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A THEORY OF IMPLICIT NUMERICAL METHODS FOR SCALAR CONSERVATION LAWS

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Abstract. We present a novel theory of implicit methods for scalar conservation laws which does to a large extent not rely on classical structures in numerical analysis, e.g. the notion of total variation stability is not employed. Thereby, for the first time rigorous theoretical results concerning the range of applicability of implicit schemes are given. A new proof of convergence is given relying on monotonicity of the considered methods and on a special discretization of the initial data. The theoretical results are confirmed by numerical tests.

Key words. conservation laws, implicit methods, monotone methods

AMS subject classifications. 35L65, 65M06, 65M12

1. Introduction. Within the literature about numerical methods for conservation laws, attributes of numerical schemes like monotonicity are generally discussed for explicit schemes, see e.g. [3, 6] and the references therein. It is usually suggested that implicit schemes have the same properties as their explicit counterparts who are identified by the use of the same numerical flux function. The exception is the formal investigation of TVD-criteria formulated by Harten, see [4, 8]. However, even this piece of an implicit theory has a significant drawback: It involves the formal inversion of the finite difference operator corresponding to an investigated implicit method. Therefore, it is not very useful, especially for the analysis of implicit methods in the nonlinear case. Furthermore, Harten assumes the flux function of the conservation law to be Lipschitz continuous in order to apply the TVD convergence strategy.

This paper deals with a novel theory of implicit methods. Thereby, we do not restrict ourselves to Lipschitz continuous fluxes in order to make possible a numerical analysis which reflects the full potential of implicit schemes. Not assuming Lipschitz continuity of the fluxes means that transport processes with infinite speed are included within the range of applicability of the described theory. As a consequence we do not rely on the TVD strategy for proving convergence since the compactness of the corresponding function space relies on the property that the solution has compact support. Instead, we describe an innovative proof of convergence for which the key ingredient is a rigorously verified implicit monotonicity criterion.

This paper is organized as follows. At first, the components of the implicit theory are developed which are centered around a monotonicity property. Three numerical methods are investigated with respect to their monotonicity. The proofs of convergence to the entropy solution are illustrated for these methods. Finally, we briefly discuss some numerical results followed by conclusive remarks and acknowledgements.

2. A Theory of Implicit Methods. In the following, u_m^l denotes the value of the numerical solution at $m\Delta x$ and $l\Delta t$. The presence of an uniform grid with a constant ratio $\lambda = \Delta t/\Delta x$ is assumed, but this is only for a more transparent notation and not essential. For simplicity, implicit 3-point-methods of the form

$$u_0^{n+1} = H(u_{-1}^{n+1}, u_0^{n+1}, u_1^{n+1}; u_0^n) \tag{2.1}$$

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are discussed. However, the whole presented theory is also valid for more general $(2k+1)$-point-methods ($k \geq 2$), meaning that the stencil can especially include the values $u_{-k}^{n+1}, \ldots, u_k^{n+1}$.

2.1. Implicit Notions. Analogously to the standard theory of explicit methods, we define the conservation form of a method and the numerical flux function.

DEFINITION 2.1 (Conservation Form, Numerical Flux Function). *A numerical scheme is in conservation form if a continuous numerical flux function $g : \mathbf{R}^2 \to \mathbf{R}$ exists, so that the method reads*

$$H(u_{-1}^{n+1}, u_0^{n+1}, u_1^{n+1}; u_0^n) = u_0^n - \lambda\big[g(u_0^{n+1}, u_1^{n+1}) - g(u_{-1}^{n+1}, u_0^{n+1})\big].$$

The transfer of the notion of consistency to the implicit case is trivial.

DEFINITION 2.2 (Consistency). *A numerical method described by the numerical flux function g is consistent if $g(v, \ldots, v) = f(v)$ holds for all $v \in \mathbf{R}$.*

The key to nonlinear stability is the notion of monotonicity.

DEFINITION 2.3 (Monotonicity). *Let two data sequences v^n and w^n be given. Let the investigated consistent and conservative numerical method produce new sequences of data v^{n+1} and w^{n+1} out of the given data v^n and w^n, respectively. Then the numerical method is monotone iff the implication*

$$v^n \geq w^n \quad \Rightarrow \quad v^{n+1} \geq w^{n+1} \tag{2.2}$$

holds in the sense of the comparison of components.

Note that the Definition captures the essence of the notion of monotonicity and is free of derivatives of H. In the following, we use the abbreviations $u_{j-1}^{n+1} =: a$, $u_j^{n+1} =: b$, $u_{j+1}^{n+1} =: c$ and $u_j^n =: d$ when appropriate to simplify the notation.

THEOREM 2.1 (Monotonicity of implicit methods). *A consistent and conservative implicit 3-point-method is monotone iff the following conditions hold:*

$$H(a + \Delta a, b, c; d) \geq H(a, b, c; d), \tag{2.3}$$

$$H(a, b, c + \Delta c; d) \geq H(a, b, c; d) \tag{2.4}$$

$$\text{and} \quad H(a, b, c; d + \Delta d) \geq H(a, b, c; d). \tag{2.5}$$

Thereby, $\Delta a, \Delta c, \Delta d \ \geq 0$.

In order to give the proof of Theorem 2.1 a clearer structure, we give at first the following Lemma.

LEMMA 2.1. *Let a consistent and conservative implicit method be given which satisfies the monotonicity conditions (2.3), (2.4) and (2.5). Furthermore, let two sequences v^n and w^n be given based on the index set $\mathbf{Z}$. Then from*

$$\exists\, k \in \mathbf{Z} : \; v_k^n > w_k^n \quad \text{and} \quad \forall\, j \in \mathbf{Z}\; (j \neq k) : \; v_j^n = w_j^n$$

follows $v^{n+1} \geq w^{n+1}$ in the sense of the comparison of components.

Proof. (of Lemma 2.1)

Let $(v^n = w^n) \Rightarrow \big(v^{n+1} = w^{n+1}\big)$ follow by the application of the considered method. By the assumption of the Lemma there exists an index $k \in \mathbf{Z}$ so that $v_k^n > w_k^n$ holds. The proof of the assertion follows by induction per $l \in \mathbf{N}_0$ for the indices $k \pm l$. Thereby, $v^{n+1} = w^{n+1}$ holds for the components not touched before the induction step.

Beginning of the induction: $l = 0$
The statement is true since $H(a, b, c; v_k^n) \geq H(a, b, c; w_k^n)$ by (2.5).
Assumption:
The statement is true for arbitrary but fixed $l > 0$.
Induction step: $l \mapsto l + 1$
Let the statement be true for the subsets $(v_{k-l}^{n+1}, \ldots, v_{k+l}^{n+1})$ and $(w_{k-l}^{n+1}, \ldots, w_{k+l}^{n+1})$ of the sequences v^{n+1} and w^{n+1}, i.e. in particular it holds $v_{k-l}^{n+1} \geq w_{k-l}^{n+1}$ and $v_{k+l}^{n+1} \geq w_{k+l}^{n+1}$.

By construction, the sequences v^{n+1} and w^{n+1} are up to now identical outside the subsets $(v_{k-l}^{n+1}, \ldots, v_{k+l}^{n+1})$ and $(w_{k-l}^{n+1}, \ldots, w_{k+l}^{n+1})$. Then it holds

$$H(a, v_{k-(l+1)}^{n+1}, v_{k-l}^{n+1}; d) \geq H(a, w_{k-(l+1)}^{n+1}, w_{k-l}^{n+1}; d) \quad \text{by} \quad (2.4) \quad \text{and} \tag{2.6}$$

$$H(v_{k+l}^{n+1}, v_{k+(l+1)}^{n+1}, c; d) \geq H(w_{k+l}^{n+1}, w_{k-(l+1)}^{n+1}, c; d) \quad \text{by} \quad (2.3). \tag{2.7}$$

Using (2.1), (2.6) and (2.7) it follows $v_{k-(l+1)}^{n+1} \geq w_{k-(l+1)}^{n+1}$ and $v_{k+(l+1)}^{n+1} \geq w_{k-(l+1)}^{n+1}$. $\square$
Herewith, the preparation for the proof of Theorem 2.1 is finished.

Proof. (of Theorem 2.1)
Let two sequences v^n, w^n be given with $v^n \mapsto v^{n+1}$ and $w^n \mapsto w^{n+1}$ by application of the considered consistent and conservative numerical method. Furthermore, let $(v^n = w^n) \Rightarrow (v^{n+1} = w^{n+1})$ hold.
"'$\Rightarrow$"':
Let the method be monotone, i.e. (2.2) holds by application of the method. Let $v^n \geq w^n$ hold in the sense of comparison of components. By the assumed monotonicity of the scheme follows $v^{n+1} \geq w^{n+1}$. To verify is now the validity of the conditions (2.3), (2.4) and (2.5).
To condition (2.3):
Let (a, b, c) be an arbitrarily chosen but fixed part of an admissible sequence v^{n+1}. Assume that for $\Delta a \geq 0$ it does not hold in general

$$H(a + \Delta a, b, c; d) \quad \geq \quad H(a, b, c; d).$$

Then there exist two tupel $(a_1, b_1, c_*; d_*)$ and $(a_2, b_2, c_*; d_*)$ with $a_1 > a_2$ and

$$H(a_1, b_1, c_*; d_*) < H(a_2, b_2, c_*; d_*).$$

By (2.1) it follows that $b_1 < b_2$ has to be valid. On the other hand there is $(a_1, b_1) \geq (a_2, b_2)$ in the sense of comparison of components by the assumed monotonicity of the method, and so the assumption is wrong and the validity of (2.3) is verified.
To condition (2.4):
The proof can be done analogously.
To condition (2.5):
This condition holds by the form of the method, see (2.1).
"'$\Leftarrow$"':
To prove is the validity of (2.2). Therefore, let $\mathbf{I}$ be the set of indices k with $v_k^n > w_k^n$, $v_k^n \in v^n$, $w_k^n \in w^n$. There are only a few possibilities for the composition of $\mathbf{I}$: It may consist of the empty set or a finite or infinite subset of $\mathbf{Z}$. The proof of the assertion follows by diversion of cases and induction by content of $\mathbf{I}$.
Case: $\mathbf{I} = \mathbf{I_0} = \emptyset$
The validity of the assertion is trivial for $\mathbf{I_0} = \emptyset$ since v^n is equal to w^n in that case.

Case: $\mathbf{I} \neq \emptyset$
As already indicated, the proof is done by induction.

Beginning of the induction: $\sharp\mathbf{I_1} = 1$ (i.e. $\mathbf{I} = \mathbf{I_1}$)
Let k be the index in the arbitrarily chosen but fixed index set $\mathbf{I_1}$. Then the validity of the monotonicity condition follows by application of Lemma 2.1.

Assumption:
Consider $\sharp\mathbf{I_m} = m$ with $m > 1$, $m \in \mathbf{N}$ (i.e. $\mathbf{I} = \mathbf{I_m}$). Let the scheme under consideration be monotone with respect to the proceeding for all subsets of $\mathbf{I_m}$.

Induction step: $m \mapsto m+1$
Now, $\sharp\mathbf{I_{m+1}} = m+1$ is considered. Thereby, $\mathbf{I_m} \subset \mathbf{I_{m+1}}$ holds without limitation of generality. Let m_1 be an arbitrary but fixed index with $m_1 \in (\mathbf{I_m} \subset \mathbf{I_{m+1}})$. Moreover, let m_2 be the index by which $m_2 \in (\mathbf{I_{m+1}} \setminus \mathbf{I_m})$ holds. By the assumption of the induction, the scheme is monotone with respect to positive changes in values corresponding to the index set $\mathbf{I_m}$. This means in particular that a positive change in $v_{m_1}^n$ together with positive changes in other values v_k^n, $k \in (\mathbf{I_m} \setminus \{m_1\})$, leads to non-negative changes in the sequence v^{n+1}.

Now a simultaneous positive change in in $v_{m_1}^n$ and $v_{m_2}^n$ is considered while in the background there are arbitrary but fixed positive changes in the values corresponding to $\bar{\mathbf{I}}_{\mathbf{m}} := \mathbf{I_{m+1}} \setminus \{m_1, m_2\}$, $\bar{\mathbf{I}}_{\mathbf{m}} \subset \mathbf{I_m}$. By the assumption of the induction, the application of the considered numerical scheme has the effect that monotonicity in the sense of Definition 2.3 holds for positive changes in v_k^n, $k \in \bar{\mathbf{I}}_{\mathbf{m}}$.

Let the data computed by positive changes in v_k^n, $k \in \bar{\mathbf{I}}_{\mathbf{m}}$, be denoted by $\bar{v}^{n+1}$ (i.e. $\bar{v}^{n+1} \geq w^{n+1}$ holds by the assumption of the induction step). Moreover, let Δ_j^1 be a change in $\bar{v}_j^{n+1}$ induced by a positive change in $v_{m_1}^n$. Then Δ_j^1 is always non-negative by the assumption of the induction. Analogously, let Δ_j^2 a change in $\bar{v}_j^{n+1}$ induced by a positive change in $v_{m_2}^n$. The change Δ_j^2 is also non-negative because of the assumed validity of condition (2.5); the proof of this assertion follows analogously to the proof of Lemma 2.1.

For the mutual effects of such changes in data corresponding to an arbitrary but fixed index i, $i \notin \{m_1, m_2\}$, there are only two possibilities because of $m_1 \neq m_2$ and since (2.1) implies that it makes no sense to investigate changes in the second argument of H:

- $H(\bar{v}_{i-1}^{n+1} + \Delta_{i-1}^1, \bar{v}_i^{n+1}, \bar{v}_{i+1}^{n+1} + \Delta_{i+1}^2; v_i^n) \overset{(2.3),(2.4)}{\geq} \bar{v}_i^{n+1}$,
- $H(\bar{v}_{i-1}^{n+1} + \Delta_{i-1}^2, \bar{v}_i^{n+1}, \bar{v}_{i+1}^{n+1} + \Delta_{i+1}^1; v_i^n) \overset{(2.3),(2.4)}{\geq} \bar{v}_i^{n+1}$.

Note the arbitrary choice of m_1 and m_2 by a simultaneous change in the data corresponding to the index set $\bar{\mathbf{I}}_{\mathbf{m}}$. Since there are no limitations concerning the choice of $\mathbf{I_m}$, the proceeding as a whole is well defined and the proof is finished. □

By monotonicity, one can easily derive the following stability statement. This is not the nonlinear stability we seek, but it is needed as a technical assertion in the convergence proof.

THEOREM 2.2 (L_∞-Stability). *Let an implicit method be given which is also conservative and monotone. Then the numerical solution satisfies*

$$||u^n||_{L_\infty} \quad \leq \quad ||u_0||_{L_\infty}.$$

Proof. Let a sequence $u^n \in L_\infty$ be given. With $a_k := \inf_{j\in\mathbf{Z}}(u_j^k)$ and $b_k := \sup_{j\in\mathbf{Z}}(u_j^k)$ we define sequences a^n, a^{n+1} and b^n, b^{n+1} by $a_j^k := a_k$ and $b_j^k := b_k$ for all

$j \in \mathbf{Z}$. Since the method is conservative,

$$a_{n+1} \;=\; u_j^n - \lambda\{g(a_{n+1}, a_{n+1}) - g(a_{n+1}, a_{n+1})\} \;\geq\; a_n \tag{2.8}$$

$$\text{and} \quad b_{n+1} \;=\; u_j^n - \lambda\{g(b_{n+1}, b_{n+1}) - g(b_{n+1}, b_{n+1})\} \;\leq\; b_n \tag{2.9}$$

holds for the sequences a^{n+1} and b^{n+1}, respectively. Consequently, by the assumed monotonicity follows $(b^n \geq u^n) \Rightarrow \left(b^{n+1} \geq u^{n+1}\right)$ and $(u^n \geq a^n) \Rightarrow \left(u^{n+1} \geq a^{n+1}\right)$ which implies the L_∞-stability of the method by (2.8) and (2.9). □

Concerning convergence to the entropy solution, the following Definition is useful.

DEFINITION 2.4 (Entropy consistency). *An implicit numerical scheme is consistent with the entropy condition of Kruzhkov if there exists a continuous numerical entropy flux G which satisfies for all $l \in \mathbf{R}$ the following assertions:*

1. *Consistency with the entropy flux of Kruzhkov*

$$G(v, v; l) = F(v; l) \;\forall v \text{ with } F(v; l) = sgn(v - l)\,[f(v) - f(l)]\,. \tag{2.10}$$

2. *Validity of a discrete entropy inequality*

$$\frac{U(u_j^{n+1}; l) - U(u_j^n; l)}{\Delta t} \leq -\frac{G(u_j^{n+1}, u_{j+1}^{n+1}; l) - G(u_{j-1}^{n+1}, u_j^{n+1}; l)}{\Delta x} \tag{2.11}$$

where $U(v; l) = |v - l|$ is chosen due to Kruzhkov.

For the sake of brevity, let

$$a \vee b := \max(a, b) \quad \text{and} \quad a \wedge b := \min(a, b) \tag{2.12}$$

hold. The important connection between the numerical entropy flux G and the numerical flux function g is now established.

LEMMA 2.2. *Let a consistent and conservative numerical scheme be given with a numerical flux function g. Then the numerical entropy flux defined by*

$$G(v, w; l) \;:=\; g(v \vee l, w \vee l) - g(v \wedge l, w \wedge l)$$

is consistent with the entropy flux of Kruzhkov.

Proof. Because the numerical scheme is consistent and conservative, the statement

$$G(v, v; l) = g(v \vee l, v \vee l) - g(v \wedge l, v \wedge l) = sgn(v - l)[f(v) - f(l)]$$

holds by use of (2.12) for all $l \in \mathbf{R}$. □

Now follows one of the main assertions described in this paper. The proceeding within the proof is a variation of a procedure employed by Crandall and Majda [2].

THEOREM 2.3. *Let an implicit scheme be given which is also consistent, conservative and monotone. Then the scheme is also consistent with the entropy condition of Kruzhkov.*

Proof. Since the method is assumed to be consistent and conservative, there exists a numerical flux function g so that one can construct a numerical entropy flux by Lemma 2.2. Thereby, the consistency with the entropy flux of Kruzhkov is given. It is left to show the validity of a discrete entropy inequality in the sense of (2.11). Therefore, let $l \in \mathbf{R}$ be chosen arbitrarily but fixed. Using (2.12), one can derive

$$\begin{aligned}
&-\lambda\big\{G(u_j^{n+1}, u_{j+1}^{n+1}; l) - G(u_{j-1}^{n+1}, u_j^{n+1}; l)\big\} = \\
&H\left(u_{j-1}^{n+1} \vee l, u_j^{n+1} \vee l, u_{j+1}^{n+1} \vee l; u_j^n \vee l\right) \\
&-H\left(u_{j-1}^{n+1} \wedge l, u_j^{n+1} \wedge l, u_{j+1}^{n+1} \wedge l; u_j^n \wedge l\right) - \left|u_j^n - l\right|.
\end{aligned} \tag{2.13}$$

One can easily show the validity of

$$H\left(u_{j-1}^{n+1} \vee l, u_j^{n+1} \vee l, u_{j+1}^{n+1} \vee l; u_j^n \vee l\right) \;\geq\; u_j^{n+1} \vee l \tag{2.14}$$

$$\text{and} \quad H(u_{j-1}^{n+1} \wedge l, u_j^{n+1} \wedge l, u_{j+1}^{n+1} \wedge l; u_j^n \wedge l) \quad \leq \quad u_j^{n+1} \wedge l. \tag{2.15}$$

For the proof of (2.14) and (2.15) the monotonicity conditions (2.3), (2.4) and (2.5) have to be used. The assertions then follow straightforwardly by diversion of the cases $u_j^{n+1} \geq l$ and $u_j^{n+1} < l$; see [1] for details. Using $|a-b| = a \vee b - a \wedge b$, one can derive from (2.13) via the estimates (2.14) and (2.15) the desired discrete entropy inequality. □

3. Examples of Implicit Methods. We now investigate three implicit numerical schemes. Since the convergence proofs are rather technical and lengthy, convergence to the entropy solution is illustrated in sufficient detail for only one method while we hint at the required procedure in the other cases. For more details, see [1].

3.1. The implicit Upwind method. The implicit formulation of the Upwind scheme reads

$$u_j^{n+1} \;=\; u_j^n - \lambda\big[f(u_j^{n+1}) - f(u_{j-1}^{n+1})\big]. \tag{3.1}$$

Monotonicity. We investigate the method (3.1) with respect to monotonicity. Again the arguments a, b, c and d are used in order to enhance the readability.
To condition (2.3):

$$\begin{aligned} &H(a+\Delta a, b, c; d) - H(a, b, c; d) = \\ &\big[d - \lambda\big[f(b) - f(a+\Delta a)\big]\big] - \big[d - \lambda\big[f(b) - f(a)\big]\big] = \lambda\left[f(a+\Delta a) - f(a)\right]. \end{aligned} \tag{3.2}$$

The condition (2.3) is only valid if f grows monotoneously.
To condition (2.4):

$$\begin{aligned} &H(a, b, c+\Delta c; d) - H(a, b, c; d) = \\ &\big[d - \lambda\big[f(b) - f(a)\big]\big] - \big[d - \lambda\big[f(b) - f(a)\big]\big] = 0 \;\; (\geq 0). \end{aligned} \tag{3.3}$$

To condition (2.5):

$$\begin{aligned} &H(a, b, c; d+\Delta d) - H(a, b, c; d) = \\ &\big[d + \Delta d - \lambda\big[f(b) - f(a)\big]\big] - \big[d - \lambda\big[f(b) - f(a)\big]\big] = \Delta d \;\; (\geq 0). \end{aligned} \tag{3.4}$$

Because of (3.2), (3.3) and (3.4) the implicit Upwind scheme is monotone if f grows monotoneously. Note that f does not need to be Lipschitz continuous.

Convergence. Now we sketch the proof of convergence. The basic idea is the following: Corresponding to a sequence $\Delta x_k \downarrow 0$, we construct a monotoneously growing sequence of discrete initial data. Then by the monotonicity of the method we get a monotoneously growing sequence of numerical solutions. Since we multiply the initial function u_0 with an arbitrarily chosen but fixed testfunction with compact support, we only have to consider u_0 over a finite domain. Because of the assumption $u_0 \in L_\infty$ and since we have L_∞-Stability, the corresponding function sequence is integrable and bounded from above. Then we can use the well-known theorem of monotone convergence of Beppo Levi to show convergence (almost everywhere) to a limit function. More formally, we state the following

THEOREM 3.1 (Convergence of the implicit Upwind method). *Let $u_0(x)$ be in $\in L_\infty^{loc}(\mathbf{R})$. Let the flux function of a given conservation law grow monotoneously and be at least continuous. Consider a sequence of nested grids indexed by $k = 1, 2, \ldots$, with mesh parameters $\Delta t_k, \Delta x_k \to 0$ as $k \to \infty$, and let $u_k(x,t)$ be the numerical approximation obtained via the implicit Upwind scheme. Then u_k converges to the unique entropy solution of the given conservation law as $k \to \infty$.*

Proof. At first, the convergence to a weak solution of the conservation law is established, followed by the verification that this weak solution is the entropy solution. For brevity of the notation, we omit the arguments (x,t) when appropriate.

The most important technical detail is the special discretization of the initial condition $u_0 \in L_\infty^{loc}(\mathbf{R})$. We use nested grids in order to compare data sets of values, i.e. refined grids always inherit cell borders. Without loss of generality, we assume that $u_0 \geq 0$ holds in the sense of pointwise comparison.

After a suitable manipulation of u_0 on a set of Lebesgue measure zero if necessary, the initial condition is discretized on cell j by

$$u_j^0 := \inf_{x \in (\Delta x_0(j-1), j\Delta x_0]} u_0(x). \tag{3.5}$$

Corresponding to the initial data we also define a piecewise continuous function

$$u_k(x,0) := u_j^0 \quad \text{for} \quad (j-1)\Delta x_k < x \leq j\Delta x_k. \tag{3.6}$$

It is a simple matter of classical analysis not described in this paper to verify that the discretization (3.5) together with (3.6) gives on any compact intervall on the x-axis a monotoneously growing function sequence with

$$\lim_{k\to\infty} u_k(x,0) \;=\; u_0(x) \text{ almost everywhere} \tag{3.7}$$

by application of the theorem of monotone convergence. Similarly, we extract discrete test elements ϕ_j^0 out of a given test function $\phi \in C_0^\infty(\mathbf{R} \times \mathbf{R}_+)$ by setting

$$\begin{aligned} \phi_j^0 \;&:=\; \phi(x_j, 0) \quad \text{and} \\ \phi_k(x,0) &= \phi_j^0 \quad \text{for} \quad x_j - \Delta x_k < x \leq x_j. \end{aligned} \tag{3.8}$$

Also, we define for $n \geq 1$ the step function

$$u_k(x,t) = u_j^n \quad \text{for} \quad x_j - \Delta x_k < x \leq x_j \quad \text{and} \quad t^{n-1} < t \leq t^n.$$

Let the test function ϕ be chosen arbitrarily but fixed. Multiplication of the implicit Upwind scheme (3.1) with $\Delta x_k \Delta t_k$ as well as with the discrete test element ϕ_j^{n+1}, summation over $j \in \mathbf{Z}$ and $n \geq 0$ and finally summation by parts yields

$$\Delta x_k \Delta t_k \sum_{j\in\mathbf{Z}} \sum_{n\geq 0} \left[u_j^n \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t_k} + f(u_j^{n+1}) \frac{\phi_{j+1}^{n+1} - \phi_j^{n+1}}{\Delta x_k} \right] = -\Delta x_k \sum_{j\in\mathbf{Z}} u_j^0 \phi_j^0. \tag{3.9}$$

By (3.6) and (3.8), the equation (3.9) is equivalent to

$$\begin{aligned} -\int_{\mathbf{R}} u_k(x,0)\phi_k(x,0)\,dx = \int_{\mathbf{R}_+} \int_{\mathbf{R}} \Bigg[u_k(x,t) \frac{\phi_k(x, t+\Delta t_k) - \phi_k(x,t)}{\Delta t_k} \\ + f(u_k(x, t+\Delta t_k)) \frac{\phi_k(x+\Delta x_k, t+\Delta t_k) - \phi_k(x, t+\Delta t_k)}{\Delta x_k} \Bigg] \,dxdt. \end{aligned} \tag{3.10}$$

We now prove convergence of (3.10) to the form which implies that u is a weak solution of the original problem, namely

$$\int_0^\infty \int_{-\infty}^\infty [u\phi_t + f(u)\phi_x] dxdt = -\int_{-\infty}^\infty u_0\phi_0 dx.$$

We first investigate the left hand side of (3.9). Therefore, let

$$K_\phi := [support(\phi) \cap \{(x,t) \,|\, t = 0\}] \quad \text{and}$$
$$K := \{x \,|\, \exists y \in K_\phi : \, y - \Delta x_0 \leq x \leq y + \Delta x_0\}.$$

By construction, K is compact and gives the largest possible interval on the x-axis where non-zero discrete data may occur. Adding zeroes, we now cast the problem into a more suitable form, namely

$$\int_{\mathbf{R}} u_k(x,0)\phi_k(x,0)\, dx = \int_K u_0(x)\phi(x,0)\, dx$$
$$+\int_K u_k(x,0)\left[\phi_k(x,0) - \phi(x,0)\right]\, dx + \int_K \left[u_k(x,0) - u_0(x)\right]\phi(x,0)\, dx. \tag{3.11}$$

Because of $u_0 \in L^\infty(\mathbf{R})$ and by our construction (3.5) and (3.6), we can estimate the absolute of the first term of (3.11) by the help of a constant $M_u < \infty$:

$$\left| \int_K u_k(x,0)\left[\phi_k(x,0) - \phi(x,0)\right]\, dx \right| \leq M_u \int_K \left|\phi_k(x,0) - \phi(x,0)\right|\, dx. \tag{3.12}$$

Since ϕ is a smooth testfunction, it is a simple but technical exercise to show

$$\sup_{x\in K} |\phi_k(x,0) - \phi(x,0)| \overset{k\to\infty}{\longrightarrow} 0,$$

i.e. $||\phi_k(x,0) - \phi(x,0)||_\infty \to 0$ for $k \to \infty$. Using this, we derive from (3.12)

$$M_u \int_K \left|\phi_k(x,0) - \phi(x,0)\right|\, dx \leq M_u|K| \sup_{x\in K} \left|\phi_k(x,0) - \phi(x,0)\right|. \tag{3.13}$$

The estimations (3.12) and (3.13) imply that the Limes for $k \to \infty$ of the investigated term goes to zero. Since ϕ is continuous, we can estimate the absolute of the second term of (3.11) with the help of a constant $M_\phi < \infty$ by

$$\left| \int_K \left[u_k(x,0) - u_0(x)\right]\phi(x,0)\, dx \right| \leq M_\phi \int_K \left|u_k(x,0) - u_0(x)\right|\, dx. \tag{3.14}$$

By our construction (3.5), (3.6), $u_k(x,0)$ approaches $u_0(x)$ from below, i.e.

$$M_\phi \int_K \left|u_k(x,0) - u_0(x)\right|\, dx = M_\phi \int_K u_0(x) - u_k(x,0)\, dx \quad \forall k. \tag{3.15}$$

Using the theorem of monotone convergence and (3.7) implies that

$$\int_K u_0(x) - u_k(x,0)\, dx \tag{3.16}$$

vanishes in the limit for $k \to \infty$. Because of (3.14), (3.15) and (3.16), the second term in (3.11) goes to zero for $k \to \infty$. To condense these results, we get by the described procedure

$$\lim_{k\to\infty} \int_{\mathbf{R}} u_k(x,0)\phi_k(x,0)\,dx = \int_{\mathbf{R}} u_0(x)\phi(x,0)\,dx. \tag{3.17}$$

By analogously defining a compact domain S including the support of ϕ in space and time, adding zeroes and using the attributes of test functions, it remains to show

$$\lim_{k\to\infty} \int_S \big|u(x,t) - u_k(x,t)\big|\,|\phi_t(x,t)|\;dxdt \;\overset{k\to\infty}{\longrightarrow}\; 0 \quad \text{and} \tag{3.18}$$

$$\lim_{k\to\infty} \int_S \big|f(u(x,t)) - f(u_k(x,t+\Delta t_k))\big|\,|\phi_x(x,t)|\;dxdt \;\overset{k\to\infty}{\longrightarrow}\; 0 \tag{3.19}$$

in order to prove convergence to a weak solution. Since ϕ_t is continuous on S, we can estimate $|\phi_t|$ in (3.18) by a constant $M_t < \infty$. Since $u_k(x,t)$ grows monotoneously with $k \to \infty$ in the sense of pointwise comparison, and since it is positive and bounded from above because of $u_0 \in L_\infty(S)$ and the monotonicity of the method, the function sequence $(u_k(x,t))_{k\in\mathbf{N}}$ converges almost everywhere to an integrable limit function on S by the theorem of monotone convergence due to Levi. We set

$$u(x,t) \;:=\; \lim_{k\to\infty} u_k(x,t).$$

Introducing exactly this limit function as the function $u(x,t)$ used up to now, the corresponding term in (3.18) becomes zero in the limit:

$$\lim_{k\to\infty} \int_S \big|u(x,t) - u_k(x,t)\big|\,|\phi_t(x,t)|\;dxdt \le M_t \int_S u(x,t) - \lim_{k\to\infty} u_k(x,t)\,dxdt = 0.$$

Note that the pointwise convergence $u_k \to u$ almost everywhere is now established and can be used in the following proofs. For proving (3.19), we need some further manipulations. We estimate $|\phi_x|$ from above by a constant $M_x < \infty$, add zero and use the triangle inequality to obtain

$$\begin{aligned}
&\int_S \big|f(u(x,t)) - f(u_k(x,t+\Delta t_k))\big|\,|\phi_x(x,t)|\;dxdt \le \\
&M_x \int_S |f(u_k(x,t)) - f(u(x,t))|\;dxdt \\
&+M_x \int_S |f(u_k(x,t+\Delta t_k)) - f(u_k(x,t))|\;dxdt.
\end{aligned} \tag{3.20}$$

We now discuss the first term in (3.20). Since by construction u_k and u are in $L_\infty(S)$, we can estimate $|f(u_k(x,t+\Delta t_k)) - f(u_k(x,t))|$ from above by a constant $M_f < \infty$ because of the continuity of f on the compact set of possible values. Then the function

$$M_f(x,t) := \left\{ \begin{array}{ccc} M_f & : & (x,t) \in S \\ 0 & : & else \end{array} \right. ,$$

is in $L_1(\mathbf{R} \times \mathbf{R}_+)$ and dominates $|f(u_k(x,t)) - f(u(x,t))|$ for all k. Since we have already shown pointwise convergence $u_k \to u$ a.e., we can apply the Theorem of dominated convergence by Lebesgue to obtain

$$\lim_{k\to\infty} M_x \int_S |f(u_k(x,t)) - f(u(x,t))| \, dxdt = \tag{3.21}$$

$$M_x \int_S \lim_{k\to\infty} |f(u_k(x,t)) - f(u(x,t))| \, dxdt = 0. \tag{3.22}$$

Now we discuss the second term in (3.20). Since by construction u_k is a step function with finite values on the compact domain S, u_k is in $L_1(S)$. Since f is continuous, then also $f \circ u_k$ is in $L_1(S)$. By the continuity in the mean of L_1-functions, there exists a $\delta(\epsilon)$ for all $\epsilon > 0$ with

$$\int_S |f(u_k(x,t+\Delta t_k)) - f(u_k(x,t))| \, dxdt \; < \; \epsilon$$

if $\Delta t_k < \delta(\epsilon)$. Since $\Delta t_k \downarrow 0$ for $k \to \infty$, ϵ can be chosen arbitrarily small, i.e. it holds

$$M_x \int_S |f(u_k(x,t+\Delta t_k)) - f(u_k(x,t))| \, dxdt \; \to \; 0 \;\; \text{for} \;\; k \to \infty. \tag{3.23}$$

By (3.22) and (3.23) the assertion (3.19) is proven. Since the test element ϕ was chosen arbitrarily, convergence to a weak solution is established.

We have now to show that exactly this weak solution is the unique entropy solution in the sense of Kruzhkov.

In a similar fashion as in the derivation of the weak form of the numerical method (3.9), we derive the weak form of the discrete entropy condition (2.10) connected with the implicit Upwind scheme using Lemma 2.1 and Theorem 2.3.

The weak form of the discrete entropy condition composed via the implicit Upwind scheme reads

$$-\Delta x \sum_{j\in\mathbf{Z}} |u_j^0 - l|\phi_j^0 \; \le \; \Delta x \Delta t \sum_{j\in\mathbf{Z}} \sum_{n\ge 0} \Bigg[|u_j^n - l| \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} \tag{3.24}$$

$$+ sgn\left(u_j^{n+1} - l\right) \left[f(u_j^{n+1}) - f(l) \right] \frac{\phi_{j+1}^{n+1} - \phi_j^{n+1}}{\Delta x} \Bigg]. \tag{3.25}$$

Using then the established convergence $u_k \to u$ a.e. of the function sequence generated by the numerical method for $\Delta t_k, \Delta x_k \downarrow 0$, we now prove convergence of (3.25) towards the weak form of the entropy condition due to Kruzhkov

$$\int_0^\infty \int_{-\infty}^\infty \Big[|u - l|\phi_t + sgn\,(u - l)\,[f(u) - f(l)]\,\phi_x \Big] dxdt \ge -\int_{-\infty}^\infty |u_0 - l|\phi_0 dx.$$

Therefore, we have to consider arbitrarily chosen but fixed test elements which consist of tupels (ϕ, l) composed of a test function ϕ with $\phi \ge 0$, $\phi \in C_0^\infty(\mathbf{R} \times \mathbf{R}_+)$, and a test number $l \in \mathbf{R}$.

Using the same notation and applying a similar procedure as in the case of the convergence proof to a weak solution, we first have a look at the expression

$$M_\phi \int_K \big| |u_k(x,0) - l| - |u_0(x) - l| \big| \, dx.$$

Since l is fixed and $u_k(x,0)$ and u_0 bounded, one can find a constant function over the compact intervall K which dominates $\big||u_k(x,0) - l| - |u_0(x) - l|\big|$ for all k. Then we use the already established convergence $u_k(x,0) \to u_0(x)$ a.e. and the Theorem of dominated convergence by Lebesgue to prove

$$\lim_{k\to\infty} M_\phi \int_K \big| |u_k(x,0) - l| - |u_0(x) - l| \big| \, dx = 0.$$

The other terms we have to look for are

$$\int_S |u_k(x,t) - l| \, \phi_t(x,t) \, dxdt \quad \text{and}$$
$$\int_S sgn \left[u_k(x,t+\Delta t_k) - l\right] \left[f(u_k(x,t+\Delta t_k)) - f(l)\right] \phi_x(x,t) \, dxdt.$$

The procedure is the same in both cases. Since the occuring derivatives of ϕ are continuous, we can estimate these over the compact domain S by finite constants. Since l is a fixed value (and so is $f(l)$), since $u_k(x,t)$ is bounded and f is continuous over the bounded intervall of possible values of u_k, we can also give constants which estimate all the expressions in u_k from above. Using the product of these finite constants as dominating function over S as well as $u_k \to u$ a.e., we employ again the Theorem of dominated convergence to receive the desired result. □

3.2. The implicit Lax-Friedrichs method. We investigate the implicit Lax-Friedrichs scheme

$$u_j^{n+1} = u_j^n + \frac{1}{2}\left[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right] - \frac{\lambda}{2}\left[f(u_{j+1}^{n+1}) - f(u_{j-1}^{n+1})\right]. \tag{3.26}$$

Monotonicity. Note that condition (2.5) is automatically satisfied.
To condition (2.3):

$$H(a+\Delta a, b, c; d) - H(a,b,c;d) = \frac{1}{2}\Delta a + \frac{\lambda}{2}\left[f(a+\Delta a) - f(a)\right]. \tag{3.27}$$

This expression is not positive or equal to zero without additional requirements.
To condition (2.4):

$$H(a,b,c+\Delta c; d) - H(a,b,c;d) = \frac{1}{2}\Delta c - \frac{\lambda}{2}\left[f(c+\Delta c) - f(c)\right]. \tag{3.28}$$

Again this expression is not positive or equal to zero without additional requirements. The mentioned requirements established by (3.27) and (3.28) can be combined to

$$|f(x+\Delta x) - f(x)| \;\le\; \frac{1}{\lambda}\Delta x.$$

Therefore, the implicit Lax-Friedrichs scheme is monotone only for Lipschitz-continuous flux functions with a Lipschitz constant $L \le (1/\lambda)$. This is a quite surprising result because it is normally suggested that the numerical characteristics include the whole domain in the case of implicit methods. Because of this, we discuss the situation for the case of the linear advection equation. In the case of a linear flux $f(u) = au$, $a \in \mathbf{R}$, the nonlinear system defined by (3.26) degenerates to a linear system. This linear system for the unknowns u_k^{n+1} is given through

$$\left[-\frac{1}{2} - a\frac{\lambda}{2}\right] u_{j-1}^{n+1} + 2u_j^{n+1} + \left[-\frac{1}{2} + a\frac{\lambda}{2}\right] = u_j^n. \tag{3.29}$$

Let an initial function $u_0(x)$ be given by

$$u_0(x) = \begin{cases} 0 & : \quad x < -1 \\ 1 & : \quad -1 \le x \le 0 \\ 0 & : \quad x > 0. \end{cases} \tag{3.30}$$

We have to investigate the structure of the tridiagonal matrix

$$(a_{ij}) := \begin{pmatrix} 2 & \frac{a\lambda-1}{2} & & & & \\ \frac{-a\lambda-1}{2} & 2 & \frac{a\lambda-1}{2} & & \mathbf{0} & \\ & \frac{-a\lambda-1}{2} & 2 & \frac{a\lambda-1}{2} & & \\ & \cdots & \cdots & \cdots & & \\ \mathbf{0} & & \frac{-a\lambda-1}{2} & 2 & \frac{a\lambda-1}{2} \\ & & & \frac{-a\lambda-1}{2} & 2 \end{pmatrix} \tag{3.31}$$

defined by (3.29). Therefore, let a be positive with $a > (1/\lambda)$ without loss of generality. Note that the entries in the lower diagonal $a_{i+1,i}$ always take on negative values while the entries in the upper diagonal $a_{i,i+1}$ are always positive with this assumption.

If we apply the usual procedure for solving the arising linear system with the tridiagonal matrix (3.31), we at first eliminate the entries in the lower diagonal $a_{i+1,i}$. The diagonal entries of the matrix have to be modified accordingly, i.e. the diagonal entry in the i-th row is modified via

$$a_{ii}^{new} = a_{ii}^{old} - \frac{a_{i,i-1}}{a_{i-1,i-1}} a_{i-1,i}.$$

Thereby, note that we always have the situation

$$a_{i,i-1} < 0,\ a_{i-1,i-1} > 0 \ \text{ and } \ a_{i-1,i} > 0,$$

so that $a_{ii}^{new} > a_{ii}^{old}$ is always satisfied. Since the right hand side (b_i) of the investigated system featuring the given data is modified via

$$b_i = u_i^n - \frac{a_{i,i-1}}{a_{i-1,i-1}} b_{i-1},$$

data sets with $u_k^n \ge 0\ \forall k$ imply only positive possible changes in the values b_i. Note that the values in the upper diagonal $a_{i,i+1}$ remain unchanged and positive which will play a crucial role in the following.

We now investigate what happens at a jump in the given data u_k^n from values 0 to 1 as in the discretized Riemann-problem (3.30) when backward elimination is applied in order to solve our system. Therefore, let us fix $u_j^n := 0 \forall j < i$ and $u_j^n := 1 \forall j \ge i$. By the described procedure, it is clear that the corresponding entries on the right hand side also show a jump from 0 to 1 after the modification due to elimination of the lower diagonal since $b_{i-1} = u_{i-1}^n = 0$ so that no positive update in b_i takes place. Particularly, backward elimination results in

$$u_{i-1}^{n+1} = \underbrace{\frac{1}{a_{i-1,i-1}^{new}}}_{>0} \Big(\underbrace{u_{i-1}^n}_{=0} - \underbrace{a_{i-1,i}}_{>0} \underbrace{u_i^n}_{=1} \Big) < 0,$$

so that the monotonicity is violated, as expected. The same behaviour can also be observed at jumps to lower values, see Figure 3.1.

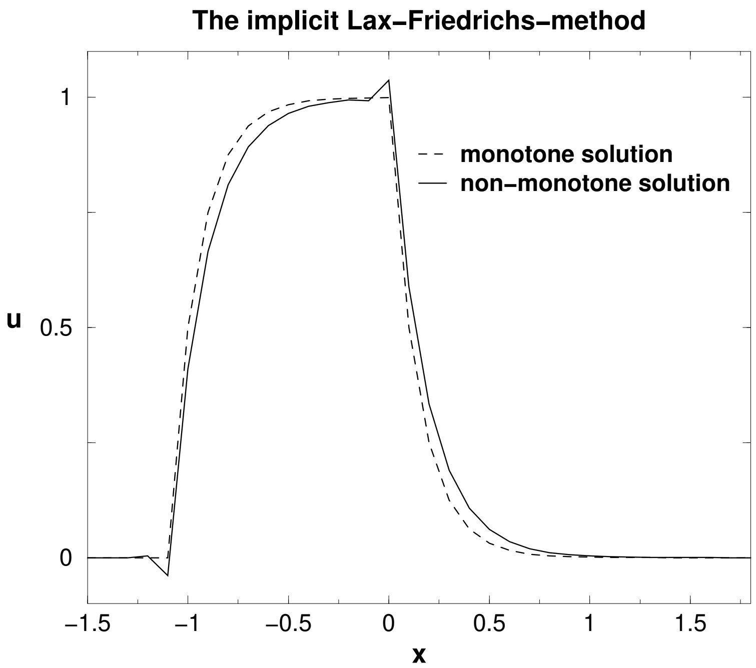

FIG. 3.1. *Numerical solutions of the linear advection equation using the initial data defined via (3.30). For simplicity, we set $\Delta x = \Delta t = 0.1$. For the displayed solutions, only one time step is computed with the implicit Lax-Friedrichs method, once with $a = 1$ – satisfying the monotonicity condition – and once with $a = 1.5$, giving monotonicity-violating oscillations near the original jumps in the Riemann-problem. The same oscillatory behaviour is given for smaller $a > 1$, resulting in lesser amplitudes than for $a < 1.5$ which was chosen in order to show a reasonable figure.*

Convergence. For shortness of the presentation, we use results from the convergence proof already done for the implicit Upwind method.

THEOREM 3.2 (Convergence of the implicit Lax-Friedrichs method). *Let $u_0(x)$ be in $\in L_\infty^{loc}(\mathbf{R})$. Let the flux function of a given conservation law be Lipschitz continuous with a Lipschitz constant lesser or equal to $1/\lambda$. Consider a sequence of nested grids indexed by $k = 1, 2, \ldots$, with mesh parameters $\Delta t_k, \Delta x_k \to 0$ as $k \to \infty$, and let $u_k(x,t)$ be the numerical approximation obtained via the implicit Lax-Friedrichs scheme. Then u_k converges to the unique entropy solution of the given conservation law as $k \to \infty$.*

Proof. Because of the assumed Lipschitz continuity with a Lipschitz constant $L \leq 1/\lambda$ of the flux function, the method is monotone.

With respect to the implicit Upwind method, the difference in the corresponding weak forms reads

$$(3.32)\quad -\frac{\Delta t_k}{2}\int_S u_k(x, t+\Delta t_k)\hat{\phi}\,dxdt \text{ and } -\frac{\Delta t_k}{2}\int_S |u_k(x,t+\Delta t_k) - l|\,\hat{\phi}\,dxdt,$$

respectively. Thereby, $\hat{\phi}$ converges in the L_∞-Norm to $\phi_{xx}(x,t)$ which is continuous since $\phi \in C_0^\infty(\mathbf{R}\times\mathbf{R}_+)$. Thus, the corresponding term can be estimated from above by a constant over the compact domain S. Since l is fixed and $u_k(x,t)$ is bounded as usual, the expressions in (3.32) vanish with $\Delta t_k \downarrow 0$. ☐

3.3. An implicit Godunov-type method. In the scalar case, a closed form of the exact solution of a Riemann-problem was described by Osher [7]. Using this, a numerical scheme can be defined via the numerical flux function

$$g_G(v,w) \;=\; \begin{cases} \min_{v\le u\le w} f(u) & : \quad v \le w, \\ \max_{w\le u\le v} f(u) & : \quad v > w. \end{cases}$$

Monotonicity. Since the relative values of the test variables have to be compared within the scheme, diversions by cases have to be employed. The validity of (2.5) is again trivial.

To condition (2.3):

Generally and before any diversions of cases,

$$H(a+\Delta a,b,c;d) - H(a,b,c;d) = \lambda\big[g_G(a+\Delta a,b) - g_G(a,b)\big]$$

holds. Since only the values b, a and $a+\Delta a$ are of importance, it is necessary to investigate three cases.

1. Case: $b \le a \le a+\Delta a$

$$\lambda\big[g_G(a+\Delta a,b) - g_G(a,b)\big] = \lambda\left[\max_{b\le u\le a+\Delta a} f(u) - \max_{b\le u\le a} f(u)\right] \ge 0.$$

2. Case: $a \le b \le a+\Delta a$

$$\lambda\big[g_G(a+\Delta a,b) - g_G(a,b)\big] = \lambda\left[\max_{b\le u\le a+\Delta a} f(u) - \min_{a\le u\le b} f(u)\right] \ge 0.$$

3. Case: $a \le a+\Delta a \le b$

$$\lambda\big[g_G(a+\Delta a,b) - g_G(a,b)\big] = \lambda\left[\min_{a+\Delta a\le u\le b} f(u) - \min_{a\le u\le b} f(u)\right] \ge 0.$$

This investigation shows that the validity of the condition (2.3) is guaranteed without any additional condition on the flux function. This can be verified analogously with respect to condition (2.4), so that the investigated Godunov-type scheme is monotone for general continuous flux functions.

Convergence. We state the following

THEOREM 3.3 (Convergence of the implicit Godunov-type method). *Let $u_0(x)$ be in $\in L^{loc}_\infty(\mathbf{R})$. Let the flux function of a given conservation law be continuous. Consider a sequence of nested grids indexed by $k = 1,2,\ldots$, with mesh parameters $\Delta t_k, \Delta x_k \to 0$ as $k \to \infty$, and let $u_k(x,t)$ be the numerical approximation obtained via the described implicit Godunov-type scheme. Then u_k converges to the unique entropy solution of the given conservation law as $k \to \infty$.*

Proof. Using a similar procedure as in the case of the implicit Lax-Friedrichs methods, we write down the differences in the weak forms to the weak forms occuring in the case of the implicit Upwind method. These are

$$\big[g_G(u_j^{n+1},u_{j+1}^{n+1}) - f(u_j^{n+1})\big]\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x_k} \quad \text{and} \tag{3.33}$$

$$\Big(\big[g_G(u_j^{n+1}\vee l, u_{j+1}^{n+1}\vee l) - g_G(u_j^{n+1}\wedge l, u_{j+1}^{n+1}\wedge l)\big] \\ -sgn\left(u_j^{n+1}-l\right)\big[f(u_j^{n+1}) - f(l)\big]\Big)\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x_k}. \tag{3.34}$$

Since g_G is continuous in the components and $u_k(x,t) \in L_1(S)$, $g_G \circ u_k$ is also in $L_1(S)$. After introducing step functions as usual, the expression incorporating g_G in (3.33) gives values $f(\xi)$ with

$$\begin{array}{ll} & \xi \in [u_k(x, t + \Delta t_k), u_k(x + \Delta x_k, t + \Delta t_k)] \\ \text{or} & \xi \in [u_k(x + \Delta x_k, t + \Delta t_k), u_k(x, t + \Delta t_k)]\,, \end{array}$$

respectively. The integrals over the terms corresponding to (3.33) then go to zero with $k \to \infty$ because of the continuity in the mean of $g_G \circ u_k$. The idea for proving convergence to zero concerning the integral of the expressions corresponding to (3.34) is the same. ☐

4. Numerical Tests. In order to show the power of the novel theory, we investigate a truly non-standard conservation law developed by Kruzhkov and Panov [5]. The conservation law under consideration is definde by the flux function

$$f(u) = \frac{|u|^\alpha}{\alpha} \text{ with } \alpha \in (0,1),$$

i.e. f features a pole at $u = 0$ which means that information is propagated with infinite speed. We take the initial condition

$$u_0(x) = \begin{cases} 0 & : \quad x < -1, \\ 1 & : \quad -1 \le x \le 0, \\ 0 & : \quad x > 0. \end{cases}$$

The exact solution is given in [5] and reads

$$u(x,t) = \begin{cases} 0 & : \quad t > \alpha(x+1), \\ 1 & : \quad x < t \le \alpha(x+1), \\ \left(\dfrac{t}{x}\right)^{1/(1-\alpha)} & : \quad t \le x. \end{cases}$$

Note the rarefaction wave which expands to infinity after arbitrarily small time. For the numerical solution we use a grid of 1000 points with $\Delta x = 0.005$ and $\Delta t = 0.00075$. We set $\alpha = 0.5$ within the flux function described above. The arising nonlinear systems of equations are solved by very simple iterative methods, namely by bisection in the case of the implicit Upwind scheme and by a simple relaxation procedure in the case of the implicit Godunov-type method, see [1] for details.

As expected, the implicit Lax-Friedrichs scheme does not give a sensible solution because the considered problem does not feature a Lipschitz continuous flux function.

Since the numerical solution of the other two schemes are nearly identical (which could be expected since the Godunov-type method can be interpreted as a generalized Upwind method with entropy fix [6]), we only display exemplarily one solution obtained by the implicit Godunov-type scheme, see Figure 4.1. The ansatz of the rarefaction wave is a bit smeared which can be explained by the monotonicity property (i.e. the method is of first order) and the employed relatively coarse grid.

5. Conclusive Remarks. We have described the fundament of a theory of implicit methods for conservation laws, extending the rigorously verified range of applicability of numerical methods. By numerical experiments we have shown the validity and usefulness of our theoretical results.

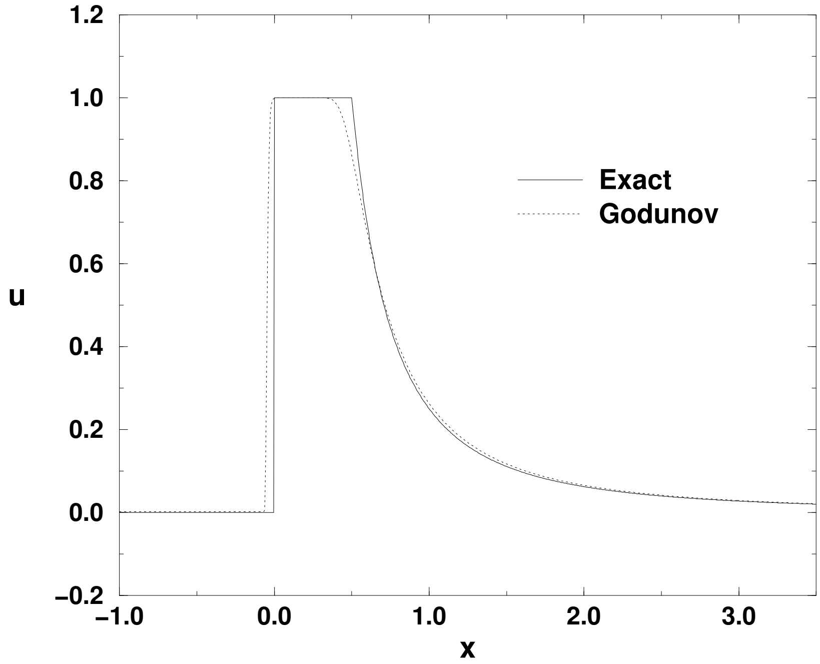

FIG. 4.1. *Numerical approximation of the described example due to Kruzhkov, obtained by the implicit Godunov-type method.*

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REFERENCES

[1] M. BREUSS, *Numerical Methods for Conservation Laws in Non-Standard-Situations*, Doctoral Thesis, University of Hamburg, Germany, in german.
[2] M. G. CRANDALL, A. MAJDA, *Monotone difference approximations for scalar conservation laws*, Math. Comp., **34** (1980), pp. 1–21.
[3] E. GODLEWSKI, P.-A. RAVIART, *Hyperbolic systems of conservation laws*, Edition Marketing (1991).
[4] A. HARTEN, *On a class of high resolution total-variation-stable finite-difference schemes*, SIAM J. Num. Anal., **21** (1984), pp. 1–23.
[5] S. N. KRUZHKOV, E. Y. PANOV, *Conservative quasilinear first-order laws with an infinite domain of dependence on the initial data*, Soviet Math. Doklady, **42** (1991), pp. 316–321.
[6] R. J. LEVEQUE, *Numerical Methods for Conservation Laws*, Birkh"auser Verlag, 2nd Edition (1992).
[7] S. OSHER, *Riemann solvers, the entropy condition, and difference approximations*, SIAM J. Num. Anal., **21** (1984), pp. 217–235.
[8] H. C. YEE, *A class of high-resolution explicit and implicit shock-capturing methods*, In: High Resolution (Upwind and TVD) Methods for the Compressible Flow Equations, VKI (1994).