primes in their natural ordering). Note that p_r is the quotient of m(1^r) and m(1^{r-1}). In each line the prime factors of the numbers m(T) are successively replaced with larger ones, and clearly m(r) is the minimal, m(1^r) the maximal value; but it is not true that these numbers increase monotonically, for example T = (4,1,1) and S = (3,3) are immediate successors, but m(T) = 240 and m(S) = 216. (This is the first occurrence of this phenomenon). More generally, a type ending with (….a, 1,...,1), with a > s+1 in the r-th position and having s components 1, has the lexicographic successor (….a–1, s+1) , and for the m(T) this means that a product

$$p_r \ \dots \ p_{r+1}\, p_{r+s} \quad \text{is replaced by} \quad p_{r+1}^{\,s+1} ,$$

and clearly the difference can become arbitrary large. A characteristic feature of the concept of type can be read off already from this small sample. Whereas the types form a natural scheme with a natural ordering, their minimal realizations move through this scheme along a path, for which of course several restrictions can be formulated, but which essentially reflects the irregularity of the sequence of primes as well as of the prime decompositions.

3.1. The set $\mathcal{T}$ of types carries various orderings; the one most appropriate for our present purposes is an ordering which may be called „arithmetical“ and which relates it to the set of the minimal realizations, as follows. For types T = (a,b,c …), T′ = (a′,b′,c′...) having the same breadth and written with decreasing exponents let us write

$$T \ \leq \ T' \ \text{ if and only if } \ a \leq a', \ b \leq b' , c \leq c', \dots ;$$

for T and S of arbitrary breadth define T ≤ S if T ≤ some initial piece of S; this is the lexicographic order comprising all lengths. Then T ≤ S if and only if m(T) divides m(S), and clearly l(T) ≤ l(S) and ω(T) ≤ ω(S) in this case. Note that S is an immediate successor of T in this ordering if and only if m(S)/m(T) is a prime. Beware: this is *not* the ordering considered usually in the combinatorial theory, when the types are viewed as partitions. The relation between partitions (of the same set) is that of „finer“, e.g. (1,1,1) is finer than (2,1), but these types are unrelated arithmetically. In fact the two relations can never obtain simultaneously because if S < T arithmetically, then l(S) < l(T). Note however that if S is finer than T, then also m(S) > m(T) (but usually m(S) is no multiple of m(T)).

Since the map T → m(T) is injective (by uniqueness of prime decomposition), this identifies $\mathcal{T}$ with a sub-poset of the natural numbers ordered by divisibility; in fact with a sub-lattice, because it is easy to see that the gcd and the lcm of two m-numbers are again m-numbers, with obvious definitions of gcd and lcm for types. The set of m-numbers is also closed under multiplication, m(S) m(T) = m(S,T) for an obvious "superposition" (S,T) of types. We refer to $\mathcal{T}$ equipped with this ordering as the *type graph*. However, this order structure of the type graph is quite different from that of ℕ, in spite of being a substructure; for example, is has only one atom, namely (1), and for m(S) dividing m(T) the complementary factor is not usually an m-number.

3.2. In order to determine the order of a type T as a vertex in the type graph, write (in unique fashion)

$T = (a(1)^{r(1)},..., a(s)^{r(s)})$, where a^r stands for a,...,a (r-times) and $a(1) > a(2) > ... a(s) \geq 1$;

in this notation $\omega(T) = \sum r(i)$ and $l(T) = \sum r(i)a(i)$. The number s = s(T) may be viewed as the number of „steps" in the type T; if T is viewed as a partition of $l(T)$, s is the number of *different* summands. The immediate successors of T are now obtained by replacing each step a^r with $(a+1, a^{r-1})$, and by addding 1 at the end; so there are s+1 immediate successors, and one sees in the same way there there are s immediate predecessors, so the order of the vertex T is 2s +1.

3.3. The above description of single types suggests to visualize the type T as the graph of a decreasing step function, defined on the interval $[0, \omega(T)]$ and having steps at r(1), r(1) + r(2), and so on; then $l(T)$ is the area under its graph. Reflecting this graph at the axis x = y , one gets a new type T^{op} with invariants

$$l(T^{op}) = l(T),\ s(T^{op}) = s(T),\ \omega(T^{op}) = a(1)$$

and explicit description

$$T^{op} = ((r(1)+..+r(s))^{a(s)}, ... , r(1)^{a(1)-a(2)}).$$

Clearly, $(T^{op})^{op} = T$. It follows immediately from the geometric description of the map $T \to T^{op}$ that, if $S < T$ are immediate successors, then so are S^{op} and T^{op} ; thus this map is an automorphism of the type graph. Viewed combinatorially, the partition T^{op} is known as the *conjugate* of T; but note that conjugation does not respect the combinatorial ordering. I wonder whether the map $T \to T^{op}$ can be given a meaningful arithmetic interpretation.

From the description of T as decreasing step function we also read off the expression of m(T) as a product of numbers P(r), namely

$$m(T) = P(r(s))^{a(s)}\ P(r(s-1))^{a(s-1)-a(s)} ...\ P(r(1))^{a(1)-a(2)}.$$

It is easy to see that m(T) is *uniquely* such a product; conversely, every such product is an m-number. Thus the numbers P(r) serve as „prime elements" for the m-numbers.

3.4. The set $\mathcal{T}$ is the union of the sets $\mathcal{T}(n)$, and the numbers m(T) are exactly the arguments where the step function $t(x)$: = cardinality of $\mathcal{T}(x)$ increases (by one unit). The $\mathcal{T}(n)$ are no lattices (differing in this respect from the lattices of divisors of a fixed number, which are even boolean), since they contain the infimum (gcd), but not in general the supremum (lcm) of two elements. Here is a procedure for determining the maximal m(T)´s (maximal with respect to divisibility) occurring in $\mathcal{T}(n)$ and thereby the whole poset: for a fixed breadth r (bounded from above by r(n): = largest r such that P(r) $\leq$ n) find the largest a such that $(a,1^{r-1})$ belongs to $\mathcal{T}(n)$; this is the smallest maximal element of breadth r. If $(a_1,...,a_s,1,...,1)$ is any maximal element of $\mathcal{T}(n)$, then $(a_1,...,a_s,2,...,1)$ cannot belong to $\mathcal{T}(n)$; so one must try out how many factors p_s, p_{s-1},..one has to remove in order to be able to add one more factor p_{s+1}, respecting the growth condition. This depends on the growth of the prime number sequence at this point (and on n, of course). Here is the list of maximal types in $\mathcal{T}(1000)$, sorted according to breadth, together with their minimal realizations:

(9)	(8,1)	(6,2)	(5,3)	(6,1,1,)	(4,2,1)	(2,2,2)	(3,1,1,1) ;
512	768	576	864	960	720	900	840

inspection of this list reveals how the procedure just described works.

3.5. Since types which are comparable in the arithmetical ordering have their minimal realizations differing by at least a factor of 2, we see that T is maximal in $\mathcal{T}(n)$ if and only if $n/2 < m(T) \leq n$; so the „upper half" (in terms of the m-numbers) of $\mathcal{T}(n)$ consists entirely of maximals, the upper half of the lower half of the maximals in $\mathcal{T}(n/2)$ and so on. Denoting the set of maximals in $\mathcal{T}(n)$ by $\mathcal{T}_{max}(n)$, having order $t_{max}(n)$, we obtain

$$t(n) = t_{max}(n) + t_{max}(n/2) + t_{max}(n/4) ...$$

Incidentally, this offers another strategy for the determination of $\mathcal{T}_{max}(n)$: simply calculate, once and for all, a sufficiently large list of all m-numbers and check which of them lie between n/2 and n. Note the embedding of $\mathcal{T}_{max}(n/2)$ into $\mathcal{T}_{max}(n)$, mapping the type T = $(a_1,...,a_s)$ to the type (2,T) = $(a_1+1,...,a_s)$. Evidently $\mathcal{T}_{max}(n)$ is a maximal set of pairwise incomparable elements of $\mathcal{T}(n)$ (an „antichain"), and the (arithmetical) order relation can obtain only between the different „layers" $\mathcal{T}_{max}(n)$, $\mathcal{T}_{max}(n/2)$, ... I would like to see a meaningful arithmetic interpretation of Dilworth´ theorem, which says in our case that if one partitions $\mathcal{T}(n)$ into disjoint chains, then $t_{max}(n)$ is the minimal number of chains needed. The reader will find it instructive to rearrange $\mathcal{T}(200)$ following the above principle. Here is a short list of values of $t_{max}(n)$:

n	10	100	200	1000	2000	10.000	100.000
$t_{max}(n)$	2	4	5	8	12	17	24 .

The function $t_{max}(n)$ is still monotonically increasing. If n is even, nothing can be lost when passing to n+1, because the smallest m-number in $\mathcal{T}_{max}(n)$ is at least n/2 + 1, which is still larger than (n+1)/2. If n is odd, and (n+1)/2 is an m-number, then n+1 is also one, so the loss is compensated. The function increases by 1 if n+1 is an m-number (so n must be odd), but (n+1)/2 is not.

3.6. The only element in $\mathcal{T}_{max}(n)$ of breadth 1 is (a), where a is the largest power of 2 smaller than n, i.e. $a = [\log_2 n]$. It is also the element having maximal length; the lengths in $\mathcal{T}_{max}(n)$ move downwards as the breadth increases. Sorting the elements of $\mathcal{T}_{max}(n)$ according to breadth I found in all examples I have checked that this sequence first increases, then decreases monotonically; for n = 1000 the sequence is 1,3,3,1, for n = 10.000 it is 1,4,6,5,1, and for n = 100.000 one obtains 1,3,7,6,6,1. This is reminiscent of many sequences of combinatorial origin [2], but I have no general proof for it.

3.7. A basic question is to characterize the sets $\mathcal{T}(n)$ in the type graph; clearly it suffices to characterize the sets $\mathcal{T}_{max}(n)$. Now a purely type-theoretical characterization is too much to ask for; for example, because the concept of type may be applied to the ideals of an arbitrary Dedekind domain. The link between $\mathcal{T}$ and the natural numbers is the function „minimal realization"; using it, we can now show that a set $M = \{T_1,...,T_r\}$ is a $\mathcal{T}_{max}(n)$ if and only if it is a *maximal* set of types satisfying

$$(*) \qquad m(T_i)/m(T_j) < 2 \qquad \text{for all } i,j.$$

Proof. Suppose $M = \mathcal{T}_{max}(n)$; we may assume $m(T_1) \leq .... \leq m(T_r)$ as well as $n/2 + 1 = m(T_1)$ and $n = m(T_r)$ (recall that m-numbers are always even). It is clear from the previous discussion that the elements of M satisfy the inequality. If T is any type which can be added to M such that (*) still holds, then $m(T_r)/m(T) < 2$, whence $m(T) > n/2$, further $m(T)/m(T_1) < 2$, whence $m(T) < n + 2$ and therefore $m(T) \leq n$. But then T belongs to $\mathcal{T}_{max}(n)$ as seen previously. Conversely, fo any maximal set M satisfying (*) and ordered as above, set $n = m(T_r)$. Then $m(T_i) > n/2$ and clearly $m(T_i) \leq n$, all i, so M is contained in $\mathcal{T}_{max}(n)$; by maximality, these sets must coincide.

As the reader will have noted, the function $n \rightarrow \mathcal{T}_{max}(n)$ is not injective. If M is as above then $M = \mathcal{T}_{max}(n)$ exactly if $n/2 < m(T_1)$ and at least $m(T_r)$, but smaller than the next m-number, which is at most $2m(T_1) = m(2,T_1)$. E.g. for m = 1000 (see the table given above), the m-number following 960 is m(10) = 1024, so $\mathcal{T}_{max}$ is constant on the interval [960, 1023].

The „philosophy" of the concept of the $\mathcal{T}_{max}(n)$ now comes within sight. If one orders the types „naturally", i.e. combinatorially, then the sequence of m-numbers is erratic; if one orders the m-numbers naturally, then the type sequence becomes erratic. The sets $\mathcal{T}_{max}(n)$ seem to offer a „middle road", since they contain maximal elements in both senses. Simple as it appears, this basic set-up of types and m-numbers may well contain arithmetically interesting information.

4. Problems concerning the sequence of types (or m-numbers): Asymptotics. The first question is to ask for some asymptotic formula for the order $t(n)$. Here is one for the order $t_r(n)$ of $\mathcal{T}_r(n)$ = types of breadth r occurring up to n. A type T occurs in $\mathcal{T}(n)$ if and only if $m(T) \leq n$; taking logarithms we see that $T = (a_1,...,a_r)$ occurs if and only if

$$a_1 \log 2 + a_2 \log 3 + \ldots + a_r \log p_r \leq \log n.$$

In other words, $t_r(n)$ is the number of lattice points in the r-dimensional simplex

$$V = \{(x_1,\ldots,x_r) \mid x_1 \geq x_2 \geq \ldots \geq x_r \text{ and } x_1 \log 2 + x_2 \log 3 + \ldots + x_r \log p_r \leq \log n\},$$

minus the points on the boundary. The volume of V is

$$\text{vol } V = (r!\ \log 2\ (\log 2 + \log 3)....(\log 2 + \log 3 + \ldots + \log p_r)^{-1} (\log n)^r =: c(r) (\log n)^r,$$

and by a well-known argument we obtain

$$t_r(n) = c(r) (\log n)^r + O((\log n)^{r-1}),$$

note that the boundary points are absorbed by the error term. Summing over r, this would give a formula for $t(n)$ which, however, is uninformative because the error term for the maximal r is dominates the remaining terms. But we do obtain a meaningful expression if we ask for the number of types of breadth *at most* r (that is, we include the boundary points) and then let r = r(n) = maximal breadth which can occur. What is still missing here is a decent asymptotic for r(n), which evidently is a very slowly growing function of n. The well known results of Cebyshev give only an asymptotic for the product P(r) itself.

4.1. Another strategy for obtaining an upper bound for $t(n)$ is as follows. Denote by p(n) the number of partitions of n and suppose we have a function f(r) such that

$$f(r) \geq p(1) + \ldots + p(r-1).$$

Then, if $t(n) > f(r)$, $\mathcal{T}(n)$ must contain a type of length r, which implies $n \geq 2^r$. By contraposition, if $n < 2^r$, then $t(n) \leq f(r)$. Taking $r = [\log_2 n] + 1$, we obtain $t(n) \leq f([\log_2 n] + 1)$. Here is a heuristic and sketchy procedure for obtaining a suitable function f : using Rademacher´s asymptotic formula [3]

$$p(n) \sim \exp(K\sqrt{n}) / cn$$

with explicitely given constants K and c, one can evaluate the sum $p(1) + \ldots + p(r-1)$ with the aid of the Euler summation formula, and with some coarse estimates one arrives at an asymptotic inequality

$$t(n) \leq A \exp(K \sqrt{\log_2 n]}) \text{ (A a constant)},$$

which (if true) would suffice to show that $t(n)$ grows asympotically slowlier than any positive power of n.

4.2. Gaps. Inspection of (the beginning of) the sequence of the m-numbers indicates a considerable irregularity; for example, there are (comparatively) large gaps between the successive m-numbers m(5,5) = 7776, m(13) = 8192 and m(6,3,1) = 8640, but m(10,2)

= 9214 and m(3,1,1,1,1) = 9240. Yet one may ask whether the gaps become longer and longer, at least for special families of types. One has to show that if m(T) – m(S) remains bounded, then T and S belong to a finite set of types. To illustrate the kind of problems arising here, let us first consider types of the form T = (a,b), S = (c,d) (a ≥ b, c ≥ d). Suppose a > c > d > b. Then

$$m(T) - m(S) = 2^a 3^b - 2^c 3^d = 2^c 3^b (2^{a-c} - 3^{d-b}) .$$

If this expression remains bounded for some sequence of types S,T, then first c and b must be bounded, hence also d, hence d–b, hence a–c, finally a, so that there are only finitely many possibilities for S and T; the argument for other constellations of a,b,c,d is similar. Going to another extreme, consider $S = (1^r)$ and T = (s), producing

$$m(S) - m(T) = 2(p_2...p_r - 2^{s-1}).$$

If this expression remains bounded for infinitely many S,T, then there must be a value, say c, such that infinitely often $p_2...p_r - 2^{s-1} = c$; we can view s-1 as a function f(r) of r. Now if p is a prime divisor c, then for large r p divides the product on the left-hand side (c ist odd), which leads to a contradiction; hence c = ±1. For c = 1, the equation $p_2...p_r - 2^{f(r)} = 1$ implies that – 1 is congruent to some power of 2 modulo the primes $p_2 ,..., p_r$, which is not the case for $p_4 = 7$. For c = – 1, we obtain that f(r) must be divisible by the order of 2 mod p_i , which is 12 for $p_6 = 13$; but $2^6 - 1 = 63$, and hence 9 divides $2^{f(r)} - 1$, which therefore is not squarefree. Note the curious logical structure of the argument. Starting with the assumption, that a certain equation has infinitely many solutions, we have concluded that there can only finitely many (in fact, the only cases are 3 – 2 = 1 and 3×5 – 16 = – 1).

In the general case, write T = (a(1),...,a(n)), S = (b(1),...,b(m)), assuming n ≥ m . Then (writing min(i) for min(a(i), b(i))

$$m(T) - m(S) =$$

$$p_1^{\min(1)}...p_m^{\min(m)} (p_1^{a(1)-\min(1)}...p_m^{a(m)-\min(m)}\ p_{m+1}^{a(m+1)}...p_n^{a(n)} - p_1^{b(1)-\min(1)}...p_m^{b(m)-\min(m)}) .$$

If this is to be bounded, then m is bounded (since all a(i), b(j) are > 0, and the expression in brackets cannot vanish), and so are the min´s; but there seems to be no easy argument disposing of the factor in brackets, except in special cases as above. At least there should be some (more or less) elementary argument constructing arbitrary long gaps in some explicit way, as in the case of prime numbers. Here is a nonconstructive one: if this were false, then there would be an absolute upper bound for the gaps, and this would imply that the order $t(n)$ of $\mathcal{T}(n)$ would grow at least proportionally with n, which is absurd.

4.3. In contrast, the *relative* length of the gaps is easily bounded by „Bertrands postulate for types“: given n, put $r = [\log_2 n] + 1$, then $n < 2^r$, so (r) does not occur in $\mathcal{T}(n)$, but $2^r \leq 2n$, so there always occurs a new type between n and 2n. One can also use the fact that twice an m-number is again one. It is amusing to compare this trivial argument with the proof of Bertrands original postulate for primes which is quite tricky.

5. Problems concerning the numbers having a fixed type: Asymptotics. The most natural problem of this kind is the determination of the counting function

$$\pi_T(x) := \text{number of natural } n \leq x \text{ with } t(n) = T$$

for a given type T and real argument x. Note the obvious equation

$$n = \Sigma\, \pi_T(n)\,,$$

where the sum can be thought of as extending over all T; clearly only those T contribute for which $m(T) \leq n$. The equation is a sort of closure relation for the functions π_T.

For $T = (1)$ we obtain the usual prime number function, $\pi_{(1)} = \pi$, and what I would like to see is a generalization of the prime number theorem to arbitrary types. In fact, it should be possible to derive such a theorem from the prime number theorem. For example, it is plain that for $T = (r)$

$$\pi_T(x) = \pi(x^{1/r})\,.$$

For $T = (1,1)$ we argue as follows: the sum

$$\pi(x/2) + \pi(x/3) + \ldots + \pi(x/p_m), \text{ with } p_m = \text{maximal prime } \leq x/2$$

counts the products $p_i p_j \leq x$ with different factors twice (for every prime $p \neq 2$ counted by $\pi(x/2)$ we have $2p$ in our set of numbers, and so on) and the $p^2 \leq x$ once; therefore,

$$\pi_T(n) = 1/2\, (\Sigma\, \pi(n/p) - \pi(n^{1/2})\,,$$

the sum extending over the primes $p \leq n/2$. We would like to compare $\pi_T(n)$ with $\pi(n)$ and argue heuristically, substituting $x/\log x$ for $\pi(x)$ and neglecting the error term. A small calculation gives

$$\Sigma\ (n/p)/\log(n/p) = n/\log n\ \Sigma\ 1/p\ \ 1/(1 - \log p/\log n)\,,$$

the summation still being the same; as is well known, the sum on the right tends to infinity at least like log log n. This indicates that the quotient $\pi_T(x)/\pi(x)$ tends to infinity, which conforms to the intuition: as the sequence of primes $\leq x$ becomes thinner, there are more possibilities to combine two of them to a product still $\leq x$ than there are members of that sequence. In fact, a first count shows $\pi_T(100) = 31$, $\pi_T(200) = 55$, whereas $\pi(100) = 25$, $\pi(200) = 45$. Applying the same kind of reasoning to the type $T = (2,1)$, we obtain a factor

$$\Sigma\ 1/p^2\ \ 1/(1 - \log n/\log p^2)\,,$$

sum over p $\leq$ $(n/2)^{1/2}$, which remains bounded, and expect this type to produce less realizations than the prime type (1); indeed $\pi_{(2,1)}(100) = 14$, $\pi_{(2,1)}(200) = 24$. There should be a decent way to express the growth of the counting function π_T for given T in terms of the exponents of T. In particular I would like to know which types have asymptotically more realizations than the prime type. And are there any different types having the same asymptotic growth of their counting functions?

5.1. Successive occurrences. A second natural question is: given T, can it happen that t(n) = t(n+1) = T? (This seems the only aspect of types which has received some attention.) Clearly not, if $T = (a_1,...,a_r)$ and $\gcd(a_1,...,a_r) = d > 1$, for in that case the numbers having type T are a subset of the set of d-th powers (not to mention T = (1)). Things change drastically if the gcd is one; the type (1,1) has 16 successive occurrences up to 200, and (2,1) still 5 (the first being 44, 45, the last 171,172); no other types have any in this interval. If such repetitions are possible, what is the maximal length b such that t(n) = t(n+1) = ... t(n+b – 1) = T? The type T = (1,1) has four triple repetitions up to 200 (the first beginning with 33, the last with 141), but T = (2,1) has none. However, T = (1,1) cannot have quadruple repetions, in fact no four successive numbers can be squarefree, because one is divisible by four. The argument is easily generalized to arbitrary types: if a is the maximal exponent occurring in T, then no 2^{a+1} successive numbers can have type T. This upper bound is not optimal, e.g. no 6 successive numbers can have type (2,1), because one of them would be divisible by 6, so would have to be either 12 or 18, which does not work. In fact, no type (a,b) can occur 6 times in a row: if n is the one divisible by 6, we must have $n = 2^a3^b$ or $= 3^a2^b$, and I leave it to the reader to figure out that this is impossible (use the fact that no two even numbers n and n+2 can have the same 2-exponent). Doubtless this type of argument can be carried further. I would very much like to know whether arbitrary long repetitions can be realized with suitable types. If not, then there would exist an absolute constant C such that among C successive natural numbers there occur at least two different types, and of course we would like to know this constant. The longest chains I know of (and which I owe to a computer search by H.Bachmann) are a quintuple repetition of the type (2,1,1), starting with 204323 and a septuple repetition of the type (2,1,1,1), starting with 440738966073.

5.2. If repetitions are possible, can there be infinitely many? The answer is no if all a_i are ≥ 3 and the abc-conjecture holds. Namely, the conjecture predicts that for every $\varepsilon > 0$ there is a constant $c(\varepsilon)$ such that

$$n + 1 \leq c(\varepsilon)\, \mathrm{rad}(n(n + 1))^{1+\varepsilon} ;$$

but if t(n) = t(n+1) = T, the hypothesis on the exponents of the prime decomposition implies

$$\mathrm{rad}(n) \leq n^{1/3} ,$$

likewise for n+1, and clearly rad(n(n+1)) = rad(n)rad(n+1), so we obtain the inequality

$$n +1 \leq c(\varepsilon)(n^{2/3} \sqrt{1 + 1/n}\,)^{1+\varepsilon} ,$$

which can hold only finitely often if ε is small.

5.3. Let us concentrate on T = (1,1) and ask if there are infinitely many repetitions, explicitely equations

$$p_1p_2 = q_1q_2 + 1$$

with primes $p_1 \neq p_2$, $q_1 \neq q_2$. Evidently one of the primes must be 2, so let us assume $p_1 = 2$; we sharpen the question further by assuming $q_1 = 3$, so we ask if there are infinitely many equations $2p = 3q + 1$ with prime numbers p and q. One sees that q must be $\equiv 3 \bmod 4$ and $p \equiv 2 \bmod 3$, and testing the first q, one finds that q = 7,11,19,31,47, 59,67,71,79,87 all work (the majority of primes $\equiv 3 \bmod 4$ less than 100), so prospectives seem to be bright. Another question of this type occurs (as is well known) in connection with Artin´s conjecture on primitive roots: if p is a prime $\equiv 3 \bmod 4$ such that $q = 2p + 1$ is again a prime, then -2 is a primitive root mod q. Both questions can be viewed as special cases of the following: let a, m, b, n be natural numbers and f the bijection of residue classes

$$a + m\mathbb{Z} \quad \to \quad b + n\mathbb{Z}, \quad x \quad \to \quad b + n((x - a)/m),$$

transferring the parameter. If gcd(b,n) = 1, then we know by Dirichlet´s theorem that f(x) is a prime infinitely often. If in addition gcd(a,m) = 1, it makes sense to ask whether infinitely often f(prime) is prime [4]. This would be, so to speak, „Dirichlet´s theorem squared“ and certainly would require (if true) entirely different methods. Note that it includes the problem of twin primes, which is the case of $1 + 2\mathbb{Z} \to 3 + 2\mathbb{Z}$.

5.4. More questions. Somewhat more generally we may ask for the minimal distance of numbers having a type T, and whether this minimal distance occurs infinitely often; note that this too includes the problem of twin primes. Further: for which types T does exist a natural number b such that t(n) = t(n+b) = T infinitely often? If for some T no such b exists, then for every C > 0 one can have

$$t(n) = t(m) = T \text{ and } |m - n| < C$$

only finitely often, in other words, the gaps between the T-numbers tend to infinity. One might suspect this behaviour not only for numbers of the same type, but also if the types are different, but satisfy suitable conditions. Numbers having a type with all exponents >1 have been called „powerful“, and it has been conjectured that there can be no three successive such numbers. That there do exist infinitely many successive pairs is confirmed by the fact that the Pell equation

$$x^2 - d\,y^2 = 1,$$

with d natural and powerful, but not a square, has infinitely many solutions. The argument given in 5.2. (using the abc-conjecture) also works if n and n+1 have different types, as long as the condition on the exponents is satisfied, explicitly: if S and T are types having all their exponents > 2, and if the abc-conjecture is true, then it can happen only finitely often that t(n) = S, t(n+1) = T. So one has to expect that the types of the solutions of such Pell equations become larger and larger. In this context one also thinks of the (proven) Catalan conjecture, which in the language of types can be formulated thus: the equation $9 - 8 = 1$ shows the only case in which two natural numbers, each having a type with all exponents divisible by a number > 1, differ by 1.

5.5. Another question which can be transferred from the prime type to arbitrary types is what might be called the Euler-Goldbach-Hilbert-Waring question: given T, does there exist a constant k such that every sufficiently large natural number is a sum of at most k numbers having type T, or, somewhat weaker, having types in a prescribed set of types? For the prime type, the well known result of Vinogradov gives a positive answer with k = 4. The Hilbert-Waring theorem settles the weaker version, with the set of types being the one of all l-th powers (for fixed exponent l). Testing T = (1,1) with k = 2, I have found no gap after 80 up to 150 (but clearly this does not justify any conjectures).

5.6. After so many unanswered questions, let us conclude this section by proving what might appear difficult at first sight, namely Dirichlet´s theorem for arbitrary types. Let gcd(a,m) = 1 and assume that the congruence

$$a \equiv x_1^{a(1)} \ldots x_r^{a(r)} \bmod m$$

is solvable. Then there are infinitely many numbers $n \equiv a \bmod m$ with t(n) = (a(1),...,a(r)): simply choose arbitrary primes $p_i \equiv x_i \bmod m$ and build up n accordingly. Note that the solvability of the congruence is a necessary condition for the existence of a single such number; and if gcd(a(1),...,a(r)) = 1, it is no condition at all (easy exercise in group theory). It is left to the reader to show (by an equally trivial argument) that the class of a mod m also contains numbers of infinitely many different types, and gcd(a,m) = 1 is not even needed for this. So a theorem of „Dirichlet type" holds not only „horizontally", namely for the sequence of numbers of certain types, but also „vertically", for the sequence of types. Statements on equidistribution over classes (if true at all), however, are presumably more difficult to prove .

6. Arithmetical functions. Among the familiar arithmetical functions some depend on the actual primes entering in the prime decomposition of the argument, like Euler´s totient or the divisor sums, but others depend only on the type, i.e. factorize over the type map $n \to t(n)$; most notably ω, the Möbius μ-function, and Liouville´s function $\lambda = (-1)^l$, where we abbreviate $l(n) = l(t(n))$. I wonder which of the type-dependent arithmetical functions can be expressed in terms of these three (of course, one must specify the meaning of „expression").

The three above-mentioned „basic" type-dependent functions are interrelated by various identities. Let us write $f * g$ for the Dirichlet convolution of arithmetic functions,

$$(f * g)(n) = \Sigma f(d)\, g(n/d) ,$$

and ε for the unit element for the Dirichlet product (the characteristic function of 1). Then we have $\mu * 1 = \varepsilon$; this is the basic property of μ, immediately implying Möbius inversion. Using multiplicativity and checking prime powers, one calculates

$$2^{\omega} * \lambda = 1 \quad (= \text{constant function } 1), \text{ whence } \mu * 2^{\omega} * \lambda = \mu * 1 = \varepsilon ,$$

showing that $\mu * 2^{\omega}$ ($= \mu^2$) is the $*$ -inverse of λ. Similarly, one can check that

$$2^{\omega} * 2^{\omega}\lambda = \varepsilon, \text{ whence } \mu * \lambda = 2^{\omega}\lambda \text{ and } \lambda = 1 * 2^{\omega}\lambda .$$

Another relation of this type is

$$\omega * \lambda = -1 * (\lambda\omega)$$

(proof omitted). The most elementary type-dependent functions, however, are the characteristic functions χ_T of the types T, or δ -functions on the set of types, namely

$$\chi_T(n) = 1 \text{ if } t(n) = T, \ = 0 \text{ otherwise;}$$

every type-dependent arithmetical function f can be written as

$$f = \Sigma\, f(T)\, \chi_T ,$$

summing over all types; the infinite sum of course makes perfect sense. Now we claim that

$$\chi_{(1)} = \mu * \omega .$$

Proof: on the one hand $\mu * \omega * 1 = \omega$, on the other hand $\chi_{(1)} * 1 = \omega$, as one sees at once, and the equation follows since the constant function 1 is Dirichlet-invertible. (A similar argument shows that the function $l * \mu$ is the characteristic function of the set of prime powers; call this function χ and observe that

$$(\chi * 1)(n) = \Sigma\, \nu_p(n) = l(n),$$

where the sum is over the prime divisors of n.)

Next we calculate, for n with prime divisors $q_1,..q_r$,

$$(\chi_{(1)} * \chi_{(1)})(n) = \Sigma\, \chi_{(1)}(n/q_i) = \begin{cases} 2 & \text{if } n = q_1q_2 \\ 1 & \text{if } n = q_1^2 \\ 0 & \text{otherwise} \end{cases} ,$$

from which one reads off the equation

$$\chi_{(1)} * \chi_{(1)} = 2\, \chi_{(1,1)} + \chi_{(2)} .$$

In order to express the right-hand side in terms of μ and ω, we calculate

$(\chi_{(1,1)} * 1)(n)$ = cardinality of $\{(p,q) \mid p,q$ prime divisors of n, $p \neq q\} = \omega(n)(\omega(n) - 1)/2$,

whence by Möbius inversion we get

$$\chi_{(1,1)} \quad = \quad \mu \square \omega(\omega - 1)/2$$

and thereby also an expression of $\chi_{(2)}$ in terms of μ and ω (note that both the Dirichlet as the usual valuewise product are involved here). We now sketch a proof that all functions χ_T have such an expression. (Remark: it is well known that $1 * \lambda$ is the characteristic function of the set of all squares of natural numbers. By Möbius inversion, this yields

$$\lambda \;=\; \mu \square \Sigma \chi_T$$

with the sum over all types of squares, that is all $(a_1,...,a_r)$ with all a_i even. Hence if we can express all such χ_T in terms of μ and ω, we also get such an expression for λ , but as an infinite sum!) Defining

$$\omega^{(k)}(n) = \text{number of } p|n \text{ with } \nu_p(n) = k, \;\; \omega_k(n) = \text{number of } p|n \text{ with } \nu_p(n) \geq k,$$

(so $\omega_1 = \omega$) we have the equations

$$\omega_k - \omega_{k+1} = \omega^{(k)} \quad \text{and} \quad \omega_k = \chi_{(k)} * 1.$$

A straightforward calculation shows

$$\chi_{(k)} * (-1)^{\omega} = (-1)^{\omega}(\omega_{k+1} - \omega^{(k)}) = (-1)^{\omega}(2\,\omega_{k+1} - \omega_k),$$

and together these equations imply inductively that all functions $\chi_{(k)}$ have an expression in terms of μ and ω. (The function $(-1)^{\omega}$ seems to play a key role, which should be elucidated further; for example, it enters in the characteristic function of the set of all powerful numbers.)

Before we proceed, we have to take a closer look at the (Dirichlet-) multiplication of the functions χ_T, which is quite interesting. Here are some more products:

$$\chi_{(1)} * \chi_{(2)} = \chi_{(2,1)} + \chi_{(3)}\,, \quad \chi_{(1)} * \chi_{(1,1)} = 3\chi_{(1,1,1)} + \chi_{(2,1)}\,,$$

$$\chi_{(1,1)} * \chi_{(1,1)} = 6\chi_{(1,1,1,1)} + 2\chi_{(2,1,1)} + \chi_{(2,2)}\,.$$

Of particular interest are the $*$ - powers of $\chi_{(1)}$. We have calculated $\chi_{(1)}^{*2}$ above; the next two are

$$\chi_{(1)}^{*3} = 6\chi_{(1,1,1)} + 3\,\chi_{(2,1)} + \chi_{(3)}\,,$$

$$\chi_{(1)}^{*4} = 24\chi_{(1,1,1,1)} + 12\,\chi_{(1,1,2)} + 6\,\chi_{(2,2)} + 4\chi_{(1,3)} + \chi_{(4)} \ .$$

The n-th power $\chi_{(1)}^{*n}$ involves all and only the types of length n ; the type 1^n appears with coefficient n!, the type (n) with coefficient 1 (proofs by induction on n).

The general case reads as follows: let $T = (a_1,...,a_r)$, $S = (b_1,...,b_s)$. Then

$$\chi_T \ * \ \chi_S \ = \ \Sigma \ m(T, S; R) \ \chi_R \ ,$$

where R runs over all types $R = (c_1,..., c_t)$ such that each c is either an a or a b or a sum of an a and a b, and all a´s and b´s are used up, informally: R arises by shoving S and T one upon another; clearly S,T < R in the arithmetic order. The multiplicity m(T, S; R) is the number of possibilities to obtain R from S and T in this way; these multiplicities should be related to the orders of the types R,S,T involved in the type graph. Note that $l(R) = l(T) + l(S)$ for such R; so the ℤ-algebra generated by the χ_T is graduated by length. Also, every product $\chi_T * \chi_S$ involves at least two summands; no χ_T is a product of other such functions.

Now we can finish the proof that the χ_T can be expressed in terms of μ and ω by an induction along ω(n). The case of ω(n) = 1 has already been settled. For arbitrary T, we have an equation

$$\chi_T \ * \ \chi_{(k)} \ = \ \Sigma \ m(T, k; R)\, \chi_R \ + \chi_{(T,k)} \ ,$$

where (T,k) denotes the type obtained from T by adding an exponent k, and the sum is over types R with ω(R) = ω(T). Since any type S with ω(S) = ω(T) + 1 can be written in the form S = (T,k), this completes the induction.

An amazing thing here is the following: if we view the types as partitions, then we see that thereby we have made the set of all partitions (of all numbers) the basis of an infinite-dimensional, associative and commutative graduated algebra, which would not be easy to see from the point of view of partitions alone. It might be interesting to describe (finite-dimensional?) representations of this algebra or images modulo primes. In any case this algebra should be of use for the theory of partitions.

7. Zeta functions of types. With every type T we can associate its zeta function

$$\zeta_T\,(s) : = \ L(s, \chi_T) \ = \ \Sigma \ n^{-s} \ ,$$

where the sum is over all n such that t(n) = T. Clearly the series converges for Re s > 1, and

$$\Sigma \ \zeta_T(s) \ = \ \zeta\ (s) \ ,$$

with the sum over all types (including T = (0), having $\chi_{(0)} = \varepsilon$), is the Riemann zeta function. This equation can be viewed as an additive analogue of the Euler product decomposition. The exact abscissa of convergence of $\zeta_T(s)$ is

$$\sigma_T := 1/\min(a(i)) ,$$

where as usual $T = (a(1),...,a(r))$. To prove this, let us assume that a(r) is minimal. It is easy to see that

$$\sigma_T \leq \sigma_{(a(r),...,a(r))}$$

(compare the summands of the series in question); on the other hand for $s = 1/a(r)$ the series $\zeta_T(s)$ diverges, because it contains subseries of the form

$$q_1^{-a(1)/a(r)} ... q_{r-1}^{-a(r-1)/a(r)} \; \Sigma\, p^{-1}$$

with fixed primes q_i and the sum running over the remaining primes; so actually $\sigma_T = \sigma_{(a(r),...,a(r))}$. The proof of $\sigma_{(a,...,a)} = 1/a$ is left to the reader. The standard formula for the abscissa of convergence of a Dirichlet series [5] reads in our case

$$\sigma_T = \inf \{a > 0 \mid \pi_T(x) = O(x^a) \} ,$$

so comparing the two expressions we obtain

$$1/\min(a(i)) = \inf \{a > 0 \mid \pi_T(x) = O(x^a) \},$$

which however is a rather trivial statement when looked upon more closely. The equation

$$L(s,f * g) = L(s,f)\, L(s,g)$$

can be used to derive equations among the functions χ_T ; for instance, the equation

$$\chi_{(1)} * \chi_{(1)} = 2\, \chi_{(1,1)} + \chi_{(2)}$$

also follows from the evident identity

$$2\, \zeta_{(1,1)} = \zeta_{(1)}{}^2 - \zeta_{(2)} .$$

7.1. One also may ask for Euler products. It is easy to see that a single $\zeta_T(s)$ cannot have one, but suitable (infinite) sums of such zetas can; simply start with such a product

$$\Pi\, (1 + p^{-a(1,p)s} + p^{-a(2,p)s} + ...)$$

with natural exponents $0 < a(1,p) < a(2,p) <$ ...(the sequence may or may not terminate) and look what happens. If this is to be a sum of functions $\zeta_T(s)$, then the exponents a(i,p) = a(i) must be independent of p, and then exactly those types occur which have all their exponents in the set of the a(i)´s, and the product is the sum of the corresponding zeta functions. For example,

$$\prod (1+p^{-s}) = L(s,\mu^2) = \zeta(s)/\zeta(2s)$$

is the sum of the zetas of all sqarefree types, plus 1, and

$$\prod (1+p^{-2s}) = L(s, (\mu * \mu^2)^2) = \zeta(2s)/\zeta(4s)$$

is 1 plus the sum of zetas of all types (2,...,2) (exercise). It would be interesting to know which functions of this kind can be expressed by the Riemann zeta function, as products and quotients of functions $\zeta(as)$ for various natural a. Operating with the Euler factors, one quickly sees that the problem boils down to the following: Let H be the subgroup of the multiplicative group of $\mathbb{Q}(t)$ generated by all $1 - t^n$. Then we want to know which elements of H have all their coefficients = 0 or 1 when expanded as a power series in t, and which sequences of 0,1 can arise in this way. For example,

$$1/(1 - t^n) = 1 + t^n + t^{2n} + \ldots.$$

yields the types of all n-th powers. If the power series is actually a polynomial, then all of its roots must be roots of unity; so e.g. $1 + t^2 + t^3$ cannot occur, and the Dirichlet series $1 + \Sigma\ \zeta_T(s)$, where the sum is over all types having exponents 2 or 3, is not an aggregate of functions $\zeta(as)$, but

$$1 + t + t^3 + t^4 = (1 + t)(1 + t^3) = (1 - t^2)/(1 - t)\ (1 - t^6)/(1 - t^3)$$

does occur, and the corresponding Dirichlet series equals $\zeta(s)\zeta(3s)/\zeta(2s)\zeta(6s)$. Note that the Dirichlet series arising in this way can be motivic zeta functions only in very special cases (are there others than X = {pt}?), since the polynomials defining the Euler factors are independent of the prime; so for example Dedekind zeta functions of number fields other than $\mathbb{Q}$ cannot occur [6].

7.2. Analytic continuation will not generally be possible, at least not for single functions ζ_T. In the standard proof of Dirichlet´s theorem one derives the formula

$$\log\ \zeta(s) \sim \zeta_{(1)}(s),$$

expressing the fact that the difference of both sides is holomorphic in a neighbourhood of 1. (I would like to see analogous formulas for other types.) Now the function on the left cannot be viewed as meromorphic at 1 (the derivation of the formula is valid only in the half-plane re s > 1). (Elementary argument: if $\log \zeta(s)$ were meromorphic at 1, then it would have a pole of order at least one. But using the well known formula

$$\log\ \zeta(s) \sim \log 1/(s - 1)$$

one sees that, for every $a > 0$, if s tends to 1+, then

$$\lim\ (s - 1)^a \log\ \zeta(s) = \lim\ (s - 1)^a \log 1/(s - 1) = 0,$$

which is a contradiction.) So one cannot speak of a pole of $\zeta_{(1)}(s)$ at $s = 1$ in the usual sense; nevertheless it is possible to establish a calculus of orders for such functions, by virtue of the following construction[7].

We consider functions $f,g : (1, 1 + \varepsilon) \rightarrow \mathbb{R}^{\times}$ defined in some (possibly varying) interval right to 1 and define $f \approx g$ if the limit of $f(s)/g(s)$, for s tending to $1+$, exists and is $\neq 0$. This is an equivalence relation; we denote the class of f by o(f) and define addition and ordering of the classes by

$$o(f) + o(g) = o(fg), \quad o(f) < o(g) \text{ if } \lim g(s)/f(s) = 0.$$

This gives an ordered semigroup, which we denote $O^{+}(1)$; its elements may be called „(right) orders of real functions at 1". Note the „ultrametric" property: if $o(f) > o(g)$, then $o(f + g) = o(g)$, which follows from the trivial fact that $\lim f(s)/g(s) = 0$ implies $\lim(f(s) + g(s))/g(s) = 1$. The map $a \rightarrow o((s-1)^{a})$ is an embedding of the additive group of real numbers into $O^{+}(1)$. By what we have seen above,

$$0 := o(1) > o(\log \zeta(s)) > -a$$

for all positive a; the function $\log \zeta(s)$ has at $s = 1$ a pole of positive but infinitesimally small order; we put

$$\delta := o(\log \zeta(s)) = o(\zeta_{(1)}(s))$$

and claim: for any type T, $o(\zeta_T)$ equals δ times the number of 1´s among the exponents of T. The proof is by induction on this number. If no exponent is 1, then, as we have seen above, ζ_T is holomorphic and $\neq 0$ in a neighbourhood of 1, having order 0. For arbitrary T consider the equation

$$\chi_T * \chi_{(1)} = \chi_{(T,1)} + \Sigma\, \chi_S ,$$

and the sum contains only types S having at most as many 1´s as T. Using the inductive hypothesis and the ultrametric property, we conclude

$$o(\zeta_{(T,1)}) = o(\zeta_T * \zeta_{(1)}) = o(\zeta_T) + o(\zeta_{(1)}) ,$$

which implies our claim. Thinking speculatively, one might expect that $o(\zeta_T)$ has some impact on the growth order of the counting function π_T (which, nota bene, cannot depend on this order alone); this would require a new sort of „Tauberian" theorem.

Note also that

$$\zeta_{(1)}(s) = L(s, \chi_{(1)}) = L(s, \mu)\, L(s, \omega) = \zeta(s)^{-1} L(s, \omega),$$

so $L(s, \omega)$ has a pole of order $1 - \delta$ at $s = 1$. We see that zetas of single types do play a role, but the role is different from the one played by the motivic zeta functions. Certainly there remain things worthy to discover.

Conclusion. I have explained elsewhere [8] in some detail how a certain incompatibility of the additive and the multiplicative structure of the natural numbers can be viewed as the ultimate source for many problems of number theory. Both structures are easy to understand if viewed in isolation; it is their simultaneous presence, with the distributive law as a link between them, which causes those problems. The set of natural numbers, as codified by the Peano axioms, is generically of an additive structure, and so is our primary intuition of it: just continue adding the unit. It is a natural problem to understand how multiplicative complexity unfolds itself as one moves (additively) along the number sequence. For this understanding the concept of multiplicative type (of which the concept of prime number is just a special case) is evidently fundamental; as are some of the problems formulated here. We have also seen that some of the natural questions arising with this concept are easily answered, others require more effort, but can be answered with existing methods, but others will (probably) require new methods. In any case one should be aware of the old dictum of Siegel [9], valid in all of mathematics, but particularly in number theory, „daß man über die wirklichen Schwierigkeiten eines Problems nichts aussagen kann, bevor man es gelöst hat."

Notes and References

1 Writing this up I was made aware by Henrik Bachmann (Hamburg) of some websites which contain numerical material, e.g. a list with successive numbers having the same type (the type is called „prime signature" there). He has also carried out a computer search which seems to confirm the expectations concerning the growth of $\pi_{(1,1)}$ and $\pi_{(2,1)}$ (section 5), and found the quintuple and septuple repetitions mentioned in section 5.1. Furthermore, he pointed out to me the argument using the Pell equation in section 5.4. I thank him heartily for his commitment to these problems.

2 See M.Aigner, Combinatorial Theory, Springer 1979, p.76. Such sequences are called „unimodal".
3 T.M.Apostol, Introduction to Analytic Number Theory, Springer 1976, p.316.

4 Our first example is the case of $3 + 4\mathbb{Z} \to 5 + 6\mathbb{Z}$, the example connected with Artin´s conjecture is $1 + 2\mathbb{Z} \to 3 + 4\mathbb{Z}$. One must exclude the case where a and m are odd and b is even (or vice versa), because in this case k must be even if $a + mk$ is to be prime, but then $b + nk$ is even. This seems to be the only obstruction for the general question to be meaningful.

5 See Don B.Zagier, Zetafunktionen und quadratische Zahlkörper, Springer 1981, p.5.

6 Lists of L-series of arithmetical functions expressed as zeta-aggregates can be found in the above-mentioned books by Zagier, p.14 and Apostol, pp. 231, 247.

7 I owe the idea to F.Waismann, Einführung in das mathematische Denken, München 1970, p. 196 ss. No doubt it has been worked out „canonically" somewhere.

8 See my essay „Über Addition und Multiplikation“, in: E.K., Studien zur Struktur und Methode der Mathematik, Leipziger Universitätsverlag 2012.

9 C.L.Siegel, Transzendente Zahlen, Mannheim 1967, S.72.