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Half-life period and decay laws

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Half-life period and decay laws

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1 Half-life period

It is well known that the law of radioactive decay is given by

$$u(t) = u_0 \exp(-\alpha t), \tag{1}$$

where α is the decay rate. As a consequence one finds the half-life

$$T = \frac{\ln(2)}{\alpha},$$

which satisfies the relation

$$u(t+T) = \frac{1}{2}u(t) \tag{2}$$

for all $t \in \mathbb{R}$. Usually the rate of decay is indicated by the half-life and not by α. So it seems fair to ask, whether relation (2) implies a law of decay as in (1). Of course the answer is no, we find two ways to construct counter examples. Let T be given, w.l.o.g. $T = 1$ ie $\alpha = \ln(2)$:

1. choosing a 1-periodic function w and considering $u(t) = w(t)\exp(-\ln(2)t)$ or by
2. choosing an arbitrary function u on $[0,1]$ with $u(1) = u(0)/2$ and defining $u(t)$ outside of this interval by $u(t) = 2^m u(t')$ with $t' = t + m \in [0,1)$ and $m \in \mathbb{Z}$.

While for the first method the degree of smoothness of u is determined by that of w, the second method, which we want to focus on, leads to $u \in C^n([0,1],\mathbb{R})$ if and only if

$$u_n^{(j)}(1) = \frac{1}{2}\,u_n^{(j)}(0),\ j = 0,1,\ldots,n.$$

We will construct polynomial decay laws which converge to the decay law (2).

2 Polynomial decay laws

One simple way of constructing examples with half-life is to construct a polynomial p on $[0,1]$ with $p(1) = \frac{1}{2}p(0)$. The simplest choice is $p(x) = 1 - \frac{1}{2}x$. The continuation procedure

$$Cp(x) = 2^n p(x') \text{ with } x' = x + n \in [0,1), n \in \mathbb{Z} \tag{3}$$

gives a function on $\mathbb{R}$ with half-life, which is continuous, however not differentiable. In order to get differentiability we have to require the conditions

$$p(1) = \frac{1}{2}p(0),\ p'(1) = \frac{1}{2}p'(0).$$

It is easy to check, that

$$p(x) = 1 - \frac{2}{3}x + \frac{1}{6}x^2$$

does the job and moreover, if we normalize to $p(0) = 1$ this is the unique quadratic polynomial satisfying these conditions. It is therefore tempting to construct polynomials $p_n, n \in \mathbb{N}$ satisfying

$$\left.\begin{array}{l} 1.\ p_n(0) = 1; \\[2mm] 2.\ \deg(p_n) = n; \\[2mm] 3.\ p_n^{(j)}(1) = \frac{1}{2}p_n^{(j)}(0),\ j = 0,1,\ldots,n-1. \end{array}\right\} \text{(C)}$$

Then the continuation procedure (3) gives a C^{n-1}-function. Again it is rather easy to see that such polynomials exist and are in fact unique, see also Section 3. The natural question to ask is, does the limit

$$\lim_{n\to\infty} p_n$$

exist, and if so what is it? The second question is easy to answer, using a computer and plotting the graph of p_n on $[0,1]$. It shows that p_n converges to $e^{-\ln(2)x}$ uniformly on $[0,1]$. In fact the convergence is extremely fast.
Figure 1 shows the error $e^{-\ln(2)x} - p_n(x)$ for $n = 2, 10$.
In the next two sections we will present two different methods to explain this behavior. The first one is based on Linear Algebra, while the second one is reducing the problem to Functional Analysis. For some technical reasons it is easier to study the corresponding growth problem, rather than the decay problem. So we are looking for polynomials P_n with

1. $P_n(0) = 1$;
2. $\deg(P_n) = n$;
3. $P_n^{(j)}(1) = 2P_n^{(j)}(0),\ j = 0,1,\ldots,n-1.$

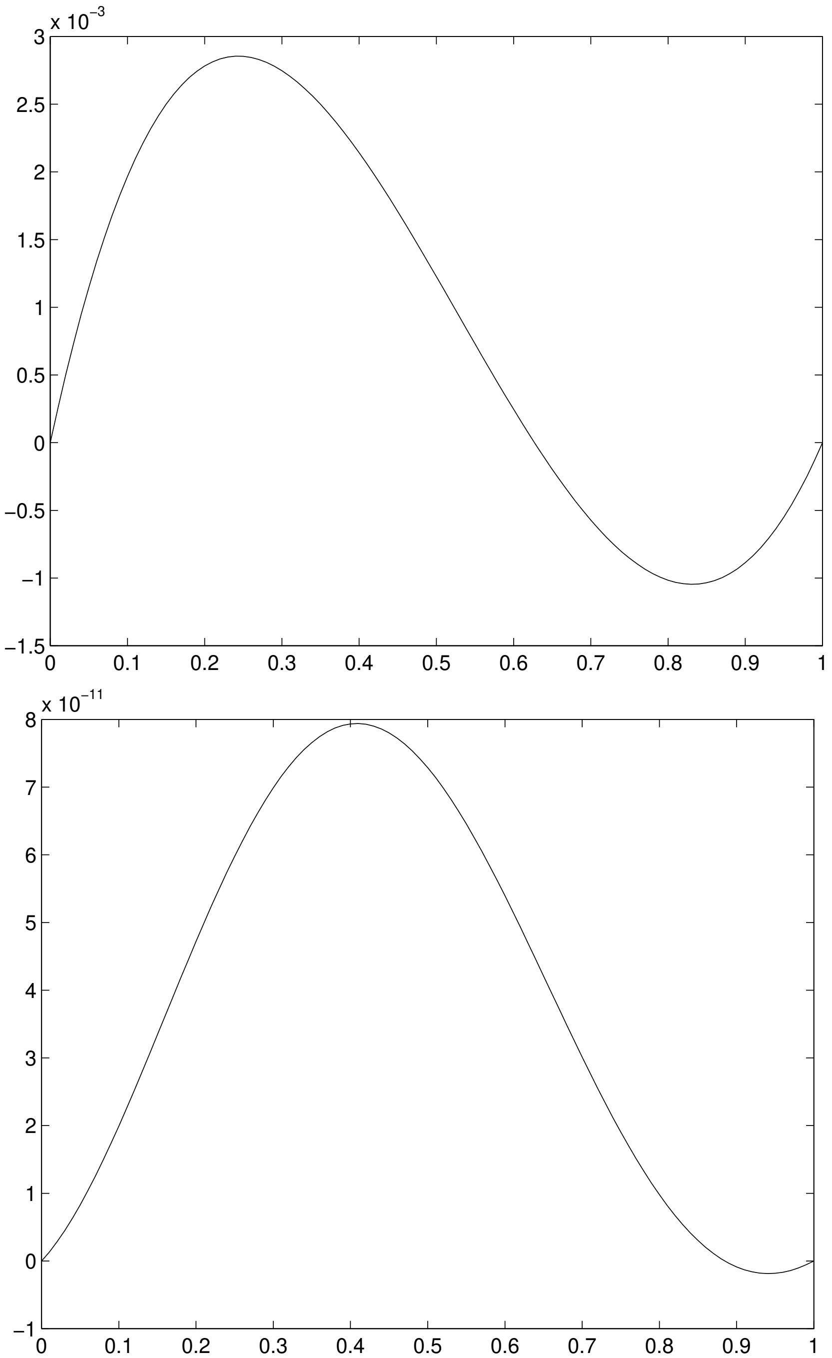

Figure 1: $p_n - e^{-\ln(2)x}$ for $n = 2, 10$

3 A linear algebra approach

Our problem can be formulated as an interpolation problem for polynomials: We are looking for a polynomial P_n of degree n or less satisfying

$$2P_n^{(j)}(0) - P_n^{(j)}(1) = 0,\ j = 0, 1, ..., n-1. \tag{4}$$

Making an *Ansatz*

$$P_n(t) = \sum_{k=0}^{n} \frac{\alpha_k^{(n)}}{k!} t^k \tag{5}$$

it is easy to see[1] that for each given $\alpha_n^{(n)}$, (4) is a linear system with n equations for n unknowns $\alpha_k^{(n)}, k = 0, 1, ..., n-1$ which can recursively be solved by

$$\alpha_k^{(n)} = \alpha_{k+1}^{(n)} + \frac{1}{2!}\alpha_{k+2}^{(n)} + \cdots + \frac{1}{(n-k)!}\alpha_n^{(n)}, k = n-1, n-2, \ldots, 0. \tag{6}$$

Hence, (4) has a unique solution with positive $\alpha_k^{(n)}$ up to a positive multiple contained in $\alpha_n^{(n)}$ which will be fixed from now on by the natural requirement $\alpha_0^{(n)} = 1$, or $P_n(0) = 1$ (see condition (C)).

The aim is to show that in a certain sense which has to be made precise, $P_n \to v$ for $n \to \infty$ where

$$v(t) = e^{\gamma t}, \quad \gamma = \log 2 \approx 0.6931.$$

Noting that

$$v(t) = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} t^k,$$

and setting

$$P_n^\star(t) = \sum_{k=0}^{n} \frac{\gamma^k}{k!} t^k, \quad \alpha^\star := (\gamma^k)_{k=0,1,...,n-1},$$

the convergence property can be stated as

$$|\alpha_k^{(n)} - \alpha_k^\star| \to 0 \text{ for fixed } k \text{ and } n \to \infty.$$

Qualitatively, we know already from (6) that $\alpha_k^{(n)}$ shares the properties $1 = \alpha_0^{(n)} > \cdots > \alpha_n^{(n)} > 0$ with $\alpha_k^\star := \gamma^k$.

[1] $P_n^{(j)}(0) = \alpha_j^{(n)}, P_n^{(j)}(1) = \sum_{k=j}^n \frac{\alpha_k^{(n)}}{(k-j)!}$ respectively $P_n^{(j)}(1) - 2P_n^{(j)}(0) = \sum_{k=j+1}^n \frac{\alpha_k^{(n)}}{(k-j)!} - \alpha_j^{(n)}$

To prove convergence, we notice that $\alpha^{(n)} = \left(\alpha_k^{(n)}\right)_{k=1,\dots,n}$ is the solution of $A_n\alpha^{(n)} = e^1$ with unit vector $e^1 \in \mathbf{R}^n$ and

$$A_n = \begin{pmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{n!} \\ -1 & 1 & \frac{1}{2!} & \dots & \frac{1}{(n-1)!} \\ 0 & -1 & 1 & \dots & \frac{1}{(n-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{7}$$

The most efficient way to solve $A_n\alpha^{(n)} = e^1$ is the *marching method*: Prescribe any $\alpha_n^{(n)} \neq 0$, march through (6) and update $\alpha_n^{(n)}$ such that $\alpha_0^{(n)} = 1$.
Plugging $\alpha^\star$ instead of $\alpha^{(n)}$ into $A_n\alpha^{(n)} - e^1 = 0$, we get a residual

$$A_n\alpha^\star - e^1 = \left((P_n^\star)^{(j)}(1) - 2(P_n^\star)^{(j)}(0)\right)_{j=0,1,\dots,n-1}. \tag{8}$$

Since

$$v^{(j)}(1) - 2v^{(j)}(0) = 0,\ j = 0,1,...,n-1,$$

and

$$v^{(j)}(0) = (P_n^\star)^{(j)}(0), j = 0,1,...,n-1,$$

the residual vector in (8) is

$$r^{(n)} = -\left((r_n^\star)^{(j)}(1)\right)_{j=0,1,\dots,n-1},$$

where

$$r_n^\star(t) = \sum_{k=n+1}^{\infty} \frac{\gamma^k}{k!}t^k$$

is the truncation error of the exponential series.
Now

$$\left.\begin{aligned} (r_n^\star)^{(j)}(1) &= \sum_{k=n+1}^{\infty} \frac{\gamma^k}{(k-j)!} \\ &= \gamma^j \sum_{k=n+1-j}^{\infty} \frac{\gamma^k}{k!} \\ &= \gamma^j \Big(e^\gamma - \sum_{k=0}^{n-j} \frac{\gamma^k}{k!}\Big) \\ &= \gamma^j \Big(\frac{\gamma^{n-j+1}}{(n-j+1)!}e^{\gamma_j}\Big) \\ &= \frac{\gamma^{n+1}}{(n-j+1)!}e^{\gamma_j} \end{aligned}\right\} \tag{9}$$

where γ_j are certain numbers satisfying $0 \leq \gamma_j \leq \gamma, j = 0,1,...,n-1$.

Since $e^{\gamma} = 2$ and

$$0 \leq \frac{\gamma^{n+1}}{(n-j+1)!} e^{\gamma_j} \leq 2\gamma^{n+1},$$

we have with the maximum norm

$$\|A_n(\alpha^\star - \alpha^{(n)})\| \leq 2\gamma^{n+1}. \tag{10}$$

Hence we finish, if $\|A_n^{-1}\|$ is uniformly bounded wrt n.
Numerical computations yield the bound $\|A_n^{-1}\| \leq 2.26$ for all n. We prove the more pessimistic inequality:

Lemma 3.1 *For A_n defined in (7) we have*

$$\|A_n^{-1}\| \leq 10.$$

Proof. For any right hand side b, the solution x of $A_n x = b$ can be again characterized and computed by the marching process: Determine x_n such that $x_0 = 0$ where

$$x_k := x_{k+1} + \frac{1}{2!} x_{k+2} + \cdots + \frac{1}{(n-k)!} x_n - b_{k+1},\ k = n-1, n-2, ..., 0. \tag{11}$$

We are going to show that $|x_k| \leq 10$ for all k whenever $|b_k| \leq 1$ for all k. The most important aspect is that at the end of the marching process $x_0 = 0$ has to be satisfied which prevents the former members $|x_k|$ in (11) from becoming too large.
To this end we redefine $x_k := x_{n-k}, k = 0, 1, ..., n$. Then we have to prove that for any sequence (x_k) with initial value x_0, $|x_0| \leq 1$ defined by

$$x_{k+1} := x_k + \frac{1}{2!} x_{k-1} + \cdots + \frac{1}{k!} x_0 - b_k,\ k = 0, 1, ..., n-1 \tag{12}$$

with arbitrary perturbations b_k, $|b_k| \leq 1$ and satisfying $x_n = 0$, we have $|x_k| < 10$ for any $k \leq n$.
Assume that for a certain $k < n$ we have $|x_k| = M \geq 10$. Choose k as small as possible and assume (without restriction) that $x_k = M$. We will show that it is impossible to fulfill $x_n = 0$ whatever b_j with $|b_j| \leq 1$ are chosen for $j \geq k+1$.

We are going to use the following very crude estimates.

$$\frac{1}{j!} + \cdots + \frac{1}{k!} \leq e - \sum_{i=0}^{j-1} \frac{1}{i!}, \tag{13}$$

$|x_j| \leq M$ for $j \leq k$, $e \leq 2.8$ and $1 \leq 0.1 \cdot M$.

First we notice using (12) with $k-1$ instead of k

$$x_{k-1} = x_k - \frac{1}{2!}x_{k-2} - \cdots - \frac{1}{k!}x_0 - b_{k-1}$$

from where – using $M \geq 10$, $|x_j| \leq 10$ for $j < k$ and (13) for $j = 2$ –

$$x_{k-1} \geq M - (e-2) \cdot M - 1 \geq (3 - e - 0.1)M \geq 0.1 \cdot M \ (> 0).$$

Now

$$x_{k+1} = x_k + \frac{1}{2!}x_{k-1} + \cdots + \frac{1}{k!}x_0 - b_k$$

$$x_{k+1} \geq M + \frac{1}{2!}0.1 \cdot M - (\frac{1}{3!} + \frac{1}{4!} + \cdots)M - 1,$$

$$x_{k+1} \geq M + 0.05 \cdot M - (e - 2.5)M - 0.1 \cdot M \geq 0.65 \cdot M,$$

$$x_{k+2} = x_{k+1} + \frac{1}{2!}x_k + \cdots + \frac{1}{(k+1)!}x_0 - b_{k+1},$$

$$x_{k+2} \geq 0.65 \cdot M + \frac{1}{2!}M + \frac{1}{3!}0.1 \cdot M - (\frac{1}{4!} + \frac{1}{5!} \cdots)M - 1,$$

$$x_{k+2} \geq 0.65 \cdot M + 0.5 \cdot M + 0.015 \cdot M - (e - 2.5 - 1/6)M - 0.1 \cdot M \geq 0.9 \cdot M.$$

Now it follows easily by induction that $x_j \geq M$ for all $j \geq k+3$ which contradicts $x_n = 0$, since also $x_{k+1} \geq 0.65 \cdot M > 1$ and $x_{k+2} \geq 0.9 \cdot M > 1$. □

Now it follows from (10):

Theorem 3.2 *Let $\alpha_k^{(n)}, k = 0, 1, ..., n$, be the coefficients of our interpolating polynomial represented in (7) and normalized by $\alpha_0^{(n)} = 1$. Then*

$$|\gamma^k - \alpha_k^{(n)}| \leq 20\gamma^{n+1},\ k = 1, ..., n. \tag{14}$$

Of course, the factor 20 in (14) is far too pessimistic. But the essential fact is that $\gamma := \ln(2) \approx 0.6931$ determines an upper bound for the asymptotic convergence rate.
Since $\gamma^k = (P_n^\star)^{(k)}(0)$ and $\alpha_k^{(n)} = (P_n)^{(k)}(0)$, as a consequence from Theorem 3.2 we have that $\|P_n - P_n^\star\|_k \to 0$, where $\|\cdot\|_k$ is the strong C^k-norm on $C^k([0,1], \mathbb{R})$.

4 A functional analytic approach: the half-life operator

We begin again with the decay problem and return to the doubling problem later on. We introduce the (nonlinear) half-life operator on the space

$$X = C([0,1], \mathbb{R}) \cap \left\{ f \;\middle|\; \int_0^1 f(r)dr \neq 0 \right\}$$

by

$$\mathcal{H}f(x) = 1 - \frac{1}{2F(1)} F(x)$$

where F denotes the primitive

$$F(x) = \int_0^x f(s)ds.$$

$\mathcal{H}$ has the following elementary properties

1. $\mathcal{H}f(1) = \mathcal{H}f(0)/2$ for $f \in X$.
2. If $f \in X$ and $f \geq 0$, then $\mathcal{H}f$ is monotonically decreasing.
3. $\mathcal{H}$ maps $A = \left\{ f \in C([0,1], \mathbb{R}) \mid \frac{1}{2} \leq f \leq 1 \right\}$ into itself.
4. $\mathcal{H}$ is compact.

What makes it interesting for our problem is the following observation:

Lemma 4.1 $\mathcal{H}p_n = p_{n+1}$

Proof. By induction. Set $p_0 = 1$. Then $\mathcal{H}p_0(x) = 1 - \frac{1}{2}x = p_1(x)$. While the initial value p_0 of our iteration does not satisfy the three conditions (C), all the iterates $p_1, p_2, \ldots$ will do. If p is a polynomial of degree n, then $\mathcal{H}p$ is a polynomial of degree $n+1$. Therefore $\deg(\mathcal{H}p_n) = n+1$. Moreover $\mathcal{H}p_n(0) = 1$. If we show that $\mathcal{H}p_n$ satisfies the conditions

$$\mathcal{H}p_n^{(j)}(1) = \frac{1}{2}\mathcal{H}p_n^{(j)}(0), j = 0, 1, \ldots, n,$$

then by uniqueness we have $\mathcal{H}p_n = p_{n+1}$. For any function f we find immediately the following expression for the derivative of $\mathcal{H}f$

$$(\mathcal{H}f)'(x) = -\frac{F'(x)}{2F(1)} = -\frac{f(x)}{2F(1)}$$

and for higher derivatives we get similarly

$$(\mathcal{H}f)^{(j)}(x) = -\frac{f^{(j-1)}(x)}{2F(1)}.$$

Applying this formula to p_n, we see by induction immediately the relation between the values of the $j-$th derivative at 0 and 1 respectively. Hence the lemma is established. $\square$

We see that the sequence of functions $p_n, n \in \mathbb{N}$ is an orbit of $p_0 = 1$ under iteration of $\mathcal{H}$. Before we go on, we note that the exponential function is a fixed point under $\mathcal{H}$, ie

Lemma 4.2 *Let $u(x) = \exp(-\ln(2)\, x)$, then $\mathcal{H}u(x) = u(x)$.*

Proof.

$$\mathcal{H}u(x) = 1 - \frac{-\frac{1}{\ln(2)}(u(x) - u(0))}{-2\frac{1}{\ln(2)}\exp(-\ln(2))} = u(x).$$

$\square$

4.1 Contraction

Now, in order to finish our calculations we replace the operator $\mathcal{H}$ by the operator $\mathcal{D}$,

$$\mathcal{D}f(x) = 1 + \frac{F(x)}{F(1)},$$

where F is defined as before. Now, $\mathcal{D}$ is a doubling operator, since $\mathcal{D}f(0) = 1$ and $\mathcal{D}f(1) = 2$, moreover, with $v = e^{\ln(2)x}$, we have

$$\mathcal{D}v(x) = v(x)$$

and we find a corresponding sequence P_n with $\mathcal{D}P_n = P_{n+1}$. Then $p_n \to u$ is equivalent to $P_n \to v$ by applying the reversal operators

$$R_1(f)(x) = 2f(1-x) \text{ and } R_2(f)(x) = \frac{1}{2}f(1-x),$$

respectively. Now, we want to show that P_n converges to v.

Theorem 4.3 *$\mathcal{D}$ is contraction on a neighborhood U of v in X.*

Proof. Consider the neighborhood

$$U = \left\{ w \in C([0,1], \mathbb{R}) \mid 1 + \frac{x}{2} \leq w(x) \leq 2 \right\}.$$

Clearly, for functions $w \in U$, we have

$$\int_0^1 w(s)ds \neq 0$$

and $\mathcal{D}$ is defined. Obviously for any positive function w the function $\mathcal{D}w$ is monotonically increasing and therefore $\mathcal{D}w \geq 1$. On the other hand $\mathcal{D}w(1) = 2$ and hence $\mathcal{D}w \leq 2$. Therefore the set

$$\{w \in C([0,1], \mathbb{R}) \mid 1 \leq w(x) \leq 2\}.$$

is globally attracting for all positive initial values. Moreover if $w \geq 1$, then

$$\mathcal{D}w(x) \geq 1 + \frac{W(x)}{2} \geq 1 + \frac{x}{2},$$

implying $\mathcal{D}U \subset U$. Let $w_1, w_2 \in U$ be two such functions with primitives W_1, W_2, set $K = W_1(1)W_2(1)$ then

$$\begin{aligned}
|\mathcal{D}w_1(x) - \mathcal{D}w_2(x)| &= \left|1 + \frac{W_1(x)}{W_1(1)} - 1 - \frac{W_2(x)}{W_2(1)}\right| \\
&= \frac{|W_2(1)W_1(x) - W_1(1)W_2(x)|}{W_1(1)W_2(1)} \\
&\leq \frac{1}{K}\left|\int_0^1 w_2(t)dt \int_0^x w_1(s)ds - \int_0^1 w_1(t)dt \int_0^x w_2(s)ds\right| \\
&= \frac{1}{K}\left|\int_0^1 \int_0^x w_1(s)w_2(t)dsdt - \int_0^1 \int_0^x w_1(t)w_2(s)dsdt\right| \\
&= \frac{1}{K}\left|\int_x^1 \int_0^x \left(w_1(s)w_2(t) - w_1(t)w_2(s)\right) dsdt\right| \\
&\leq \frac{1}{K}\left|\int_x^1 \int_0^x \left(w_1(s)w_2(t) - w_1(s)w_1(t)\right.\right. \\
&\qquad \left.\left. + w_1(s)w_1(t) - w_1(t)w_2(s)\right) dtds \right| \\
&\leq \frac{1}{K}\int_x^1 \int_0^x |w_1(s)w_2(t) - w_1(s)w_1(t)| \\
&\qquad + |w_1(s)w_1(t) - w_1(t)w_2(s)|\, dtds \\
&\leq \frac{1}{K}\left(\int_x^1 \int_0^x w_1(s)\|w_1 - w_2\|_\infty dtds\right. \\
&\qquad \left. + \int_x^1 \int_0^x w_1(t)\|w_1 - w_2\|_\infty dsdt\right)
\end{aligned}$$

$$\leq \frac{2}{K}\|w_1 - w_2\|_\infty \left(\int\limits_x^1 \int\limits_0^x ds dt + \int\limits_x^1 \int\limits_0^x ds dt \right).$$

$$\leq \frac{1}{K}\|w_1 - w_2\|_\infty.$$

Now we need to estimate $K = W_1(1)W_2(1)$. We do this by observing, that for $w \in U$ we have $\mathcal{D}w(x) \geq 1 + \frac{x}{2}$. So for $w \in U$ one can estimate $W(1) \geq \frac{5}{4}$ and

$$W_1(1)W_2(1) \geq \left(\frac{5}{4}\right)^2$$

and we obtain

$$\|\mathcal{D}w_1(x) - \mathcal{D}w_2(x)\|_\infty \leq \frac{16}{25}\|w_1 - w_2\|_\infty.$$

This completes the proof. □

Corollary 4.4 *P_n converges to v.*

Proof. $P_1 = 1 + x \in U$. □

4.2 Linearization

To obtain a better rate of convergence towards the fixed point v of $\mathcal{D}$ we look at the derivative $d\mathcal{D}(v)$ and its operator norm. We have seen that the convergence of P_n towards $v(x) := e^{\gamma x}$ can be studied by means of the contraction of the operator $\mathcal{D}$ which can be considered as an operator mapping the Banach space $C^k([0,1], \mathbb{R})$ (equipped with the C^k-norm) into itself having the Fréchet derivative

$$(d\mathcal{D}(f))(h)(x) = \frac{\int\limits_0^1 f(s)ds \cdot \int\limits_0^x h(s)ds - \int\limits_0^x f(s)ds \cdot \int\limits_0^1 h(s)ds}{\left(\int\limits_0^1 f(s)ds\right)^2}$$

For the fixed point $v(x)$ we know that

$$\int\limits_0^1 v(s)ds = \frac{1}{\gamma}, \quad \int\limits_0^x v(s)ds = \frac{e^{\gamma x} - 1}{\gamma}$$

such that – setting $A := d\mathcal{D}(v)$ –

$$(A(h))(x) = \gamma \left(\int\limits_0^x h(s)ds - (e^{\gamma x} - 1) \int\limits_0^1 h(s)ds \right).$$

We rewrite $A(h)$ in the following way:

$$A(h)(x) = \gamma \left(\int_0^x h(s)ds\,[1 - (e^{\gamma x} - 1)] - (e^{\gamma x} - 1) \int_x^1 h(s)ds \right).$$

This yields an immediate estimate

$$\begin{aligned} \|A(h)\|_\infty &\leq \gamma \left(\|x\|_\infty \|h\|_\infty + \|(e^{\gamma x} - 1)(1 - x)\|_\infty \|h\|_\infty \right) \\ &\leq \gamma \|h\|_\infty. \end{aligned}$$

With a bit more work its possible to improve this estimate considerably. First we use the intermediate value theorem to obtain for values $\xi_1 \in [0, x]$ and $\xi_2 \in [x, 1]$

$$A(h)(x) = \gamma \left(x h(\xi_1)(2 - e^{\gamma x}) - (e^{\gamma x} - 1)(1 - x) h(\xi_2) \right).$$

This allows us to estimate

$$\begin{aligned} \|A(h)\|_\infty &\leq \gamma \| \left(x(2 - e^{\gamma x}) + (e^{\gamma x} - 1)(1 - x) \right) \|_\infty \|h\|_\infty \\ &= \gamma \| \left(3x - 1 - 2x e^{\gamma x} + e^{\gamma x} \right) \|_\infty \|h\|_\infty. \end{aligned}$$

It is not easy to discuss this last expression

$$k(x) = 3x - 1 - 2x e^{\gamma x} + e^{\gamma x}$$

analytically, however it is easy to plot k to obtain a bound $\|k\|_\infty \leq 0.505$, see Figure 2. Using $\gamma \cdot 0.505 \approx 0.35$ we have

$$\|A(h)\|_\infty \leq 0.35 \|h\|_\infty.$$

From the analytic point of view, it can be said that k has one maximum, between $\frac{1}{2}$ and $\frac{2}{3}$, and that k is monotonically increasing on $[0, \frac{1}{2}]$ and $k(\frac{1}{2}) = \frac{1}{2}$. For $x \in [\frac{1}{2}, 1]$ we write

$$k(x) = 3x - 1 + (1 - 2x)2^x \leq 3x - 1 + (1 - 2x) = x$$

Observing that k is monotonically decreasing on $[2/3, 1]$ we can explicitely give a bound $\|k\|_\infty \leq \frac{2}{3}$. Altogether we get a local convergence rate which is better than any of the previous rates.

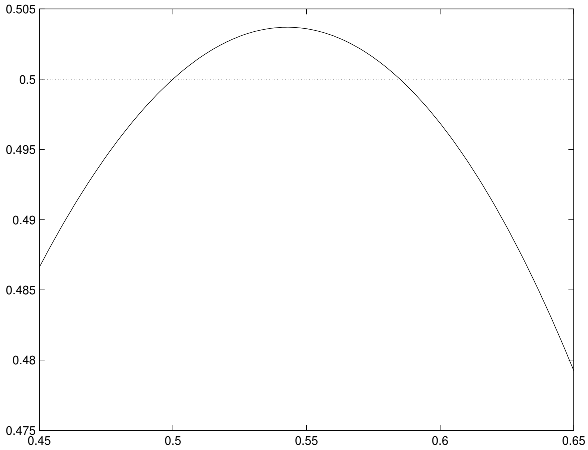

Figure 2: This depicts the graph of k indicating a bound 0.505