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- A Principle in Discretization Procedures -

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Continuous Convergence of Relations
– A Principle in Discretization Procedures –

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There are situations in Numerical Analysis concerning the convergence of discretization procedures which seem to be of different type but can uniformly be described by a property introduced here, called *continuous convergence of relations* . Historical examples demonstrate this standardization.

1 Introduction

As far as numerical procedures occur in Applied Analysis, i.e. replacement of a problem by a sequence of approximate problems, a fundamental property expected to be an inherent part of the methods is the convergence of the solutions of the approximate problems to a solution of the original problem under consideration.

This includes Approximation Theory, methods for the computation of approximate solutions of linear or nonlinear systems of ordinary or partial differential equations completed by initial and/or boundary conditions and by entropy conditions in the case of lost uniqueness, quadrature formulae etc.

Relations expected or guaranteed to be fulfilled by the solutions of the original problem have to be fulfilled in an at least similar manner by the approximate solutions. This leads to the necessity that the operators describing the original equations as well as additional relations known from the very beginning have to be approximated and have to converge in a suitable way to the original relations.

In this context, an abstract concept of so-called *continuous convergence* does occur in different situations of Numerical Analysis. This will here be demonstrated by means of historical examples.

2 Continuous Convergence

Roughly speaking, continuous convergnce of objects P_n acting on elements u^n $(n = 1, 2, \cdots)$ to an object P acting on an element u means the implcation

$$u^n \to u \quad \wedge \quad P_n u^n \to v \quad \Longrightarrow \quad Pu = v \quad . \tag{1}$$

We write

$$P_n \xrightarrow{c} P \ .$$

Continuous Convergence was at the first time introduced by du Bois-Reymond [2] in 1886 in a paper on the integration of series. Courant [3] used this concept 1914 in the theory of conformal mappings and Rinow ([13], p. 64) 1961 in connection with the convergence of sequences $\{C_n\}$ of continuous operators −mapping a metric space into a metric space− to a continuous limit operator C. Rinow proved the following theorem:

Theorem: (Rinow): *Let $(\mathcal{V},\rho)$ and $(\mathcal{W},\sigma)$ be metric spaces and $C:\mathcal{V}\to\mathcal{W}$ a continuous operator. Let $\{C_n|C_n:\mathcal{V}\to\mathcal{W}\}$ be a sequence of continuous operators. Then the following two properties are necessary and sufficient for $C_n \overset{c}{\to} C$:*

a) *the operators C_n are equicontinuous*

b) *$\{C_n\}$ converges pointwise to C on a subset $\tilde{\mathcal{V}}$ which is dense in $\mathcal{V}$.*

If $\mathcal{V}$ and $\mathcal{W}$ are Banach spaces and if the operators C and $C_n \quad (n=1,2,\cdots)$ are linear, equicontinuity of the operators C_n and the property of the operators to be uniformly bounded, coincide, and Rinow's theorem coincides with the Banach-Steinhaus theorem [1] so that in this case already pointwise convergence of $\{C_n\}$ to C on $\mathcal{V}$ implies continuous convergence.

Completeness or linearity of the spaces are not required for the validity of Rinow's theorem !

A particular application of the Banach-Steinhaus theorem is the Lax-Richtmyer theorem [9] on the convergence of consistent finite difference methods for linear partial evolution equations [1].

Another application of the Banach-Steinhaus theorem is the convergence (hence, even continuous convergence) of Gauss-type quadrature formulas.

Let us now generally assume that there is a metric space $\mathcal{V}$, an index set $\mathcal{J}$ and a relation R so that for ordered pairs $(u,\Phi)\in\mathcal{V}\times\mathcal{J}$ the question can be answered whether (u,Φ) belongs to $R\subset\mathcal{V}\times\mathcal{J}$ or not. If the relation is fulfilled, i.e. if $(u,\Phi)\in R$, we write

$$uR\Phi\,. \tag{2}$$

Moreover, assume that there are subsets $\mathcal{V}_n\subset\mathcal{V}$ $(n=1,2,\cdots)$ and relations R_n concerning ordered pairs $(u^n,\Phi)\in\mathcal{V}_n\times\mathcal{J}$ $(n=1,2,\cdots)$ so that the question arises whether or not u^n is related to Φ. If the answer is *yes*, i.e. if $(u^n,\Phi)\in R_n$, we write

$$u^nR_n\Phi\,. \tag{3}$$

Definition: We call the sequence $\{R_n\}$ of relations *continuously convergent* to the relation R with respect to the triple $(\mathcal{V},\{\mathcal{V}_n\},\mathcal{J})$ if the following implication holds:

$$\forall\Phi\in\mathcal{J}:\ \{u^n|u^n\in\mathcal{V}_n\ (n=1,2,\cdots)\,;\,u^nR_n\Phi\}\to u \quad\Longrightarrow\quad uR\Phi\,.$$

We then write

$$R_n\overset{c}{\to}R\,. \tag{4}$$

Trivial example:

Let $\mathcal{V}=\mathcal{R}$ (real numbers) and $\mathcal{J}=\{\Phi\}$ with a particular $\Phi\in\mathcal{R}$.

Define: $\quad uR\Phi\Longleftrightarrow u\le\Phi$.

[1] see subsection 3.2 of this paper

Let $\mathcal{V}_n \subset \mathcal{R} \quad (n = 1, 2, \cdots)$ and define R_n by

$$u^n R_n \Phi \quad \Longleftrightarrow \quad u^n \leq \Phi + \frac{1}{n} \ .$$

Then, obviously,

$$\{u^n \,|\, u^n R_n \Phi\} \to u \quad \text{leads to} \quad u \leq \Phi \,,$$

i.e. to

$$u R \Phi \ .$$

3 Realizations

If a problem to be treated consists in finding a solution u that fulfills uRf for given elements f of a certain set, if the approximate solutions u^n fulfill the relations $u^n R_n f \quad (n = 1, 2, \cdots)$, if −moreover− an existence theorem for the solution u is available as well as a convergence theorem $u^n \to u$ of the numerical method, $R_n \stackrel{c}{\to} R$ does only reflect this convergence theorem.

The next subsection shows an example:

3.1 Approximation Theory

Consider the situation $\mathcal{V} = C([0,1])$ equipped with the Tschebychef norm $\|u\|_\infty = \max\limits_{0 \leq t \leq 1} |u(t)|$.

Let $\mathcal{J} = \mathcal{V}$ and $\mathcal{V}_n = \mathcal{P}_n$ (polynomials of maximal degree n).

Define R by

$$uRf \quad \Longleftrightarrow \quad u \equiv f$$

and R_n by

$$u^n R_n f \quad \Longleftrightarrow \quad u^n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} f\left(\frac{\nu}{n}\right) x^\nu (1-x)^{n-\nu} \quad (n = 1, 2, \cdots) \ . \tag{5}$$

Here, the right hand side of (5) is monotone with respect to f for each fixed $n \in \mathcal{N}$, and because of

$$u^n(x) = \begin{cases} 1 & \text{for} \quad f(t) \equiv 1 \\ x & \text{for} \quad f(t) = t \\ x^2 + x\frac{1-x}{n} & \text{for} \quad f(t) = t^2 \end{cases} \ , \tag{6}$$

$$\lim_{n \to \infty} \|u^n - f\|_\infty = 0 \tag{7}$$

holds in all three cases.

But from Korovkin's theorem [7], (6) leads to the fact that (7) does not only hold in these special cases but **for all** $f \in \mathcal{J}$.

Hence,

$$\{u^n | u^n R_n f\} \to u$$

yields $u \equiv f$, i.e. uRf .

In other words, $R_n \xrightarrow{c} R$ in the context considered here is just another formulation of Korovkin's result by which the Weierstrass Approximation Theorem [14] follows immediately, and this ends this example.

Another situation is the following one:
Sometimes it can be shown that (4) holds provided that the numerical solutions u^n generated by a certain procedure represented by the requirement that u^n fulfills the relation $u^n R_n f$, converge to a limit $\tilde{u}$.

The proof of the validity of (4) then guarantees that $\tilde{u}$ is a solution u of the original problem which is represented by the requirement that u fulfills uRf.

In other words: The proof of $u^n \to \tilde{u}$ together with the proof of the validity of (4) guarantees the existence of a solution of the original problem (existence proof) as well as the convergene of the numerical procedure under consideration.

The next subsection shows an example:

3.2 Lax-Richtmyer Theory

Assume $\mathcal{B}$ to be a Banach space and $A : \mathcal{B} \to \mathcal{B}$ the *infinitesimal generator* of the semigroup of linear operators $E(t) : \mathcal{B} \to \mathcal{B}$ $(0 \le t \le T)$, i.e.: the one-parameter family

$$\{u(t) \mid 0 \le t \le T \; ; \; u(0) = u_0\}$$

generated by

$$u(t) = E(t)u_0 \quad (0 \le t \le T) \tag{8}$$

is the generalized unique solution of the linear autonomous initial value problem

$$u_t = Au \quad , \quad u(0) = u_0 \quad , \tag{9}$$

and this for arbitrary $u_0 \in \mathcal{B}$. Thus, the existence problem is already answered.

For each $t \in [0, T]$, let

$$\tilde{\mathcal{V}}(t) = \{\tilde{u}(t) = E(t)v \mid v \in \mathcal{B}\} \tag{10}$$

Let $\mathcal{J} = \mathcal{B}$ and let the initial value problem (9) now be discretized together with a suitable reconstruction procedure with respect to the spatial variable as far as such variables occur besides the time variable t. This reconstruction is assumed to extend the discrete approximations on each time level to the spatial inter grid points in such a way that the reconstructed functions become elements of $\mathcal{B}$. The spatial step sizes are chosen proportional to the (e.g. equidistant) time step size $h = \frac{T}{n}$ $(n = 0, 1, \cdots)$, and for convenience we restrict ourselves to a one-step finite difference equation

$$u_\nu^n = C(h)u_{\nu-1}^n = C^\nu(h)u_0 \quad (\nu = 1, 2, \cdots, n) \tag{11}$$

with $u_0^n = u_0 \quad \forall n \in \mathcal{N}_0$ and with $C(0) = I$ (identity).

The linear difference operators $C(h)$, including the reconstruction procedure, map $\mathcal{B}$ into $\mathcal{B}$, ν counts the time steps.

$u_\nu^n \in \mathcal{B}$ is expected to approximate $u(t_\nu)$ where $t_\nu = \nu h \quad (\nu \in \{0, 1, 2, \cdots, n\})$.

For each $\nu \in \{0, 1, \cdots, n\}$, let

$$\tilde{\mathcal{V}}_n(t_\nu) = \{\tilde{u}_\nu^n = C^\nu(h)v \mid v \in \mathcal{B}\} \tag{12}$$

and let the relation R between $\mathcal{V} := \bigcup_{0\le t\le T} \tilde{\mathcal{V}}(t)$ and $\mathcal{J}$ be defined by

$$\tilde{u}(t) R u_0 \quad \Longleftrightarrow \quad \tilde{u} = u \quad \text{fulfills (8)} \quad (0 \le t \le T) \tag{13}$$

as well as the relations R_n between $\mathcal{V}_n := \bigcup_{\nu=0,1,\cdots,n} \tilde{\mathcal{V}}_n(t_\nu)$ and $\mathcal{J}$ $(n = 0, 1, \cdots)$ by

$$\tilde{u}^n_\nu R_n u_0 \quad \Longleftrightarrow \quad \tilde{u}^n_\nu = u^n_\nu \quad \text{fulfills (11)} \quad , \quad (\nu = 0, 1, \cdots, n) \; . \tag{14}$$

Provided that $\{\tilde{u}^n_\nu = C^\nu(h)v \; , \; (\nu = 0, 1, \cdots, n)\}$ converges for $n \to \infty$, $\nu h \to t$ to a certain limit $\tilde{u}(t) \in \mathcal{B}$, and this for each fixed $t \in [0, T]$ and for each fixed $v \in \mathcal{B}$, the Banach-Steinhaus theorem yields

$$\exists \kappa < \infty \; : \; \|C^\nu(h)\| \le \kappa \; , \; \forall \, \nu \in \mathcal{N}_0 \; , \; \forall h \in [0, h_0] \tag{15}$$

with a certain $h_0 > 0$ and with $\nu h \le T$, i.e. *numerical stability* of the procedure (11).

In order to guarantee that $\tilde{u}(t)$ is the generalized solution (8), i.e. $\tilde{u}(t) = E(t)u_0$, one needs an additional property of the numerical procedure, namely that it is *consistent* with the original problem (9), i.e. : $R_n \overset{c}{\to} R$ finally follows from the proof that the procedure was constructed in such a way that

$$\left\| \left\{ E\left(\frac{T}{n}\right) - C\left(\frac{T}{n}\right) \right\} u \right\| = o\left(\frac{1}{n}\right) \quad (\text{for } n \to \infty) \quad , \quad \text{for each fixed } u \in \hat{\mathcal{V}} \; , \tag{16}$$

holds with a suitable set $\hat{\mathcal{V}} \underset{\text{dense}}{\subset} \mathcal{V}$.

There exist several generalizations of the Lax-Richtmyer theorem to non-linear problems [15], some of them by means of Rinow's theorem. In particular, finite difference methods for semi-linear evolution equations with Lipschitz continuous nonlinearity could be included so that Dahlquist's theorem [4] on the convergence of finite multistep difference methods for ordinary initial value problems became a special case of the generalized Lax-Richtmyer theory.

We end this example and are going to consider in the next subsection another situation, namely a nonlinear case where one only knows that a limit of the sequence $\{u^n\}$ is a (weak) solution of the given original problem **provided that** convergence of this sequence takes place.

3.3 Systems of Conservation Laws

We ask for vector valued solutions $\mathbf{V} = \mathbf{V}(x, t)$ of the genuinely nonlinear[2] system of conservation laws[3]

$$\begin{aligned} \partial_t \mathbf{V} + \partial_x \mathbf{f}(\mathbf{V}) = \mathbf{0} \quad &\text{on} \quad \Omega = \{(x, t) \,|\, x \in \mathcal{R} \, , \, t \in [0, T]\} \\ \mathbf{V}(x, 0) = \mathbf{V}_0(x) & \end{aligned} \tag{17}$$

where the flux function $\mathbf{f}$ is continuously differentiable. Classic existence theorems for nonlinear partial initial value problems guarantee the existence of unique genuine solutions only locally, i.e. in a certain neighbourhood of the initial manifold. But physicists, engineers etc. are often interested in global

[2] see e.g. [16], p. 86

[3] for convenience, we restrict ourselves to only one spatial variable

solutions, at least if the process modeled by the differntial equation problem and started from the initial values leads to a long lasting measurable effect.

Thus, a concept of *weak solutions* has to be created.

In order to establish a suitable definition of a weak solution, assume temporarily that there is a smooth genuine solution of the problem (17) on Ω. Put this solution into (17), then multiply (17) by an arbitrary *test function* $\Phi \in C_0^1(\Omega)$ (set of the functions continuously differentiable on Ω and with compact support).

Particularly, each of the functions Φ vanishes on the boundary of its support, possibly with the exception of such parts of this boundary which belong to the x-axis.

Now integrate by parts over Ω. This leads to

$$\int_{\Omega} \{\mathbf{V}\,\partial_t\,\Phi + \mathbf{f}(\mathbf{V})\,\partial_x\,\Phi\} d(x,t) + \int_{-\infty}^{\infty} \mathbf{V}_0(x)\,\Phi(x,0)\,dx = \mathbf{0}\ ,\ \forall\,\Phi \in C_0^1(\Omega)\ . \tag{18}$$

Let us now forget about the way we ended with formula (18), and let us – vice versa– ask for functions $\mathbf{V} = (v_1, v_2, \cdots, v_m)^T$ with

$$v_\mu \in L_1^{\text{loc}}(\Omega) \quad (\mu = 1, 2, \cdots, m)\ ,$$

which fulfill problem (18). Such functions will be called weak solutions of the original problem (17). They are no longer necessarily smooth: e.g. shocks can occur as it is well known from mathematical fluiddynamics.

This concept of weak solutions yields the advantage that the set of candidates for solutions can be extended considerably. But there is also a remarkable disadvantage: The uniqueness of the solutions is not guaranteed and, as a matter of fact, there are often more than one solution of problem (18).

Let $\mathcal{S}(\mathbf{V}_0)$ be the set solutions of (18) provided that the components of $\mathbf{V}_0$ are piecewise continuous. Obviously,

$$\mathcal{S}(\mathbf{V}_0) \subset \left(L_1^{\text{loc}}(\Omega)\right)^m := \mathcal{V}\ . \tag{19}$$

With

$$\|\mathbf{V}\|_{\mathcal{V}} := \max_{\mu=1,2,\cdots m} \|v_\mu\|_{L_1}$$

the space $\mathcal{V}$ becomes a normed space, where the L_1-norm is defined by

$$\|v_\mu\|_{L_1} = \int_K |v_\mu(x,t)|\,dx\,dt \tag{20}$$

with a sufficiently large compact rectangle including all points (x,t) of interest.

Assume R to be a relation between elements $\mathbf{V} \in \mathcal{V}$ and $\mathbf{V}_0 \in \mathcal{J}$ with

$$\mathcal{J} := \{\mathbf{V}_0\ :\ \mathcal{R} \to \mathcal{R}^m \,|\, \mathbf{V}_0 \text{ componentwise piecewise continuous}\}$$

where R is defined by

$$\mathbf{V} R \mathbf{V}_0 \quad \Longleftrightarrow \quad \mathbf{V} \in \mathcal{S}(\mathbf{V}_0)\ . \tag{21}$$

Again we try to solve (18) by a finite difference one-step method which again can be represented by a formula like (11), namely

$$\mathbf{V}^\nu = C^\nu(h)\,\mathbf{V}_0 \ .$$

But now, the operators $C(h)$ $(0 \leq h \leq h_0)$ are nonlinear and of conservation form (cf. e.g. [16] , p. 173). The method is assumed to be a **T**otal **V**ariation **D**eminishing method (TVD method; cf. [6]), consistent with (17) (cf. e.g. [16], p. 145), and the reconstruction rule consists in

$$\mathbf{V}^n(x,t) = \mathbf{V}^n(x_i, t_\nu) \quad \text{for} \quad x_i - \frac{\Delta x}{2} \leq x < x_i + \frac{\Delta x}{2} \quad , \quad t_\nu \leq t < t_{\nu+1} \tag{22}$$

$(i = 0, \pm 1, \pm 2, \cdots)$, $(\nu = 0, 1, 2, \cdots)$ with

$$\mathbf{V}^n(x,0) = \frac{1}{\Delta x} \int\limits_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} \mathbf{V}_0(\xi)\, d\xi \quad \text{for} \quad x_i - \frac{\Delta x}{2} \leq x < x_i + \frac{\Delta x}{2} \tag{23}$$

$(i = 0, \pm 1, \pm 2, \cdots)$. Here, Δx represents the step size in spatial direction and $x_i = i\,\Delta x$ $(i = 0, \pm 1, \pm 2, \cdots)$.

Denote by $\mathcal{S}_n(\mathbf{V}_0) \subset \left(L_1^{\text{loc}}(\Omega)\right)^m$ the set of solutions of the discretization method under consideration, and define the relations R_n between

$$\mathcal{V}_n = \{\mathbf{V}^n \,|\, \mathbf{V}^n \text{ piecewise constant on the intergrid points according to (22)}\} \subset \mathcal{V}$$

and $\mathcal{V}$ by

$$\mathbf{V}^n R_n \mathbf{V}_0 \quad \Longleftrightarrow \quad \mathbf{V}^n \in \mathcal{S}_n(\mathbf{V}_0) \quad . \tag{24}$$

The continuous convergence $R_n \stackrel{c}{\to} R$ then follows from the Lax-Wendroff theorem [10] provided that convergence $\mathbf{V}^n \to \mathbf{V}$ for a certain limit $\mathbf{V}$ takes place.

As a final realization of the concept of *Continuous convergence of relations* we are going to treat the occurence of entropy conditions.

3.4 Entropy Conditions

As already mentioned in subsection 3.3, weak solutions are often not unique. Moreover, the Lax-Wendroff theorem did not state the existence of a weak solution in this situation; it only guarantees that convergence of a suitable finite difference method −if convergence is ensured− leads to one of the weak solutions.

Thus, in situations like this, the question arises: Which of the weak solutions is the relevant one from the point of view of the application under consideration.

Particularly in Physics, systems of conservation laws normally include the First Main Theorem of Thermodynamics, namely the conservation of energy, but does not automatically also respect the Second Main Theorem, namely the fact that processes in closed systems behave in such a way that the entropy does not decrease. Consideration of this physical law leads to an additional inequality relation the relevant solution has to respect, and this relation, called the entropy condition, can be used in order to pick out of the set of weak solutions the physically relevant one, called the *entropy solution*. Olga Oleinik [12] and Peter Lax ([11], pp. 603 - 634) generalized this idea to any case where quasi-linear partial evolution equations of so-called conservation form occur.

In order to guarantee that a convergent discretization procedure for the numerical approximation of an entropy solution does really generate a sequence of approximations that converges for decreasing step sizes to the entropy solution and not to one of the other weak solutions, one often has also to discretize the additional entropy condition in a suitable way. This yields a sequence of *numerical entropy conditions*.

We are now going to describe in a first glance these connections in a more general setting [4].

Assume that there is a given problem

$$Au = v \tag{25}$$

where A maps a metric space $\mathcal{V}$ into a metric space $\mathcal{W}$ with a given right hand side $v \in \mathcal{W}$. We ask for solutions $u \in \mathcal{V}$ of problem (25) [5]. Let

$$\mathcal{S} = \{u \in \mathcal{V} \mid u \quad \text{solves} \quad (25)\} \quad . \tag{26}$$

Provided that $\mathcal{S}$ contains more than one element, we ask for a unique entropy solution $u_E \in \mathcal{S}$. For this reason, we take the ordered pairs $\{(u, \Phi) \,|\, u \in \int , \, \Phi \in \mathcal{J}\}$ into account where $\mathcal{J}$ is a suitable index set, and let R be a relation between elements of $\mathcal{S}$ and $\mathcal{J}$ so that there is **at most** one element in $\mathcal{S}$, denoted by u_E (if it exists), which fulfills the *entropy condition*

$$u_E R \Phi \quad , \quad \forall \, \Phi \in \mathcal{J} \quad . \tag{27}$$

Hence, the complete problem to be solved consists in finding a solution u_E of the problem (25) which fulfills additionally the relation (27).

Let $\mathcal{V}_n \subset \mathcal{V} \quad (n = 1, 2, \cdots)$ and $\{v^n \mid v^n \in \mathcal{V}_n \, , \, (n = 1, 2, \cdots)\} \subset \mathcal{V}$ a given sequence with

$$\lim_{n \to \infty} v^n = v \, . \tag{28}$$

The numerical procedure in order to solve (25) approximately then consists in the construction of operators $A_n \, : \, \mathcal{V}_n \to \mathcal{W} \quad , \quad (n = 1, 2, \cdots)$ so that each of the problems

$$A_n u^n = v^n \quad , \quad (n = 1, 2, \cdots) \tag{29}$$

has at least one solution [6], i.e.

$$\mathcal{S}_n := \{u^n \in \mathcal{V}_n \mid u^n \text{ solves } (26)\} \neq \emptyset \tag{30}$$

for each fixed $n \in \mathcal{N}$.

Here it can happen that $\mathcal{S}_n$ contains more than one element, e.g. if implicit finite difference equations combined with suitable reconstruction for the extension of the numerical solutions to the inter grid points are used.

Assume that the method (29) is convergent, i.e.

$$\{\mathcal{S}_n \mid (n = 1, 2, \cdots)\} \longrightarrow \mathcal{S} \tag{31}$$

[4]see also [17]

[5]*solution* can also mean *weak solution* of a given problem of which (25) is a weak formulation.

[6]Numerical equations which do not have a solution do not make sense.

in the sense of set-convergence. Set-convergence means in this context that each sequence

$$\{u^n \mid u^n \in \mathcal{S}_n \ , \ (n = 1, 2, \cdots)\} \subset \mathcal{V} \tag{32}$$

contains a convergent subsequence with its limit in $\mathcal{S}$ [7].

Theorem: *For each fixed $n \in \mathcal{N}$, let R_n be a relation between elements of $\mathcal{S}_n$ and of $\mathcal{J}$ and assume that*

- *there is at least one element $u_E^n \in \mathcal{S}_n$ with $u_E^n R_n \Phi \ , \ \forall \, \Phi \in \mathcal{J} \ , \ (n = 1, 2, \cdots)$*
- $R_n \xrightarrow{c} R$.

Then there is a unique entropy solution u_E of the problem {(25),(27)}, *and each full sequence*

$$\{u_E^n \mid u_E^n \in \mathcal{S}_n \ , \ (n = 1, 2, \cdots)\} \tag{33}$$

converges to u_E .

Proof: Because of the assumptions, there is at least one sequence of the type (33), and each of these sequences is of the type (32). Take one of the convergent subsequences of a particular type-(33)-sequence and denote its limit as u_E . $\mathcal{S}_n \to \mathcal{S}$ implies $u_E \in \mathcal{S}$, and $R_n \xrightarrow{c} R$ yields

$$u_E R \Phi \ , \ \forall \, \Phi \in \mathcal{J} \ .$$

But because there is at most ome element $u_E \in \mathcal{S}$ which fulfills (27), u_E is unique and does neither depend on the particular type-(33)-sequence nor on the special convergent subsequence of this sequence. And because of the uniqueness of u_E, not only subsequences converge to this limit but the full sequence behaves so.

Example:

Let us again consider a hyperbolic conservation law initial value problem, but for convenience we restrict ourselves to a one-dimensional scalar problem [8] on $\Omega = \{(x,t) \in \mathcal{R}^2 \mid -\infty < x < \infty \ , \ t \in [0,T]\}$ formulated in its weak form:

Find $u \in \mathcal{V} := L_1^{\text{loc}}(\Omega)$ with

$$\int\limits_{\Omega} [\partial_t \Phi(x,t) \, u(x,t) \ + \ \partial_x \Phi(x,t) \, f(u(x,t))] \, d\,\Omega \ + \int\limits_{\mathcal{R}} \Phi(x,0) \, u_0(x) \, d\,x = 0 \ , \quad \forall \Phi \in C_0^1(\Omega) \ . \tag{34}$$

The existence of weak solutions is given by a paper of Kruzhkov [8].

Let

$$\mathcal{J} := \left\{ \hat{\Phi} = (\Phi, c) \mid \Phi \in \, C_0^1(\Omega) \, , \, c \in \mathcal{R} \right\} \ .$$

Expect $u_\nu^n(x_i)$ to become an approximation of $u(i \, \Delta \, x \, , \, m \, h)$, $(i = 0, \pm 1, \pm 2, \cdots, n)$ by means of a general 3-point scheme

$$u_{\nu+1}^n(x_i) = u_\nu^n(x_i) - \sigma \, \{g(u_\nu^n(x_{i+1}), u_\nu^n(x_i)) - g(u_\nu^n(x_i), u_\nu^n(x_{i-1})\} \tag{35}$$

with the step sizes $\Delta \, x$ in spatial direction and h in time direction discretizing the underlying original problem

$$\partial_t \, u + \partial_x \, f(u) = 0 \ , \quad u(x,0) = u_0(x) \ , \tag{36}$$

[7] A convergence theorem that leads to (31) is at the same time an existence theorem for the original problem (25) but does not guarantee the uniqueness of the solution.

[8] It should be mentioned that also multi-dimensional problems are concerned

$f(u) \in C^1(\mathcal{R})$ nonnegative and strictly convex.

Here, the step size ratio $\sigma = \frac{h}{\Delta x}$ is assumed to be constant and to fulfill the CFL-*condition*

$$\sigma < \frac{1}{|f'|^*_\infty} \quad , \quad |f'|^*_\infty := \max\left\{|f'(u)| \ , \ \forall\, |u| \leq \|u_0\|_{L_\infty}\right\} \ .$$

The *numerical flux* $g(\cdot\,,\cdot) : \mathcal{R}^2 \to \mathcal{R}$ has to be constructed in such a way that it fulfills the *consistency condition*

$$g(v,v) = f(v) \quad , \quad \forall\, v \in \mathcal{R} \ ,$$

and the numerical initial values together with the reconstruction rule are chosen as in formulas (22), (23).

Now, let the entropy condition of the original problem (34) be described by means of the relation R between elements of $\mathcal{V}$ and $\mathcal{J}$, namely by

$$uR\hat{\Phi} \iff \int_\Omega \{\partial_t \Phi(x,t)\, V\,(u(x,t);c) + \partial_x\, \Phi(x,t)\, F\,(u(x,t);c)\}\; d\,\Omega + \int_{\mathcal{R}} \Phi(x,0)\, V(u_0(x);\, c)\, d\,x \geq 0 \ , \ \forall\, \hat{\Phi} \in \mathcal{J} \ . \tag{37}$$

Here, we choose a particular one-parameter family $\{V(\cdot;c)\,|\, c \in \mathcal{R}\}$ of real valued functions which are continuous, convex and piecewise differentiable with respect to x for each fixed $c \in \mathcal{R}$, e.g.

$$V(u;c) = |u - c| \ ,$$

called the *entropy functional*, and an *entropy flux* F that belongs to V as a solution of

$$\partial_x\, F(u(x,t);c) = -\partial_t\, V(u(x,t);c)$$

for all $c \in \mathcal{R}$ and for each smooth solution u of (36).

Kruzhkov did not only show that there is at least one solution of the problem (34) but that there is at most one of them which additionally fulfills the inequality relation (37).

Let us now introduce a *numerical flux function* $G \ : \ \mathcal{R}^3 \ \to \ \mathcal{R}$ by

$$G(\alpha, \beta; c) := F_+(\alpha\,, c) + F_-(\beta\,, c)$$

with

$$F_+(\alpha\,, c) = \begin{cases} f(\alpha) & \text{for} \quad \alpha \geq c \\ 0 & \text{for} \quad \alpha < c \end{cases} \quad , \quad F_-(\beta\,, c) = \begin{cases} 0 & \text{for} \quad \beta \geq c \\ -f(\beta) & \text{for} \quad \beta < c \end{cases} \quad .$$

If the numerical entropy conditions will then be introduced by

$$u^n R_n \hat{\Phi} \iff \int_\Omega \Phi(x,t) \left\{ \begin{array}{l} \frac{V(u^n(x,t+h);c) - V(u^n(x,t);c)}{h} \\ + \frac{G(u^n(x,t),u^n(x+\Delta x,t);c) - G(u^n(x-\Delta x,t),u^n(x,t);c)}{\Delta x} \end{array} \right\} d\,\Omega \leq 0 \ , \quad \forall\, \hat{\Phi} \in \mathcal{J} \ , \tag{38}$$

the assumptions of the Theorem of this section are fulfilled., i.e. there is a unique entropy solution of the problem (34), and each sequence which fulfills the inequality (38) converges to it, and there are sequences of this type (cf. [17]).

A well known method of this type is the monotone Engquist-Osher TVD-scheme [5] , where the TVD-property [6] (**T**otal-**V**ariation-**D**eminishing) imitates the fact that the entropy solution of the original problem (34) shows this TVD behaviour.

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