Hamburger Beiträge
zur Angewandten Mathematik

Mehdi Foroughi and Jens Struckmeier

An One-Dimensional Asymptotic Model
for Nonlinear Accoustics

Reihe A
Preprint 173
April 2003

Hamburger Beiträge zur Angewandten Mathematik

Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

An One–Dimensional Asymptotic Model for Nonlinear Accoustics

Mehdi Foroughi* and Jens Struckmeier
Department of Mathematics
University of Hamburg
Bundesstr. 55
D–20146 Hamburg
Germany

Abstract

We consider a simplified accoustic model to describe nonlinear phenomena occuring in loudspeakers. The simplification is that we restrict to the one–dimensional isentropic Euler equations in a slab, where on the right end a membrane is moving periodically with frequency ω and maximal displacement $\varepsilon \ll 1$. The asymptotic model based on the small parameter ε yields hyperbolic first order systems, which are investigated numerically for two different frequencies ω.

1 Introduction

Modern bass–reflex loudspeakers should combine a small bass-reflex enclosure together with a high power and an excellent sound quality. The design and optimization of such moving iron loudspeakers requires the accurate prediction of the generated sound fields. During the operation the change in the enclosure volume due to the moving membrane is often no longer negligible. In such cases, the linear wave equation is no longer an appropriate mathematical model.

To take into account nonlinear phenomena in the sound field one should go back to the fundamental equations from gas dynamics, like the Euler equations for the density, pressure and velocity of the enclosed gas. Nevertheless, numerical simulations for the nonlinear equations from gas dynamics, which include the moving membrane as a moving boundary, are quite difficult, because the maximal displacement of the membrane is still small compared to the typical dimension of the enclosure.

In the present work we consider a simplified model from gas dynamics, namely the one–dimensional isentropic Euler equations in a slab. On the right boundary of the slab there is a moving membrane with frequency ω and a maximal displacement $\varepsilon \ll 1$. Transforming this time–dependent domain to a fixed interval results in a modified nonlinear system for the density and velocity, which contains the small parameter ε.

An asymptotic expansion of this system yields a hierarchy of hyperbolic systems, which are coupled via nonlinear right hand sides. We show that the second order system exactly describes the first harmonic in the sound field and is therefore appropriate to include nonlinear phenomena.

*Grant sponsor: Fraunhofer–Institute for Industrial Mathematics, Kaiserslautern, Germany.

Our work is connected with the description of oscillations of a gas contained in a tube closed at one end and driven by a piston located at the other end, see for example the recently published work by Cox, Mortell and Reck [1] and the references given there; in particular, Klein and Peters [2]. An advantage of our approach might be that it can be easily extended to the multi–dimensional case.

2 The Acoustic Model

The starting point for our acoustic model are the isentropic Euler equations in one–space dimensions given by

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0 \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial x}p = 0 \tag{2.2}$$

with equation of state

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^\gamma, \tag{2.3}$$

Here, ρ, u and p denote the mass density, the velocity and pressure, respectively, which are functions of the space variable x and the time $t > 0$. The value γ in (2.3) defines the ratio of specific heat, ρ_0 und p_0 are the reference values for the density and the pressure. We consider the above system on the time–dependent domain $\Omega \subset \mathbb{R}$, with

$$\Omega = [0, 1 + \varepsilon h(t; \omega))$$

where $\varepsilon \ll 1$ denotes the maximal displacement of the moving membrane and $h(t; \omega)$ is the oscillatory movement with frequency ω.
Applying the transformation

$$x_1 = \frac{x}{1 + \varepsilon h(t; \omega)}, \quad t_1 = t,$$

the system above transform to

$$(1 + \varepsilon h)\, \rho_t - \varepsilon x \dot{h}\, \rho_x + (\rho u)_x = 0 \tag{2.4}$$

$$(1 + \varepsilon h)\, (\rho u)_t - \varepsilon x \dot{h}\, (\rho u)_x + (\rho u^2)_x + p_x = 0 \tag{2.5}$$

with $x \in [0, 1]$.

2.1 The asymptotic model

On the right boundary of Ω we have by Taylor–expansion

$$u(t, 1 + \varepsilon h) = u(t, 1) + \varepsilon h(t) u_x(t, 1) + O(\varepsilon^2) \tag{2.6}$$

On the other hand, assuming $\omega = O(1)$, the moving membrane yields

$$u(t, 1 + \varepsilon h) = \frac{d}{dt}(1 + \varepsilon h(t)) = \varepsilon \dot{h} = O(\varepsilon) \tag{2.7}$$

From (2.6) and (2.7) we obtain $u(1,t) = O(\varepsilon)$ and we assume for the following that $u = O(\varepsilon)$ on the whole domain.
Then the asymptotic expansions reads

$$\begin{aligned} \rho &= \rho^{(0)} + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + o(\varepsilon^3) \\ u &= \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + o(\varepsilon^3) \\ p &= p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + o(\varepsilon^3) \end{aligned}$$

Substituting the Ansatz into the system (2.4), (2.5) yields the zeroth order approximation

$$\rho_t^{(0)} = 0, \quad p_x^{(0)} = 0$$

i.e. the zeroth order density is independent of t, $\rho^{(0)} = \rho^{(0)}(x)$, and the pressure $p^{(0)}$ only depends on t, $p^{(0)} = p^{(0)}(t)$.
From the equation of state (2.3) we have by Taylor expansion

$$\begin{aligned} p &= p_0\rho_0^{-\gamma}\Big[\rho^{(0)\gamma} + \varepsilon\Big(\gamma\rho^{(0)\gamma-1}\rho^{(1)}\Big) \\ &\quad + \varepsilon^2\Big(\gamma\rho^{(0)\gamma-1}\rho^{(2)} + \frac{\gamma(\gamma-1)}{2}\rho^{(0)\gamma-2}\rho^{(1)2}\Big) + O(\varepsilon^3)\Big] \end{aligned}$$

from which we conclude that

$$p^{(0)} = p_0\rho_0^{-\gamma}\rho^{(0)\gamma} \tag{2.8}$$

$$p^{(1)} = \gamma p_0\rho_0^{-\gamma}\rho^{(0)\gamma-1}\rho^{(1)} \tag{2.9}$$

$$p^{(2)} = p_0\rho_0^{-\gamma}\Big(\gamma\rho^{(0)\gamma-1}\rho^{(2)} + \frac{\gamma(\gamma-1)}{2}\rho^{(0)\gamma-2}\rho^{(1)2}\Big) \tag{2.10}$$

2.2 First order system

From (2.8) it follows that the zeroth order density and pressure are constant in space and time and with $\rho^{(0)} = \rho_0$ relations (2.9), (2.10) simplify to

$$p^{(1)} = c^2\rho^{(1)} \tag{2.11}$$

$$p^{(2)} = c^2\rho^{(2)} + c^2\frac{\gamma-1}{2\rho_0}(\rho^{(1)})^2 \tag{2.12}$$

Then, the first order system reads

$$\frac{\partial}{\partial t}p^{(1)} + c^2\rho_0\frac{\partial}{\partial x}u^{(1)} = 0 \tag{2.13}$$

$$\frac{\partial}{\partial t}u^{(1)} + \frac{1}{\rho_0}\frac{\partial}{\partial x}p^{(1)} = 0 \tag{2.14}$$

where

$$c = \Big(\frac{\gamma p_0}{\rho_0}\Big)^{\frac{1}{2}}$$

denotes the sound speed.
The linear system (2.13), (2.14) is strictly hyperbolic with characteristics speeds $\lambda_{+/-} = \pm c$ and invariants $p + c\rho_0 u$ and $p - c\rho_0 u$, respectively.
From the moving membrane we have

$$u^{(1)}(t,1) = \dot{h}$$

On the left boundary we use the corresponding condition

$$u^{(1)}(t,0) = 0$$

One may even decouple the system into two homogeneous wave equations, namely

$$p^{(1)}_{tt} - c^2 p^{(1)}_{xx} = 0$$

and

$$u^{(1)}_{tt} - c^2 u^{(1)}_{xx} = 0$$

but then additional boundary conditions are necessary in order to define a well–posed problem.

2.3 Second order system

Collecting the terms of order ε^2 yields the inhomogeneous system

$$\begin{aligned} \rho^{(2)}_t + \rho_0 u^{(2)}_x &= -\Big(h\rho^{(1)}_t - x\dot{h}\rho^{(1)}_x + (\rho^{(1)}u^{(1)})_x\Big) \\ u^{(2)}_t + \frac{1}{\rho_0} p^{(2)}_x &= -\Big(hu^{(1)}_t + \frac{1}{\rho_0}(\rho^{(1)}u^{(1)})_t - x\dot{h}u^{(1)}_x + \Big((u^{(1)})^2\Big)_x\Big) \end{aligned}$$

and using (2.11), (2.12) we obtain

$$p^{(2)}_t + c^2\rho_0 u^{(2)}_x = \Big(x\dot{h} - u^{(1)}\Big)p^{(1)}_x + \left(\frac{\gamma}{c^2\rho_0}p^{(1)} - h\right)p^{(1)}_t \tag{2.15}$$

$$u^{(2)}_t + \frac{1}{\rho_0}p^{(2)}_x = \Big(x\dot{h} - u^{(1)}\Big)u^{(1)}_x - \Big(\frac{1}{c^2\rho_0}p^{(1)} + h\Big)u^{(1)}_t \tag{2.16}$$

The boundary conditions for the second order velocity are now

$$u^{(2)}(t,0) = u^{(2)}(t,1) = 0$$

i.e. a nonvanishing velocity is generated only by the inhomogeneous part of the system. The system (2.15), (2.16) may again written as a wave equation: differentiating the first equation with respect to t, the second with respect to x yields

$$p^{(2)}_{tt} - c^2 p^{(2)}_{xx} = c^2\rho_0\Big((u^{(1)})^2\Big)_{xx} - 2hp^{(1)}_{tt} + x\ddot{h}p^{(1)}_x + 2x\dot{h}p^{(1)}_{xt} + \frac{\gamma-1}{c^2\rho_0}\Big(p^{(1)}_{tt}p^{(1)} + (p^{(1)}_t)^2\Big)$$

A similar inhomogeneous wave equation may be derived for the second order velocity $u^{(2)}$.

2.4 Higher order systems and validity of the asymptotic expansion

The asymptotic procedure given above can be easily extended to higher order systems. The corresponding equations will have the same structure like Eqns. (2.15), (2.16), i.e. the left hand side is a linear hyperbolic system with characteristic speeds $\lambda_{+/-} = \pm c$ and a nonlinear inhomogeneous right hand side which couples the system with the lower order models.

We will see in the next section that the second order model corresponds to the first harmonic of the nonlinear accoustic model. Hence the higher order systems will describe the behaviour of the higher order harmonics, respectively.

Our asymptotic approach is certainly restricted to the case $\omega = O(1)$. If $\omega \ll 1$, e.g. $\omega = O(\varepsilon)$, then the moving membrane will give a velocity $u(t, 1 + \varepsilon h) = O(\varepsilon^2)$ and the asymptotic expansion will look different.

For $\omega \gg 1$, e.g. $\omega = O(1/\varepsilon)$, we will have $u(t, 1 + \varepsilon h) = O(1)$ and the asymptotic approach becomes more difficult, because alreads the zeroth system is nonlinear. The resulting equation are even more complicated than Euler's equation, because the model contains the moving membrane in the equations itself. The advantage remains that the model can be considered on a fixed spatial domain and it is worthwhile to consider this asymptotic case separately.

Even for $\omega = O(1)$ the current approach will break down in the long time behaviour when considering a resonant frequency ω. For this case it is known from linear theory that the resonance vibration will yield an amplitude which is linearly increasing in time. Hence, for $t = O(1/\varepsilon)$ the velocity $u(t, x)$ will no longer be of the order $O(\varepsilon)$.

3 Periodic Excitation

In this section we study numerically the behaviour of the first and second order pressure and velocity. We assume that the membrane moves periodically according to

$$h(t; \omega) = \cos(\omega t)$$

The numerical simulations use the finite–difference approach based on a Lax–Friedrich discretization of the flux terms [3]. For simplicity we use $c^2 = \rho_0 = 1$ and $\gamma = 5/3$. The simulations are performed with a grid spacing of 10^{-3} and time step $\Delta t = 5 \cdot 10^{-4}$, such that the CFL–condition is satisfied.

The first order system is given by the initial–boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t} p^{(1)} + c^2 \rho_0 \frac{\partial}{\partial x} u^{(1)} &= 0 \\ \frac{\partial}{\partial t} u^{(1)} + \frac{1}{\rho_0} \frac{\partial}{\partial x} p^{(1)} &= 0 \end{aligned}$$

with initial conditions $p^{(1)}(0, x) = u^{(1)}(0, x) = 0$ and boundary conditions

$$u^{(1)}(t, 0) = 0, \qquad u^{(1)}(t, 1) = -\omega \sin(\omega t)$$

The second order system reads

$$\begin{aligned} p_t^{(2)} + c^2 \rho_0 u_x^{(2)} &= -\Big(x\omega \sin(\omega t) + u^{(1)}\Big) p_x^{(1)} + \left(\frac{\gamma}{c^2 \rho_0} p^{(1)} - \cos(\omega t)\right) p_t^{(1)} \\ u_t^{(2)} + \frac{1}{\rho_0} p_x^{(2)} &= -\Big(x\omega \sin(\omega t) + u^{(1)}\Big) u_x^{(1)} - \Big(\frac{1}{c^2 \rho_0} p^{(1)} + \cos(\omega t)\Big) u_t^{(1)} \end{aligned}$$

with initial conditions $p^{(2)}(0, x) = u^{(2)}(0, x) = 0$ and boundary conditions

$$u^{(2)}(t, 0) = u^{(2)}(t, 1) = 0$$

3.1 Excitation frequency $\omega = 12$

Figures 3.1–3.4 show the first and second order pressure and velocity profiles at time $t = 1$ and $t = 2$. The moving membrane on the right end of the slab generates waves moving from right to the left with characteristic speed $c = -1$.
At time $t = 1$ the wave exactly arrives at the left end of the slab. Due to the numerical diffusion of the Lax–Friedrichs discretization the first order pressure wave already interacts with the left boundary, see Fig. 3.1.
The second order system yields pressure and velocity waves with doubled frequency and increasing amplitude when moving from right to the left. The amplitude close to the left boundary is two orders of magnitude larger than the excitation amplitude, see Fig 3.2. The double frequency belongs to the first harmonic, which seems to be strongly excited.

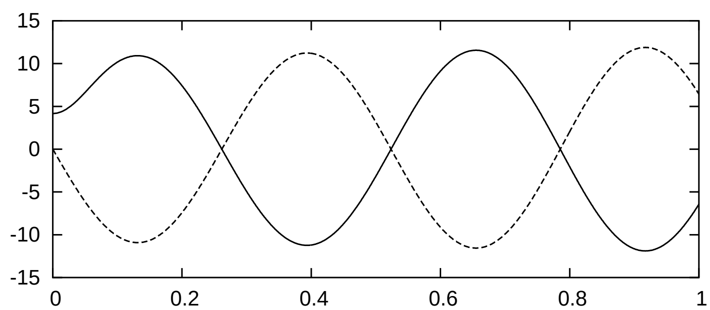

Fig. 3.1: 1st order pressure (solid), velocity (dash) at $t = 1$.

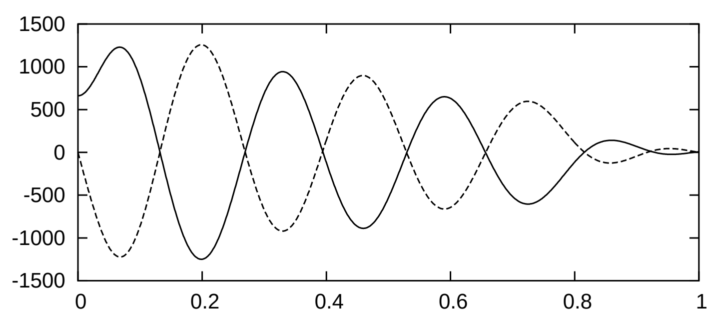

Fig. 3.2: 2nd order pressure (solid), velocity (dash) at $t = 1$.

At $t = 1$ the incoming waves are reflected at the left boundary and afterwards the whole system starts to oscillate. Figures 3.3 and 3.4 show the intermediate state of the system of first and second order, respectively, at time $t = 2$.
One observes a small phase shift between the pressure and velocity profiles. The second order system describes the behaviour of the first harmonic, where the amplitude is again two orders of magnitude larger than the one of the ground frequency.

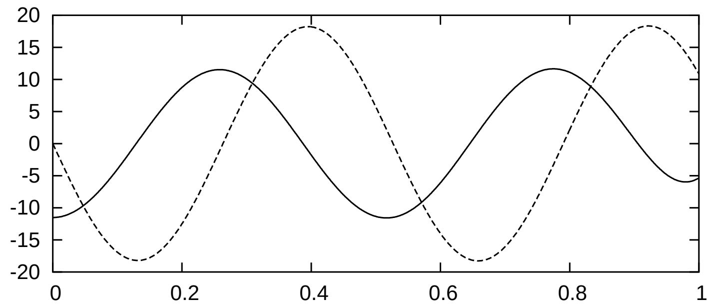

Fig. 3.3: 1st order pressure (solid), velocity (dash) at $t = 2$.

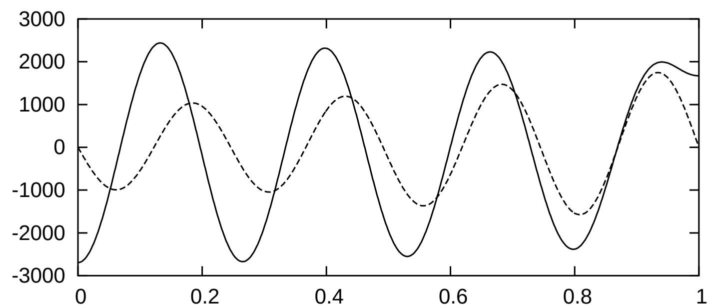

Fig. 3.4: 2nd order pressure (solid), velocity (dash) at $t = 2$.

3.2 Excitation frequency $\omega = \pi$

In the next example we consider the resonant frequency $\omega = \pi$. From linear theory one expects that the first order system generates a standing oscillation with an amplitude linearly increasing in time. This is exactly reproduced by our simulations shown in Figs. 3.5 and 3.6, where we show the intermediate state at $t = 1$ and $t = 2$. Morover we detect a phase shift between the pressure and the velocity wave of $\pi/4$.

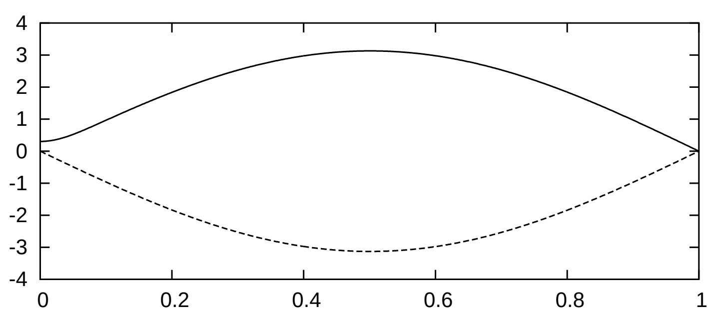

Fig. 3.5: 1st order pressure (solid), velocity (dash) at $t = 1$.

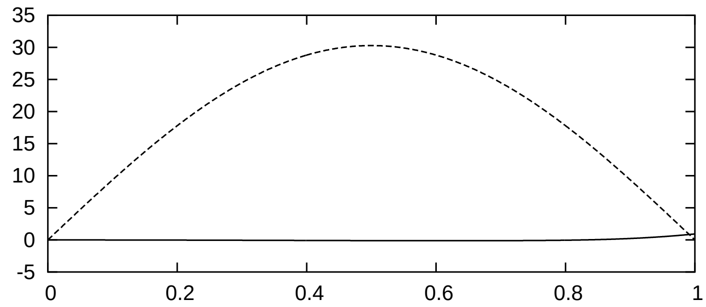

Fig. 3.6: 1st order pressure (solid), velocity (dash) at $t = 10$.

The numerical diffusion again results in a small deviation from the true behaviour of the pressure at the left boundary in Fig. 3.5 and the right boundary in Fig. 3.6, respectively.

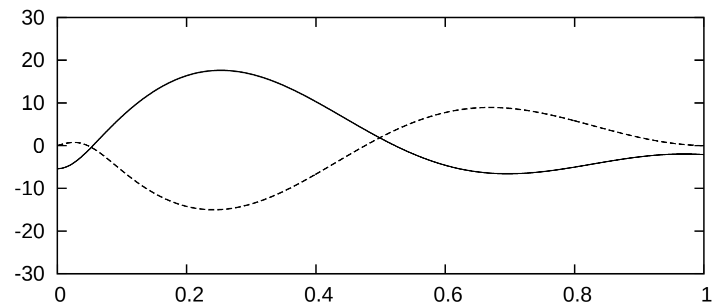

Fig. 3.7: 2nd order pressure (solid), velocity (dash) at $t = 1$.

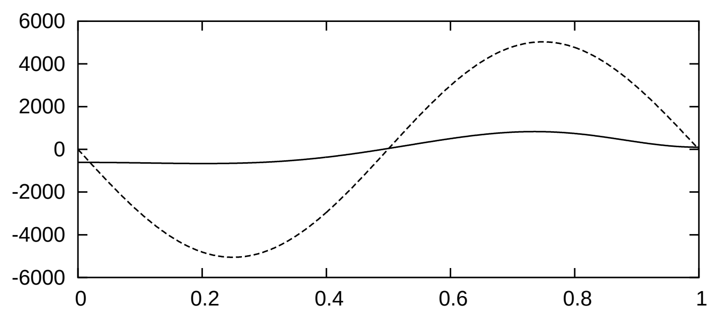

Fig. 3.8: 2nd order pressure (solid), velocity (dash) at $t = 10$.

The results for the second order system are shown in Figs. 3.7 and 3.8. After some initial phase, see Fig. 3.7, even the second order system produces a standing oscillation with an increasing amplitude in time. Fig 3.8 indicates that we again have the first harmonic, i.e. the oscillation has the frequency $\omega_1 = 2\pi$.

The phase shift between pressure and velocity profile is again about $\pi/4$, although the pressure at time $t = 10$ is not exactly in rest position, like in the first order system. This again might be due to some numerical defects.

time	1st order	2nd order
3	9.32	153
4	12.38	364
5	15.42	675
6	18.44	1161
7	21.43	1793
8	24.40	2655
9	27.35	3794
10	30.28	5032

Tab. 3.1: Amplitude of the velocity wave vs. time

Tab. 3.1 shows the velocity amplitudes between $t = 3$ and $t = 10$. Whereas the first order system shows the expected linear increase in time, the amplitude of the second order system increases cubically in time.

4 Conclusion

In the present paper we investigated a simplified nonlinear accoustic model based on the one–dimensional isentropic Euler equations from gas dynamics. The equations were considered in a slab, where on the right boundary there is a moving membrane with frequency ω and maximal displacement $\varepsilon \ll 1$. Transforming the moving domain to a fixed interval we derived nonlinear equations containing the small parameter ε.
An asymptotic expansion procedure was used to define a hierarchy of inhomogeneous hyperbolic equations, which describe the higher order harmonics in the sound field. Numerical simulations were performed for the first and second order system, which show a cubically increase in time of the amplitude of the first harmonic, when the membrane moves with a resonant frequency. In the non–resonant case the amplitude of the first harmonic was two orders of magnitude larger then the excitation amplitude, which indicates that the first harmonic is strongly excited.

Acknowledgement

The second author would like to thank Nicoletta Ianiro and the people from the Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Universita La Sapienza, Rome, Italy, for their kind hospitality, where parts of the research were done.

References

[1] E.A. Cox, M.P. Mortell and S. Reck, Nonlinear standing and resonantly forced oscillations in a tube with slowly changing length, *SIAM J. Appl. Math.*, Vol. 62, No. 3, 965–989, 2002.

[2] R. Klein, N. Peters, Cumulative effects of weak pressure waves during the induction period of a thermal explosion in a closed cylinder, *J. Fluid Mech.*, 187, 197–230, 1988.

[3] R.J. Leveque, *Numerical methods for conservation laws*, Basel, Birkhäuser, 1992.