DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 83-075
August 1983

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ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

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* Supported by the German Academic Exchange Service (DAAD).
** Supported by FINEP (Brasil) and FAPESP (São Paulo-Brasil).

Abstract

We introduce a Dirac-Kähler model for the two dimensional Wess-Zumino N=2 Lagrangean. We can show that in this model, when we go to the euclidean space-time lattice, we have no energy doubling, the action has no lattice surface terms (contrary to other authors), while the Hamiltonians (when time is continuous) present lattice surface terms.

The problem of constructing a supersymmetric theory on the lattice has been the subject of some recent articles /1, 2,3,4/ whith the special attention to the two dimensional Wess-Zumino theory. The authors of References 2,3 and 4 make use of the Nicolai mapping as a tool for the formulation of supersymmetric euclidean actions on the space-time lattice, while in References 1 and 4 use is made of geometric fermions, in the sense of Dirac-Kähler formalism, in order to obatin Hamiltonians on the space lattice.

The purpose of the present letter is to write down an explicit Dirac-Kähler (D-K) model for the supersymmetric N=2 Wess-Zumino (W-Z) two dimensional theory. We shall write down the Lagrangean and the corresponding differential equations in differential form. From these we obtain the conserved supersymmetric currents and extend these considerations to the space-time lattice. For this purpose we shall use extensively the formalism developped in Reference 5 on the D-K equation.

We have found that, in our D-K model, it is possible to have an action in the space-time lattice which is invariant under the supersymmetric transformations (as defined below) without the additional surface lattice terms which are present in the corresponding actions of References 2 and 4. However, our corresponding Hamiltonians, in order to preserve the invariance under the supersymmetric transformations, will necessarily contain the space lattice surface terms as in Reference 2 and 4.

We shall firstly formulate our model in the two dimensional Minkowski space with the mapping /5/

$$\gamma^{\mu} \longleftrightarrow dx^{\mu} \vee \qquad (1)$$

($\vee$= Clifford product), where we choose $\gamma^2 = \sigma_2$ and $\gamma^1 = i\sigma_1$ (σ_1, σ_2 being the Pauli matrices; our time index is denoted by 2). We will assume that in our D-K model the fermions are described by the differential forms:

$$\Psi = f_0 + \frac{f_{\mu\nu}}{2!} dx^{\mu} \wedge dx^{\nu} = f_0 + f_{12}\, dx^{12} \qquad (2)$$

$$\Psi^{*} = f_0^{*} + \frac{f_{\mu\nu}^{*}}{2!} dx^{\mu} \wedge dx^{\nu} = f_0^{*} + f_{12}^{*}\, dx^{12} \qquad (3)$$

with $\mu, \nu = 1, 2$, $dx^{12} = dx^1 \wedge dx^2$ and f_0^{*}, f_{12}^{*} denoting the complex conjugates of the Grassmann variables f_0 , f_{12} . We will impose the following equal time anticomutation relations:

$$\begin{aligned} \{f_0^{*}(x), f_0(y)\} &= \{f_{12}^{*}(x), f_{12}(y)\} = \frac{1}{2}\,\delta(x-y) \\ \{f_0, f_{12}\} &= \{f_0^{*}, f_{12}\} = \{f_0, f_{12}^{*}\} = \{f_0^{*}, f_{12}^{*}\} = 0 \end{aligned} \qquad (4)$$

The boson field is assumed to be described by the differential forms:

$$\Phi = \varphi_\mu dx^\mu \quad , \quad \Phi^* = \varphi^*_\mu dx^\mu \qquad (5)$$

where

$$\varphi_\mu = \partial_\mu \varphi \quad , \quad \varphi^*_\mu = \partial_\mu \varphi^* \qquad (6)$$

and φ^* is the complex conjugate of the complex scalar field φ.

We shall propose the following Lagrangean density:

$$\mathcal{L} = \mathcal{P}_0 \Big\{ \Phi^* \vee \Phi + 2i\ \Psi^* \vee (d-\delta) dx^2 \vee \Psi - W'^* W' -$$

$$- i\ \Psi^* \vee dx^2 \vee \Psi \vee (dx^{12}+1) \vee dx^2 W'' - i\ \Psi^* \vee dx^2 \vee \Psi \vee (dx^{12}-1) \vee dx^2 W''^* \Big\} \qquad (7)$$

where W depends only on φ and W^* only on φ^*, being assumed as zero forms.

$\mathcal{P}_0$ projects zero form components from the above Clifford products, and $W' = \frac{dW}{d\varphi}$, $W'' = \frac{d^2W}{d\varphi^2}$, $W'^* = \frac{dW^*}{d\varphi^*}$, $W''^* = \frac{d^2W^*}{d\varphi^{*2}}$ etc... .

It is easy to see that, by making the identification $\chi_1 = f_0 + f_{12}$, $\chi_2 = f_0 - f_{12}$, which are the components of a spinor in two dimensions, our Lagrangean (7) corresponds exactly to the usual N=2 Wess-Zumino model in two dimensions. Let us observe that these two components, in the notation of Reference 5, have different flavours.

From (7) we obtain the following D-K equations:

$$(d-\delta)\Phi = -W''^* W' - i\ \mathcal{P}_0 \left(\Psi^* \vee dx^2 \vee \Psi \vee (dx^{12}-1) \vee dx^2 \right) W'''^* \qquad (8)$$

$$(d-\delta)\Psi' = \frac{1}{2} dx^2 \vee \Psi \vee (dx^{12}+1) W'' + \frac{1}{2} dx^2 \vee \Psi \vee (dx^{12}-1) W''^* \qquad (9)$$

where we have introduced $\Psi' = dx^2 \vee \Psi \vee dx^2 = \mathcal{B}\Psi = f_0 - f_{12}\, dx^{12}$, $\mathcal{B}$ being the antiautomorphism defined in Ref. 5. In components, eq.(9) write as:

$$(\partial_1 + \partial_2)(f_0 + f_{12}) = -(f_0 - f_{12}) W''^*$$
$$(\partial_1 - \partial_2)(f_0 - f_{12}) = -(f_0 + f_{12}) W'' \qquad (10)$$

Consider now the 1 scalar product of Kähler /5/, constructed from the boson and fermi differential forms $(\Phi, \Psi')_1$.

By making use of the identity /5/:

$$d(\Phi, \Psi')_1 = \left((d-\delta)\Phi, \Psi' \right)_0 + \left(\Phi, (d-\delta)\Psi' \right)_0 \qquad (11)$$

it follows after use of eqs. (8) and (9):

$$\frac{1}{2} d \left((\Phi, \Psi')_1 + (\Psi', \Phi)_1 \right) = \Big(-W''^* W' f_0 + \frac{1}{2} (\varphi_1 + \varphi_2)(f_0 + f_{12}) W'' +$$

$$+ \frac{1}{2} (\varphi_1 - \varphi_2)(f_0 - f_{12}) W''^* \Big) \mathcal{E} \qquad (12)$$

where $\mathcal{E} = \frac{1}{2} \mathcal{E}_{\mu\nu}\, dx^\mu \wedge dx^\nu$ denotes the volume element (our convention is $\mathcal{E}_{12} = -\mathcal{E}_{21} = 1$).

Now our Lagrangean (7), or the corresponding equa_ tions (8) and (9) are invariant under the "dual transformations":

$$\Psi \to \tilde{\Psi} = \Psi V dx^2 \quad (\text{or } f_0 \leftrightarrow f_{12})$$

$$W'' \to -W'' \,, \quad W''^* \to -W''^* , \quad \varphi \to -\varphi , \quad \varphi^* \to -\varphi^* \qquad (13)$$

$$W' \to W' \quad , \quad W'^* \to W'^*$$

Therefore we obtain the dual relation of (12) by making the substitutions (13):

$$\frac{1}{2} d\left(\widetilde{(\Phi, \Psi')}_1 + \widetilde{(\Psi', \Phi)}_1 \right) = \left(W''^* W' f_{12} + \frac{1}{2}(\varphi_1+\varphi_2)(f_0+f_{12})W'' - \frac{1}{2}(\varphi_1-\varphi_2)(f_0-f_{12})W''^* \right)\varepsilon \qquad (14)$$

Adding expressions (12) and (14) we obtain:

$$d\left\{ \frac{1}{2}(\Phi,\Psi')_1 + \frac{1}{2}(\Psi',\Phi)_1 + \frac{1}{2}\widetilde{(\Phi,\Psi')} + \frac{1}{2}\widetilde{(\Psi',\Phi)}_1 - W'(\Psi' - \tilde{\Psi}')V dx^2 \right\} = 0 \qquad (15)$$

where use was made of the relation:

$$d\left(W'\Psi' V dx^2\right) - d\left(W'\tilde{\Psi}' V dx^2\right) = (f_0+f_{12})(\partial_1+\partial_2)W' - W'W''^*(f_0-f_{12}) \qquad (16)$$

which follows from the equation of motion (10).

From expression (15) it follows that we have a conserved supercurrent with components:

$$j_1 = (\varphi_1+\varphi_2)(f_0-f_{12}) + (f_0+f_{12})W'$$
$$j_2 = (\varphi_1+\varphi_2)(f_0-f_{12}) - (f_0+f_{12})W' \qquad (17)$$

If instead of Φ in expression (15) we use Φ^*, we shall obtain in a similar way a conserved supercurrent with components:

$$j_1' = (\varphi_1^*-\varphi_2^*)(f_0+f_{12}) + (f_0-f_{12})W'^*$$
$$j_2' = -(\varphi_1^*-\varphi_2^*)(f_0+f_{12}) + (f_0-f_{12})W'^* \qquad (18)$$

The complex conjugates of (17) and (18) also give conserved supercurrents with components j_1^*, j_2^* and $j_1'^*$, $j_2'^*$. Integrating over the space the time components of these currents we obtain the corresponding charges Q, Q', Q^* and Q'^*, which will induce on the field components (after the use of expressions (4) and the corresponding equal time commutators for the boson field) the following transformations:

by Q:

$$\delta\varphi = 0 \qquad \delta\varphi^* = -i(f_0-f_{12})$$
$$\delta f_0 = 0 \qquad \delta f_0^* = \frac{1}{2}(\varphi_1+\varphi_2) - \frac{1}{2}W' \qquad (19)$$
$$\delta f_{12} = 0 \qquad \delta f_{12}^* = -\frac{1}{2}(\varphi_1+\varphi_2) - \frac{1}{2}W'$$

by Q':

$$\delta'\varphi = -i(f_0+f_{12}) \qquad \delta'\varphi^* = 0$$
$$\delta' f_0 = 0 \qquad \delta' f_0^* = -\frac{1}{2}(\varphi_1^* - \varphi_2^*) + \frac{1}{2}w'^* \tag{20}$$
$$\delta' f_{12} = 0 \qquad \delta' f_{12}^* = -\frac{1}{2}(\varphi_1^* - \varphi_2^*) - \frac{1}{2}w'^*$$

by Q^*:

$$\delta^*\varphi = -i(f_0^* - f_{12}^*) \qquad \delta^*\varphi^* = 0$$
$$\delta^* f_0 = \frac{1}{2}(\varphi_1^* + \varphi_2^*) - \frac{1}{2}w'^* \qquad \delta^* f_0^* = 0 \tag{21}$$
$$\delta^* f_{12} = -\frac{1}{2}(\varphi_1^* + \varphi_2^*) - \frac{1}{2}w'^* \qquad \delta^* f_{12}^* = 0$$

by Q'^*:

$$\delta'^*\varphi = 0 \qquad \delta'^*\varphi^* = -i(f_0^* + f_{12}^*)$$
$$\delta'^* f_0 = -\frac{1}{2}(\varphi_1 - \varphi_2) + \frac{1}{2}w' \qquad \delta'^* f_0^* = 0 \tag{22}$$
$$\delta'^* f_{12} = -\frac{1}{2}(\varphi_1 - \varphi_2) - \frac{1}{2}w' \qquad \delta'^* f_{12}^* = 0$$

By going to the space-time lattice, our Lagrangean density (7) writes as:

$$\begin{aligned}\mathcal{L}_0 = {} & \Delta_\mu^+\varphi^*\Delta^{\mu+}\varphi + 2i f_0^*(\Delta_2^- f_0 + \Delta_1^- f_{12}) + 2i f_{12}^*(\Delta_1^+ f_0 + \Delta_2^+ f_{12}) - \\ & - w'^* w' - i\,[f_0^* f_0 - f_{12}^* f_{12} + f_0^* f_{12} - f_{12}^* f_0]\,w'' + \\ & + i\,[-f_0^* f_{12} + f_{12}^* f_0 + f_0^* f_0 - f_{12}^* f_{12}]\,w''^* \end{aligned} \tag{23}$$

This expression was obtained from (7) after the substitution $d \to \Delta^+$, $\delta \to \Delta^-$, where Δ^+ corresponds to the boun dary operator $\check{\Delta}$ and Δ^- to the coboundary operator $\check{\nabla}$ of Refe rence 5.

The equations of motion which follow from (23) are:

$$\Delta_2^- f_0 + \Delta_1^- f_{12} = \frac{1}{2}\left[(f_0+f_{12})w'' - (f_0 - f_{12})w''^*\right] \tag{24a}$$
$$\Delta_1^+ f_0 + \Delta_2^+ f_{12} = \frac{1}{2}\left[-(f_0+f_{12})w'' - (f_0 - f_{12})w''^*\right] \tag{24b}$$
$$\Delta_2^+ f_0^* + \Delta_1^- f_{12}^* = \frac{1}{2}\left[-(f_0^* - f_{12}^*)w'' + (f_0^* + f_{12}^*)w''^*\right] \tag{24c}$$
$$\Delta_1^+ f_0^* + \Delta_2^- f_{12}^* = \frac{1}{2}\left[-(f_0^* - f_{12}^*)w'' - (f_0^* + f_{12}^*)w''^*\right] \tag{24d}$$

$$\Delta_\mu^- \Delta^{\mu+}\varphi^* = -w'^* w'' - i\,[\quad]\,w''' \tag{25a}$$
$$\Delta_\mu^- \Delta^{\mu+}\varphi = -w''^* w' + i\,[\quad]\,w'''^* \tag{25b}$$

Where the $[\ ]$ which multiplies w''' in (25a) is the same which multiplies w'' in expression (23). Correspondingly the $[\ ]$ which multiplies w'''^* in (25b) is the same which multiplies w''^* in (23).

The supertransformations which correspond to Q, Q', Q^* and Q'^* in the space-time lattice will be defined by the transformations (19), (20), (21), (22) respectively where we make the substitution $\varphi_1 \to \Delta_1^+\varphi$ and $\varphi_2 \to \Delta_2^+\varphi$.

It can be seen, fron eqs. (24a)-(24d) and (25), that when we go to the euclidean space we have no energy doubling.

In order to study the transformation properties of the action and the Hamiltonian under supertransformations, the expressions $[\ ]w''$ and $[\ ]w''^*$ which appear in expression (23) should be taken symmetrically, that is $\frac{1}{2}[\ \ ]w''+\frac{1}{2}w''[\ \ ]$ and $\frac{1}{2}[\ \ ]w''^{*}+\frac{1}{2}w''^{*}[\ \ ]$. In equations (25) we shall have a corresponding symmetrization. With this symmetrization in mind, we can show that the action which corresponds to the Lagrangean density (23) is invariant under the supertransformations (19)-(22), as defined above in the space-time lattice without any addition of further surface lattice terms. This is contrary to previous statements of References 2 and 4, which present surface lattice terms in the action.

By making now the time continuous, keeping only the space lattice, than the supertransformations (19)-(22), with $\varphi_1 = \Delta_1^+\varphi$, are now consequences of the form of the charges Q , Q', Q^* and Q'^* and the equal time canonical commutation relations for the boson field and the relations (4) for the fermion field. In this case it is possible to define an Hamiltonian corresponding to the Lagrangean density (23) given by:

$$H = \sum_x \mathcal{H} = \sum_x \mathcal{H}_B + \sum_x \mathcal{H}_F \qquad (26)$$

where the sum extends over the lattice points and where

$$\mathcal{H}_B = (\Delta_1^+A)^2 + (\partial_2 A)^2 + (\Delta_1^+B)^2 + (\partial_2 B)^2 + U^2 + V^2 \qquad (27a)$$

$$\mathcal{H}_F = -4i\alpha_0\,\Delta_1^-\alpha_{12} - 4i\beta_0\,\Delta_1^-\beta_{12} + 4i\frac{\delta U}{\delta A}\alpha_0\alpha_{12} - 4i\frac{\delta V}{\delta A}\alpha_0\beta_0 + 4i\frac{\delta U}{\delta B}\beta_{12}\alpha_{12} - 4i\frac{\delta V}{\delta B}\beta_{12}\beta_0 \qquad (27b)$$

where we introduced the definitions: $\varphi = A+iB$, $f_0=\alpha_0+i\beta_0$, $f_{12}=\alpha_{12}+i\beta_{12}$, $W'=U+iV$. The Hamiltonian (26) is not any more invariant under the supersymmetric transformations (19)-(22) in the space lattice, giving rise to different lattice surface terms. Separating the supercharges Q , Q' , Q^* , Q'^* into its real and imaginary parts $Q=Q_1+iQ_2$, $Q'=Q'_1+iQ'_2$, and defining $q_1=Q_1+Q'_1$, $q_2=Q_2+Q'_2$, $q_3=Q_1-Q'_1$, $q_4=Q_2-Q'_2$ its possible to show that:

$$2q_1^2 = 2q_2^2 = \sum_x \left(\mathcal{H} + U\Delta_1^+A - V\Delta_1^+B\right) \qquad (28a)$$

$$2q_3^2 = 2q_4^2 = \sum_x \left(\mathcal{H} - U\Delta_1^+A + V\Delta_1^+B\right) \qquad (28b)$$

We see therefore that in expressions (28a) and (28b) we have two different classes of Hamiltonian on the space lattice, differing in the surface lattice terms. Going to the space continuum these Hamiltonians coincide.

In the space lattice it holds:

$$\{q_1,q_4\} = -\{q_2,q_3\} = \sum_x \left(V(x)\,\Delta_1^+A(x) + U(x)\,\Delta_1^+B(x)\right) \qquad (29)$$

which goes to zero in the continuum space. In this case we have also:

$$\{q_1,q_3\} = \{q_2,q_4\} = P \qquad (30)$$

where P is the momentum operator of the system. Introducing now $Q_1^1 = q_1 - q_3$, $Q_2^1 = q_1 + q_3$, $Q_1^2 = q_2 - q_4$, $Q_2^2 = q_2 + q_4$, in the continuum space-time, (28a), (28b), (29) and (30) can be written as:

$$\{Q_\alpha^a, Q_\beta^b\} = 2\,\delta^{ab}\left(\delta_{\alpha\beta} H - (\sigma_3)_{\alpha\beta} P\right) \qquad (31)$$

, $a, b = 1, 2$ and $\alpha, \beta = 1, 2$. Finally, in a way similar to Reference 2, we can consider the operator $T = \sum_x \left(\alpha_0(x)\beta_0(x) + \alpha_{12}(x)\beta_{12}(x)\right)$. It is simple to see that $[T, q_1] = -q_2$, $[T, q_2] = q_1$, and therefore T is the generator of an O_2 group in the space (q_1, q_2). Similarly T is also a generator of an O_2 group in the space (q_3, q_4) , since $[T, q_3] = -q_4$ and $[T, q_4] = q_3$. Therefore the two classes (q_1, q_2) and (q_3, q_4) are on the same footing, although in the space lattice we cannot consider them simultaneously because of expression (29). Since T makes the changes $\alpha_0 \to \beta_0$, $\beta_0 \to -\alpha_0$, $\alpha_{12} \to \beta_{12}$ and $\beta_{12} \to -\alpha_{12}$, we see that it leaves the Hamiltonians (28a) and (28b) in the space lattice invariant.

In conclusion we have succeed in constructing a D-K model, for the two dimensional Wess-Zumino N=2 Lagrangean, which when extended to the space-time lattice presents some nice features. Evidently other D-K models can be constructed for the N=2 two dimensional Wess-Zumino Lagrangean and which will present similar interesting properties.

Acknowledgements:

We thank Prof. H.Joos for the suggestion of the topic of the present research, for his constant advice and interest. To the Desy Theory Group our thanks for their kind hospitality.

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